Dynamic System Response

- Solution of Linear, Constant-Coefficient, Ordinary Differential Equations
  - Classical Operator Method
  - Laplace Transform Method
- Laplace Transform Properties
- 1st-Order Dynamic System Time and Frequency Response
- 2nd-Order Dynamic System Time and Frequency Response

Laplace Transform Methods

- A basic mathematical model used in many areas of engineering is the linear ordinary differential equation with constant coefficients:

$$a_n \frac{d^n q_o}{dt^n} + a_{n-1} \frac{d^{n-1} q_o}{dt^{n-1}} + \cdots + a_1 \frac{dq_o}{dt} + a_0 q_o =$$

$$b_m \frac{d^m q_i}{dt^m} + b_{m-1} \frac{d^{m-1} q_i}{dt^{m-1}} + \cdots + b_1 \frac{dq_i}{dt} + b_0 q_i$$

- q_o is the output (response) variable of the system
- q_i is the input (excitation) variable of the system
- a_n and b_m are the physical parameters of the system

- Straightforward analytical solutions are available no matter how high the order n of the equation.
- Review of the classical operator method for solving linear differential equations with constant coefficients will be useful. When the input $q_i(t)$ is specified, the right hand side of the equation becomes a known function of time, $f(t)$.
- The classical operator method of solution is a three-step procedure:
  - Find the complimentary (homogeneous) solution q_{oc} for the equation with $f(t) = 0$.
  - Find the particular solution q_{op} with $f(t)$ present.
  - Get the complete solution $q_o = q_{oc} + q_{op}$ and evaluate the constants of integration by applying known initial conditions.

- *Step 1*
  - To find q_{oc}, rewrite the differential equation using the differential operator notation $D = d/dt$, treat the equation as if it were algebraic, and write the system characteristic equation as:

$$a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0 = 0$$

  - Treat this equation as an algebraic equation in the unknown *D* and solve for the *n* roots (eigenvalues) $s_1, s_2, ..., s_n$. Since root finding is a rapid computerized operation, we assume all the roots are available and now we state rules that allow one to immediately write down q_{oc}:
  - Real, unrepeated root s_1:

$$q_{oc} = c_1 e^{s_1 t}$$

  - Real root s_2 repeated *m* times:

$$q_{oc} = c_0 e^{s_2 t} + c_1 t e^{s_2 t} + c_2 t^2 e^{s_2 t} + \cdots + c_m t^m e^{s_2 t}$$

- When the *a's* are real numbers, then any complex roots that might appear always come in pairs $a \pm ib$:

$$q_{oc} = ce^{at}\sin(bt + \phi)$$

- For repeated root pairs $a \pm ib$, $a \pm ib$, and so forth, we have:

$$q_{oc} = c_0 e^{at}\sin(bt + \phi_0) + c_1 t e^{at}\sin(bt + \phi_1) + \cdots$$

- The *c's* and *ϕ's* are constants of integration whose numerical values cannot be found until the last step.

- *Step 2*
  - The particular solution q_{op} takes into account the "forcing function" *f(t)* and methods for getting the particular solution depend on the form of *f(t)*.
  - The method of undetermined coefficients provides a simple method of getting particular solutions for most *f(t)'s* of practical interest.
  - To check whether this approach will work, differentiate *f(t)* over and over. If repeated differentiation ultimately leads to zeros, or else to repetition of a finite number of different time functions, then the method will work.
  - The particular solution will then be a sum of terms made up of each different type of function found in *f(t)* and all its derivatives, each term multiplied by an unknown constant (undetermined coefficient).

- If *f(t)* or one of its derivatives contains a term identical to a term in q_{oc}, the corresponding terms should be multiplied by *t*.
- This particular solution is then substituted into the differential equation making it an identity. Gather like terms on each side, equate their coefficients, and obtain a set of simultaneous algebraic equations that can be solved for all the undetermined coefficients.

- ***Step 3***
  - We now have q_{oc} (with *n* unknown constants) and q_{op} (with no unknown constants). The complete solution $q_o = q_{oc} + q_{op}$. The initial conditions are then applied to find the *n* unknown constants.

- Certain advanced analysis methods are most easily developed through the use of the Laplace Transform.
- A transformation is a technique in which a function is transformed from dependence on one variable to dependence on another variable. Here we will transform relationships specified in the time domain into a new domain wherein the axioms of algebra can be applied rather than the axioms of differential or difference equations.
- The transformations used are the Laplace transformation (differential equations) and the Z transformation (difference equations).
- The Laplace transformation results in functions of the time variable *t* being transformed into functions of the frequency-related variable *s*.

- The Z transformation is a direct outgrowth of the Laplace transformation and the use of a modulated train of impulses to represent a sampled function mathematically.
- The Z transformation allows us to apply the frequency-domain analysis and design techniques of continuous control theory to discrete control systems.
- One use of the Laplace Transform is as an alternative method for solving linear differential equations with constant coefficients. Although this method will not solve any equations that cannot be solved also by the classical operator method, it presents certain advantages:
  - Separate steps to find the complementary solution, particular solution, and constants of integration are not used. The complete solution, including initial conditions, is obtained at once.

- There is never any question about which initial conditions are needed. In the classical operator method, the initial conditions are evaluated at $t = 0^+$, a time just after the input is applied. For some kinds of systems and inputs, these initial conditions are not the same as those before the input is applied, so extra work is required to find them. The Laplace Transform method uses the conditions before the input is applied; these are generally physically known and are often zero, simplifying the work.
- For inputs that cannot be described by a single formula for their entire course, but must be defined over segments of time, the classical operator method requires a piecewise solution with tedious matching of final conditions of one piece with initial conditions of the next. The Laplace Transform method handles such discontinuous inputs very neatly.
- The Laplace Transform method allows the use of graphical techniques for predicting system performance without actually solving system differential equations.

- All theorems and techniques of the Laplace Transform derive from the fundamental definition for the direct Laplace Transform *F(s)* of the time function *f(t)*:

$$L\left[f(t)\right] = F(s) = \int_0^\infty f(t)e^{-st}dt \quad t > 0$$

$$s = \text{ a complex variable } = \sigma + i\omega$$

- This integral cannot be evaluated for all *f(t)'s*, but when it can, it establishes a unique pair of functions, *f(t)* in the time domain and its companion *F(s)* in the *s* domain. Comprehensive tables of Laplace Transform pairs are available. Signals we can physically generate always have corresponding Laplace transforms. When we use the Laplace Transform to solve differential equations, we must transform entire equations, not just isolated *f(t)* functions, so several theorems are necessary for this.

- Linearity Theorem:

$$L\left[a_1 f_1(t) + a_2 f_2(t)\right] = L\left[a_1 f_1(t)\right] + L\left[a_2 f_2(t)\right] = a_1 F_1(s) + a_2 F_2(s)$$

- Differentiation Theorem:

$$L\left[\frac{df}{dt}\right] = sF(s) - f(0)$$

$$L\left[\frac{d^2 f}{dt^2}\right] = s^2 F(s) - sf(0) - \frac{df}{dt}(0)$$

$$L\left[\frac{d^n f}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2}\frac{df}{dt}(0) - \cdots - \frac{d^{n-1} f}{dt^{n-1}}(0)$$

  – *f(0), (df/dt)(0),* etc., are initial values of *f(t)* and its derivatives evaluated numerically at a time instant *before* the driving input is applied.

- Integration Theorem:

$$L\left[\int f(t)dt\right] = \frac{F(s)}{s} + \frac{f^{(-1)}(0)}{s}$$

$$L\left[f^{(-n)}(t)\right] = \frac{F(s)}{s^n} + \sum_{k=1}^{n} \frac{f^{(-k)}(0)}{s^{n-k+1}}$$

where $f^{(-n)}(t) = \int \cdots \int f(t)(dt)^n$ and $f^{(-0)}(t) = f(t)$

  – Again, the initial values of *f(t)* and its integrals are evaluated numerically at a time instant *before* the driving input is applied.

- Delay Theorem:
  - The Laplace Transform provides a theorem useful for the dynamic system element called *dead time (transport lag)* and for dealing efficiently with discontinuous inputs.

$$\left.\begin{aligned} u(t) &= 1.0 \Rightarrow t > 0 \\ u(t) &= 0 \Rightarrow t < 0 \\ u(t-a) &= 1.0 \Rightarrow t > a \\ u(t-a) &= 0 \Rightarrow t < a \end{aligned}\right\} \quad L\left[f(t-a)u(t-a)\right] = e^{-as}F(s)$$

- Final Value Theorem:
  - If we know $Q_0(s)$, $q_0(\infty)$ can be found quickly without doing the complete inverse transform by use of the *final value theorem.*

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

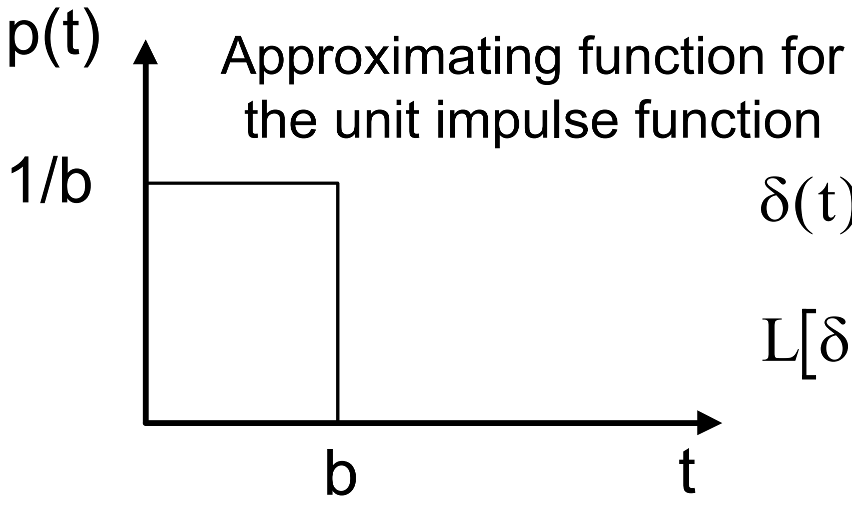
  - This is true if the system and input are such that the output approaches a constant value as t approaches ∞.
- Initial Value Theorem:
  - This theorem is helpful for finding the value of *f(t)* just after the input has been applied, i.e., at $t = 0^+$. In getting the *F(s)* needed to apply this theorem, our usual definition of initial conditions as those *before* the input is applied must be used.

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

- Impulse Function $\delta(t)$

$$\delta(t) = 0 \Rightarrow t \neq 0$$

$$\int_{-\varepsilon}^{+\varepsilon} \delta(t)dt = 1 \Rightarrow \varepsilon > 0$$



$$\delta(t) = \lim_{b \to 0} p(t)$$

$$L[\delta(t)] = L\left[\frac{du}{dt}\right] = sU(s) = s\frac{1}{s} = 1.0$$

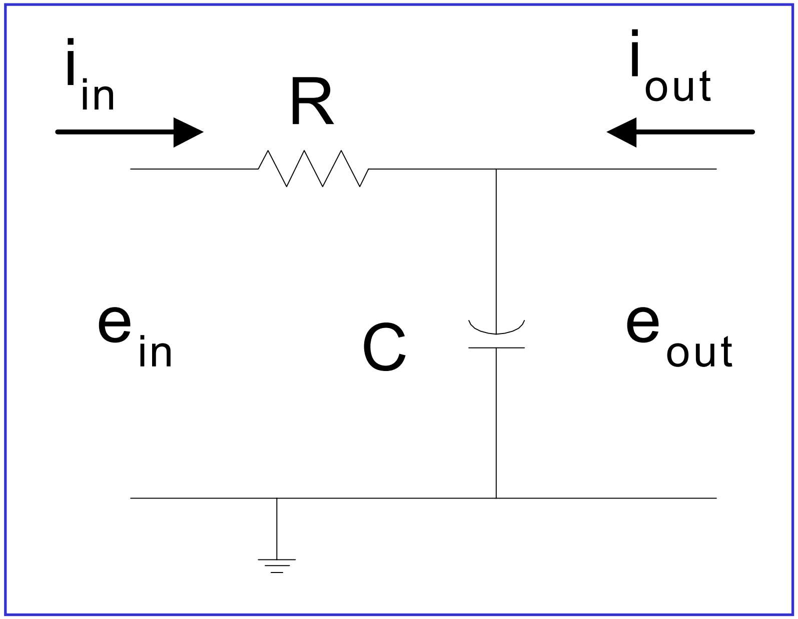
- The step function is the integral of the impulse function, or conversely, the impulse function is the derivative of the step function.
- When we multiply the impulse function by some number, we increase the "strength of the impulse", but "strength" now means area, not height as it does for "ordinary" functions.

- An impulse that has an infinite magnitude and zero duration is mathematical fiction and does not occur in physical systems.
- If, however, the magnitude of a pulse input to a system is very large and its duration is very short compared to the system time constants, then we can approximate the pulse input by an impulse function.
- The impulse input supplies energy to the system in an infinitesimal time.

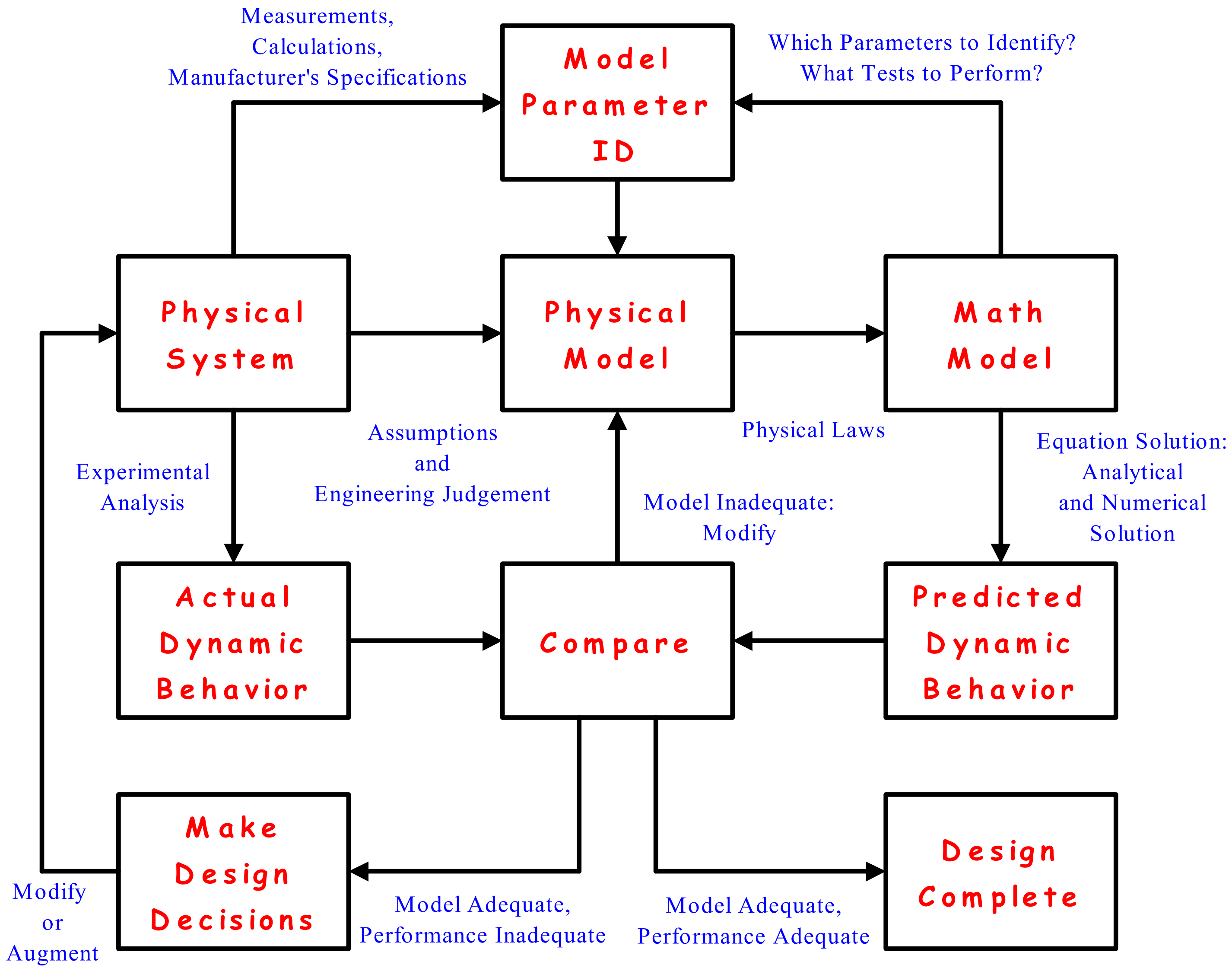
- Inverse Laplace Transformation
  - A convenient method for obtaining the inverse Laplace transform is to use a table of Laplace transforms. In this case, the Laplace transform must be in a form immediately recognizable in such a table.
  - If a particular transform F(s) cannot be found in a table, then we may expand it into partial fractions and write F(s) in terms of simple functions of *s* for which inverse Laplace transforms are already known.
  - These methods for finding inverse Laplace transforms are based on the fact that the unique correspondence of a time function and its inverse Laplace transform holds for any continuous time function.

Mechatronics
Time Response & Frequency Response
1st-Order Dynamic System

Example: RC Low-Pass Filter



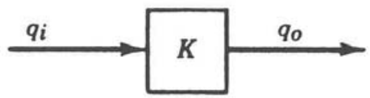
Dynamic System Investigation
of the
RC Low-Pass Filter



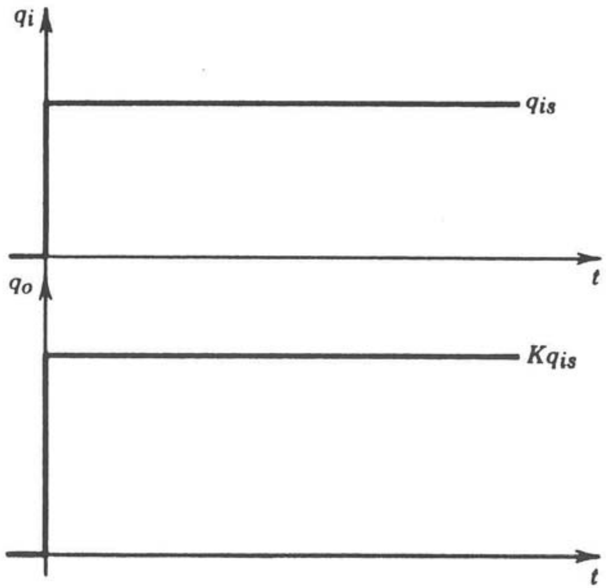
Dynamic System Investigation

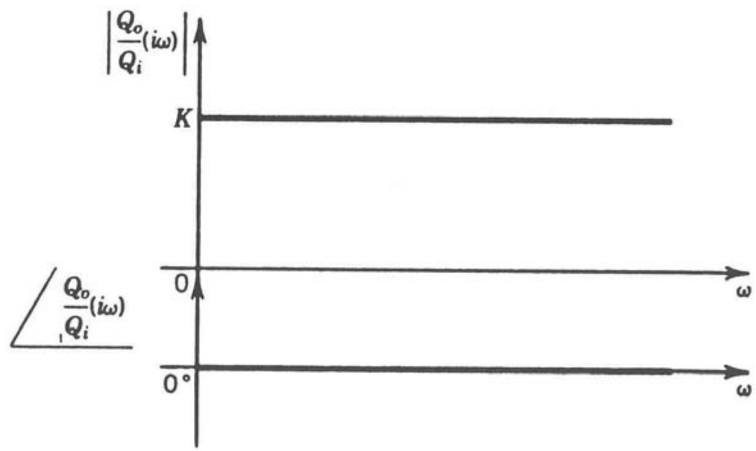
Zero-Order Dynamic System Model

$$q_o = K\,q_i$$

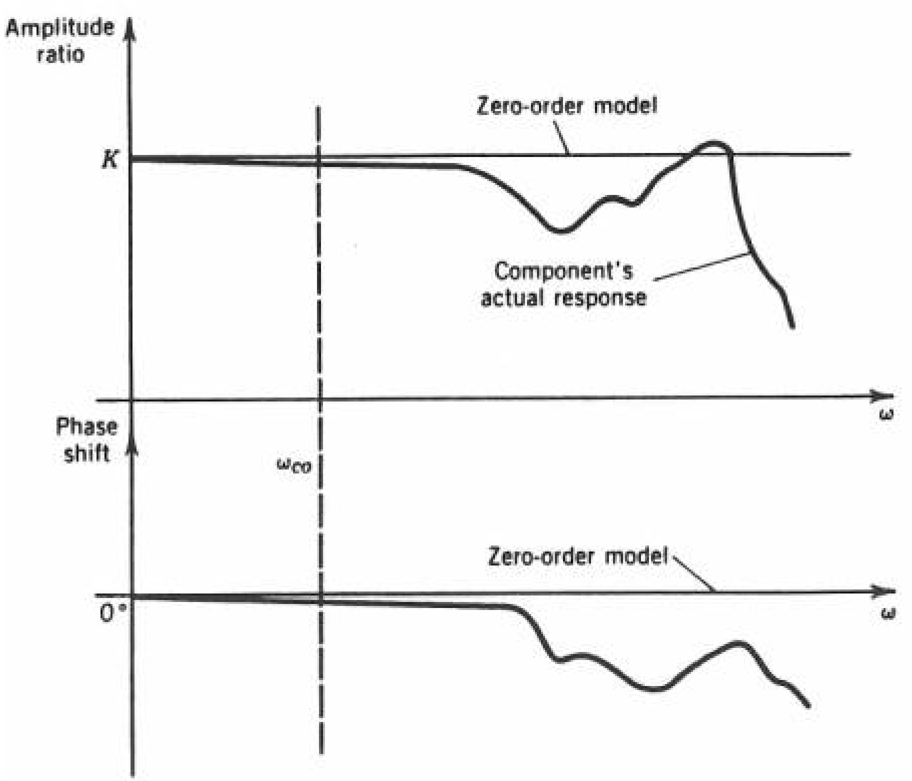


$$\frac{Q_o}{Q_i}(s) = K \qquad \frac{Q_o}{Q_i}(i\omega) = K\angle 0^\circ$$





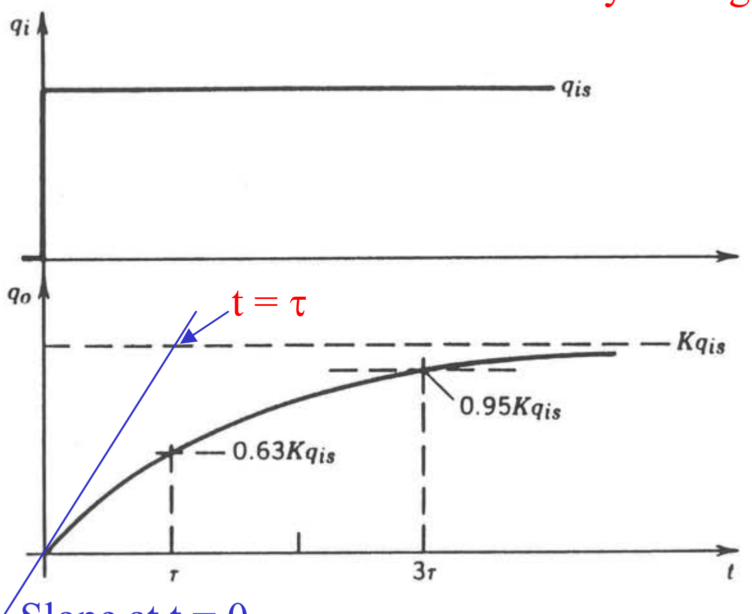
Validation of a Zero-Order Dynamic System Model



1st-Order Dynamic System Model

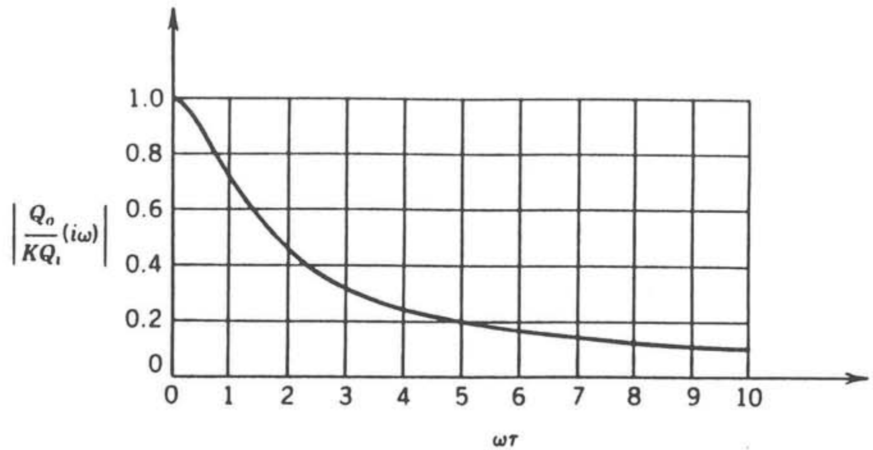
$$\tau \frac{dq_o}{dt} + q_o = Kq_i$$

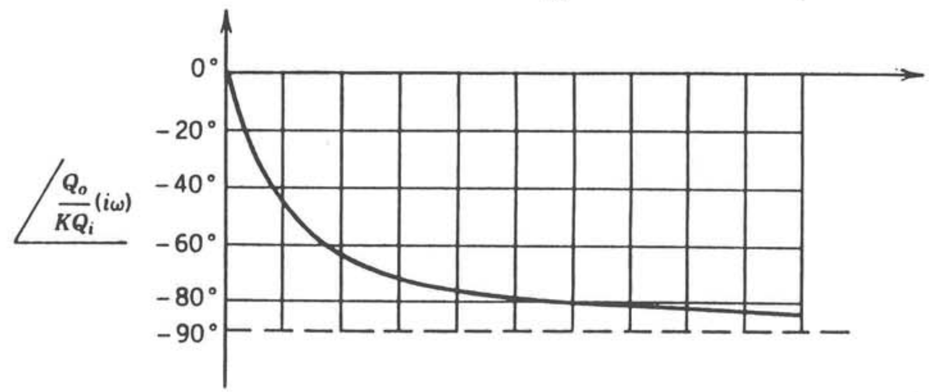
τ = time constant
K = steady-state gain



$$q_o = Kq_{is}(1 - e^{-(t/\tau)})$$

$$\dot{q}_o = \frac{Kq_{is}}{\tau} e^{-\frac{t}{\tau}} \quad \Rightarrow \quad \dot{q}_o\Big|_{t=0} = \frac{Kq_{is}}{\tau}$$





$$\frac{Q_o}{Q_i}(s) = \frac{K}{\tau s + 1} \qquad \frac{Q_o}{Q_i}(i\omega) = \frac{K}{\sqrt{(\omega\tau)^2 + 1}} \angle -\tan^{-1}\omega\tau$$

- How would you determine if an experimentally-determined step response of a system could be represented by a first-order system step response?

$$q_o(t) = Kq_{is}\left(1 - e^{-\frac{t}{\tau}}\right)$$

$$\frac{q_o(t) - Kq_{is}}{Kq_{is}} = -e^{-\frac{t}{\tau}}$$

$$1 - \frac{q_o(t)}{Kq_{is}} = e^{-\frac{t}{\tau}}$$

$$\log_{10}\left[1 - \frac{q_o(t)}{Kq_{is}}\right] = -\frac{t}{\tau}\log_{10} e = -0.4343\frac{t}{\tau}$$

Straight-Line Plot:

$$\log_{10}\left[1 - \frac{q_o(t)}{Kq_{is}}\right] \text{ vs. } t$$

Slope = $-0.4343/\tau$

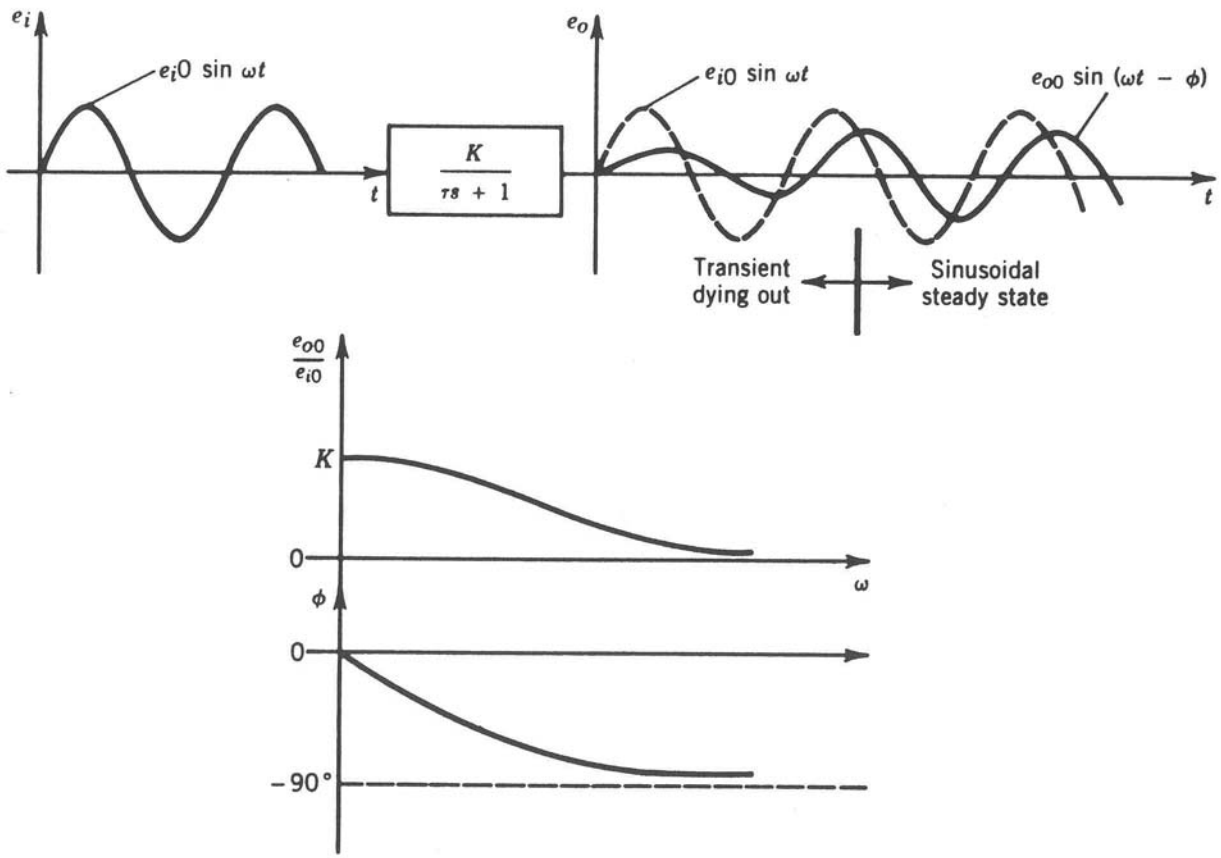
- This approach gives a more accurate value of τ since the best line through all the data points is used rather than just two points, as in the 63.2% method. Furthermore, if the data points fall nearly on a straight line, we are assured that the instrument is behaving as a first-order type. If the data deviate considerably from a straight line, we know the system is not truly first order and a τ value obtained by the 63.2% method would be quite misleading.
- An even stronger verification (or refutation) of first-order dynamic characteristics is available from frequency-response testing. If the system is truly first-order, the amplitude ratio follows the typical low- and high-frequency asymptotes (slope 0 and –20 dB/decade) and the phase angle approaches -90° asymptotically.

– If these characteristics are present, the numerical value of τ is found by determining ω (rad/sec) at the breakpoint and using $\tau = 1/\omega_{break}$. Deviations from the above amplitude and/or phase characteristics indicate non-first-order behavior.

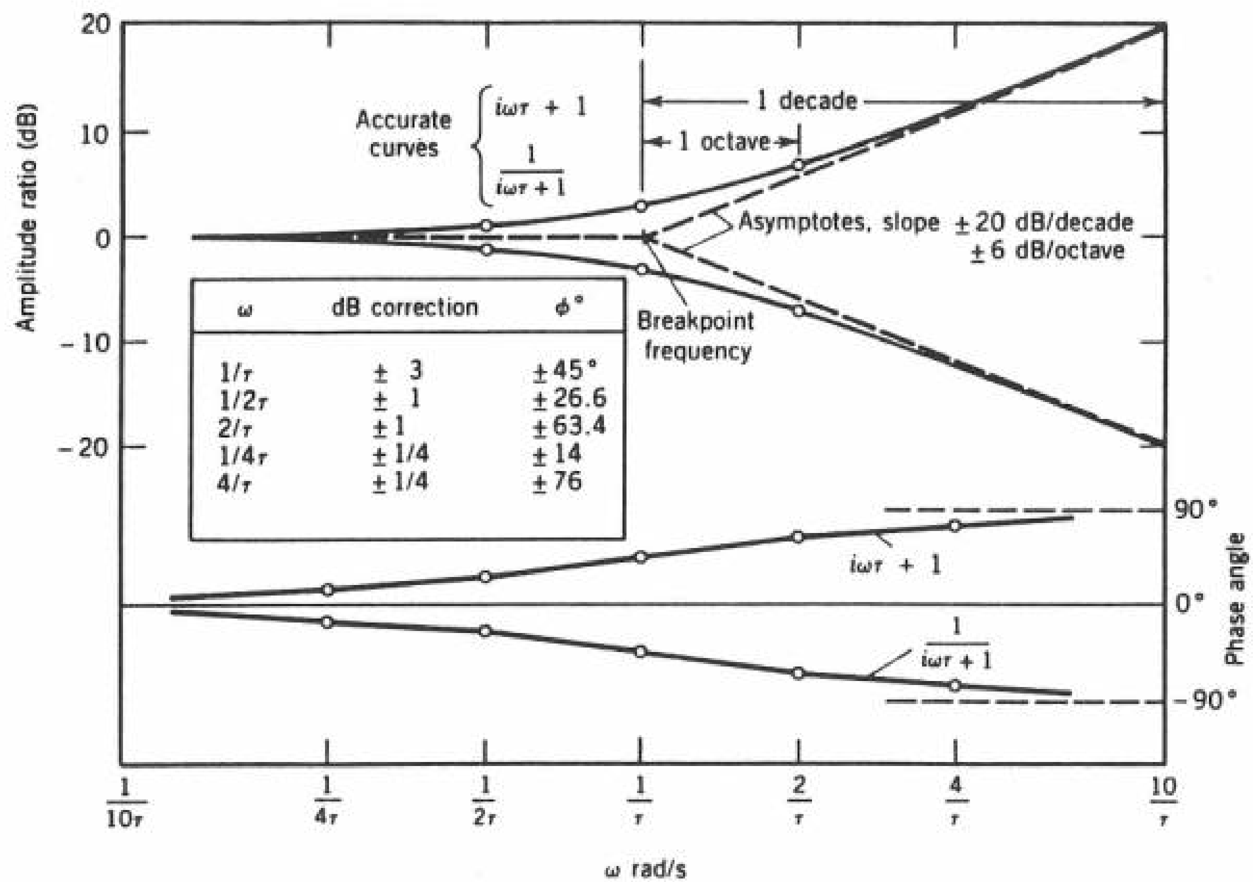
- What is the relationship between the unit-step response and the unit-ramp response and between the unit-impulse response and the unit-step response?
  - For a linear time-invariant system, the response to the derivative of an input signal can be obtained by differentiating the response of the system to the original signal.
  - For a linear time-invariant system, the response to the integral of an input signal can be obtained by integrating the response of the system to the original signal and by determining the integration constants from the zero-output initial condition.

- Unit-Step Input is the derivative of the Unit-Ramp Input.
- Unit-Impulse Input is the derivative of the Unit-Step Input.
- Once you know the unit-step response, take the derivative to get the unit-impulse response and integrate to get the unit-ramp response.

System Frequency Response



Bode Plotting of 1^{st}-Order Frequency Response

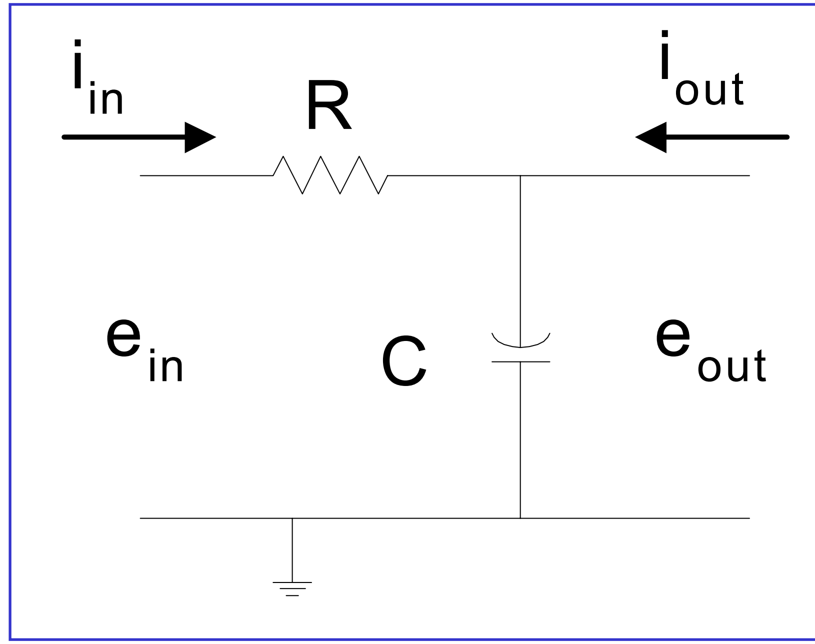


dB = 20 $\log_{10}$ (amplitude ratio)

decade = 10 to 1 frequency change

octave = 2 to 1 frequency change

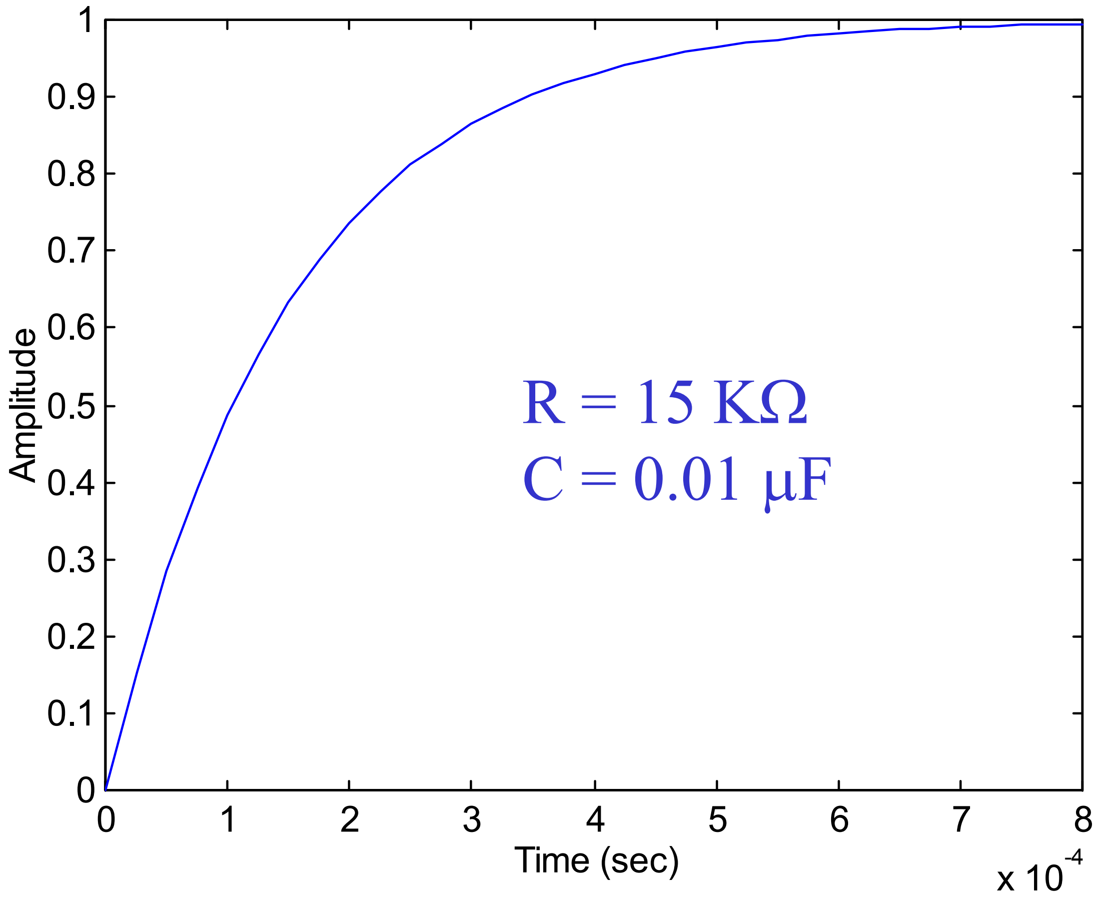
Analog Electronics:
RC Low-Pass Filter
Time Response &
Frequency Response



$$\begin{bmatrix} e_{in} \\ i_{in} \end{bmatrix} = \begin{bmatrix} RCs+1 & -R \\ Cs & -1 \end{bmatrix} \begin{bmatrix} e_{out} \\ i_{out} \end{bmatrix}$$

$$\frac{e_{out}}{e_{in}} = \frac{1}{RCs+1} = \frac{1}{\tau s+1} \quad \text{when } i_{out} = 0$$

Time Response to Unit Step Input



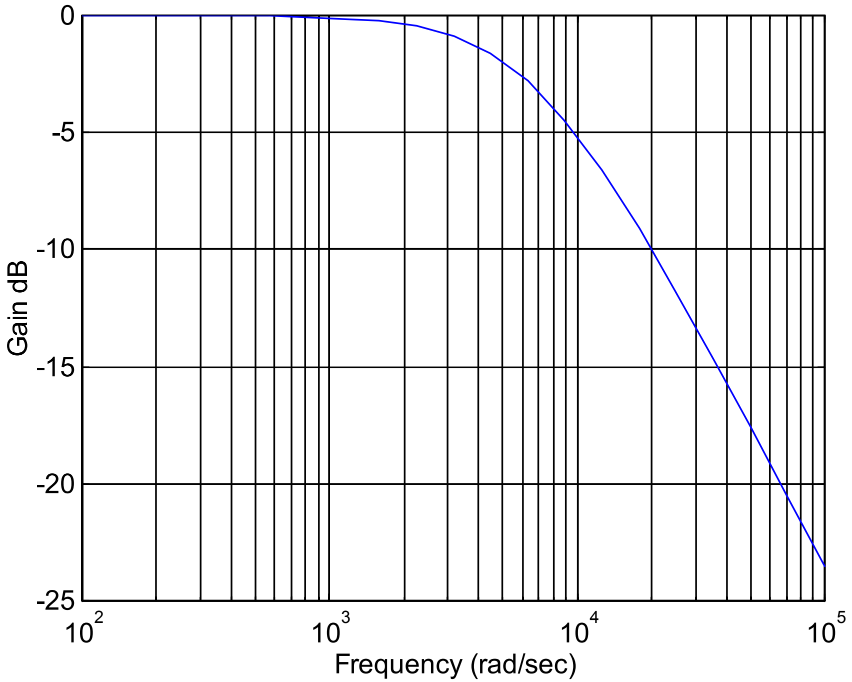
Time Constant τ = RC

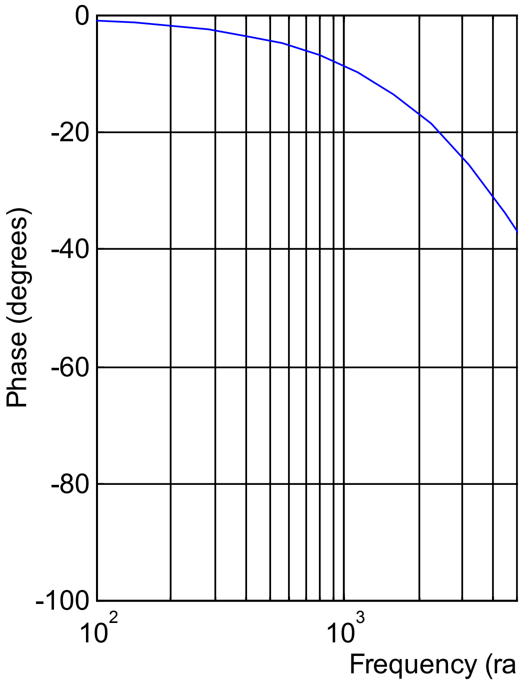
- Time Constant τ
  - Time it takes the step response to reach 63% of the steady-state value
- Rise Time $T_r = 2.2\ \tau$
  - Time it takes the step response to go from 10% to 90% of the steady-state value
- Delay Time $T_d = 0.69\ \tau$
  - Time it takes the step response to reach 50% of the steady-state value

Frequency Response

$R = 15\ K\Omega$

$C = 0.01\ \mu F$





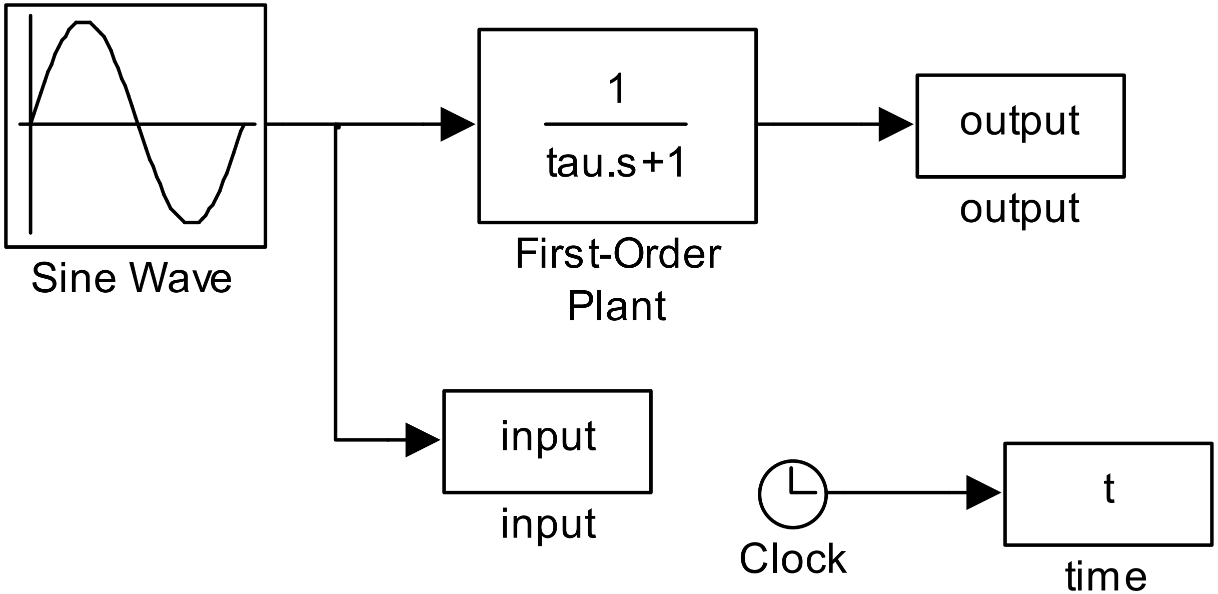
Bandwidth = $1/\tau$

$$\frac{e_{out}}{e_{in}}(i\omega) = \frac{K}{i\omega\tau + 1} = \frac{K\angle 0^\circ}{\sqrt{(\omega\tau)^2 + 1^2}\angle \tan^{-1}\omega\tau} = \frac{K}{\sqrt{(\omega\tau)^2 + 1^2}}\angle -\tan^{-1}\omega\tau$$

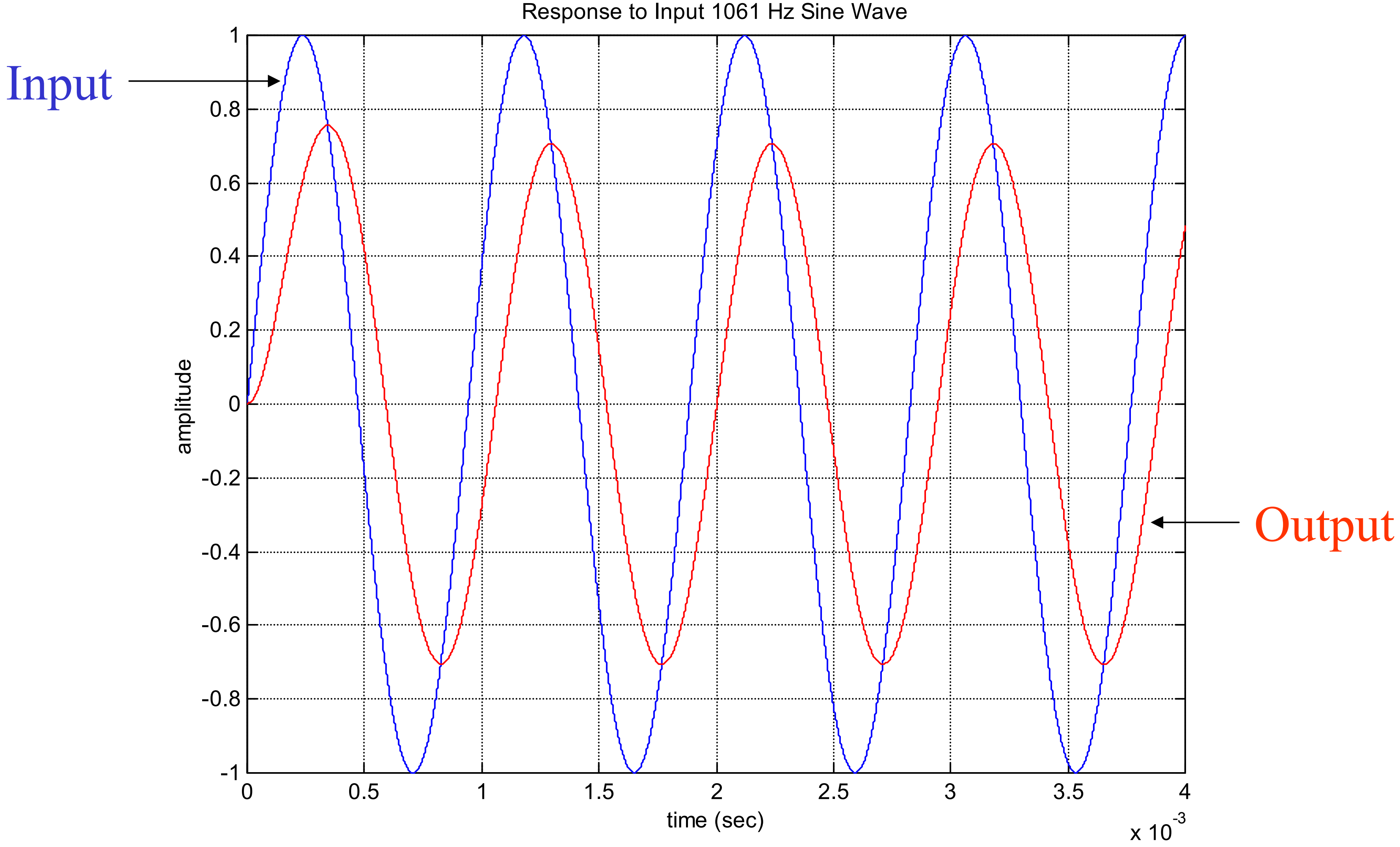
- Bandwidth
  - The bandwidth is the frequency where the amplitude ratio drops by a factor of 0.707 = -3dB of its gain at zero or low-frequency.
  - For a 1st -order system, the bandwidth is equal to $1/\tau$.
  - The larger (smaller) the bandwidth, the faster (slower) the step response.
  - Bandwidth is a direct measure of system susceptibility to noise, as well as an indicator of the system speed of response.

MatLab / Simulink Diagram
Frequency Response for 1061 Hz Sine Input

τ = 1.5E-4 sec

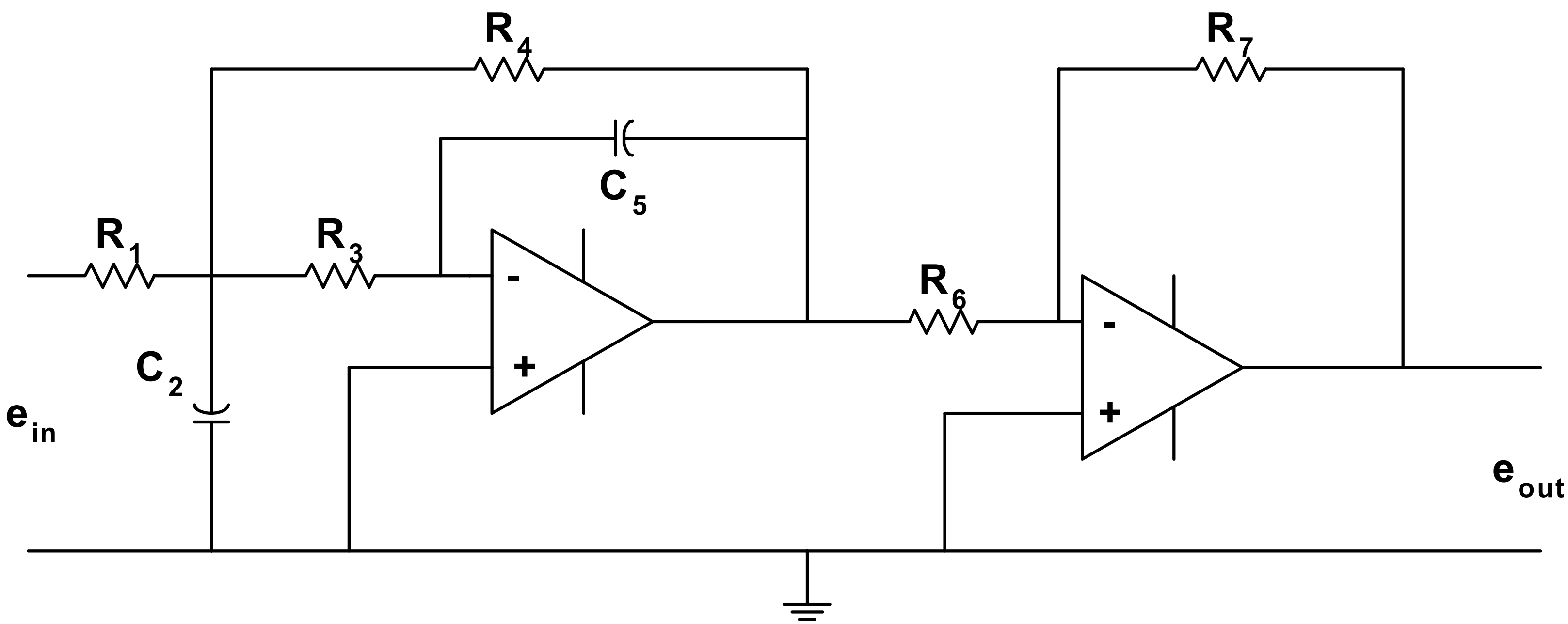


Amplitude Ratio = 0.707 = -3 dB   Phase Angle = -45°

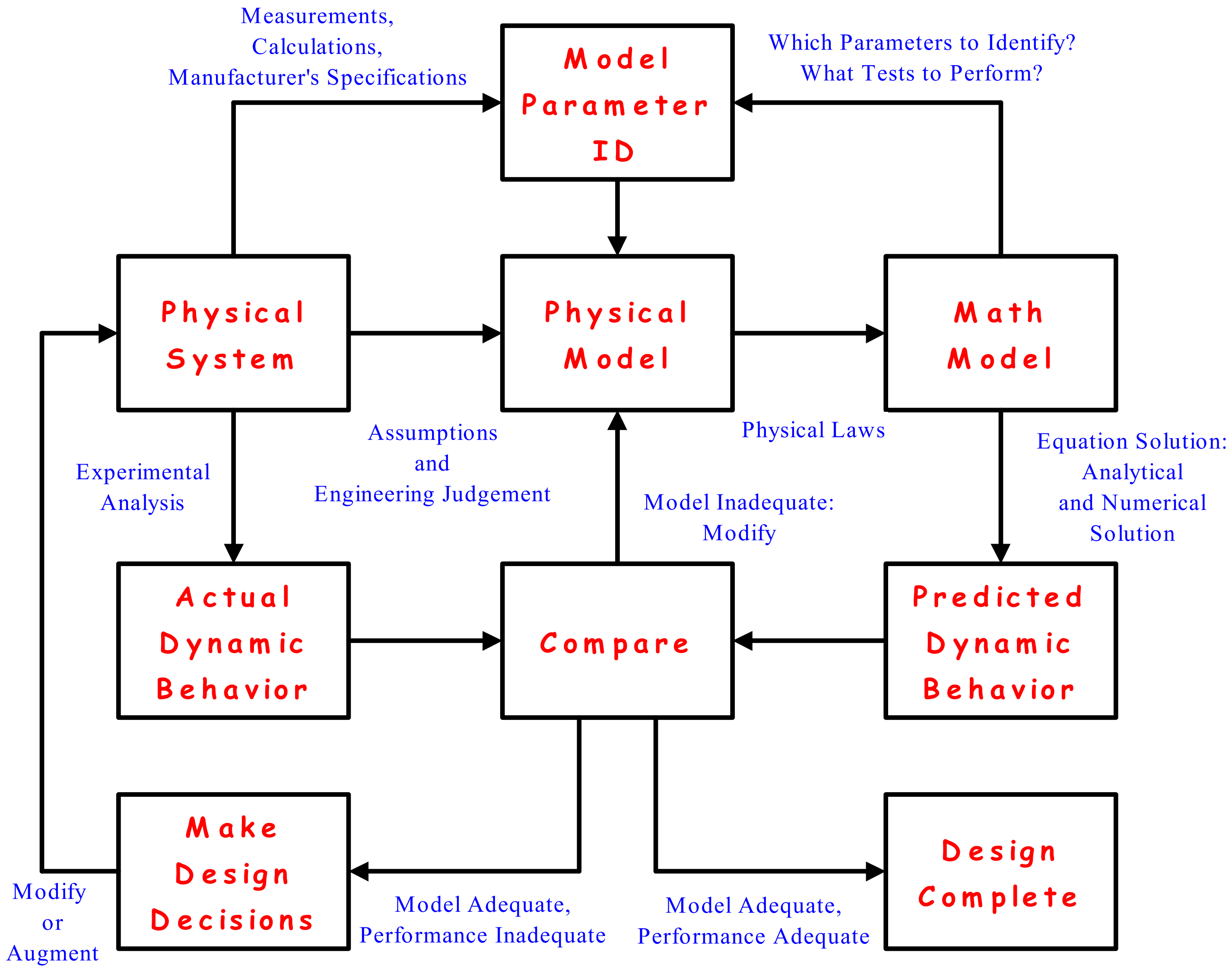


Mechatronics
Time Response & Frequency Response
2nd-Order Dynamic System

Example: 2-Pole, Low-Pass, Active Filter



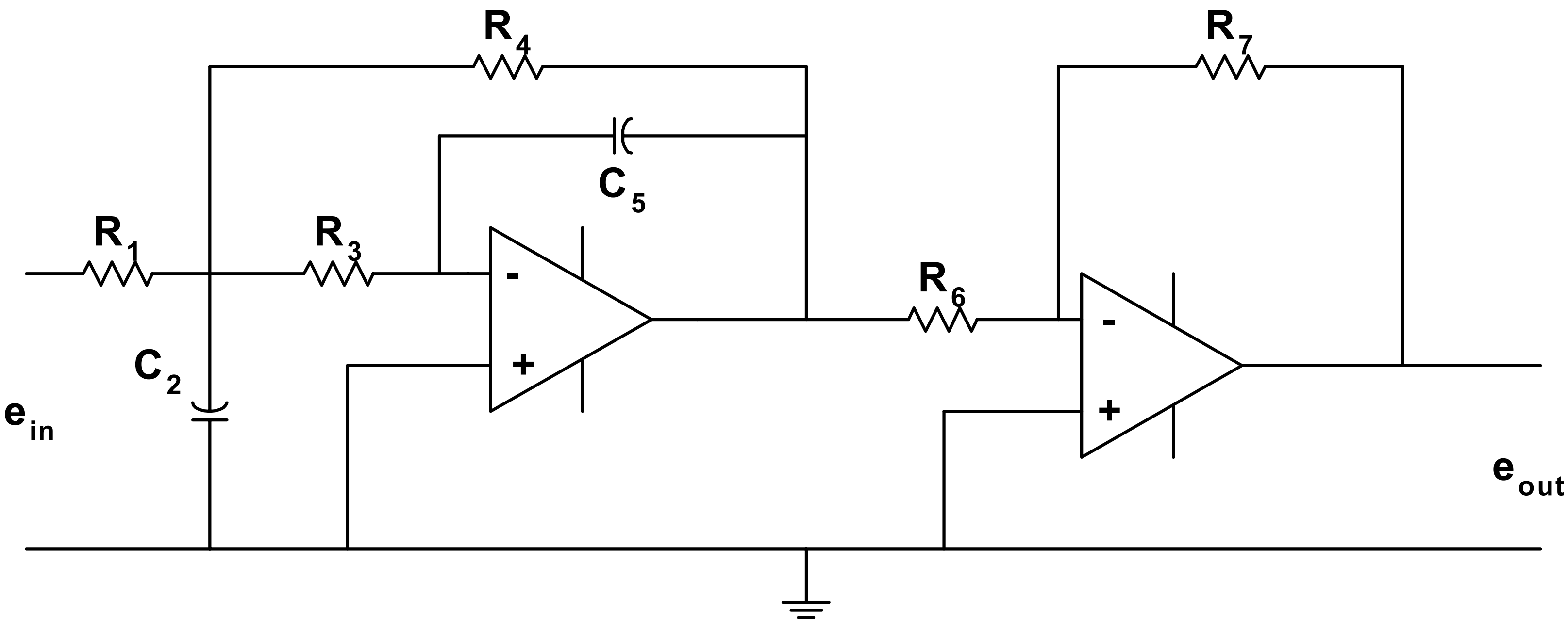
Dynamic System Investigation
of the Two-Pole, Low-Pass, Active Filter



Dynamic System Investigation

Physical Model Ideal Transfer Function

$$\frac{e_{out}}{e_{in}}(s) = \frac{\left(\frac{R_7}{R_6}\right)\left(\frac{1}{R_1R_3C_2C_5}\right)}{s^2 + \left(\frac{1}{R_3C_2} + \frac{1}{R_1C_2} + \frac{1}{R_4C_2}\right)s + \frac{1}{R_3R_4C_2C_5}}$$



2^{nd}-Order Dynamic System Model

$$a_2 \frac{d^2 q_0}{dt^2} + a_1 \frac{dq_0}{dt} + a_0 q_0 = b_0 q_i$$

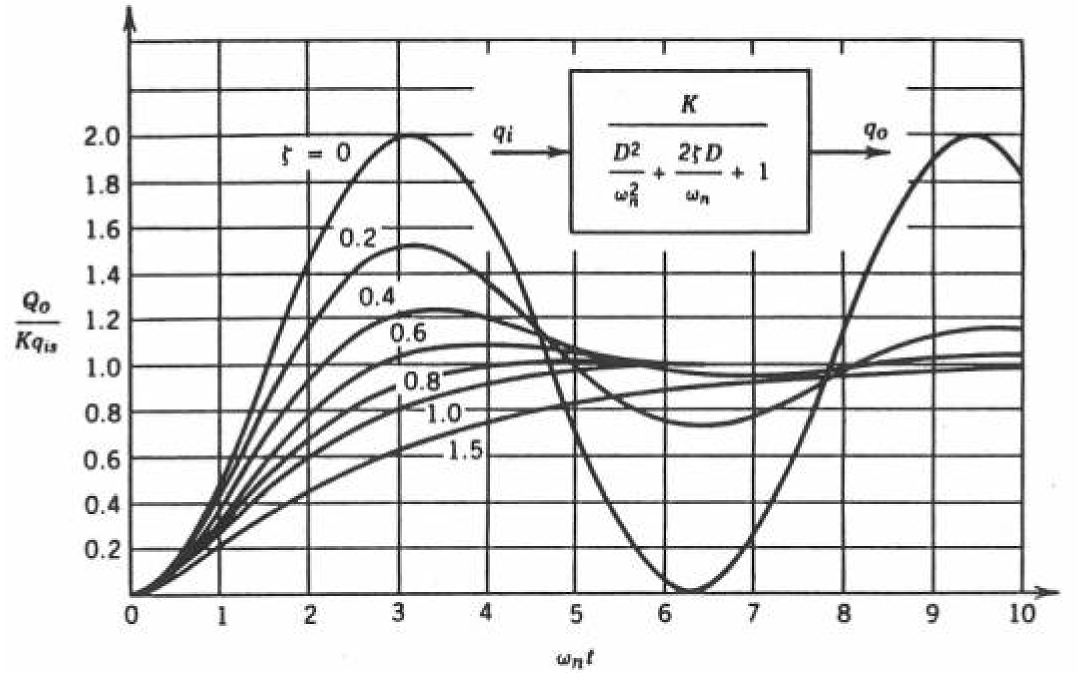
$$\frac{1}{\omega_n^2}\frac{d^2 q_0}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dq_0}{dt} + q_0 = Kq_i$$

$$\omega_n \triangleq \sqrt{\frac{a_0}{a_2}} = \text{undamped natural frequency}$$

$$\zeta \triangleq \frac{a_1}{2\sqrt{a_2 a_0}} = \text{damping ratio}$$

$$K \triangleq \frac{b_0}{a_0} = \text{steady-state gain}$$

Step Response of a 2^{nd}-Order System



Step Response of a 2nd-Order System

$$\frac{1}{\omega_n^2}\frac{d^2q_0}{dt^2}+\frac{2\varsigma}{\omega_n}\zeta\frac{dq_0}{dt}+q_0=Kq_i$$

Underdamped

$$q_o=Kq_{is}\left[1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t+\sin^{-1}\sqrt{1-\zeta^2}\right)\right]\qquad \zeta<1$$

Critically Damped

$$q_o=Kq_{is}\left[1-\left(1+\omega_n t\right)e^{-\omega_n t}\right]\qquad \zeta=1$$

Over-damped

$$q_o=Kq_{is}\left[1-\frac{\zeta+\sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}}e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t}+\frac{\zeta-\sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}}e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}\right]\qquad \zeta>1$$

Frequency Response of a 2nd-Order System

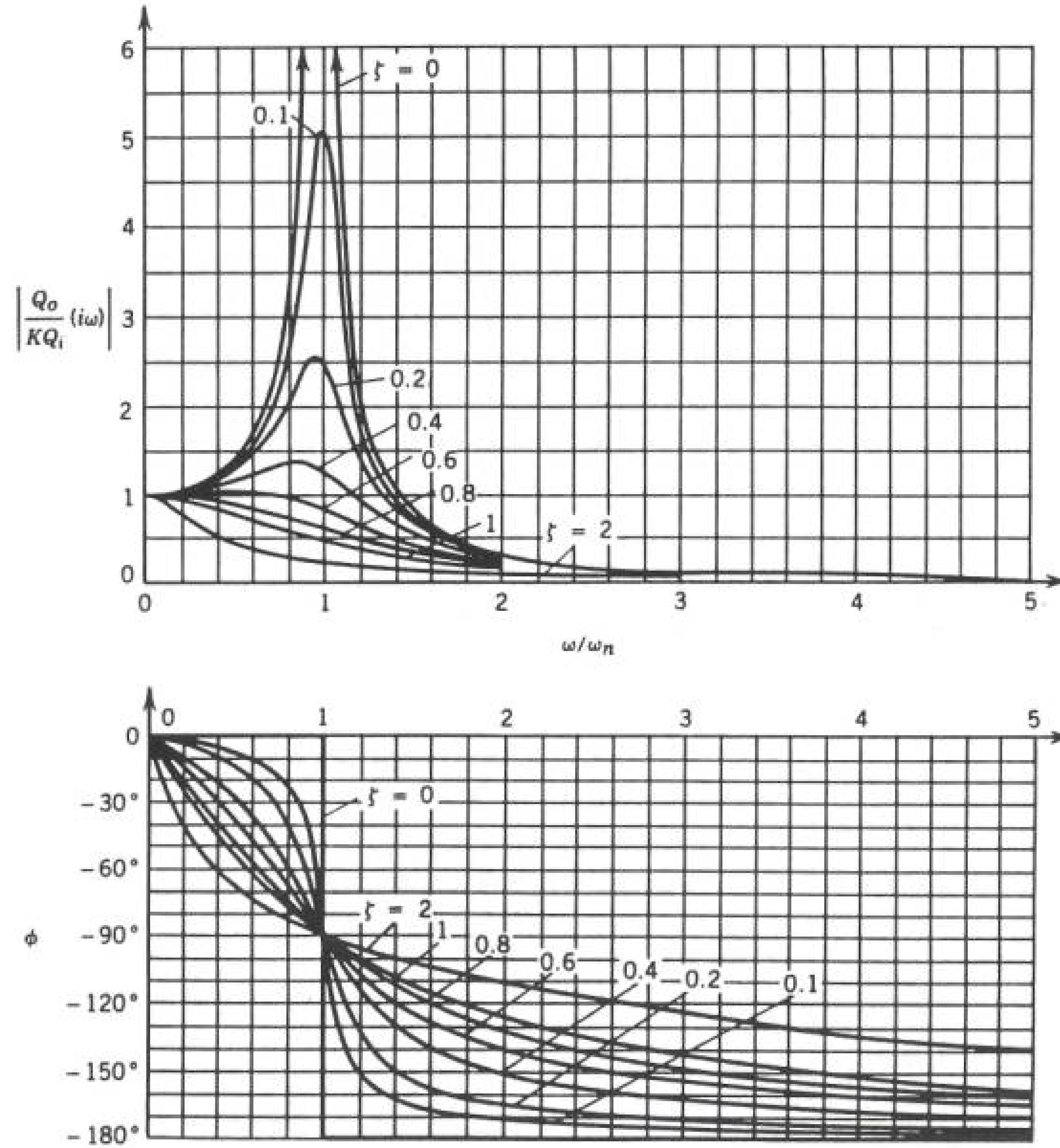
Laplace Transfer Function

$$\frac{Q_o}{Q_i}(s) = \frac{K}{\dfrac{s^2}{\omega_n^2} + \dfrac{2\zeta s}{\omega_n} + 1}$$

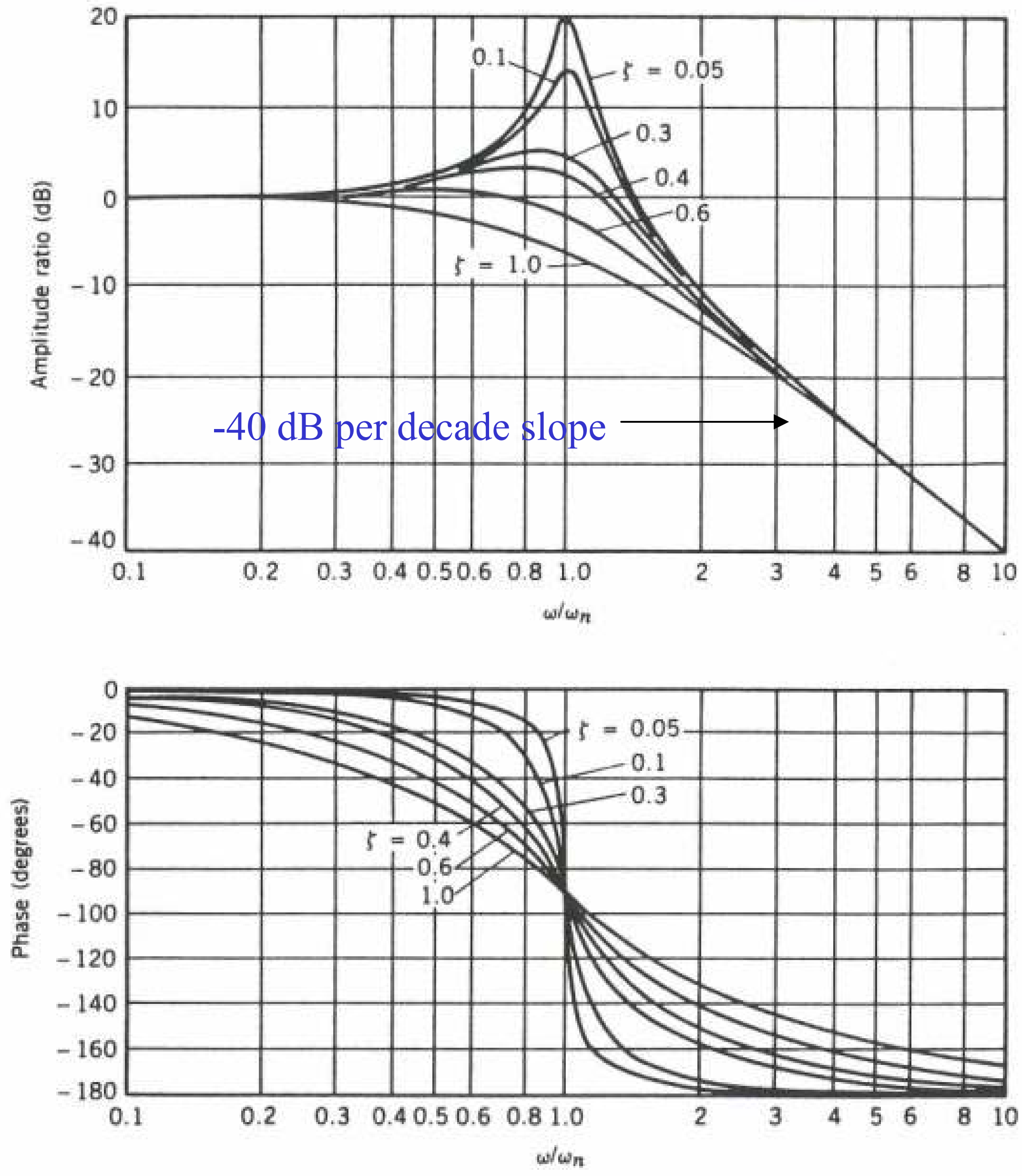
Sinusoidal Transfer Function

$$\frac{Q_o}{Q_i}(i\omega) = \frac{K}{\sqrt{\left[1 - \left(\dfrac{\omega}{\omega_n}\right)^2\right]^2 + \dfrac{4\zeta^2\omega^2}{\omega_n^2}}} \angle \tan^{-1} \frac{2\zeta}{\left(\dfrac{\omega}{\omega_n} - \dfrac{\omega_n}{\omega}\right)}$$

Frequency Response of a 2nd-Order System



Frequency Response of a 2^{nd}-Order System



Some Observations

- When a physical system exhibits a natural oscillatory behavior, a 1st-order model (or even a cascade of several 1st-order models) cannot provide the desired response. The simplest model that does possess that possibility is the 2nd-order dynamic system model.
- This system is very important in control design.
  - System specifications are often given assuming that the system is 2nd order.
  - For higher-order systems, we can often use dominant pole techniques to approximate the system with a 2nd-order transfer function.

- Damping ratio ζ clearly controls oscillation; $\zeta < 1$ is required for oscillatory behavior.
- The undamped case ($\zeta = 0$) is not physically realizable (total absence of energy loss effects) but gives us, mathematically, a sustained oscillation at frequency ω_n.
- Natural oscillations of damped systems are at the damped natural frequency ω_d, and not at ω_n.

$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

- In hardware design, an optimum value of $\zeta = 0.64$ is often used to give maximum response speed without excessive oscillation.

- Undamped natural frequency ω_n is the major factor in response speed. For a given ζ response speed is directly proportional to ω_n.
- Thus, when 2nd-order components are used in feedback system design, large values of ω_n (small lags) are desirable since they allow the use of larger loop gain before stability limits are encountered.
- For frequency response, a resonant peak occurs for $\zeta < 0.707$. The peak frequency is ω_p and the peak amplitude ratio depends only on ζ.

$$\omega_p = \omega_n \sqrt{1-2\zeta^2} \qquad \text{peak amplitude ratio} = \frac{K}{2\zeta\sqrt{1-\zeta^2}}$$

- Bandwidth
  - The bandwidth is the frequency where the amplitude ratio drops by a factor of 0.707 = -3dB of its gain at zero or low-frequency.
  - For a 1^{st} -order system, the bandwidth is equal to $1/\tau$.
  - The larger (smaller) the bandwidth, the faster (slower) the step response.
  - Bandwidth is a direct measure of system susceptibility to noise, as well as an indicator of the system speed of response.
  - For a 2^{nd}-order system:

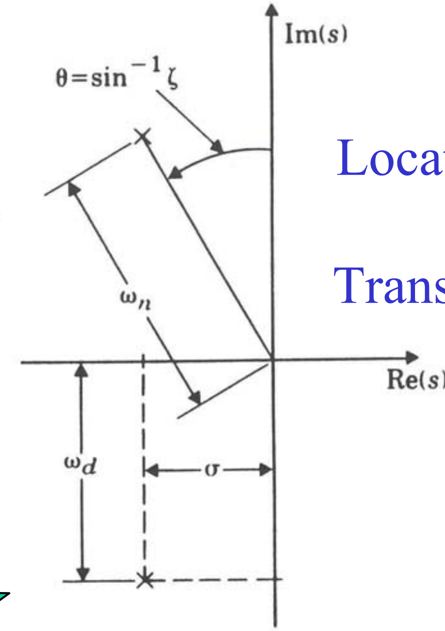
$$\mathrm{BW} = \omega_n \sqrt{1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4}}$$

– As ζ varies from 0 to 1, BW varies from $1.55\omega_n$ to $0.64\omega_n$. For a value of $\zeta = 0.707$, BW = ω_n. For most design considerations, we assume that the bandwidth of a 2nd-order all pole system can be approximated by ω_n.

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\varsigma\omega_n s + \omega_n^2}$$

$$s_{1,2} = -\varsigma\omega_n \pm i\omega_n\sqrt{1-\varsigma^2}$$

$$s_{1,2} = -\sigma \pm i\omega_d$$



Location of Poles Of Transfer Function

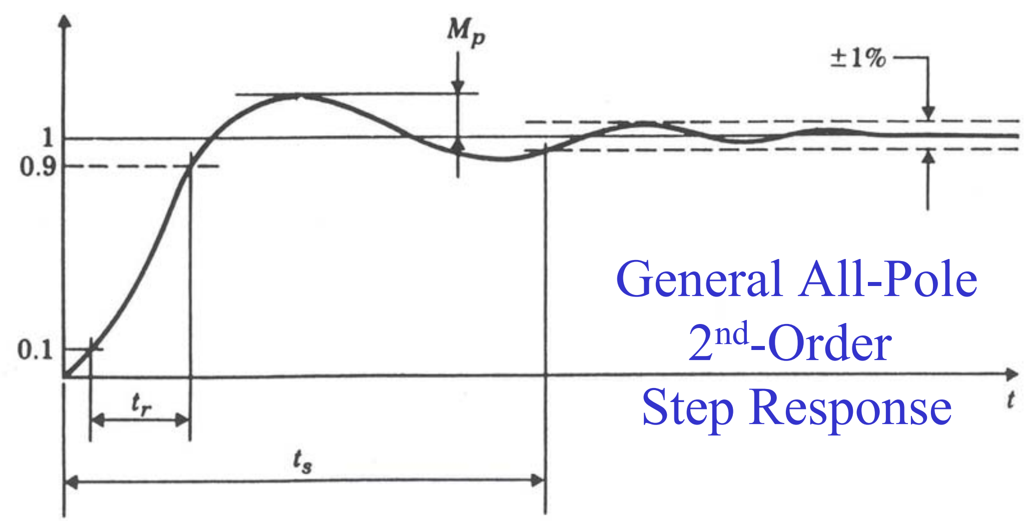
$$y(t) = 1 - e^{-\sigma t}\left(\cos\omega_d t + \frac{\sigma}{\omega_d}\sin\omega_d t\right)$$

$$t_r \approx \frac{1.8}{\omega_n} \quad \text{rise time}$$

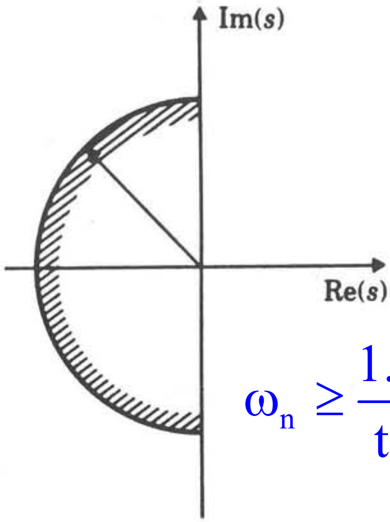
$$t_s \approx \frac{4.6}{\varsigma\omega_n} \quad \text{settling time}$$

$$M_p = e^{\frac{-\pi\varsigma}{\sqrt{1-\varsigma^2}}} \quad (0 \le \zeta < 1) \quad \text{overshoot}$$

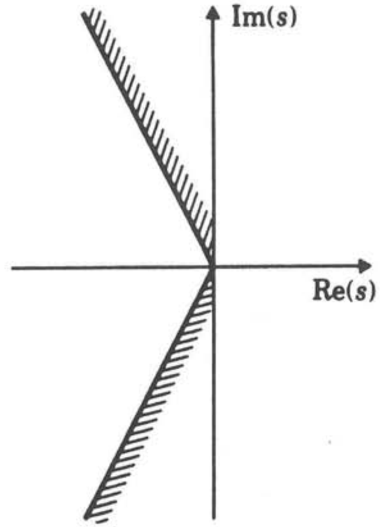
$$\approx \left(1 - \frac{\zeta}{0.6}\right) \qquad (0 \le \zeta \le 0.6)$$

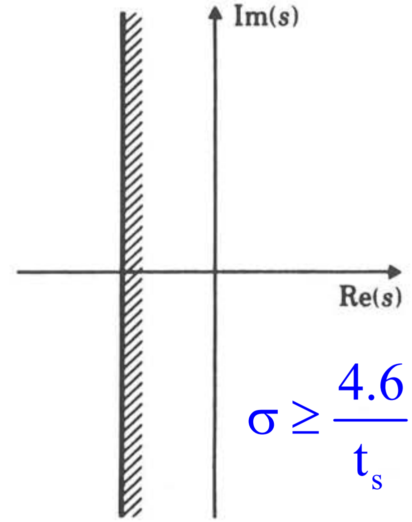


General All-Pole 2^{nd}-Order Step Response



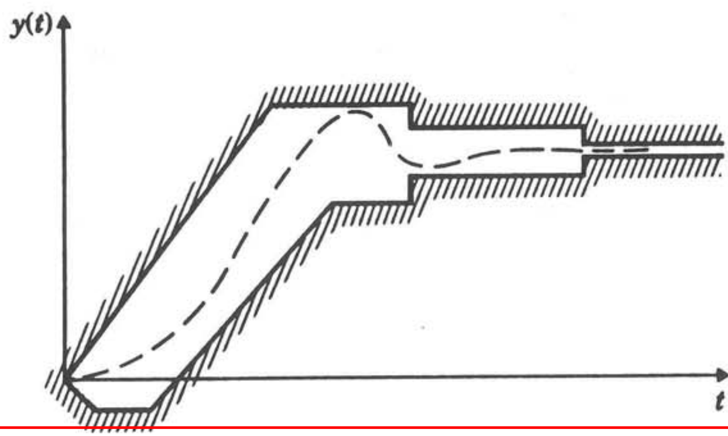
$$\omega_n \geq \frac{1.8}{t_r}$$

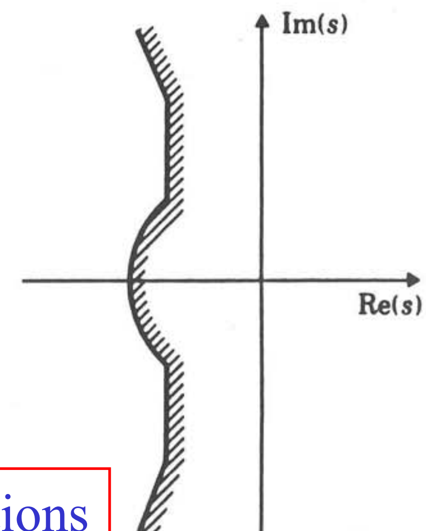




$$\sigma \geq \frac{4.6}{t_s}$$

$$\zeta \geq 0.6\left(1 - M_p\right) \qquad 0 \leq \zeta \leq 0.6$$





Time-Response Specifications vs. Pole-Location Specifications

Experimental Determination of ζ and ω_n

- ζ and ω_n can be obtained in a number of ways from step or frequency-response tests.
- For an underdamped second-order system, the values of ζ and ω_n may be found from the relations:

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \Rightarrow \zeta = \sqrt{\frac{1}{\left(\frac{\pi}{\log_e\left(M_p\right)}\right)^2 + 1}}$$

$$T = \frac{2\pi}{\omega_d} \qquad \omega_d = \omega_n\sqrt{1-\zeta^2} \Rightarrow \omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = \frac{2\pi}{T\sqrt{1-\zeta^2}}$$

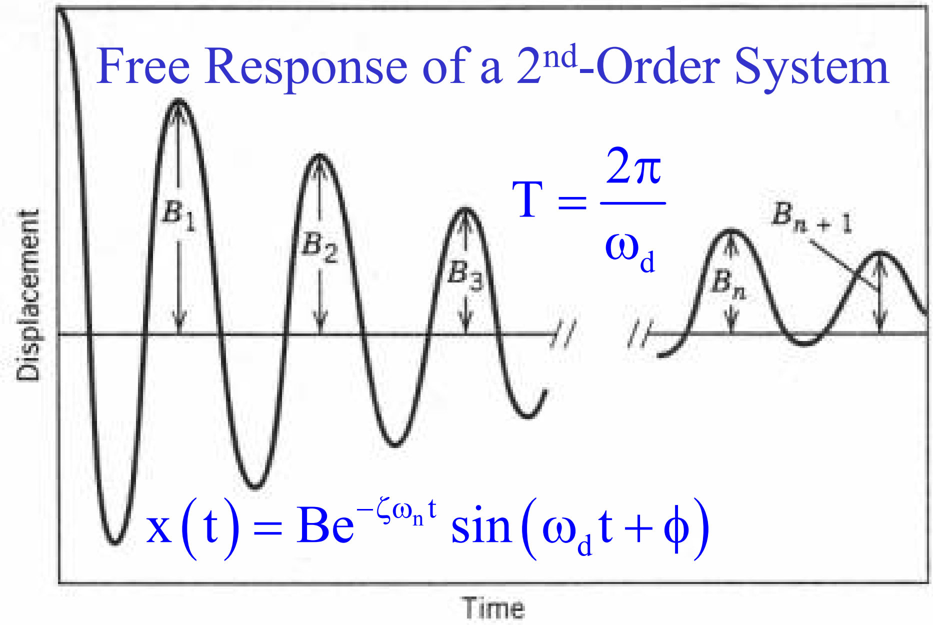
- Logarithmic Decrement δ is the natural logarithm of the ratio of two successive amplitudes.

$$\delta = \ln\left(\frac{x(t)}{x(t+T)}\right) = \ln\left(e^{\zeta\omega_n T}\right) = \zeta\omega_n T$$

$$= \frac{\zeta\omega_n 2\pi}{\omega_d} = \frac{\zeta\omega_n 2\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}$$

$$\delta = \frac{1}{n}\ln\frac{B_1}{B_{n+1}}$$



- If several cycles of oscillation appear in the record, it is more accurate to determine the period T as the average of as many distinct cycles as are available rather than from a single cycle.
- If a system is strictly linear and second-order, the value of n is immaterial; the same value of ζ will be found for any number of cycles. Thus if ζ is calculated for, say, $n = 1, 2, 4$, and 6 and different numerical values of ζ are obtained, we know that the system is not following the postulated mathematical model.

- For overdamped systems ($\zeta > 1.0$), no oscillations exist, and the determination of ζ and ω_n becomes more difficult. Usually it is easier to express the system response in terms of two time constants.

– For the overdamped step response:

$$q_o = Kq_{is}\left[1-\frac{\zeta+\sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}}e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t}+\frac{\zeta-\sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}}e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}\right] \qquad \zeta>1$$

$$\frac{q_o}{Kq_{is}}=\frac{\tau_1}{\tau_2-\tau_1}e^{-\frac{t}{\tau_1}}-\frac{\tau_2}{\tau_2-\tau_1}e^{-\frac{t}{\tau_2}}+1$$

– where $\tau_1 \triangleq \dfrac{1}{\left(\zeta-\sqrt{\zeta^2-1}\right)\omega_n}$ $\qquad \tau_2 \triangleq \dfrac{1}{\left(\zeta+\sqrt{\zeta^2-1}\right)\omega_n}$

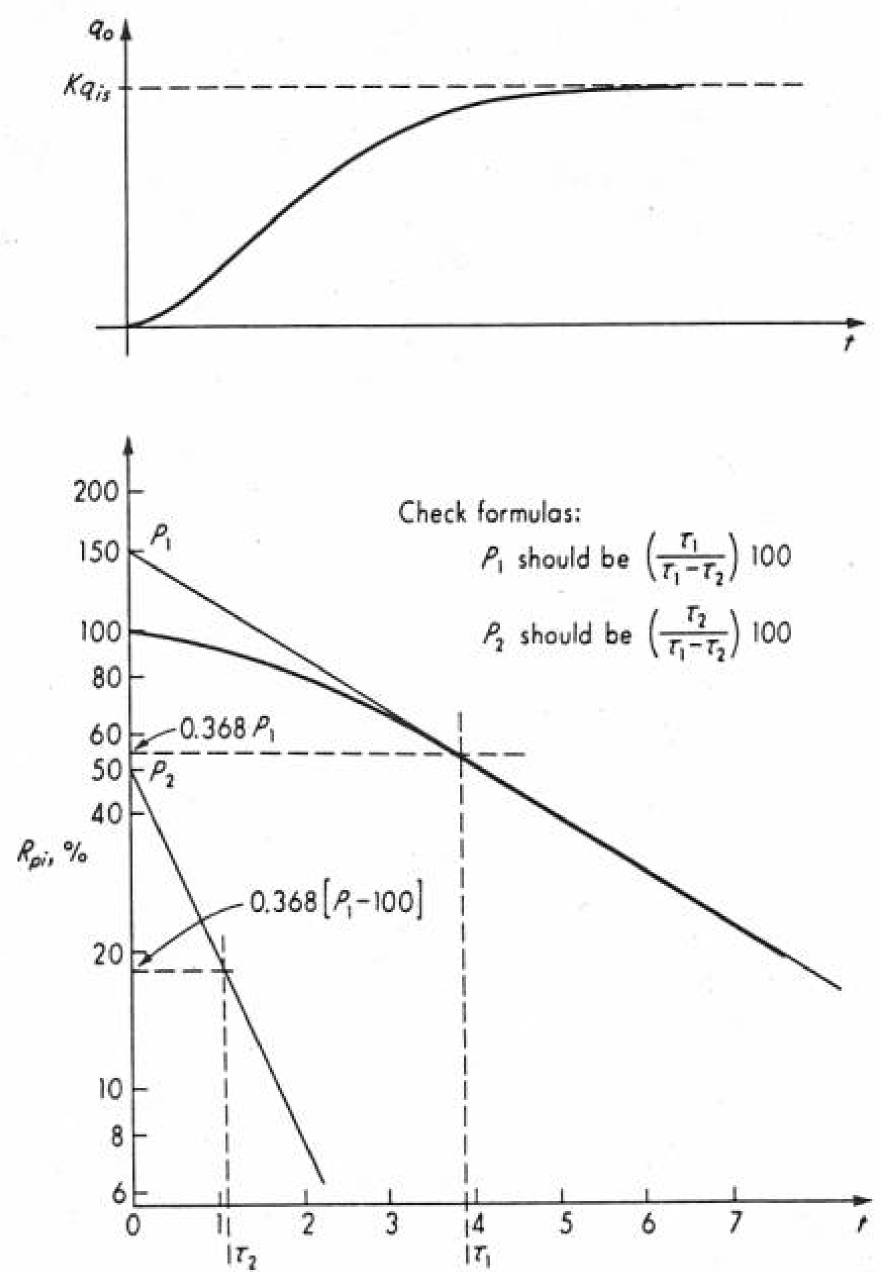
– To find τ_1 and τ_2 from a step-function response curve, we may proceed as follows:

- Define the percent incomplete response R_{pi} as:

$$R_{pi} \triangleq \left(1 - \frac{q_o}{Kq_{is}}\right)100$$

- Plot R_{pi} on a logarithmic scale versus time *t* on a linear scale. This curve will approach a straight line for large *t* if the system is second-order. Extend this line back to t = 0, and note the value P_1 where this line intersects the R_{pi} scale. Now, τ_1 is the time at which the straight-line asymptote has the value $0.368P_1$.
- Now plot on the same graph a new curve which is the difference between the straight-line asymptote and R_{pi}. If this new curve is not a straight line, the system is not second-order. If it is a straight line, the time at which this line has the value $0.368(P_1\text{-}100)$ is numerically equal to τ_2.
- Frequency-response methods may also be used to find τ_1 and τ_2.

Step-Response Test for Overdamped Second-Order Systems



Frequency-Response Test of Second-Order Systems