Speed-Accuracy Trade-off in Collective Decision Making

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Collective Decision Making in two alternative choice tasks


Birds deciding whether to migrate or not

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Leader election in a two party system

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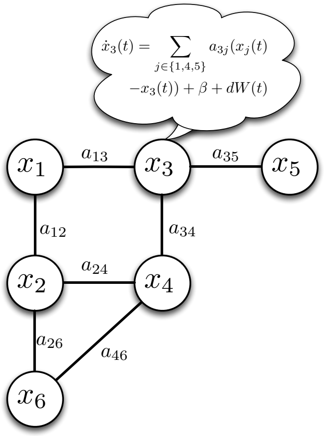
Social information assimilation + decision-making = Socio-Cognitive Networks

Collective Decision Making in Socio-Cognitive Networks

Alternative 1
versus
Alternative 2



Collective Decision Making in Socio-Cognitive Networks

Alternative 1
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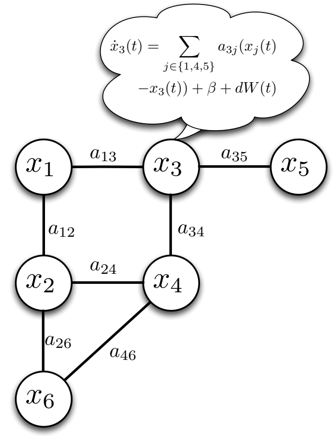


- Ideal group versus Condorcet group

R. D. Sorkin, C. J. Hays, and R. West. Signal-detection analysis of group decision making. *Psychological review*, 108(1):183, 2001

P. Braca, S. Marano, V. Matta, and P. Willett. Asymptotic optimality of running consensus in testing binary hypotheses. *IEEE Transactions on Signal Processing*, 58(2):814–825, 2010

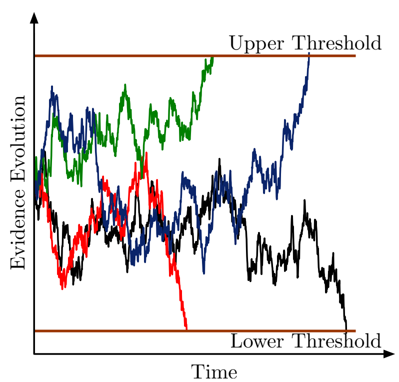
Drift Diffusion Model and the Free Response Paradigm

- Models human decision making in two alternative choice tasks
- Evidence evolution in a two alternative choice task is modeled by

$$dx(t) = \beta dt + dW(t), \quad x(t) = x_0$$

- Decision process at time τ is

$$\begin{cases} x(\tau) > \eta, & \text{choose alternative 1,} \\ x(\tau) < -\eta, & \text{choose alternative 2,} \\ \text{else,} & \text{collect more evidence.} \end{cases}$$



R. Bogacz, E. Brown, J. Moehlis, P. Holmes, and J. D. Cohen. The physics of optimal decision making: A formal analysis of performance in two-alternative forced choice tasks. *Psychological Review*, 113(4):700–765, 2006

Social Interaction and the DeGroot Model

- $\mathbf{p}$: vector of opinions in a network
- A: row stochastic matrix
- models consensus seeking in a social network by

$$\mathbf{p}(t+1) = A\mathbf{p}(t).$$

- same as the celebrated consensus dynamics in multi-agent systems
- Continuous time consensus seeking in a social network modeled by

$$\dot{\mathbf{p}}(t) = -L\mathbf{p}(t), \quad \mathbf{p}(0) = \mathbf{p}_0 \qquad L = \text{Laplacian Matrix}$$

M. H. DeGroot. Reaching a consensus. *Journal of the American Statistical Association*, 69(345):118–121, 1974

J. N. Tsitsiklis. *Problems in Decentralized Decision Making and Computation*. PhD thesis, Massachusetts Institute of Technology, November 1984

A. Jadbabaie, J. Lin, and A. S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*, 48(6):988–1001, 2003

R. Olfati-Saber, J. A. Fax, and R. M. Murray. Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95(1):215–233, 2007

Coupled Drift Diffusion Model

- n decision-makers collect noisy signals and interact with each other
- the evidence aggregation process well modeled by

$$d\mathbf{x}(t) = \underbrace{-L\mathbf{x}(t)dt}_{\text{social term}} + \underbrace{\beta\mathbf{1}_n dt + \sigma dW(t)}_{\text{noisy signal}}, \quad \mathbf{x}(0) = \mathbf{0}_n. \qquad (1)$$

Quantities of interest:

- Expected decision times
- Error rates (probability of wrong decision)

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Standard approach:

- solve first passage time associated with the FP equation for (1)
- an elliptic PDE with n variables

I. Poulakakis, L. Scardovi, and N. E. Leonard. Node classification in networks of stochastic evidence accumulators. *arXiv preprint arXiv:1210.4235*, October 2012

Asymptotic Optimality of the Coupled DDM

- **Evidence vector:** $\mathbf{x}(t) = x_{\text{cen}}(t)\mathbf{1}_n + \boldsymbol{\epsilon}(t)$

$$\mathrm{d}x_{\text{cen}}(t) = \beta\mathrm{d}t + \frac{1}{n}\mathbf{1}_n^\top \mathrm{d}\mathbf{W}(t), \; x_{\text{cen}}(0) = 0$$

$$\mathrm{d}\boldsymbol{\epsilon}(t) = -L\boldsymbol{\epsilon}(t)\mathrm{d}t + (I_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^\top)\mathrm{d}\mathbf{W}_n(t), \boldsymbol{\epsilon}(0) = \mathbf{0}_n.$$

- $\epsilon_k(t) \to \mathcal{N}(0, 1/\mu_k), \qquad \frac{1}{\mu_k} = \sum_{p=2}^n \frac{1}{2\lambda_p} {u_k^{(p)}}^2$
- μ_k is a certainty index determined purely by the interaction graph

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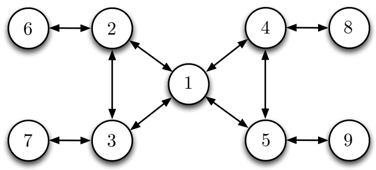
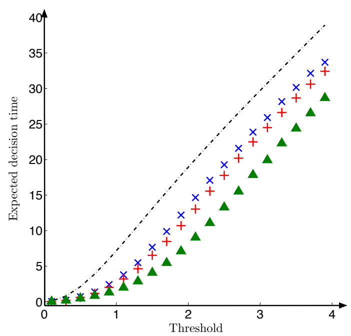
Asymptotic optimality

$$\frac{x_k(t) - \beta t}{\sqrt{t}} = \frac{x_{\text{cen}}(t) - \beta t}{\sqrt{t}} + \frac{\epsilon_k(t)}{\sqrt{t}} \implies x_k(t) = x_{\text{cen}}(t) + o(1)$$

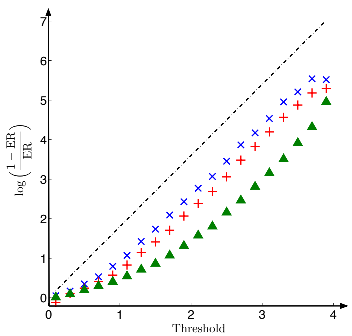
Numerical Illustration: Asymptotic Optimality



Interaction graph



Expected decision time



Log odds of error rates

Decoupled Approximation to the Coupled DDM

- decoupled approximation to $\boldsymbol{\epsilon}(t)$

$$\mathrm{d}\boldsymbol{\epsilon}(t) = -L\boldsymbol{\epsilon}(t)\mathrm{d}t + (I_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^\top)\mathrm{d}\mathbf{W}_n(t), \boldsymbol{\epsilon}(0) = \mathbf{0}_n$$

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- approximate $\epsilon_k(t)$ by the O-U process

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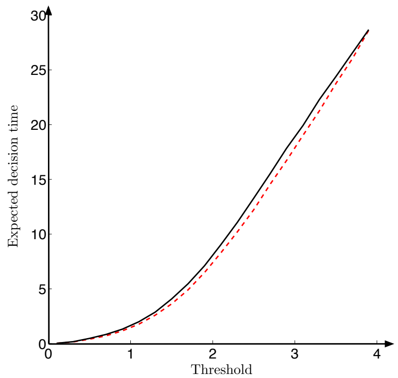
Efficiency of approximation

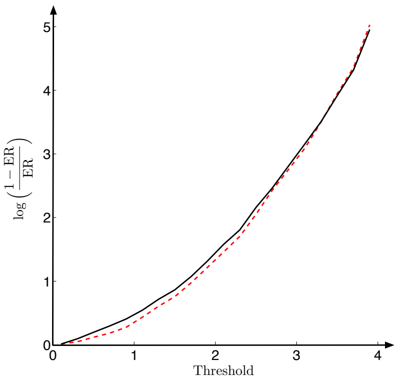
$$\lim_{t\to+\infty} \mathrm{corr}(\epsilon_k(t), \varepsilon_k(t)) = \mu_k \sum_{p=1}^{n} \frac{1}{2\mathrm{eig}_p(L + \mathrm{diag}(\boldsymbol{\mu}/2))}(\tilde{u}_k^{(p)})^2 - \frac{2}{n}$$

1. approximate evidence at node k: $x_{\mathrm{cen}}(t) + \varepsilon_k(t)$
2. **Decision time and Error Rate:** need to solve n elliptic PDEs with two variables opposed to a PDE with n variables earlier

Numerical Illustration: Decoupled Approximation



Expected decision time



Log odds of error rates

The reduced DDM approximates the coupled DDM well

Further Approximations

- bound the contribution by the O-U process $\varepsilon_k(t)$
- for sufficiently large K, with high probability

$$\max_{s\in[0,t]} |\varepsilon_k(t)| \leq \frac{K}{\sqrt{\mu_k}}$$

- effective threshold for the centralized DDM belongs to the set $(\eta - K/\sqrt{\mu_k}, \eta + K/\sqrt{\mu_k})$

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Bounds on Decision Time and Error Rates

$$\frac{\eta_k - \frac{K}{\sqrt{\mu_k}}}{\beta} \tanh\left(\beta n\left(\eta_k - \frac{K}{\sqrt{\mu_k}}\right)\right) \leq \mathrm{ET}_k \leq \frac{\eta_k + \frac{K}{\sqrt{\mu_k}}}{\beta} \tanh\left(\beta n\left(\eta_k + \frac{K}{\sqrt{\mu_k}}\right)\right)$$

$$\frac{1}{1+\exp\left(2\beta n(\eta_k + \frac{K}{\sqrt{\mu}})\right)} \leq \mathrm{ER}_k \leq \frac{1}{1+\exp\left(2\beta n(\eta_k - \frac{K}{\sqrt{\mu}})\right)}$$

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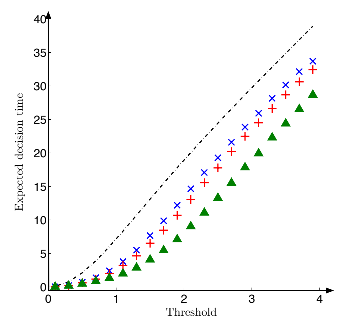
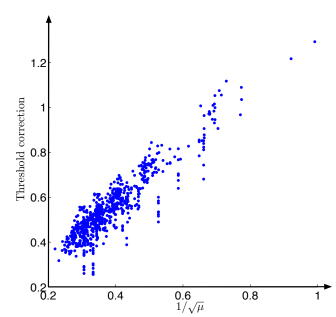
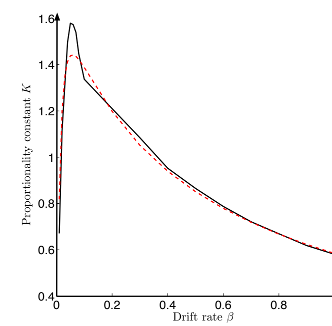
Empirical Estimates for Threshold Correction



Expected decision time



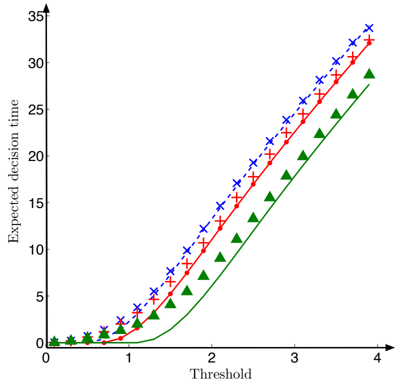
Log odds of error rates



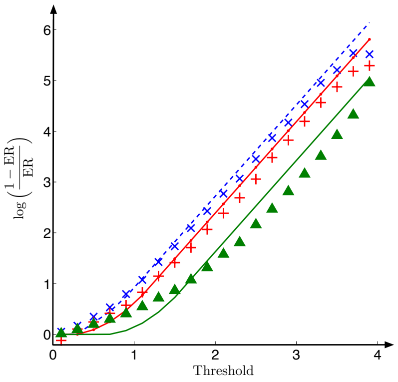
Log odds of error rates

1. coupled DDM at each node well approximated by centralized DDM with a modified threshold
2. effective threshold at node $k = \eta - \frac{K(\beta)}{\sqrt{\mu_k}}$, $\quad K(\beta) = \frac{e^{-\frac{1}{4\sqrt{\beta}}}}{\sqrt{\beta}(1+\beta/3)}$

Numerical Illustration: Threshold Corrected Centralized DDM



Expected decision time



Log odds of error rates

The centralized DDM with corrected thresholds approximates the coupled DDM well

Conclusions and Future Directions

Conclusions:

1. towards rigorous modeling and analysis of socio-cognitive networks
2. coupled DDM as model for social decision-making in 2-AC tasks
3. a computationally tractable decoupled approximation to coupled DDM
4. further approximation by the threshold corrected centralized DDM
5. ideas extend to multi-alternative choice tasks
   and 2-AC tasks with recency effect

Future Directions:

1. relaxing the continuous communication assumption
2. heterogeneous individuals
3. general decision-making tasks, e.g., multi-armed bandits