COLLEGE ALGEBRA

VERSION $\lfloor \pi \rfloor$

PENN STATE ALTOONA – MATH 021 EDITION[1]

BY

Carl Stitz, Ph.D.
Lakeland Community College

Jeff Zeager, Ph.D.
Lorain County Community College

Fall 2016

[1]This is a shorter and slightly modified version of the original manuscript. It only covers the material from chapters 0, 1, and 2, with one additional section: section 2.2 (adapted from sec. 8.1). This file was compiled by Prof. J. Gil (Penn State Altoona). For the original version of the book and more information about the authors, visit http://www.stitz-zeager.com/

ACKNOWLEDGEMENTS

While the cover of this textbook lists only two names, the book as it stands today would simply not exist if not for the tireless work and dedication of several people. First and foremost, we wish to thank our families for their patience and support during the creative process. We would also like to thank our students - the sole inspiration for the work. Among our colleagues, we wish to thank Rich Basich, Bill Previts, and Irina Lomonosov, who not only were early adopters of the textbook, but also contributed materials to the project. Special thanks go to Katie Cimperman, Terry Dykstra, Frank LeMay, and Rich Hagen who provided valuable feedback from the classroom. Thanks also to David Stumpf, Ivana Gorgievska, Jorge Gerszonowicz, Kathryn Arocho, Heather Bubnick, and Florin Muscutariu for their unwaivering support (and sometimes defense!) of the book. From outside the classroom, we wish to thank Don Anthan and Ken White, who designed the electric circuit applications used in the text, as well as Drs. Wendy Marley and Marcia Ballinger for the Lorain CCC enrollment data used in the text. The authors are also indebted to the good folks at our schools' bookstores, Gwen Sevtis (Lakeland CC) and Chris Callahan (Lorain CCC), for working with us to get printed copies to the students as inexpensively as possible. We would also like to thank Lakeland folks Jeri Dickinson, Mary Ann Blakeley, Jessica Novak, and Corrie Bergeron for their enthusiasm and promotion of the project. The administrations at both schools have also been very supportive of the project, so from Lakeland, we wish to thank Dr. Morris W. Beverage, Jr., President, Dr. Fred Law, Provost, Deans Don Anthan and Dr. Steve Oluic, and the Board of Trustees. From Lorain County Community College, we which to thank Dr. Roy A. Church, Dr. Karen Wells, and the Board of Trustees. From the Ohio Board of Regents, we wish to thank former Chancellor Eric Fingerhut, Darlene McCoy, Associate Vice Chancellor of Affordability and Efficiency, and Kelly Bernard. From OhioLINK, we wish to thank Steve Acker, John Magill, and Stacy Brannan. We also wish to thank the good folks at WebAssign, most notably Chris Hall, COO, and Joel Hollenbeck (former VP of Sales.) Last, but certainly not least, we wish to thank all the folks who have contacted us over the interwebs, most notably Dimitri Moonen and Joel Wordsworth, who gave us great feedback, and Antonio Olivares who helped debug the source code.

Table of Contents

Preface

Thank you for your interest in our book, but more importantly, thank you for taking the time to read the Preface. I always read the Prefaces of the textbooks which I use in my classes because I believe it is in the Preface where I begin to understand the authors - who they are, what their motivation for writing the book was, and what they hope the reader will get out of reading the text. Pedagogical issues such as content organization and how professors and students should best use a book can usually be gleaned out of its Table of Contents, but the reasons behind the choices authors make should be shared in the Preface. Also, I feel that the Preface of a textbook should demonstrate the authors' love of their discipline and passion for teaching, so that I come away believing that they really want to help students and not just make money. Thus, I thank my fellow Preface-readers again for giving me the opportunity to share with you the need and vision which guided the creation of this book and passion which both Carl and I hold for Mathematics and the teaching of it.

Carl and I are natives of Northeast Ohio. We met in graduate school at Kent State University in 1997. I finished my Ph.D in Pure Mathematics in August 1998 and started teaching at Lorain County Community College in Elyria, Ohio just two days after graduation. Carl earned his Ph.D in Pure Mathematics in August 2000 and started teaching at Lakeland Community College in Kirtland, Ohio that same month. Our schools are fairly similar in size and mission and each serves a similar population of students. The students range in age from about 16 (Ohio has a Post-Secondary Enrollment Option program which allows high school students to take college courses for free while still in high school.) to over 65. Many of the "non-traditional" students are returning to school in order to change careers. A majority of the students at both schools receive some sort of financial aid, be it scholarships from the schools' foundations, state-funded grants or federal financial aid like student loans, and many of them have lives busied by family and job demands. Some will be taking their Associate degrees and entering (or re-entering) the workforce while others will be continuing on to a four-year college or university. Despite their many differences, our students share one common attribute: they do not want to spend $200 on a College Algebra book.

The challenge of reducing the cost of textbooks is one that many states, including Ohio, are taking quite seriously. Indeed, state-level leaders have started to work with faculty from several of the colleges and universities in Ohio and with the major publishers as well. That process will take considerable time so Carl and I came up with a plan of our own. We decided that the best way to help our students right now was to write our own College Algebra book and give it away electronically for free. We were granted sabbaticals from our respective institutions for the Spring semester of 2009 and actually began writing the textbook on December 16, 2008. Using an open-source text editor called TexNicCenter and an open-source distribution of LaTeX called MikTex 2.7, Carl and I wrote and edited all of the text, exercises and answers and created all of the graphs (using Metapost within LaTeX) for Version $0.\overline{9}$ in about eight months. (We choose to create a text in only black and white to keep printing costs to a minimum for those students who prefer a printed edition. This somewhat Spartan page layout stands in sharp relief to the explosion of colors found in most other College Algebra texts, but neither Carl nor I believe the four-color print adds anything of value.) I used the book in three sections of College Algebra at Lorain County Community College in the Fall of 2009 and Carl's colleague, Dr. Bill Previts, taught a section of College Algebra at Lakeland with the book that semester as well. Students had the option of downloading the book as a .pdf file from our website www.stitz-zeager.com or buying a low-cost printed version from our colleges' respective bookstores. (By giving this book away for free electronically, we end the cycle of new editions appearing every 18 months to curtail the used book market.) During Thanksgiving break in November 2009, many additional exercises written by Dr. Previts were added and the typographical errors found by our students and others were corrected. On December 10, 2009, Version $\sqrt{2}$ was released. The book remains free for download at our website and by using Lulu.com as an on-demand printing service, our bookstores are now able to provide a printed edition for just under $19. Neither Carl nor I have, or will ever, receive any royalties from the printed editions. As a contribution back to the open-source community, all of the LaTeX files used to compile the book are available for free under a Creative Commons License on our website as well. That way, anyone who would like to rearrange or edit the content for their classes can do so as long as it remains free.

The only disadvantage to not working for a publisher is that we don't have a paid editorial staff. What we have instead, beyond ourselves, is friends, colleagues and unknown people in the open-source community who alert us to errors they find as they read the textbook. What we gain in not having to report to a publisher so dramatically outweighs the lack of the paid staff that we have turned down every offer to publish our book. (As of the writing of this Preface, we've had three offers.) By maintaining this book by ourselves, Carl and I retain all creative control and keep the book our own. We control the organization, depth and rigor of the content which means we can resist the pressure to diminish the rigor and homogenize the content so as to appeal to a mass market. A casual glance through the Table of Contents of most of the major publishers' College Algebra books reveals nearly isomorphic content in both order and depth. Our Table of Contents shows a different approach, one that might be labeled "Functions First." To truly use The Rule of Four, that is, in order to discuss each new concept algebraically, graphically, numerically and verbally, it seems completely obvious to us that one would need to introduce functions first. (Take a moment and compare our ordering to the classic "equations first, then the Cartesian Plane and THEN functions" approach seen in most of the major players.) We then introduce a class of functions and discuss the equations, inequalities (with a heavy emphasis on sign diagrams) and applications which involve functions in that class. The material is presented at a level that definitely prepares a student for Calculus while giving them relevant Mathematics which can be used in other classes as well. Graphing calculators are used sparingly and only as a tool to enhance the Mathematics, not to replace it. The answers to nearly all of the computational homework exercises are given in the text and we have gone to great lengths to write some very thought provoking discussion questions whose answers are not given. One will notice that our exercise sets are much shorter than the traditional sets of nearly 100 “drill and kill” questions which build skill devoid of understanding. Our experience has been that students can do about 15-20 homework exercises a night so we very carefully chose smaller sets of questions which cover all of the necessary skills and get the students thinking more deeply about the Mathematics involved.

Critics of the Open Educational Resource movement might quip that “open-source is where bad content goes to die,” to which I say this: take a serious look at what we offer our students. Look through a few sections to see if what we’ve written is bad content in your opinion. I see this open-source book not as something which is “free and worth every penny”, but rather, as a high quality alternative to the business as usual of the textbook industry and I hope that you agree. If you have any comments, questions or concerns please feel free to contact me at jeff@stitz-zeager.com or Carl at carl@stitz-zeager.com.

Jeff Zeager
Lorain County Community College
January 25, 2010

Chapter 0

Prerequisites

The authors would like nothing more than to dive right into the sheer excitement of Precalculus. However, experience - our own as well as that of our colleagues - has taught us that is it beneficial, if not completely necessary, to review what students should know before embarking on a Precalculus adventure. The goal of Chapter 0 is exactly that: to review the concepts, skills and vocabulary we believe are prerequisite to a rigorous, college-level Precalculus course. This review is not designed to teach the material to students who have never seen it before thus the presentation is more succinct and the exercise sets are shorter than those usually found in an Intermediate Algebra text. An outline of the chapter is given below.

Section 0.1 (Basic Set Theory and Interval Notation) contains a brief summary of the set theory terminology used throughout the text including sets of real numbers and interval notation.

Section 0.2 (Real Number Arithmetic) lists the properties of real number arithmetic.

Section 0.3 (Linear Equations and Inequalities) focuses on solving linear equations and linear inequalities from a strictly algebraic perspective. The geometry of graphing lines in the plane is deferred until Section 2.1 (Linear Functions).

Section 0.4 (Absolute Value Equations and Inequalities) begins with a definition of absolute value as a distance. Fundamental properties of absolute value are listed and then basic equations and inequalities involving absolute value are solved using the 'distance definition' and those properties. Absolute value is revisited in much greater depth in Section 2.3 (Absolute Value Functions).

Section 0.5 (Polynomial Arithmetic) covers the addition, subtraction, multiplication and division of polynomials as well as the vocabulary which is used extensively when the graphs of polynomials are studied in Chapter 3 (Polynomials).

Section 0.6 (Factoring) covers basic factoring techniques and how to solve equations using those techniques along with the Zero Product Property of Real Numbers.

Section 0.7 (Quadratic Equations) discusses solving quadratic equations using the technique of 'completing the square' and by using the Quadratic Formula. Equations which are 'quadratic in form' are also discussed.

Section 0.8 (Rational Expressions and Equations) starts with the basic arithmetic of rational expressions and the simplifying of compound fractions. Solving equations by clearing denominators and the handling negative integer exponents are presented but the graphing of rational functions is deferred until Chapter 4 (Rational Functions).

Section 0.9 (Radicals and Equations) covers simplifying radicals as well as the solving of basic equations involving radicals.

0.1 Basic Set Theory and Interval Notation

0.1.1 Some Basic Set Theory Notions

Like all good Math books, we begin with a definition.

Definition 0.1. A **set** is a well-defined collection of objects which are called the 'elements' of the set. Here, 'well-defined' means that it is possible to determine if something belongs to the collection or not, without prejudice.

The collection of letters that make up the word "smolko" is well-defined and is a set, but the collection of the worst Math teachers in the world is **not** well-defined and therefore is **not** a set.[1] In general, there are three ways to describe sets and those methods are listed below.

Ways to Describe Sets

1. **The Verbal Method:** Use a sentence to define the set.
2. **The Roster Method:** Begin with a left brace '$\{$', list each element of the set *only once* and then end with a right brace '$\}$'.
3. **The Set-Builder Method:** A combination of the verbal and roster methods using a "dummy variable" such as x.

For example, let S be the set described *verbally* as the set of letters that make up the word "smolko". A **roster** description of S is $\{s, m, o, l, k\}$. Note that we listed 'o' only once, even though it appears twice in the word "smolko". Also, the *order* of the elements doesn't matter, so $\{k, l, m, o, s\}$ is also a roster description of S. Moving right along, a **set-builder** description of S is: $\{x \mid x \text{ is a letter in the word "smolko"}\}$. The way to read this is 'The set of elements x such that x is a letter in the word "smolko".' In each of the above cases, we may use the familiar equals sign '=' and write $S = \{s, m, o, l, k\}$ or $S = \{x \mid x \text{ is a letter in the word "smolko"}\}$.

Notice that m is in S but many other letters, such as q, are not in S. We express these ideas of set inclusion and exclusion mathematically using the symbols $m \in S$ (read 'm is in S') and $q \notin S$ (read 'q is not in S'). More precisely, we have the following.

Definition 0.2. Let A be a set.

- If x is an element of A then we write $x \in A$ which is read 'x is in A'.
- If x is *not* an element of A then we write $x \notin A$ which is read 'x is not in A'.

Now let's consider the set $C = \{x \mid x \text{ is a consonant in the word "smolko"}\}$. A roster description of C is $C = \{s, m, l, k\}$. Note that by construction, every element of C is also in S. We express

[1] For a more thought-provoking example, consider the collection of all things that do not contain themselves - this leads to the famous Russell's Paradox.

this relationship by stating that the set C is a **subset** of the set S, which is written in symbols as $C \subseteq S$. The more formal definition is given below.

Definition 0.3. Given sets A and B, we say that the set A is a **subset** of the set B and write '$A \subseteq B$' if every element in A is also an element of B.

Note that in our example above $C \subseteq S$, but not vice-versa, since $o \in S$ but $o \notin C$. Additionally, the set of vowels $V = \{a, e, i, o, u\}$, while it does have an element in common with S, is not a subset of S. (As an added note, S is not a subset of V, either.) We could, however, *build* a set which contains both S and V as subsets by gathering all of the elements in both S and V together into a single set, say $U = \{s, m, o, l, k, a, e, i, u\}$. Then $S \subseteq U$ and $V \subseteq U$. The set U we have built is called the **union** of the sets S and V and is denoted $S \cup V$. Furthermore, S and V aren't completely *different* sets since they both contain the letter 'o.' The **intersection** of two sets is the set of elements (if any) the two sets have in common. In this case, the intersection of S and V is $\{o\}$, written $S \cap V = \{o\}$. We formalize these ideas below.

Definition 0.4. Suppose A and B are sets.

- The **intersection** of A and B is $A \cap B = \{x \,|\, x \in A \text{ and } x \in B\}$
- The **union** of A and B is $A \cup B = \{x \,|\, x \in A \text{ or } x \in B \text{ (or both)}\}$

The key words in Definition 0.4 to focus on are the conjunctions: 'intersection' corresponds to 'and' meaning the elements have to be in *both* sets to be in the intersection, whereas 'union' corresponds to 'or' meaning the elements have to be in one set, or the other set (or both). In other words, to belong to the union of two sets an element must belong to *at least one* of them.

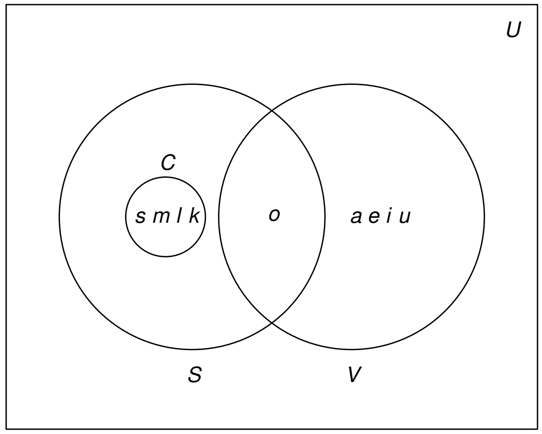
Returning to the sets C and V above, $C \cup V = \{s, m, l, k, a, e, i, o, u\}$.[2] When it comes to their intersection, however, we run into a bit of notational awkwardness since C and V have no elements in common. While we could write $C \cap V = \{\}$, this sort of thing happens often enough that we give the set with no elements a name.

Definition 0.5. The **Empty Set** $\emptyset$ is the set which contains no elements. That is,

$$\emptyset = \{\} = \{x \,|\, x \neq x\}.$$

As promised, the empty set is the set containing no elements since no matter what 'x' is, '$x = x$.' Like the number '0,' the empty set plays a vital role in mathematics.[3] We introduce it here more as a symbol of convenience as opposed to a contrivance.[4] Using this new bit of notation, we have for the sets C and V above that $C \cap V = \emptyset$. A nice way to visualize relationships between sets and set operations is to draw a **Venn Diagram**. A Venn Diagram for the sets S, C and V is drawn at the top of the next page.

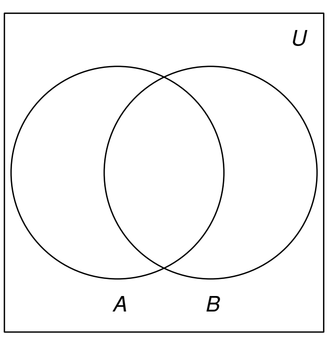
[2]Which just so happens to be the same set as $S \cup V$.
[3]Sadly, the full extent of the empty set's role will not be explored in this text.
[4]Actually, the empty set can be used to generate numbers - mathematicians can create something from nothing!



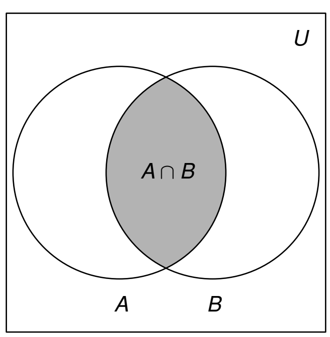
A Venn Diagram for C, S and V.

In the Venn Diagram above we have three circles - one for each of the sets C, S and V. We visualize the area enclosed by each of these circles as the elements of each set. Here, we've spelled out the elements for definitiveness. Notice that the circle representing the set C is completely inside the circle representing S. This is a geometric way of showing that $C \subseteq S$. Also, notice that the circles representing S and V overlap on the letter 'o'. This common region is how we visualize $S \cap V$. Notice that since $C \cap V = \emptyset$, the circles which represent C and V have no overlap whatsoever.

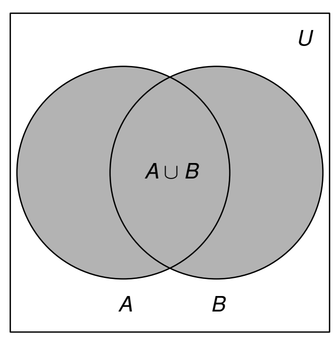
All of these circles lie in a rectangle labeled U (for 'universal' set). A universal set contains all of the elements under discussion, so it could always be taken as the union of all of the sets in question, or an even larger set. In this case, we could take $U = S \cup V$ or U as the set of letters in the entire alphabet. The reader may well wonder if there is an ultimate universal set which contains *everything*. The short answer is 'no' and we refer you once again to Russell's Paradox. The usual triptych of Venn Diagrams indicating generic sets A and B along with $A \cap B$ and $A \cup B$ is given below.



Sets A and B.



$A \cap B$ is shaded.



$A \cup B$ is shaded.

0.1.2 Sets of Real Numbers

The playground for most of this text is the set of **Real Numbers**. Many quantities in the 'real world' can be quantified using real numbers: the temperature at a given time, the revenue generated by selling a certain number of products and the maximum population of Sasquatch which can inhabit a particular region are just three basic examples. A succinct, but nonetheless incomplete[5] definition of a real number is given below.

Definition 0.6. A **real number** is any number which possesses a decimal representation. The set of real numbers is denoted by the character $\mathbb{R}$.

Certain subsets of the real numbers are worthy of note and are listed below. In fact, in more advanced texts,[6] the real numbers are *constructed* from some of these subsets.

Special Subsets of Real Numbers

1. The **Natural Numbers**: $\mathbb{N} = \{1, 2, 3, \ldots\}$ The periods of ellipsis '...' here indicate that the natural numbers contain 1, 2, 3 'and so forth'.
2. The **Whole Numbers**: $\mathbb{W} = \{0, 1, 2, \ldots\}$.
3. The **Integers**: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$.[a]
4. The **Rational Numbers**: $\mathbb{Q} = \left\{ \frac{a}{b} \,|\, a \in \mathbb{Z} \text{ and } b \in \mathbb{Z} \right\}$. Rational numbers are the ratios of integers where the denominator is not zero. It turns out that another way to describe the rational numbers[b] is:

   $$\mathbb{Q} = \{x \,|\, x \text{ possesses a repeating or terminating decimal representation}\}$$

5. The **Irrational Numbers**: $\mathbb{P} = \{x \,|\, x \in \mathbb{R} \text{ but } x \notin \mathbb{Q}\}$.[c] That is, an irrational number is a real number which isn't rational. Said differently,

   $$\mathbb{P} = \{x \,|\, x \text{ possesses a decimal representation which neither repeats nor terminates}\}$$

[a] The symbol $\pm$ is read 'plus or minus' and it is a shorthand notation which appears throughout the text. Just remember that $x = \pm 3$ means $x = 3$ or $x = -3$.

[b] See Section 9.2.

[c] Examples here include number π (See Section 10.1), $\sqrt{2}$ and 0.101001000100001 ....

Note that every natural number is a whole number which, in turn, is an integer. Each integer is a rational number (take $b = 1$ in the above definition for $\mathbb{Q}$) and since every rational number is a real number[7] the sets $\mathbb{N}$, $\mathbb{W}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are nested like Matryoshka dolls. More formally, these sets form a subset chain: $\mathbb{N} \subseteq \mathbb{W} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$. The reader is encouraged to sketch a Venn Diagram depicting $\mathbb{R}$ and all of the subsets mentioned above. It is time for an example.

[5] Math pun intended!

[6] See, for instance, Landau's Foundations of Analysis.

[7] Thanks to long division!

Example 0.1.1.

1. Write a roster description for $P = \{2^n \mid n \in \mathbb{N}\}$ and $E = \{2n \mid n \in \mathbb{Z}\}$.

2. Write a verbal description for $S = \{x^2 \mid x \in \mathbb{R}\}$.

3. Let $A = \{-117, \frac{4}{5}, 0.20\overline{2002}, 0.202002000200002\ldots\}$.

   (a) Which elements of A are natural numbers? Rational numbers? Real numbers?

   (b) Find $A \cap \mathbb{W}$, $A \cap \mathbb{Z}$ and $A \cap \mathbb{P}$.

4. What is another name for $\mathbb{N} \cup \mathbb{Q}$? What about $\mathbb{Q} \cup \mathbb{P}$?

Solution.

1. To find a roster description for these sets, we need to list their elements. Starting with $P = \{2^n \mid n \in \mathbb{N}\}$, we substitute natural number values n into the formula 2^n. For $n = 1$ we get $2^1 = 2$, for $n = 2$ we get $2^2 = 4$, for $n = 3$ we get $2^3 = 8$ and for $n = 4$ we get $2^4 = 16$. Hence P describes the powers of 2, so a roster description for P is $P = \{2, 4, 8, 16, \ldots\}$ where the '...' indicates the that pattern continues.[8]

   Proceeding in the same way, we generate elements in $E = \{2n \mid n \in \mathbb{Z}\}$ by plugging in integer values of n into the formula $2n$. Starting with $n = 0$ we obtain $2(0) = 0$. For $n = 1$ we get $2(1) = 2$, for $n = -1$ we get $2(-1) = -2$ for $n = 2$, we get $2(2) = 4$ and for $n = -2$ we get $2(-2) = -4$. As n moves through the integers, $2n$ produces all of the *even* integers.[9] A roster description for E is $E = \{0, \pm 2, \pm 4, \ldots\}$.

2. One way to verbally describe S is to say that S is the 'set of all squares of real numbers'. While this isn't incorrect, we'd like to take this opportunity to delve a little deeper.[10] What makes the set $S = \{x^2 \mid x \in \mathbb{R}\}$ a little trickier to wrangle than the sets P or E above is that the dummy variable here, x, runs through all *real* numbers. Unlike the natural numbers or the integers, the real numbers cannot be listed in any methodical way.[11] Nevertheless, we can select some real numbers, square them and get a sense of what kind of numbers lie in S. For $x = -2$, $x^2 = (-2)^2 = 4$ so 4 is in S, as are $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$ and $(\sqrt{117})^2 = 117$. Even things like $(-\pi)^2$ and $(0.101001000100001\ldots)^2$ are in S.

   So suppose $s \in S$. What can be said about s? We know there is some real number x so that $s = x^2$. Since $x^2 \geq 0$ for any real number x, we know $s \geq 0$. This tells us that everything

[8] This isn't the most *precise* way to describe this set - it's always dangerous to use '...' since we assume that the pattern is clearly demonstrated and thus made evident to the reader. Formulas are more precise because the pattern is clear.

[9] This shouldn't be too surprising, since an even integer is *defined* to be an integer multiple of 2.

[10] Think of this as an opportunity to stop and smell the mathematical roses.

[11] This is a nontrivial statement. Interested readers are directed to a discussion of Cantor's Diagonal Argument.

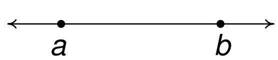
in S is a non-negative real number.[12] This begs the question: are all of the non-negative real numbers in S? Suppose n is a non-negative real number, that is, $n \geq 0$. If n were in S, there would be a real number x so that $x^2 = n$. As you may recall, we can solve $x^2 = n$ by 'extracting square roots': $x = \pm\sqrt{n}$. Since $n \geq 0$, $\sqrt{n}$ is a real number.[13] Moreover, $(\sqrt{n})^2 = n$ so n is the square of a real number which means $n \in S$. Hence, S is the set of non-negative real numbers.

3. (a) The set A contains no natural numbers.[14] Clearly, $\frac{4}{5}$ is a rational number as is -117 (which can be written as $\frac{-117}{1}$). It's the last two numbers listed in A, $0.20\overline{2002}$ and $0.202002000200002\ldots$, that warrant some discussion. First, recall that the 'line' over the digits 2002 in $0.20\overline{2002}$ (called the vinculum) indicates that these digits repeat, so it is a rational number.[15] As for the number $0.202002000200002\ldots$, the '...' indicates the pattern of adding an extra '0' followed by a '2' is what defines this real number. Despite the fact there is a *pattern* to this decimal, this decimal is *not repeating*, so it is not a rational number - it is, in fact, an irrational number. All of the elements of A are real numbers, since all of them can be expressed as decimals (remember that $\frac{4}{5} = 0.8$).

   (b) The set $A \cap \mathbb{W} = \{x \,|\, x \in A \text{ and } x \in \mathbb{W}\}$ is another way of saying we are looking for the set of numbers in A which are whole numbers. Since A contains no whole numbers, $A \cap \mathbb{W} = \emptyset$. Similarly, $A \cap \mathbb{Z}$ is looking for the set of numbers in A which are integers. Since -117 is the only integer in A, $A \cap \mathbb{Z} = \{-117\}$. As for the set $A \cap \mathbb{P}$, as discussed in part (a), the number $0.202002000200002\ldots$ is irrational, so $A \cap \mathbb{P} = \{0.202002000200002\ldots\}$.

4. The set $\mathbb{N} \cup \mathbb{Q} = \{x \,|\, x \in \mathbb{N} \text{ or } x \in \mathbb{Q}\}$ is the union of the set of natural numbers with the set of rational numbers. Since every natural number is a rational number, $\mathbb{N}$ doesn't contribute any new elements to $\mathbb{Q}$, so $\mathbb{N} \cup \mathbb{Q} = \mathbb{Q}$.[16] For the set $\mathbb{Q} \cup \mathbb{P}$, we note that every real number is either rational or not, hence $\mathbb{Q} \cup \mathbb{P} = \mathbb{R}$, pretty much by the definition of the set $\mathbb{P}$. □

As you may recall, we often visualize the set of real numbers $\mathbb{R}$ as a line where each point on the line corresponds to one and only one real number. Given two different real numbers a and b, we write $a < b$ if a is located to the left of b on the number line, as shown below.



The real number line with two numbers a and b where $a < b$.

While this notion seems innocuous, it is worth pointing out that this convention is rooted in two deep properties of real numbers. The first property is that $\mathbb{R}$ is complete. This means that there

[12] This means S is a subset of the non-negative real numbers.
[13] This is called the 'square root closed' property of the non-negative real numbers.
[14] Carl was tempted to include $0.\overline{9}$ in the set A, but thought better of it. See Section 9.2 for details.
[15] So $0.20\overline{2002} = 0.20200220022002\ldots$.
[16] In fact, anytime $A \subseteq B$, $A \cup B = B$ and vice-versa. See the exercises.

are no ‘holes’ or ‘gaps’ in the real number line.[17] Another way to think about this is that if you choose any two distinct (different) real numbers, and look between them, you’ll find a solid line segment (or interval) consisting of infinitely many real numbers. The next result tells us what types of numbers we can expect to find.

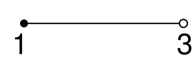
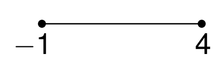
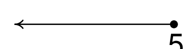
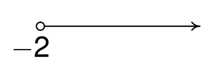
Density Property of $\mathbb{Q}$ and $\mathbb{P}$ in $\mathbb{R}$
Between any two distinct real numbers, there is at least one rational number and irrational number. It then follows that between any two distinct real numbers there will be infinitely many rational and irrational numbers.

The root word ‘dense’ here communicates the idea that rationals and irrationals are ‘thoroughly mixed’ into $\mathbb{R}$. The reader is encouraged to think about how one would find both a rational and an irrational number between, say, 0.9999 and 1. Once you’ve done that, try doing the same thing for the numbers $0.\overline{9}$ and 1. (‘Try’ is the operative word, here.[18])

The second property $\mathbb{R}$ possesses that lets us view it as a line is that the set is totally ordered. This means that given any two real numbers a and b, either $a < b$, $a > b$ or $a = b$ which allows us to arrange the numbers from least (left) to greatest (right). You may have heard this property given as the ‘Law of Trichotomy’.

Law of Trichotomy
If a and b are real numbers then exactly one of the following statements is true:
$a < b$ $\qquad$ $a > b$ $\qquad$ $a = b$

Segments of the real number line are called **intervals**. They play a huge role not only in this text but also in the Calculus curriculum so we need a concise way to describe them. We start by examining a few examples of the **interval notation** associated with some specific sets of numbers.

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid 1 \leq x < 3\}$	$[1, 3)$	
$\{x \mid -1 \leq x \leq 4\}$	$[-1, 4]$	
$\{x \mid x \leq 5\}$	$(-\infty, 5]$	
$\{x \mid x > -2\}$	$(-2, \infty)$	

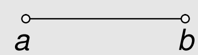
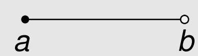
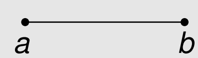
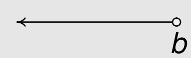
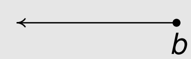
As you can glean from the table, for intervals with finite endpoints we start by writing ‘left endpoint, right endpoint’. We use square brackets, ‘[’ or ‘]’, if the endpoint is included in the interval. This

[17] Alas, this intuitive feel for what it means to be ‘complete’ is as good as it gets at this level. Completeness does get a much more precise meaning later in courses like Analysis and Topology.
[18] Again, see Section 9.2 for details.

corresponds to a ‘filled-in’ or ‘closed’ dot on the number line to indicate that the number is included in the set. Otherwise, we use parentheses, ‘(’ or ‘)’ that correspond to an ‘open’ circle which indicates that the endpoint is not part of the set. If the interval does not have finite endpoints, we use the symbol $-\infty$ to indicate that the interval extends indefinitely to the left and the symbol ∞ to indicate that the interval extends indefinitely to the right. Since infinity is a concept, and not a number, we always use parentheses when using these symbols in interval notation, and use the appropriate arrow to indicate that the interval extends indefinitely in one or both directions. We summarize all of the possible cases in one convenient table below.[19]

Interval Notation

Let a and b be real numbers with $a < b$.

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid a < x < b\}$	(a, b)	
$\{x \mid a \leq x < b\}$	$[a, b)$	
$\{x \mid a < x \leq b\}$	$(a, b]$	
$\{x \mid a \leq x \leq b\}$	$[a, b]$	
$\{x \mid x < b\}$	$(-\infty, b)$	
$\{x \mid x \leq b\}$	$(-\infty, b]$	
$\{x \mid x > a\}$	(a, ∞)	
$\{x \mid x \geq a\}$	$[a, \infty)$	
$\mathbb{R}$	$(-\infty, \infty)$	

We close this section with an example that ties together several concepts presented earlier. Specifically, we demonstrate how to use interval notation along with the concepts of ‘union’ and ‘intersection’ to describe a variety of sets on the real number line.

[19]The importance of understanding interval notation in Calculus cannot be overstated so please do yourself a favor and memorize this chart.

Example 0.1.2.

1. Express the following sets of numbers using interval notation.

   (a) $\{x \mid x \leq -2 \text{ or } x \geq 2\}$

   (b) $\{x \mid x \neq 3\}$

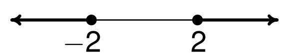
   (c) $\{x \mid x \neq \pm 3\}$

   (d) $\{x \mid -1 < x \leq 3 \text{ or } x = 5\}$

2. Let $A = [-5, 3)$ and $B = (1, \infty)$. Find $A \cap B$ and $A \cup B$.

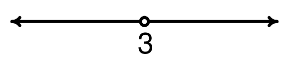
Solution.

1. (a) The best way to proceed here is to graph the set of numbers on the number line and glean the answer from it. The inequality $x \leq -2$ corresponds to the interval $(-\infty, -2]$ and the inequality $x \geq 2$ corresponds to the interval $[2, \infty)$. The 'or' in $\{x \mid x \leq -2 \text{ or } x \geq 2\}$ tells us that we are looking for the union of these two intervals, so our answer is $(-\infty, -2] \cup [2, \infty)$.



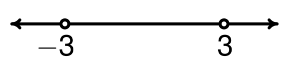
$(-\infty, -2] \cup [2, \infty)$

   (b) For the set $\{x \mid x \neq 3\}$, we shade the entire real number line except $x = 3$, where we leave an open circle. This divides the real number line into two intervals, $(-\infty, 3)$ and $(3, \infty)$. Since the values of x could be in one of these intervals *or* the other, we once again use the union symbol to get $\{x \mid x \neq 3\} = (-\infty, 3) \cup (3, \infty)$.



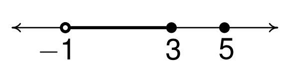
$(-\infty, 3) \cup (3, \infty)$

   (c) For the set $\{x \mid x \neq \pm 3\}$, we proceed as before and exclude both $x = 3$ and $x = -3$ from our set. (Do you remember what we said back on 6 about $x = \pm 3$?) This breaks the number line into *three* intervals, $(-\infty, -3)$, $(-3, 3)$ and $(3, \infty)$. Since the set describes real numbers which come from the first, second *or* third interval, we have $\{x \mid x \neq \pm 3\} = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.



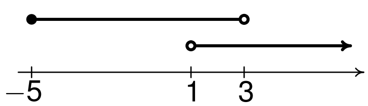
$(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

   (d) Graphing the set $\{x \mid -1 < x \leq 3 \text{ or } x = 5\}$ yields the interval $(-1, 3]$ along with the single number 5. While we *could* express this single point as $[5, 5]$, it is customary to write a single point as a 'singleton set', so in our case we have the set $\{5\}$. Thus our final answer is $\{x \mid -1 < x \leq 3 \text{ or } x = 5\} = (-1, 3] \cup \{5\}$.

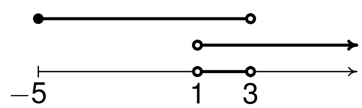


$(-1, 3] \cup \{5\}$

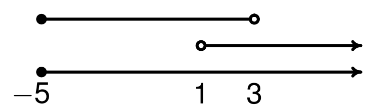
2. We start by graphing $A = [-5, 3)$ and $B = (1, \infty)$ on the number line. To find $A \cap B$, we need to find the numbers in common to both A and B, in other words, the overlap of the two intervals. Clearly, everything between 1 and 3 is in both A and B. However, since 1 is in A but not in B, 1 is not in the intersection. Similarly, since 3 is in B but not in A, it isn't in the intersection either. Hence, $A \cap B = (1, 3)$. To find $A \cup B$, we need to find the numbers in at least one of A or B. Graphically, we shade A and B along with it. Notice here that even though 1 isn't in B, it is in A, so it's the union along with all the other elements of A between -5 and 1. A similar argument goes for the inclusion of 3 in the union. The result of shading both A and B together gives us $A \cup B = [-5, \infty)$.



$A = [-5, 3)$, $B = (1, \infty)$



$A \cap B = (1, 3)$



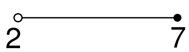
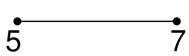
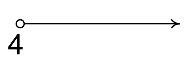
$A \cup B = [-5, \infty)$ □

0.1.3 Exercises

1. Find a verbal description for $O = \{2n - 1 \mid n \in \mathbb{N}\}$
2. Find a roster description for $X = \{z^2 \mid z \in \mathbb{Z}\}$
3. Let $A = \left\{-3, -1.02, -\frac{3}{5}, 0.57, 1.\overline{23}, \sqrt{3}, 5.2020020002\ldots, \frac{20}{10}, 117\right\}$

    (a) List the elements of A which are natural numbers.
    (b) List the elements of A which are irrational numbers.
    (c) Find $A \cap \mathbb{Z}$
    (d) Find $A \cap \mathbb{Q}$

4. Fill in the chart below.

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid -1 \leq x < 5\}$		
	$[0, 3)$	
		
$\{x \mid -5 < x \leq 0\}$		
	$(-3, 3)$	
		
$\{x \mid x \leq 3\}$		
	$(-\infty, 9)$	
		
$\{x \mid x \geq -3\}$		

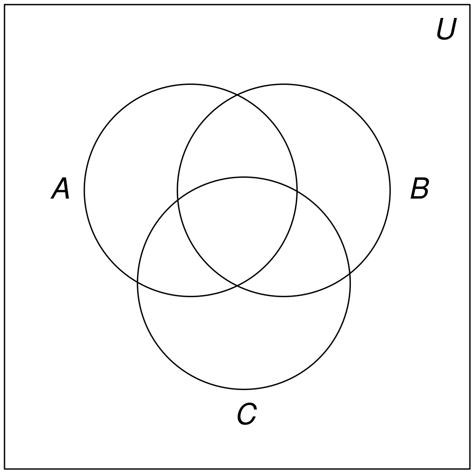
In Exercises 5 - 10, find the indicated intersection or union and simplify if possible. Express your answers in interval notation.

5. $(-1, 5] \cap [0, 8)$
6. $(-1, 1) \cup [0, 6]$
7. $(-\infty, 4] \cap (0, \infty)$
8. $(-\infty, 0) \cap [1, 5]$
9. $(-\infty, 0) \cup [1, 5]$
10. $(-\infty, 5] \cap [5, 8)$

In Exercises 11 - 22, write the set using interval notation.

11. $\{x \mid x \neq 5\}$
12. $\{x \mid x \neq -1\}$
13. $\{x \mid x \neq -3, 4\}$
14. $\{x \mid x \neq 0, 2\}$
15. $\{x \mid x \neq 2, -2\}$
16. $\{x \mid x \neq 0, \pm 4\}$
17. $\{x \mid x \leq -1 \text{ or } x \geq 1\}$
18. $\{x \mid x < 3 \text{ or } x \geq 2\}$
19. $\{x \mid x \leq -3 \text{ or } x > 0\}$
20. $\{x \mid x \leq 5 \text{ or } x = 6\}$
21. $\{x \mid x > 2 \text{ or } x = \pm 1\}$
22. $\{x \mid -3 < x < 3 \text{ or } x = 4\}$

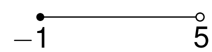
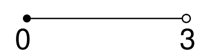
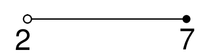
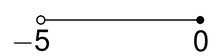
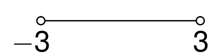
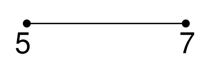
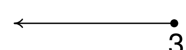
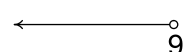
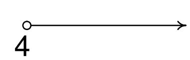
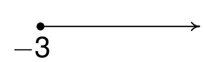
For Exercises 23 - 28, use the blank Venn Diagram below A, B, and C as a guide for you to shade the following sets.



23. $A \cup C$
24. $B \cap C$
25. $(A \cup B) \cup C$
26. $(A \cap B) \cap C$
27. $A \cap (B \cup C)$
28. $(A \cap B) \cup (A \cap C)$
29. Explain how your answers to problems 27 and 28 show $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Phrased differently, this shows 'intersection *distributes* over union.' Discuss with your classmates if 'union' distributes over 'intersection.' Use a Venn Diagram to support your answer.
30. Discuss with your classmates how many numbers are in the interval $(0, 1)$.

0.1.4 Answers

1. O is the odd natural numbers.
2. $X = \{0, 1, 4, 9, 16, ...\}$
3. (a) $\frac{20}{10} = 2$ and 117
   (b) $\sqrt{3}$ and 5.2020020002
   (c) $\left\{-3, \frac{20}{10}, 117\right\}$
   (d) $\left\{-3, -1.02, -\frac{3}{5}, 0.57, 1.\overline{23}, \frac{20}{10}, 117\right\}$
4.

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid -1 \leq x < 5\}$	$[-1, 5)$	
$\{x \mid 0 \leq x < 3\}$	$[0, 3)$	
$\{x \mid 2 < x \leq 7\}$	$(2, 7]$	
$\{x \mid -5 < x \leq 0\}$	$(-5, 0]$	
$\{x \mid -3 < x < 3\}$	$(-3, 3)$	
$\{x \mid 5 \leq x \leq 7\}$	$[5, 7]$	
$\{x \mid x \leq 3\}$	$(-\infty, 3]$	
$\{x \mid x < 9\}$	$(-\infty, 9)$	
$\{x \mid x > 4\}$	$(4, \infty)$	
$\{x \mid x \geq -3\}$	$[-3, \infty)$	

5. $(-1,5] \cap [0,8) = [0,5]$

6. $(-1,1) \cup [0,6] = (-1,6]$

7. $(-\infty,4] \cap (0,\infty) = (0,4]$

8. $(-\infty,0) \cap [1,5] = \emptyset$

9. $(-\infty,0) \cup [1,5] = (-\infty,0) \cup [1,5]$

10. $(-\infty,5] \cap [5,8) = \{5\}$

11. $(-\infty,5) \cup (5,\infty)$

12. $(-\infty,-1) \cup (-1,\infty)$

13. $(-\infty,-3) \cup (-3,4) \cup (4,\infty)$

14. $(-\infty,0) \cup (0,2) \cup (2,\infty)$

15. $(-\infty,-2) \cup (-2,2) \cup (2,\infty)$

16. $(-\infty,-4) \cup (-4,0) \cup (0,4) \cup (4,\infty)$

17. $(-\infty,-1] \cup [1,\infty)$

18. $(-\infty,\infty)$

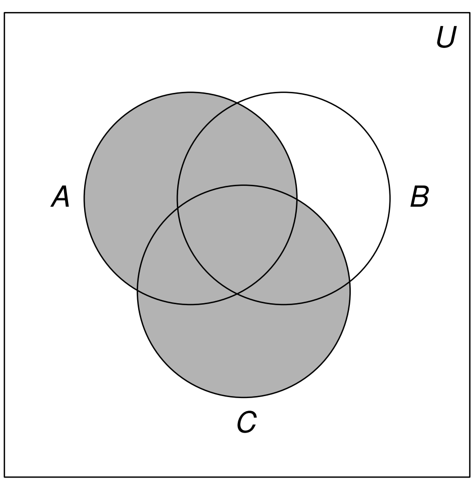
19. $(-\infty,-3] \cup (0,\infty)$

20. $(-\infty,5] \cup \{6\}$

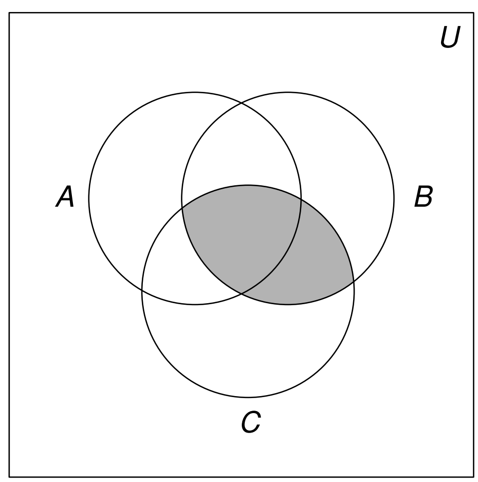
21. $\{-1\} \cup \{1\} \cup (2,\infty)$

22. $(-3,3) \cup \{4\}$

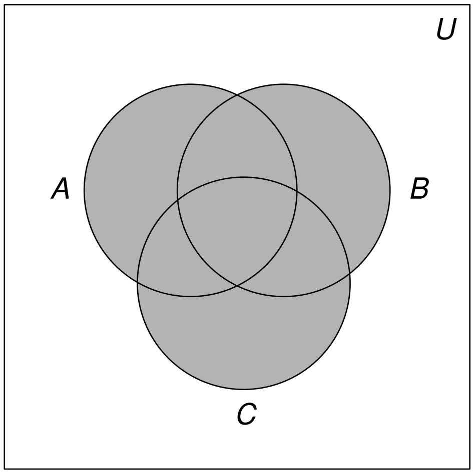
23. $A \cup C$



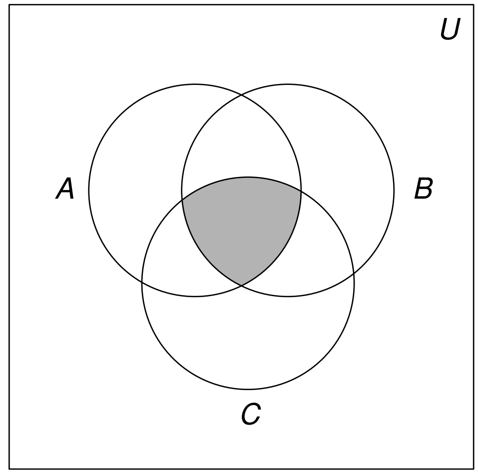
24. $B \cap C$



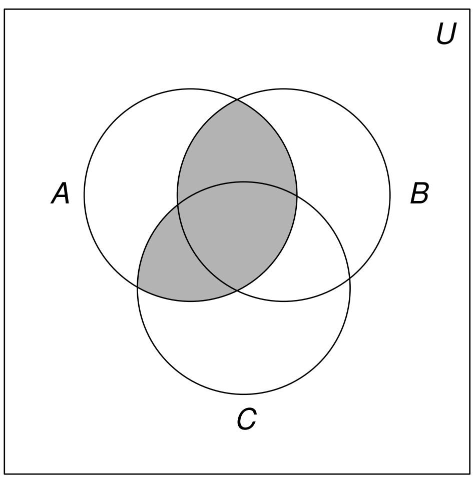
25. $(A \cup B) \cup C$



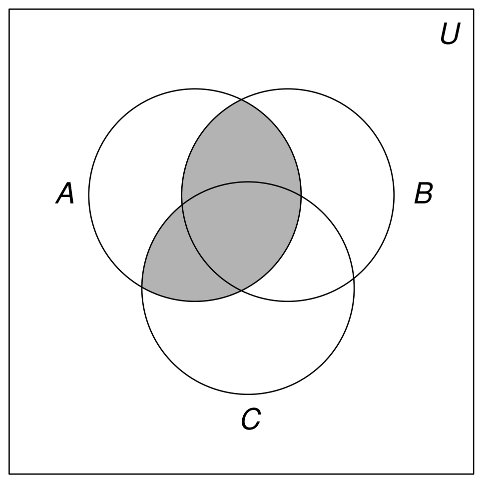
26. $(A \cap B) \cap C$



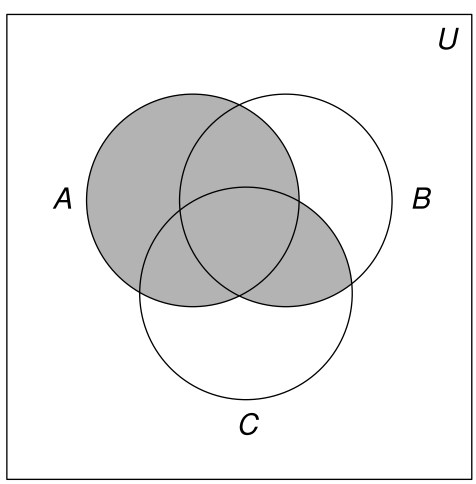
27. $A \cap (B \cup C)$



28. $(A \cap B) \cup (A \cap C)$



29. Yes, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.



0.2 Real Number Arithmetic

In this section we list the properties of real number arithmetic. This is meant to be a succinct, targeted review so we'll resist the temptation to wax poetic about these axioms and their subtleties and refer the interested reader to a more formal course in Abstract Algebra. There are two (primary) operations one can perform with real numbers: addition and multiplication.

Properties of Real Number Addition

- **Closure:** For all real numbers a and b, $a + b$ is also a real number.
- **Commutativity:** For all real numbers a and b, $a + b = b + a$.
- **Associativity:** For all real numbers a, b and c, $a + (b + c) = (a + b) + c$.
- **Identity:** There is a real number '0' so that for all real numbers a, $a + 0 = a$.
- **Inverse:** For all real numbers a, there is a real number $-a$ such that $a + (-a) = 0$.
- **Definition of Subtraction:** For all real numbers a and b, $a - b = a + (-b)$.

Next, we give real number multiplication a similar treatment. Recall that we may denote the product of two real numbers a and b a variety of ways: ab, $a \cdot b$, $a(b)$, $(a)(b)$ and so on. We'll refrain from using $a \times b$ for real number multiplication in this text with one notable exception in Definition 0.7.

Properties of Real Number Multiplication

- **Closure:** For all real numbers a and b, ab is also a real number.
- **Commutativity:** For all real numbers a and b, $ab = ba$.
- **Associativity:** For all real numbers a, b and c, $a(bc) = (ab)c$.
- **Identity:** There is a real number '1' so that for all real numbers a, $a \cdot 1 = a$.
- **Inverse:** For all real numbers $a \neq 0$, there is a real number $\dfrac{1}{a}$ such that $a\left(\dfrac{1}{a}\right) = 1$.
- **Definition of Division:** For all real numbers a and $b \neq 0$, $a \div b = \dfrac{a}{b} = a\left(\dfrac{1}{b}\right)$.

While most students and some faculty tend to skip over these properties or give them a cursory glance at best,[1] it is important to realize that the properties stated above are what drive the symbolic manipulation for all of Algebra. When listing a tally of more than two numbers, $1 + 2 + 3$ for example, we don't need to specify the order in which those numbers are added. Notice though, try as we might, we can add only two numbers at a time and it is the associative property of addition which assures us that we could organize this sum as $(1 + 2) + 3$ or $1 + (2 + 3)$. This brings up a

[1] Not unlike how Carl approached all the Elven poetry in The Lord of the Rings.

note about 'grouping symbols'. Recall that parentheses and brackets are used in order to specify which operations are to be performed first. In the absence of such grouping symbols, multiplication (and hence division) is given priority over addition (and hence subtraction). For example, $1 + 2 \cdot 3 = 1 + 6 = 7$, but $(1 + 2) \cdot 3 = 3 \cdot 3 = 9$. As you may recall, we can 'distribute' the 3 across the addition if we really wanted to do the multiplication first: $(1 + 2) \cdot 3 = 1 \cdot 3 + 2 \cdot 3 = 3 + 6 = 9$. More generally, we have the following.

The Distributive Property and Factoring

For all real numbers a, b and c:

- **Distributive Property:** $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$.
- **Factoring:**[a] $ab + ac = a(b + c)$ and $ac + bc = (a + b)c$.

[a]Or, as Carl calls it, 'reading the Distributive Property from right to left.'

It is worth pointing out that we didn't really need to list the Distributive Property both for $a(b + c)$ (distributing from the left) and $(a + b)c$ (distributing from the right), since the commutative property of multiplication gives us one from the other. Also, 'factoring' really is the same equation as the distributive property, just read from right to left. These are the first of many redundancies in this section, and they exist in this review section for one reason only - in our experience, many students *see* these things differently so we will list them as such.

It is hard to overstate the importance of the Distributive Property. For example, in the expression $5(2 + x)$, without knowing the value of x, we cannot perform the addition inside the parentheses first; we must rely on the distributive property here to get $5(2 + x) = 5 \cdot 2 + 5 \cdot x = 10 + 5x$. The Distributive Property is also responsible for combining 'like terms'. Why is $3x + 2x = 5x$? Because $3x + 2x = (3 + 2)x = 5x$.

We continue our review with summaries of other properties of arithmetic, each of which can be derived from the properties listed above. First up are properties of the additive identity 0.

Properties of Zero

Suppose a and b are real numbers.

- **Zero Product Property:** $ab = 0$ if and only if $a = 0$ or $b = 0$ (or both)

  Note: This not only says that $0 \cdot a = 0$ for any real number a, it also says that the *only* way to get an answer of '0' when multiplying two real numbers is to have one (or both) of the numbers be '0' in the first place.

- **Zeros in Fractions:** If $a \neq 0$, $\dfrac{0}{a} = 0 \cdot \left(\dfrac{1}{a}\right) = 0$.

  Note: The quantity $\dfrac{a}{0}$ is undefined.[a]

[a]The expression $\frac{0}{0}$ is technically an 'indeterminant form' as opposed to being strictly 'undefined' meaning that with Calculus we can make some sense of it in certain situations. We'll talk more about this in Chapter 4.

The Zero Product Property drives most of the equation solving algorithms in Algebra because it allows us to take complicated equations and reduce them to simpler ones. For example, you may recall that one way to solve $x^2 + x - 6 = 0$ is by factoring[2] the left hand side of this equation to get $(x - 2)(x + 3) = 0$. From here, we apply the Zero Product Property and set each factor equal to zero. This yields $x - 2 = 0$ or $x + 3 = 0$ so $x = 2$ or $x = -3$. This application to solving equations leads, in turn, to some deep and profound structure theorems in Chapter 3.

Next up is a review of the arithmetic of 'negatives'. On page 18 we first introduced the dash which we all recognize as the 'negative' symbol in terms of the additive inverse. For example, the number -3 (read 'negative 3') is defined so that $3 + (-3) = 0$. We then defined subtraction using the concept of the additive inverse again so that, for example, $5 - 3 = 5 + (-3)$. In this text we do not distinguish typographically between the dashes in the expressions '$5-3$' and '-3' even though they are mathematically quite different.[3] In the expression '$5 - 3$,' the dash is a *binary* operation (that is, an operation requiring *two* numbers) whereas in '-3', the dash is a *unary* operation (that is, an operation requiring only one number). You might ask, 'Who cares?' Your calculator does - that's who! In the text we can write $-3 - 3 = -6$ but that will not work in your calculator. Instead you'd need to type $^{-}3 - 3$ to get -6 where the first dash comes from the '$+/-$' key.

Properties of Negatives

Given real numbers a and b we have the following.

- **Additive Inverse Properties:** $-a = (-1)a$ and $-(-a) = a$
- **Products of Negatives:** $(-a)(-b) = ab$.
- **Negatives and Products:** $-ab = -(ab) = (-a)b = a(-b)$.
- **Negatives and Fractions:** If b is nonzero, $-\dfrac{a}{b} = \dfrac{-a}{b} = \dfrac{a}{-b}$ and $\dfrac{-a}{-b} = \dfrac{a}{b}$.
- **'Distributing' Negatives:** $-(a + b) = -a - b$ and $-(a - b) = -a + b = b - a$.
- **'Factoring' Negatives:**[a] $-a - b = -(a + b)$ and $b - a = -(a - b)$.

[a] Or, as Carl calls it, reading 'Distributing' Negatives from right to left.

An important point here is that when we 'distribute' negatives, we do so across addition or subtraction only. This is because we are really distributing a factor of -1 across each of these terms: $-(a + b) = (-1)(a + b) = (-1)(a) + (-1)(b) = (-a) + (-b) = -a - b$. Negatives do not 'distribute' across multiplication: $-(2\cdot 3) \neq (-2)\cdot(-3)$. Instead, $-(2\cdot 3) = (-2)\cdot(3) = (2)\cdot(-3) = -6$. The same sort of thing goes for fractions: $-\frac{3}{5}$ can be written as $\frac{-3}{5}$ or $\frac{3}{-5}$, but not $\frac{-3}{-5}$. Speaking of fractions, we now review their arithmetic.

[2] Don't worry. We'll review this in due course. And, yes, this is our old friend the Distributive Property!

[3] We're not just being lazy here. We looked at many of the big publishers' Precalculus books and none of them use different dashes, either.

Properties of Fractions

Suppose a, b, c and d are real numbers. Assume them to be nonzero whenever necessary; for example, when they appear in a denominator.

- **Identity Properties:** $a = \frac{a}{1}$ and $\frac{a}{a} = 1$.
- **Fraction Equality:** $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$.
- **Multiplication of Fractions:** $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$. In particular: $\frac{a}{b} \cdot c = \frac{a}{b} \cdot \frac{c}{1} = \frac{ac}{b}$

  Note: A common denominator is **not** required to **multiply** fractions!
- **Division**[a] **of Fractions:** $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$.

  In particular: $1 \div \frac{a}{b} = \frac{b}{a}$ and $\frac{a}{b} \div c = \frac{a}{b} \div \frac{c}{1} = \frac{a}{b} \cdot \frac{1}{c} = \frac{a}{bc}$

  Note: A common denominator is **not** required to **divide** fractions!
- **Addition and Subtraction of Fractions:** $\frac{a}{b} \pm \frac{c}{b} = \frac{a \pm c}{b}$.

  Note: A common denominator **is** required to **add or subtract** fractions!
- **Equivalent Fractions:** $\frac{a}{b} = \frac{ad}{bd}$, since $\frac{a}{b} = \frac{a}{b} \cdot 1 = \frac{a}{b} \cdot \frac{d}{d} = \frac{ad}{bd}$

  Note: The *only* way to change the denominator is to multiply both it and the numerator by the same nonzero value because we are, in essence, multiplying the fraction by 1.
- **'Reducing'**[b] **Fractions:** $\frac{a\cancel{d}}{b\cancel{d}} = \frac{a}{b}$, since $\frac{ad}{bd} = \frac{a}{b} \cdot \frac{d}{d} = \frac{a}{b} \cdot 1 = \frac{a}{b}$.

  In particular, $\frac{ab}{b} = a$ since $\frac{ab}{b} = \frac{ab}{1 \cdot b} = \frac{a\cancel{b}}{1 \cdot \cancel{b}} = \frac{a}{1} = a$ and $\frac{b-a}{a-b} = \frac{(-1)\cancel{(a-b)}}{\cancel{(a-b)}} = -1$.

  Note: We may only cancel common **factors** from both numerator and denominator.

[a] The old 'invert and multiply' or 'fraction gymnastics' play.

[b] Or 'Canceling' Common Factors - this is really just reading the previous property 'from right to left'.

Students make so many mistakes with fractions that we feel it is necessary to pause a moment in the narrative and offer you the following example.

Example 0.2.1. Perform the indicated operations and simplify. By 'simplify' here, we mean to have the final answer written in the form $\frac{a}{b}$ where a and b are integers which have no common factors. Said another way, we want $\frac{a}{b}$ in 'lowest terms'.

1. $\dfrac{1}{4}+\dfrac{6}{7}$
2. $\dfrac{5}{12}-\left(\dfrac{47}{30}-\dfrac{7}{3}\right)$
3. $\dfrac{\frac{7}{3-5}-\frac{7}{3-5.21}}{5-5.21}$
4. $\dfrac{\frac{12}{5}-\frac{7}{24}}{1+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)}$
5. $\dfrac{(2(2)+1)(-3-(-3))-5(4-7)}{4-2(3)}$
6. $\left(\dfrac{3}{5}\right)\left(\dfrac{5}{13}\right)-\left(\dfrac{4}{5}\right)\left(-\dfrac{12}{13}\right)$

Solution.

1. It may seem silly to start with an example this basic but experience has taught us not to take much for granted. We start by finding the lowest common denominator and then we rewrite the fractions using that new denominator. Since 4 and 7 are **relatively prime**, meaning they have no factors in common, the lowest common denominator is $4 \cdot 7 = 28$.

$$\begin{aligned}
\frac{1}{4}+\frac{6}{7} &= \frac{1}{4}\cdot\frac{7}{7}+\frac{6}{7}\cdot\frac{4}{4} && \text{Equivalent Fractions}\\
&= \frac{7}{28}+\frac{24}{28} && \text{Multiplication of Fractions}\\
&= \frac{31}{28} && \text{Addition of Fractions}
\end{aligned}$$

The result is in lowest terms because 31 and 28 are relatively prime so we're done.

2. We could begin with the subtraction in parentheses, namely $\frac{47}{30}-\frac{7}{3}$, and then subtract that result from $\frac{5}{12}$. It's easier, however, to first distribute the negative across the quantity in parentheses and then use the Associative Property to perform all of the addition and subtraction in one step.[4] The lowest common denominator[5] for all three fractions is 60.

$$\begin{aligned}
\frac{5}{12}-\left(\frac{47}{30}-\frac{7}{3}\right) &= \frac{5}{12}-\frac{47}{30}+\frac{7}{3} && \text{Distribute the Negative}\\
&= \frac{5}{12}\cdot\frac{5}{5}-\frac{47}{30}\cdot\frac{2}{2}+\frac{7}{3}\cdot\frac{20}{20} && \text{Equivalent Fractions}\\
&= \frac{25}{60}-\frac{94}{60}+\frac{140}{60} && \text{Multiplication of Fractions}\\
&= \frac{71}{60} && \text{Addition and Subtraction of Fractions}
\end{aligned}$$

The numerator and denominator are relatively prime so the fraction is in lowest terms and we have our final answer.

[4]See the remark on page 18 about how we add $1 + 2 + 3$.

[5]We could have used $12 \cdot 30 \cdot 3 = 1080$ as our common denominator but then the numerators would become unnecessarily large. It's best to use the *lowest* common denominator.

3. What we are asked to simplify in this problem is known as a 'complex' or 'compound' fraction. Simply put, we have fractions within a fraction.[6] The longest division line[7] acts as a grouping symbol, quite literally dividing the compound fraction into a numerator (containing fractions) and a denominator (which in this case does not contain fractions). The first step to simplifying a compound fraction like this one is to see if you can simplify the little fractions inside it. To that end, we clean up the fractions in the numerator as follows.

$$\begin{aligned}
\frac{\frac{7}{3-5}-\frac{7}{3-5.21}}{5-5.21} &= \frac{\frac{7}{-2}-\frac{7}{-2.21}}{-0.21} \\
&= \frac{-\left(-\frac{7}{2}+\frac{7}{2.21}\right)}{0.21} && \text{Properties of Negatives} \\
&= \frac{\frac{7}{2}-\frac{7}{2.21}}{0.21} && \text{Distribute the Negative}
\end{aligned}$$

We are left with a compound fraction with decimals. We could replace 2.21 with $\frac{221}{100}$ but that would make a mess.[8] It's better in this case to eliminate the decimal by multiplying the numerator and denominator of the fraction with the decimal in it by 100 (since $2.21 \cdot 100 = 221$ is an integer) as shown below.

$$\frac{\frac{7}{2}-\frac{7}{2.21}}{0.21} = \frac{\frac{7}{2}-\frac{7\cdot 100}{2.21\cdot 100}}{0.21} = \frac{\frac{7}{2}-\frac{700}{221}}{0.21}$$

We now perform the subtraction in the numerator and replace 0.21 with $\frac{21}{100}$ in the denominator. This will leave us with one fraction divided by another fraction. We finish by performing the 'division by a fraction is multiplication by the reciprocal' trick and then cancel any factors that we can.

$$\begin{aligned}
\frac{\frac{7}{2}-\frac{700}{221}}{0.21} &= \frac{\frac{7}{2}\cdot\frac{221}{221}-\frac{700}{221}\cdot\frac{2}{2}}{\frac{21}{100}} = \frac{\frac{1547}{442}-\frac{1400}{442}}{\frac{21}{100}} \\
&= \frac{\frac{147}{442}}{\frac{21}{100}} = \frac{147}{442}\cdot\frac{100}{21} = \frac{14700}{9282} = \frac{350}{221}
\end{aligned}$$

The last step comes from the factorizations $14700 = 42 \cdot 350$ and $9282 = 42 \cdot 221$.

[6] Fractionception, perhaps?
[7] Also called a 'vinculum'.
[8] Try it if you don't believe us.

4. We are given another compound fraction to simplify and this time both the numerator and denominator contain fractions. As before, the longest division line acts as a grouping symbol to separate the numerator from the denominator.

$$\frac{\frac{12}{5} - \frac{7}{24}}{1 + \left(\frac{12}{5}\right)\left(\frac{7}{24}\right)} = \frac{\left(\frac{12}{5} - \frac{7}{24}\right)}{\left(1 + \left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)}$$

Hence, one way to proceed is as before: simplify the numerator and the denominator then perform the 'division by a fraction is the multiplication by the reciprocal' trick. While there is nothing wrong with this approach, we'll use our Equivalent Fractions property to rid ourselves of the 'compound' nature of this fraction straight away. The idea is to multiply both the numerator and denominator by the lowest common denominator of each of the 'smaller' fractions - in this case, $24 \cdot 5 = 120$.

$$\begin{aligned}
\frac{\left(\frac{12}{5} - \frac{7}{24}\right)}{\left(1 + \left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)} &= \frac{\left(\frac{12}{5} - \frac{7}{24}\right) \cdot 120}{\left(1 + \left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right) \cdot 120} && \text{Equivalent Fractions} \\
&= \frac{\left(\frac{12}{5}\right)(120) - \left(\frac{7}{24}\right)(120)}{(1)(120) + \left(\frac{12}{5}\right)\left(\frac{7}{24}\right)(120)} && \text{Distributive Property} \\
&= \frac{\frac{12 \cdot 120}{5} - \frac{7 \cdot 120}{24}}{120 + \frac{12 \cdot 7 \cdot 120}{5 \cdot 24}} && \text{Multiply fractions} \\
&= \frac{\frac{12 \cdot 24 \cdot \cancel{5}}{\cancel{5}} - \frac{7 \cdot 5 \cdot \cancel{24}}{\cancel{24}}}{120 + \frac{12 \cdot 7 \cdot \cancel{5} \cdot \cancel{24}}{\cancel{5} \cdot \cancel{24}}} && \text{Factor and cancel} \\
&= \frac{(12 \cdot 24) - (7 \cdot 5)}{120 + (12 \cdot 7)} \\
&= \frac{288 - 35}{120 + 84} \\
&= \frac{253}{204}
\end{aligned}$$

Since $253 = 11 \cdot 23$ and $204 = 2 \cdot 2 \cdot 3 \cdot 17$ have no common factors our result is in lowest terms which means we are done.

5. This fraction may look simpler than the one before it, but the negative signs and parentheses mean that we shouldn't get complacent. Again we note that the division line here acts as a grouping symbol. That is,

$$\frac{(2(2)+1)(-3-(-3))-5(4-7)}{4-2(3)} = \frac{((2(2)+1)(-3-(-3))-5(4-7))}{(4-2(3))}$$

This means that we should simplify the numerator and denominator first, then perform the division last. We tend to what's in parentheses first, giving multiplication priority over addition and subtraction.

$$\begin{aligned}
\frac{(2(2)+1)(-3-(-3))-5(4-7)}{4-2(3)} &= \frac{(4+1)(-3+3)-5(-3)}{4-6} \\
&= \frac{(5)(0)+15}{-2} \\
&= \frac{15}{-2} \\
&= -\frac{15}{2} && \text{Properties of Negatives}
\end{aligned}$$

Since $15 = 3 \cdot 5$ and 2 have no common factors, we are done.

6. In this problem, we have multiplication and subtraction. Multiplication takes precedence so we perform it first. Recall that to multiply fractions, we do *not* need to obtain common denominators; rather, we multiply the corresponding numerators together along with the corresponding denominators. Like the previous example, we have parentheses and negative signs for added fun!

$$\begin{aligned}
\left(\frac{3}{5}\right)\left(\frac{5}{13}\right)-\left(\frac{4}{5}\right)\left(-\frac{12}{13}\right) &= \frac{3\cdot 5}{5\cdot 13}-\frac{4\cdot(-12)}{5\cdot 13} && \text{Multiply fractions} \\
&= \frac{15}{65}-\frac{-48}{65} \\
&= \frac{15}{65}+\frac{48}{65} && \text{Properties of Negatives} \\
&= \frac{15+48}{65} && \text{Add numerators} \\
&= \frac{63}{65}
\end{aligned}$$

Since $64 = 3 \cdot 3 \cdot 7$ and $65 = 5 \cdot 13$ have no common factors, our answer $\frac{63}{65}$ is in lowest terms and we are done. □

Of the issues discussed in the previous set of examples none causes students more trouble than simplifying compound fractions. We presented two different methods for simplifying them: one in which we simplified the overall numerator and denominator and then performed the division and one in which we removed the compound nature of the fraction at the very beginning. We encourage the reader to go back and use both methods on each of the compound fractions presented. Keep in mind that when a compound fraction is encountered in the rest of the text it will usually be simplified using only one method and we may not choose your favorite method. Feel free to use the other one in your notes.

Next, we review exponents and their properties. Recall that $2 \cdot 2 \cdot 2$ can be written as 2^3 because exponential notation expresses repeated multiplication. In the expression 2^3, 2 is called the **base** and 3 is called the **exponent**. In order to generalize exponents from natural numbers to the integers, and eventually to rational and real numbers, it is helpful to think of the exponent as a count of the number of factors of the base we are multiplying by 1. For instance,

$$2^3 = 1 \cdot (\text{three factors of two}) = 1 \cdot (2 \cdot 2 \cdot 2) = 8.$$

From this, it makes sense that

$$2^0 = 1 \cdot (\text{zero factors of two}) = 1.$$

What about 2^{-3}? The '$-$' in the exponent indicates that we are 'taking away' three factors of two, essentially dividing by three factors of two. So,

$$2^{-3} = 1 \div (\text{three factors of two}) = 1 \div (2 \cdot 2 \cdot 2) = \frac{1}{2 \cdot 2 \cdot 2} = \frac{1}{8}.$$

We summarize the properties of integer exponents below.

Properties of Integer Exponents

Suppose a and b are nonzero real numbers and n and m are integers.

- **Product Rules:** $(ab)^n = a^n b^n$ and $a^n a^m = a^{n+m}$.
- **Quotient Rules:** $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ and $\frac{a^n}{a^m} = a^{n-m}$.
- **Power Rule:** $(a^n)^m = a^{nm}$.
- **Negatives in Exponents:** $a^{-n} = \frac{1}{a^n}$.

  In particular, $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n = \frac{b^n}{a^n}$ and $\frac{1}{a^{-n}} = a^n$.
- **Zero Powers:** $a^0 = 1$.

  Note: The expression 0^0 is an indeterminate form.[a]
- **Powers of Zero:** For any *natural* number n, $0^n = 0$.

  Note: The expression 0^n for integers $n \leq 0$ is not defined.

[a] See the comment regarding '$\frac{0}{0}$' on page 19.

While it is important the state the Properties of Exponents, it is also equally important to take a moment to discuss one of the most common errors in Algebra. It is true that $(ab)^2 = a^2b^2$ (which some students refer to as 'distributing' the exponent to each factor) but you cannot do this sort of thing with addition. That is, in general, $(a+b)^2 \neq a^2 + b^2$. (For example, take $a = 3$ and $b = 4$.) The same goes for any other powers.

With exponents now in the mix, we can now state the Order of Operations Agreement.

Order of Operations Agreement

When evaluating an expression involving real numbers:

1. Evaluate any expressions in **p**arentheses (or other grouping symbols.)
2. Evaluate **e**xponents.
3. Evaluate **m**ultiplication and **d**ivision as you read from left to right.
4. Evaluate **a**ddition and **s**ubtraction as you read from left to right.

We note that there are many useful mnemonic device for remembering the order of operations.[a]

[a] Our favorite is '**P**lease **e**ntertain **m**y **d**ear **a**uld **S**asquatch.'

For example, $2 + 3 \cdot 4^2 = 2 + 3 \cdot 16 = 2 + 48 = 50$. Where students get into trouble is with things like -3^2. If we think of this as $0 - 3^2$, then it is clear that we evaluate the exponent first: $-3^2 = 0 - 3^2 = 0 - 9 = -9$. In general, we interpret $-a^n = -(a^n)$. If we want the 'negative' to also be raised to a power, we must write $(-a)^n$ instead. To summarize, $-3^2 = -9$ but $(-3)^2 = 9$.

Of course, many of the 'properties' we've stated in this section can be viewed as ways to circumvent the order of operations. We've already seen how the distributive property allows us to simplify $5(2 + x)$ by performing the indicated multiplication **before** the addition that's in parentheses. Similarly, consider trying to evaluate $2^{30172} \cdot 2^{-30169}$. The Order of Operations Agreement demands that the exponents be dealt with first, however, trying to compute 2^{30172} is a challenge, even for a calculator. One of the Product Rules of Exponents, however, allow us to rewrite this product, essentially performing the multiplication first, to get: $2^{30172-30169} = 2^3 = 8$.

Let's take a break and enjoy another example.

Example 0.2.2. Perform the indicated operations and simplify.

1. $\dfrac{(4-2)(2\cdot 4)-(4)^2}{(4-2)^2}$
2. $4(-5)(-5+3)^{-4} + 2(-5)^2(-4)(-5+3)^{-5}$
3. $\dfrac{\left(\dfrac{5\cdot 3^{51}}{4^{36}}\right)}{\left(\dfrac{5\cdot 3^{49}}{4^{34}}\right)}$
4. $\dfrac{2\left(\dfrac{5}{12}\right)^{-1}}{1-\left(\dfrac{5}{12}\right)^{-2}}$

Solution.

1. We begin working inside parentheses then deal with the exponents before working through the other operations. As we saw in Example 0.2.1, the division here acts as a grouping symbol, so we save the division to the end.

$$\frac{(4-2)(2\cdot 4)-(4)^2}{(4-2)^2} = \frac{(2)(8)-(4)^2}{(2)^2} = \frac{(2)(8)-16}{4}$$

$$= \frac{16-16}{4} = \frac{0}{4} = 0$$

2. As before, we simplify what's in the parentheses first, then work our way through the exponents, multiplication, and finally, the addition.

$$\begin{aligned}
4(-5)(-5+3)^{-4}+2(-5)^2(-4)(-5+3)^{-5} &= 4(-5)(-2)^{-4}+2(-5)^2(-4)(-2)^{-5} \\
&= 4(-5)\left(\frac{1}{(-2)^4}\right)+2(-5)^2(-4)\left(\frac{1}{(-2)^5}\right) \\
&= 4(-5)\left(\frac{1}{16}\right)+2(25)(-4)\left(\frac{1}{-32}\right) \\
&= (-20)\left(\frac{1}{16}\right)+(-200)\left(\frac{1}{-32}\right) \\
&= \frac{-20}{16}+\left(\frac{-200}{-32}\right) \\
&= \frac{-5\cdot \cancel{4}}{4\cdot \cancel{4}}+\frac{-25\cdot \cancel{8}}{-4\cdot \cancel{8}} \\
&= \frac{-5}{4}+\frac{-25}{-4} \\
&= \frac{-5}{4}+\frac{25}{4} \\
&= \frac{-5+25}{4} \\
&= \frac{20}{4} \\
&= 5
\end{aligned}$$

3. The Order of Operations Agreement mandates that we work within each set of parentheses first, giving precedence to the exponents, then the multiplication, and, finally the division. The trouble with this approach is that the exponents are so large that computation becomes a trifle unwieldy. What we observe, however, is that the bases of the exponential expressions, 3 and 4, occur in both the numerator and denominator of the compound fraction, giving us

hope that we can use some of the Properties of Exponents (the Quotient Rule, in particular) to help us out. Our first step here is to invert and multiply. We see immediately that the 5's cancel after which we group the powers of 3 together and the powers of 4 together and apply the properties of exponents.

$$\begin{aligned}
\frac{\left(\dfrac{5\cdot 3^{51}}{4^{36}}\right)}{\left(\dfrac{5\cdot 3^{49}}{4^{34}}\right)} &= \frac{5\cdot 3^{51}}{4^{36}}\cdot\frac{4^{34}}{5\cdot 3^{49}} = \frac{\cancel{5}\cdot 3^{51}\cdot 4^{34}}{\cancel{5}\cdot 3^{49}\cdot 4^{36}} = \frac{3^{51}}{3^{49}}\cdot\frac{4^{34}}{4^{36}} \\
&= 3^{51-49}\cdot 4^{34-36} = 3^2\cdot 4^{-2} = 3^2\cdot\left(\frac{1}{4^2}\right) \\
&= 9\cdot\left(\frac{1}{16}\right) = \frac{9}{16}
\end{aligned}$$

4. We have yet another instance of a compound fraction so our first order of business is to rid ourselves of the compound nature of the fraction like we did in Example 0.2.1. To do this, however, we need to tend to the exponents first so that we can determine what common denominator is needed to simplify the fraction.

$$\begin{aligned}
\frac{2\left(\dfrac{5}{12}\right)^{-1}}{1-\left(\dfrac{5}{12}\right)^{-2}} &= \frac{2\left(\dfrac{12}{5}\right)}{1-\left(\dfrac{12}{5}\right)^{2}} = \frac{\left(\dfrac{24}{5}\right)}{1-\left(\dfrac{12^2}{5^2}\right)} = \frac{\left(\dfrac{24}{5}\right)}{1-\left(\dfrac{144}{25}\right)} \\
&= \frac{\left(\dfrac{24}{5}\right)\cdot 25}{\left(1-\dfrac{144}{25}\right)\cdot 25} = \frac{\left(\dfrac{24\cdot 5\cdot\cancel{5}}{\cancel{5}}\right)}{\left(1\cdot 25-\dfrac{144\cdot\cancel{25}}{\cancel{25}}\right)} = \frac{120}{25-144} \\
&= \frac{120}{-119} = -\frac{120}{119}
\end{aligned}$$

Since 120 and 119 have no common factors, we are done. □

One of the places where the properties of exponents play an important role is in the use of **Scientific Notation**. The basis for scientific notation is that since we use decimals (base ten numerals) to represent real numbers, we can adjust where the decimal point lies by multiplying by an appropriate power of 10. This allows scientists and engineers to focus in on the 'significant' digits[9] of a number - the nonzero values - and adjust for the decimal places later. For instance, $-621 = -6.21\times 10^2$ and $0.023 = 2.3\times 10^{-2}$. Notice here that we revert to using the familiar '$\times$' to indicate multiplication.[10] In general, we arrange the real number so exactly one non-zero digit appears to the left of the decimal point. We make this idea precise in the following:

[9] Awesome pun!

[10] This is the 'notable exception' we alluded to earlier.

Definition 0.7. A real number is written in **Scientific Notation** if it has the form $\pm n.d_1d_2\ldots \times 10^k$ where n is a natural number, d_1, d_2, etc., are whole numbers, and k is an integer.

On calculators, scientific notation may appear using an 'E' or 'EE' as opposed to the $\times$ symbol. For instance, while we will write 6.02×10^{23} in the text, the calculator may display 6.02 E 23 or 6.02 EE 23.

Example 0.2.3. Perform the indicated operations and simplify. Write your final answer in scientific notation, rounded to two decimal places.

1. $\dfrac{\left(6.626 \times 10^{-34}\right)\left(3.14 \times 10^{9}\right)}{1.78 \times 10^{23}}$
2. $\left(2.13 \times 10^{53}\right)^{100}$

Solution.

1. As mentioned earlier, the point of scientific notation is to separate out the 'significant' parts of a calculation and deal with the powers of 10 later. In that spirit, we separate out the powers of 10 in both the numerator and the denominator and proceed as follows

$$\begin{aligned}
\frac{\left(6.626 \times 10^{-34}\right)\left(3.14 \times 10^{9}\right)}{1.78 \times 10^{23}} &= \frac{(6.626)(3.14)}{1.78} \cdot \frac{10^{-34} \cdot 10^{9}}{10^{23}} \\
&= \frac{20.80564}{1.78} \cdot \frac{10^{-34+9}}{10^{23}} \\
&= 11.685\ldots \cdot \frac{10^{-25}}{10^{23}} \\
&= 11.685\ldots \times 10^{-25-23} \\
&= 11.685\ldots \times 10^{-48}
\end{aligned}$$

   We are asked to write our final answer in scientific notation, rounded to two decimal places. To do this, we note that $11.685\ldots = 1.1685\ldots \times 10^{1}$, so

$$11.685\ldots \times 10^{-48} = 1.1685\ldots \times 10^{1} \times 10^{-48} = 1.1685\ldots \times 10^{1-48} = 1.1685\ldots \times 10^{-47}$$

   Our final answer, rounded to two decimal places, is 1.17×10^{-47}.

   We could have done that whole computation on a calculator so why did we bother doing any of this by hand in the first place? The answer lies in the next example.

2. If you try to compute $\left(2.13 \times 10^{53}\right)^{100}$ using most hand-held calculators, you'll most likely get an 'overflow' error. It is possible, however, to use the calculator in combination with the properties of exponents to compute this number. Using properties of exponents, we get:

$$\begin{aligned}
\left(2.13 \times 10^{53}\right)^{100} &= (2.13)^{100}\left(10^{53}\right)^{100} \\
&= \left(6.885\ldots \times 10^{32}\right)\left(10^{53\times 100}\right) \\
&= \left(6.885\ldots \times 10^{32}\right)\left(10^{5300}\right) \\
&= 6.885\ldots \times 10^{32} \cdot 10^{5300} \\
&= 6.885\ldots \times 10^{5332}
\end{aligned}$$

To two decimal places our answer is 6.88×10^{5332}. □

We close our review of real number arithmetic with a discussion of roots and radical notation. Just as subtraction and division were defined in terms of the inverse of addition and multiplication, respectively, we define roots by undoing natural number exponents.

Definition 0.8. Let a be a real number and let n be a natural number. If n is odd, then the **principal n^{th} root** of a (denoted $\sqrt[n]{a}$) is the unique real number satisfying $\left(\sqrt[n]{a}\right)^n = a$. If n is even, $\sqrt[n]{a}$ is defined similarly provided $a \geq 0$ and $\sqrt[n]{a} \geq 0$. The number n is called the **index** of the root and the the number a is called the **radicand**. For $n = 2$, we write $\sqrt{a}$ instead of $\sqrt[2]{a}$.

The reasons for the added stipulations for even-indexed roots in Definition 0.8 can be found in the Properties of Negatives. First, for all real numbers, $x^{\text{even power}} \geq 0$, which means it is never negative. Thus if a is a *negative* real number, there are no real numbers x with $x^{\text{even power}} = a$. This is why if n is even, $\sqrt[n]{a}$ only exists if $a \geq 0$. The second restriction for even-indexed roots is that $\sqrt[n]{a} \geq 0$. This comes from the fact that $x^{\text{even power}} = (-x)^{\text{even power}}$, and we require $\sqrt[n]{a}$ to have just one value. So even though $2^4 = 16$ and $(-2)^4 = 16$, we require $\sqrt[4]{16} = 2$ and ignore -2.

Dealing with odd powers is much easier. For example, $x^3 = -8$ has one and only one real solution, namely $x = -2$, which means not only does $\sqrt[3]{-8}$ exist, there is only one choice, namely $\sqrt[3]{-8} = -2$. Of course, when it comes to solving $x^{5213} = -117$, it's not so clear that there is one and only one real solution, let alone that the solution is $\sqrt[5213]{-117}$. Such pills are easier to swallow once we've thought a bit about such equations graphically,[11] and ultimately, these things come from the completeness property of the real numbers mentioned earlier.

We list properties of radicals below as a 'theorem' since they can be justified using the properties of exponents.

Theorem 0.1. Properties of Radicals: Let a and b be real numbers and let m and n be natural numbers. If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then

- **Product Rule:** $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$
- **Quotient Rule:** $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$, provided $b \neq 0$.
- **Power Rule:** $\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

[11]See Chapter 3.

The proof of Theorem 0.1 is based on the definition of the principal n^{th} root and the Properties of Exponents. To establish the product rule, consider the following. If n is odd, then by definition $\sqrt[n]{ab}$ is the unique real number such that $(\sqrt[n]{ab})^n = ab$. Given that $(\sqrt[n]{a}\,\sqrt[n]{b})^n = (\sqrt[n]{a})^n(\sqrt[n]{b})^n = ab$ as well, it must be the case that $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$. If n is even, then $\sqrt[n]{ab}$ is the unique non-negative real number such that $(\sqrt[n]{ab})^n = ab$. Note that since n is even, $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are also non-negative thus $\sqrt[n]{a}\sqrt[n]{b} \geq 0$ as well. Proceeding as above, we find that $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$. The quotient rule is proved similarly and is left as an exercise. The power rule results from repeated application of the product rule, so long as $\sqrt[n]{a}$ is a real number to start with. We leave that as an exercise as well.

We pause here to point out one of the most common errors students make when working with radicals. Obviously $\sqrt{9} = 3$, $\sqrt{16} = 4$ and $\sqrt{9+16} = \sqrt{25} = 5$. Thus we can clearly see that $5 = \sqrt{25} = \sqrt{9+16} \neq \sqrt{9} + \sqrt{16} = 3 + 4 = 7$ because we all know that $5 \neq 7$. The authors urge you to never consider 'distributing' roots or exponents. It's wrong and no good will come of it because in general $\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b}$.

Since radicals have properties inherited from exponents, they are often written as such. We define rational exponents in terms of radicals in the box below.

Definition 0.9. Let a be a real number, let m be an integer and let n be a natural number.

- $a^{\frac{1}{n}} = \sqrt[n]{a}$ whenever $\sqrt[n]{a}$ is a real number.[a]
- $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$ whenever $\sqrt[n]{a}$ is a real number.

[a] If n is even we need $a \geq 0$.

It would make life really nice if the rational exponents defined in Definition 0.9 had all of the same properties that integer exponents have as listed on page 26 - but they don't. Why not? Let's look at an example to see what goes wrong. Consider the Product Rule which says that $(ab)^n = a^n b^n$ and let $a = -16$, $b = -81$ and $n = \frac{1}{4}$. Plugging the values into the Product Rule yields the equation $((-16)(-81))^{1/4} = (-16)^{1/4}(-81)^{1/4}$. The left side of this equation is $1296^{1/4}$ which equals 6 but the right side is undefined because neither root is a real number. Would it help if, when it comes to even roots (as signified by even denominators in the fractional exponents), we ensure that everything they apply to is non-negative? That works for some of the rules – we leave it as an exercise to see which ones - but does not work for the Power Rule.

Consider the expression $\left(a^{\frac{2}{3}}\right)^{\frac{3}{2}}$. Applying the usual laws of exponents, we'd be tempted to simplify this as $\left(a^{\frac{2}{3}}\right)^{\frac{3}{2}} = a^{\frac{2}{3}\cdot\frac{3}{2}} = a^1 = a$. However, if we substitute $a = -1$ and apply Definition 0.9, we find $(-1)^{\frac{2}{3}} = \left(\sqrt[3]{-1}\right)^2 = (-1)^2 = 1$ so that $\left((-1)^{\frac{2}{3}}\right)^{\frac{3}{2}} = 1^{\frac{3}{2}} = \left(\sqrt{1}\right)^3 = 1^3 = 1$. Thus in this case we have $\left(a^{\frac{2}{3}}\right)^{\frac{3}{2}} \neq a$ even though all of the roots were defined. It is true, however, that $\left(a^{\frac{3}{2}}\right)^{\frac{2}{3}} = a$ and we leave this for the reader to show. The moral of the story is that when simplifying powers of rational exponents where the base is negative or worse, unknown, it's usually best to rewrite them as radicals.[12]

[12] Much to Jeff's chagrin. He's fairly traditional and therefore doesn't care much for radicals.

Example 0.2.4. Perform the indicated operations and simplify.

1. $\dfrac{-(-4)-\sqrt{(-4)^2-4(2)(-3)}}{2(2)}$

2. $\dfrac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{\sqrt{3}}{3}\right)^2}$

3. $(\sqrt[3]{-2}-\sqrt[3]{-54})^2$

4. $2\left(\frac{9}{4}-3\right)^{\frac{1}{3}}+2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{9}{4}-3\right)^{-\frac{2}{3}}$

Solution.

1. We begin in the numerator and note that the radical here acts a grouping symbol,[13] so our first order of business is to simplify the radicand.

$$\begin{aligned}\frac{-(-4)-\sqrt{(-4)^2-4(2)(-3)}}{2(2)} &= \frac{-(-4)-\sqrt{16-4(2)(-3)}}{2(2)}\\ &= \frac{-(-4)-\sqrt{16-4(-6)}}{2(2)}\\ &= \frac{-(-4)-\sqrt{16-(-24)}}{2(2)}\\ &= \frac{-(-4)-\sqrt{16+24}}{2(2)}\\ &= \frac{-(-4)-\sqrt{40}}{2(2)}\end{aligned}$$

As you may recall, 40 can be factored using a perfect square as $40 = 4 \cdot 10$ so we use the product rule of radicals to write $\sqrt{40} = \sqrt{4 \cdot 10} = \sqrt{4}\sqrt{10} = 2\sqrt{10}$. This lets us factor a '2' out of both terms in the numerator, eventually allowing us to cancel it with a factor of 2 in the denominator.

$$\begin{aligned}\frac{-(-4)-\sqrt{40}}{2(2)} &= \frac{-(-4)-2\sqrt{10}}{2(2)} &&= \frac{4-2\sqrt{10}}{2(2)}\\ &= \frac{2\cdot 2-2\sqrt{10}}{2(2)} &&= \frac{2(2-\sqrt{10})}{2(2)}\\ &= \frac{\cancel{2}(2-\sqrt{10})}{\cancel{2}(2)} &&= \frac{2-\sqrt{10}}{2}\end{aligned}$$

Since the numerator and denominator have no more common factors,[14] we are done.

[13] The line extending horizontally from the square root symbol '$\sqrt{}$' is, you guessed it, another vinculum.

[14] Do you see why we aren't 'canceling' the remaining 2's?

2. Once again we have a compound fraction, so we first simplify the exponent in the denominator to see which factor we'll need to multiply by in order to clean up the fraction.

$$\begin{aligned}
\frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{\sqrt{3}}{3}\right)^2} &= \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{(\sqrt{3})^2}{3^2}\right)} &&= \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{3}{9}\right)} \\
&= \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{1\cdot\cancel{3}}{3\cdot\cancel{3}}\right)} &&= \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{1}{3}\right)} \\
&= \frac{2\left(\frac{\sqrt{3}}{3}\right)\cdot 3}{\left(1-\left(\frac{1}{3}\right)\right)\cdot 3} &&= \frac{\frac{2\cdot\sqrt{3}\cdot\cancel{3}}{\cancel{3}}}{1\cdot 3-\frac{1\cdot\cancel{3}}{\cancel{3}}} \\
&= \frac{2\sqrt{3}}{3-1} &&= \frac{\cancel{2}\sqrt{3}}{\cancel{2}} = \sqrt{3}
\end{aligned}$$

3. Working inside the parentheses, we first encounter $\sqrt[3]{-2}$. While the -2 isn't a perfect cube,[15] we may think of $-2=(-1)(2)$. Since $(-1)^3=-1$, -1 *is* a perfect cube, and we may write $\sqrt[3]{-2}=\sqrt[3]{(-1)(2)}=\sqrt[3]{-1}\sqrt[3]{2}=-\sqrt[3]{2}$. When it comes to $\sqrt[3]{54}$, we may write it as $\sqrt[3]{(-27)(2)}=\sqrt[3]{-27}\sqrt[3]{2}=-3\sqrt[3]{2}$. So,

$$\sqrt[3]{-2}-\sqrt[3]{-54}=-\sqrt[3]{2}-(-3\sqrt[3]{2})=-\sqrt[3]{2}+3\sqrt[3]{2}.$$

At this stage, we can simplify $-\sqrt[3]{2}+3\sqrt[3]{2}=2\sqrt[3]{2}$. You may remember this as being called 'combining like radicals,' but it is in fact just another application of the distributive property:

$$-\sqrt[3]{2}+3\sqrt[3]{2}=(-1)\sqrt[3]{2}+3\sqrt[3]{2}=(-1+3)\sqrt[3]{2}=2\sqrt[3]{2}.$$

Putting all this together, we get:

$$\begin{aligned}
(\sqrt[3]{-2}-\sqrt[3]{-54})^2 &= (-\sqrt[3]{2}+3\sqrt[3]{2})^2 &&= (2\sqrt[3]{2})^2 \\
&= 2^2(\sqrt[3]{2})^2 = 4\sqrt[3]{2^2} &&= 4\sqrt[3]{4}
\end{aligned}$$

Since there are no perfect integer cubes which are factors of 4 (apart from 1, of course), we are done.

[15] Of an integer, that is!

4. We start working in parentheses and get a common denominator to subtract the fractions:

$$\frac{9}{4} - 3 = \frac{9}{4} - \frac{3 \cdot 4}{1 \cdot 4} = \frac{9}{4} - \frac{12}{4} = \frac{-3}{4}$$

Since the denominators in the fractional exponents are odd, we can proceed using the properties of exponents:

$$\begin{aligned}
2\left(\frac{9}{4} - 3\right)^{1/3} + 2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{9}{4} - 3\right)^{-2/3} &= 2\left(\frac{-3}{4}\right)^{1/3} + 2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{-3}{4}\right)^{-2/3} \\
&= 2\left(\frac{(-3)^{1/3}}{(4)^{1/3}}\right) + 2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{4}{-3}\right)^{2/3} \\
&= 2\left(\frac{(-3)^{1/3}}{(4)^{1/3}}\right) + 2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{(4)^{2/3}}{(-3)^{2/3}}\right) \\
&= \frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{2 \cdot 9 \cdot 1 \cdot 4^{2/3}}{4 \cdot 3 \cdot (-3)^{2/3}} \\
&= \frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{\cancel{2} \cdot 3 \cdot \cancel{3} \cdot 4^{2/3}}{2 \cdot \cancel{2} \cdot \cancel{3} \cdot (-3)^{2/3}} \\
&= \frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{3 \cdot 4^{2/3}}{2 \cdot (-3)^{2/3}}
\end{aligned}$$

At this point, we could start looking for common denominators but it turns out that these fractions reduce even further. Since $4 = 2^2$, $4^{1/3} = (2^2)^{1/3} = 2^{2/3}$. Similarly, $4^{2/3} = (2^2)^{2/3} = 2^{4/3}$. The expressions $(-3)^{1/3}$ and $(-3)^{2/3}$ contain negative bases so we proceed with caution and convert them back to radical notation to get: $(-3)^{1/3} = \sqrt[3]{-3} = -\sqrt[3]{3} = -3^{1/3}$ and $(-3)^{2/3} = (\sqrt[3]{-3})^2 = (-\sqrt[3]{3})^2 = (\sqrt[3]{3})^2 = 3^{2/3}$. Hence:

$$\begin{aligned}
\frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{3 \cdot 4^{2/3}}{2 \cdot (-3)^{2/3}} &= \frac{2 \cdot (-3^{1/3})}{2^{2/3}} + \frac{3 \cdot 2^{4/3}}{2 \cdot 3^{2/3}} \\
&= \frac{2^1 \cdot (-3^{1/3})}{2^{2/3}} + \frac{3^1 \cdot 2^{4/3}}{2^1 \cdot 3^{2/3}} \\
&= 2^{1-2/3} \cdot (-3^{1/3}) + 3^{1-2/3} \cdot 2^{4/3-1} \\
&= 2^{1/3} \cdot (-3^{1/3}) + 3^{1/3} \cdot 2^{1/3} \\
&= -2^{1/3} \cdot 3^{1/3} + 3^{1/3} \cdot 2^{1/3} \\
&= 0
\end{aligned}$$

□

0.2.1 Exercises

In Exercises 1 - 40, perform the indicated operations and simplify.

1. $5-2+3$
2. $5-(2+3)$
3. $\dfrac{2}{3}-\dfrac{4}{7}$
4. $\dfrac{3}{8}+\dfrac{5}{12}$
5. $\dfrac{5-3}{-2-4}$
6. $\dfrac{2(-3)}{3-(-3)}$
7. $\dfrac{2(3)-(4-1)}{2^2+1}$
8. $\dfrac{4-5.8}{2-2.1}$
9. $\dfrac{1-2(-3)}{5(-3)+7}$
10. $\dfrac{5(3)-7}{2(3)^2-3(3)-9}$
11. $\dfrac{2((-1)^2-1)}{((-1)^2+1)^2}$
12. $\dfrac{(4-(-2)-6}{(-2)^2-4}$
13. $\dfrac{3-\frac{4}{9}}{-2-(-3)}$
14. $\dfrac{\frac{2}{3}-\frac{4}{5}}{4-\frac{7}{10}}$
15. $\dfrac{2\left(\frac{4}{3}\right)}{1-\left(\frac{4}{3}\right)^2}$
16. $\dfrac{1-\left(\frac{5}{3}\right)\left(\frac{3}{5}\right)}{1+\left(\frac{5}{3}\right)\left(\frac{3}{5}\right)}$
17. $\left(\dfrac{2}{3}\right)^{-5}$
18. $3^{-1}-4^{-2}$
19. $\dfrac{1+2^{-3}}{3-4^{-1}}$
20. $\dfrac{3\cdot 5^{100}}{12\cdot 5^{98}}$
21. $\sqrt{121}$
22. $(\sqrt{2})^8$
23. $\sqrt[3]{80}$
24. $\sqrt[4]{\dfrac{32}{625}}$
25. $64^{1/3}$
26. $(-216)^{1/3}$
27. $10000^{1/2}$
28. $(-8)^{-4/3}$
29. $\sqrt{3^2+4^2}$
30. $\sqrt{12}-\sqrt{75}$
31. $(-8)^{2/3}-9^{-3/2}$
32. $\left(-\frac{32}{9}\right)^{-3/5}$
33. $\sqrt{(3-4)^2+(5-2)^2}$
34. $\sqrt{(2-(-1))^2+\left(\frac{1}{2}-3\right)^2}$
35. $\sqrt{(\sqrt{5}-2\sqrt{5})^2+(\sqrt{18}-\sqrt{8})^2}$
36. $\dfrac{-12+\sqrt{18}}{21}$
37. $\dfrac{-2-\sqrt{(2)^2-4(3)(-1)}}{2(3)}$
38. $\dfrac{-(-4)+\sqrt{(-4)^2-4(1)(-1)}}{2(1)}$
39. $2(-5)(-5+1)^{-1}+(-5)^2(-1)(-5+1)^{-2}$
40. $3\sqrt{2(4)+1}+3(4)\left(\frac{1}{2}\right)(2(4)+1)^{-1/2}(2)$

In Exercises 41 - 48, simplify the algebraic expressions.

41. $(y^5)^9$
42. $(w^{-2})^{-1}$
43. $(3x^2)^6$
44. $(16x^2)(4x^5)$
45. $\dfrac{(4x^3)(2x^5)}{6x^4}$
46. $\dfrac{24t^3}{15t^4}\cdot\dfrac{5t^8}{3t^6}$
47. $\dfrac{(ab^3c^3)^2(a^2b)^3}{a^{-3}b^2c}$
48. $\dfrac{u^4(uv^3)^{-2}(uv)^5}{v^{-3}(u^2v)^3}$

0.2.2 Answers

1. 6	2. 0	3. $\frac{2}{21}$	4. $\frac{19}{24}$
5. $-\frac{1}{3}$	6. -1	7. $\frac{3}{5}$	8. 18
9. $-\frac{7}{8}$	10. Undefined.	11. 0	12. Undefined.
13. $\frac{23}{9}$	14. $-\frac{4}{99}$	15. $-\frac{24}{7}$	16. 0
17. $\frac{243}{32}$	18. $\frac{13}{48}$	19. $\frac{9}{22}$	20. $\frac{25}{4}$
21. 11	22. 16	23. $2\sqrt[3]{10}$	24. $\frac{2}{5}\sqrt[4]{2}$
25. 4	26. -6	27. 100	28. $\frac{1}{16}$
29. 5	30. $-3\sqrt{3}$	31. $\frac{107}{27}$	32. $-\frac{3\sqrt[5]{3}}{8} = -\frac{3^{6/5}}{8}$
33. $\sqrt{10}$	34. $\frac{\sqrt{61}}{2}$	35. $\sqrt{7}$	36. $\frac{-4+\sqrt{2}}{7}$
37. -1	38. $2+\sqrt{5}$	39. $\frac{15}{16}$	40. 13
41. y^{45}	42. w^2	43. $729x^{12}$	44. $64x^7$
45. $\frac{4x^4}{3}$	46. $\frac{8t}{3}$	47. $a^{11}b^7c^5$	48. $\frac{u}{v}$

0.3 Linear Equations and Inequalities

In the introduction to this chapter we said that we were going to review "the concepts, skills and vocabulary we believe are prerequisite to a rigorous, college-level Precalculus course." So far, we've presented a lot of vocabulary and concepts but we haven't done much to refresh the skills needed to survive in the Precalculus wilderness. Thus over the course of the next few sections we will focus our review on the Algebra skills needed to solve basic equations and inequalities. In general, equations and inequalities fall into one of three categories: conditional, identity or contradiction, depending on the nature of their solutions. A **conditional** equation or inequality is true for only *certain* real numbers. For example, $2x + 1 = 7$ is true precisely when $x = 3$, and $w - 3 \leq 4$ is true precisely when $w \leq 7$. An **identity** is an equation or inequality that is true for *all* real numbers. For example, $2x - 3 = 1 + x - 4 + x$ or $2t \leq 2t + 3$. A **contradiction** is an equation or inequality that is *never* true. Examples here include $3x - 4 = 3x + 7$ and $a - 1 > a + 3$.

As you may recall, solving an equation or inequality means finding all of the values of the variable, if any exist, which make the given equation or inequality true. This often requires us to manipulate the given equation or inequality from its given form to an easier form. For example, if we're asked to solve $3 - 2(x - 3) = 7x + 3(x + 1)$, we get $x = \frac{1}{2}$, but not without a fair amount of algebraic manipulation. In order to obtain the correct answer(s), however, we need to make sure that whatever maneuvers we apply are reversible in order to guarantee that we maintain a chain of **equivalent** equations or inequalities. Two equations or inequalities are called **equivalent** if they have the same solutions. We list these 'legal moves' below.

Procedures which Generate Equivalent Equations

- Add (or subtract) the same real number to (from) both sides of the equation.
- Multiply (or divide) both sides of the equation by the same **nonzero** real number.[a]

Procedures which Generate Equivalent Inequalities

- Add (or subtract) the same real number to (from) both sides of the equation.
- Multiply (or divide) both sides of the equation by the same **positive** real number.[b]

[a] Multiplying both sides of an equation by 0 collapses the equation to $0 = 0$, which doesn't do anybody any good.

[b] Remember that if you multiply both sides of an inequality by a negative real number, the inequality sign is reversed: $3 \leq 4$, but $(-2)(3) \geq (-2)(4)$.

0.3.1 Linear Equations

The first type of equations we need to review are **linear** equations as defined below.

Definition 0.10. An equation is said to be **linear** in a variable X if it can be written in the form $AX = B$ where A and B are expressions which do not involve X and $A \neq 0$.

One key point about Definition 0.10 is that the exponent on the unknown 'X' in the equation is 1, that is $X = X^1$. Our main strategy for solving linear equations is summarized below.

Strategy for Solving Linear Equations

In order to solve an equation which is linear in a given variable, say X:

1. Isolate all of the terms containing X on one side of the equation, putting all of the terms not containing X on the other side of the equation.

2. Factor out the X and divide both sides of the equation by its coefficient.

We illustrate this process with a collection of examples below.

Example 0.3.1. Solve the following equations for the indicated variable. Check your answer.

1. Solve for x: $3x - 6 = 7x + 4$
2. Solve for t: $3 - 1.7t = \frac{t}{4}$
3. Solve for a: $\frac{1}{18}(7 - 4a) + 2 = \frac{a}{3} - \frac{4 - a}{12}$
4. Solve for y: $8y\sqrt{3} + 1 = 7 - \sqrt{12}(5 - y)$
5. Solve for x: $\frac{3x - 1}{2} = x\sqrt{50} + 4$
6. Solve for y: $x(4 - y) = 8y$

Solution.

1. The variable we are asked to solve for is x so our first move is to gather all of the terms involving x on one side and put the remaining terms on the other.[1]

$$
\begin{array}{rcll}
3x - 6 & = & 7x + 4 & \\
(3x - 6) - 7x + 6 & = & (7x + 4) - 7x + 6 & \text{Subtract } 7x\text{, add } 6 \\
3x - 7x - 6 + 6 & = & 7x - 7x + 4 + 6 & \text{Rearrange terms} \\
-4x & = & 10 & 3x - 7x = (3 - 7)x = -4x \\
\frac{-4x}{-4} & = & \frac{10}{-4} & \text{Divide by the coefficient of } x \\
x & = & -\frac{5}{2} & \text{Reduce to lowest terms}
\end{array}
$$

To check our answer, we substitute $x = -\frac{5}{2}$ into each side of the **orginial** equation to see the equation is satisfied. Sure enough, $3\left(-\frac{5}{2}\right) - 6 = -\frac{27}{2}$ and $7\left(-\frac{5}{2}\right) + 4 = -\frac{27}{2}$.

[1] In the margin notes, when we speak of operations, e.g.,'Subtract $7x$,' we mean to subtract $7x$ from **both** sides of the equation. The 'from both sides of the equation' is omitted in the interest of spacing.

2. In our next example, the unknown is t and we not only have a fraction but also a decimal to wrangle. Fortunately, with equations we can multiply both sides to rid us of these computational obstacles:

$$\begin{array}{rcll}
3-1.7t &=& \dfrac{t}{4} & \\
40(3-1.7t) &=& 40\left(\dfrac{t}{4}\right) & \text{Multiply by 40} \\
40(3)-40(1.7t) &=& \dfrac{40t}{4} & \text{Distribute} \\
120-68t &=& 10t & \\
120 &=& 10t+68t & \text{Add } 68t \text{ to both sides} \\
120 &=& 78t & 68t+10t=(68+10)t=78t \\
\dfrac{120}{78} &=& t & \text{Divide by the coefficient of } t \\
\dfrac{20}{13} &=& t & \text{Reduce to lowest terms}
\end{array}$$

To check, we again substitute $t=\frac{20}{13}$ into each side of the original equation. We find that $3-1.7\left(\frac{20}{13}\right)=3-\left(\frac{17}{10}\right)\left(\frac{20}{13}\right)=\frac{5}{13}$ and $\frac{(20/13)}{4}=\frac{20}{13}\cdot\frac{1}{4}=\frac{5}{13}$ as well.

3. To solve this next equation, we begin once again by clearing fractions. The least common denominator here is 36:

$$\begin{array}{rcll}
\dfrac{1}{18}(7-4a)+2 &=& \dfrac{a}{3}-\dfrac{4-a}{12} & \\
36\left(\dfrac{1}{18}(7-4a)+2\right) &=& 36\left(\dfrac{a}{3}-\dfrac{4-a}{12}\right) & \text{Multiply by 36} \\
\dfrac{36}{18}(7-4a)+(36)(2) &=& \dfrac{36a}{3}-\dfrac{36(4-a)}{12} & \text{Distribute} \\
2(7-4a)+72 &=& 12a-3(4-a) & \text{Distribute} \\
14-8a+72 &=& 12a-12+3a & \\
86-8a &=& 15a-12 & 12a+3a=(12+3)a=15a \\
(86-8a)+8a+12 &=& (15a-12)+8a+12 & \text{Add } 8a \text{ and } 12 \\
86+12-8a+8a &=& 15a+8a-12+12 & \text{Rearrange terms} \\
98 &=& 23a & 15a+8a=(15+8)a=23a \\
\dfrac{98}{23} &=& \dfrac{23a}{23} & \text{Divide by the coefficient of } a \\
\dfrac{98}{23} &=& a &
\end{array}$$

The check, as usual, involves substituting $a=\frac{98}{23}$ into both sides of the original equation. The reader is encouraged to work through the (admittedly messy) arithmetic. Both sides work out to $\frac{199}{138}$.

4. The square roots may dishearten you but we treat them just like the real numbers they are. Our strategy is the same: get everything with the variable (in this case y) on one side, put everything else on the other and divide by the coefficient of the variable. We've added a few steps to the narrative that we would ordinarily omit just to help you see that this equation is indeed linear.

$$\begin{array}{rcll}
8y\sqrt{3}+1 & = & 7-\sqrt{12}(5-y) & \\
8y\sqrt{3}+1 & = & 7-\sqrt{12}(5)+\sqrt{12}\,y & \text{Distribute} \\
8y\sqrt{3}+1 & = & 7-(2\sqrt{3})5+(2\sqrt{3})y & \sqrt{12}=\sqrt{4\cdot 3}=2\sqrt{3} \\
8y\sqrt{3}+1 & = & 7-10\sqrt{3}+2y\sqrt{3} & \\
8y\sqrt{3}-2y\sqrt{3} & = & 6-10\sqrt{3} & \text{Subtract 1 and } 2y\sqrt{3} \\
(8\sqrt{3}-2\sqrt{3})y & = & 6-10\sqrt{3} & \text{Factor} \\
6y\sqrt{3} & = & 6-10\sqrt{3} & \text{See note below} \\
y & = & \dfrac{6-10\sqrt{3}}{6\sqrt{3}} & \text{Divide by } 6\sqrt{3} \\
y & = & \dfrac{\not{2}\left(3-5\sqrt{3}\right)}{\not{2}\cdot 3\sqrt{3}} & \text{Factor and cancel} \\
y & = & \dfrac{3-5\sqrt{3}}{3\sqrt{3}} &
\end{array}$$

In the list of computations above we marked the row $6y\sqrt{3}=6-10\sqrt{3}$ with a note. That's because we wanted to draw your attention to this line without breaking the flow of the manipulations. The equation $6y\sqrt{3}=6-10\sqrt{3}$ is in fact linear according to Definition 0.10: the variable is y, the value of A is $6\sqrt{3}$ and $B=6-10\sqrt{3}$. Checking the solution, while not trivial, is good mental exercise. Each side works out to be $\frac{27-40\sqrt{3}}{3}$.

5. Proceeding as before, we simplify radicals and clear denominators. Once we gather all of the terms containing x on one side and move the other terms to the other, we factor out x to identify its coefficient then divide to get our answer.

$$\begin{array}{rcll}
\dfrac{3x-1}{2} & = & x\sqrt{50}+4 & \\
\dfrac{3x-1}{2} & = & 5x\sqrt{2}+4 & \sqrt{50}=5\sqrt{2} \\
3x-1 & = & 2\left(5x\sqrt{2}+4\right) & \text{Multiply by 2} \\
3x-1 & = & 10x\sqrt{2}+8 & \text{Distribute} \\
3x-10x\sqrt{2} & = & 9 & \text{Subtract } 10x\sqrt{2}\text{, add 1} \\
(3-10\sqrt{2})x & = & 9 & \text{Factor } x
\end{array}$$

Finally, dividing both sides by the coefficient of x, we arrive at

$$x = \frac{9}{3 - 10\sqrt{2}}$$

The reader is encouraged to check this solution - it isn't as bad as it looks if you're careful! Each side works out to be $\frac{12+5\sqrt{2}}{3-10\sqrt{2}}$.

6. If we were instructed to solve our last equation for x, we'd be done in one step: divide both sides by $(4 - y)$ - assuming $4 - y \neq 0$, that is. Alas, we are instructed to solve for y, which means we have some more work to do.

$$\begin{array}{rcll} x(4 - y) & = & 8y & \\ 4x - xy & = & 8y & \text{Distribute} \\ (4x - xy) + xy & = & 8y + xy & \text{Add } xy \\ 4x & = & (8 + x)y & \text{Factor} \end{array}$$

In order to finish the problem, we need to divide both sides of the equation by the coefficient of y which in this case is $8+x$. Since this expression contains a variable, we need to stipulate that we may perform this division only if $8 + x \neq 0$, or, in other words, $x \neq -8$. Hence, we write our solution as:

$$y = \frac{4x}{8 + x}, \quad \text{provided } x \neq -8$$

What happens if $x = -8$? Substituting $x = -8$ into the original equation gives $(-8)(4-y) = 8y$ or $-32 + 8y = 8y$. This reduces to $-32 = 0$, which is a contradiction. This means there is no solution when $x = -8$, so we've covered all the bases. Checking our answer requires some Algebra we haven't reviewed yet in this text, but the necessary skills *should* be lurking somewhere in the mathematical mists of your mind. The adventurous reader is invited to show that both sides work out to $\dfrac{32x}{x+8}$. □

0.3.2 Linear Inequalities

We now turn our attention to linear inequalities. Unlike linear equations which admit at most one solution, the solutions to linear inequalities are generally intervals of real numbers. While the solution strategy for solving linear inequalities is the same as with solving linear equations, we need to remind ourselves that, should we decide to multiply or divide both sides of an inequality by a **negative** number, we need to reverse the direction of the inequality. (See page 38.) In the example below, we work not only some 'simple' linear inequalities in the sense there is only one inequality present, but also some 'compound' linear inequalities which require us to use the notions of intersection and union.

Example 0.3.2. Solve the following inequalities for the indicated variable.

1. Solve for x: $\dfrac{7-8x}{2} \geq 4x+1$
2. Solve for y: $\dfrac{3}{4} \leq \dfrac{7-y}{2} < 6$
3. Solve for t: $2t-1 \leq 4-t < 6t+1$
4. Solve for x: $5+\sqrt{7}x \leq 4x+1 \leq 8$
5. Solve for w: $2.1-0.01w \leq -3$ or $2.1-0.01w \geq 3$

Solution.

1. We begin by clearing denominators and gathering all of the terms containing x to one side of the inequality and putting the remaining terms on the other.

$$\begin{array}{rcll}
\dfrac{7-8x}{2} & \geq & 4x+1 & \\
2\left(\dfrac{7-8x}{2}\right) & \geq & 2(4x+1) & \text{Multiply by 2} \\
\dfrac{\cancel{2}(7-8x)}{\cancel{2}} & \geq & 2(4x)+2(1) & \text{Distribute} \\
7-8x & \geq & 8x+2 & \\
(7-8x)+8x-2 & \geq & 8x+2+8x-2 & \text{Add } 8x\text{, subtract 2} \\
7-2-8x+8x & \geq & 8x+8x+2-2 & \text{Rearrange terms} \\
5 & \geq & 16x & 8x+8x=(8+8)x=16x \\
\dfrac{5}{16} & \geq & \dfrac{16x}{16} & \text{Divide by the coefficient of } x \\
\dfrac{5}{16} & \geq & x &
\end{array}$$

We get $\frac{5}{16} \geq x$ or, said differently, $x \leq \frac{5}{16}$. We express this set[2] of real numbers as $\left(-\infty, \frac{5}{16}\right]$. Though not required to do so, we could partially check our answer by substituting $x = \frac{5}{16}$ and a few other values in our solution set ($x = 0$, for instance) to make sure the inequality holds. (It also isn't a bad idea to choose an $x > \frac{5}{16}$, say $x = 1$, to see that the inequality *doesn't* hold there.) The only real way to actually show that our answer works for *all* values in our solution set is to start with $x \leq \frac{5}{16}$ and reverse all of the steps in our solution procedure to prove it is equivalent to our original inequality.

2. We have our first example of a 'compound' inequality. The solutions to

$$\frac{3}{4} \leq \frac{7-y}{2} < 6$$

must satisfy

$$\frac{3}{4} \leq \frac{7-y}{2} \quad \underline{\text{and}} \quad \frac{7-y}{2} < 6$$

[2]Using set-builder notation, our 'set' of solutions here is $\{x \mid x \leq \frac{5}{16}\}$.

One approach is to solve each of these inequalities separately, then intersect their solution sets. While this method works (and will be used later for more complicated problems), since our variable y appears only in the middle expression, we can proceed by essentially working both inequalities at once:

$$\begin{array}{rccr}
\frac{3}{4} \leq & \frac{7-y}{2} & < 6 & \\
4\left(\frac{3}{4}\right) \leq & 4\left(\frac{7-y}{2}\right) & < 4(6) & \text{Multiply by 4} \\
\frac{\cancel{4}\cdot 3}{\cancel{4}} \leq & \frac{\overset{2}{\cancel{4}}(7-y)}{\cancel{2}} & < 24 & \\
3 \leq & 2(7-y) & < 24 & \\
3 \leq & 14-2y & < 24 & \text{Distribute} \\
3-14 \leq & (14-2y)-14 & < 24-14 & \text{Subtract 14} \\
-11 \leq & -2y & < 10 & \\
\frac{-11}{-2} \geq & \frac{-2y}{-2} & > \frac{10}{-2} & \text{Divide by the coefficient of } y \\
 & & & \text{Reverse inequalities} \\
\frac{11}{2} \geq & y & > -5 &
\end{array}$$

Our final answer is $\frac{11}{2} \geq y > -5$, or, said differently, $-5 < y \leq \frac{11}{2}$. In interval notation, this is $\left(-5, \frac{11}{2}\right]$. We could check the reasonableness of our answer as before, and the reader is encouraged to do so.

3. We have another compound inequality and what distinguishes this one from our previous example is that 't' appears on both sides of both inequalities. In this case, we need to create two separate inequalities and find all of the real numbers t which satisfy both $2t-1 \leq 4-t$ *and* $4-t < 6t+1$. The first inequality, $2t-1 \leq 4-t$, reduces to $3t \leq 5$ or $t \leq \frac{5}{3}$. The second inequality, $4-t < 6t+1$, becomes $3 < 7t$ which reduces to $t > \frac{3}{7}$. Thus our solution is all real numbers t with $t \leq \frac{5}{3}$ *and* $t > \frac{3}{7}$, or, writing this as a compound inequality, $\frac{3}{7} < t \leq \frac{5}{3}$. Using interval notation,[3] we express our solution as $\left(\frac{3}{7}, \frac{5}{3}\right]$.

4. As before, with this inequality we have no choice but to solve each inequality individually and intersect the solution sets. Starting with the leftmost inequality, we first note that the in the term $\sqrt{7}x$, the vinculum of the square root extends over the 7 only, meaning the x is not part of the radicand. In order to avoid confusion, we will write $\sqrt{7}x$ as $x\sqrt{7}$.

$$\begin{array}{rcll}
5+x\sqrt{7} & \leq & 4x+1 & \\
(5+x\sqrt{7})-4x-5 & \leq & (4x+1)-4x-5 & \text{Subtract } 4x \text{ and } 5 \\
x\sqrt{7}-4x+5-5 & \leq & 4x-4x+1-5 & \text{Rearrange terms} \\
x(\sqrt{7}-4) & \leq & -4 & \text{Factor}
\end{array}$$

[3]If we intersect the solution sets of the two individual inequalities, we get the answer, too: $\left(-\infty, \frac{5}{3}\right] \cap \left(\frac{3}{7}, \infty\right) = \left(\frac{3}{7}, \frac{5}{3}\right]$.

At this point, we need to exercise a bit of caution because the number $\sqrt{7}-4$ is negative.[4] When we divide by it the inequality reverses:

$$\begin{array}{rcll} x(\sqrt{7}-4) & \leq & -4 & \\ \dfrac{x(\sqrt{7}-4)}{\sqrt{7}-4} & \geq & \dfrac{-4}{\sqrt{7}-4} & \text{Divide by the coefficient of } x \\ & & & \text{Reverse inequalities} \\ x & \geq & \dfrac{-4}{\sqrt{7}-4} & \\ x & \geq & \dfrac{-4}{-(4-\sqrt{7})} & \\ x & \geq & \dfrac{4}{4-\sqrt{7}} & \end{array}$$

We're only half done because we still have the rightmost inequality to solve. Fortunately, that one seems rather mundane: $4x+1 \leq 8$ reduces to $x \leq \frac{7}{4}$ without too much incident. Our solution is $x \geq \frac{4}{4-\sqrt{7}}$ *and* $x \leq \frac{7}{4}$. We may be tempted to write $\frac{4}{4-\sqrt{7}} \leq x \leq \frac{7}{4}$ and call it a day but that would be nonsense! To see why, notice that $\sqrt{7}$ is between 2 and 3 so $\frac{4}{4-\sqrt{7}}$ is between $\frac{4}{4-2}=2$ and $\frac{4}{4-3}=4$. In particular, we get $\frac{4}{4-\sqrt{7}} > 2$. On the other hand, $\frac{7}{4} < 2$. This means that our 'solutions' have to be simultaneously greater than 2 AND less than 2 which is impossible. Therefore, this compound inequality has no solution, which means we did all that work for nothing.[5]

5. Our last example is yet another compound inequality but here, instead of the two inequalities being connected with the conjunction '*and*', they are connected with '*or*', which indicates that we need to find the *union* of the results of each. Starting with $2.1-0.01w \leq -3$, we get $-0.01w \leq -5.1$, which gives[6] $w \geq 510$. The second inequality, $2.1-0.01w \geq 3$, becomes $-0.01w \geq 0.9$, which reduces to $w \leq -90$. Our solution set consists of all real numbers w with $w \geq 510$ *or* $w \leq -90$. In interval notation, this is $(-\infty,-90] \cup [510,\infty)$. □

[4]Since $4<7<9$, it stands to reason that $\sqrt{4}<\sqrt{7}<\sqrt{9}$ so $2<\sqrt{7}<3$.

[5]Much like how people walking on treadmills get nowhere. Math is the endurance cardio of the brain, folks!

[6]Don't forget to flip the inequality!

0.3.3 Exercises

In Exercises 1 - 24, solve the given linear equation and check your answer.

1. $3x - 4 = 2 - 4(x - 3)$
2. $\dfrac{3 - 2t}{4} = 7t + 1$
3. $\dfrac{2(w - 3)}{5} = \dfrac{4}{15} - \dfrac{3w + 1}{9}$
4. $2 - 3x = \frac{8}{9} - x$
5. $\frac{6}{7} - 2x + 6x = \frac{202}{7}$
6. $0.1(x + 2x + 7x) = \dfrac{8 - x}{10}$
7. $\frac{3}{8} + \frac{x}{2} = -2x - \frac{37}{8}$
8. $\frac{2}{5}x + 1 = \frac{17}{35} + x$
9. $\frac{1}{7}x + 3 = -\frac{18}{35} + 6x$
10. $6(-\frac{2}{3}x + 8) = 48$
11. $-\sqrt{5}(4x + 8) = -12\sqrt{5}$
12. $7(-\frac{6}{7}x - 8) = -74$
13. $\dfrac{5w + 3}{\sqrt{8}} = \sqrt{2}$
14. $w - \sqrt{11} = 1 - \sqrt{11}\,w$
15. $\dfrac{14 + \pi w}{100} = 3.14$
16. $0.3y + 0.7y = \sqrt{7} - y$
17. $\dfrac{20 - 4x}{4} + x = 3\left(x - \frac{100}{3}\right)$
18. $\sqrt{3}\,t + \sqrt{5}\,t = t$
19. $-0.02y + 1000 = 0$
20. $\dfrac{49w - 14}{7} = 3w - (2 - 4w)$
21. $7 - (4 - x) = \dfrac{2x - 3}{2}$
22. $3t\sqrt{7} + 5 = 0$
23. $\sqrt{50}\,y = \dfrac{6 - \sqrt{8}\,y}{3}$
24. $4 - (2x + 1) = \dfrac{x\sqrt{7}}{9}$

In equations 25 - 34, solve each equation for the indicated variable.

25. Solve for y: $3x + 2y = 4$
26. Solve for x: $3x + 2y = 4$
27. Solve for C: $F = \frac{9}{5}C + 32$
28. Solve for x: $p = -2.5x + 15$
29. Solve for y: $x(y - 3) = 2y + 1$
30. Solve for π: $C = 2\pi r$
31. Solve for V: $PV = nRT$
32. Solve for R: $PV = nRT$
33. Solve for g: $E = mgh$
34. Solve for m: $E = \frac{1}{2}mv^2$

In Exercises 35 - 52, solve the given inequality. Write your answer using interval notation.

35. $3 - 4x \geq 0$
36. $2t - 1 < 3 - (4t - 3)$
37. $\dfrac{7 - y}{4} \geq 3y + 1$
38. $0.05R + 1.2 > 0.8 - 0.25R$
39. $7 - (2 - x) \leq x + 3$
40. $\dfrac{10m + 1}{5} \geq 2m - \dfrac{1}{2}$

41. $x\sqrt{12} - \sqrt{3} > \sqrt{3}\,x + \sqrt{27}$

42. $2t - 7 \leq \sqrt[3]{18}\,t$

43. $117y \geq y\sqrt{2} - 7y\sqrt[4]{8}$

44. $-\frac{1}{2} \leq 5x - 3 \leq \frac{1}{2}$

45. $-\frac{3}{2} \leq \frac{4 - 2t}{10} < \frac{7}{6}$

46. $-0.1 \leq \frac{5 - x}{3} - 2 < 0.1$

47. $2y \leq 3 - y < 7$

48. $3x \geq 4 - x \geq 3$

49. $6 - 5t > \frac{4t}{3} \geq t - 2$

50. $2x + 1 \leq -1$ or $2x + 1 \geq 1$

51. $4 - x \leq 0$ or $2x + 7 < x$

52. $\frac{5 - 2x}{3} > x$ or $2x + 5 \geq 1$

0.3.4 Answers

1. $x = \frac{18}{7}$
2. $t = -\frac{1}{30}$
3. $w = \frac{61}{33}$
4. $x = \frac{5}{9}$
5. $x = 7$
6. $x = \frac{8}{11}$
7. $x = -2$
8. $x = \frac{6}{7}$
9. $x = \frac{3}{5}$
10. $x = 0$
11. $x = 1$
12. $x = 3$
13. $w = \frac{1}{5}$
14. $w = 1$
15. $w = \frac{300}{\pi}$
16. $y = \frac{\sqrt{7}}{2}$
17. $x = 35$
18. $t = 0$
19. $y = 50000$
20. All real numbers.
21. No solution.
22. $t = -\dfrac{5}{3\sqrt{7}}$
23. $y = \dfrac{6}{17\sqrt{2}}$
24. $x = \dfrac{27}{18+\sqrt{7}}$
25. $y = \dfrac{4-3x}{2}$ or $y = -\frac{3}{2}x + 2$
26. $x = \dfrac{4-2y}{3}$ or $x = -\frac{2}{3}y + \frac{4}{3}$
27. $C = \frac{5}{9}(F - 32)$ or $C = \frac{5}{9}F - \frac{160}{9}$
28. $x = \dfrac{15-p}{2.5}$ or $x = -\frac{2}{5}p + 6$.
29. $y = \dfrac{3x+1}{x-2}$, provided $x \neq 2$.
30. $\pi = \dfrac{C}{2r}$, provided $r \neq 0$.
31. $V = \dfrac{nRT}{P}$, provided $P \neq 0$.
32. $R = \dfrac{PV}{nT}$, provided $n \neq 0$, $T \neq 0$.
33. $g = \dfrac{E}{mh}$, provided $m \neq 0$, $h \neq 0$.
34. $m = \dfrac{2E}{v^2}$, provided $v^2 \neq 0$ (so $v \neq 0$).
35. $\left(-\infty, \frac{3}{4}\right]$
36. $\left(-\infty, \frac{7}{6}\right)$
37. $\left(-\infty, \frac{3}{13}\right]$
38. $\left(-\frac{4}{3}, \infty\right)$
39. No solution.
40. $(-\infty, \infty)$
41. $(4, \infty)$
42. $\left[\frac{7}{2-\sqrt[3]{18}}, \infty\right)$
43. $[0, \infty)$
44. $\left[\frac{1}{2}, \frac{7}{10}\right]$
45. $\left(-\frac{23}{6}, \frac{19}{2}\right]$
46. $\left(-\frac{13}{10}, -\frac{7}{10}\right]$
47. $(-4, 1]$
48. $\{1\} = [1, 1]$
49. $\left[-6, \frac{18}{19}\right)$
50. $(-\infty, -1] \cup [0, \infty)$
51. $(-\infty, -7) \cup [4, \infty)$
52. $(-\infty, \infty)$

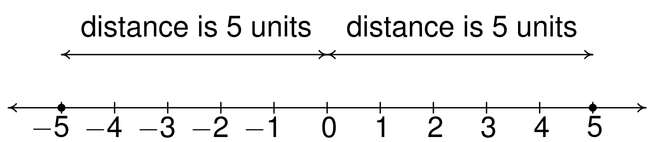
0.4 Absolute Value Equations and Inequalities

In this section, we review some basic concepts involving the absolute value of a real number x. There are a few different ways to define absolute value and in this section we choose the following definition. (Absolute value will be revisited in much greater depth in Section 2.3 where we present what one can think of as the "precise" definition.)

Definition 0.11. Absolute Value as Distance: For every real number x, the **absolute value** of x, denoted $|x|$, is the distance between x and 0 on the number line. More generally, if x and c are real numbers, $|x-c|$ is the distance between the numbers x and c on the number line. As a consequence, we have

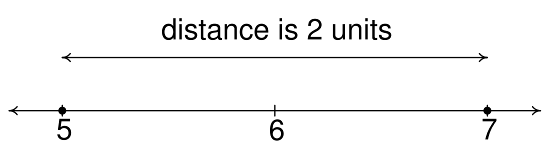
$$|x-c| = \begin{cases} -x+c, & \text{if } \ x-c<0 \\ x-c, & \text{if } \ x-c \geq 0 \end{cases} \qquad (0.11')$$

For example, $|5| = 5$ and $|-5| = 5$, since each is 5 units from 0 on the number line:



Graphically why $|-5| = 5$ and $|5| = 5$

Computationally, the absolute value 'makes negative numbers positive', though we need to be a little cautious with this description. While $|-7| = 7$, $|5-7| \neq 5+7$. The absolute value acts as a grouping symbol, so $|5-7| = |-2| = 2$, which makes sense since 5 and 7 are two units away from each other on the number line:



Graphically why $|5-7| = 2$

We list some of the operational properties of absolute value below.

Theorem 0.2. Properties of Absolute Value: Let a, b and x be real numbers and let n be an integer.[a] Then

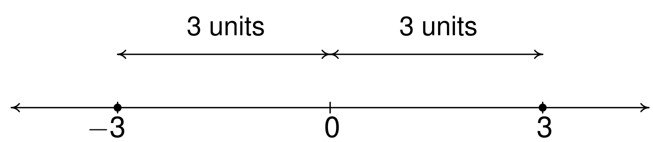
- **Product Rule:** $|ab| = |a||b|$
- **Power Rule:** $|a^n| = |a|^n$ whenever a^n is defined
- **Quotient Rule:** $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$, provided $b \neq 0$

[a]See page 6 if you don't remember what an integer is.

The proof of Theorem 0.2 is difficult, but not impossible, using the distance definition of absolute value or even the 'it makes negatives positive' notion. It is, however, much easier if one uses the "precise" definition given in Section 2.3 so we will revisit the proof then. For now, let's focus on how to solve basic equations and inequalities involving the absolute value.

0.4.1 Absolute Value Equations

Thinking of absolute value in terms of distance gives us a geometric way to interpret equations. For example, to solve $|x| = 3$, we are looking for all real numbers x whose distance from 0 is 3 units. If we move three units to the right of 0, we end up at $x = 3$. If we move three units to the left, we end up at $x = -3$. Thus the solutions to $|x| = 3$ are $x = \pm 3$.



The solutions to $|x| = 3$ are $x = \pm 3$.

Thinking this way gives us the following.

Theorem 0.3. Absolute Value Equations: Suppose A and B are real numbers.

- $|A| \geq 0$ for every A, and $|A| = 0$ if and only if $A = 0$.
- If $B > 0$ then $|A| = B$ if and only if $A = B$ or $A = -B$.
- For $B < 0$, $|A| = B$ has no solution.
- $|A| = |B|$ if and only if $A = B$ or $A = -B$.
  (That is, if two numbers have the same absolute values, they are either the same number or exact opposites.)

Theorem 0.3 is our main tool in solving equations involving the absolute value, since it allows us a way to rewrite such equations as compound linear equations.

Strategy for Solving Equations Involving Absolute Value

In order to solve an equation involving the absolute value of a quantity $|A|$:

1. Isolate the absolute value on one side of the equation so it has the form $|A| = B$.
2. Apply Theorem 0.3.

The techniques we use to 'isolate the absolute value' are precisely those we used in Section 0.3 to isolate the variable when solving linear equations. Time for some practice.

Example 0.4.1. Solve each of the following equations.

1. $|3x-1| = 6$
2. $\dfrac{3-|y+5|}{2} = 1$
3. $3|2t+1| - \sqrt{5} = 0$
4. $4 - |5w+3| = 5$
5. $\left|3 - x\sqrt[3]{12}\right| = |4x+1|$
6. $|t-1| - 3|t+1| = 0$

Solution.

1. The equation $|3x-1| = 6$ is of already in the form $|X| = c$, so we know $3x-1 = 6$ or $3x-1 = -6$. Solving the former gives us at $x = \frac{7}{3}$ and solving the latter yields $x = -\frac{5}{3}$. We may check both of these solutions by substituting them into the original equation and showing that the arithmetic works out.

2. We begin solving $\frac{3-|y+5|}{2} = 1$ by isolating the absolute value to put it in the form $|X| = c$.

$$\begin{array}{rcll} \dfrac{3-|y+5|}{2} & = & 1 & \\ 3-|y+5| & = & 2 & \text{Multiply by 2} \\ -|y+5| & = & -1 & \text{Subtract 3} \\ |y+5| & = & 1 & \text{Divide by } -1 \end{array}$$

At this point, we have $y+5 = 1$ or $y+5 = -1$, so our solutions are $y = -4$ or $y = -6$. We leave it to the reader to check both answers in the original equation.

3. As in the previous example, we first isolate the absolute value. Don't let the $\sqrt{5}$ throw you off - it's just another real number, so we treat it as such:

$$\begin{array}{rcll} 3|2t+1| - \sqrt{5} & = & 0 & \\ 3|2t+1| & = & \sqrt{5} & \text{Add } \sqrt{5} \\ |2t+1| & = & \dfrac{\sqrt{5}}{3} & \text{Divide by 3} \end{array}$$

From here, we have that $2t+1 = \frac{\sqrt{5}}{3}$ or $2t+1 = -\frac{\sqrt{5}}{3}$. The first equation gives $t = \frac{\sqrt{5}-3}{6}$ while the second gives $t = \frac{-\sqrt{5}-3}{6}$ thus we list our answers as $t = \frac{-3\pm\sqrt{5}}{6}$. The reader should enjoy the challenge of substituting both answers into the original equation and following through the arithmetic to see that both answers work.

4. Upon isolating the absolute value in the equation $4 - |5w+3| = 5$, we get $|5w+3| = -1$. At this point, we know there cannot be any real solution. By definition, the absolute value is a *distance*, and as such is never negative. We write 'no solution' and carry on.

5. Our next equation already has the absolute value expressions (plural) isolated, so we work from the principle that if $|x| = |y|$, then $x = y$ or $x = -y$. Thus from $\left|3 - x\sqrt[3]{12}\right| = |4x+1|$ we get two equations to solve:

$$3 - x\sqrt[3]{12} = 4x+1, \qquad \text{and} \qquad 3 - x\sqrt[3]{12} = -(4x+1)$$

Notice that the right side of the second equation is $-(4x+1)$ and not simply $-4x+1$. Remember, the expression $4x+1$ represents a single real number so in order to negate it we need to negate the *entire* expression $-(4x+1)$. Moving along, when solving $3 - x\sqrt[3]{12} = 4x+1$, we obtain $x = \frac{2}{4+\sqrt[3]{12}}$ and the solution to $3 - x\sqrt[3]{12} = -(4x+1)$ is $x = \frac{4}{\sqrt[3]{12}-4}$. As usual, the reader is invited to check these answers by substituting them into the original equation.

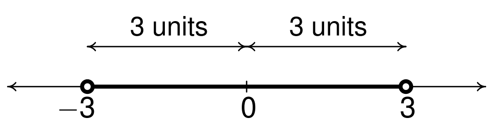
6. We start by isolating one of the absolute value expressions: $|t-1| - 3|t+1| = 0$ gives $|t-1| = 3|t+1|$. While this *resembles* the form $|x| = |y|$, the coefficient 3 in $3|t+1|$ prevents it from being an exact match. Not to worry - since 3 is positive, $3 = |3|$ so

$$3|t+1| = |3||t+1| = |3(t+1)| = |3t+3|.$$

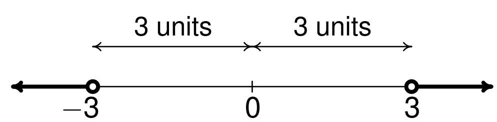
Hence, our equation becomes $|t-1| = |3t+3|$ which results in the two equations: $t-1 = 3t+3$ and $t-1 = -(3t+3)$. The first equation gives $t = -2$ and the second gives $t = -\frac{1}{2}$. The reader is encouraged to check both answers in the original equation. □

0.4.2 Absolute Value Inequalities

We now turn our attention to solving some basic inequalities involving the absolute value. Suppose we wished to solve $|x| < 3$. Geometrically, we are looking for all of the real numbers whose distance from 0 is *less* than 3 units. We get $-3 < x < 3$, or in interval notation, $(-3, 3)$. Suppose we are asked to solve $|x| > 3$ instead. Now we want the distance between x and 0 to be *greater* than 3 units. Moving in the positive direction, this means $x > 3$. In the negative direction, this puts $x < -3$. Our solutions would then satisfy $x < -3$ *or* $x > 3$. In interval notation, we express this as $(-\infty, -3) \cup (3, \infty)$.



The solution to $|x| < 3$ is $(-3, 3)$



The solution to $|x| > 3$ is $(-\infty, -3) \cup (3, \infty)$

Generalizing this notion, we get the following:

Theorem 0.4. Inequalities Involving Absolute Value.

If B is a real number such that $B > 0$, then

- $|A| < B$ is equivalent to $-B < A < B$.
- $|A| > B$ is equivalent to $A > B$ or $A < -B$.

Recall that $|A| \geq 0$ for all values of A. In particular, $|A| < B$ has no solution if $B \leq 0$.

If the inequality we're faced with involves '$\leq$' or '$\geq$,' we can combine the results of Theorem 0.4 with Theorem 0.3 as needed.

Strategy for Solving Inequalities Involving Absolute Value

In order to solve an inequality involving the absolute value of a quantity $|A|$:

1. Isolate the absolute value on one side of the inequality.
2. Apply Theorem 0.4.

Example 0.4.2. Solve the following inequalities.

1. $\left|x - \sqrt{5}\right| > 1$
2. $\dfrac{4 - 2|2x + 1|}{4} \geq -\sqrt{3}$
3. $|2x - 1| \leq 3|4 - 8x| - 10$
4. $|2x - 1| \leq 3|4 - 8x| + 10$
5. $2 < |x - 1| \leq 5$
6. $|10x - 5| + |10 - 5x| \leq 0$

Solution.

1. From Theorem 0.4, $\left|x - \sqrt{5}\right| > 1$ is equivalent to $x - \sqrt{5} < -1$ or $x - \sqrt{5} > 1$. Solving this compound inequality, we get $x < -1 + \sqrt{5}$ or $x > 1 + \sqrt{5}$. Our answer, in interval notation, is: $\left(-\infty, -1 + \sqrt{5}\right) \cup \left(1 + \sqrt{5}, \infty\right)$. As with linear inequalities, we can partially check our answer by selecting values of x both inside and outside the solution intervals to see which values of x satisfy the original inequality and which do not.

2. Our first step in solving $\frac{4-2|2x+1|}{4} \geq -\sqrt{3}$ is to isolate the absolute value.

$$
\begin{array}{rcll}
\dfrac{4 - 2|2x + 1|}{4} & \geq & -\sqrt{3} & \\
4 - 2|2x + 1| & \geq & -4\sqrt{3} & \text{Multiply by 4} \\
-2|2x + 1| & \geq & -4 - 4\sqrt{3} & \text{Subtract 4} \\
|2x + 1| & \leq & \dfrac{-4 - 4\sqrt{3}}{-2} & \text{Divide by } -2\text{, reverse the inequality} \\
|2x + 1| & \leq & 2 + 2\sqrt{3} & \text{Reduce}
\end{array}
$$

Since we're dealing with '$\leq$' instead of just '$<$,' we can combine Theorems 0.4 and 0.3 to rewrite this last inequality as:[1] $-(2 + 2\sqrt{3}) \leq 2x + 1 \leq 2 + 2\sqrt{3}$. Subtracting the '1' across both inequalities gives $-3 - 2\sqrt{3} \leq 2x \leq 1 + 2\sqrt{3}$, which reduces to $\frac{-3-2\sqrt{3}}{2} \leq x \leq \frac{1+2\sqrt{3}}{2}$. In interval notation this reads as $\left[\frac{-3-2\sqrt{3}}{2}, \frac{1+2\sqrt{3}}{2}\right]$.

[1] Note the use of parentheses: $-(2 + 2\sqrt{3})$ as opposed to $-2 + 2\sqrt{3}$.

3. There are two absolute values in $|2x-1| \leq 3|4-8x|-10$, so it is unclear how we are to proceed. However, before jumping in and trying to apply (or misapply) Theorem 0.4, we note that $|4-8x| = |(-4)(2x-1)|$. Using this, we get:

$$\begin{array}{rcll} |2x-1| & \leq & 3|4-8x|-10 & \\ |2x-1| & \leq & 3|(-4)(2x-1)|-10 & \text{Factor} \\ |2x-1| & \leq & 3|-4||2x-1|-10 & \text{Product Rule} \\ |2x-1| & \leq & 12|2x-1|-10 & \\ -11|2x-1| & \leq & -10 & \text{Subtract } 12|2x-1| \\ |2x-1| & \geq & \dfrac{10}{11} & \text{Divide by } -11 \text{ and reduce} \end{array}$$

At this point, we invoke Theorems 0.3 and 0.4 and write the equivalent compound inequality: $2x-1 \leq -\frac{10}{11}$ *or* $2x-1 \geq \frac{10}{11}$. We get $x \leq \frac{1}{22}$ *or* $x \geq \frac{21}{22}$, which, in interval notation reads $\left(-\infty, \frac{1}{22}\right] \cup \left[\frac{21}{22}, \infty\right)$.

4. The inequality $|2x-1| \leq 3|4-8x|+10$ differs from the previous example in exactly one respect: on the right side of the inequality, we have '+10' instead of '-10.' The steps to isolate the absolute value here are identical to those in the previous example, but instead of obtaining $|2x-1| \geq \frac{10}{11}$ as before, we obtain $|2x-1| \geq -\frac{10}{11}$. This latter inequality is *always* true. (Absolute value is, by definition, a distance and hence always 0 or greater.) Thus our solution to this inequality is all real numbers, $(-\infty, \infty)$.

5. To solve $2 < |x-1| \leq 5$, we rewrite it as the compound inequality: $2 < |x-1|$ *and* $|x-1| \leq 5$. The first inequality, $2 < |x-1|$, can be re-written as $|x-1| > 2$ so it is equivalent to $x-1 < -2$ *or* $x-1 > 2$. Thus the solution to $2 < |x-1|$ is $x < -1$ or $x > 3$, which in interval notation is $(-\infty, -1) \cup (3, \infty)$. For $|x-1| \leq 5$, we combine the results of Theorems 0.3 and 0.4 to get $-5 \leq x-1 \leq 5$ so that $-4 \leq x \leq 6$, or $[-4, 6]$. Our solution to $2 < |x-1| \leq 5$ is comprised of values of x which satisfy both parts of the inequality, so we intersect $(-\infty, -1) \cup (3, \infty)$ with $[-4, 6]$ to get our final answer $[-4, -1) \cup (3, 6]$.

6. Our first hope when encountering $|10x-5| + |10-5x| \leq 0$ is that we can somehow combine the two absolute value quantities as we'd done in earlier examples. We leave it to the reader to show, however, that no matter what we try to factor out of the absolute value quantities, what remains inside the absolute values will always be different. At this point, we take a step back and look at the equation in a more general way: we are adding two absolute values together and wanting the result to be less than or equal to 0. Since the absolute value of anything is always 0 or greater, there are no solutions to: $|10x-5| + |10-5x| < 0$. Is it possible that $|10x-5| + |10-5x| = 0$? Only if there is an x where $|10x-5| = 0$ and $|10-5x| = 0$ at the same time.[2] The first equation holds only when $x = 1$, while the second holds only when $x = 2$. Alas, we have no solution.[3] □

[2] Do you see why?

[3] Not for lack of trying, however!

We close this section with an example of how the properties in Theorem 0.2 are used in Calculus. Here, 'ε' is the Greek letter 'epsilon' and it represents a positive real number. Those of you who will be taking Calculus in the future should become *very* familiar with this type of algebraic manipulation.

$$\begin{aligned}
\left|\frac{8-4x}{3}\right| &< \varepsilon \\
\frac{|8-4x|}{|3|} &< \varepsilon && \text{Quotient Rule} \\
\frac{|-4(x-2)|}{3} &< \varepsilon && \text{Factor} \\
\frac{|-4||x-2|}{3} &< \varepsilon && \text{Product Rule} \\
\frac{4|x-2|}{3} &< \varepsilon \\
\frac{3}{4}\cdot\frac{4|x-2|}{3} &< \frac{3}{4}\cdot\varepsilon && \text{Multiply by } \frac{3}{4} \\
|x-2| &< \frac{3}{4}\varepsilon
\end{aligned}$$

0.4.3 Exercises

In Exercises 1 - 18, solve the equation.

1. $|x| = 6$
2. $|3t - 1| = 10$
3. $|4 - w| = 7$
4. $4 - |y| = 3$
5. $2|5m + 1| - 3 = 0$
6. $|7x - 1| + 2 = 0$
7. $\dfrac{5 - |x|}{2} = 1$
8. $\frac{2}{3}|5 - 2w| - \frac{1}{2} = 5$
9. $|3t - \sqrt{2}| + 4 = 6$
10. $\dfrac{|2v + 1| - 3}{4} = \frac{1}{2} - |2v + 1|$
11. $\dfrac{|3 - 2y| + 4}{2} = 2 - |3 - 2y|$
12. $|x - 5| + |1 - 5x| = 0$
13. $|3t - 2| = |2t + 7|$
14. $|3x + 1| = |4x|$
15. $|1 - \sqrt{2}\,y| = |y + 1|$
16. $|4 - x| - |x + 2| = 0$
17. $|2 - 5z| = 5|z + 1|$
18. $\sqrt{3}|w - 1| = 2|w + 1|$

In Exercises 19 - 30, solve the inequality. Write your answer using interval notation.

19. $|3x - 5| \leq 4$
20. $|7t + 2| > 10$
21. $|2w + 1| - 5 < 0$
22. $|2 - y| - 4 \geq -3$
23. $|3z + 5| + 2 < 1$
24. $2|7 - v| + 4 > 1$
25. $3 - |x + \sqrt{5}| < -3$
26. $|5t| \leq |t| + 3$
27. $|w - 3| < |3 - w|$
28. $2 \leq |4 - y| < 7$
29. $1 < |2w - 9| \leq 3$
30. $\pi > |\pi - x| \geq 1$

0.4.4 Answers

1. $x = -6$ or $x = 6$
2. $t = -3$ or $t = \frac{11}{3}$
3. $w = -3$ or $w = 11$
4. $y = -1$ or $y = 1$
5. $m = -\frac{1}{2}$ or $m = \frac{1}{10}$
6. No solution
7. $x = -3$ or $x = 3$
8. $w = -\frac{13}{8}$ or $w = \frac{53}{8}$
9. $t = \frac{\sqrt{2} \pm 2}{3}$
10. $v = -1$ or $v = 0$
11. $y = \frac{3}{2}$
12. No solution
13. $t = -1$ or $t = 9$
14. $x = -\frac{1}{7}$ or $x = 1$
15. $y = 0$ or $y = \frac{2}{\sqrt{2} - 1}$
16. $x = 1$
17. $z = -\frac{3}{10}$
18. $w = \frac{\sqrt{3} \pm 2}{\sqrt{3} \mp 2}$ See footnote[4]
19. $\left[\frac{1}{3}, 3\right]$
20. $\left(-\infty, -\frac{12}{7}\right) \cup \left(\frac{8}{7}, \infty\right)$
21. $(-3, 2)$
22. $(-\infty, 1] \cup [3, \infty)$
23. No solution
24. $(-\infty, \infty)$
25. $(-\infty, -6 - \sqrt{5}) \cup (6 - \sqrt{5}, \infty)$
26. $\left[-\frac{3}{4}, \frac{3}{4}\right]$
27. No solution
28. $(-3, 2] \cup [6, 11)$
29. $[3, 4) \cup (5, 6]$
30. $(0, \pi - 1] \cup [\pi + 1, 2\pi)$

[4] That is, $w = \frac{\sqrt{3} + 2}{\sqrt{3} - 2}$ or $w = \frac{\sqrt{3} - 2}{\sqrt{3} + 2}$

0.5 Polynomial Arithmetic

In this section, we review the arithmetic of **polynomials**. What precisely is a polynomial?

Definition 0.12. A **polynomial** is a sum of terms each of which is a real number or a real number multiplied by one or more variables to natural number powers.

Some examples of polynomials are $x^2 + x\sqrt{3} + 4$, $27x^2y + \frac{7x}{2}$ and 6. Things like $3\sqrt{x}$, $4x - \frac{2}{x+1}$ and $13x^{2/3}y^2$ are **not** polynomials. (Do you see why not?) Below we review some of the terminology associated with polynomials.

Definition 0.13. Polynomial Vocabulary

- **Constant Terms:** Terms in polynomials without variables are called **constant** terms.
- **Coefficient:** In non-constant terms, the real number factor in the expression is called the **coefficient** of the term.
- **Degree:** The **degree** of a non-constant term is the sum of the exponents on the variables in the term; non-zero constant terms are defined to have degree 0. The degree of a polynomial is the highest degree of the nonzero terms.
- **Like Terms:** Terms in a polynomial are called **like** terms if they have the same variables each with the same corresponding exponents.
- **Simplified:** A polynomial is said to be **simplified** if all arithmetic operations have been completed and there are no longer any like terms.
- **Classification by Number of Terms:** A simplified polynomial is called a
  - **monomial** if it has exactly one nonzero term
  - **binomial** if it has exactly two nonzero terms
  - **trinomial** if it has exactly three nonzero terms

For example, $x^2 + x\sqrt{3} + 4$ is a trinomial of degree 2. The coefficient of x^2 is 1 and the constant term is 4. The polynomial $27x^2y + \frac{7x}{2}$ is a binomial of degree 3 ($x^2y = x^2y^1$) with constant term 0.

The concept of 'like' terms really amounts to finding terms which can be combined using the Distributive Property. For example, in the polynomial $17x^2y - 3xy^2 + 7xy^2$, $-3xy^2$ and $7xy^2$ are like terms, since they have the same variables with the same corresponding exponents. This allows us to combine these two terms as follows:

$$17x^2y - 3xy^2 + 7xy^2 = 17x^2y + (-3)xy^2 + 7xy^2 + 17x^2y + (-3+7)xy^2 = 17x^2y + 4xy^2$$

Note that even though $17x^2y$ and $4xy^2$ have the same variables, they are not like terms since in the first term we have x^2 and $y = y^1$ but in the second we have $x = x^1$ and $y = y^2$ so the corresponding exponents aren't the same. Hence, $17x^2y + 4xy^2$ is the simplified form of the polynomial.

There are four basic operations we can perform with polynomials: addition, subtraction, multiplication and division. The first three of these operations follow directly from properties of real number arithmetic and will be discussed together first. Division, on the other hand, is a bit more complicated and will be discussed separately.

0.5.1 Polynomial Addition, Subtraction and Multiplication.

Adding and subtracting polynomials comes down to identifying like terms and then adding or subtracting the coefficients of those like terms. Multiplying polynomials comes to us courtesy of the Generalized Distributive Property.

Theorem 0.5. Generalized Distributive Property: To multiply a quantity of n terms by a quantity of m terms, multiply each of the n terms of the first quantity by each of the m terms in the second quantity and add the resulting $n \cdot m$ terms together.

In particular, Theorem 0.5 says that, before combining like terms, a product of an n-term polynomial and an m-term polynomial will generate $(n \cdot m)$-terms. For example, a binomial times a trinomial will produce six terms some of which may be like terms. Thus the simplified end result may have fewer than six terms but you will start with six terms.

A special case of Theorem 0.5 is the famous **F.O.I.L.**, listed here:[1]

Theorem 0.6. F.O.I.L: The terms generated from the product of two binomials: $(a + b)(c + d)$ can be verbalized as follows "Take the sum of:

- the product of the **F**irst terms a and c, ac
- the product of the **O**uter terms a and d, ad
- the product of the **I**nner terms b and c, bc
- the product of the **L**ast terms b and d, bd."

That is, $(a + b)(c + d) = ac + ad + bc + bd$.

Theorem 0.5 is best proved using the technique known as Mathematical Induction which is covered in Section 9.3. The result is really nothing more than repeated applications of the Distributive Property so it seems reasonable and we'll use it without proof for now. The other major piece of polynomial multiplication is one of the Power Rules of Exponents from page 26 in Section 0.2, namely $a^n a^m = a^{n+m}$. The Commutative and Associative Properties of addition and multiplication are also used extensively. We put all of these properties to good use in the next example.

[1] We caved to peer pressure on this one. Apparently all of the cool Precalculus books have FOIL in them even though it's redundant once you know how to distribute multiplication across addition. In general, we don't like mechanical shortcuts that interfere with a student's understanding of the material and FOIL is one of the worst.

Example 0.5.1. Perform the indicated operations and simplify.

1. $\left(3x^2-2x+1\right)-(7x-3)$
2. $4xz^2-3z(xz-x+4)$
3. $(2t+1)(3t-7)$
4. $\left(3y-\sqrt[3]{2}\right)\left(9y^2+3\sqrt[3]{2}y+\sqrt[3]{4}\right)$
5. $\left(4w-\dfrac{1}{2}\right)^2$
6. $\left[2(x+h)-(x+h)^2\right]-\left(2x-x^2\right)$

Solution.

1. We begin 'distributing the negative' as indicated on page 20 in Section 0.2, then we rearrange and combine like term:

$$\begin{array}{rcll} \left(3x^2-2x+1\right)-(7x-3) & = & 3x^2-2x+1-7x+3 & \text{Distribute} \\ & = & 3x^2-2x-7x+1+3 & \text{Rearrange terms} \\ & = & 3x^2-9x+4 & \text{Combine like terms} \end{array}$$

Our answer is $3x^2-9x+4$.

2. Following in our footsteps from the previous example, we first distribute the $-3z$ through, then rearrange and combine like terms.

$$\begin{array}{rcll} 4xz^2-3z(xz-x+4) & = & 4xz^2-3z(xz)+3z(x)-3z(4) & \text{Distribute} \\ & = & 4xz^2-3xz^2+3xz-12z & \text{Multiply} \\ & = & xz^2+3xz-12z & \text{Combine like terms} \end{array}$$

We get our final answer: $xz^2+3xz-12z$

3. At last, we have a chance to use our F.O.I.L. technique:

$$\begin{array}{rcll} (2t+1)(3t-7) & = & (2t)(3t)+(2t)(-7)+(1)(3t)+(1)(-7) & \text{F.O.I.L.} \\ & = & 6t^2-14t+3t-7 & \text{Multiply} \\ & = & 6t^2-11t-7 & \text{Combine like terms} \end{array}$$

We get $6t^2-11t-7$ as our final answer.

4. We use the Generalized Distributive Property here, multiplying each term in the second quantity first by $3y$, then by $-\sqrt[3]{2}$:

$$\begin{array}{rcl} \left(3y-\sqrt[3]{2}\right)\left(9y^2+3\sqrt[3]{2}y+\sqrt[3]{4}\right) & = & 3y\left(9y^2\right)+3y\left(3\sqrt[3]{2}y\right)+3y\left(\sqrt[3]{4}\right) \\ & & -\sqrt[3]{2}\left(9y^2\right)-\sqrt[3]{2}\left(3\sqrt[3]{2}y\right)-\sqrt[3]{2}\left(\sqrt[3]{4}\right) \\ & = & 27y^3+9y^2\sqrt[3]{2}+3y\sqrt[3]{4}-9y^2\sqrt[3]{2}-3y\sqrt[3]{4}-\sqrt[3]{8} \\ & = & 27y^3+9y^2\sqrt[3]{2}-9y^2\sqrt[3]{2}+3y\sqrt[3]{4}-3y\sqrt[3]{4}-2 \\ & = & 27y^3-2 \end{array}$$

To our surprise and delight, this product reduces to $27y^3-2$.

5. Since exponents do **not** distribute across powers,[2] $\left(4w - \frac{1}{2}\right)^2 \neq (4w)^2 - \left(\frac{1}{2}\right)^2$. (We know you knew that.) Instead, we proceed as follows:

$$\begin{array}{rcll}
\left(4w - \dfrac{1}{2}\right)^2 & = & \left(4w - \dfrac{1}{2}\right)\left(4w - \dfrac{1}{2}\right) & \\
& = & (4w)(4w) + (4w)\left(-\dfrac{1}{2}\right) + \left(-\dfrac{1}{2}\right)(4w) + \left(-\dfrac{1}{2}\right)\left(-\dfrac{1}{2}\right) & \text{Distribute} \\
& = & 16w^2 - 2w - 2w + \dfrac{1}{4} & \text{Multiply} \\
& = & 16w^2 - 4w + \dfrac{1}{4} & \text{Combine like terms}
\end{array}$$

Our (correct) final answer is $16w^2 - 4w + \frac{1}{4}$.

6. Our last example has two levels of grouping symbols. We begin simplifying the quantity inside the brackets, squaring out the binomial $(x + h)^2$ in the same way we expanded the square in our last example:

$$(x+h)^2 = (x+h)(x+h) = (x)(x) + (x)(h) + (h)(x) + (h)(h) = x^2 + 2xh + h^2$$

When we substitute this into our expression, we envelope it in parentheses, as usual, so we don't forget to distribute the negative.

$$\begin{array}{rcll}
\left[2(x+h) - (x+h)^2\right] - \left(2x - x^2\right) & = & \left[2(x+h) - \left(x^2 + 2xh + h^2\right)\right] - \left(2x - x^2\right) & \text{Substitute} \\
& = & \left[2x + 2h - x^2 - 2xh - h^2\right] - \left(2x - x^2\right) & \text{Distribute} \\
& = & 2x + 2h - x^2 - 2xh - h^2 - 2x + x^2 & \text{Distribute} \\
& = & 2h - 2xh - h^2 & \text{Combine like terms}
\end{array}$$

We find no like terms in $2h - 2xh - h^2$ so we are finished. □

We conclude our discussion of polynomial multiplication by showcasing two special products which happen often enough they should be committed to memory.

Theorem 0.7. Special Products: Let a and b be real numbers:

- **Perfect Square:** $(a+b)^2 = a^2 + 2ab + b^2$ and $(a-b)^2 = a^2 - 2ab + b^2$
- **Difference of Two Squares:** $(a-b)(a+b) = a^2 - b^2$

The formulas in Theorem 0.7 can be verified by working through the multiplication.[3]

---

[2]See the remarks following the Properties of Exponents on 26.

[3]These are both special cases of F.O.I.L.

0.5.2 Polynomial Long Division.

We now turn our attention to polynomial long division. Dividing two polynomials follows the same algorithm, in principle, as dividing two natural numbers so we review that process first. Suppose we wished to divide 2585 by 79. The standard division tableau is given below.

$$\begin{array}{rr} & 32 \\ 79 & \overline{)2585} \\ & -\underline{237}\downarrow \\ & 215 \\ & -\underline{158} \\ & 57 \end{array}$$

In this case, 79 is called the **divisor**, 2585 is called the **dividend**, 32 is called the **quotient** and 57 is called the **remainder**. We can check our answer by showing:

$$\text{dividend} = (\text{divisor})(\text{quotient}) + \text{remainder}$$

or in this case, $2585 = (79)(32) + 57\checkmark$. We hope that the long division tableau evokes warm, fuzzy memories of your formative years as opposed to feelings of hopelessness and frustration. If you experience the latter, keep in mind that the Division Algorithm essentially is a two-step process, iterated over and over again. First, we guess the number of times the divisor goes into the dividend and then we subtract off our guess. We repeat those steps with what's left over until what's left over (the remainder) is less than what we started with (the divisor). That's all there is to it!

The division algorithm for polynomials has the same basic two steps but when we subtract polynomials, we must take care to subtract *like terms* only. As a transition to polynomial division, let's write out our previous division tableau in expanded form.

$$\begin{array}{rr} & 3\cdot 10+2 \\ 7\cdot 10+9 & \overline{)2\cdot 10^3+5\cdot 10^2+8\cdot 10+5} \\ & -\underline{(2\cdot 10^3+3\cdot 10^2+7\cdot 10)}\ \downarrow \\ & 2\cdot 10^2+1\cdot 10+5 \\ & -\underline{(1\cdot 10^2+5\cdot 10+8)} \\ & 5\cdot 10+7 \end{array}$$

Written this way, we see that when we line up the digits we are really lining up the coefficients of the corresponding powers of 10 - much like how we'll have to keep the powers of x lined up in the same columns. The big difference between polynomial division and the division of natural numbers is that the value of x is an unknown quantity. So unlike using the known value of 10, when we subtract there can be no regrouping of coefficients as in our previous example. (The subtraction $215 - 158$ requires us to 'regroup' or 'borrow' from the tens digit, then the hundreds digit.) This actually makes polynomial division easier.[4] Before we dive into examples, we first state a theorem telling us when we can divide two polynomials, and what to expect when we do so.

> **Theorem 0.8. Polynomial Division:** Let d and p be nonzero polynomials where the degree of p is greater than or equal to the degree of d. There exist two unique polynomials, q and r, such that $p = d \cdot q + r$, where either $r = 0$ or the degree of r is strictly less than the degree of d.

Essentially, Theorem 0.8 tells us that we can divide polynomials whenever the degree of the divisor is less than or equal to the degree of the dividend. We know we're done with the division when the polynomial left over (the remainder) has a degree strictly less than the divisor. It's time to walk through a few examples to refresh your memory.

Example 0.5.2. Perform the indicated division. Check your answer by showing

$$\text{dividend} = (\text{divisor})(\text{quotient}) + \text{remainder}$$

1. $\left(x^3 + 4x^2 - 5x - 14\right) \div (x - 2)$
2. $(2t + 7) \div (3t - 4)$
3. $\left(6y^2 - 1\right) \div (2y + 5)$
4. $\left(w^3\right) \div \left(w^2 - \sqrt{2}\right)$.

Solution.

1. To begin $\left(x^3 + 4x^2 - 5x - 14\right) \div (x - 2)$, we divide the first term in the dividend, namely x^3, by the first term in the divisor, namely x, and get $\frac{x^3}{x} = x^2$. This then becomes the first term in the quotient. We proceed as in regular long division at this point: we multiply the entire divisor, $x - 2$, by this first term in the quotient to get $x^2(x - 2) = x^3 - 2x^2$. We then subtract this result from the dividend.

$$\begin{array}{r|l}
 & x^2 \\ \hline
x-2 & x^3 + 4x^2 - 5x - 14 \\
 & -\underline{\left(x^3 - 2x^2\right)} \quad \downarrow \\
 & \qquad\quad 6x^2 - 5x
\end{array}$$

Now we 'bring down' the next term of the quotient, namely $-5x$, and repeat the process. We divide $\frac{6x^2}{x} = 6x$, and add this to the quotient polynomial, multiply it by the divisor (which yields $6x(x - 2) = 6x^2 - 12x$) and subtract.

$$\begin{array}{r|l}
 & x^2 + 6x \\ \hline
x-2 & x^3 + 4x^2 - 5x - 14 \\
 & -\underline{\left(x^3 - 2x^2\right)} \qquad\quad \downarrow \\
 & \qquad\quad 6x^2 - 5x \qquad \downarrow \\
 & \qquad -\underline{\left(6x^2 - 12x\right)} \quad \downarrow \\
 & \qquad\qquad\qquad 7x - 14
\end{array}$$

[4] In our opinion - you can judge for yourself.

Finally, we 'bring down' the last term of the dividend, namely -14, and repeat the process. We divide $\frac{7x}{x} = 7$, add this to the quotient, multiply it by the divisor (which yields $7(x-2) = 7x-14$) and subtract.

$$\begin{array}{r|rrrr} & x^2 & +\ 6x & +\ 7 & \\ \hline x-2 & x^3 & +\ 4x^2 & -\ 5x & -\ 14 \\ & -\left(x^3\right. & \left.-\ 2x^2\right) & & \\ \hline & & 6x^2 & -\ 5x & \\ & & -\left(6x^2\right. & \left.-\ 12x\right) & \\ \hline & & & 7x & -\ 14 \\ & & & -\ (7x & -\ 14) \\ \hline & & & & 0 \end{array}$$

In this case, we get a quotient of x^2+6x+7 with a remainder of 0. To check our answer, we compute

$$(x-2)\left(x^2+6x+7\right)+0 = x^3+6x^2+7x-2x^2-12x-14 = x^3+4x^2-5x-14 \checkmark$$

2. To compute $(2t+7) \div (3t-4)$, we start as before. We find $\frac{2t}{3t} = \frac{2}{3}$, so that becomes the first (and only) term in the quotient. We multiply the divisor $(3t-4)$ by $\frac{2}{3}$ and get $2t - \frac{8}{3}$. We subtract this from the divided and get $\frac{29}{3}$.

$$\begin{array}{r|rr} & & \frac{2}{3} \\ \hline 3t-4 & 2t & +\ 7 \\ & -\left(2t\right. & \left.-\ \frac{8}{3}\right) \\ \hline & & \frac{29}{3} \end{array}$$

Our answer is $\frac{2}{3}$ with a remainder of $\frac{29}{3}$. To check our answer, we compute

$$(3t-4)\left(\frac{2}{3}\right)+\frac{29}{3} = 2t-\frac{8}{3}+\frac{29}{3} = 2t+\frac{21}{3} = 2t+7 \checkmark$$

3. When we set-up the tableau for $\left(6y^2-1\right) \div (2y+5)$, we must first issue a 'placeholder' for the 'missing' y-term in the dividend, $6y^2-1 = 6y^2+0y-1$. We then proceed as before. Since $\frac{6y^2}{2y} = 3y$, $3y$ is the first term in our quotient. We multiply $(2y+5)$ times $3y$ and subtract

it from the dividend. We bring down the -1, and repeat.

$$\begin{array}{rrrrr} & 3y & - & \frac{15}{2} \\ \cline{2-4} 2y+5 \big) & 6y^2 + & 0y & - & 1 \\ & -\left(6y^2 + 15y\right) & & \downarrow \\ \cline{2-2} & & -15y & - & 1 \\ & & -\left(-15y - \frac{75}{2}\right) \\ \cline{3-4} & & & & \frac{73}{2} \end{array}$$

Our answer is $3y - \frac{15}{2}$ with a remainder of $\frac{73}{2}$. To check our answer, we compute:

$$(2y+5)\left(3y - \frac{15}{2}\right) + \frac{73}{2} = 6y^2 - 15y + 15y - \frac{75}{2} + \frac{73}{2} = 6y^2 - 1 \checkmark$$

4. For our last example, we need 'placeholders' for both the divisor $w^2 - \sqrt{2} = w^2 + 0w - \sqrt{2}$ and the dividend $w^3 = w^3 + 0w^2 + 0w + 0$. The first term in the quotient is $\frac{w^3}{w^2} = w$, and when we multiply and subtract this from the dividend, we're left with just $0w^2 + w\sqrt{2} + 0 = w\sqrt{2}$.

$$\begin{array}{rllll} & & & w & \\ \cline{2-5} w^2+0w-\sqrt{2} \big) & w^3 & +0w^2 + & 0w & +0 \\ - & \left(w^3\right. & +0w^2 - & \left. w\sqrt{2}\right) & \downarrow \\ \cline{2-4} & & 0w^2 + & w\sqrt{2} & +0 \end{array}$$

Since the degree of $w\sqrt{2}$ (which is 1) is less than the degree of the divisor (which is 2), we are done.[5] Our answer is w with a remainder of $w\sqrt{2}$. To check, we compute:

$$\left(w^2 - \sqrt{2}\right) w + w\sqrt{2} = w^3 - w\sqrt{2} + w\sqrt{2} = w^3 \checkmark$$

□

[5] Since $\frac{0w^2}{w^2} = 0$, we could proceed, write our quotient as $w + 0$, and move on... but even pedants have limits.

0.5.3 Exercises

In Exercises 1 - 15, expand and combine like terms.

1. $(4-3x)+(3x^2+2x+7)$
2. $t^2+4t-2(3-t)$
3. $q(200-3q)-(5q+500)$
4. $(3y-1)(2y+1)$
5. $\left(3-\frac{x}{2}\right)(2x+5)$
6. $-(4t+3)(t^2-2)$
7. $2(w-\sqrt{5})(w+\sqrt{5})$
8. $(5a^2-3)(25a^4+15a^2+9)$
9. $(x^2-2x+3)(x^2+2x+3)$
10. $(\sqrt{7}-z)(\sqrt{7}+z)$
11. $(x-\sqrt[3]{5})^3$
12. $(x-\sqrt[3]{5})(x^2+x\sqrt[3]{5}+\sqrt[3]{25})$
13. $(w-3)^2-(w^2+9)$
14. $(x+h)^2-2(x+h)-(x^2-2x)$
15. $(x-2-\sqrt{5})(x-2+\sqrt{5})$

In Exercises 16 - 27, perform the indicated division. Check your answer by showing

$$\text{dividend} = (\text{divisor})(\text{quotient}) + \text{remainder}$$

16. $(5x^2-3x+1)\div(x+1)$
17. $(3y^2+6y-7)\div(y-3)$
18. $(6w-3)\div(2w+5)$
19. $(2x+1)\div(3x-4)$
20. $(t^2-4)\div(2t+1)$
21. $(w^3-8)\div(5w-10)$
22. $(2x^2-x+1)\div(3x^2+1)$
23. $(4y^4+3y^2+1)\div(2y^2-y+1)$
24. $w^4\div(w^3-2)$
25. $(5t^3-t+1)\div(t^2+4)$
26. $(t^3-4)\div(t-\sqrt[3]{4})$
27. $(x^2-2x-1)\div(x-[1-\sqrt{2}])$

In Exercises 28 - 33 verify the given formula by showing the left hand side of the equation simplifies to the right hand side of the equation.

28. **Perfect Cube:** $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$
29. **Difference of Cubes:** $(a-b)(a^2+ab+b^2) = a^3-b^3$
30. **Sum of Cubes:** $(a+b)(a^2-ab+b^2) = a^3+b^3$
31. **Perfect Quartic:** $(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$
32. **Difference of Quartics:** $(a-b)(a+b)(a^2+b^2) = a^4-b^4$
33. **Sum of Quartics:** $(a^2+ab\sqrt{2}+b^2)(a^2-ab\sqrt{2}+b^2) = a^4+b^4$
34. With help from your classmates, determine under what conditions $(a+b)^2 = a^2+b^2$. What about $(a+b)^3 = a^3+b^3$? In general, when does $(a+b)^n = a^n+b^n$ for a natural number $n \geq 2$?

0.5.4 Answers

1. $3x^2 - x + 11$
2. $t^2 + 6t - 6$
3. $-3q^2 + 195q - 500$
4. $6y^2 + y - 1$
5. $-x^2 + \frac{7}{2}x + 15$
6. $-4t^3 - 3t^2 + 8t + 6$
7. $2w^2 - 10$
8. $125a^6 - 27$
9. $x^4 + 2x^2 + 9$
10. $7 - z^2$
11. $x^3 - 3x^2\sqrt[3]{5} + 3x\sqrt[3]{25} - 5$
12. $x^3 - 5$
13. $-6w$
14. $h^2 + 2xh - 2h$
15. $x^2 - 4x - 1$
16. quotient: $5x - 8$, remainder: 9
17. quotient: $3y + 15$, remainder: 38
18. quotient: 3, remainder: -18
19. quotient: $\frac{2}{3}$, remainder: $\frac{11}{3}$
20. quotient: $\frac{t}{2} - \frac{1}{4}$, remainder: $-\frac{15}{4}$
21. quotient: $\frac{w^2}{5} + \frac{2w}{5} + \frac{4}{5}$, remainder: 0
22. quotient: $\frac{2}{3}$, remainder: $-x + \frac{1}{3}$
23. quotient: $2y^2 + y + 1$, remainder: 0
24. quotient: w, remainder: $2w$
25. quotient: $5t$, remainder: $-21t + 1$
26. quotient:[6] $t^2 + t\sqrt[3]{4} + 2\sqrt[3]{2}$, remainder: 0
27. quotient: $x - 1 - \sqrt{2}$, remainder: 0

[6]Note: $\sqrt[3]{16} = 2\sqrt[3]{2}$.

0.6 Factoring

Now that we have reviewed the basics of polynomial arithmetic it's time to review the basic techniques of factoring polynomial expressions. Our goal is to apply these techniques to help us solve certain specialized classes of non-linear equations. Given that 'factoring' literally means to resolve a product into its factors, it is, in the purest sense, 'undoing' multiplication. If this sounds like division to you then you've been paying attention. Let's start with a numerical example.

Suppose we are asked to factor 16337. We could write $16337 = 16337 \cdot 1$, and while this is technically a factorization of 16337, it's probably not an answer the poser of the question would accept. Usually, when we're asked to factor a natural number, we are being asked to resolve it into to a product of so-called 'prime' numbers.[1] Recall that **prime numbers** are defined as natural numbers whose only (natural number) factors are themselves and 1. They are, in essence, the 'building blocks' of natural numbers as far as multiplication is concerned. Said differently, we can build - via multiplication - any natural number given enough primes. So how do we find the prime factors of 16337? We start by dividing each of the primes: 2, 3, 5, 7, etc., into 16337 until we get a remainder of 0. Eventually, we find that $16337 \div 17 = 961$ with a remainder of 0, which means $16337 = 17 \cdot 961$. So factoring and division are indeed closely related - factors of a number are precisely the divisors of that number which produce a zero remainder.[2] We continue our efforts to see if 961 can be factored down further, and we find that $961 = 31 \cdot 31$. Hence, 16337 can be 'completely factored' as $17 \cdot 31^2$. (This factorization is called the **prime factorization** of 16337.)

In factoring natural numbers, our building blocks are prime numbers, so to be completely factored means that every number used in the factorization of a given number is prime. One of the challenges when it comes to factoring polynomial expressions is to explain what it means to be 'completely factored'. In this section, our 'building blocks' for factoring polynomials are 'irreducible' polynomials as defined below.

Definition 0.14. A polynomial is said to be **irreducible** if it cannot be written as the product of polynomials of lower degree.

While Definition 0.14 seems straightforward enough, sometimes a greater level of specificity is required. For example, $x^2 - 3 = (x - \sqrt{3})(x + \sqrt{3})$. While $x - \sqrt{3}$ and $x + \sqrt{3}$ are perfectly fine polynomials, factoring which requires irrational numbers is usually saved for a more advanced treatment of factoring.[3] For now, we will restrict ourselves to factoring using rational coefficients. So, while the polynomial $x^2 - 3$ can be factored using irrational numbers, it is called irreducible **over the rationals**, since there are no polynomials with *rational* coefficients of smaller degree which can be used to factor it.[4]

Since polynomials involve terms, the first step in any factoring strategy involves pulling out factors which are common to all of the terms. For example, in the polynomial $18x^2y^3 - 54x^3y^2 - 12xy^2$,

[1] As mentioned in Section 0.2, this is possible, in only one way, thanks to the Fundamental Theorem of Arithmetic.
[2] We'll refer back to this when we get to Section 3.2.
[3] See Section 3.3.
[4] If this isn't immediately obvious, don't worry - in some sense, it shouldn't be. We'll talk more about this later.

each coefficient is a multiple of 6 so we can begin the factorization as $6(3x^2y^3 - 9x^3y^2 - 2xy^2)$. The remaining coefficients: 3, 9 and 2, have no common factors so 6 was the greatest common factor. What about the variables? Each term contains an x, so we can factor an x from each term. When we do this, we are effectively dividing each term by x which means the exponent on x in each term is reduced by 1: $6x(3xy^3 - 9x^2y^2 - 2y^2)$. Next, we see that each term has a factor of y in it. In fact, each term has at least *two* factors of y in it, since the lowest exponent on y in each term is 2. This means that we can factor y^2 from each term. Again, factoring out y^2 from each term is tantamount to dividing each term by y^2 so the exponent on y in each term is reduced by *two*: $6xy^2(3xy - 9x^2 - 2)$. Just like we checked our division by multiplication in the previous section, we can check our factoring here by multiplication, too. $6xy^2(3xy - 9x^2 - 2) = (6xy^2)(3xy) - (6xy^2)(9x^2) - (6xy^2)(2) = 18x^2y^3 - 54x^3y^2 - 12xy^2$ ✓. We summarize how to find the Greatest Common Factor (G.C.F.) of a polynomial expression below.

Finding the G.C.F. of a Polynomial Expression

- If the coefficients are integers, find the G.C.F. of the coefficients.

  NOTE 1: If all of the coefficients are negative, consider the negative as part of the G.C.F..

  NOTE 2: If the coefficients involve fractions, get a common denominator, combine numerators, reduce to lowest terms and apply this step to the polynomial in the numerator.

- If a variable is common to all of the terms, the G.C.F. contains that variable to the smallest exponent which appears among the terms.

For example, to factor $-\frac{3}{5}z^3 - 6z^2$, we would first get a common denominator and factor as:

$$-\frac{3}{5}z^3 - 6z^2 = \frac{-3z^3 - 30z^2}{5} = \frac{-3z^2(z+10)}{5} = -\frac{3z^2(z+10)}{5}$$

We now list some common factoring formulas, each of which can be verified by multiplying out the right side of the equation. While they all should look familiar - this is a review section after all - some should look more familiar than others since they appeared as 'special product' formulas in the previous section.

Common Factoring Formulas

- **Perfect Square Trinomials:** $a^2 + 2ab + b^2 = (a+b)^2$ and $a^2 - 2ab + b^2 = (a-b)^2$

- **Difference of Two Squares:** $a^2 - b^2 = (a-b)(a+b)$

  NOTE: In general, the sum of squares, $a^2 + b^2$ is irreducible over the rationals.

- **Sum of Two Cubes:** $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

  NOTE: In general, $a^2 - ab + b^2$ is irreducible over the rationals.

- **Difference of Two Cubes:** $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

  NOTE: In general, $a^2 + ab + b^2$ is irreducible over the rationals.

Our next example gives us practice with these formulas.

Example 0.6.1. Factor the following polynomials completely over the rationals. That is, write each polynomial as a product polynomials of lowest degree which are irreducible over the rationals.

1. $18x^2 - 48x + 32$
2. $64y^2 - 1$
3. $75t^4 + 30t^3 + 3t^2$
4. $w^4z - wz^4$
5. $81 - 16t^4$
6. $x^6 - 64$

Solution.

1. Our first step is to factor out the G.C.F. which in this case is 2. To match what is left with one of the special forms, we rewrite $9x^2 = (3x)^2$ and $16 = 4^2$. Since the 'middle' term is $-24x = -2(4)(3x)$, we see that we have a perfect square trinomial.

$$\begin{array}{rcll} 18x^2 - 48x + 32 & = & 2(9x^2 - 24x + 16) & \text{Factor out G.C.F.} \\ & = & 2((3x)^2 - 2(4)(3x) + (4)^2) & \\ & = & 2(3x - 4)^2 & \text{Perfect Square Trinomial: } a = 3x,\ b = 4 \end{array}$$

   Our final answer is $2(3x - 4)^2$. To check, we multiply out $2(3x - 4)^2$ to show that it equals $18x^2 - 48x + 32$.

2. For $64y^2 - 1$, we note that the G.C.F. of the terms is just 1, so there is nothing (of substance) to factor out of both terms. Since $64y^2 - 1$ is the difference of two terms, one of which is a square, we look to the Difference of Squares Formula for inspiration. By identifying $64y^2 = (8y)^2$ and $1 = 1^2$, we get

$$\begin{array}{rcll} 64y^2 - 1 & = & (8y)^2 - 1^2 & \\ & = & (8y - 1)(8y + 1) & \text{Difference of Squares, } a = 8y,\ b = 1 \end{array}$$

   As before, we can check our answer by multiplying out $(8y - 1)(8y + 1)$ to show that it equals $64y^2 - 1$.

3. The G.C.F. of the terms in $75t^4 + 30t^3 + 3t^2$ is $3t^2$, so we factor that out first. We identify what remains as a perfect square trinomial:

$$\begin{array}{rcll} 75t^4 + 30t^3 + 3t^2 & = & 3t^2(25t^2 + 10t + 1) & \text{Factor out G.C.F.} \\ & = & 3t^2((5t)^2 + 2(1)(5t) + 1^2) & \\ & = & 3t^2(5t + 1)^2 & \text{Perfect Square Trinomial, } a = 5t,\ b = 1 \end{array}$$

   Our final answer is $3t^2(5t + 1)^2$, which the reader is invited to check.

4. For $w^4z - wz^4$, we identify the G.C.F. as wz and once we factor it out a difference of cubes is revealed:

$$\begin{array}{rcll} w^4z - wz^4 & = & wz(w^3 - z^3) & \text{Factor out G.C.F.} \\ & = & wz(w - z)(w^2 + wz + z^2) & \text{Difference of Cubes, } a = w,\ b = z \end{array}$$

   Our final answer is $wz(w - z)(w^2 + wz + z^2)$. The reader is strongly encouraged to multiply this out to see that it reduces to $w^4z - wz^4$.

5. The G.C.F. of the terms in $81 - 16t^4$ is just 1 so there is nothing of substance to factor out from both terms. With just a difference of two terms, we are limited to fitting this polynomial into either the Difference of Two Squares or Difference of Two Cubes formula. Since the variable here is t^4, and 4 is a multiple of 2, we can think of $t^4 = (t^2)^2$. This means that we can write $16t^4 = (4t^2)^2$ which is a perfect square. (Since 4 is not a multiple of 3, we cannot write t^4 as a perfect cube of a polynomial.) Identifying $81 = 9^2$ and $16t^4 = (4t^2)^2$, we apply the Difference of Squares Formula to get:

$$\begin{aligned} 81 - 16t^4 &= 9^2 - (4t^2)^2 \\ &= (9 - 4t^2)(9 + 4t^2) \quad \text{Difference of Squares, } a = 9,\ b = 4t^2 \end{aligned}$$

At this point, we have an opportunity to proceed further. Identifying $9 = 3^2$ and $4t^2 = (2t)^2$, we see that we have another difference of squares in the first quantity, which we can reduce. (The sum of two squares in the second quantity <u>cannot</u> be factored over the rationals.)

$$\begin{aligned} 81 - 16t^4 &= (9 - 4t^2)(9 + 4t^2) \\ &= (3^2 - (2t)^2)(9 + 4t^2) \\ &= (3 - 2t)(3 + 2t)(9 + 4t^2) \quad \text{Difference of Squares, } a = 3,\ b = 2t \end{aligned}$$

As always, the reader is encouraged to multiply out $(3-2t)(3+2t)(9+4t^2)$ to check the result.

6. With a G.C.F. of 1 and just two terms, $x^6 - 64$ is a candidate for both the Difference of Squares and the Difference of Cubes formulas. Notice that we can identify $x^6 = (x^3)^2$ and $64 = 8^2$ (both perfect squares), but also $x^6 = (x^2)^3$ and $64 = 4^3$ (both perfect cubes). If we follow the Difference of Squares approach, we get:

$$\begin{aligned} x^6 - 64 &= (x^3)^2 - 8^2 \\ &= (x^3 - 8)(x^3 + 8) \quad \text{Difference of Squares, } a = x^3 \text{ and } b = 8 \end{aligned}$$

At this point, we have an opportunity to use both the Difference and Sum of Cubes formulas:

$$\begin{aligned} x^6 - 64 &= (x^3 - 2^3)(x^3 + 2^3) && \text{Sum/Difference of Cubes} \\ &= (x - 2)(x^2 + 2x + 2^2)(x + 2)(x^2 - 2x + 2^2) && \text{with } a = x \text{ and } b = 2 \\ &= (x - 2)(x + 2)(x^2 - 2x + 4)(x^2 + 2x + 4) && \text{Rearrange factors} \end{aligned}$$

From this approach, our final answer is $(x - 2)(x + 2)(x^2 - 2x + 4)(x^2 + 2x + 4)$.

Following the Difference of Cubes Formula approach, we get

$$\begin{aligned} x^6 - 64 &= (x^2)^3 - 4^3 \\ &= (x^2 - 4)((x^2)^2 + 4x^2 + 4^2) \quad \text{Difference of Cubes, } a = x^2,\ b = 4 \\ &= (x^2 - 4)(x^4 + 4x^2 + 16) \end{aligned}$$

At this point, we recognize $x^2 - 4$ as a difference of two squares:

$$\begin{aligned} x^6 - 64 &= (x^2 - 2^2)(x^4 + 4x^2 + 16) \\ &= (x - 2)(x + 2)(x^4 + 4x^2 + 16) \quad \text{Difference of Squares, } a = x,\ b = 2 \end{aligned}$$

Unfortunately, the remaining factor x^4+4x^2+16 is not a perfect square trinomial - the middle term would have to be $8x^2$ for this to work - so our final answer using this approach is $(x-2)(x+2)(x^4+4x^2+16)$. This isn't as factored as our result from the Difference of Squares approach which was $(x-2)(x+2)(x^2-2x+4)(x^2+2x+4)$. While it is true that $x^4+4x^2+16=(x^2-2x+4)(x^2+2x+4)$, there is no 'intuitive' way to motivate this factorization at this point.[5] The moral of the story? When given the option between using the Difference of Squares and Difference of Cubes, start with the Difference of Squares. Our final answer to this problem is $(x-2)(x+2)(x^2-2x+4)(x^2+2x+4)$. The reader is strongly encouraged to show that this reduces down to x^6-64 after performing all of the multiplication. □

The formulas on page 69, while useful, can only take us so far, so we need to review some more advanced factoring strategies.

Advanced Factoring Formulas

- **'un-F.O.I.L.ing':** Given a trinomial Ax^2+Bx+C, try to reverse the F.O.I.L. process.

  That is, find a, b, c and d such that $Ax^2+Bx+C=(ax+b)(cx+d)$.

  NOTE: This means $ac=A$, $bd=C$ and $B=ad+bc$.

- **Factor by Grouping:** If the expression contains four terms with no common factors among the four terms, try 'factor by grouping':

$$ac+bc+ad+bd=(a+b)c+(a+b)d=(a+b)(c+d)$$

The techniques of 'un-F.O.I.L.ing' and 'factoring by grouping' are difficult to describe in general but should make sense to you with enough practice. Be forewarned - like all 'Rules of Thumb', these strategies work just often enough to be useful, but you can be sure there are exceptions which will defy any advice given here and will require some 'inspiration' to solve.[6] Even though Chapter 3 will give us more powerful factoring methods, we'll find that, in the end, there is no single algorithm for factoring which works for every polynomial. In other words, there will be times when you just have to try something and see what happens.

Example 0.6.2. Factor the following polynomials completely over the integers.[7]

1. x^2-x-6
2. $2t^2-11t+5$
3. $36-11y-12y^2$
4. $18xy^2-54xy-180x$
5. $2t^3-10t^2-3t+15$
6. x^4+4x^2+16

[5]Of course, this begs the question, "How do we know x^2-2x+4 and x^2+2x+4 are irreducible?" (We were told so on page 69, but no reason was given.) Stay tuned! We'll get back to this in due course.

[6]Jeff will be sure to pepper the Exercises with these.

[7]This means that all of the coefficients in the factors will be integers. In a rare departure from form, Carl decided to avoid fractions in this set of examples. Don't get complacent, though, because fractions will return with a vengeance soon enough.

Solution.

1. The G.C.F. of the terms $x^2 - x - 6$ is 1 and $x^2 - x - 6$ isn't a perfect square trinomial (Think about why not.) so we try to reverse the F.O.I.L. process and look for integers a, b, c and d such that $(ax + b)(cx + d) = x^2 - x - 6$. To get started, we note that $ac = 1$. Since a and c are meant to be integers, that leaves us with either a and c both being 1, or a and c both being -1. We'll go with $a = c = 1$, since we can factor[8] the negatives into our choices for b and d. This yields $(x + b)(x + d) = x^2 - x - 6$. Next, we use the fact that $bd = -6$. The product is negative so we know that one of b or d is positive and the other is negative. Since b and d are integers, one of b or d is ± 1 and the other is ∓ 6 OR one of b or d is ± 2 and the other is ∓ 3. After some guessing and checking,[9] we find that $x^2 - x - 6 = (x + 2)(x - 3)$.

2. As with the previous example, we check the G.C.F. of the terms in $2t^2 - 11t + 5$, determine it to be 1 and see that the polynomial doesn't fit the pattern for a perfect square trinomial. We now try to find integers a, b, c and d such that $(at + b)(ct + d) = 2t^2 - 11t + 5$. Since $ac = 2$, we have that one of a or c is 2, and the other is 1. (Once again, we ignore the negative options.) At this stage, there is nothing really distinguishing a from c so we choose $a = 2$ and $c = 1$. Now we look for b and d so that $(2t + b)(t + d) = 2t^2 - 11t + 5$. We know $bd = 5$ so one of b or d is ± 1 and the other ± 5. Given that bd is positive, b and d must have the same sign. The negative middle term $-11t$ guides us to guess $b = -1$ and $d = -5$ so that we get $(2t - 1)(t - 5) = 2t^2 - 11t + 5$. We verify our answer by multiplying.[10]

3. Once again, we check for a nontrivial G.C.F. and see if $36 - 11y - 12y^2$ fits the pattern of a perfect square. Twice disappointed, we rewrite $36 - 11y - 12y^2 = -12y^2 - 11y + 36$ for notational convenience. We now look for integers a, b, c and d such that $-12y^2 - 11y + 36 = (ay + b)(cy + d)$. Since $ac = -12$, we know that one of a or c is ± 1 and the other ± 12 OR one of them is ± 2 and the other is ± 6 OR one of them is ± 3 while the other is ± 4. Since their product is -12, however, we know one of them is positive, while the other is negative. To make matters worse, the constant term 36 has its fair share of factors, too. Our answers for b and d lie among the pairs ± 1 and ± 36, ± 2 and ± 18, ± 4 and ± 9, or ± 6. Since we know one of a or c will be negative, we can simplify our choices for b and d and just look at the positive possibilities. After some guessing and checking,[11] we find $(-3y + 4)(4y + 9) = -12y^2 - 11y + 36$.

4. Since the G.C.F. of the terms in $18xy^2 - 54xy - 180x$ is $18x$, we begin the problem by factoring it out first: $18xy^2 - 54xy - 180x = 18x(y^2 - 3y - 10)$. We now focus our attention on $y^2 - 3y - 10$. We can take a and c to both be 1 which yields $(y + b)(y + d) = y^2 - 3y - 10$. Our choices for b and d are among the factor pairs of -10: ± 1 and ± 10 or ± 2 and ± 5, where

[8] Pun intended!

[9] The authors have seen some strange gimmicks that allegedly help students with this step. We don't like them so we're sticking with good old-fashioned guessing and checking.

[10] That's the 'checking' part of 'guessing and checking'.

[11] Some of these guesses can be more 'educated' than others. Since the middle term is relatively 'small,' we don't expect the 'extreme' factors of 36 and 12 to appear, for instance.

one of b or d is positive and the other is negative. We find $(y-5)(y+2) = y^2 - 3y - 10$. Our final answer is $18xy^2 - 54xy - 180x = 18x(y-5)(y+2)$.

5. Since $2t^3 - 10t^2 - 3t + 15$ has four terms, we are pretty much resigned to factoring by grouping. The strategy here is to factor out the G.C.F. from two *pairs* of terms, and see if this reveals a common factor. If we group the first two terms, we can factor out a $2t^2$ to get $2t^3 - 10t^2 = 2t^2(t-5)$. We now try to factor something out of the last two terms that will leave us with a factor of $(t-5)$. Sure enough, we can factor out a -3 from both: $-3t+15 = -3(t-5)$. Hence, we get

$$2t^3 - 10t^2 - 3t + 15 = 2t^2(t-5) - 3(t-5) = (2t^2 - 3)(t-5)$$

Now the question becomes can we factor $2t^2 - 3$ over the integers? This would require integers a, b, c and d such that $(at+b)(ct+d) = 2t^2 - 3$. Since $ab = 2$ and $cd = -3$, we aren't left with many options - in fact, we really have only four choices: $(2t-1)(t+3)$, $(2t+1)(t-3)$, $(2t-3)(t+1)$ and $(2t+3)(t-1)$. None of these produces $2t^2 - 3$ - which means it's irreducible over the integers - thus our final answer is $(2t^2 - 3)(t-5)$.

6. Our last example, x^4+4x^2+16, is our old friend from Example 0.6.1. As noted there, it is not a perfect square trinomial, so we could try to reverse the F.O.I.L. process. This is complicated by the fact that our highest degree term is x^4, so we would have to look at factorizations of the form $(x+b)(x^3+d)$ as well as $(x^2+b)(x^2+d)$. We leave it to the reader to show that neither of those work. This is an example of where 'trying something' pays off. Even though we've stated that it is not a perfect square trinomial, it's pretty close. Identifying $x^4 = (x^2)^2$ and $16 = 4^2$, we'd have $(x^2+4)^2 = x^4 + 8x^2 + 16$, but instead of $8x^2$ as our middle term, we only have $4x^2$. We could add in the extra $4x^2$ we need, but to keep the balance, we'd have to subtract it off. Doing so produces and unexpected opportunity:

$$\begin{array}{rcll} x^4+4x^2+16 & = & x^4+4x^2+16+(4x^2-4x^2) & \text{Adding and subtracting the same term} \\ & = & x^4+8x^2+16-4x^2 & \text{Rearranging terms} \\ & = & (x^2+4)^2-(2x)^2 & \text{Factoring perfect square trinomial} \\ & = & [(x^2+4)-2x][(x^2+4)+2x] & \text{Difference of Squares: } a=(x^2+4),\ b=2x \\ & = & (x^2-2x+4)(x^2+2x+4) & \text{Rearraging terms} \end{array}$$

We leave it to the reader to check that neither $x^2 - 2x + 4$ nor $x^2 + 2x + 4$ factor over the integers, so we are done. □

0.6.1 Solving Equations by Factoring

Many students wonder why they are forced to learn how to factor. Simply put, factoring is our main tool for solving the non-linear equations which arise in many of the applications of Mathematics.[12] We use factoring in conjunction with the Zero Product Property of Real Numbers which was first stated on page 19 and is given here again for reference.

[12] Also known as 'story problems' or 'real-world examples'.

The Zero Product Property of Real Numbers: If a and b are real numbers with $ab = 0$ then either $a = 0$ or $b = 0$ or both.

For example, consider the equation $6x^2 + 11x = 10$. To see how the Zero Product Property is used to help us solve this equation, we first set the equation equal to zero and then apply the techniques from Example 0.6.2:

$$\begin{array}{rcll} 6x^2+11x & = & 10 & \\ 6x^2+11x-10 & = & 0 & \text{Subtract 10 from both sides} \\ (2x+5)(3x-2) & = & 0 & \text{Factor} \\ 2x+5=0 & \text{or} & 3x-2=0 & \text{Zero Product Property} \\ x=-\frac{5}{2} & \text{or} & x=\frac{2}{3} & a=2x+5,\ b=3x-2 \end{array}$$

The reader should check that both of these solutions satisfy the original equation.

It is critical that you see the importance of setting the expression equal to 0 before factoring. Otherwise, we'd get:

$$\begin{array}{rcll} 6x^2+11x & = & 10 & \\ x(6x+11) & = & 10 & \text{Factor} \end{array}$$

What we **cannot** deduce from this equation is that $x = 10$ or $6x + 11 = 10$ or that $x = 2$ and $6x + 11 = 5$, etc.. (It's wrong and you should feel bad if you do it.) It is precisely because 0 plays such a special role in the arithmetic of real numbers (as the Additive Identity) that we can assume a factor is 0 when the product is 0. No other real number has that ability.

We summarize the **correct** equation solving strategy below.

Strategy for Solving Non-linear Equations

1. Put all of the nonzero terms on one side of the equation so that the other side is 0.
2. Factor.
3. Use the Zero Product Property of Real Numbers and set each factor equal to 0.
4. Solve each of the resulting equations.

Let's finish the section with a collection of examples in which we use this strategy.

Example 0.6.3. Solve the following equations.

1. $3x^2 = 35 - 16x$
2. $t = \dfrac{1+4t^2}{4}$
3. $(y-1)^2 = 2(y-1)$
4. $\dfrac{w^4}{3} = \dfrac{8w^3-12}{12} - \dfrac{w^2-4}{4}$
5. $z(z(18z+9)-50) = 25$
6. $x^4 - 8x^2 - 9 = 0$

Solution.

1. We begin by gathering all of the nonzero terms to one side getting 0 on the other and then we proceed to factor and apply the Zero Product Property.

$$\begin{array}{rclr} 3x^2 & = & 35-16x & \\ 3x^2+16x-35 & = & 0 & \text{Add } 16x\text{, subtract } 35 \\ (3x-5)(x+7) & = & 0 & \text{Factor} \\ 3x-5=0 & \text{or} & x+7=0 & \text{Zero Product Property} \\ x=\frac{5}{3} & \text{or} & x=-7 & \end{array}$$

We check our answers by substituting each of them into the original equation. Plugging in $x=\frac{5}{3}$ yields $\frac{25}{3}$ on both sides while $x=-7$ gives 147 on both sides.

2. To solve $t=\frac{1+4t^2}{4}$, we first clear fractions then move all of the nonzero terms to one side of the equation, factor and apply the Zero Product Property.

$$\begin{array}{rclr} t & = & \dfrac{1+4t^2}{4} & \\ 4t & = & 1+4t^2 & \text{Clear fractions (multiply by 4)} \\ 0 & = & 1+4t^2-4t & \text{Subtract 4} \\ 0 & = & 4t^2-4t+1 & \text{Rearrange terms} \\ 0 & = & (2t-1)^2 & \text{Factor (Perfect Square Trinomial)} \end{array}$$

At this point, we get $(2t-1)^2=(2t-1)(2t-1)=0$, so, the Zero Product Property gives us $2t-1=0$ in both cases.[13] Our final answer is $t=\frac{1}{2}$, which we invite the reader to check.

3. Following the strategy outlined above, the first step to solving $(y-1)^2=2(y-1)$ is to gather the nonzero terms on one side of the equation with 0 on the other side and factor.

$$\begin{array}{rclr} (y-1)^2 & = & 2(y-1) & \\ (y-1)^2-2(y-1) & = & 0 & \text{Subtract } 2(y-1) \\ (y-1)[(y-1)-2] & = & 0 & \text{Factor out G.C.F.} \\ (y-1)(y-3) & = & 0 & \text{Simplify} \\ y-1=0 & \text{or} & y-3=0 & \\ y=1 & \text{or} & y=3 & \end{array}$$

Both of these answers are easily checked by substituting them into the original equation.

An alternative method to solving this equation is to begin by dividing both sides by $(y-1)$ to simplify things outright. As we saw in Example 0.3.1, however, whenever we divide by

[13] More generally, given a positive power p, the only solution to $X^p=0$ is $X=0$.

a variable quantity, we make the explicit assumption that this quantity is nonzero. Thus we must stipulate that $y - 1 \neq 0$.

$$\begin{aligned} \frac{(y-1)^2}{(y-1)} &= \frac{2(y-1)}{(y-1)} && \text{Divide by } (y-1) \text{ - this assumes } (y-1) \neq 0 \\ y-1 &= 2 \\ y &= 3 \end{aligned}$$

Note that in this approach, we obtain the $y = 3$ solution, but we 'lose' the $y = 1$ solution. How did that happen? Assuming $y - 1 \neq 0$ is equivalent to assuming $y \neq 1$. This is an issue because $y = 1$ is a solution to the original equation and it was 'divided out' too early. The moral of the story? If you decide to divide by a variable expression, double check that you aren't excluding any solutions.[14]

4. Proceeding as before, we clear fractions, gather the nonzero terms on one side of the equation, have 0 on the other and factor.

$$\begin{aligned} \frac{w^4}{3} &= \frac{8w^3-12}{12} - \frac{w^2-4}{4} \\ 12\left(\frac{w^4}{3}\right) &= 12\left(\frac{8w^3-12}{12} - \frac{w^2-4}{4}\right) && \text{Multiply by 12} \\ 4w^4 &= (8w^3-12) - 3(w^2-4) && \text{Distribute} \\ 4w^4 &= 8w^3 - 12 - 3w^2 + 12 && \text{Distribute} \\ 0 &= 8w^3 - 12 - 3w^2 + 12 - 4w^4 && \text{Subtract } 4w^4 \\ 0 &= 8w^3 - 3w^2 - 4w^4 && \text{Gather like terms} \\ 0 &= w^2(8w - 3 - 4w^2) && \text{Factor out G.C.F.} \end{aligned}$$

At this point, we apply the Zero Product Property to deduce that $w^2 = 0$ or $8w - 3 - 4w^2 = 0$. From $w^2 = 0$, we get $w = 0$. To solve $8w - 3 - 4w^2 = 0$, we rearrange terms and factor: $-4w^2 + 8w - 3 = (2w - 1)(-2w + 3) = 0$. Applying the Zero Product Property again, we get $2w - 1 = 0$ (which gives $w = \frac{1}{2}$), or $-2w + 3 = 0$ (which gives $w = \frac{3}{2}$). Our final answers are $w = 0$, $w = \frac{1}{2}$ and $w = \frac{3}{2}$. The reader is encouraged to check each of these answers in the original equation. (You need the practice with fractions!)

5. For our next example, we begin by subtracting the 25 from both sides then work out the indicated operations before factoring by grouping.

$$\begin{aligned} z(z(18z+9)-50) &= 25 \\ z(z(18z+9)-50) - 25 &= 0 && \text{Subtract 25} \\ z(18z^2+9z-50) - 25 &= 0 && \text{Distribute} \\ 18z^3 + 9z^2 - 50z - 25 &= 0 && \text{Distribute} \\ 9z^2(2z+1) - 25(2z+1) &= 0 && \text{Factor} \\ (9z^2-25)(2z+1) &= 0 && \text{Factor} \end{aligned}$$

[14] You will see other examples throughout this text where dividing by a variable quantity does more harm than good. Keep this basic one in mind as you move on in your studies - it's a good cautionary tale.

At this point, we use the Zero Product Property and get $9z^2 - 25 = 0$ or $2z + 1 = 0$. The latter gives $z = -\frac{1}{2}$ whereas the former factors as $(3z - 5)(3z + 5) = 0$. Applying the Zero Product Property again gives $3z - 5 = 0$ (so $z = \frac{5}{3}$) or $3z + 5 = 0$ (so $z = -\frac{5}{3}$.) Our final answers are $z = -\frac{1}{2}$, $z = \frac{5}{3}$ and $z = -\frac{5}{3}$, each of which good fun to check.

6. The nonzero terms of the equation $x^4 - 8x^2 - 9 = 0$ are already on one side of the equation so we proceed to factor. This trinomial doesn't fit the pattern of a perfect square so we attempt to reverse the F.O.I.L.ing process. With an x^4 term, we have two possible forms to try: $(ax^2 + b)(cx^2 + d)$ and $(ax^3 + b)(cx + d)$. We leave it to you to show that $(ax^3 + b)(cx + d)$ does not work and we show that $(ax^2 + b)(cx^2 + d)$ does.

   Since the coefficient of x^4 is 1, we take $a = c = 1$. The constant term is -9 so we know b and d have opposite signs and our choices are limited to two options: either b and d come from ± 1 and ± 9 OR one is 3 while the other is -3. After some trial and error, we get $x^4 - 8x^2 - 9 = (x^2 - 9)(x^2 + 1)$. Hence $x^4 - 8x^2 - 9 = 0$ reduces to $(x^2 - 9)(x^2 + 1) = 0$. The Zero Product Property tells us that either $x^2 - 9 = 0$ or $x^2 + 1 = 0$. To solve the former, we factor: $(x - 3)(x + 3) = 0$, so $x - 3 = 0$ (hence, $x = 3$) or $x + 3 = 0$ (hence, $x = -3$). The equation $x^2 + 1 = 0$ has no (real) solution, since for any real number x, x^2 is always 0 or greater. Thus $x^2 + 1$ is always positive. Our final answers are $x = 3$ and $x = -3$. As always, the reader is invited to check both answers in the original equation. □

0.6.2 Exercises

In Exercises 1 - 30, factor completely over the integers. Check your answer by multiplication.

1. $2x - 10x^2$
2. $12t^5 - 8t^3$
3. $16xy^2 - 12x^2y$
4. $5(m+3)^2 - 4(m+3)^3$
5. $(2x-1)(x+3) - 4(2x-1)$
6. $t^2(t-5) + t - 5$
7. $w^2 - 121$
8. $49 - 4t^2$
9. $81t^4 - 16$
10. $9z^2 - 64y^4$
11. $(y+3)^2 - 4y^2$
12. $(x+h)^3 - (x+h)$
13. $y^2 - 24y + 144$
14. $25t^2 + 10t + 1$
15. $12x^3 - 36x^2 + 27x$
16. $m^4 + 10m^2 + 25$
17. $27 - 8x^3$
18. $t^6 + t^3$
19. $x^2 - 5x - 14$
20. $y^2 - 12y + 27$
21. $3t^2 + 16t + 5$
22. $6x^2 - 23x + 20$
23. $35 + 2m - m^2$
24. $7w - 2w^2 - 3$
25. $3m^3 + 9m^2 - 12m$
26. $x^4 + x^2 - 20$
27. $(x^2+1)^2 - 4$
28. $(y^2 - 36)^3$
29. $4(t^2-1)^2 + 3(t^2-1) - 10$
30. $(w^4+3)^2 + 2(w^4+3) + 1$

In Exercises 31 - 36, factor completely over the rationals.

31. $\frac{1}{5}x^2 - x - \frac{14}{5}$
32. $\frac{1}{3}y^2 - 4y + 9$
33. $\frac{3}{4}t^2 + 4t + \frac{5}{4}$
34. $\frac{12}{5}x^2 - \frac{46}{5}x + 8$
35. $\frac{35}{2} + y - \frac{1}{2}y^2$
36. $3t - \frac{6}{7}t^2 - \frac{9}{7}$

In Exercises 37 - 51, find all rational number solutions. Check your answers.

37. $(7x+3)(x-5) = 0$
38. $(2t-1)^2(t+4) = 0$
39. $(y^2+4)(3y^2+y-10) = 0$
40. $4t = t^2$
41. $y + 3 = 2y^2$
42. $26x = 8x^2 + 21$
43. $16x^4 = 9x^2$
44. $w(6w+11) = 10$
45. $2w^2+5w+2 = -3(2w+1)$
46. $x^2(x-3) = 16(x-3)$
47. $(2t+1)^3 = (2t+1)$
48. $a^4 + 4 = 6 - a^2$
49. $\dfrac{8t^2}{3} = 2t + 3$
50. $\dfrac{x^3+x}{2} = \dfrac{x^2+1}{3}$
51. $\dfrac{y^4}{3} - y^2 = \dfrac{3}{2}(y^2+3)$

0.6.3 Answers

1. $2x(1-5x)$
2. $4t^3(3t^2-2)$
3. $4xy(4y-3x)$
4. $-(m+3)^2(4m+7)$
5. $(2x-1)(x-1)$
6. $(t-5)(t^2+1)$
7. $(w-11)(w+11)$
8. $(7-2t)(7+2t)$
9. $(3t-2)(3t+2)(9t^2+4)$
10. $(3z-8y^2)(3z+8y^2)$
11. $-3(y-3)(y+1)$
12. $(x+h)(x+h-1)(x+h+1)$
13. $(y-12)^2$
14. $(5t+1)^2$
15. $3x(2x-3)^2$
16. $(m^2+5)^2$
17. $(3-2x)(9+6x+4x^2)$
18. $t^3(t+1)(t^2-t+1)$
19. $(x-7)(x+2)$
20. $(y-9)(y-3)$
21. $(3t+1)(t+5)$
22. $(2x-5)(3x-4)$
23. $(7-m)(5+m)$
24. $(-2w+1)(w-3)$
25. $3m(m-1)(m+4)$
26. $(x-2)(x+2)(x^2+5)$
27. $(x-1)(x+1)(x^2+3)$
28. $(y-6)^3(y+6)^3$
29. $(2t-3)(2t+3)(t^2+1)$
30. $(w^4+4)^2$
31. $\frac{1}{5}(x-7)(x+2)$
32. $\frac{1}{3}(y-9)(y-3)$
33. $\frac{1}{4}(3t+1)(t+5)$
34. $\frac{2}{5}(2x-5)(3x-4)$
35. $\frac{1}{2}(7-y)(5+y)$
36. $\frac{3}{7}(-2t+1)(t-3)$
37. $x=-\dfrac{3}{7}$ or $x=5$
38. $t=\dfrac{1}{2}$ or $t=-4$
39. $y=\dfrac{5}{3}$ or $y=-2$
40. $t=0$ or $t=4$
41. $y=-1$ or $y=\dfrac{3}{2}$
42. $x=\dfrac{3}{2}$ or $x=\dfrac{7}{4}$
43. $x=0$ or $x=\pm\dfrac{3}{4}$
44. $w=-\dfrac{5}{2}$ or $w=\dfrac{2}{3}$
45. $w=-5$ or $w=-\dfrac{1}{2}$
46. $x=3$ or $x=\pm 4$
47. $t=-1$, $t=-\dfrac{1}{2}$, or $t=0$
48. $a=\pm 1$
49. $t=-\dfrac{3}{4}$ or $t=\dfrac{3}{2}$
50. $x=\dfrac{2}{3}$
51. $y=\pm 3$

0.7 Quadratic Equations

In Section 0.6.1, we reviewed how to solve basic non-linear equations by factoring. The astute reader should have noticed that all of the equations in that section were carefully constructed so that the polynomials could be factored using the integers. To demonstrate just how contrived the equations had to be, we can solve $2x^2 + 5x - 3 = 0$ by factoring, $(2x - 1)(x + 3) = 0$, from which we obtain $x = \frac{1}{2}$ and $x = -3$. If we change the 5 to a 6 and try to solve $2x^2 + 6x - 3 = 0$, however, we find that this polynomial doesn't factor over the integers and we are stuck. It turns out that there are two real number solutions to this equation, but they are *irrational* numbers, and our aim in this section is to review the techniques which allow us to find these solutions.[1] In this section, we focus our attention on **quadratic** equations.

> **Definition 0.15.** An equation is said to be **quadratic** in a variable X if it can be written in the form $AX^2 + BX + C = 0$ where A, B and C are expressions which do not involve X and $A \neq 0$.

Think of quadratic equations as equations that are one degree up from linear equations - instead of the highest power of X being just $X = X^1$, it's X^2. The simplest class of quadratic equations to solve are the ones in which $B = 0$. In that case, we have the following.

> **Solving Quadratic Equations by Extracting Square Roots**
>
> If c is a real number with $c \geq 0$, the solutions to $X^2 = c$ are $X = \pm\sqrt{c}$.
>
> **Note:** If $c < 0$, $X^2 = c$ has no real number solutions.

There are a couple different ways to see why Extracting Square Roots works, both of which are demonstrated by solving the equation $x^2 = 3$. If we follow the procedure outlined in the previous section, we subtract 3 from both sides to get $x^2 - 3 = 0$ and we now try to factor $x^2 - 3$. As mentioned in the remarks following Definition 0.14, we could think of $x^2 - 3 = x^2 - (\sqrt{3})^2$ and apply the Difference of Squares formula to factor $x^2 - 3 = (x - \sqrt{3})(x + \sqrt{3})$. We solve $(x - \sqrt{3})(x + \sqrt{3}) = 0$ by using the Zero Product Property as before by setting each factor equal to zero: $x - \sqrt{3} = 0$ and $x + \sqrt{3} - 0$. We get the answers $x = \pm\sqrt{3}$. In general, if $c \geq 0$, then $\sqrt{c}$ is a real number, so $x^2 - c = x^2 - (\sqrt{c})^2 = (x - \sqrt{c})(x + \sqrt{c})$. Replacing the '3' with 'c' in the above discussion gives the general result.

Another way to view this result is to visualize 'taking the square root' of both sides: since $x^2 = c$, $\sqrt{x^2} = \sqrt{c}$. How do we simplify $\sqrt{x^2}$? We have to exercise a bit of caution here. Note that $\sqrt{(5)^2}$ and $\sqrt{(-5)^2}$ both simplify to $\sqrt{25} = 5$. In both cases, $\sqrt{x^2}$ returned a *positive* number, since the negative in -5 was 'squared away' *before* we took the square root. In other words, $\sqrt{x^2}$ is x if x is positive, or, if x is negative, we make x positive - that is, $\sqrt{x^2} = |x|$, the absolute value of x. So from $x^2 = 3$, we 'take the square root' of both sides of the equation to get $\sqrt{x^2} = \sqrt{3}$. This simplifies to $|x| = \sqrt{3}$, which by Theorem 0.3 is equivalent to $x = \sqrt{3}$ or $x = -\sqrt{3}$. Replacing the '3' in the previous argument with 'c,' gives the general result.

[1] While our discussion in this section departs from factoring, we'll see in Chapter 3 that the same correspondence between factoring and solving equations holds whether or not the polynomial factors over the integers.

As you might expect, Extracting Square Roots can be applied to more complicated equations. Consider the equation below. We can solve it by Extracting Square Roots provided we first isolate the perfect square quantity:

$$
\begin{array}{rcll}
2\left(x+\frac{3}{2}\right)^2-\frac{15}{2} & = & 0 & \\
2\left(x+\frac{3}{2}\right)^2 & = & \frac{15}{2} & \text{Add } \frac{15}{2} \\
\left(x+\frac{3}{2}\right)^2 & = & \frac{15}{4} & \text{Divide by 2} \\
x+\frac{3}{2} & = & \pm\sqrt{\frac{15}{4}} & \text{Extract Square Roots} \\
x+\frac{3}{2} & = & \pm\frac{\sqrt{15}}{2} & \text{Property of Radicals} \\
x & = & -\frac{3}{2}\pm\frac{\sqrt{15}}{2} & \text{Subtract } \frac{3}{2} \\
x & = & -\frac{3\pm\sqrt{15}}{2} & \text{Add fractions}
\end{array}
$$

Let's return to the equation $2x^2+6x-3=0$ from the beginning of the section. We leave it to the reader to show that

$$2\left(x+\frac{3}{2}\right)^2-\frac{15}{2}=2x^2+6x-3.$$

(Hint: Expand the left side.) In other words, we can solve $2x^2+6x-3=0$ by *transforming* into an equivalent equation. This process, you may recall, is called 'Completing the Square.' We'll revisit Completing the Square in Section 2.4 in more generality and for a different purpose but for now we revisit the steps needed to complete the square to solve a quadratic equation.

Solving Quadratic Equations: Completing the Square

To solve a quadratic equation $AX^2+BX+C=0$ by Completing the Square:

1. Subtract the constant C from both sides.
2. Divide both sides by A, the coefficient of X^2. (Remember: $A \neq 0$.)
3. Add $\left(\frac{B}{2A}\right)^2$ to both sides of the equation. (That's half the coefficient of X, squared.)
4. Factor the left hand side of the equation as $\left(X+\frac{B}{2A}\right)^2$.
5. Extract Square Roots.
6. Subtract $\frac{B}{2A}$ from both sides.

To refresh our memories, we apply this method to solve $3x^2 - 24x + 5 = 0$:

$$\begin{array}{rcll} 3x^2 - 24x + 5 & = & 0 & \\ 3x^2 - 24x & = & -5 & \text{Subtract } C = 5 \\ x^2 - 8x & = & -\frac{5}{3} & \text{Divide by } A = 3 \\ x^2 - 8x + 16 & = & -\frac{5}{3} + 16 & \text{Add } \left(\frac{B}{2A}\right)^2 = (-4)^2 = 16 \\ (x-4)^2 & = & \frac{43}{3} & \text{Factor: Perfect Square Trinomial} \\ x - 4 & = & \pm\sqrt{\frac{43}{3}} & \text{Extract Square Roots} \\ x & = & 4 \pm \sqrt{\frac{43}{3}} & \text{Add } 4 \end{array}$$

At this point, we use properties of fractions and radicals to 'rationalize' the denominator:[2]

$$\sqrt{\frac{43}{3}} = \sqrt{\frac{43 \cdot 3}{3 \cdot 3}} = \frac{\sqrt{129}}{\sqrt{9}} = \frac{\sqrt{129}}{3}$$

We can now get a common (integer) denominator which yields:

$$x = 4 \pm \sqrt{\frac{43}{3}} = 4 \pm \frac{\sqrt{129}}{3} = \frac{12 \pm \sqrt{129}}{3}$$

The key to Completing the Square is that the procedure always produces a perfect square trinomial. To see why this works *every single time*, we start with $AX^2 + BX + C = 0$ and follow the procedure:

$$\begin{array}{rcll} AX^2 + BX + C & = & 0 & \\ AX^2 + BX & = & -C & \text{Subtract } C \\ X^2 + \frac{BX}{A} & = & -\frac{C}{A} & \text{Divide by } A \neq 0 \\ X^2 + \frac{BX}{A} + \left(\frac{B}{2A}\right)^2 & = & -\frac{C}{A} + \left(\frac{B}{2A}\right)^2 & \text{Add } \left(\frac{B}{2A}\right)^2 \end{array}$$

(Hold onto the line above for a moment.) Here's the heart of the method - we need to show that

$$X^2 + \frac{BX}{A} + \left(\frac{B}{2A}\right)^2 = \left(X + \frac{B}{2A}\right)^2$$

To show this, we start with the right side of the equation and apply the Perfect Square Formula from Theorem 0.7

$$\left(X + \frac{B}{2A}\right)^2 = X^2 + 2\left(\frac{B}{2A}\right)X + \left(\frac{B}{2A}\right)^2 = X^2 + \frac{BX}{A} + \left(\frac{B}{2A}\right)^2 \checkmark$$

[2]Recall that this means we want to get a denominator with rational (more specifically, integer) numbers.

With just a few more steps we can solve the general equation $AX^2 + BX + C = 0$ so let's pick up the story where we left off. (The line on the previous page we told you to hold on to.)

$$\begin{aligned}
X^2 + \frac{BX}{A} + \left(\frac{B}{2A}\right)^2 &= -\frac{C}{A} + \left(\frac{B}{2A}\right)^2 & & \\
\left(X + \frac{B}{2A}\right)^2 &= -\frac{C}{A} + \frac{B^2}{4A^2} & & \text{Factor: Perfect Square Trinomial} \\
\left(X + \frac{B}{2A}\right)^2 &= -\frac{4AC}{4A^2} + \frac{B^2}{4A^2} & & \text{Get a common denominator} \\
\left(X + \frac{B}{2A}\right)^2 &= \frac{B^2 - 4AC}{4A^2} & & \text{Add fractions} \\
X + \frac{B}{2A} &= \pm\sqrt{\frac{B^2 - 4AC}{4A^2}} & & \text{Extract Square Roots} \\
X + \frac{B}{2A} &= \pm\frac{\sqrt{B^2 - 4AC}}{2A} & & \text{Properties of Radicals} \\
X &= -\frac{B}{2A} \pm \frac{\sqrt{B^2 - 4AC}}{2A} & & \text{Subtract } \frac{B}{2A} \\
X &= \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} & & \text{Add fractions.}
\end{aligned}$$

Lo and behold, we have derived the legendary **Quadratic Formula**!

Theorem 0.9. Quadratic Formula: The solution to $AX^2 + BX + C = 0$ with $A \neq 0$ is:

$$X = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

We can check our earlier solutions to $2x^2 + 6x - 3 = 0$ and $3x^2 - 24x + 5 = 0$ using the Quadratic Formula. For $2x^2 + 6x - 3 = 0$, we identify $A = 2$, $B = 6$ and $C = -3$. The quadratic formula gives:

$$x = \frac{-6 \pm \sqrt{6^2 - 4(2)(-3)}}{2(2)} - \frac{-6 \pm \sqrt{36 + 24}}{4} = \frac{-6 \pm \sqrt{60}}{4}$$

Using properties of radicals ($\sqrt{60} = 2\sqrt{15}$), this reduces to $\frac{2(-3\pm\sqrt{15})}{4} = \frac{-3\pm\sqrt{15}}{2}$. We leave it to the reader to show these two answers are the same as $-\frac{3\pm\sqrt{15}}{2}$, as required.[3]

For $3x^2 - 24x + 5 = 0$, we identify $A = 3$, $B = -24$ and $C = 5$. Here, we get:

$$x = \frac{-(-24) \pm \sqrt{(-24)^2 - 4(3)(5)}}{2(3)} = \frac{24 \pm \sqrt{516}}{6}$$

Since $\sqrt{516} = 2\sqrt{129}$, this reduces to $x = \frac{12\pm\sqrt{129}}{3}$.

[3] Think about what $-(3 \pm \sqrt{15})$ is really telling you.

It is worth noting that the Quadratic Formula applies to all quadratic equations - even ones we could solve using other techniques. For example, to solve $2x^2+5x-3=0$ we identify $A=2$, $B=5$ and $C=-3$. This yields:

$$x = \frac{-5 \pm \sqrt{5^2-4(2)(-3)}}{2(2)} = \frac{-5 \pm \sqrt{49}}{4} = \frac{-5 \pm 7}{4}$$

At this point, we have $x = \frac{-5+7}{4} = \frac{1}{2}$ and $x = \frac{-5-7}{4} = \frac{-12}{4} = -3$ - the same two answers we obtained factoring. We can also use it to solve $x^2 = 3$, if we wanted to. From $x^2 - 3 = 0$, we have $A = 1$, $B = 0$ and $C = -3$. The Quadratic Formula produces

$$x = \frac{-0 \pm \sqrt{0^2-4(1)(3)}}{2(1)} = \frac{\pm\sqrt{12}}{2} = \pm\frac{2\sqrt{3}}{2} = \pm\sqrt{3}$$

As this last example illustrates, while the Quadratic Formula *can* be used to solve every quadratic equation, that doesn't mean it *should* be used. Many times other methods are more efficient. We now provide a more comprehensive approach to solving Quadratic Equations.

Strategies for Solving Quadratic Equations

- If the variable appears in the squared term only, isolate it and Extract Square Roots.
- Otherwise, put the nonzero terms on one side of the equation so that the other side is 0.
  - Try factoring.
  - If the expression doesn't factor easily, use the Quadratic Formula.

The reader is encouraged to pause for a moment to think about why 'Completing the Square' doesn't appear in our list of strategies despite the fact that we've spent the majority of the section so far talking about it.[4] Let's get some practice solving quadratic equations, shall we?

Example 0.7.1. Find all real number solutions to the following equations.

1. $3-(2w-1)^2 = 0$
2. $5x - x(x-3) = 7$
3. $(y-1)^2 = 2 - \frac{y+2}{3}$
4. $5(25-21x) = \frac{59}{4} - 25x^2$
5. $-4.9t^2 + 10t\sqrt{3} + 2 = 0$
6. $2x^2 = 3x^4 - 6$

Solution.

1. Since $3-(2w-1)^2 = 0$ contains a perfect square, we isolate it first then extract square roots:

$$\begin{aligned}
3-(2w-1)^2 &= 0 & \\
3 &= (2w-1)^2 & \text{Add } (2w-1)^2 \\
\pm\sqrt{3} &= 2w-1 & \text{Extract Square Roots} \\
1\pm\sqrt{3} &= 2w & \text{Add } 1 \\
\frac{1\pm\sqrt{3}}{2} &= w & \text{Divide by } 2
\end{aligned}$$

[4] Unacceptable answers include "Jeff and Carl are mean" and "It was one of Carl's Pedantic Rants".

We find our two answers $w = \frac{1 \pm \sqrt{3}}{2}$. The reader is encouraged to check both answers by substituting each into the original equation.[5]

2. To solve $5x - x(x-3) = 7$, we begin performing the indicated operations and getting one side equal to 0.

$$\begin{array}{rclr} 5x - x(x-3) & = & 7 & \\ 5x - x^2 + 3x & = & 7 & \text{Distribute} \\ -x^2 + 8x & = & 7 & \text{Gather like terms} \\ -x^2 + 8x - 7 & = & 0 & \text{Subtract } 7 \end{array}$$

At this point, we attempt to factor and find $-x^2 + 8x - 7 = (x-1)(-x+7)$. Using the Zero Product Property, we get $x - 1 = 0$ or $-x + 7 = 0$. Our answers are $x = 1$ or $x = 7$, both of which are easy to check.

3. Even though we have a perfect square in $(y-1)^2 = 2 - \frac{y+2}{3}$, Extracting Square Roots won't help matters since we have a y on the other side of the equation. Our strategy here is to perform the indicated operations (and clear the fraction for good measure) and get 0 on one side of the equation.

$$\begin{array}{rclr} (y-1)^2 & = & 2 - \dfrac{y+2}{3} & \\ y^2 - 2y + 1 & = & 2 - \dfrac{y+2}{3} & \text{Perfect Square Trinomial} \\ 3(y^2 - 2y + 1) & = & 3\left(2 - \dfrac{y+2}{3}\right) & \text{Multiply by } 3 \\ 3y^2 - 6y + 3 & = & 6 - 3\left(\dfrac{y+2}{3}\right) & \text{Distribute} \\ 3y^2 - 6y + 3 & = & 6 - (y+2) & \\ 3y^2 - 6y + 3 - 6 + (y+2) & = & 0 & \text{Subtract 6, Add } (y+2) \\ 3y^2 - 5y - 1 & = & 0 & \end{array}$$

A cursory attempt at factoring bears no fruit, so we run this through the Quadratic Formula with $A = 3$, $B = -5$ and $C = -1$.

$$\begin{array}{rcl} y & = & \dfrac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(-1)}}{2(3)} \\ y & = & \dfrac{5 \pm \sqrt{25 + 12}}{6} \\ y & = & \dfrac{5 \pm \sqrt{37}}{6} \end{array}$$

Since 37 is prime, we have no way to reduce $\sqrt{37}$. Thus, our final answers are $y = \frac{5 \pm \sqrt{37}}{6}$. The reader is encouraged to supply the details of the challenging verification of the answers.

[5]It's excellent practice working with radicals fractions so we really, *really* want you to take the time to do it.

4. We proceed as before; our aim is to gather the nonzero terms on one side of the equation.

$$\begin{array}{rcll}
5(25-21x) & = & \dfrac{59}{4}-25x^2 & \\
125-105x & = & \dfrac{59}{4}-25x^2 & \text{Distribute} \\
4(125-105x) & = & 4\left(\dfrac{59}{4}-25x^2\right) & \text{Multiply by 4} \\
500-420x & = & 59-100x^2 & \text{Distribute} \\
500-420x-59+100x^2 & = & 0 & \text{Subtract 59, Add } 100x^2 \\
100x^2-420x+441 & = & 0 & \text{Gather like terms}
\end{array}$$

With highly composite numbers like 100 and 441, factoring seems inefficient at best,[6] so we apply the Quadratic Formula with $A = 100$, $B = -420$ and $C = 441$:

$$\begin{array}{rcl}
x & = & \dfrac{-(-420)\pm\sqrt{(-420)^2-4(100)(441)}}{2(100)} \\
 & = & \dfrac{420\pm\sqrt{176000-176400}}{200} \\
 & = & \dfrac{420\pm\sqrt{0}}{200} \\
 & = & \dfrac{420\pm 0}{200} \\
 & = & \dfrac{420}{200} \\
 & = & \dfrac{21}{10}
\end{array}$$

To our surprise and delight we obtain just one answer, $x = \frac{21}{10}$.

5. Our next equation $-4.9t^2 + 10t\sqrt{3} + 2 = 0$, already has 0 on one side of the equation, but with coefficients like -4.9 and $10\sqrt{3}$, factoring with integers is not an option. We could make things a *bit* easier on the eyes by clearing the decimal (by multiplying through by 10) to get $-49t^2+100t\sqrt{3}+20 = 0$ but we simply cannot rid ourselves of the irrational number $\sqrt{3}$. The Quadratic Formula is our only recourse. With $A = -49$, $B = 100\sqrt{3}$ and $C = 20$ we get:

[6]This is actually the Perfect Square Trinomial $(10x-21)^2$.

$$\begin{aligned}
t &= \frac{-100\sqrt{3} \pm \sqrt{(100\sqrt{3})^2 - 4(-49)(20)}}{2(-49)} \\
&= \frac{-100\sqrt{3} \pm \sqrt{30000 + 3920}}{-98} \\
&= \frac{-100\sqrt{3} \pm \sqrt{33920}}{-98} \\
&= \frac{-100\sqrt{3} \pm 8\sqrt{530}}{-98} \\
&= \frac{2(-50\sqrt{3} \pm 4\sqrt{530})}{2(-49)} \\
&= \frac{-50\sqrt{3} \pm 4\sqrt{530}}{-49} && \text{Reduce} \\
&= \frac{-(-50\sqrt{3} \pm 4\sqrt{530})}{49} && \text{Properties of Negatives} \\
&= \frac{50\sqrt{3} \mp 4\sqrt{530}}{49} && \text{Distribute}
\end{aligned}$$

You'll note that when we 'distributed' the negative in the last step, we changed the '$\pm$' to a '$\mp$.' While this is technically correct, at the end of the day both symbols mean 'plus or minus',[7] so we can write our answers as $t = \frac{50\sqrt{3} \pm 4\sqrt{530}}{49}$. Checking these answers are a true test of arithmetic mettle.

6. At first glance, the equation $2x^2 = 3x^4 - 6$ seems misplaced. The highest power of the variable x here is 4, not 2, so this equation isn't a quadratic equation - at least not in terms of the variable x. It is, however, an example of an equation that is quadratic 'in disguise.'[8] We introduce a new variable u to help us see the pattern - specifically we let $u = x^2$. Thus $u^2 = (x^2)^2 = x^4$. So in terms of the variable u, the equation $2x^2 = 3x^4 - 6$ is $2u = 3u^2 - 6$. The latter is a quadratic equation, which we can solve using the usual techniques:

$$\begin{aligned}
2u &= 3u^2 - 6 \\
0 &= 3u^2 - 2u - 6 && \text{Subtract } 2u
\end{aligned}$$

After a few attempts at factoring, we resort to the Quadratic Formula with $A = 3$, $B = -2$,

[7]There are instances where we need both symbols, however. For example, the Sum and Difference of Cubes Formulas (page 69) can be written as a single formula: $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$. In this case, all of the 'top' symbols are read to give the sum formula; the 'bottom' symbols give the difference formula.

[8]More formally, **quadratic in form.** Carl likes 'Quadratics in Disguise' since it reminds him of the tagline of one of his beloved childhood cartoons and toy lines.

$C = -6$ and get:

$$\begin{aligned} u &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-6)}}{2(3)} \\ &= \frac{2 \pm \sqrt{4+72}}{6} \\ &= \frac{2 \pm \sqrt{76}}{6} \\ &= \frac{2 \pm \sqrt{4 \cdot 19}}{6} \\ &= \frac{2 \pm 2\sqrt{19}}{6} && \text{Properties of Radicals} \\ &= \frac{2(1 \pm \sqrt{19})}{2(3)} && \text{Factor} \\ &= \frac{1 \pm \sqrt{19}}{3} && \text{Reduce} \end{aligned}$$

We've solved the equation for u, but what we still need to solve the original equation[9] - which means we need to find the corresponding values of x. Since $u = x^2$, we have two equations:

$$x^2 = \frac{1+\sqrt{19}}{3} \quad \text{or} \quad x^2 = \frac{1-\sqrt{19}}{3}$$

We can solve the first equation by extracting square roots to get $x = \pm\sqrt{\frac{1+\sqrt{19}}{3}}$. The second equation, however, has no real number solutions because $\frac{1-\sqrt{19}}{3}$ is a negative number. For our final answers we can rationalize the denominator[10] to get:

$$x = \pm\sqrt{\frac{1+\sqrt{19}}{3}} = \pm\sqrt{\frac{1+\sqrt{19}}{3} \cdot \frac{3}{3}} = \pm\frac{\sqrt{3+3\sqrt{19}}}{3}$$

As with the previous exercise, the very challenging check is left to the reader. □

Our last example above, the 'Quadratic in Disguise', hints that the Quadratic Formula is applicable to a wider class of equations than those which are strictly quadratic. We give some general guidelines to recognizing these beasts in the wild on the next page.

[9] Or, you've solved the equation for 'you' (u), now you have to solve it for your instructor (x).
[10] We'll say more about this technique in Section 0.9.

Identifying Quadratics in Disguise

An equation is a 'Quadratic in Disguise' if it can be written in the form: $AX^{2m} + BX^m + C = 0$.
In other words:

- There are exactly three terms, two with variables and one constant term.
- The exponent on the variable in one term is *exactly twice* the variable on the other term.

To transform a Quadratic in Disguise to a quadratic equation, let $u = X^m$ so $u^2 = (X^m)^2 = X^{2m}$. This transforms the equation into $Au^2 + Bu + C = 0$.

For example, $3x^6 - 2x^3 + 1 = 0$ is a Quadratic in Disguise, since $6 = 2 \cdot 3$. If we let $u = x^3$, we get $u^2 = (x^3)^2 = x^6$, so the equation becomes $3u^2 - 2u + 1 = 0$. However, $3x^6 - 2x^2 + 1 = 0$ is *not* a Quadratic in Disguise, since $6 \neq 2 \cdot 2$. The substitution $u = x^2$ yields $u^2 = (x^2)^2 = x^4$, not x^6 as required. We'll see more instances of 'Quadratics in Disguise' in later sections.

We close this section with a review of the **discriminant** of a quadratic equation as defined below.

Definition 0.16. The Discriminant: Given a quadratic equation $AX^2 + BX + C = 0$, the quantity $B^2 - 4AC$ is called the **discriminant** of the equation.

The discriminant is the radicand of the square root in the quadratic formula:

$$X = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

It *discriminates* between the nature and number of solutions we get from a quadratic equation. The results are summarized below.

Theorem 0.10. Discriminant Theorem: Given a Quadratic Equation $AX^2 + BX + C = 0$, let $D = B^2 - 4AC$ be the discriminant.

- If $D > 0$, there are two distinct real number solutions to the equation.
- If $D = 0$, there is one repeated real number solution.

  Note: 'Repeated' here comes from the fact that 'both' solutions $\frac{-B \pm 0}{2A}$ reduce to $-\frac{B}{2A}$.
- If $D < 0$, there are no real solutions.

For example, $x^2 + x - 1 = 0$ has two real number solutions since the discriminant works out to be $(1)^2 - 4(1)(-1) = 5 > 0$. This results in a $\pm\sqrt{5}$ in the Quadratic Formula, generating two different answers. On the other hand, $x^2 + x + 1 = 0$ has no real solutions since here, the discriminant is $(1)^2 - 4(1)(1) = -3 < 0$ which generates a $\pm\sqrt{-3}$ in the Quadratic Formula. The equation $x^2 + 2x + 1 = 0$ has discriminant $(2)^2 - 4(1)(1) = 0$ so in the Quadratic Formula we get a $\pm\sqrt{0} = 0$ thereby generating just one solution. More can be said as well. For example, the discriminant of $6x^2 - x - 40 = 0$ is 961. This is a perfect square, $\sqrt{961} = 31$, which means our solutions are rational numbers. When our solutions are rational numbers, the quadratic actually factors nicely. In our example $6x^2 - x - 40 = (2x + 5)(3x - 8)$. Admittedly, if you've already computed the discriminant, you're most of the way done with the problem and probably wouldn't take the time to experiment with factoring the quadratic at this point – but we'll see another use for this analysis of the discriminant in the next section.[11]

[11] Specifically in Example 0.8.1.

0.7.1 Exercises

In Exercises 1 - 33, find all real solutions. Check your answers, as directed by your instructor.

1. $x^2 - 44 = 0$
2. $5x^2 = 32$
3. $(x-1)^2 + 49 = 0$
4. $3\left(x - \dfrac{1}{2}\right)^2 = \dfrac{5}{12}$
5. $4 - (5t+3)^2 = 3$
6. $3(y^2-3)^2 - 2 = 10$
7. $x^2 + x - 1 = 0$
8. $3w^2 = 2 - w$
9. $y(y+4) = 1$
10. $x^2 - 6x + 9 = 16$
11. $4w^2 - 4\sqrt{3}\,w + 3 = 0$
12. $9y^2 + 6\sqrt{2}\,y + 2 = 0$
13. $x^2 + \sqrt{2}\,x - \frac{3}{2} = 0$
14. $x^2 + 7x = 0$
15. $x^2 - 6x + 4 = 0$
16. $2x^2 - 5x - 3 = 0$
17. $\frac{2}{5}x^2 - \frac{4}{5}x - \frac{6}{5} = 0$
18. $3x^2 - 4x - 4 = 0$
19. $\dfrac{z}{2} = 4z^2 - 1$
20. $0.1v^2 + 0.2v = 0.3$
21. $x^2 = x - 1$
22. $3 - t = 2(t+1)^2$
23. $(x-3)^2 = x^2 + 9$
24. $(3y-1)(2y+1) = 5y$
25. $w^4 + 3w^2 - 1 = 0$
26. $2x^4 + x^2 = 3$
27. $(2-y)^4 = 3(2-y)^2 + 1$
28. $3x^4 + 6x^2 = 15x^3$
29. $6p + 2 = p^2 + 3p^3$
30. $10v = 7v^3 - v^5$
31. $y^2 - \sqrt{8}\,y = \sqrt{18}\,y - 1$
32. $x^2\sqrt{3} = x\sqrt{6} + \sqrt{12}$
33. $\dfrac{v^2}{3} = \dfrac{v\sqrt{3}}{2} + 1$

In Exercises 34 - 42, complete the square and write each expression in the form $a(x-p)^2 + q$.

34. $x^2 - 10x$
35. $x^2 + x$
36. $x^2 - \frac{1}{3}x + \frac{1}{6}$
37. $x^2 + 6x + 6$
38. $x^2 - x - 1$
39. $2x^2 + 6x + 5$
40. $3x^2 + 12x + 1$
41. $6x^2 - 3x + 1$
42. $\frac{1}{3}(x^2 - 2x - 5)$

43. Prove that for every nonzero number p, $x^2 + xp + p^2 = 0$ has no real solutions.
44. Solve for t: $-\dfrac{1}{2}gt^2 + vt + h = 0$. Assume $g > 0$, $v \geq 0$ and $h \geq 0$.

0.7.2 Answers

1. $x = \pm 2\sqrt{11}$
2. $x = \pm\dfrac{4\sqrt{10}}{5}$
3. No real solution.
4. $x = \dfrac{3 \pm \sqrt{5}}{6}$
5. $t = -\dfrac{4}{5}, -\dfrac{2}{5}$
6. $y = \pm 1, \pm\sqrt{5}$
7. $x = \dfrac{-1 \pm \sqrt{5}}{2}$
8. $w = -1, \dfrac{2}{3}$
9. $y = -2 \pm \sqrt{5}$
10. $x = -1, 7$
11. $w = \dfrac{\sqrt{3}}{2}$
12. $y = \dfrac{-\sqrt{2}}{3}$
13. $x = \dfrac{-3\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}$
14. $x = -7, 0$
15. $x = 3 \pm \sqrt{5}$
16. $x = -\dfrac{1}{2}, 3$
17. $x = -1, 3$
18. $x = -\dfrac{2}{3}, 2$
19. $z = \dfrac{1 \pm \sqrt{65}}{16}$
20. $v = -3, 1$
21. No real solution.
22. $t = \dfrac{-5 \pm \sqrt{33}}{4}$
23. $x = 0$
24. $y = \dfrac{2 \pm \sqrt{10}}{6}$
25. $w = \pm\sqrt{\dfrac{\sqrt{13} - 3}{2}}$
26. $x = \pm 1$
27. $y = \dfrac{4 \pm \sqrt{6 + 2\sqrt{13}}}{2}$
28. $x = 0, \dfrac{5 \pm \sqrt{17}}{2}$
29. $p = -\dfrac{1}{3}, \pm\sqrt{2}$
30. $v = 0, \pm\sqrt{2}, \pm\sqrt{5}$
31. $y = \dfrac{5\sqrt{2} \pm \sqrt{46}}{2}$
32. $x = \dfrac{\sqrt{2} \pm \sqrt{10}}{2}$
33. $v = -\dfrac{\sqrt{3}}{2}, 2\sqrt{3}$
34. $(x - 5)^2 - 25$
35. $(x + \frac{1}{2})^2 - \frac{1}{4}$
36. $(x - \frac{1}{6})^2 + \frac{5}{36}$
37. $(x + 3)^2 - 3$
38. $(x - \frac{1}{2})^2 - \frac{5}{4}$
39. $2(x + \frac{3}{2})^2 + \frac{1}{2}$
40. $3(x + 2)^2 - 11$
41. $6(x - \frac{1}{4})^2 + \frac{5}{8}$
42. $\frac{1}{3}(x - 1)^2 - 2$
43. The discriminant is: $D = p^2 - 4p^2 = -3p^2 < 0$. Since $D < 0$, there are no real solutions.
44. $t = \dfrac{v \pm \sqrt{v^2 + 2gh}}{g}$

0.8 Rational Expressions and Equations

We now turn our attention to rational expressions - that is, algebraic fractions - and equations which contain them. The reader is encouraged to keep in mind the properties of fractions listed on page 20 because we will need them along the way. Before we launch into reviewing the basic arithmetic operations of rational expressions, we take a moment to review how to simplify them properly. As with numeric fractions, we 'cancel common *factors*,' not common *terms*. That is, in order to simplify rational expressions, we first *factor* the numerator and denominator. For example:

$$\frac{x^4+5x^3}{x^3-25x} \neq \frac{x^4+5\cancel{x^3}}{\cancel{x^3}-25x}$$

but, rather

$$\begin{aligned}
\frac{x^4+5x^3}{x^3-25x} &= \frac{x^3(x+5)}{x(x^2-25)} && \text{Factor G.C.F.}\\
&= \frac{x^3(x+5)}{x(x-5)(x+5)} && \text{Difference of Squares}\\
&= \frac{\cancelto{x^2}{x^3}\cancel{(x+5)}}{\cancel{x}(x-5)\cancel{(x+5)}} && \text{Cancel common factors}\\
&= \frac{x^2}{x-5}
\end{aligned}$$

This equivalence holds provided the factors being canceled aren't 0. Since a factor of x and a factor of $x+5$ were canceled, $x \neq 0$ and $x+5 \neq 0$, so $x \neq -5$. We usually stipulate this as:

$$\frac{x^4+5x^3}{x^3-25x} = \frac{x^2}{x-5}, \qquad \text{provided } x \neq 0,\ x \neq -5$$

While we're talking about common mistakes, please notice that

$$\frac{5}{x^2+9} \neq \frac{5}{x^2}+\frac{5}{9}$$

Just like their numeric counterparts, you don't add algebraic fractions by *adding denominators* of fractions with *common numerators* - it's the other way around:[1]

$$\frac{x^2+9}{5} = \frac{x^2}{5}+\frac{9}{5}$$

It's time to review the basic arithmetic operations with rational expressions.

[1] One of the most common errors students make on college Mathematics placement tests is that they forget how to add algebraic fractions correctly. This places many students into remedial classes even though they are probably ready for college-level Math. We urge you to really study this section with great care so that you don't fall into that trap.

Example 0.8.1. Perform the indicated operations and simplify.

1. $\dfrac{2x^2-5x-3}{x^4-4} \div \dfrac{x^2-2x-3}{x^5+2x^3}$
2. $\dfrac{5}{w^2-9} - \dfrac{w+2}{w^2-9}$
3. $\dfrac{3}{y^2-8y+16} + \dfrac{y+1}{16y-y^3}$
4. $\dfrac{\dfrac{2}{4-(x+h)} - \dfrac{2}{4-x}}{h}$
5. $2t^{-3}-(3t)^{-2}$
6. $10x(x-3)^{-1}+5x^2(-1)(x-3)^{-2}$

Solution.

1. As with numeric fractions, we divide rational expressions by 'inverting and multiplying'. Before we get too carried away however, we factor to see what, if any, factors cancel.

$$\begin{aligned}
\frac{2x^2-5x-3}{x^4-4} \div \frac{x^2-2x-3}{x^5+2x^3} &= \frac{2x^2-5x-3}{x^4-4} \cdot \frac{x^5+2x^3}{x^2-2x-3} && \text{Invert and multiply} \\
&= \frac{(2x^2-5x-3)(x^5+2x^3)}{(x^4-4)(x^2-2x-3)} && \text{Multiply fractions} \\
&= \frac{(2x+1)(x-3)x^3(x^2+2)}{(x^2-2)(x^2+2)(x-3)(x+1)} && \text{Factor} \\
&= \frac{(2x+1)\cancel{(x-3)}x^3\cancel{(x^2+2)}}{(x^2-2)\cancel{(x^2+2)}\cancel{(x-3)}(x+1)} && \text{Cancel common factors} \\
&= \frac{x^3(2x+1)}{(x+1)(x^2-2)} && \text{Provided } x \neq 3
\end{aligned}$$

The '$x \neq 3$' is mentioned since a factor of $(x-3)$ was canceled as we reduced the expression. We also canceled a factor of (x^2+2). Why is there no stipulation as a result of canceling this factor? Because $x^2+2 \neq 0$. (Can you see why?) At this point, we *could* go ahead and multiply out the numerator and denominator to get

$$\frac{x^3(2x+1)}{(x+1)(x^2-2)} = \frac{2x^4+x^3}{x^3+x^2-2x-2}$$

but for most of the applications where this kind of algebra is needed (solving equations, for instance), it is best to leave things factored. Your instructor will let you know whether to leave your answer in factored form or not.[2]

[2] Speaking of factoring, do you remember why x^2-2 can't be factored over the integers?

2. As with numeric fractions we need common denominators in order to subtract. This is the case here so we proceed by subtracting the numerators.

$$\begin{aligned} \frac{5}{w^2-9} - \frac{w+2}{w^2-9} &= \frac{5-(w+2)}{w^2-9} && \text{Subtract fractions} \\ &= \frac{5-w-2}{w^2-9} && \text{Distribute} \\ &= \frac{3-w}{w^2-9} && \text{Combine like terms} \end{aligned}$$

At this point, we need to see if we can reduce this expression so we proceed to factor. It first appears as if we have no common factors among the numerator and denominator until we recall the property of 'factoring negatives' from Page 20: $3-w = -(w-3)$. This yields:

$$\begin{aligned} \frac{3-w}{w^2-9} &= \frac{-(w-3)}{(w-3)(w+3)} && \text{Factor} \\ &= \frac{-\cancel{(w-3)}}{\cancel{(w-3)}(w+3)} && \text{Cancel common factors} \\ &= \frac{-1}{w+3} && \text{Provided } w \neq 3 \end{aligned}$$

The stipulation $w \neq 3$ comes from the cancellation of the $(w-3)$ factor.

3. In this next example, we are asked to add two rational expressions with *different* denominators. As with numeric fractions, we must first find a *common denominator*. To do so, we start by factoring each of the denominators.

$$\begin{aligned} \frac{3}{y^2-8y+16} + \frac{y+1}{16y-y^3} &= \frac{3}{(y-4)^2} + \frac{y+1}{y(16-y^2)} && \text{Factor} \\ &= \frac{3}{(y-4)^2} + \frac{y+1}{y(4-y)(4+y)} && \text{Factor some more} \end{aligned}$$

To find the common denominator, we examine the factors in the first denominator and note that we need a factor of $(y-4)^2$. We now look at the second denominator to see what other factors we need. We need a factor of y and $(4+y) = (y+4)$. What about $(4-y)$? As mentioned in the last example, we can factor this as: $(4-y) = -(y-4)$. Using properties of negatives, we 'migrate' this negative out to the front of the fraction, turning the addition into subtraction. We find the (least) common denominator to be $(y-4)^2 y(y+4)$. We can now proceed to multiply the numerator and denominator of each fraction by whatever factors each is missing from their respective denominators to produce equivalent expressions with common denominators.

$$\begin{aligned} \frac{3}{(y-4)^2}+\frac{y+1}{y(4-y)(4+y)} &= \frac{3}{(y-4)^2}+\frac{y+1}{y(-(y-4))(y+4)} \\ &= \frac{3}{(y-4)^2}-\frac{y+1}{y(y-4)(y+4)} \\ &= \frac{3}{(y-4)^2}\cdot\frac{y(y+4)}{y(y+4)}-\frac{y+1}{y(y-4)(y+4)}\cdot\frac{(y-4)}{(y-4)} && \text{Equivalent Fractions} \\ &= \frac{3y(y+4)}{(y-4)^2y(y+4)}-\frac{(y+1)(y-4)}{y(y-4)^2(y+4)} && \text{Multiply Fractions} \end{aligned}$$

At this stage, we can subtract numerators and simplify. We'll keep the denominator factored (in case we can reduce down later), but in the numerator, since there are no common factors, we proceed to perform the indicated multiplication and combine like terms.

$$\begin{aligned} \frac{3y(y+4)}{(y-4)^2y(y+4)}-\frac{(y+1)(y-4)}{y(y-4)^2(y+4)} &= \frac{3y(y+4)-(y+1)(y-4)}{(y-4)^2y(y+4)} && \text{Subtract numerators} \\ &= \frac{3y^2+12y-(y^2-3y-4)}{(y-4)^2y(y+4)} && \text{Distribute} \\ &= \frac{3y^2+12y-y^2+3y+4}{(y-4)^2y(y+4)} && \text{Distribute} \\ &= \frac{2y^2+15y+4}{y(y+4)(y-4)^2} && \text{Gather like terms} \end{aligned}$$

We would like to factor the numerator and cancel factors it has in common with the denominator. After a few attempts, it appears as if the numerator doesn't factor, at least over the integers. As a check, we compute the discriminant of $2y^2+15y+4$ and get $15^2-4(2)(4)=193$. This isn't a perfect square so we know that the quadratic equation $2y^2+15y+4=0$ has irrational solutions. This means $2y^2+15y+4$ can't factor over the integers[3] so we are done.

4. In this example, we have a compound fraction, and we proceed to simplify it as we did its numeric counterparts in Example 0.2.1. Specifically, we start by multiplying the numerator and denominator of the 'big' fraction by the least common denominator of the 'little' fractions inside of it - in this case we need to use $(4-(x+h))(4-x)$ - to remove the compound nature of the 'big' fraction. Once we have a more normal looking fraction, we can proceed as we

[3] See the remarks following Theorem 0.10.

have in the previous examples.

$$\begin{aligned}
\frac{\frac{2}{4-(x+h)} - \frac{2}{4-x}}{h} &= \frac{\left(\frac{2}{4-(x+h)} - \frac{2}{4-x}\right)}{h} \cdot \frac{(4-(x+h))(4-x)}{(4-(x+h))(4-x)} && \text{Equivalent fractions} \\
&= \frac{\left(\frac{2}{4-(x+h)} - \frac{2}{4-x}\right) \cdot (4-(x+h))(4-x)}{h(4-(x+h))(4-x)} && \text{Multiply} \\
&= \frac{\frac{2(4-(x+h))(4-x)}{4-(x+h)} - \frac{2(4-(x+h))(4-x)}{4-x}}{h(4-(x+h))(4-x)} && \text{Distribute} \\
&= \frac{\frac{2\cancel{(4-(x+h))}(4-x)}{\cancel{(4-(x+h))}} - \frac{2(4-(x+h))\cancel{(4-x)}}{\cancel{(4-x)}}}{h(4-(x+h))(4-x)} && \text{Reduce} \\
&= \frac{2(4-x) - 2(4-(x+h))}{h(4-(x+h))(4-x)}
\end{aligned}$$

Now we can clean up and factor the numerator to see if anything cancels. (This why we kept the denominator factored.)

$$\begin{aligned}
\frac{2(4-x) - 2(4-(x+h))}{h(4-(x+h))(4-x)} &= \frac{2[(4-x)-(4-(x+h))]}{h(4-(x+h))(4-x)} && \text{Factor out G.C.F.} \\
&= \frac{2[4-x-4+(x+h)]}{h(4-(x+h))(4-x)} && \text{Distribute} \\
&= \frac{2[4-4-x+x+h]}{h(4-(x+h))(4-x)} && \text{Rearrange terms} \\
&= \frac{2h}{h(4-(x+h))(4-x)} && \text{Gather like terms} \\
&= \frac{2\cancel{h}}{\cancel{h}(4-(x+h))(4-x)} && \text{Reduce} \\
&= \frac{2}{(4-(x+h))(4-x)} && \text{Provided } h \neq 0
\end{aligned}$$

Your instructor will let you know if you are to multiply out the denominator or not.[4]

5. At first glance, it doesn't seem as if there is anything that can be done with $2t^{-3} - (3t)^{-2}$ because the exponents on the variables are different. However, since the exponents are

[4]We'll keep it factored because in Calculus it needs to be factored.

negative, these are actually rational expressions. In the first term, the -3 exponent applies to the t *only* but in the second term, the exponent -2 applies to *both* the 3 and the t, as indicated by the parentheses. One way to proceed is as follows:

$$\begin{aligned} 2t^{-3} - (3t)^{-2} &= \frac{2}{t^3} - \frac{1}{(3t)^2} \\ &= \frac{2}{t^3} - \frac{1}{9t^2} \end{aligned}$$

We see that we are being asked to subtract two rational expressions with different denominators, so we need to find a common denominator. The first fraction contributes a t^3 to the denominator, while the second contributes a factor of 9. Thus our common denominator is $9t^3$, so we are missing a factor of '9' in the first denominator and a factor of 't' in the second.

$$\begin{aligned} \frac{2}{t^3} - \frac{1}{9t^2} &= \frac{2}{t^3} \cdot \frac{9}{9} - \frac{1}{9t^2} \cdot \frac{t}{t} && \text{Equivalent Fractions} \\ &= \frac{18}{9t^3} - \frac{t}{9t^3} && \text{Multiply} \\ &= \frac{18 - t}{9t^3} && \text{Subtract} \end{aligned}$$

We find no common factors among the numerator and denominator so we are done.

A second way to approach this problem is by factoring. We can extend the concept of the 'Polynomial G.C.F.' to these types of expressions and we can follow the same guidelines as set forth on page 69 to factor out the G.C.F. of these two terms. The key ideas to remember are that we take out each factor with the *smallest* exponent and factoring is the same as dividing. We first note that $2t^{-3} - (3t)^{-2} = 2t^{-3} - 3^{-2}t^{-2}$ and we see that the smallest power on t is -3. Thus we want to factor out t^{-3} from both terms. It's clear that this will leave 2 in the first term, but what about the second term? Since factoring is the same as dividing, we would be dividing the second term by t^{-3} which thanks to the properties of exponents is the same as *multiplying* by $\frac{1}{t^{-3}} = t^3$. The same holds for 3^{-2}. Even though there are no factors of 3 in the first term, we can factor out 3^{-2} by multiplying it by $\frac{1}{3^{-2}} = 3^2 = 9$. We put these ideas together below.

$$\begin{aligned} 2t^{-3} - (3t)^{-2} &= 2t^{-3} - 3^{-2}t^{-2} && \text{Properties of Exponents} \\ &= 3^{-2}t^{-3}(2(3)^2 - t^1) && \text{Factor} \\ &= \frac{1}{3^2}\frac{1}{t^3}(18 - t) && \text{Rewrite} \\ &= \frac{18 - t}{9t^3} && \text{Multiply} \end{aligned}$$

While both ways are valid, one may be more of a natural fit than the other depending on the circumstances and temperament of the student.

6. As with the previous example, we show two different yet equivalent ways to approach simplifying $10x(x-3)^{-1}+5x^2(-1)(x-3)^{-2}$. First up is what we'll call the 'common denominator approach' where we rewrite the negative exponents as fractions and proceed from there.

   - *Common Denominator Approach*:

$$\begin{aligned}
10x(x-3)^{-1}+5x^2(-1)(x-3)^{-2} &= \frac{10x}{x-3}+\frac{5x^2(-1)}{(x-3)^2} \\
&= \frac{10x}{x-3}\cdot\frac{x-3}{x-3}-\frac{5x^2}{(x-3)^2} && \text{Equivalent Fractions} \\
&= \frac{10x(x-3)}{(x-3)^2}-\frac{5x^2}{(x-3)^2} && \text{Multiply} \\
&= \frac{10x(x-3)-5x^2}{(x-3)^2} && \text{Subtract} \\
&= \frac{5x(2(x-3)-x)}{(x-3)^2} && \text{Factor out G.C.F.} \\
&= \frac{5x(2x-6-x)}{(x-3)^2} && \text{Distribute} \\
&= \frac{5x(x-6)}{(x-3)^2} && \text{Combine like terms}
\end{aligned}$$

     Both the numerator and the denominator are completely factored with no common factors so we are done.

   - *'Factoring Approach'*: In this case, the G.C.F. is $5x(x-3)^{-2}$. Factoring this out of both terms gives:

$$\begin{aligned}
10x(x-3)^{-1}+5x^2(-1)(x-3)^{-2} &= 5x(x-3)^{-2}(2(x-3)^1-x) && \text{Factor} \\
&= \frac{5x}{(x-3)^2}(2x-6-x) && \text{Rewrite, distribute} \\
&= \frac{5x(x-6)}{(x-3)^2} && \text{Multiply}
\end{aligned}$$

     As expected, we got the same reduced fraction as before. □

Next, we review the solving of equations which involve rational expressions. As with equations involving numeric fractions, our first step in solving equations with algebraic fractions is to clear denominators. In doing so, we run the risk of introducing what are known as **extraneous** solutions - 'answers' which don't satisfy the original equation. As we illustrate the techniques used to solve these basic equations, see if you can find the step which creates the problem for us.

Example 0.8.2. Solve the following equations.

1. $1 + \dfrac{1}{x} = x$
2. $\dfrac{t^3 - 2t + 1}{t - 1} = \dfrac{1}{2}t - 1$
3. $\dfrac{3}{1 - w\sqrt{2}} - \dfrac{1}{2w + 5} = 0$
4. $3(x^2 + 4)^{-1} + 3x(-1)(x^2 + 4)^{-2}(2x) = 0$
5. Solve $x = \dfrac{2y + 1}{y - 3}$ for y.
6. Solve $\dfrac{1}{f} = \dfrac{1}{S_1} + \dfrac{1}{S_2}$ for S_1.

Solution.

1. Our first step is to clear the fractions by multiplying both sides of the equation by x. In doing so, we are implicitly assuming $x \neq 0$; otherwise, we would have no guarantee that the resulting equation is equivalent to our original equation.[5]

$$\begin{array}{rcll} 1 + \dfrac{1}{x} & = & x & \\ \left(1 + \dfrac{1}{x}\right) x & = & (x)x & \text{Provided } x \neq 0 \\ 1(x) + \dfrac{1}{x}(x) & = & x^2 & \text{Distribute} \\ x + \dfrac{x}{x} & = & x^2 & \text{Multiply} \\ x + 1 & = & x^2 & \\ 0 & = & x^2 - x - 1 & \text{Subtract } x\text{, subtract } 1 \\ x & = & \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} & \text{Quadratic Formula} \\ x & = & \dfrac{1 \pm \sqrt{5}}{2} & \text{Simplify} \end{array}$$

We obtain two answers, $x = \frac{1 \pm \sqrt{5}}{2}$. Neither of these are 0 thus neither contradicts our assumption that $x \neq 0$. The reader is invited to check both of these solutions.[6]

[5]See page 38.

[6]The check relies on being able to 'rationalize' the denominator - a skill we haven't reviewed yet. (Come back after you've read Section 0.9 if you want to!) Additionally, the positive solution to this equation is the famous Golden Ratio.

2. To solve the equation, we clear denominators. Here, we need to assume $t-1\neq 0$, or $t\neq 1$.

$$\begin{array}{rcll}
\dfrac{t^3-2t+1}{t-1} & = & \dfrac{1}{2}t-1 & \\
\left(\dfrac{t^3-2t+1}{t-1}\right)\cdot 2(t-1) & = & \left(\dfrac{1}{2}t-1\right)\cdot 2(t-1) & \text{Provided } t\neq 1 \\
\dfrac{(t^3-2t+1)(2\cancel{(t-1)})}{\cancel{(t-1)}} & = & \dfrac{1}{\cancel{2}}t(\cancel{2}(t-1))-1(2(t-1)) & \text{Multiply, distribute} \\
2(t^3-2t+1) & = & t^2-t-2t+2 & \text{Distribute} \\
2t^3-4t+2 & = & t^2-3t+2 & \text{Distribute, combine like terms} \\
2t^3-t^2-t & = & 0 & \text{Subtract } t^2\text{, add } 3t\text{, subtract } 2 \\
t(2t^2-t-1) & = & 0 & \text{Factor} \\
t=0 & \text{or} & 2t^2-t-1=0 & \text{Zero Product Property} \\
t=0 & \text{or} & (2t+1)(t-1)=0 & \text{Factor} \\
t=0 & \text{or} & 2t+1=0 \quad \text{or} \quad t-1=0 & \\
t & = & 0,\ -\dfrac{1}{2} \text{ or } 1 &
\end{array}$$

We assumed that $t\neq 1$ in order to clear denominators. Sure enough, the 'solution' $t=1$ doesn't check in the original equation since it causes division by 0. In this case, we call $t=1$ an *extraneous* solution. Note that $t=1$ *does* work in every equation *after* we clear denominators. In general, multiplying by variable expressions can produce these 'extra' solutions, which is why checking our answers is always encouraged.[7] The other two solutions, $t=0$ and $t=-\frac{1}{2}$, both work.

3. As before, we begin by clearing denominators. Here, we assume $1-w\sqrt{2}\neq 0$ (so $w\neq\frac{1}{\sqrt{2}}$) and $2w+5\neq 0$ (so $w\neq-\frac{5}{2}$).

$$\begin{array}{rcll}
\dfrac{3}{1-w\sqrt{2}}-\dfrac{1}{2w+5} & = & 0 & \\
\left(\dfrac{3}{1-w\sqrt{2}}-\dfrac{1}{2w+5}\right)(1-w\sqrt{2})(2w+5) & = & 0(1-w\sqrt{2})(2w+5) & w\neq\dfrac{1}{\sqrt{2}},-\dfrac{5}{2} \\
\dfrac{3\cancel{(1-w\sqrt{2})}(2w+5)}{\cancel{(1-w\sqrt{2})}}-\dfrac{1(1-w\sqrt{2})\cancel{(2w+5)}}{\cancel{(2w+5)}} & = & 0 & \text{Distribute} \\
3(2w+5)-(1-w\sqrt{2}) & = & 0 &
\end{array}$$

The result is a *linear* equation in w so we gather the terms with w on one side of the equation

[7] Contrast this with what happened in Example 0.6.3 when we divided by a variable and 'lost' a solution.

and put everything else on the other. We factor out w and divide by its coefficient.

$$\begin{aligned} 3(2w+5) - (1 - w\sqrt{2}) &= 0 & \\ 6w + 15 - 1 + w\sqrt{2} &= 0 & \text{Distribute} \\ 6w + w\sqrt{2} &= -14 & \text{Subtract 14} \\ (6+\sqrt{2})w &= -14 & \text{Factor} \\ w &= -\frac{14}{6+\sqrt{2}} & \text{Divide by } 6+\sqrt{2} \end{aligned}$$

This solution is different than our excluded values, $\frac{1}{\sqrt{2}}$ and $-\frac{5}{2}$, so we keep $w = -\frac{14}{6+\sqrt{2}}$ as our final answer. The reader is invited to check this in the original equation.

4. To solve our next equation, we have two approaches to choose from: we could rewrite the quantities with negative exponents as fractions and clear denominators, or we can factor. We showcase each technique below.

   - *Clearing Denominators Approach*: We rewrite the negative exponents as fractions and clear denominators. In this case, we multiply both sides of the equation by $(x^2+4)^2$, which is never 0. (Think about that for a moment.) As a result, we need not exclude any x values from our solution set.

$$\begin{aligned} 3(x^2+4)^{-1} + 3x(-1)(x^2+4)^{-2}(2x) &= 0 & \\ \frac{3}{x^2+4} + \frac{3x(-1)(2x)}{(x^2+4)^2} &= 0 & \text{Rewrite} \\ \left(\frac{3}{x^2+4} - \frac{6x^2}{(x^2+4)^2}\right)(x^2+4)^2 &= 0(x^2+4)^2 & \text{Multiply} \\ \frac{3\cancel{(x^2+4)^2}^{(x^2+4)}}{\cancel{(x^2+4)}} - \frac{6x^2\cancel{(x^2+4)^2}}{\cancel{(x^2+4)^2}} &= 0 & \text{Distribute} \\ 3(x^2+4) - 6x^2 &= 0 & \\ 3x^2 + 12 - 6x^2 &= 0 & \text{Distribute} \\ -3x^2 &= -12 & \text{Combine like terms, subtract 12} \\ x^2 &= 4 & \text{Divide by } -3 \\ x &= \pm\sqrt{4} = \pm 2 & \text{Extract square roots} \end{aligned}$$

   We leave it to the reader to show both $x=-2$ and $x=2$ satisfy the original equation.

   - *Factoring Approach*: Since the equation is already set equal to 0, we're ready to factor. Following the guidelines presented in Example 0.8.1, we factor out $3(x^2+4)^{-2}$ from both terms and look to see if more factoring can be done:

$$\begin{array}{rcll}
3(x^2+4)^{-1}+3x(-1)(x^2+4)^{-2}(2x) & = & 0 & \\
3(x^2+4)^{-2}((x^2+4)^1+x(-1)(2x)) & = & 0 & \text{Factor} \\
3(x^2+4)^{-2}(x^2+4-2x^2) & = & 0 & \\
3(x^2+4)^{-2}(4-x^2) & = & 0 & \text{Gather like terms} \\
3(x^2+4)^{-2}=0 & \text{or} & 4-x^2=0 & \text{Zero Product Property} \\
\dfrac{3}{x^2+4}=0 & \text{or} & 4=x^2 &
\end{array}$$

The first equation yields no solutions (Think about this for a moment.) while the second gives us $x=\pm\sqrt{4}=\pm 2$ as before.

5. We are asked to solve this equation for y so we begin by clearing fractions with the stipulation that $y-3\neq 0$ or $y\neq 3$. We are left with a linear equation in the variable y. To solve this, we gather the terms containing y on one side of the equation and everything else on the other. Next, we factor out the y and divide by its coefficient, which in this case turns out to be $x-2$. In order to divide by $x-2$, we stipulate $x-2\neq 0$ or, said differently, $x\neq 2$.

$$\begin{array}{rcll}
x & = & \dfrac{2y+1}{y-3} & \\
x(y-3) & = & \left(\dfrac{2y+1}{y-3}\right)(y-3) & \text{Provided } y\neq 3 \\
xy-3x & = & \dfrac{(2y+1)\cancel{(y-3)}}{\cancel{(y-3)}} & \text{Distribute, multiply} \\
xy-3x & = & 2y+1 & \\
xy-2y & = & 3x+1 & \text{Add } 3x\text{, subtract } 2y \\
y(x-2) & = & 3x+1 & \text{Factor} \\
y & = & \dfrac{3x+1}{x-2} & \text{Divide provided } x\neq 2
\end{array}$$

We highly encourage the reader to check the answer algebraically to see where the restrictions on x and y come into play.[8]

6. Our last example comes from physics and the world of photography.[9] We take a moment here to note that while superscripts in mathematics indicate exponents (powers), subscripts are used primarily to distinguish one or more variables. In this case, S_1 and S_2 are two *different* variables (much like x and y) and we treat them as such. Our first step is to clear

[8] It involves simplifying a compound fraction!

[9] See this article on focal length.

denominators by multiplying both sides by fS_1S_2 - provided each is nonzero. We end up with an equation which is linear in S_1 so we proceed as in the previous example.

$$\begin{aligned}
\frac{1}{f} &= \frac{1}{S_1} + \frac{1}{S_2} & \\
\left(\frac{1}{f}\right)(fS_1S_2) &= \left(\frac{1}{S_1} + \frac{1}{S_2}\right)(fS_1S_2) & \text{Provided } f \neq 0,\ S_1 \neq 0,\ S_2 \neq 0 \\
\frac{fS_1S_2}{f} &= \frac{fS_1S_2}{S_1} + \frac{fS_1S_2}{S_2} & \text{Multiply, distribute} \\
\frac{\cancel{f}S_1S_2}{\cancel{f}} &= \frac{f\cancel{S_1}S_2}{\cancel{S_1}} + \frac{fS_1\cancel{S_2}}{\cancel{S_2}} & \text{Cancel} \\
S_1S_2 &= fS_2 + fS_1 & \\
S_1S_2 - fS_1 &= fS_2 & \text{Subtract } fS_1 \\
S_1(S_2 - f) &= fS_2 & \text{Factor} \\
S_1 &= \frac{fS_2}{S_2 - f} & \text{Divide provided } S_2 \neq f
\end{aligned}$$

As always, the reader is highly encouraged to check the answer.[10] □

[10]. . . and see what the restriction $S_2 \neq f$ means in terms of focusing a camera!

0.8.1 Exercises

In Exercises 1 - 21, perform the indicated operations and simplify.

1. $\dfrac{x^2-9}{x^2}\cdot\dfrac{3x}{x^2-x-6}$

2. $\dfrac{t^2-2t}{t^2+1}\div(3t^2-2t-8)$

3. $\dfrac{4y-y^2}{2y+1}\div\dfrac{y^2-16}{2y^2-5y-3}$

4. $\dfrac{x+1}{x^2-1}\div\dfrac{x-1}{x+1}$

5. $\dfrac{x+1}{6x^2+17x+5}\cdot\dfrac{3x+1}{x+1}$

6. $\dfrac{x+y}{xy}\div\left(\dfrac{1}{x}+\dfrac{1}{y}\right)$

7. $\dfrac{x}{3x-1}-\dfrac{1-x}{3x-1}$

8. $\dfrac{2}{w-1}-\dfrac{w^2+1}{w-1}$

9. $\dfrac{2-y}{3y}-\dfrac{1-y}{3y}+\dfrac{y^2-1}{3y}$

10. $b+\dfrac{1}{b-3}-2$

11. $\dfrac{2x}{x-4}-\dfrac{1}{2x+1}$

12. $\dfrac{m^2}{m^2-4}+\dfrac{1}{2-m}$

13. $\dfrac{\frac{2}{x}-2}{x-1}$

14. $\dfrac{\frac{3}{2-h}-\frac{3}{2}}{h}$

15. $\dfrac{\frac{1}{x+h}-\frac{1}{x}}{h}$

16. $3w^{-1}-(3w)^{-1}$

17. $-2y^{-1}+2(3-y)^{-2}$

18. $3(x-2)^{-1}-3x(x-2)^{-2}$

19. $\dfrac{t^{-1}+t^{-2}}{t^{-3}}$

20. $\dfrac{2(3+h)^{-2}-2(3)^{-2}}{h}$

21. $\dfrac{(7-x-h)^{-1}-(7-x)^{-1}}{h}$

In Exercises 22 - 33, find all real solutions. Be sure to check for extraneous solutions.

22. $\dfrac{x}{5x+4}=3$

23. $\dfrac{3y-1}{y^2+1}=1$

24. $\dfrac{1}{w+3}+\dfrac{1}{w-3}=\dfrac{w^2-3}{w^2-9}$

25. $\dfrac{2x}{x-5}=3+\dfrac{1-x}{x-3}$

26. $\dfrac{2}{x-2}-\dfrac{2}{x+3}=1$

27. $t+3+\dfrac{1}{t+3}=2$

28. $\dfrac{2x+17}{x+1}=x+5$

29. $\dfrac{t^2-2t+1}{t^3+t^2-2t}=1$

30. $\dfrac{-y^3+4y}{y^2-9}=4y$

31. $w+\sqrt{3}=\dfrac{3w-w^3}{w-\sqrt{3}}$

32. $\dfrac{2}{x\sqrt{2}-1}-1=\dfrac{3}{x\sqrt{2}+1}$

33. $\dfrac{x^2}{(1+x\sqrt{3})^2}=3$

In Exercises 34 - 36, use Theorem 0.3 along with the techniques in this section to find all real solutions.

34. $\left|\dfrac{3n}{n-1}\right|=3$

35. $\left|\dfrac{2x}{x^2-1}\right|=2$

36. $\left|\dfrac{2t}{4-t^2}\right|=\left|\dfrac{2}{t-2}\right|$

In Exercises 37 - 42, solve the given equation for the indicated variable.

37. Solve for y: $\dfrac{1-2y}{y+3} = x$

38. Solve for y: $x = 3 - \dfrac{2}{1-y}$

39.[11] Solve for T_2: $\dfrac{V_1}{T_1} = \dfrac{V_2}{T_2}$

40. Solve for t_0: $\dfrac{t_0}{1 - t_0 t_1} = 2$

41. Solve for x: $\dfrac{1}{x-v} + \dfrac{1}{x+v} = 5$

42. Solve for R: $P = \dfrac{25R}{(R+4)^2}$

[11] Recall: subscripts on variables have no intrinsic mathematical meaning; they're just used to distinguish one variable from another. In other words, treat quantities like 'V_1' and 'V_2' as two different variables as you would 'x' and 'y.'

0.8.2 Answers

1. $\dfrac{3(x+3)}{x(x+2)},\ x \neq 3$
2. $\dfrac{t}{(3t+4)(t^2+1)},\ t \neq 2$
3. $-\dfrac{y(y-3)}{y+4},\ y \neq -\frac{1}{2}, 3, 4$
4. $\dfrac{x+1}{(x-1)^2},\ x \neq -1$
5. $\dfrac{1}{2x+5},\ x \neq -\frac{1}{3}$
6. $1,\ x \neq 0,\ y \neq 0$
7. $\dfrac{2x-1}{3x-1}$
8. $-w-1,\ w \neq 1$
9. $\dfrac{y}{3},\ y \neq 0$
10. $\dfrac{b^2-5b+7}{b-3}$
11. $\dfrac{4x^2+x+4}{(x-4)(2x+1)}$
12. $\dfrac{m+1}{m+2},\ m \neq 2$
13. $-\dfrac{2}{x},\ x \neq 1$
14. $\dfrac{3}{4-2h},\ h \neq 0$
15. $-\dfrac{1}{x(x+h)},\ h \neq 0$
16. $\dfrac{8}{3w}$
17. $-\dfrac{2(y^2-7y+9)}{y(y-3)^2}$
18. $-\dfrac{6}{(x-2)^2}$
19. $t^2+t,\ t \neq 0$
20. $-\dfrac{2(h+6)}{9(h+3)^2},\ h \neq 0$
21. $\dfrac{1}{(7-x)(7-x-h)},\ h \neq 0$
22. $x = -\dfrac{6}{7}$
23. $y = 1, 2$
24. $w = -1$
25. $x = \dfrac{10}{3}$
26. $x = \dfrac{-1 \pm \sqrt{65}}{2}$
27. $t = -2$
28. $x = -6, 2$
29. No solution.
30. $y = 0, \pm 2\sqrt{2}$
31. $w = -\sqrt{3}, -1$
32. $x = -\dfrac{3\sqrt{2}}{2}, \sqrt{2}$
33. $x = -\dfrac{\sqrt{3}}{2}, -\dfrac{\sqrt{3}}{4}$
34. $n = \dfrac{1}{2}$
35. $x = \dfrac{1 \pm \sqrt{5}}{2}, \dfrac{-1 \pm \sqrt{5}}{2}$
36. $t = -1$
37. $y = \dfrac{1-3x}{x+2},\ y \neq -3$
38. $y = \dfrac{x-1}{x-3},\ y \neq 1$
39. $T_2 = \dfrac{V_2 T_1}{V_1},\ T_1 \neq 0, T_2 \neq 0$
40. $t_0 = \dfrac{2}{2t_1+1}$
41. $x = \dfrac{1 \pm \sqrt{25v^2+1}}{5},\ x \neq \pm v.$
42. $R = \dfrac{(25-8P) \pm 5\sqrt{25-16P}}{2P},\ R \neq -4$

0.9 Radicals and Equations

In this section we review simplifying expressions and solving equations involving radicals. In addition to the product, quotient and power rules stated in Theorem 0.1 in Section 0.2, we present the following result which states that n^{th} roots and n^{th} powers more or less 'undo' each other.

> **Theorem 0.11. Simplifying n^{th} powers of n^{th} roots**: Suppose n is a natural number, a is a real number and $\sqrt[n]{a}$ is a real number. Then
>
> - $(\sqrt[n]{a})^n = a$
> - if n is odd, $\sqrt[n]{a^n} = a$; if n is even, $\sqrt[n]{a^n} = |a|$.

Since $\sqrt[n]{a}$ is *defined* so that $(\sqrt[n]{a})^n = a$, the first claim in the theorem is just a re-wording of Definition 0.8. The second part of the theorem breaks down along odd/even exponent lines due to how exponents affect negatives. To see this, consider the specific cases of $\sqrt[3]{(-2)^3}$ and $\sqrt[4]{(-2)^4}$.

In the first case, $\sqrt[3]{(-2)^3} = \sqrt[3]{-8} = -2$, so we have an instance of when $\sqrt[n]{a^n} = a$. The reason that the cube root 'undoes' the third power in $\sqrt[3]{(-2)^3} = -2$ is because the negative is preserved when raised to the third (odd) power. In $\sqrt[4]{(-2)^4}$, the negative 'goes away' when raised to the fourth (even) power: $\sqrt[4]{(-2)^4} = \sqrt[4]{16}$. According to Definition 0.8, the fourth root is defined to give only *non-negative* numbers, so $\sqrt[4]{16} = 2$. Here we have a case where $\sqrt[4]{(-2)^4} = 2 = |-2|$, not -2.

In general, we need the absolute values to simplify $\sqrt[n]{a^n}$ only when n is even because a negative to an even power is always positive. In particular, $\sqrt{x^2} = |x|$, not just 'x' (unless we *know* $x \geq 0$.)[1] We practice these formulas in the following example.

Example 0.9.1. Perform the indicated operations and simplify.

1. $\sqrt{x^2+1}$
2. $\sqrt{t^2 - 10t + 25}$
3. $\sqrt[3]{48x^{14}}$
4. $\sqrt[4]{\dfrac{\pi r^4}{L^8}}$
5. $2x\sqrt[3]{x^2-4} + \left(\dfrac{1}{(\sqrt[3]{x^2-4})^2}\right)(2x)$
6. $\sqrt{(\sqrt{18y} - \sqrt{8y})^2 + (\sqrt{20} - \sqrt{80})^2}$

Solution.

1. We told you back on page 32 that roots do not 'distribute' across addition and since $x^2 + 1$ cannot be factored over the real numbers, $\sqrt{x^2+1}$ cannot be simplified. It may seem silly to start with this example but it is extremely important that you understand what maneuvers are legal and which ones are not.[2]

[1] If this discussion sounds familiar, see the discussion following Definition 0.9 and the discussion following 'Extracting the Square Root' on page 81.

[2] You really do need to understand this otherwise horrible evil will plague your future studies in Math. If you say something totally wrong like $\sqrt{x^2+1} = x + 1$ then you may never pass Calculus. PLEASE be careful!

2. Again we note that $\sqrt{t^2 - 10t + 25} \neq \sqrt{t^2} - \sqrt{10t} + \sqrt{25}$, since radicals do *not* distribute across addition and subtraction.[3] In this case, however, we can factor the radicand and simplify as

$$\sqrt{t^2 - 10t + 25} = \sqrt{(t-5)^2} = |t-5|$$

Without knowing more about the value of t, we have no idea if $t-5$ is positive or negative so $|t-5|$ is our final answer.[4]

3. To simplify $\sqrt[3]{48x^{14}}$, we need to look for perfect cubes in the radicand. For the cofficient, we have $48 = 8 \cdot 6 = 2^3 \cdot 6$. To find the largest perfect cube factor in x^{14}, we divide 14 (the exponent on x) by 3 (since we are looking for a perfect *cube*). We get 4 with a remainder of 2. This means $14 = 4 \cdot 3 + 2$, so $x^{14} = x^{4 \cdot 3 + 2} = x^{4 \cdot 3} x^2 = (x^4)^3 x^2$. Putting this altogether gives:

$$\begin{aligned} \sqrt[3]{48x^{14}} &= \sqrt[3]{2^3 \cdot 6 \cdot (x^4)^3 x^2} && \text{Factor out perfect cubes} \\ &= \sqrt[3]{2^3}\sqrt[3]{(x^4)^3}\sqrt[3]{6x^2} && \text{Rearrange factors, Product Rule of Radicals} \\ &= 2x^4\sqrt[3]{6x^2} \end{aligned}$$

4. In this example, we are looking for perfect fourth powers in the radicand. In the numerator r^4 is clearly a perfect fourth power. For the denominator, we take the power on the L, namely 12, and divide by 4 to get 3. This means $L^8 = L^{2 \cdot 4} = (L^2)^4$. We get

$$\begin{aligned} \sqrt[4]{\frac{\pi r^4}{L^{12}}} &= \frac{\sqrt[4]{\pi r^4}}{\sqrt[4]{L^{12}}} && \text{Quotient Rule of Radicals} \\ &= \frac{\sqrt[4]{\pi}\sqrt[4]{r^4}}{\sqrt[4]{(L^2)^4}} && \text{Product Rule of Radicals} \\ &= \frac{\sqrt[4]{\pi}|r|}{|L^2|} && \text{Simplify} \end{aligned}$$

Without more information about r, we cannot simplify $|r|$ any further. However, we can simplify $|L^2|$. Regardless of the choice of L, $L^2 \geq 0$. Actually, $L^2 > 0$ because L is in the denominator which means $L \neq 0$. Hence, $|L^2| = L^2$. Our answer simplifies to:

$$\frac{\sqrt[4]{\pi}|r|}{|L^2|} = \frac{|r|\sqrt[4]{\pi}}{L^2}$$

5. First, we need to obtain a common denominator. Since we can view the first term as having a denominator of 1, the common denominator is precisely the denominator of the second term, namely $(\sqrt[3]{x^2 - 4})^2$. With common denominators, we proceed to add the two fractions.

[3] Let $t = 1$ and see what happens to $\sqrt{t^2 - 10t + 25}$ versus $\sqrt{t^2} - \sqrt{10t} + \sqrt{25}$.

[4] In general, $|t-5| \neq |t| - |5|$ and $|t-5| \neq t+5$ so watch what you're doing!

Our last step is to factor the numerator to see if there are any cancellation opportunities with the denominator.

$$\begin{aligned}
2x\sqrt[3]{x^2-4}+\left(\frac{1}{(\sqrt[3]{x^2-4})^2}\right)(2x) &= 2x\sqrt[3]{x^2-4}+\frac{2x}{(\sqrt[3]{x^2-4})^2} && \text{Mutiply}\\
&= (2x\sqrt[3]{x^2-4})\cdot\frac{(\sqrt[3]{x^2-4})^2}{(\sqrt[3]{x^2-4})^2}+\frac{2x}{(\sqrt[3]{x^2-4})^2} && \text{Equivalent fractions}\\
&= \frac{2x(\sqrt[3]{x^2-4})^3}{(\sqrt[3]{x^2-4})^2}+\frac{2x}{(\sqrt[3]{x^2-4})^2} && \text{Multiply}\\
&= \frac{2x(x^2-4)}{(\sqrt[3]{x^2-4})^2}+\frac{2x}{(\sqrt[3]{x^2-4})^2} && \text{Simplify}\\
&= \frac{2x(x^2-4)+2x}{(\sqrt[3]{x^2-4})^2} && \text{Add}\\
&= \frac{2x(x^2-4+1)}{(\sqrt[3]{x^2-4})^2} && \text{Factor}\\
&= \frac{2x(x^2-3)}{(\sqrt[3]{x^2-4})^2}
\end{aligned}$$

We cannot reduce this any further because x^2-3 is irreducible over the rational numbers.

6. We begin by working inside each set of parentheses, using the product rule for radicals and combining like terms.

$$\begin{aligned}
\sqrt{(\sqrt{18y}-\sqrt{8y})^2+(\sqrt{20}-\sqrt{80})^2} &= \sqrt{(\sqrt{9\cdot 2y}-\sqrt{4\cdot 2y})^2+(\sqrt{4\cdot 5}-\sqrt{16\cdot 5})^2}\\
&= \sqrt{(\sqrt{9}\sqrt{2y}-\sqrt{4}\sqrt{2y})^2+(\sqrt{4}\sqrt{5}-\sqrt{16}\sqrt{5})^2}\\
&= \sqrt{(3\sqrt{2y}-2\sqrt{2y})^2+(2\sqrt{5}-4\sqrt{5})^2}\\
&= \sqrt{(\sqrt{2y})^2+(-2\sqrt{5})^2}\\
&= \sqrt{2y+(-2)^2(\sqrt{5})^2}\\
&= \sqrt{2y+4\cdot 5}\\
&= \sqrt{2y+20}
\end{aligned}$$

To see if this simplifies any further, we factor the radicand: $\sqrt{2y+20}=\sqrt{2(y+10)}$. Finding no perfect square factors, we are done. □

Theorem 0.11 allows us to generalize the process of 'Extracting Square Roots' to 'Extracting n^{th} roots' which in turn allows us to solve equations[5] of the form $X^n=c$.

[5]Well, not entirely. The equation $x^7=1$ has seven answers: $x=1$ and six complex number solutions.

Extracting n^{th} roots:

- If c is a real number and n is odd then the real number solution to $X^n = c$ is $X = \sqrt[n]{c}$.
- If $c \geq 0$ and n is even then the real number solutions to $X^n = c$ are $X = \pm\sqrt[n]{c}$.

  Note: If $c < 0$ and n is even then $X^n = c$ has no real number solutions.

Essentially, we solve $X^n = c$ by 'taking the n^{th} root' of both sides: $\sqrt[n]{X^n} = \sqrt[n]{c}$. Simplifying the left side gives us just X if n is odd or $|X|$ if n is even. In the first case, $X = \sqrt[n]{c}$, and in the second, $X = \pm\sqrt[n]{c}$. Putting this together with the other part of Theorem 0.11, namely $\left(\sqrt[n]{a}\right)^n = a$, gives us a strategy for solving equations which involve n^{th} and n^{th} roots.

Strategies for Power and Radical Equations

- If the equation involves an n^{th} power and the variable appears in only one term, isolate the term with the n^{th} power and extract n^{th} roots.
- If the equation involves an n^{th} root and the variable appears in that n^{th} root, isolate the n^{th} root and raise both sides of the equation to the n^{th} power.

  Note: When raising both sides of an equation to an *even* power, be sure to check for extraneous solutions.

The note about 'extraneous solutions' can be demonstrated by the basic equation: $\sqrt{x} = -2$. This equation has no solution since, by definition, $\sqrt{x} \geq 0$ for all real numbers x. However, if we square both sides of this equation, we get $\left(\sqrt{x}\right)^2 = (-2)^2$ or $x = 4$. However, $x = 4$ doesn't check in the original equation, since $\sqrt{4} = 2$, not -2. Once again, the root[6] of all of our problems lies in the fact that a *negative* number to an *even* power results in a *positive* number. In other words, raising both sides of an equation to an even power does *not* produce an equivalent equation, but rather, an equation which may possess *more* solutions than the original. Hence the cautionary remark above about extraneous solutions.

Example 0.9.2. Solve the following equations.

1. $(5x+3)^4 = 16$
2. $1 - \dfrac{(5-2w)^3}{7} = 9$
3. $t + \sqrt{2t+3} = 6$
4. $\sqrt{2} - 3\sqrt[3]{2y+1} = 0$
5. $\sqrt{4x-1} + 2\sqrt{1-2x} = 1$
6. $\sqrt[4]{n^2+2} + n = 0$

For the remaining problems, assume that all of the variables represent positive real numbers.[7]

7. Solve for r: $V = \frac{4\pi}{3}(R^3 - r^3)$.
8. Solve for v: $m = \dfrac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$.

[6]Pun intended!

[7]That is, you needn't worry that you're multiplying or dividing by 0 or that you're forgetting absolute value symbols.

Solution.

1. In our first equation, the quantity containing x is already isolated, so we extract fourth roots. Since the exponent here is even, when the roots are extracted we need both the positive and negative roots.

$$\begin{array}{rcll} (5x+3)^4 & = & 16 & \\ 5x+3 & = & \pm\sqrt[4]{16} & \text{Extract fourth roots} \\ 5x+3 & = & \pm 2 & \\ 5x+3=2 & \text{or} & 5x+3=-2 & \\ x=-\dfrac{1}{5} & \text{or} & x=-1 & \end{array}$$

We leave it to the reader that both of these solutions satisfy the original equation.

2. In this example, we first need to isolate the quantity containing the variable w. Here, third (cube) roots are required and since the exponent (index) is odd, we do not need the $\pm$:

$$\begin{array}{rcll} 1-\dfrac{(5-2w)^3}{7} & = & 9 & \\ -\dfrac{(5-2w)^3}{7} & = & 8 & \text{Subtract 1} \\ (5-2w)^3 & = & -56 & \text{Multiply by } -7 \\ 5-2w & = & \sqrt[3]{-56} & \text{Extract cube root} \\ 5-2w & = & \sqrt[3]{(-8)(7)} & \\ 5-2w & = & \sqrt[3]{-8}\sqrt[3]{7} & \text{Product Rule} \\ 5-2w & = & -2\sqrt[3]{7} & \\ -2w & = & -5-2\sqrt[3]{7} & \text{Subtract 5} \\ w & = & \dfrac{-5-2\sqrt[3]{7}}{-2} & \text{Divide by } -2 \\ w & = & \dfrac{5+2\sqrt[3]{7}}{2} & \text{Properties of Negatives} \end{array}$$

The reader should check the answer because it provides a hearty review of arithmetic.

3. To solve $t+\sqrt{2t+3}=6$, we first isolate the square root, then proceed to square both sides of the equation. In doing so, we run the risk of introducing extraneous solutions so checking our answers here is a necessity.

$$\begin{array}{rcll} t+\sqrt{2t+3} & = & 6 & \\ \sqrt{2t+3} & = & 6-t & \text{Subtract } t \\ (\sqrt{2t+3})^2 & = & (6-t)^2 & \text{Square both sides} \\ 2t+3 & = & 36-12t+t^2 & \text{Perfect Square Trinomial} \\ 0 & = & t^2-14t+33 & \text{Subtract } 2t \text{ and } 3 \\ 0 & = & (t-3)(t-11) & \text{Factor} \end{array}$$

From the Zero Product Property, we know either $t-3=0$ (which gives $t=3$) or $t-11=0$ (which gives $t=11$). When checking our answers, we find $t=3$ satisfies the original equation, but $t=11$ does not.[8] So our final answer is $t=3$ only.

4. In our next example, we locate the variable (in this case y) beneath a cube root, so we first isolate that root and cube both sides.

$$
\begin{array}{rcll}
\sqrt{2}-3\sqrt[3]{2y+1} & = & 0 & \\
-3\sqrt[3]{2y+1} & = & -\sqrt{2} & \text{Subtract } \sqrt{2} \\
\sqrt[3]{2y+1} & = & \dfrac{-\sqrt{2}}{-3} & \text{Divide by } -3 \\
\sqrt[3]{2y+1} & = & \dfrac{\sqrt{2}}{3} & \text{Properties of Negatives} \\
(\sqrt[3]{2y+1})^3 & = & \left(\dfrac{\sqrt{2}}{3}\right)^3 & \text{Cube both sides} \\
2y+1 & = & \dfrac{(\sqrt{2})^3}{3^3} & \\
2y+1 & = & \dfrac{2\sqrt{2}}{27} & \\
2y & = & \dfrac{2\sqrt{2}}{27}-1 & \text{Subtract 1} \\
2y & = & \dfrac{2\sqrt{2}-27}{27} & \text{Subtract fractions} \\
y & = & \dfrac{2\sqrt{2}-27}{54} & \text{Divide by 2 } \left(\text{multiply by } \frac{1}{2}\right)
\end{array}
$$

Since we raised both sides to an *odd* power, we don't need to worry about extraneous solutions but we encourage the reader to check the solution just for the fun of it.

5. In the equation $\sqrt{4x-1}+2\sqrt{1-2x}=1$, we have not one but two square roots. We begin by isolating one of the square roots and squaring both sides.

$$
\begin{array}{rcll}
\sqrt{4x-1}+2\sqrt{1-2x} & = & 1 & \\
\sqrt{4x-1} & = & 1-2\sqrt{1-2x} & \text{Subtract } 2\sqrt{1-2x} \text{ from both sides} \\
(\sqrt{4x-1})^2 & = & (1-2\sqrt{1-2x})^2 & \text{Square both sides} \\
4x-1 & = & 1-4\sqrt{1-2x}+(2\sqrt{1-2x})^2 & \text{Perfect Square Trinomial} \\
4x-1 & = & 1-4\sqrt{1-2x}+4(1-2x) & \\
4x-1 & = & 1-4\sqrt{1-2x}+4-8x & \text{Distribute} \\
4x-1 & = & 5-8x-4\sqrt{1-2x} & \text{Gather like terms}
\end{array}
$$

[8] It is worth noting that when $t=11$ is substituted into the original equation, we get $11+\sqrt{25}=6$. If the $+\sqrt{25}$ were $-\sqrt{25}$, the solution would check. Once again, when squaring both sides of an equation, we lose track of $\pm$, which is what lets extraneous solutions in the door.

At this point, we have just one square root so we proceed to isolate it and square both sides a second time.

$$\begin{array}{rcll} 4x-1 & = & 5-8x-4\sqrt{1-2x} & \\ 12x-6 & = & -4\sqrt{1-2x} & \text{Subtract 5, add } 8x \\ (12x-6)^2 & = & (-4\sqrt{1-2x})^2 & \text{Square both sides} \\ 144x^2-144x+36 & = & 16(1-2x) & \\ 144x^2-144x+36 & = & 16-32x & \\ 144x^2-112x+20 & = & 0 & \text{Subtract 16, add } 32x \\ 4(36x^2-28x+5) & = & 0 & \text{Factor} \\ 4(2x-1)(18x-5) & = & 0 & \text{Factor some more} \end{array}$$

From the Zero Product Property, we know either $2x-1=0$ or $18x-5=0$. The former gives $x=\frac{1}{2}$ while the latter gives us $x=\frac{5}{18}$. Since we squared both sides of the equation (twice!), we need to check for extraneous solutions. We find $x=\frac{5}{18}$ to be extraneous, so our only solution is $x=\frac{1}{2}$.

6. As usual, our first step in solving $\sqrt[4]{n^2+2}+n=0$ is to isolate the radical. We then proceed to raise both sides to the fourth power to eliminate the fourth root:

$$\begin{array}{rcll} \sqrt[4]{n^2+2}+n & = & 0 & \\ \sqrt[4]{n^2+2} & = & -n & \text{Subtract } n \\ (\sqrt[4]{n^2+2})^4 & = & (-n)^4 & \text{Raise both sides to the } 4^{\text{th}} \text{ power} \\ n^2+2 & = & n^4 & \\ 0 & = & n^4-n^2-2 & \text{Subtract } n^2 \text{ and } 2 \\ 0 & = & (n^2-2)(n^2+1) & \text{Factor - `Quadratic in Disguise'} \end{array}$$

At this point, the Zero Product Property gives either $n^2-2=0$ or $n^2+1=0$. From $n^2-2=0$, we get $n^2=2$, so $n=\pm\sqrt{2}$. From $n^2+1=0$, we get $n^2=-1$, which gives no real solutions.[9] Since we raised both sides to an even (the fourth) power, we need to check for extraneous solutions. We find that $n=-\sqrt{2}$ works but $n=\sqrt{2}$ is extraneous.

7. In this problem, we are asked to solve for r. While there are a lot of letters in this equation, r appears in only one term: r^3. Our strategy is to isolate r^3 then extract the cube root.

$$\begin{array}{rcll} V & = & \frac{4\pi}{3}(R^3-r^3) & \\ 3V & = & 4\pi(R^3-r^3) & \text{Multiply by 3} \\ 3V & = & 4\pi R^3-4\pi r^3 & \text{Distribute} \\ 3V-4\pi R^3 & = & -4\pi r^3 & \text{Subtract } 4\pi R^3 \end{array}$$

[9] Why is that again?

$$\begin{aligned}
\frac{3V - 4\pi R^3}{-4\pi} &= r^3 && \text{Divide by } -4\pi \\
\frac{4\pi R^3 - 3V}{4\pi} &= r^3 && \text{Properties of Negatives} \\
\sqrt[3]{\frac{4\pi R^3 - 3V}{4\pi}} &= r && \text{Extract the cube root}
\end{aligned}$$

The check is, as always, left to the reader and highly encouraged.

8. Our last equation to solve comes from Einstein's Special Theory of Relativity and relates the mass of an object to its velocity as it moves.[10] We are asked to solve for v which is located in just one term, namely v^2, which happens to lie in a fraction underneath a square root which is itself a denominator. We have quite a lot of work ahead of us!

$$\begin{aligned}
m &= \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} && \\
m\sqrt{1 - \frac{v^2}{c^2}} &= m_0 && \text{Multiply by } \sqrt{1 - \frac{v^2}{c^2}} \\
\left(m\sqrt{1 - \frac{v^2}{c^2}}\right)^2 &= m_0^2 && \text{Square both sides} \\
m^2\left(1 - \frac{v^2}{c^2}\right) &= m_0^2 && \text{Properties of Exponents} \\
m^2 - \frac{m^2 v^2}{c^2} &= m_0^2 && \text{Distribute} \\
-\frac{m^2 v^2}{c^2} &= m_0^2 - m^2 && \text{Subtract } m^2 \\
m^2 v^2 &= -c^2(m_0^2 - m^2) && \text{Multiply by } -c^2 \; (c^2 \neq 0) \\
m^2 v^2 &= -c^2 m_0^2 + c^2 m^2 && \text{Distribute} \\
v^2 &= \frac{-c^2 m_0^2 + c^2 m^2}{m^2} && \text{Divide by } m^2 \; (m^2 \neq 0) \\
v^2 &= \frac{c^2 m^2 - c^2 m_0^2}{m^2} && \text{Rearrange} \\
v &= \sqrt{\frac{c^2 m^2 - c^2 m_0^2}{m^2}} && \text{Extract the square root, remember } v > 0
\end{aligned}$$

[10] See this article on the Lorentz Factor.

$$
\begin{aligned}
v &= \frac{\sqrt{c^2(m^2 - m_0^2)}}{\sqrt{m^2}} && \text{Properties of Radicals} \\
v &= \frac{|c|\sqrt{m^2 - m_0^2}}{|m|} \\
v &= \frac{c\sqrt{m^2 - m_0^2}}{m} && \text{since } c > 0 \text{ and } m > 0
\end{aligned}
$$

Checking the answer algebraically would earn the reader great honor and respect on the Algebra battlefield so it is highly recommended.

0.9.1 Rationalizing Denominators and Numerators

In Section 0.7, there were a few instances where we needed to 'rationalize' a denominator - that is, take a fraction with radical in the denominator and re-write it as an equivalent fraction without a radical in the denominator. There are various reasons for wanting to do this, but the most pressing reason is that rationalizing denominators - and numerators as well - gives us an opportunity for more practice with fractions and radicals. To help refresh your memory, we rationalize a denominator and then a numerator below:

$$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{4}} = \frac{\sqrt{2}}{2} \quad \text{and} \quad \frac{7\sqrt[3]{4}}{3} = \frac{7\sqrt[3]{4}\sqrt[3]{2}}{3\sqrt[3]{2}} = \frac{7\sqrt[3]{8}}{3\sqrt[3]{2}} = \frac{7\cdot 2}{3\sqrt[3]{2}} = \frac{14}{3\sqrt[3]{2}}$$

In general, if the fraction contains either a single term numerator or denominator with an undesirable n^{th} root, we multiply the numerator and denominator by whatever is required to obtain a perfect n^{th} power in the radicand that we want to eliminate. If the fraction contains two terms the situation is somewhat more complicated. To see why, consider the fraction $\frac{3}{4-\sqrt{5}}$. Suppose we wanted to rid the denominator of the $\sqrt{5}$ term. We could try as above and multiply numerator and denominator by $\sqrt{5}$ but that just yields:

$$\frac{3}{4-\sqrt{5}} = \frac{3\sqrt{5}}{(4-\sqrt{5})\sqrt{5}} = \frac{3\sqrt{5}}{4\sqrt{5}-\sqrt{5}\sqrt{5}} = \frac{3\sqrt{5}}{4\sqrt{5}-5}$$

We haven't removed $\sqrt{5}$ from the denominator - we've just shuffled it over to the other term in the denominator. As you may recall, the strategy here is to multiply both numerator and denominator by what's called the **conjugate**.

Definition 0.17. Congugate of a Square Root Expression: If a, b and c are real numbers with $c > 0$ then the quantities $(a + b\sqrt{c})$ and $(a - b\sqrt{c})$ are **conjugates** of one another.[a] Conjugates multiply according to the Difference of Squares Formula:

$$(a + b\sqrt{c})(a - b\sqrt{c}) = a^2 - (b\sqrt{c})^2 = a^2 - b^2c$$

[a] As are $(b\sqrt{c} - a)$ and $(b\sqrt{c} + a)$: $(b\sqrt{c} - a)(b\sqrt{c} + a) = b^2c - a^2$.

That is, to get the conjugate of a two-term expression involving a square root, you change the '$-$' to a '+,' or vice-versa. For example, the conjugate of $4-\sqrt{5}$ is $4+\sqrt{5}$, and when we multiply these two factors together, we get $(4-\sqrt{5})(4+\sqrt{5}) = 4^2-(\sqrt{5})^2 = 16-5 = 11$. Hence, to eliminate the $\sqrt{5}$ from the denominator of our original fraction, we multiply both the numerator and denominator by the *conjugate* of $4-\sqrt{5}$:

$$\frac{3}{4-\sqrt{5}} = \frac{3(4+\sqrt{5})}{(4-\sqrt{5})(4+\sqrt{5})} = \frac{3(4+\sqrt{5})}{4^2-(\sqrt{5})^2} = \frac{3(4+\sqrt{5})}{16-5} = \frac{12+3\sqrt{5}}{11}$$

What if we had $\sqrt[3]{5}$ instead of $\sqrt{5}$? We could try multiplying $4-\sqrt[3]{5}$ by $4+\sqrt[3]{5}$ to get

$$(4-\sqrt[3]{5})(4+\sqrt[3]{5}) = 4^2-(\sqrt[3]{5})^2 = 16-\sqrt[3]{25},$$

which leaves us with a cube root. What we need to undo the cube root is a perfect cube, which means we look to the Difference of Cubes Formula for inspiration: $a^3-b^3 = (a-b)(a^2+ab+b^2)$. If we take $a = 4$ and $b = \sqrt[3]{5}$, we multiply

$$(4-\sqrt[3]{5})(4^2+4\sqrt[3]{5}+(\sqrt[3]{5})^2) = 4^3+4^2\sqrt[3]{5}+4\sqrt[3]{5}-4^2\sqrt[3]{5}-4(\sqrt[3]{5})^2-(\sqrt[3]{5})^3 = 64-5 = 59$$

So if we were charged with rationalizing the denominator of $\frac{3}{4-\sqrt[3]{5}}$, we'd have:

$$\frac{3}{4-\sqrt[3]{5}} = \frac{3(4^2+4\sqrt[3]{5}+(\sqrt[3]{5})^2)}{(4-\sqrt[3]{5})(4^2+4\sqrt[3]{5}+(\sqrt[3]{5})^2)} = \frac{48+12\sqrt[3]{5}+3\sqrt[3]{25}}{59}$$

This sort of thing extends to n^{th} roots since $(a-b)$ is a factor of a^n-b^n for all natural numbers n, but in practice, we'll stick with square roots with just a few cube roots thrown in for a challenge.[11]

Example 0.9.3. Rationalize the indicated numerator or denominator:

1. Rationalize the denominator: $\dfrac{3}{\sqrt[3]{24x^2}}$
2. Rationalize the numerator: $\dfrac{\sqrt{9+h}-3}{h}$

Solution.

1. We are asked to rationalize the denominator, which in this case contains a third root. That means we need to work to create third powers of each of the factors of the radicand. To do so, we first factor the radicand: $24x^2 = 8\cdot 3\cdot x^2 = 2^3\cdot 3\cdot x^2$. To obtain third powers, we need

[11] To see what to do about fourth roots, use long division to find $(a^4-b^4)\div(a-b)$, and apply this to $4-\sqrt[4]{5}$.

to multiply by $3^2 \cdot x$ inside the radical.

$$\begin{aligned}
\frac{2}{\sqrt[3]{24x^2}} &= \frac{2}{\sqrt[3]{2^3 \cdot 3 \cdot x^2}} \\
&= \frac{2\sqrt[3]{3^2 \cdot x}}{\sqrt[3]{2^3 \cdot 3 \cdot x^2} \cdot \sqrt[3]{3^2 \cdot x}} && \text{Equivalent Fractions} \\
&= \frac{2\sqrt[3]{3^2 \cdot x}}{\sqrt[3]{2^3 \cdot 3 \cdot x^2 \cdot 3^2 \cdot x}} && \text{Product Rule} \\
&= \frac{2\sqrt[3]{3^2 \cdot x}}{\sqrt[3]{2^3 \cdot 3^3 \cdot x^3}} && \text{Properties of Exponents} \\
&= \frac{2\sqrt[3]{3^2 \cdot x}}{2 \cdot 3 \cdot x} && \text{Product Rule} \\
&= \frac{\cancel{2}\sqrt[3]{3^2 \cdot x}}{\cancel{2} \cdot 3 \cdot x} && \text{Reduce} \\
&= \frac{\sqrt[3]{9x}}{3x} && \text{Simplify}
\end{aligned}$$

2. Here, we are asked to rationalize the *numerator*. Since it is a two term numerator involving a square root, we multiply both numerator and denominator by the conjugate of $\sqrt{9+h}-3$, namely $\sqrt{9+h}+3$. After simplifying, we find an opportunity to reduce the fraction:

$$\begin{aligned}
\frac{\sqrt{9+h}-3}{h} &= \frac{(\sqrt{9+h}-3)(\sqrt{9+h}+3)}{h(\sqrt{9+h}+3)} && \text{Equivalent Fractions} \\
&= \frac{(\sqrt{9+h})^2-3^2}{h(\sqrt{9+h}+3)} && \text{Difference of Squares} \\
&= \frac{(9+h)-9}{h(\sqrt{9+h}+3)} && \text{Simplify} \\
&= \frac{h}{h(\sqrt{9+h}+3)} && \text{Simplify} \\
&= \frac{\overset{1}{\cancel{h}}}{\cancel{h}(\sqrt{9+h}+3)} && \text{Reduce} \\
&= \frac{1}{\sqrt{9+h}+3}
\end{aligned}$$

We close this section with an awesome example from Calculus.

Example 0.9.4. Simplify the compound fraction $\dfrac{\frac{1}{\sqrt{2(x+h)+1}}-\frac{1}{\sqrt{2x+1}}}{h}$ then rationalize the numerator of the result.

Solution. We start by multiplying the top and bottom of the 'big' fraction by $\sqrt{2x+2h+1}\sqrt{2x+1}$.

$$
\begin{aligned}
\frac{\dfrac{1}{\sqrt{2(x+h)+1}} - \dfrac{1}{\sqrt{2x+1}}}{h} &= \frac{\dfrac{1}{\sqrt{2x+2h+1}} - \dfrac{1}{\sqrt{2x+1}}}{h} \\
&= \frac{\left(\dfrac{1}{\sqrt{2x+2h+1}} - \dfrac{1}{\sqrt{2x+1}}\right)\sqrt{2x+2h+1}\sqrt{2x+1}}{h\sqrt{2x+2h+1}\sqrt{2x+1}} \\
&= \frac{\dfrac{\cancel{\sqrt{2x+2h+1}}\sqrt{2x+1}}{\cancel{\sqrt{2x+2h+1}}} - \dfrac{\sqrt{2x+2h+1}\cancel{\sqrt{2x+1}}}{\cancel{\sqrt{2x+1}}}}{h\sqrt{2x+2h+1}\sqrt{2x+1}} \\
&= \frac{\sqrt{2x+1} - \sqrt{2x+2h+1}}{h\sqrt{2x+2h+1}\sqrt{2x+1}}
\end{aligned}
$$

Next, we multiply the numerator and denominator by the conjugate of $\sqrt{2x+1} - \sqrt{2x+2h+1}$, namely $\sqrt{2x+1} + \sqrt{2x+2h+1}$, simplify and reduce:

$$
\begin{aligned}
\frac{\sqrt{2x+1} - \sqrt{2x+2h+1}}{h\sqrt{2x+2h+1}\sqrt{2x+1}} &= \frac{(\sqrt{2x+1} - \sqrt{2x+2h+1})(\sqrt{2x+1} + \sqrt{2x+2h+1})}{h\sqrt{2x+2h+1}\sqrt{2x+1}(\sqrt{2x+1} + \sqrt{2x+2h+1})} \\
&= \frac{(\sqrt{2x+1})^2 - (\sqrt{2x+2h+1})^2}{h\sqrt{2x+2h+1}\sqrt{2x+1}(\sqrt{2x+1} + \sqrt{2x+2h+1})} \\
&= \frac{(2x+1) - (2x+2h+1)}{h\sqrt{2x+2h+1}\sqrt{2x+1}(\sqrt{2x+1} + \sqrt{2x+2h+1})} \\
&= \frac{2x+1-2x-2h-1}{h\sqrt{2x+2h+1}\sqrt{2x+1}(\sqrt{2x+1} + \sqrt{2x+2h+1})} \\
&= \frac{-2\cancel{h}}{\cancel{h}\sqrt{2x+2h+1}\sqrt{2x+1}(\sqrt{2x+1} + \sqrt{2x+2h+1})} \\
&= \frac{-2}{\sqrt{2x+2h+1}\sqrt{2x+1}(\sqrt{2x+1} + \sqrt{2x+2h+1})}
\end{aligned}
$$

While the denominator is quite a bit more complicated than what we started with, we have done what was asked of us. In the interest of full disclosure, the reason we did all of this was to cancel the original 'h' from the denominator. That's an awful lot of effort to get rid of just one little h, but you'll see the significance of this in Calculus. □

0.9.2 Rational Exponents and Equations

Recall the following definition.

Definition 0.18. Let x be a real number, m an integer[a] and n a natural number.

- $x^{\frac{1}{n}} = \sqrt[n]{x}$ and is defined whenever $\sqrt[n]{x}$ is defined.
- $x^{\frac{m}{n}} = \left(\sqrt[n]{x}\right)^m = \sqrt[n]{x^m}$, whenever $\left(\sqrt[n]{x}\right)^m$ is defined.

[a]Recall this means $m = 0, \pm 1, \pm 2, \ldots$

As discussed in Section 0.2, rational exponents behave very similarly to the usual integer exponents, but there are exceptions and one has to be careful when attempting to apply the laws of exponents. For instance, even though $(x^{3/2})^{2/3} = x$, it is not true that $(x^{2/3})^{3/2} = x$. If we substitute $x = -1$ and apply Definition 0.18, we find $(-1)^{2/3} = \left(\sqrt[3]{-1}\right)^2 = (-1)^2 = 1$ so that $\left((-1)^{2/3}\right)^{3/2} = 1^{3/2} = \left(\sqrt{1}\right)^3 = 1^3 = 1$. We see in this case that $\left(x^{2/3}\right)^{3/2} \neq x$. In fact,

$$\left(x^{2/3}\right)^{3/2} = \left(\left(\sqrt[3]{x}\right)^2\right)^{3/2} = \left(\sqrt{\left(\sqrt[3]{x}\right)^2}\right)^3 = \left(\left|\sqrt[3]{x}\right|\right)^3 = \left|\left(\sqrt[3]{x}\right)^3\right| = |x|$$

In the play-by-play analysis, we see that when we canceled the 2's in multiplying $\frac{2}{3} \cdot \frac{3}{2}$, we were, in fact, attempting to cancel a square with a square root. Recall that $\sqrt{x^2} = |x|$. The moral of the story is that when simplifying fractional exponents, it's often best to rewrite them as radicals. However, in most other cases, rational exponents are preferred.

Example 0.9.5. Solve the following equations.

1. $x^{2/3} = x^{4/3} - 6$
2. $3(2-x)^{1/3} = x(2-x)^{-2/3}$

Solution.

1. To solve $x^{2/3} = x^{4/3} - 6$, we get 0 on one side and attempt to solve $x^{4/3} - x^{2/3} - 6 = 0$. Since there are three terms, and the exponent on one of the variable terms, $x^{4/3}$, is exactly twice that of the other, $x^{2/3}$, we have ourselves a 'quadratic in disguise' and we can rewrite the equation as $\left(x^{2/3}\right)^2 - x^{2/3} - 6 = 0$. If we let $u = x^{2/3}$, then in terms of u, we get $u^2 - u - 6 = 0$. Solving for u, we obtain $u = -2$ or $u = 3$. Replacing $x^{2/3}$ back in for u, we get $x^{2/3} = -2$ or $x^{2/3} = 3$. To avoid the trouble we encountered in the discussion following Definition 0.18, we now convert back to radical notation. By interpreting $x^{2/3}$ as $\sqrt[3]{x^2}$ we have $\sqrt[3]{x^2} = -2$ or $\sqrt[3]{x^2} = 3$. Cubing both sides of these equations results in $x^2 = -8$, which admits no real solution, or $x^2 = 27$, which gives the solutions $x = \pm 3\sqrt{3}$.

2. To solve $3(2-x)^{1/3} = x(2-x)^{-2/3}$, we gather all the nonzero terms on one side and obtain $3(2-x)^{1/3} - x(2-x)^{-2/3} = 0$. We set $r(x) = 3(2-x)^{1/3} - x(2-x)^{-2/3}$. As in the previous problem, the denominators of the rational exponents are odd, which means $(2-x)$ can take positive or negative values without concerns. However, the negative exponent on the second term indicates a denominator. Rewriting $r(x)$ with positive exponents, we obtain

$$r(x) = 3(2-x)^{1/3} - \frac{x}{(2-x)^{2/3}}.$$

Setting the denominator equal to zero we get $(2-x)^{2/3} = 0$, or $\sqrt[3]{(2-x)^2} = 0$. After cubing both sides, and subsequently taking square roots, we get $2 - x = 0$, or $x = 2$. Hence, x can only take values in $(-\infty, 2) \cup (2, \infty)$. There are two school of thought on how to solve $r(x) = 0$ and we demonstrate both.

- *Factoring Approach.* From $r(x) = 3(2-x)^{1/3} - x(2-x)^{-2/3}$, we note that the quantity $(2-x)$ is common to both terms. When we factor out common factors, we factor out the quantity with the *smaller* exponent. In this case, since $-\frac{2}{3} < \frac{1}{3}$, we factor $(2-x)^{-2/3}$ from both quantities. While it may seem odd to do so, we need to factor $(2-x)^{-2/3}$ *from* $(2-x)^{1/3}$, which results in subtracting the exponent $-\frac{2}{3}$ from $\frac{1}{3}$. We proceed using the usual properties of exponents.

$$\begin{aligned} r(x) &= 3(2-x)^{1/3} - x(2-x)^{-2/3} \\ &= (2-x)^{-2/3}\left[3(2-x)^{\frac{1}{3}-\left(-\frac{2}{3}\right)} - x\right] \\ &= (2-x)^{-2/3}\left[3(2-x)^{3/3} - x\right] \\ &= (2-x)^{-2/3}\left[3(2-x) - x\right] \\ &= (2-x)^{-2/3}(6-4x) \end{aligned}$$

  To solve $r(x) = 0$, we set $(2-x)^{-2/3}(6-4x) = 0$ and conclude $6 - 4x = 0$, so $x = \frac{3}{2}$.

- *Common Denominator Approach.* We rewrite

$$\begin{aligned} r(x) &= 3(2-x)^{1/3} - \frac{x}{(2-x)^{2/3}} \\ &= \frac{3(2-x)^{1/3}(2-x)^{2/3}}{(2-x)^{2/3}} - \frac{x}{(2-x)^{2/3}} \quad \text{common denominator} \\ &= \frac{3(2-x)^{\frac{1}{3}+\frac{2}{3}}}{(2-x)^{2/3}} - \frac{x}{(2-x)^{2/3}} \\ &= \frac{3(2-x) - x}{(2-x)^{2/3}} = \frac{6-4x}{(2-x)^{2/3}} \end{aligned}$$

  As before, when we set $r(x) = 0$, we obtain $x = \frac{3}{2}$.

0.9.3 Exercises

In Exercises 1 - 17, perform the indicated operations and simplify.

1. $\sqrt{9x^2}$
2. $\sqrt[3]{8t^3}$
3. $\sqrt{50y^6}$
4. $\sqrt[3]{-64x^3}$
5. $\sqrt{81ab^5}$
6. $\sqrt{75x^6y}$
7. $\dfrac{\sqrt{54y^3}}{\sqrt{3y}}$
8. $\dfrac{\sqrt[3]{54x^5}}{\sqrt[3]{2x^{-1}}}$
9. $\sqrt[5]{8w^4} \cdot \sqrt[5]{12w}$
10. $\sqrt{4t^2 + 4t + 1}$
11. $\sqrt{w^2 - 16w + 64}$
12. $\sqrt{(\sqrt{12x} - \sqrt{3x})^2 + 1}$
13. $\sqrt{\dfrac{c^2 - v^2}{c^2}}$
14. $\sqrt[3]{\dfrac{24\pi r^5}{L^3}}$
15. $\sqrt[4]{\dfrac{32\pi\varepsilon^8}{\rho^{12}}}$
16. $\sqrt{x} - \dfrac{x+1}{\sqrt{x}}$
17. $3\sqrt{1+t^2} + \left(\dfrac{1}{\sqrt{1+t^2}}\right)(-3t^2)$

In Exercises 18 - 29, find all real solutions.

18. $(2x+1)^3 + 8 = 0$
19. $\dfrac{(1-2y)^4}{3} = 27$
20. $\dfrac{1}{1+2t^3} = 4$
21. $\sqrt{3x+1} = 4$
22. $5 - \sqrt[3]{t^2+1} = 1$
23. $x + 1 = \sqrt{3x+7}$
24. $y + \sqrt{3y+10} = -2$
25. $3t + \sqrt{6-9t} = 2$
26. $2x - 1 = \sqrt{x+3}$
27. $w = \sqrt[4]{12 - w^2}$
28. $\sqrt{x-2} + \sqrt{x-5} = 3$
29. $\sqrt{2x+1} = 3 + \sqrt{4-x}$

In Exercises 30 - 33, solve each equation for the indicated variable. Assume all quantities represent positive real numbers.

30. Solve for h: $I = \dfrac{bh^3}{12}$.
31. Solve for a: $I_0 = \dfrac{5\sqrt{3}a^4}{16}$
32. Solve for g: $T = 2\pi\sqrt{\dfrac{L}{g}}$
33. Solve for v: $L = L_0\sqrt{1 - \dfrac{v^2}{c^2}}$.

In Exercises 34 - 46, rationalize the numerator or denominator, and simplify.

34. $\dfrac{1}{\sqrt{3}}$
35. $\dfrac{2\sqrt{6}}{3}$
36. $\dfrac{3}{\sqrt{7}-2}$
37. $\dfrac{4}{3-\sqrt{2}}$

38. $\dfrac{6}{2\sqrt{3}-\sqrt{10}}$
39. $\dfrac{\sqrt{5}+1}{2}$
40. $\dfrac{6}{\sqrt[3]{24x^7}}$
41. $\dfrac{6}{\sqrt[3]{9b^2}}$
42. $\dfrac{\sqrt{x}-2}{x-4}$
43. $\dfrac{\sqrt[3]{x+1}-2}{x-7}$
44. $\dfrac{\sqrt{x}-\sqrt{c}}{x-c}$
45. $\dfrac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h}$
46. $\dfrac{\sqrt{2x+2h+1}-\sqrt{2x+1}}{h}$

In Exercises 47 - 58, find all real solutions of the equation.

47. $x^{\frac{3}{2}}=8$
48. $x^{\frac{2}{3}}=4$
49. $4x^{\frac{3}{5}}-1=7$
50. $5-(4-2x)^{\frac{2}{3}}=1$
51. $x^{\frac{2}{7}}+4=4x^{\frac{1}{7}}$
52. $x^{\frac{5}{3}}+4x^{\frac{2}{3}}-21x^{-\frac{1}{3}}=0$
53. $(x+1)^{\frac{6}{5}}-(x+1)^{\frac{1}{5}}=0$
54. $(x-1)^{\frac{3}{2}}+4(x-1)^{\frac{1}{2}}=0$
55. $3(x-2)^{\frac{4}{3}}=x(x-2)^{\frac{1}{3}}$
56. $x^{\frac{7}{3}}-x^{\frac{1}{3}}=0$
57. $x^{\frac{1}{2}}(x^2-4)=0$
58. $x^{-\frac{1}{2}}(x+5)^2=0$

59. The National Weather Service uses the following formula to calculate the wind chill:

$$W=35.74+0.6215\,T_a-35.75\,V^{0.16}+0.4275\,T_a\,V^{0.16},$$

where W is the wind chill temperature in °F, T_a is the air temperature in °F, and V is the wind speed in miles per hour. Note that W is defined only for air temperatures at or lower than 50°F and wind speeds above 3 miles per hour.

(a) Suppose the air temperature is 42° and the wind speed is 7 miles per hour. Find the wind chill temperature. Round your answer to two decimal places.

(b) Suppose the air temperature is 37°F and the wind chill temperature is 30°F. Find the wind speed. Round your answer to two decimal places.

60. The Cobb-Douglas production model states that the yearly total dollar value of the production output P in an economy is a function of labor x (the total number of hours worked in a year) and capital y (the total dollar value of all of the stuff purchased in order to make things). Specifically, $P=ax^by^{1-b}$. By fixing P, we create what's known as an 'isoquant' and we can then solve for y as a function of x. Assume that the Cobb-Douglas production model for the country of Sasquatchia is $P=1.25x^{0.4}y^{0.6}$. If $P=300$ and $x=100$, what is y?

0.9.4 Answers

1. $3|x|$
2. $2t$
3. $5\sqrt{2}|y^3|$
4. $-4x$
5. $9b^2\sqrt{ab}$
6. $5|x|^3\sqrt{3y}$
7. $3\sqrt{2}\,y$
8. $3x^2$
9. $2w\sqrt[5]{3}$
10. $|2t+1|$
11. $|w-8|$
12. $\sqrt{3x+1}$
13. $\dfrac{\sqrt{c^2-v^2}}{|c|}$
14. $\dfrac{2r\sqrt[3]{3\pi r^2}}{L}$
15. $\dfrac{2\varepsilon^2\sqrt[4]{2\pi}}{|\rho^3|}$
16. $-\dfrac{1}{\sqrt{x}}$
17. $\dfrac{3}{\sqrt{1+t^2}}$
18. $x=-\dfrac{3}{2}$
19. $y=-1,2$
20. $t=-\dfrac{\sqrt[3]{3}}{2}$
21. $x=5$
22. $t=\pm 3\sqrt{7}$
23. $x=3$
24. $y=-3$
25. $t=-\dfrac{1}{3},\dfrac{2}{3}$
26. $x=\dfrac{5+\sqrt{57}}{8}$
27. $w=\sqrt{3}$
28. $x=6$
29. $x=4$
30. $h=\sqrt[3]{\dfrac{12I}{b}}$
31. $a=\dfrac{2\sqrt[4]{I_0}}{\sqrt[4]{5\sqrt{3}}}$
32. $g=\dfrac{4\pi^2 L}{T^2}$
33. $v=\dfrac{c\sqrt{L_0^2-L^2}}{L_0}$
34. $\dfrac{\sqrt{3}}{3}$
35. $\dfrac{4}{\sqrt{6}}$
36. $\sqrt{7}+2$
37. $\dfrac{4}{7}(3+\sqrt{2})$
38. $6\sqrt{3}+3\sqrt{10}$
39. $\dfrac{2}{\sqrt{5}-1}$
40. $\dfrac{\sqrt[3]{9x^2}}{x^3}$
41. $\dfrac{2\sqrt[3]{3b}}{b}$
42. $\dfrac{1}{\sqrt{x}+2}$
43. $\dfrac{1}{(\sqrt[3]{x+1})^2+2\sqrt[3]{x+1}+4}$
44. $\dfrac{1}{\sqrt{x}+\sqrt{c}}$
45. $\dfrac{1}{(\sqrt[3]{x+h})^2+\sqrt[3]{x+h}\,\sqrt[3]{x}+(\sqrt[3]{x})^2}$
46. $\dfrac{2}{\sqrt{2x+2h+1}+\sqrt{2x+1}}$

47. $x = 4$
48. $x = \pm 8$
49. $x = 2\sqrt[3]{4}$
50. $x = -2, 6$
51. $x = 128$
52. $x = -7, 3$
53. $x = -1, 0$
54. $x = 1$
55. $x = 2, 3$
56. $x = -1, 0, 1$
57. $x = 0, 2$
58. No solution.

59. (a) $W \approx 37.55^{\circ}$F.
(b) $V \approx 9.84$ miles per hour.

60. First rewrite the model as $P = 1.25x^{\frac{2}{5}}y^{\frac{3}{5}}$. Then $300 = 1.25x^{\frac{2}{5}}y^{\frac{3}{5}}$ yields $y = \left(\frac{300}{1.25x^{\frac{2}{5}}}\right)^{\frac{5}{3}}$. If $x = 100$ then $y \approx 430.21486$.

Chapter 1

Relations and Functions

1.1 The Cartesian Coordinate Plane

1.1.1 The Cartesian Coordinate Plane

In order to visualize the pure excitement that is Precalculus, we need to unite Algebra and Geometry. Simply put, we must find a way to draw algebraic things. Let's start with possibly the greatest mathematical achievement of all time: the **Cartesian Coordinate Plane**.[1]

Imagine two real number lines crossing at a right angle at 0. The horizontal number line is usually called the ***x*-axis** while the vertical number line is usually called the ***y*-axis**.[2] As with the usual number line, we imagine these axes extending off indefinitely in both directions.[3] Having two number lines allows us to locate the positions of points off of the number lines as well as points on the lines themselves.

For example, consider the point P on the next page. To use the numbers on the axes to label this point, we imagine dropping a vertical line from the x-axis to P and extending a horizontal line from the y-axis to P. This process is sometimes called 'projecting' the point P to the x- (respectively y-) axis. We then describe the point P using the **ordered pair** $(2,-4)$. The first number in the ordered pair is called the **abscissa** or ***x*-coordinate** and the second is called the **ordinate** or ***y*-coordinate**.[4] Taken together, the ordered pair $(2,-4)$ comprise the **Cartesian coordinates** of the point P. In practice, the distinction between a point and its coordinates is blurred; for example, we often speak of 'the point $(2,-4)$.' We can think of $(2,-4)$ as instructions on how to reach P from the **origin** $(0,0)$ by moving 2 units to the right and 4 units downwards. Notice that the order in the ordered pair is important – if we wish to plot the point $(-4,2)$, we would move to the left 4 units

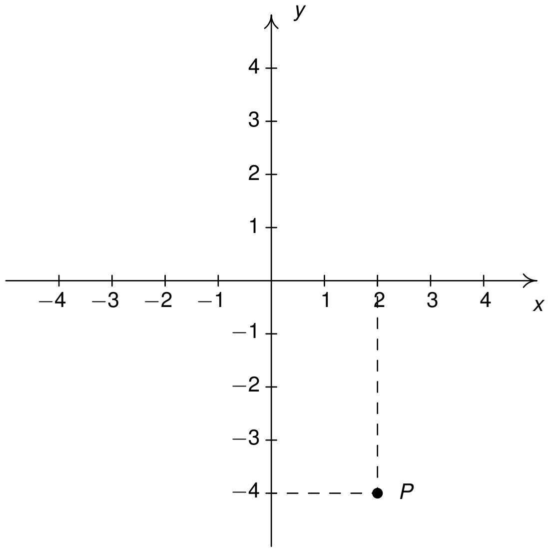
[1] So named in honor of René Descartes.

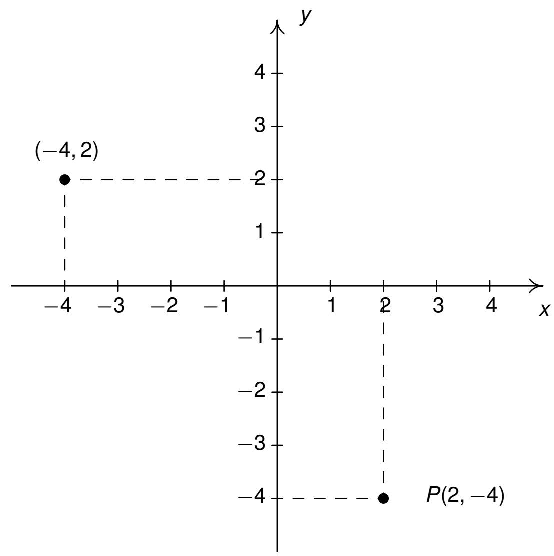
[2] The labels can vary depending on the context of application.

[3] Usually extending off towards infinity is indicated by arrows, but here, the arrows are used to indicate the *direction* of increasing values of x and y.

[4] Again, the names of the coordinates can vary depending on the context of the application. If, for example, the horizontal axis represented time we might choose to call it the t-axis. The first number in the ordered pair would then be the t-coordinate.

from the origin and then move upwards 2 units, as below on the right.





When we speak of the Cartesian Coordinate Plane, we mean the set of all possible ordered pairs (x, y) as x and y take values from the real numbers. Below is a summary of important facts about Cartesian coordinates.

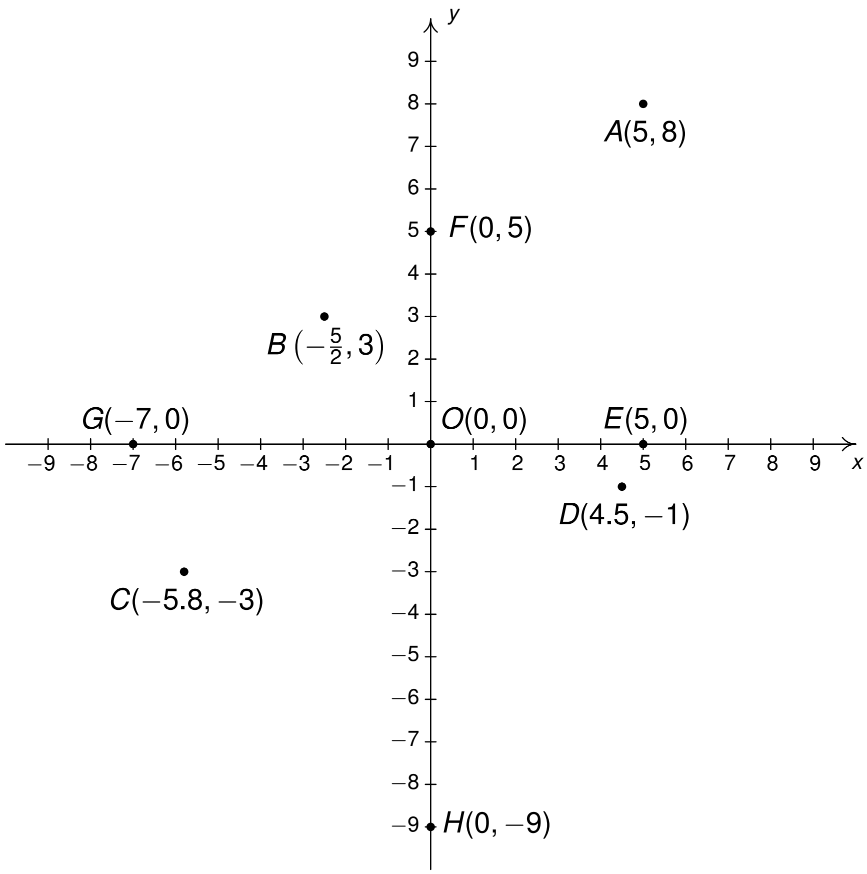
Important Facts about the Cartesian Coordinate Plane

- (a, b) and (c, d) represent the same point in the plane if and only if $a = c$ and $b = d$.
- (x, y) lies on the x-axis if and only if $y = 0$.
- (x, y) lies on the y-axis if and only if $x = 0$.
- The origin is the point $(0, 0)$. It is the only point common to both axes.

Example 1.1.1. Plot the following points: $A(5, 8)$, $B\left(-\frac{5}{2}, 3\right)$, $C(-5.8, -3)$, $D(4.5, -1)$, $E(5, 0)$, $F(0, 5)$, $G(-7, 0)$, $H(0, -9)$, $O(0, 0)$.[5]

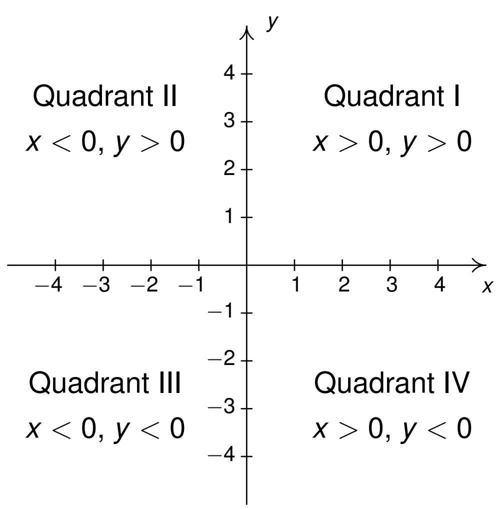
Solution. To plot these points, we start at the origin and move to the right if the x-coordinate is positive; to the left if it is negative. Next, we move up if the y-coordinate is positive or down if it is negative. If the x-coordinate is 0, we start at the origin and move along the y-axis only. If the y-coordinate is 0 we move along the x-axis only.

[5] The letter O is almost always reserved for the origin.



□

The axes divide the plane into four regions called **quadrants**. They are labeled with Roman numerals and proceed counterclockwise around the plane:



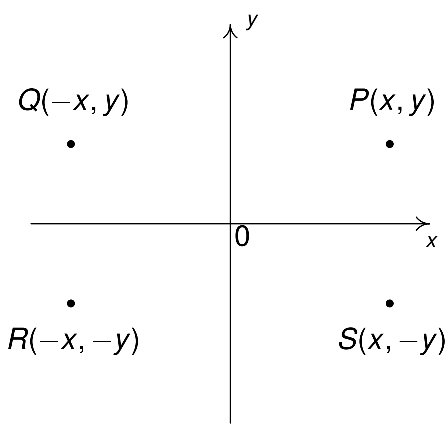
For example, $(1, 2)$ lies in Quadrant I, $(-1, 2)$ in Quadrant II, $(-1, -2)$ in Quadrant III and $(1, -2)$ in Quadrant IV. If a point other than the origin happens to lie on the axes, we typically refer to that point as lying on the positive or negative x-axis (if $y = 0$) or on the positive or negative y-axis (if $x = 0$). For example, $(0, 4)$ lies on the positive y-axis whereas $(-117, 0)$ lies on the negative x-axis. Such points do not belong to any of the four quadrants.

One of the most important concepts in all of Mathematics is **symmetry**.[6] There are many types of symmetry in Mathematics, but three of them can be discussed easily using Cartesian Coordinates.

Definition 1.1. Two points (a, b) and (c, d) in the plane are said to be

- **symmetric about the *x*-axis** if $a = c$ and $b = -d$
- **symmetric about the *y*-axis** if $a = -c$ and $b = d$
- **symmetric about the origin** if $a = -c$ and $b = -d$

Schematically,



In the above figure, P and S are symmetric about the x-axis, as are Q and R; P and Q are symmetric about the y-axis, as are R and S; and P and R are symmetric about the origin, as are Q and S.

Example 1.1.2. Let P be the point $(-2, 3)$. Find the points which are symmetric to P about the:

1. x-axis
2. y-axis
3. origin

Check your answer by plotting the points.

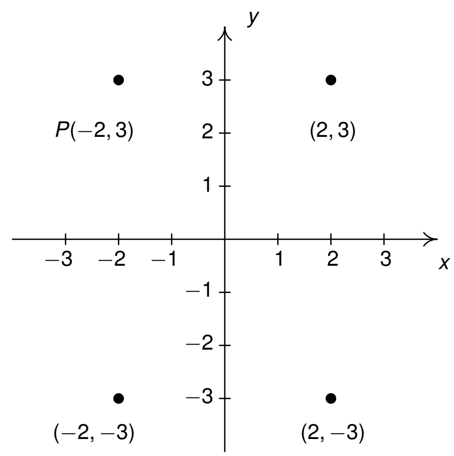
Solution. The figure after Definition 1.1 gives us a good way to think about finding symmetric points in terms of taking the opposites of the x- and/or y-coordinates of $P(-2, 3)$.

[6] According to Carl. Jeff thinks symmetry is overrated.

1. To find the point symmetric about the x-axis, we replace the y-coordinate with its opposite to get $(-2,-3)$.

2. To find the point symmetric about the y-axis, we replace the x-coordinate with its opposite to get $(2,3)$.

3. To find the point symmetric about the origin, we replace the x- and y-coordinates with their opposites to get $(2,-3)$.



□

One way to visualize the processes in the previous example is with the concept of a **reflection**. If we start with our point $(-2,3)$ and pretend that the x-axis is a mirror, then the reflection of $(-2,3)$ across the x-axis would lie at $(-2,-3)$. If we pretend that the y-axis is a mirror, the reflection of $(-2,3)$ across that axis would be $(2,3)$. If we reflect across the x-axis and then the y-axis, we would go from $(-2,3)$ to $(-2,-3)$ then to $(2,-3)$, and so we would end up at the point symmetric to $(-2,3)$ about the origin. We summarize and generalize this process below.

Reflections

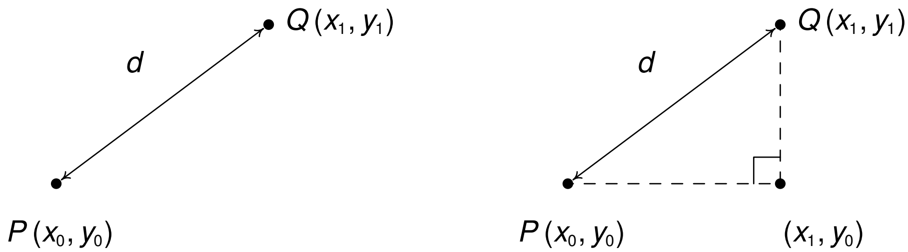
To reflect a point (x,y) about the:

- x-axis, replace y with $-y$.
- y-axis, replace x with $-x$.
- origin, replace x with $-x$ and y with $-y$.

1.1.2 Distance in the Plane

Another important concept in Geometry is the notion of length. If we are going to unite Algebra and Geometry using the Cartesian Plane, then we need to develop an algebraic understanding of what distance in the plane means. Suppose we have two points, $P(x_0,y_0)$ and $Q(x_1,y_1)$, in the plane. By the **distance** d between P and Q, we mean the length of the line segment joining P with

Q. (Remember, given any two distinct points in the plane, there is a unique line containing both points.) Our goal now is to create an algebraic formula to compute the distance between these two points. Consider the generic situation below on the left.



With a little more imagination, we can envision a right triangle whose hypotenuse has length d as drawn above on the right. From the latter figure, we see that the lengths of the legs of the triangle are $|x_1 - x_0|$ and $|y_1 - y_0|$ so the Pythagorean Theorem gives us

$$|x_1 - x_0|^2 + |y_1 - y_0|^2 = d^2$$
$$(x_1 - x_0)^2 + (y_1 - y_0)^2 = d^2$$

(Do you remember why we can replace the absolute value notation with parentheses?) By extracting the square root of both sides of the second equation and using the fact that distance is never negative, we get

Equation 1.1. The Distance Formula: The distance d between the points $P\,(x_0, y_0)$ and $Q\,(x_1, y_1)$ is:

$$d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

It is not always the case that the points P and Q lend themselves to constructing such a triangle. If the points P and Q are arranged vertically or horizontally, or describe the exact same point, we cannot use the above geometric argument to derive the distance formula. It is left to the reader in Exercise 16 to verify Equation 1.1 for these cases.

Example 1.1.3. Find and simplify the distance between $P(-2, 3)$ and $Q(1, -3)$.

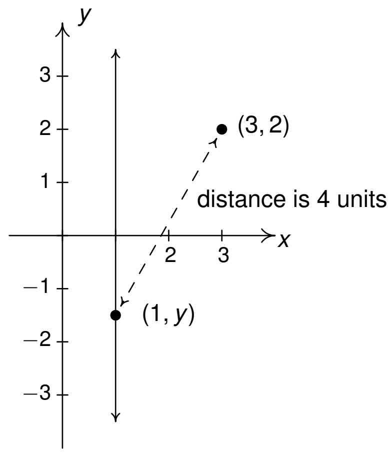
Solution.

$$\begin{aligned} d &= \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \\ &= \sqrt{(1 - (-2))^2 + (-3 - 3)^2} \\ &= \sqrt{9 + 36} \\ &= 3\sqrt{5} \end{aligned}$$

So the distance is $3\sqrt{5}$. □

Example 1.1.4. Find all of the points with x-coordinate 1 which are 4 units from the point $(3, 2)$.

Solution. We shall soon see that the points we wish to find are on the line $x = 1$, but for now we'll just view them as points of the form $(1, y)$. Visually,

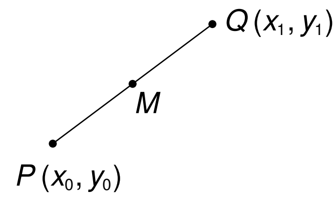


We require that the distance from $(3, 2)$ to $(1, y)$ be 4. The Distance Formula, Equation 1.1, yields

$$\begin{aligned}
d &= \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \\
4 &= \sqrt{(1-3)^2 + (y-2)^2} \\
4 &= \sqrt{4 + (y-2)^2} \\
4^2 &= \left(\sqrt{4 + (y-2)^2}\right)^2 && \text{squaring both sides} \\
16 &= 4 + (y-2)^2 \\
12 &= (y-2)^2 \\
(y-2)^2 &= 12 \\
y - 2 &= \pm\sqrt{12} && \text{extracting the square root} \\
y - 2 &= \pm 2\sqrt{3} \\
y &= 2 \pm 2\sqrt{3}
\end{aligned}$$

We obtain two answers: $(1, 2 + 2\sqrt{3})$ and $(1, 2 - 2\sqrt{3})$. The reader is encouraged to think about why there are two answers. □

Related to finding the distance between two points is the problem of finding the **midpoint** of the line segment connecting two points. Given two points, $P(x_0, y_0)$ and $Q(x_1, y_1)$, the **midpoint** M of P and Q is defined to be the point on the line segment connecting P and Q whose distance from P is equal to its distance from Q.



If we think of reaching M by going 'halfway over' and 'halfway up' we get the following formula.

Equation 1.2. The Midpoint Formula: The midpoint M of the line segment connecting $P\,(x_0, y_0)$ and $Q\,(x_1, y_1)$ is:

$$M = \left(\frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2}\right)$$

If we let d denote the distance between P and Q, we leave it as Exercise 17 to show that the distance between P and M is $d/2$ which is the same as the distance between M and Q. This suffices to show that Equation 1.2 gives the coordinates of the midpoint.

Example 1.1.5. Find the midpoint of the line segment connecting $P(-2, 3)$ and $Q(1, -3)$.

Solution.

$$\begin{aligned} M &= \left(\frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2}\right) \\ &= \left(\frac{(-2) + 1}{2}, \frac{3 + (-3)}{2}\right) = \left(-\frac{1}{2}, \frac{0}{2}\right) \\ &= \left(-\frac{1}{2}, 0\right) \end{aligned}$$

The midpoint is $\left(-\frac{1}{2}, 0\right)$. □

We close with a more abstract application of the Midpoint Formula. We will revisit the following example in Exercise 72 in Section 2.1.

Example 1.1.6. If $a \neq b$, prove that the line $y = x$ equally divides the line segment with endpoints (a, b) and (b, a).

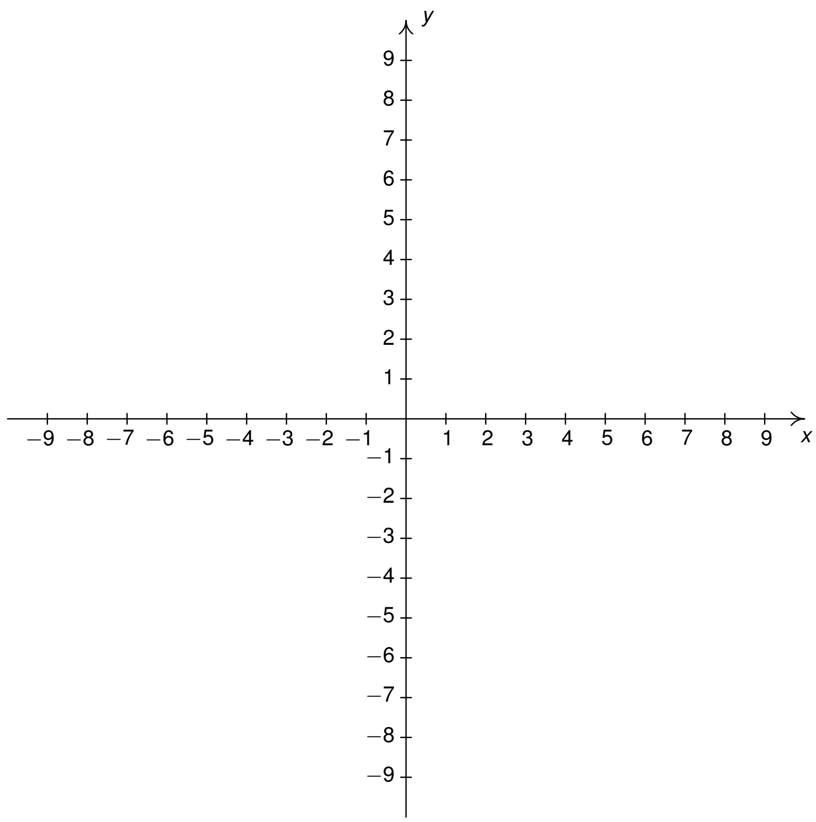
Solution. To prove the claim, we use Equation 1.2 to find the midpoint

$$\begin{aligned} M &= \left(\frac{a + b}{2}, \frac{b + a}{2}\right) \\ &= \left(\frac{a + b}{2}, \frac{a + b}{2}\right) \end{aligned}$$

Since the x and y coordinates of this point are the same, we find that the midpoint lies on the line $y = x$, as required. □

1.1.3 Exercises

1. Plot (approx.) and label the points $A(-3,-7)$, $B(1.3,-2)$, $C(\pi,\sqrt{10})$, $D(0,8)$, $E(-5.5,0)$, $F(-8,4)$, $G(9.2,-7.8)$ and $H(7,5)$ in the Cartesian Coordinate Plane given below.



2. For each point given in Exercise 1 above

   - Identify the quadrant or axis in/on which the point lies.
   - Find the point symmetric to the given point about the x-axis.
   - Find the point symmetric to the given point about the y-axis.
   - Find the point symmetric to the given point about the origin.

In Exercises 3 - 10, find the distance d between the points and the midpoint M of the line segment which connects them.

3. $(1,2)$, $(-3,5)$

4. $(3,-10)$, $(-1,2)$

5. $\left(\frac{1}{2}, 4\right)$, $\left(\frac{3}{2}, -1\right)$

6. $\left(-\frac{2}{3}, \frac{3}{2}\right)$, $\left(\frac{7}{3}, 2\right)$

7. $\left(\frac{24}{5}, \frac{6}{5}\right)$, $\left(-\frac{11}{5}, -\frac{19}{5}\right)$.

8. $\left(\sqrt{2}, \sqrt{3}\right)$, $\left(-\sqrt{8}, -\sqrt{12}\right)$

9. $\left(2\sqrt{45}, \sqrt{12}\right)$, $\left(\sqrt{20}, \sqrt{27}\right)$.

10. $(0, 0)$, (x, y)

11. Find all of the points of the form $(x, -1)$ which are 4 units from the point $(3, 2)$.

12. Find all of the points on the y-axis which are 5 units from the point $(-5, 3)$.

13. Find all of the points on the x-axis which are 2 units from the point $(-1, 1)$.

14. Find all of the points of the form $(x, -x)$ which are 1 unit from the origin.

15. Let's assume for a moment that we are standing at the origin and the positive y-axis points due North while the positive x-axis points due East. Our Sasquatch-o-meter tells us that Sasquatch is 3 miles West and 4 miles South of our current position. What are the coordinates of his position? How far away is he from us? If he runs 7 miles due East what would his new position be?

16. Verify the Distance Formula 1.1 for the cases when:

    (a) The points are arranged vertically. (Hint: Use $P(a, y_0)$ and $Q(a, y_1)$.)

    (b) The points are arranged horizontally. (Hint: Use $P(x_0, b)$ and $Q(x_1, b)$.)

    (c) The points are actually the same point. (You shouldn't need a hint for this one.)

17. Verify the Midpoint Formula by showing the distance between $P(x_1, y_1)$ and M and the distance between M and $Q(x_2, y_2)$ are both half of the distance between P and Q.

18. Show that the points A, B and C below are the vertices of a right triangle.

    (a) $A(-3, 2)$, $B(-6, 4)$, and $C(1, 8)$

    (b) $A(-3, 1)$, $B(4, 0)$ and $C(0, -3)$

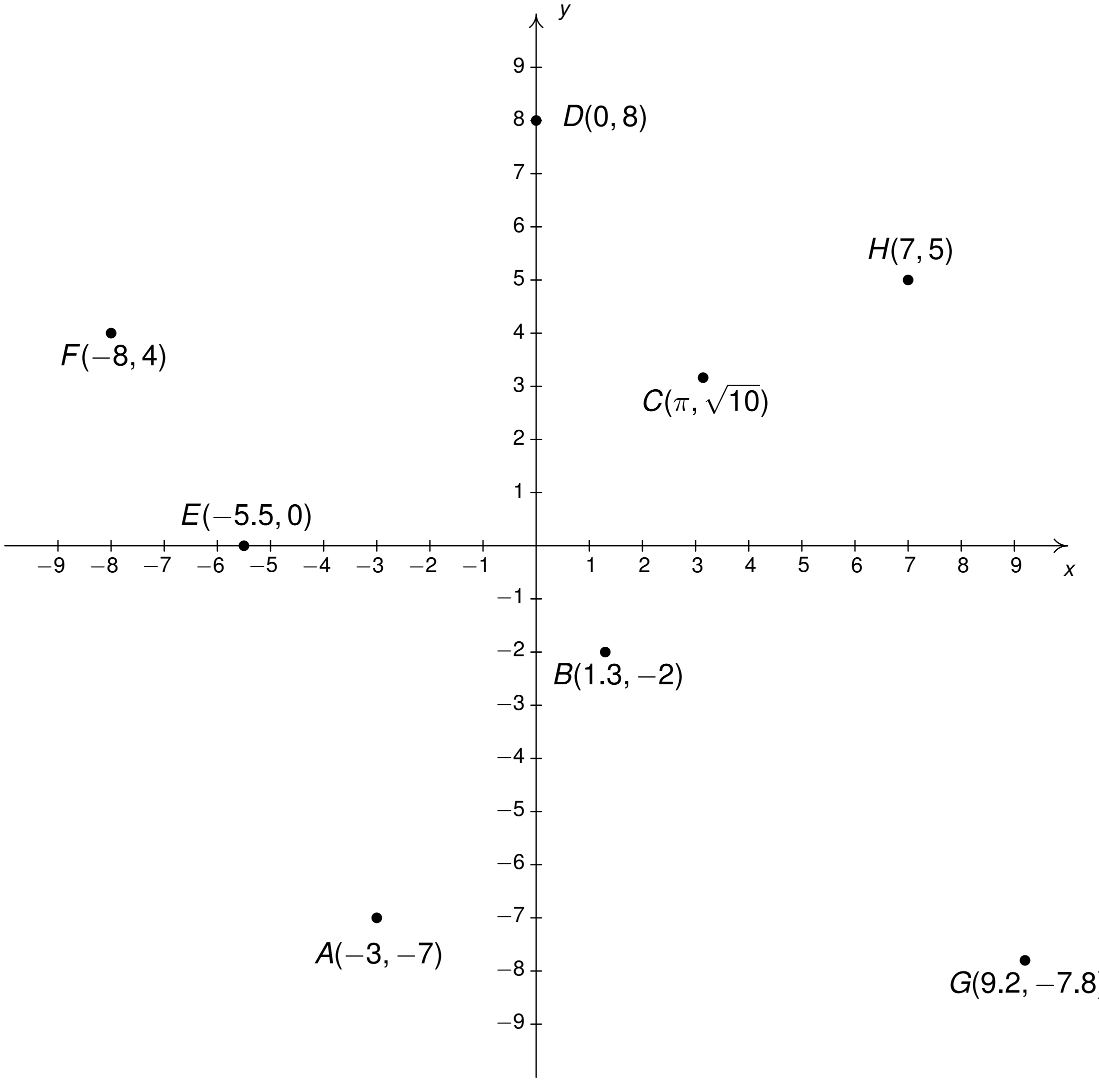
19. Find a point $D(x, y)$ such that the points $A(-3, 1)$, $B(4, 0)$, $C(0, -3)$ and D are the corners of a square. Justify your answer.

20. The world is not flat.[7] Thus the Cartesian Plane cannot possibly be the end of the story. Discuss with your classmates how you would extend Cartesian Coordinates to represent the three dimensional world. What would the Distance and Midpoint formulas look like, assuming those concepts make sense at all?

[7] There are those who disagree with this statement. Look them up on the Internet some time when you're bored.

1.1.4 Answers

1. The required points $A(-3,-7)$, $B(1.3,-2)$, $C(\pi,\sqrt{10})$, $D(0,8)$, $E(-5.5,0)$, $F(-8,4)$, $G(9.2,-7.8)$, and $H(7,5)$ are plotted in the Cartesian Coordinate Plane below.



2. (a) The point $A(-3,-7)$ is
   - in Quadrant III
   - symmetric about x-axis with $(-3,7)$
   - symmetric about y-axis with $(3,-7)$
   - symmetric about origin with $(3,7)$

   (b) The point $B(1.3,-2)$ is
   - in Quadrant IV
   - symmetric about x-axis with $(1.3,2)$
   - symmetric about y-axis with $(-1.3,-2)$
   - symmetric about origin with $(-1.3,2)$

(c) The point $C(\pi, \sqrt{10})$ is

- in Quadrant I
- symmetric about x-axis with $(\pi, -\sqrt{10})$
- symmetric about y-axis with $(-\pi, \sqrt{10})$
- symmetric about origin with $(-\pi, -\sqrt{10})$

(d) The point $D(0, 8)$ is

- on the positive y-axis
- symmetric about x-axis with $(0, -8)$
- symmetric about y-axis with $(0, 8)$
- symmetric about origin with $(0, -8)$

(e) The point $E(-5.5, 0)$ is

- on the negative x-axis
- symmetric about x-axis with $(-5.5, 0)$
- symmetric about y-axis with $(5.5, 0)$
- symmetric about origin with $(5.5, 0)$

(f) The point $F(-8, 4)$ is

- in Quadrant II
- symmetric about x-axis with $(-8, -4)$
- symmetric about y-axis with $(8, 4)$
- symmetric about origin with $(8, -4)$

(g) The point $G(9.2, -7.8)$ is

- in Quadrant IV
- symmetric about x-axis with $(9.2, 7.8)$
- symmetric about y-axis with $(-9.2, -7.8)$
- symmetric about origin with $(-9.2, 7.8)$

(h) The point $H(7, 5)$ is

- in Quadrant I
- symmetric about x-axis with $(7, -5)$
- symmetric about y-axis with $(-7, 5)$
- symmetric about origin with $(-7, -5)$

3. $d = 5$, $M = \left(-1, \frac{7}{2}\right)$
4. $d = 4\sqrt{10}$, $M = (1, -4)$
5. $d = \sqrt{26}$, $M = \left(1, \frac{3}{2}\right)$
6. $d = \frac{\sqrt{37}}{2}$, $M = \left(\frac{5}{6}, \frac{7}{4}\right)$
7. $d = \sqrt{74}$, $M = \left(\frac{13}{10}, -\frac{13}{10}\right)$
8. $d = 3\sqrt{5}$, $M = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{3}}{2}\right)$
9. $d = \sqrt{83}$, $M = \left(4\sqrt{5}, \frac{5\sqrt{3}}{2}\right)$
10. $d = \sqrt{x^2 + y^2}$, $M = \left(\frac{x}{2}, \frac{y}{2}\right)$
11. $(3 + \sqrt{7}, -1)$, $(3 - \sqrt{7}, -1)$
12. $(0, 3)$
13. $(-1 + \sqrt{3}, 0)$, $(-1 - \sqrt{3}, 0)$
14. $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
15. $(-3, -4)$, 5 miles, $(4, -4)$

18. (a) The distance from A to B is $|AB| = \sqrt{13}$, the distance from A to C is $|AC| = \sqrt{52}$, and the distance from B to C is $|BC| = \sqrt{65}$. Since $\left(\sqrt{13}\right)^2 + \left(\sqrt{52}\right)^2 = \left(\sqrt{65}\right)^2$, we are guaranteed by the converse of the Pythagorean Theorem that the triangle is a right triangle.

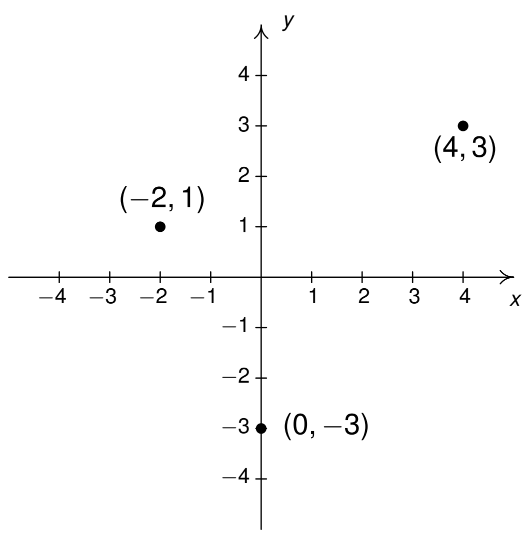
(b) Show that $|AC|^2 + |BC|^2 = |AB|^2$

1.2 Relations

From one point of view,[1] all of Precalculus can be thought of as studying sets of points in the plane. With the Cartesian Plane now fresh in our memory we can discuss those sets in more detail and as usual, we begin with a definition.

Definition 1.2. A **relation** is a set of points in the plane.

Since relations are sets, we can describe them using the techniques presented in Section 0.1. That is, we can describe a relation verbally, using the roster method, or using set-builder notation. Since the elements in a relation are points in the plane, we often try to describe the relation graphically or algebraically as well. Depending on the situation, one method may be easier or more convenient to use than another. As an example, consider the relation $R = \{(-2,1),(4,3),(0,-3)\}$. As written, R is described using the roster method. Since R consists of points in the plane, we follow our instinct and plot the points. Doing so produces the **graph** of R.



The graph of R.

In the following example, we graph a variety of relations.

Example 1.2.1. Graph the following relations.

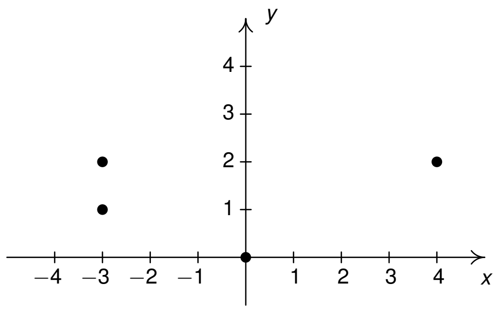
1. $A = \{(0,0),(-3,1),(4,2),(-3,2)\}$
2. $HLS_1 = \{(x,3) \mid -2 \leq x \leq 4\}$
3. $HLS_2 = \{(x,3) \mid -2 \leq x < 4\}$
4. $V = \{(3,y) \mid y \text{ is a real number}\}$
5. $H = \{(x,y) \mid y = -2\}$
6. $R = \{(x,y) \mid 1 < y \leq 3\}$

[1] Carl's, of course.

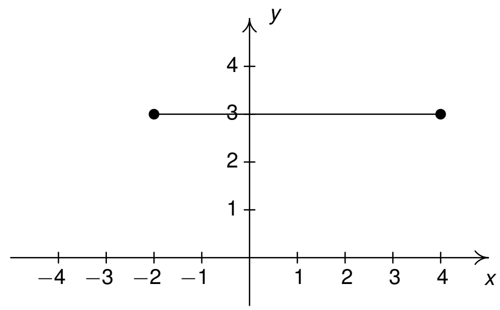
Solution.

1. To graph A, we simply plot all of the points which belong to A, as shown below on the left.

2. Don't let the notation in this part fool you. The name of this relation is HLS_1, just like the name of the relation in number 1 was A. The letters and numbers are just part of its name, just like the numbers and letters of the phrase 'King George III' were part of George's name. In words, $\{(x,3) \mid -2 \leq x \leq 4\}$ reads 'the set of points $(x,3)$ such that $-2 \leq x \leq 4$.' All of these points have the same y-coordinate, 3, but the x-coordinate is allowed to vary between -2 and 4, inclusive. Some of the points which belong to HLS_1 include some friendly points like: $(-2,3)$, $(-1,3)$, $(0,3)$, $(1,3)$, $(2,3)$, $(3,3)$, and $(4,3)$. However, HLS_1 also contains the points $(0.829,3)$, $\left(-\frac{5}{6},3\right)$, $(\sqrt{\pi},3)$, and so on. It is impossible[2] to list all of these points, which is why the variable x is used. Plotting several friendly representative points should convince you that HLS_1 describes the horizontal line segment from the point $(-2,3)$ up to and including the point $(4,3)$.



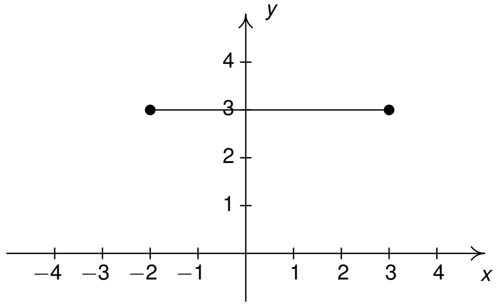
The graph of A



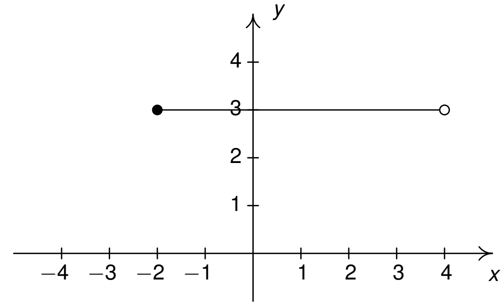
The graph of HLS_1

3. HLS_2 is hauntingly similar to HLS_1. In fact, the only difference between the two is that instead of '$-2 \leq x \leq 4$' we have '$-2 \leq x < 4$'. This means that we still get a horizontal line segment which includes $(-2,3)$ and extends to $(4,3)$, but we do *not* include $(4,3)$ because of the strict inequality $x < 4$. How do we denote this on our graph? It is a common mistake to make the graph start at $(-2,3)$ end at $(3,3)$ as pictured below on the left. The problem with this graph is that we are forgetting about the points like $(3.1,3)$, $(3.5,3)$, $(3.9,3)$, $(3.99,3)$, and so forth. There is no real number that comes 'immediately before' 4, so to describe the set of points we want, we draw the horizontal line segment starting at $(-2,3)$ and draw an open circle at $(4,3)$ as depicted below on the right.

[2]Really impossible. The interested reader is encouraged to research **countable** versus **uncountable** sets.



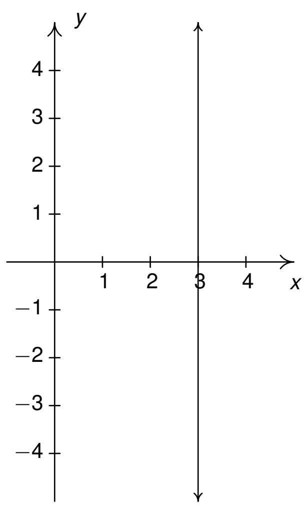
This is NOT the correct graph of HLS_2



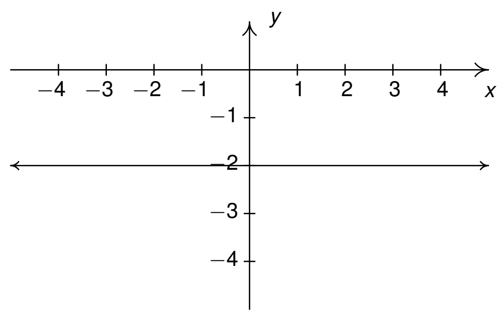
The graph of HLS_2

4. Next, we come to the relation V, described as the set of points $(3, y)$ such that y is a real number. All of these points have an x-coordinate of 3, but the y-coordinate is free to be whatever it wants to be, without restriction.[3] Plotting a few 'friendly' points of V should convince you that all the points of V lie on the vertical line[4] $x = 3$. Since there is no restriction on the y-coordinate, we put arrows on the end of the portion of the line we draw to indicate it extends indefinitely in both directions. The graph of V is below on the left.

5. Though written slightly differently, the relation $H = \{(x, y) \mid y = -2\}$ is similar to the relation V above in that only one of the coordinates, in this case the y-coordinate, is specified, leaving x to be 'free'. Plotting some representative points gives us the horizontal line $y = -2$.



The graph of V

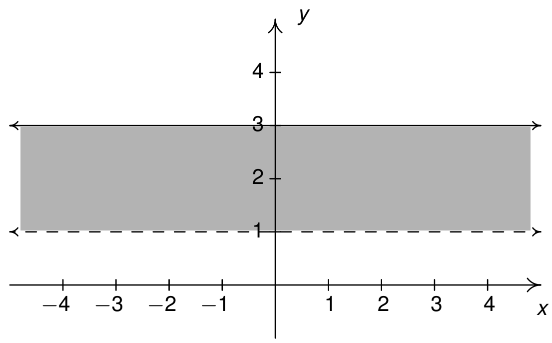


The graph of H

6. For our last example, we turn to $R = \{(x, y) \mid 1 < y \leq 3\}$. As in the previous example, x is free to be whatever it likes. The value of y, on the other hand, while not completely free, is permitted to roam between 1 and 3 excluding 1, but including 3. After plotting some[5] friendly elements of R, it should become clear that R consists of the region between the horizontal

[3]We'll revisit the concept of a 'free variable' in Section 2.2.
[4]Don't worry, we'll be refreshing your memory about vertical and horizontal lines in just a moment!
[5]The word 'some' is a relative term. It may take 5, 10, or 50 points until you see the pattern.

lines $y = 1$ and $y = 3$. Since R requires that the y-coordinates be greater than 1, but not equal to 1, we dash the line $y = 1$ to indicate that those points do not belong to R.



The graph of R □

The relations V and H in the previous example lead us to our final way to describe relations: **algebraically**. We can more succinctly describe the points in V as those points which satisfy the equation '$x = 3$'. Most likely, you have seen equations like this before. Depending on the context, '$x = 3$' could mean we have solved an equation for x and arrived at the solution $x = 3$. In this case, however, '$x = 3$' describes a set of points in the plane whose x-coordinate is 3. Similarly, the relation H above can be described by the equation '$y = -2$'. At some point in your mathematical upbringing, you probably learned the following.

Equations of Vertical and Horizontal Lines

- The graph of the equation $x = a$ is a **vertical line** through $(a, 0)$.
- The graph of the equation $y = b$ is a **horizontal line** through $(0, b)$.

Given that the very simple equations $x = a$ and $y = b$ produced lines, it's natural to wonder what shapes other equations might yield. Thus our next objective is to study the graphs of equations in a more general setting as we continue to unite Algebra and Geometry.

1.2.1 Graphs of Equations

In this section, we delve more deeply into the connection between Algebra and Geometry by focusing on graphing relations described by equations. The main idea of this section is the following.

The Fundamental Graphing Principle
The graph of an equation is the set of points which satisfy the equation. That is, a point (x, y) is on the graph of an equation if and only if x and y satisfy the equation.

Here, 'x and y satisfy the equation' means 'x and y make the equation true'. It is at this point that we gain some insight into the word 'relation'. If the equation to be graphed contains both x and y, then the equation itself is what is relating the two variables. More specifically, in the next two examples, we consider the graph of the equation $x^2 + y^3 = 1$. Even though it is not specifically spelled out, what we are doing is graphing the relation $R = \{(x, y) \mid x^2 + y^3 = 1\}$. The points (x, y) we graph belong to the *relation R* and are necessarily *related* by the equation $x^2 + y^3 = 1$, since it is those pairs of x and y which make the equation true.

Example 1.2.2. Determine whether or not $(2, -1)$ is on the graph of $x^2 + y^3 = 1$.

Solution. We substitute $x = 2$ and $y = -1$ into the equation to see if the equation is satisfied.

$$\begin{aligned} (2)^2 + (-1)^3 &\stackrel{?}{=} 1 \\ 3 &\neq 1 \end{aligned}$$

Hence, $(2, -1)$ is **not** on the graph of $x^2 + y^3 = 1$. □

We could spend hours randomly guessing and checking to see if points are on the graph of the equation. A more systematic approach is outlined in the following example.

Example 1.2.3. Graph $x^2 + y^3 = 1$.

Solution. To efficiently generate points on the graph of this equation, we first solve for y

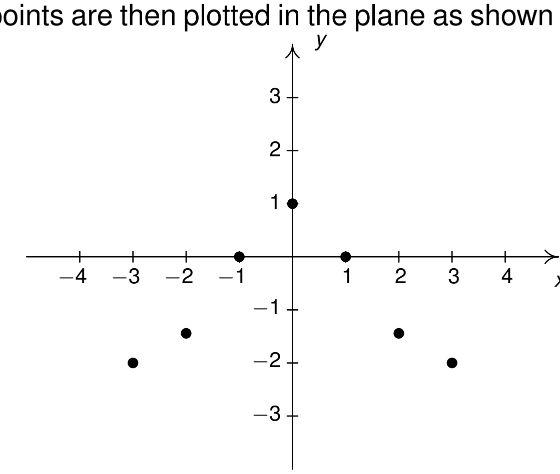
$$\begin{aligned} x^2 + y^3 &= 1 \\ y^3 &= 1 - x^2 \\ \sqrt[3]{y^3} &= \sqrt[3]{1 - x^2} \\ y &= \sqrt[3]{1 - x^2} \end{aligned}$$

We now substitute a value in for x, determine the corresponding value y, and plot the resulting point (x, y). For example, substituting $x = -3$ into the equation yields

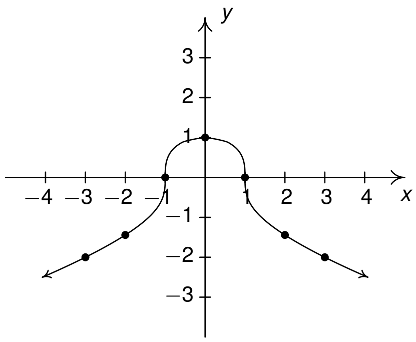
$$y = \sqrt[3]{1 - x^2} = \sqrt[3]{1 - (-3)^2} = \sqrt[3]{-8} = -2,$$

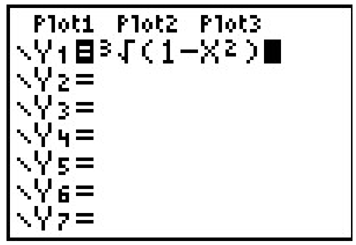
so the point $(-3, -2)$ is on the graph. Continuing in this manner, we generate a table of points which are on the graph of the equation. These points are then plotted in the plane as shown below.

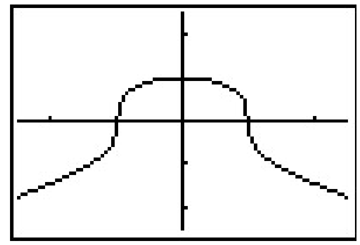
x	y	(x, y)
-3	-2	$(-3, -2)$
-2	$-\sqrt[3]{3}$	$(-2, -\sqrt[3]{3})$
-1	0	$(-1, 0)$
0	1	$(0, 1)$
1	0	$(1, 0)$
2	$-\sqrt[3]{3}$	$(2, -\sqrt[3]{3})$
3	-2	$(3, -2)$



Remember, these points constitute only a small sampling of the points on the graph of this equation. To get a better idea of the shape of the graph, we could plot more points until we feel comfortable 'connecting the dots'. Doing so would result in a curve similar to the one pictured below on the far left.







Don't worry if you don't get all of the little bends and curves just right − Calculus is where the art of precise graphing takes center stage. For now, we will settle with our naive 'plug and plot' approach to graphing. If you feel like all of this tedious computation and plotting is beneath you, then you can reach for a graphing calculator, input the formula as shown above, and graph. □

Of all of the points on the graph of an equation, the places where the graph crosses or touches the axes hold special significance. These are called the **intercepts** of the graph. Intercepts come in two distinct varieties: x-intercepts and y-intercepts. They are defined below.

Definition 1.3. Suppose the graph of an equation is given.

- A point on a graph which is also on the x-axis is called an ***x*-intercept** of the graph.
- A point on a graph which is also on the y-axis is called an ***y*-intercept** of the graph.

In our previous example the graph had two x-intercepts, $(-1, 0)$ and $(1, 0)$, and one y-intercept, $(0, 1)$. The graph of an equation can have any number of intercepts, including none at all! Since x-intercepts lie on the x-axis, we can find them by setting $y = 0$ in the equation. Similarly, since y-intercepts lie on the y-axis, we can find them by setting $x = 0$ in the equation. Keep in mind, intercepts are *points* and therefore must be written as ordered pairs. To summarize,

Finding the Intercepts of the Graph of an Equation

Given an equation involving x and y, we find the intercepts of the graph as follows:

- x-intercepts have the form $(x, 0)$; set $y = 0$ in the equation and solve for x.
- y-intercepts have the form $(0, y)$; set $x = 0$ in the equation and solve for y.

Another fact which you may have noticed about the graph in the previous example is that it seems to be symmetric about the y-axis. To actually prove this analytically, we assume (x, y) is a generic point on the graph of the equation. That is, we assume $x^2 + y^3 = 1$ is true. As we learned in Section 1.1, the point symmetric to (x, y) about the y-axis is $(-x, y)$. To show that the graph is symmetric about the y-axis, we need to show that $(-x, y)$ satisfies the equation $x^2 + y^3 = 1$, too. Substituting $(-x, y)$ into the equation gives

$$\begin{aligned} (-x)^2 + (y)^3 &\stackrel{?}{=} 1 \\ x^2 + y^3 &\stackrel{\checkmark}{=} 1 \end{aligned}$$

Since we are assuming the original equation $x^2 + y^3 = 1$ is true, we have shown that $(-x, y)$ satisfies the equation (since it leads to a true result) and hence is on the graph. In this way, we can check whether the graph of a given equation possesses any of the symmetries discussed in Section 1.1. We summarize the procedure in the following result.

Testing the Graph of an Equation for Symmetry

To test the graph of an equation for symmetry

- about the y-axis – substitute $(-x, y)$ into the equation and simplify. If the result is equivalent to the original equation, the graph is symmetric about the y-axis.
- about the x-axis – substitute $(x, -y)$ into the equation and simplify. If the result is equivalent to the original equation, the graph is symmetric about the x-axis.
- about the origin - substitute $(-x, -y)$ into the equation and simplify. If the result is equivalent to the original equation, the graph is symmetric about the origin.

Intercepts and symmetry are two tools which can help us sketch the graph of an equation analytically, as demonstrated in the next example.

Example 1.2.4. Find the x- and y-intercepts (if any) of the graph of $(x-2)^2 + y^2 = 1$. Test for symmetry. Plot additional points as needed to complete the graph.

Solution. To look for x-intercepts, we set $y = 0$ and solve

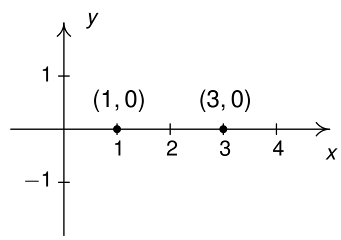
$$\begin{aligned} (x-2)^2 + y^2 &= 1 \\ (x-2)^2 + 0^2 &= 1 \\ (x-2)^2 &= 1 \\ \sqrt{(x-2)^2} &= \sqrt{1} \qquad \text{extract square roots} \\ x - 2 &= \pm 1 \\ x &= 2 \pm 1 \\ x &= 3, 1 \end{aligned}$$

We get two answers for x which correspond to two x-intercepts: $(1, 0)$ and $(3, 0)$. Turning our attention to y-intercepts, we set $x = 0$ and solve

$$\begin{aligned} (x-2)^2 + y^2 &= 1 \\ (0-2)^2 + y^2 &= 1 \\ 4 + y^2 &= 1 \\ y^2 &= -3 \end{aligned}$$

Since there is no real number which squares to a negative number (Do you remember why?), we are forced to conclude that the graph has no y-intercepts.

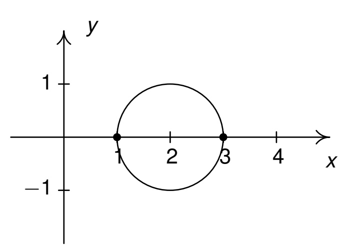
Plotting the data we have so far, we get



Moving along to symmetry, we can immediately dismiss the possibility that the graph is symmetric about the y-axis or the origin. If the graph possessed either of these symmetries, then the fact that $(1,0)$ is on the graph would mean $(-1,0)$ would have to be on the graph. (Why?) Since $(-1,0)$ would be another x-intercept (and we've found all of these), the graph can't have y-axis or origin symmetry. The only symmetry left to test is symmetry about the x-axis. To that end, we substitute $(x,-y)$ into the equation and simplify

$$\begin{aligned} (x-2)^2 + y^2 &= 1 \\ (x-2)^2 + (-y)^2 &\stackrel{?}{=} 1 \\ (x-2)^2 + y^2 &\stackrel{\checkmark}{=} 1 \end{aligned}$$

Since we have obtained our original equation, we know the graph is symmetric about the x-axis. This means we can cut our 'plug and plot' time in half: whatever happens below the x-axis is reflected above the x-axis, and vice-versa. Proceeding as we did in the previous example, we obtain



□

A couple of remarks are in order. First, it is entirely possible to choose a value for x which does not correspond to a point on the graph. For example, in the previous example, if we solve for y as is our custom, we get

$$y = \pm\sqrt{1-(x-2)^2}.$$

Upon substituting $x = 0$ into the equation, we would obtain

$$y = \pm\sqrt{1-(0-2)^2} = \pm\sqrt{1-4} = \pm\sqrt{-3},$$

which is not a real number. This means there are no points on the graph with an x-coordinate of 0. When this happens, we move on and try another point. This is another drawback of the 'plug-and-plot' approach to graphing equations. Luckily, we will devote much of the remainder of this book to developing techniques which allow us to graph entire families of equations quickly.[6] Second, it is instructive to show what would have happened had we tested the equation in the last example for symmetry about the y-axis. Substituting $(-x, y)$ into the equation yields

$$\begin{array}{rcl} (x-2)^2 + y^2 & = & 1 \\ (-x-2)^2 + y^2 & \stackrel{?}{=} & 1 \\ ((-1)(x+2))^2 + y^2 & \stackrel{?}{=} & 1 \\ (x+2)^2 + y^2 & \stackrel{?}{=} & 1. \end{array}$$

This last equation does not *appear* to be equivalent to our original equation. However, to actually prove that the graph is not symmetric about the y-axis, we need to find a point (x, y) on the graph whose reflection $(-x, y)$ is not. Our x-intercept $(1, 0)$ fits this bill nicely, since if we substitute $(-1, 0)$ into the equation we get

$$\begin{array}{rcl} (x-2)^2 + y^2 & \stackrel{?}{=} & 1 \\ (-1-2)^2 + 0^2 & \neq & 1 \\ 9 & \neq & 1. \end{array}$$

This proves that $(-1, 0)$ is not on the graph.

[6]Without the use of a calculator, if you can believe it!

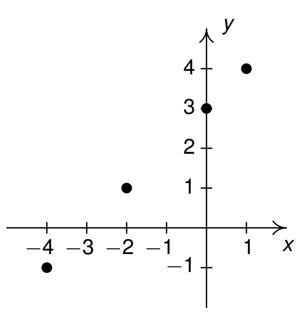
1.2.2 Exercises

In Exercises 1 - 20, graph the given relation.

1. $\{(-3,9),\ (-2,4),\ (-1,1),\ (0,0),\ (1,1),\ (2,4),\ (3,9)\}$
2. $\{(-2,0),\ (-1,1),\ (-1,-1),\ (0,2),\ (0,-2),\ (1,3),\ (1,-3)\}$
3. $\{(m,2m) \mid m = 0, \pm 1, \pm 2\}$
4. $\left\{\left(\frac{6}{k}, k\right) \mid k = \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6\right\}$
5. $\left\{\left(n, 4-n^2\right) \mid n = 0, \pm 1, \pm 2\right\}$
6. $\left\{\left(\sqrt{j}, j\right) \mid j = 0, 1, 4, 9\right\}$
7. $\{(x,-2) \mid x > -4\}$
8. $\{(x,3) \mid x \leq 4\}$
9. $\{(-1,y) \mid y > 1\}$
10. $\{(2,y) \mid y \leq 5\}$
11. $\{(-2,y) \mid -3 < y \leq 4\}$
12. $\{(3,y) \mid -4 \leq y < 3\}$
13. $\{(x,2) \mid -2 \leq x < 3\}$
14. $\{(x,-3) \mid -4 < x \leq 4\}$
15. $\{(x,y) \mid x > -2\}$
16. $\{(x,y) \mid x \leq 3\}$
17. $\{(x,y) \mid y < 4\}$
18. $\{(x,y) \mid x \leq 3,\ y < 2\}$
19. $\{(x,y) \mid x > 0,\ y < 4\}$
20. $\{(x,y) \mid -\sqrt{2} \leq x \leq \frac{2}{3},\ \pi < y \leq \frac{9}{2}\}$

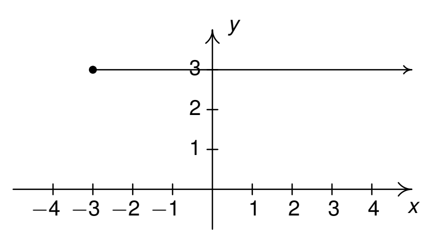
In Exercises 21 - 30, describe the given relation using either the roster or set-builder method.

21.



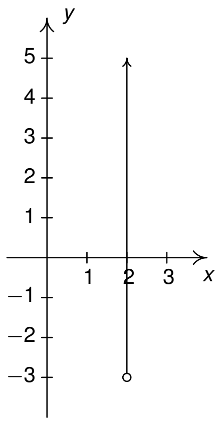
Relation A

22.



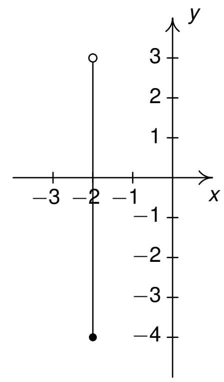
Relation B

23.



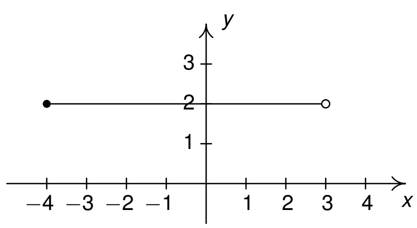
Relation *C*

24.



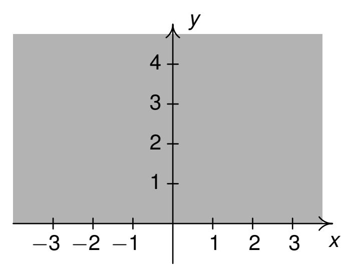
Relation *D*

25.



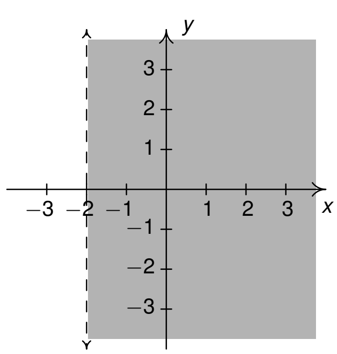
Relation *E*

26.



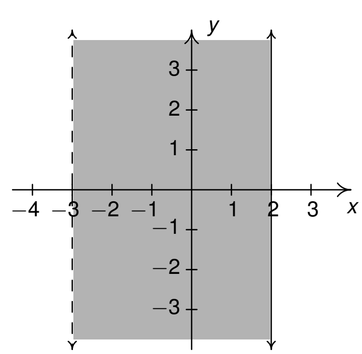
Relation *F*

27.



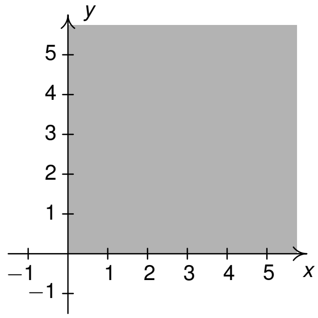
Relation *G*

28.



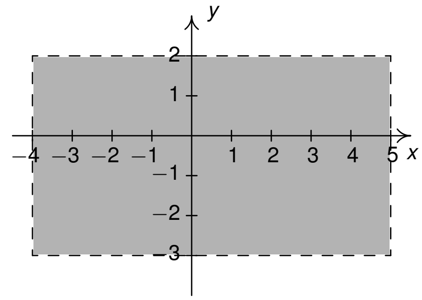
Relation *H*

29.



Relation I

30.



Relation J

In Exercises 31 - 36, graph the given line.

31. $x = -2$
32. $x = 3$
33. $y = 3$
34. $y = -2$
35. $x = 0$
36. $y = 0$

Some relations are fairly easy to describe in words or with the roster method but are rather difficult, if not impossible, to graph. Discuss with your classmates how you might graph the relations given in Exercises 37 - 40. Please note that in the notation below we are using the ellipsis, $\ldots$, to denote that the list does not end, but rather, continues to follow the established pattern indefinitely. For the relations in Exercises 37 and 38, give two examples of points which belong to the relation and two points which do not belong to the relation.

37. $\{(x, y) \mid x \text{ is an odd integer, and } y \text{ is an even integer.}\}$
38. $\{(x, 1) \mid x \text{ is an irrational number }\}$
39. $\{(1, 0), (2, 1), (4, 2), (8, 3), (16, 4), (32, 5), \ldots\}$
40. $\{\ldots, (-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9), \ldots\}$

For each equation given in Exercises 41 - 52:

- Find the x- and y-intercept(s) of the graph, if any exist.
- Follow the procedure in Example 1.2.3 to create a table of sample points on the graph of the equation.
- Plot the sample points and create a rough sketch of the graph of the equation.
- Test for symmetry. If the equation appears to fail any of the symmetry tests, find a point on the graph of the equation whose reflection fails to be on the graph as was done at the end of Example 1.2.4

41. $y = x^2 + 1$

42. $y = x^2 - 2x - 8$

43. $y = x^3 - x$

44. $y = \frac{x^3}{4} - 3x$

45. $y = \sqrt{x - 2}$

46. $y = 2\sqrt{x + 4} - 2$

47. $3x - y = 7$

48. $3x - 2y = 10$

49. $(x + 2)^2 + y^2 = 16$

50. $x^2 - y^2 = 1$

51. $4y^2 - 9x^2 = 36$

52. $x^3 y = -4$

The procedures which we have outlined in the Examples of this section and used in Exercises 41 through 52 all rely on the fact that the equations were "well-behaved". Not everything in Mathematics is quite so tame, as the following equations will show you. Discuss with your classmates how you might approach graphing the equations given in Exercises 53 - 56. What difficulties arise when trying to apply the various tests and procedures given in this section? For more information, including pictures of the curves, each curve name is a link to its page at www.wikipedia.org. For a much longer list of fascinating curves, click here.

53. $x^3 + y^3 - 3xy = 0$ Folium of Descartes

54. $x^4 = x^2 + y^2$ Kampyle of Eudoxus

55. $y^2 = x^3 + 3x^2$ Tschirnhausen cubic

56. $(x^2 + y^2)^2 = x^3 + y^3$ Crooked egg

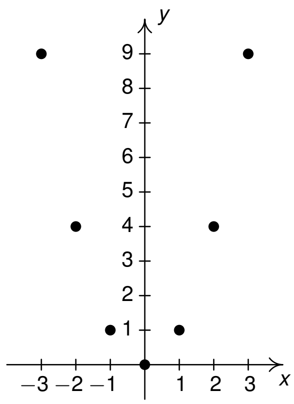
57. With the help of your classmates, find examples of equations whose graphs possess

- symmetry about the x-axis only
- symmetry about the y-axis only
- symmetry about the origin only
- symmetry about the x-axis, y-axis, and origin

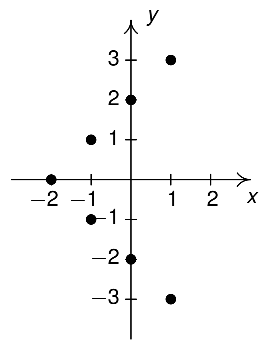
Can you find an example of an equation whose graph possesses exactly *two* of the symmetries listed above? Why or why not?

1.2.3 Answers

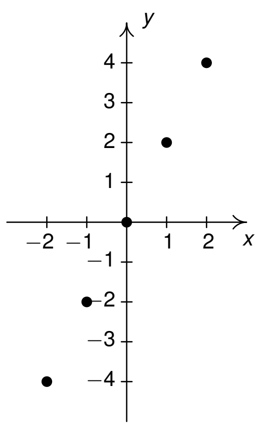
1.



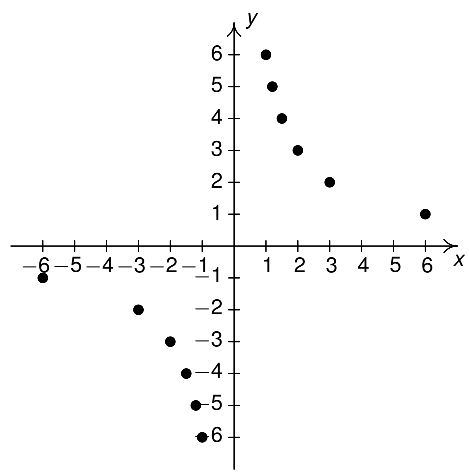
2.



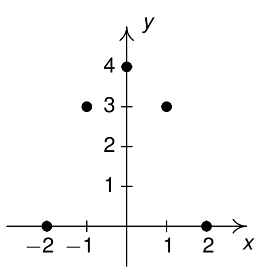
3.



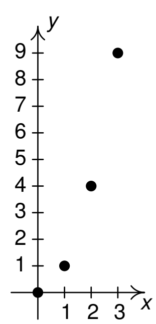
4.



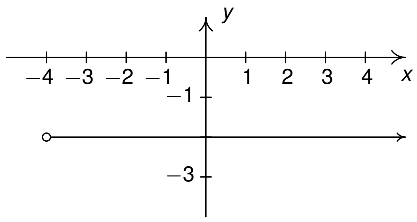
5.



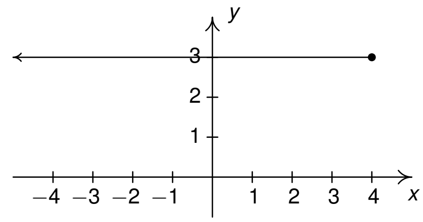
6.



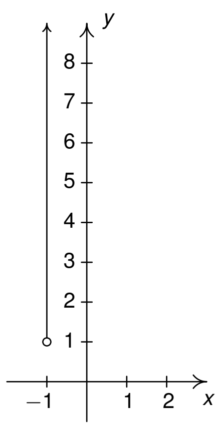
7.


y
−4 −3 −2 −1 1 2 3 4 x
−1
−3

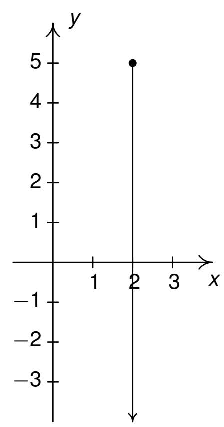
8.


y
3
2
1
−4 −3 −2 −1 1 2 3 4 x

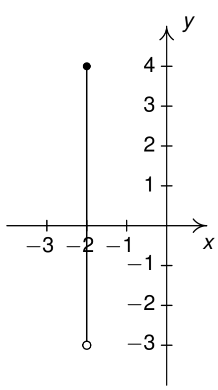
9.


y
8
7
6
5
4
3
2
1
−1 1 2 x

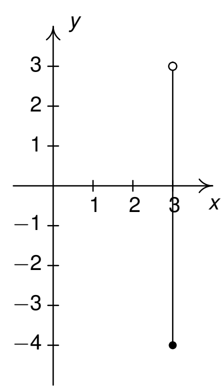
10.


y
5
4
3
2
1
1 2 3 x
−1
−2
−3

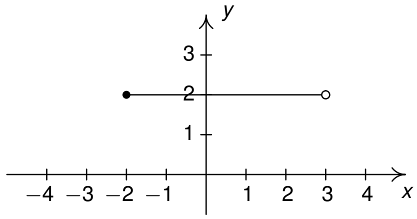
11.


y
4
3
2
1
−3 −2 −1 x
−1
−2
−3

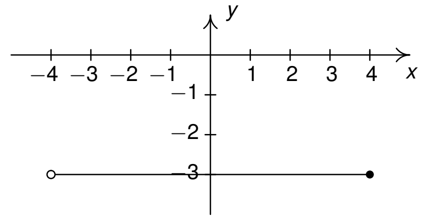
12.


y
3
2
1
1 2 3 x
−1
−2
−3
−4

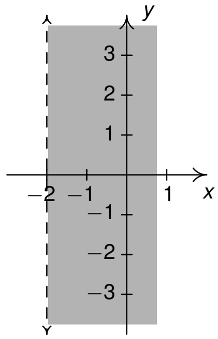
13.



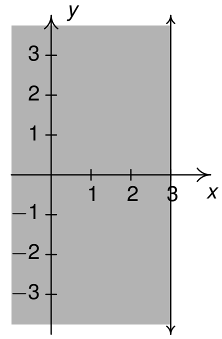
14.



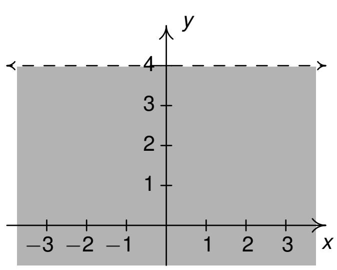
15.



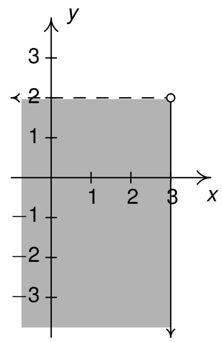
16.



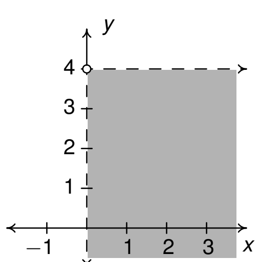
17.



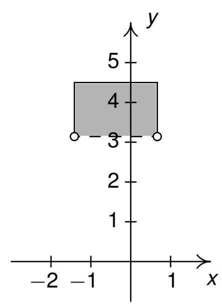
18.



19.



20.



21. $A = \{(-4,-1),(-2,1),(0,3),(1,4)\}$

22. $B = \{(x,3) \mid x \geq -3\}$

23. $C = \{(2,y) \mid y > -3\}$

24. $D = \{(-2,y) \mid -4 \leq y < 3\}$

25. $E = \{(x,2) \mid -4 < x \leq 3\}$

26. $F = \{(x,y) \mid y \geq 0\}$

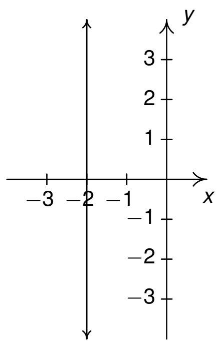
27. $G = \{(x,y) \mid x > -2\}$

28. $H = \{(x,y) \mid -3 < x \leq 2\}$

29. $I = \{(x,y) \mid x \geq 0, y \geq 0\}$

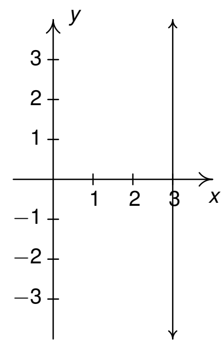
30. $J = \{(x,y) \mid -4 < x < 5,\ -3 < y < 2\}$

31.



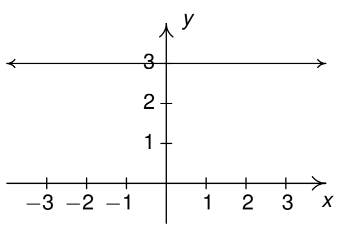
The line $x = -2$

32.



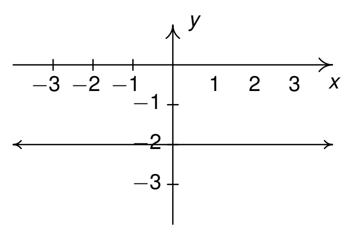
The line $x = 3$

33.



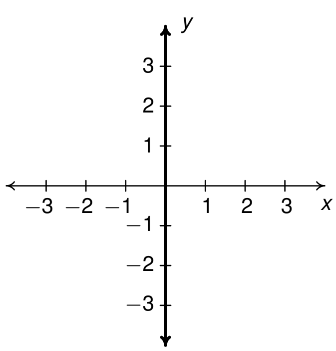
The line $y = 3$

34.



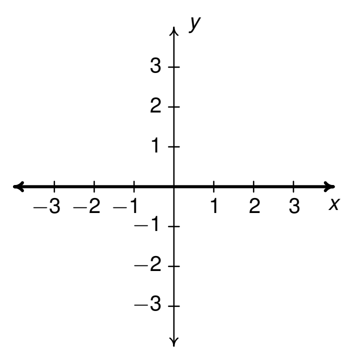
The line $y = -2$

35.



The line $x = 0$ is the y-axis

36.



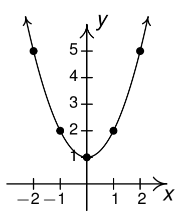
The line $y = 0$ is the x-axis

41. $y = x^2 + 1$

The graph has no x-intercepts

y-intercept: $(0, 1)$

x	y	(x, y)
-2	5	$(-2, 5)$
-1	2	$(-1, 2)$
0	1	$(0, 1)$
1	2	$(1, 2)$
2	5	$(2, 5)$



The graph is not symmetric about the x-axis (e.g. $(2, 5)$ is on the graph but $(2, -5)$ is not)

The graph is symmetric about the y-axis

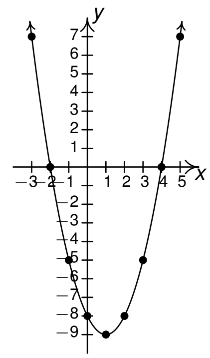
The graph is not symmetric about the origin (e.g. $(2, 5)$ is on the graph but $(-2, -5)$ is not)

42. $y = x^2 - 2x - 8$

x-intercepts: $(4, 0)$, $(-2, 0)$

y-intercept: $(0, -8)$

x	y	(x, y)
-3	7	$(-3, 7)$
-2	0	$(-2, 0)$
-1	-5	$(-1, -5)$
0	-8	$(0, -8)$
1	-9	$(1, -9)$
2	-8	$(2, -8)$
3	-5	$(3, -5)$
4	0	$(4, 0)$
5	7	$(5, 7)$



The graph is not symmetric about the x-axis (e.g. $(-3, 7)$ is on the graph but $(-3, -7)$ is not)

The graph is not symmetric about the y-axis (e.g. $(-3, 7)$ is on the graph but $(3, 7)$ is not)

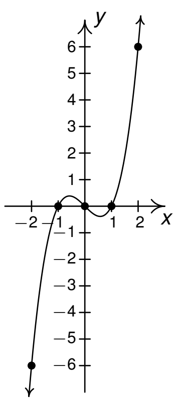
The graph is not symmetric about the origin (e.g. $(-3, 7)$ is on the graph but $(3, -7)$ is not)

43. $y = x^3 - x$

x-intercepts: $(-1,0), (0,0), (1,0)$

y-intercept: $(0,0)$

x	y	(x,y)
-2	-6	$(-2,-6)$
-1	0	$(-1,0)$
0	0	$(0,0)$
1	0	$(1,0)$
2	6	$(2,6)$



The graph is not symmetric about the x-axis. (e.g. $(2,6)$ is on the graph but $(2,-6)$ is not)

The graph is not symmetric about the y-axis. (e.g. $(2,6)$ is on the graph but $(-2,6)$ is not)

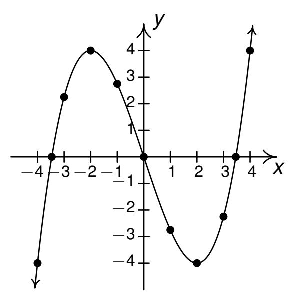
The graph is symmetric about the origin.

44. $y = \frac{x^3}{4} - 3x$

x-intercepts: $\left(\pm 2\sqrt{3}, 0\right), (0,0)$

y-intercept: $(0,0)$

x	y	(x,y)
-4	-4	$(-4,-4)$
-3	$\frac{9}{4}$	$\left(-3, \frac{9}{4}\right)$
-2	4	$(-2,4)$
-1	$\frac{11}{4}$	$\left(-1, \frac{11}{4}\right)$
0	0	$(0,0)$
1	$-\frac{11}{4}$	$\left(1, -\frac{11}{4}\right)$
2	-4	$(2,-4)$
3	$-\frac{9}{4}$	$\left(3, -\frac{9}{4}\right)$
4	4	$(4,4)$



The graph is not symmetric about the x-axis (e.g. $(-4,-4)$ is on the graph but $(-4,4)$ is not)

The graph is not symmetric about the y-axis (e.g. $(-4,-4)$ is on the graph but $(4,-4)$ is not)

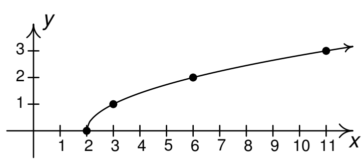
The graph is symmetric about the origin

45. $y = \sqrt{x-2}$

x-intercept: $(2, 0)$

The graph has no y-intercepts

x	y	(x, y)
2	0	$(2, 0)$
3	1	$(3, 1)$
6	2	$(6, 2)$
11	3	$(11, 3)$



The graph is not symmetric about the x-axis (e.g. $(3, 1)$ is on the graph but $(3, -1)$ is not)

The graph is not symmetric about the y-axis (e.g. $(3, 1)$ is on the graph but $(-3, 1)$ is not)

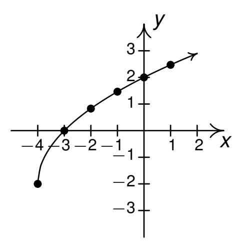
The graph is not symmetric about the origin (e.g. $(3, 1)$ is on the graph but $(-3, -1)$ is not)

46. $y = 2\sqrt{x+4} - 2$

x-intercept: $(-3, 0)$

y-intercept: $(0, 2)$

x	y	(x, y)
-4	-2	$(-4, -2)$
-3	0	$(-3, 0)$
-2	$2\sqrt{2}-2$	$\left(-2, \sqrt{2}-2\right)$
-1	$2\sqrt{3}-2$	$\left(-2, \sqrt{3}-2\right)$
0	2	$(0, 2)$
1	$2\sqrt{5}-2$	$\left(-2, \sqrt{5}-2\right)$



The graph is not symmetric about the x-axis (e.g. $(-4, -2)$ is on the graph but $(-4, 2)$ is not)

The graph is not symmetric about the y-axis (e.g. $(-4, -2)$ is on the graph but $(4, -2)$ is not)

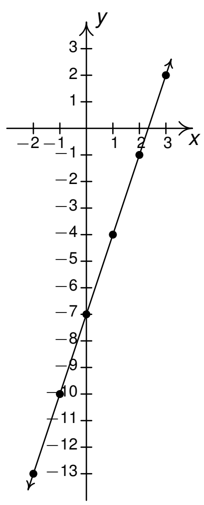
The graph is not symmetric about the origin (e.g. $(-4, -2)$ is on the graph but $(4, 2)$ is not)

47. $3x - y = 7$
Re-write as: $y = 3x - 7$.

x-intercept: $(\frac{7}{3}, 0)$

y-intercept: $(0, -7)$

x	y	(x, y)
-2	-13	$(-2, -13)$
-1	-10	$(-1, -10)$
0	-7	$(0, -7)$
1	-4	$(1, -4)$
2	-1	$(2, -1)$
3	2	$(3, 2)$



The graph is not symmetric about the x-axis (e.g. $(3, 2)$ is on the graph but $(3, -2)$ is not)

The graph is not symmetric about the y-axis (e.g. $(3, 2)$ is on the graph but $(-3, 2)$ is not)

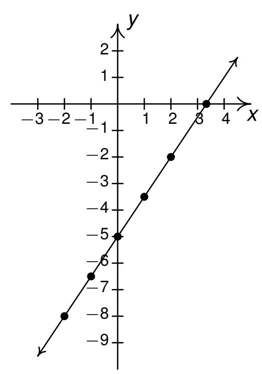
The graph is not symmetric about the origin (e.g. $(3, 2)$ is on the graph but $(-3, -2)$ is not)

48. $3x - 2y = 10$
Re-write as: $y = \frac{3x-10}{2}$.

x-intercepts: $\left(\frac{10}{3}, 0\right)$

y-intercept: $(0, -5)$

x	y	(x, y)
-2	-8	$(-2, -8)$
-1	$-\frac{13}{2}$	$\left(-1, -\frac{13}{2}\right)$
0	-5	$(0, -5)$
1	$-\frac{7}{2}$	$\left(1, -\frac{7}{2}\right)$
2	-2	$(2, -2)$



The graph is not symmetric about the x-axis (e.g. $(2, -2)$ is on the graph but $(2, 2)$ is not)

The graph is not symmetric about the y-axis (e.g. $(2, -2)$ is on the graph but $(-2, -2)$ is not)

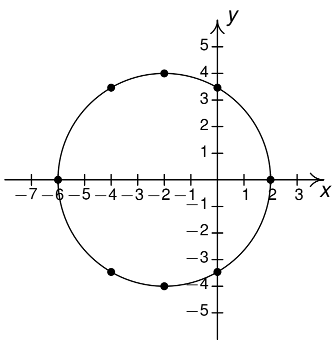
The graph is not symmetric about the origin (e.g. $(2, -2)$ is on the graph but $(-2, 2)$ is not)

49. $(x+2)^2+y^2=16$
Re-write as $y=\pm\sqrt{16-(x+2)^2}$.

x-intercepts: $(-6,0)$, $(2,0)$

y-intercepts: $\left(0,\pm 2\sqrt{3}\right)$

x	y	(x,y)
-6	0	$(-6,0)$
-4	$\pm 2\sqrt{3}$	$\left(-4,\pm 2\sqrt{3}\right)$
-2	± 4	$(-2,\pm 4)$
0	$\pm 2\sqrt{3}$	$\left(0,\pm 2\sqrt{3}\right)$
2	0	$(2,0)$



The graph is symmetric about the x-axis

The graph is not symmetric about the y-axis (e.g. $(-6,0)$ is on the graph but $(6,0)$ is not)

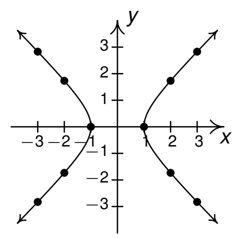
The graph is not symmetric about the origin (e.g. $(-6,0)$ is on the graph but $(6,0)$ is not)

50. $x^2-y^2=1$
Re-write as: $y=\pm\sqrt{x^2-1}$.

x-intercepts: $(-1,0),(1,0)$

The graph has no y-intercepts

x	y	(x,y)
-3	$\pm\sqrt{8}$	$(-3,\pm\sqrt{8})$
-2	$\pm\sqrt{3}$	$(-2,\pm\sqrt{3})$
-1	0	$(-1,0)$
1	0	$(1,0)$
2	$\pm\sqrt{3}$	$(2,\pm\sqrt{3})$
3	$\pm\sqrt{8}$	$(3,\pm\sqrt{8})$



The graph is symmetric about the x-axis

The graph is symmetric about the y-axis

The graph is symmetric about the origin

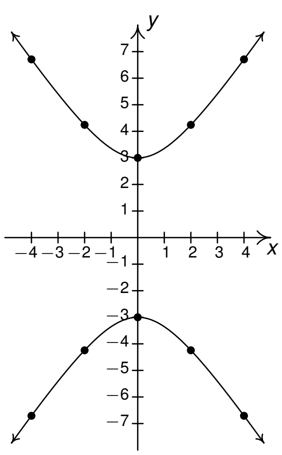
51. $4y^2 - 9x^2 = 36$

Re-write as: $y = \pm\frac{\sqrt{9x^2+36}}{2}$.

The graph has no x-intercepts

y-intercepts: $(0, \pm 3)$

x	y	(x, y)
-4	$\pm 3\sqrt{5}$	$\left(-4, \pm 3\sqrt{5}\right)$
-2	$\pm 3\sqrt{2}$	$\left(-2, \pm 3\sqrt{2}\right)$
0	± 3	$(0, \pm 3)$
2	$\pm 3\sqrt{2}$	$\left(2, \pm 3\sqrt{2}\right)$
4	$\pm 3\sqrt{5}$	$\left(4, \pm 3\sqrt{5}\right)$



The graph is symmetric about the x-axis

The graph is symmetric about the y-axis

The graph is symmetric about the origin

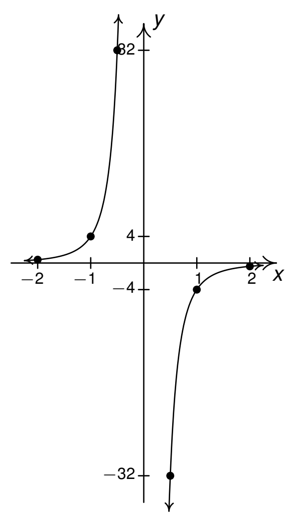
52. $x^3 y = -4$

Re-write as: $y = -\dfrac{4}{x^3}$.

The graph has no x-intercepts

The graph has no y-intercepts

x	y	(x, y)
-2	$\frac{1}{2}$	$(-2, \frac{1}{2})$
-1	4	$(-1, 4)$
$-\frac{1}{2}$	32	$(-\frac{1}{2}, 32)$
$\frac{1}{2}$	-32	$(\frac{1}{2}, -32)$
1	-4	$(1, -4)$
2	$-\frac{1}{2}$	$(2, -\frac{1}{2})$



The graph is not symmetric about the x-axis (e.g. $(1, -4)$ is on the graph but $(1, 4)$ is not)

The graph is not symmetric about the y-axis (e.g. $(1, -4)$ is on the graph but $(-1, -4)$ is not)

The graph is symmetric about the origin

1.3 Introduction to Functions

One of the core concepts in College Algebra is the **function**. There are many ways to describe a function and we begin by defining a function as a special kind of relation.

Definition 1.4. A relation in which each x-coordinate is matched with only one y-coordinate is said to describe y as a **function** of x.

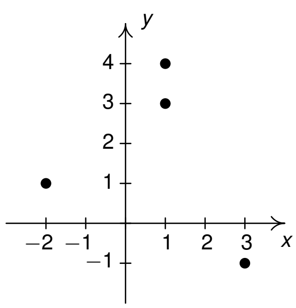
Example 1.3.1. Which of the following relations describe y as a function of x?

1. $R_1 = \{(-2, 1), (1, 3), (1, 4), (3, -1)\}$
2. $R_2 = \{(-2, 1), (1, 3), (2, 3), (3, -1)\}$

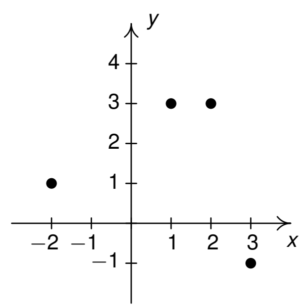
Solution. A quick scan of the points in R_1 reveals that the x-coordinate 1 is matched with two *different* y-coordinates: namely 3 and 4. Hence in R_1, y is not a function of x. On the other hand, every x-coordinate in R_2 occurs only once which means each x-coordinate has only one corresponding y-coordinate. So, R_2 does represent y as a function of x. □

Note that in the previous example, the relation R_2 contained two different points with the same y-coordinates, namely $(1, 3)$ and $(2, 3)$. Remember, in order to say y is a function of x, we just need to ensure the same x-coordinate isn't used in more than one point.[1]

To see what the function concept means geometrically, we graph R_1 and R_2 in the plane.



The graph of R_1



The graph of R_2

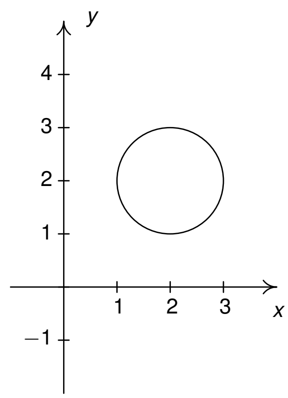
The fact that the x-coordinate 1 is matched with two different y-coordinates in R_1 presents itself graphically as the points $(1, 3)$ and $(1, 4)$ lying on the same vertical line, $x = 1$. If we turn our attention to the graph of R_2, we see that no two points of the relation lie on the same vertical line. We can generalize this idea as follows

Theorem 1.1. The Vertical Line Test: A set of points in the plane represents y as a function of x if and only if no two points lie on the same vertical line.

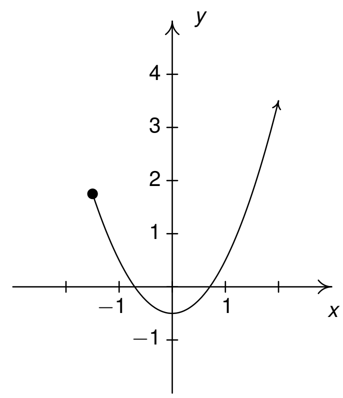
[1] We will have occasion later in the text to concern ourselves with the concept of x being a function of y. In this case, R_1 represents x as a function of y; R_2 does not.

It is worth taking some time to meditate on the Vertical Line Test; it will check to see how well you understand the concept of 'function' as well as the concept of 'graph'.

Example 1.3.2. Use the Vertical Line Test to determine which of the following relations describes y as a function of x.



The graph of R

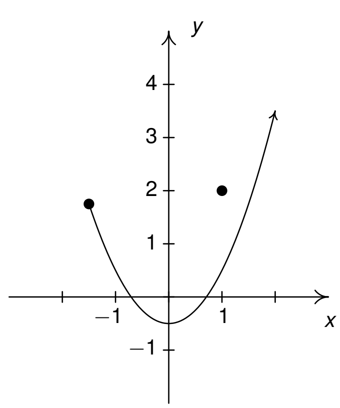


The graph of S

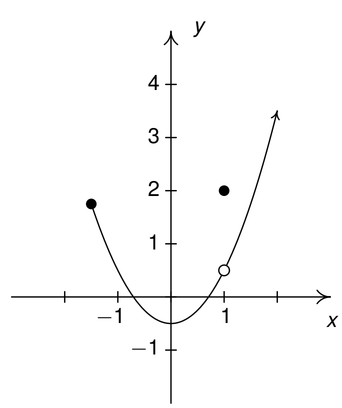
Solution. Looking at the graph of R, we can easily imagine a vertical line crossing the graph more than once. Hence, R does not represent y as a function of x. However, in the graph of S, every vertical line crosses the graph at most once, so S does represent y as a function of x. □

In the previous test, we say that the graph of the relation R **fails** the Vertical Line Test, whereas the graph of S **passes** the Vertical Line Test. Note that in the graph of R there are infinitely many vertical lines which cross the graph more than once. However, to fail the Vertical Line Test, all you need is one vertical line that fits the bill, as the next example illustrates.

Example 1.3.3. Use the Vertical Line Test to determine which of the following relations describes y as a function of x.

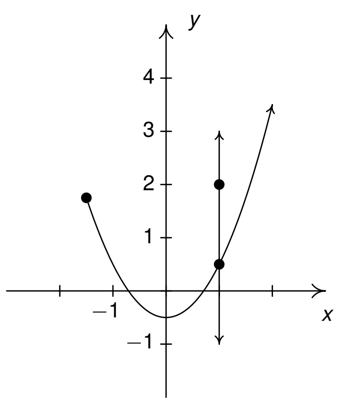


The graph of S_1

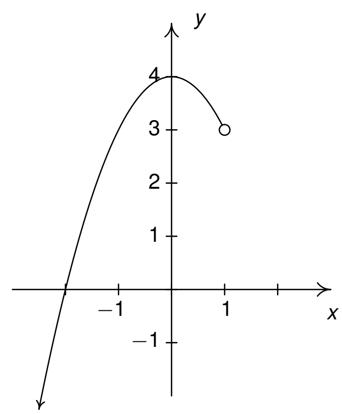


The graph of S_2

Solution. Both S_1 and S_2 are slight modifications to the relation S in the previous example whose graph we determined passed the Vertical Line Test. In both S_1 and S_2, it is the addition of the point $(1, 2)$ which threatens to cause trouble. In S_1, there is a point on the curve with x-coordinate 1 just below $(1, 2)$, which means that both $(1, 2)$ and this point on the curve lie on the vertical line $x = 1$. (See the picture below and the left.) Hence, the graph of S_1 fails the Vertical Line Test, so y is not a function of x here. However, in S_2 notice that the point with x-coordinate 1 on the curve has been omitted, leaving an 'open circle' there. Hence, the vertical line $x = 1$ crosses the graph of S_2 only at the point $(1, 2)$. Indeed, any vertical line will cross the graph at most once, so we have that the graph of S_2 passes the Vertical Line Test. Thus it describes y as a function of x. □



S_1 and the line $x = 1$



The graph of G for Ex. 1.3.4

Suppose a relation F describes y as a function of x. The sets of x- and y-coordinates are given special names which we define below.

Definition 1.5. Suppose F is a relation which describes y as a function of x.

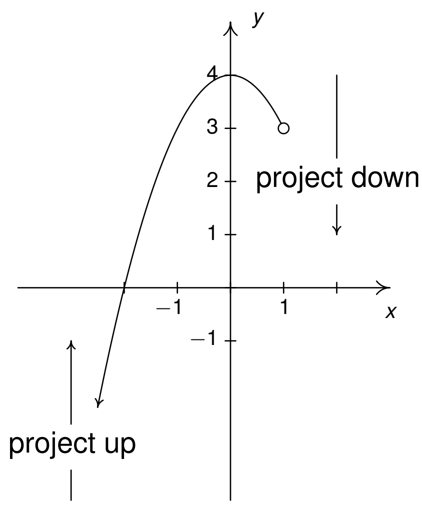
- The set of the x-coordinates of the points in F is called the **domain** of F.
- The set of the y-coordinates of the points in F is called the **range** of F.

We demonstrate finding the domain and range of functions given to us either graphically or via the roster method in the following example.

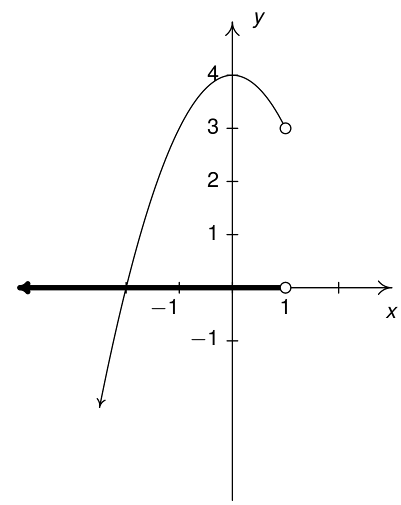
Example 1.3.4. Find the domain and range of the function $F = \{(-3, 2), (0, 1), (4, 2), (5, 2)\}$ and of the function G whose graph is given above on the right.

Solution. The domain of F is the set of the x-coordinates of the points in F, namely $\{-3, 0, 4, 5\}$ and the range of F is the set of the y-coordinates, namely $\{1, 2\}$.

To determine the domain and range of G, we need to determine which x and y values occur as coordinates of points on the given graph. To find the domain, it may be helpful to imagine collapsing the curve to the x-axis and determining the portion of the x-axis that gets covered. This is called **projecting** the curve to the x-axis. Before we start projecting, we need to pay attention to two subtle notations on the graph: the arrowhead on the lower left corner of the graph indicates that the graph continues to curve downwards to the left forever more; and the open circle at $(1,3)$ indicates that the point $(1,3)$ isn't on the graph, but all points on the curve leading up to that point are.

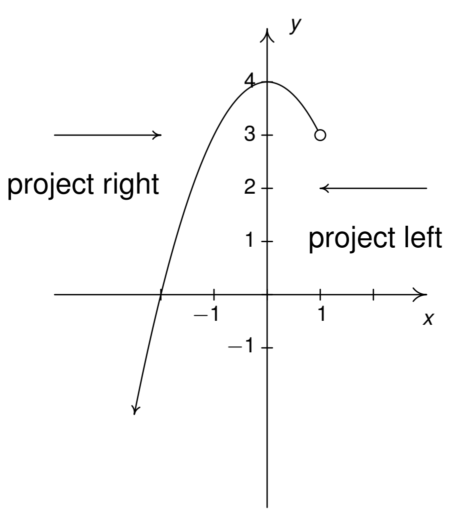


The graph of G

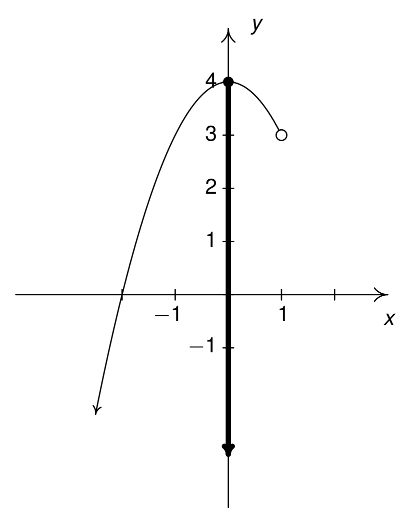


The graph of G

We see from the figure that if we project the graph of G to the x-axis, we get all real numbers less than 1. Using interval notation, we write the domain of G as $(-\infty, 1)$. To determine the range of G, we project the curve to the y-axis as follows:



The graph of G



The graph of G

Note that even though there is an open circle at $(1,3)$, we still include the y value of 3 in our range, since the point $(-1,3)$ is on the graph of G. We see that the range of G is all real numbers less than or equal to 4, or, in interval notation, $(-\infty, 4]$. □

All functions are relations, but not all relations are functions. Thus the equations which described the relations in Section 1.2 may or may not describe y as a function of x. The algebraic representation of functions is possibly the most important way to view them so we need a process for determining whether or not an equation of a relation represents a function. (We delay the discussion of finding the domain of a function given algebraically until Section 1.4.)

Example 1.3.5. Determine which equations represent y as a function of x.

1. $x^3 + y^2 = 1$
2. $x^2 + y^3 = 1$
3. $x^2 y = 1 - 3y$

Solution. For each of these equations, we solve for y and determine whether each choice of x will determine only one corresponding value of y.

1.
$$\begin{aligned} x^3 + y^2 &= 1 \\ y^2 &= 1 - x^3 \\ \sqrt{y^2} &= \sqrt{1 - x^3} \quad \text{extract square roots} \\ y &= \pm\sqrt{1 - x^3} \end{aligned}$$

   If we substitute $x = 0$ into our equation for y, we get $y = \pm\sqrt{1 - 0^3} = \pm 1$, so that $(0, 1)$ and $(0, -1)$ are on the graph of this equation. Hence, this equation does not represent y as a function of x.

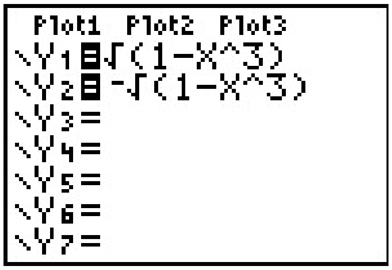
2.
$$\begin{aligned} x^2 + y^3 &= 1 \\ y^3 &= 1 - x^2 \\ \sqrt[3]{y^3} &= \sqrt[3]{1 - x^2} \\ y &= \sqrt[3]{1 - x^2} \end{aligned}$$

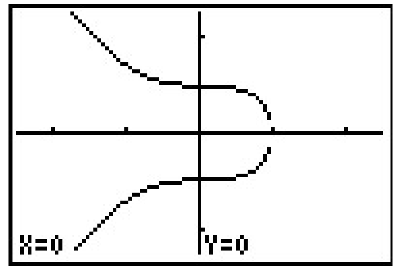
   For every choice of x, the equation $y = \sqrt[3]{1 - x^2}$ returns only **one** value of y. Hence, this equation describes y as a function of x.

3.
$$\begin{aligned} x^2 y &= 1 - 3y \\ x^2 y + 3y &= 1 \\ y\left(x^2 + 3\right) &= 1 \quad \text{factor} \\ y &= \frac{1}{x^2 + 3} \end{aligned}$$

For each choice of x, there is only one value for y, so this equation describes y as a function of x. □

We could try to use our graphing calculator to verify our responses to the previous example, but we immediately run into trouble. The calculator's "Y=" menu requires that the equation be of the form 'y = some expression of x'. If we wanted to verify that the first equation in Example 1.3.5 does not represent y as a function of x, we would need to enter two separate expressions into the calculator: one for the positive square root and one for the negative square root we found when solving the equation for y. As predicted, the resulting graph shown below clearly fails the Vertical Line Test, so the equation does not represent y as a function of x.





Thus in order to use the calculator to show that $x^3 + y^2 = 1$ does not represent y as a function of x we needed to know *analytically* that y was not a function of x so that we could use the calculator properly. There are more advanced graphing utilities out there which can do implicit function plots, but you need to know even more Algebra to make them work properly. Do you get the point we're trying to make here? We believe it is in your best interest to learn the analytic way of doing things so that you are always smarter than your calculator.

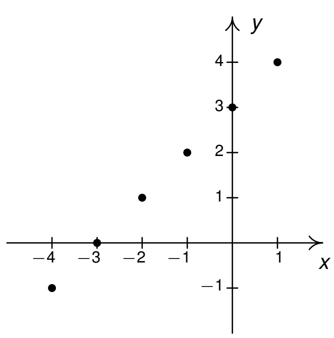
1.3.1 Exercises

In Exercises 1 - 12, determine whether or not the relation represents y as a function of x. Find the domain and range of those relations which are functions.

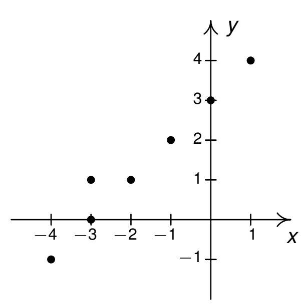
1. $\{(-3,9),\ (-2,4),\ (-1,1),\ (0,0),\ (1,1),\ (2,4),\ (3,9)\}$
2. $\{(-3,0),(1,6),(2,-3),(4,2),(-5,6),(4,-9),(6,2)\}$
3. $\{(-3,0),(-7,6),(5,5),(6,4),(4,9),(3,0)\}$
4. $\{(1,2),(4,4),(9,6),(16,8),(25,10),(36,12),...\}$
5. $\{(x,y)\,|\,x \text{ is an odd integer, and } y \text{ is an even integer}\}$
6. $\{(x,1)\,|\,x \text{ is an irrational number}\}$
7. $\{(1,0),\ (2,1),\ (4,2),\ (8,3),\ (16,4),\ (32,5),\ \ldots\}$
8. $\{...,\ (-3,9),\ (-2,4),\ (-1,1),\ (0,0),\ (1,1),\ (2,4),\ (3,9),\ \ldots\}$
9. $\{(-2,y)\,|\,-3 < y < 4\}$
10. $\{(x,3)\,|\,-2 \leq x < 4\}$
11. $\{(t,t^2)\ |\ t \text{ is a real number}\}$
12. $\{(t^2,t)\ |\ t \text{ is a real number}\}$

In Exercises 13 - 32, determine whether or not the relation represents y as a function of x. Find the domain and range of those relations which are functions.

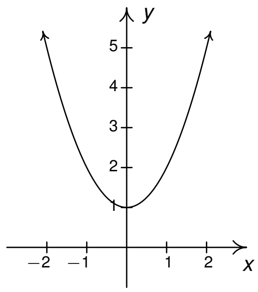
13.



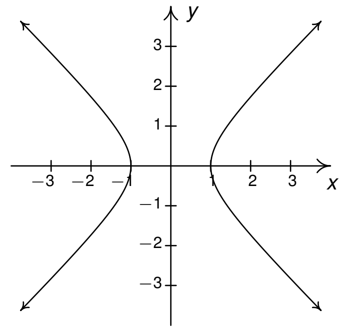
14.



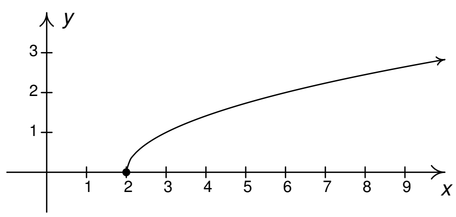
15.



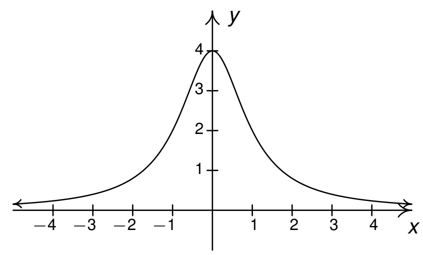
16.



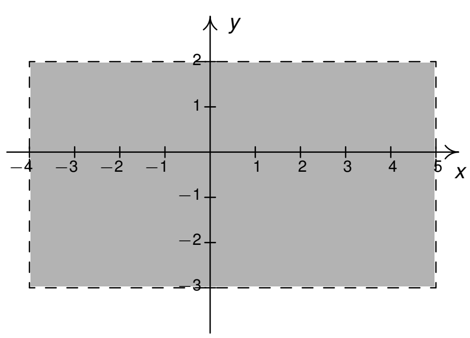
17.



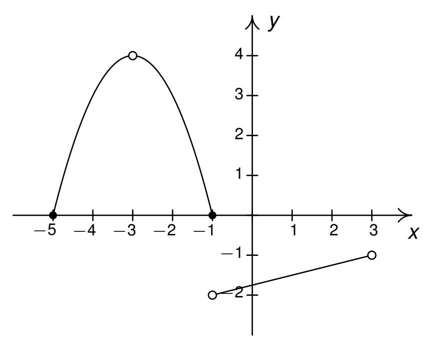
18.



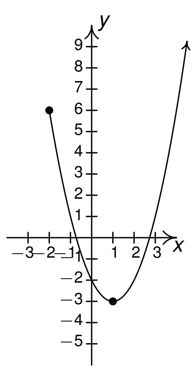
19.



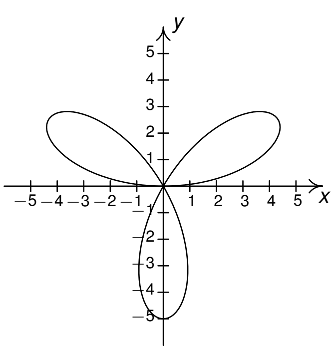
20.



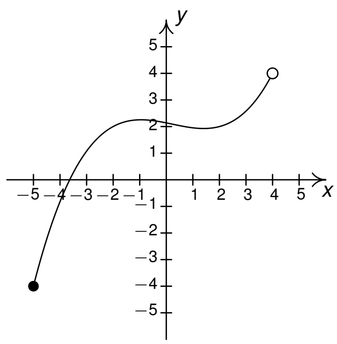
21.



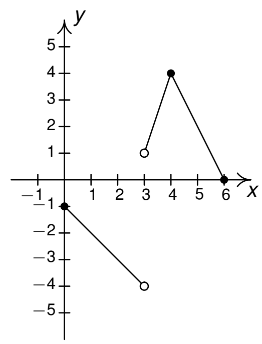
22.



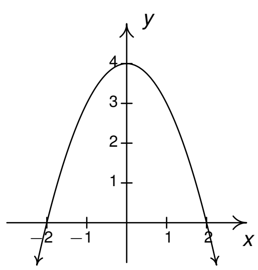
23.



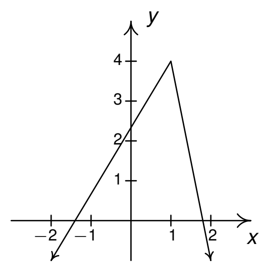
24.



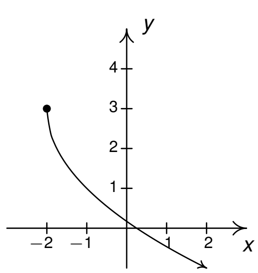
25.



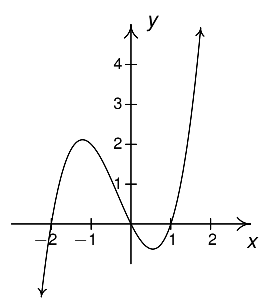
26.



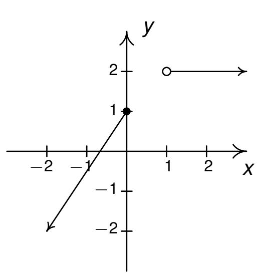
27.



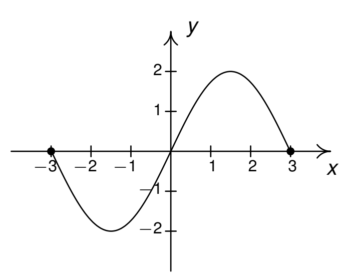
28.



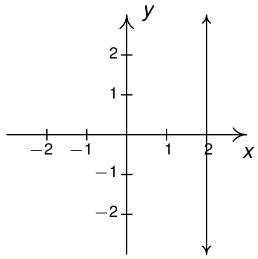
29.



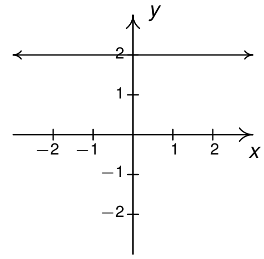
30.



31.



32.



In Exercises 33 - 47, determine whether or not the equation represents y as a function of x.

33. $y = x^3 - x$

34. $y = \sqrt{x - 2}$

35. $x^3 y = -4$

36. $x^2 - y^2 = 1$

37. $y = \dfrac{x}{x^2 - 9}$

38. $x = -6$

39. $x = y^2 + 4$

40. $y = x^2 + 4$

41. $x^2 + y^2 = 4$

42. $y = \sqrt{4 - x^2}$

43. $x^2 - y^2 = 4$

44. $x^3 + y^3 = 4$

45. $2x + 3y = 4$

46. $2xy = 4$

47. $x^2 = y^2$

48. Explain why the population P of Sasquatch in a given area is a function of time t. What would be the range of this function?

49. Explain why the relation between your classmates and their email addresses may not be a function. What about phone numbers and Social Security Numbers?

The process given in Example 1.3.5 for determining whether an equation of a relation represents y as a function of x breaks down if we cannot solve the equation for y in terms of x. However, that does not prevent us from proving that an equation fails to represent y as a function of x. What we really need is two points with the same x-coordinate and different y-coordinates which both satisfy the equation so that the graph of the relation would fail the Vertical Line Test 1.1. Discuss with your classmates how you might find such points for the relations given in Exercises 50 - 53.

50. $x^3 + y^3 - 3xy = 0$

51. $x^4 = x^2 + y^2$

52. $y^2 = x^3 + 3x^2$

53. $(x^2 + y^2)^2 = x^3 + y^3$

1.3.2 Answers

1. Function
   domain = $\{-3, -2, -1, 0, 1, 2, 3\}$
   range = $\{0, 1, 4, 9\}$

2. Not a function

3. Function
   domain = $\{-7, -3, 3, 4, 5, 6\}$
   range = $\{0, 4, 5, 6, 9\}$

4. Function
   domain = $\{1, 4, 9, 16, 25, 36, ...\}$
   = $\{x \mid x$ is a perfect square$\}$
   range = $\{2, 4, 6, 8, 10, 12, ...\}$
   = $\{y \mid y$ is a positive even integer$\}$

5. Not a function

6. Function
   domain = $\{x \mid x$ is irrational$\}$
   range = $\{1\}$

7. Function
   domain = $\{x \mid x = 2^n$ for some whole number $n\}$
   range = $\{y \mid y$ is any whole number$\}$

8. Function
   domain = $\{x \mid x$ is any integer$\}$
   range = $\{y \mid y = n^2$ for some integer $n\}$

9. Not a function

10. Function
    domain = $[-2, 4)$, range = $\{3\}$

11. Function
    domain = $(-\infty, \infty)$
    range = $[0, \infty)$

12. Not a function

13. Function
    domain = $\{-4, -3, -2, -1, 0, 1\}$
    range = $\{-1, 0, 1, 2, 3, 4\}$

14. Not a function

15. Function
    domain = $(-\infty, \infty)$
    range = $[1, \infty)$

16. Not a function

17. Function
    domain = $[2, \infty)$
    range = $[0, \infty)$

18. Function
    domain = $(-\infty, \infty)$
    range = $(0, 4]$

19. Not a function

20. Function
    domain = $[-5, -3) \cup (-3, 3)$
    range = $(-2, -1) \cup [0, 4)$

21. Function
domain = $[-2, \infty)$
range = $[-3, \infty)$

22. Not a function

23. Function
domain = $[-5, 4)$
range = $[-4, 4)$

24. Function
domain = $[0, 3) \cup (3, 6]$
range = $(-4, -1] \cup [0, 4]$

25. Function
domain = $(-\infty, \infty)$
range = $(-\infty, 4]$

26. Function
domain = $(-\infty, \infty)$
range = $(-\infty, 4]$

27. Function
domain = $[-2, \infty)$
range = $(-\infty, 3]$

28. Function
domain = $(-\infty, \infty)$
range = $(-\infty, \infty)$

29. Function
domain = $(-\infty, 0] \cup (1, \infty)$
range = $(-\infty, 1] \cup \{2\}$

30. Function
domain = $[-3, 3]$
range = $[-2, 2]$

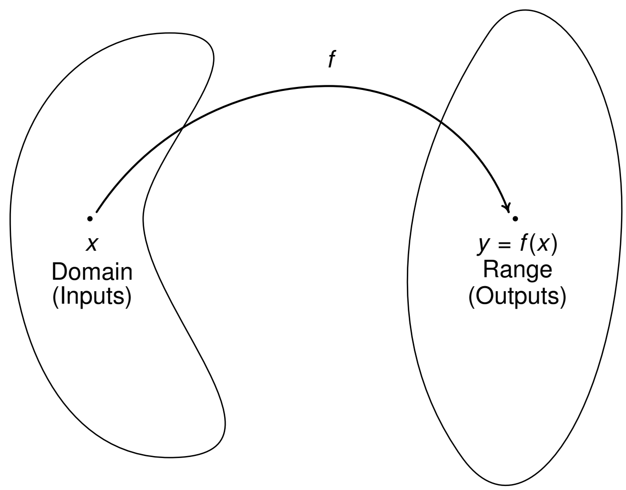
31. Not a function

32. Function
domain = $(-\infty, \infty)$
range = $\{2\}$

33. Function
34. Function
35. Function
36. Not a function
37. Function
38. Not a function
39. Not a function
40. Function
41. Not a function
42. Function
43. Not a function
44. Function
45. Function
46. Function
47. Not a function

1.4 Function Notation

In Definition 1.4, we described a function as a special kind of relation – one in which each x-coordinate is matched with only one y-coordinate. In this section, we focus more on the **process** by which the x is matched with the y. If we think of the domain of a function as a set of **inputs** and the range as a set of **outputs**, we can think of a function f as a process by which each input x is matched with only one output y. Since the output is completely determined by the input x and the process f, we symbolize the output with **function notation**: '$f(x)$', read 'f **of** x.' In other words, $f(x)$ is the output which results by applying the process f to the input x. In this case, the parentheses here do not indicate multiplication, as they do elsewhere in Algebra. This can cause confusion if the context is not clear, so you must read carefully. This relationship is typically visualized using a diagram similar to the one below.



The value of y is completely dependent on the choice of x. For this reason, x is often called the **independent variable**, or **argument** of f, whereas y is often called the **dependent variable**.

As we shall see, the process of a function f is usually described using an algebraic formula. For example, suppose a function f takes a real number and performs the following two steps, in sequence

1. multiply by 3
2. add 4

If we choose 5 as our input, in step 1 we multiply by 3 to get $(5)(3) = 15$. In step 2, we add 4 to our result from step 1 which yields $15 + 4 = 19$. Using function notation, we would write $f(5) = 19$ to indicate that the result of applying the process f to the input 5 gives the output 19. In general, if we use x for the input, applying step 1 produces $3x$. Following with step 2 produces $3x + 4$ as our final output. Hence for an input x, we get the output $f(x) = 3x + 4$. Notice that to check our formula for the case $x = 5$, we replace the occurrence of x in the formula for $f(x)$ with 5 to get $f(5) = 3(5) + 4 = 15 + 4 = 19$, as required.

Example 1.4.1. Suppose a function g is described by applying the following steps, in sequence

1. add 4
2. multiply by 3

Determine $g(5)$ and find an expression for $g(x)$.

Solution. Starting with 5, step 1 gives $5 + 4 = 9$. Continuing with step 2, we get $(3)(9) = 27$. To find a formula for $g(x)$, we start with our input x. Step 1 produces $x + 4$. We now wish to multiply this entire quantity by 3, so we use a parentheses: $3(x + 4) = 3x + 12$. Hence, $g(x) = 3x + 12$. We can check our formula by replacing x with 5 to get $g(5) = 3(5) + 12 = 15 + 12 = 27$ ✓. □

Most of the functions we will encounter in College Algebra will be described using formulas like the ones we developed for $f(x)$ and $g(x)$ above. Evaluating formulas using this function notation is a key skill for success in this and many other Math courses.

Example 1.4.2. Let $f(x) = -x^2 + 3x + 4$

1. Find and simplify the following.
   (a) $f(-1)$, $f(0)$, $f(2)$
   (b) $f(2x)$, $2f(x)$
   (c) $f(x + 2)$, $f(x) + 2$, $f(x) + f(2)$
2. Solve $f(x) = 4$.

Solution.

1. (a) To find $f(-1)$, we replace every occurrence of x in the expression $f(x)$ with -1

$$\begin{aligned} f(-1) &= -(-1)^2 + 3(-1) + 4 \\ &= -(1) + (-3) + 4 \\ &= 0 \end{aligned}$$

   Similarly, $f(0) = -(0)^2 + 3(0) + 4 = 4$, and $f(2) = -(2)^2 + 3(2) + 4 = -4 + 6 + 4 = 6$.

   (b) To find $f(2x)$, we replace every occurrence of x with the quantity $2x$

$$\begin{aligned} f(2x) &= -(2x)^2 + 3(2x) + 4 \\ &= -(4x^2) + (6x) + 4 \\ &= -4x^2 + 6x + 4 \end{aligned}$$

The expression $2f(x)$ means we multiply the expression $f(x)$ by 2

$$\begin{aligned} 2f(x) &= 2\left(-x^2+3x+4\right) \\ &= -2x^2+6x+8 \end{aligned}$$

(c) To find $f(x+2)$, we replace every occurrence of x with the quantity $x+2$

$$\begin{aligned} f(x+2) &= -(x+2)^2+3(x+2)+4 \\ &= -\left(x^2+4x+4\right)+(3x+6)+4 \\ &= -x^2-4x-4+3x+6+4 \\ &= -x^2-x+6 \end{aligned}$$

To find $f(x)+2$, we add 2 to the expression for $f(x)$

$$\begin{aligned} f(x)+2 &= \left(-x^2+3x+4\right)+2 \\ &= -x^2+3x+6 \end{aligned}$$

From our work above, we see $f(2)=6$ so that

$$\begin{aligned} f(x)+f(2) &= \left(-x^2+3x+4\right)+6 \\ &= -x^2+3x+10 \end{aligned}$$

2. Since $f(x)=-x^2+3x+4$, the equation $f(x)=4$ is equivalent to $-x^2+3x+4=4$. Solving we get $-x^2+3x=0$, or $x(-x+3)=0$. We get $x=0$ or $x=3$, and we can verify these answers by checking that $f(0)=4$ and $f(3)=4$. □

A few notes about Example 1.4.2 are in order. First note the difference between the answers for $f(2x)$ and $2f(x)$. For $f(2x)$, we are multiplying the *input* by 2; for $2f(x)$, we are multiplying the *output* by 2. As we see, we get entirely different results. Along these lines, note that $f(x+2)$, $f(x)+2$ and $f(x)+f(2)$ are three *different* expressions as well. Even though function notation uses parentheses, as does multiplication, there is *no* general 'distributive property' of function notation. Finally, note the practice of using parentheses when substituting one algebraic expression into another; we highly recommend this practice as it will reduce careless errors.

Suppose now we wish to find $r(3)$ for $r(x)=\frac{2x}{x^2-9}$. Substitution gives

$$r(3)=\frac{2(3)}{(3)^2-9}=\frac{6}{0},$$

which is undefined. (Why is this, again?) The number 3 is not an allowable input to the function r; in other words, 3 is not in the domain of r. Which other real numbers are forbidden in this formula? We think back to arithmetic. The reason $r(3)$ is undefined is because substitution results in a division by 0. To determine which other numbers result in such a transgression, we set the denominator equal to 0 and solve

$$\begin{aligned} x^2 - 9 &= 0 \\ x^2 &= 9 \\ \sqrt{x^2} &= \sqrt{9} \quad \text{extract square roots} \\ x &= \pm 3 \end{aligned}$$

As long as we substitute numbers other than 3 and -3, the expression $r(x)$ is a real number. Hence, we write our domain in interval notation[1] as $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$. When a formula for a function is given, we assume that the function is valid for all real numbers which make arithmetic sense when substituted into the formula. This set of numbers is often called the **implied domain**[2] of the function. At this stage, there are only two mathematical sins we need to avoid: division by 0 and extracting even roots of negative numbers. The following example illustrates these concepts.

Example 1.4.3. Find the domain[3] of the following functions.

1. $g(x) = \sqrt{4 - 3x}$
2. $h(x) = \sqrt[5]{4 - 3x}$
3. $f(x) = \dfrac{2}{1 - \dfrac{4x}{x-3}}$
4. $F(x) = \dfrac{\sqrt[4]{2x+1}}{x^2 - 1}$
5. $r(t) = \dfrac{4}{6 - \sqrt{t+3}}$
6. $I(x) = \dfrac{3x^2}{x}$

Solution.

1. The potential disaster for g is if the radicand[4] is negative. To avoid this, we set $4 - 3x \geq 0$. From this, we get $3x \leq 4$ or $x \leq \frac{4}{3}$. What this shows is that as long as $x \leq \frac{4}{3}$, the expression $4 - 3x \geq 0$, and the formula $g(x)$ returns a real number. Our domain is $\left(-\infty, \frac{4}{3}\right]$.

2. The formula for $h(x)$ is hauntingly close to that of $g(x)$ with one key difference – whereas the expression for $g(x)$ includes an even indexed root (namely a square root), the formula for $h(x)$ involves an odd indexed root (the fifth root). Since odd roots of real numbers (even negative real numbers) are real numbers, there is no restriction on the inputs to h. Hence, the domain is $(-\infty, \infty)$.

3. In the expression for f, there are two denominators. We need to make sure neither of them is 0. To that end, we set each denominator equal to 0 and solve. For the 'small' denominator, we get $x - 3 = 0$ or $x = 3$. For the 'large' denominator

[1] See the Exercises for Section 1.1.
[2] or, 'implicit domain'
[3] The word 'implied' is, well, implied.
[4] The 'radicand' is the expression 'inside' the radical.

$$\begin{aligned} 1 - \frac{4x}{x-3} &= 0 \\ 1 &= \frac{4x}{x-3} \\ (1)(x-3) &= \left(\frac{4x}{\cancel{x-3}}\right)\cancel{(x-3)} \quad \text{clear denominators} \\ x - 3 &= 4x \\ -3 &= 3x \\ -1 &= x \end{aligned}$$

So we get two real numbers which make denominators 0, namely $x = -1$ and $x = 3$. Our domain is all real numbers except -1 and 3: $(-\infty, -1) \cup (-1, 3) \cup (3, \infty)$.

4. In finding the domain of F, we notice that we have two potentially hazardous issues: not only do we have a denominator, we have a fourth (even-indexed) root. Our strategy is to determine the restrictions imposed by each part and select the real numbers which satisfy both conditions. To satisfy the fourth root, we require $2x + 1 \geq 0$. From this we get $2x \geq -1$ or $x \geq -\frac{1}{2}$. Next, we round up the values of x which could cause trouble in the denominator by setting the denominator equal to 0. We get $x^2 - 1 = 0$, or $x = \pm 1$. Hence, in order for a real number x to be in the domain of F, $x \geq -\frac{1}{2}$ but $x \neq \pm 1$. In interval notation, this set is $\left[-\frac{1}{2}, 1\right) \cup (1, \infty)$.

5. Don't be put off by the 't' here. It is an independent variable representing a real number, just like x does, and is subject to the same restrictions. As in the previous problem, we have double danger here: we have a square root and a denominator. To satisfy the square root, we need a non-negative radicand so we set $t + 3 \geq 0$ to get $t \geq -3$. Setting the denominator equal to zero gives $6 - \sqrt{t+3} = 0$, or $\sqrt{t+3} = 6$. Squaring both sides gives $t + 3 = 36$, or $t = 33$. Since we squared both sides in the course of solving this equation, we need to check our answer.[5] Sure enough, when $t = 33$, $6 - \sqrt{t+3} = 6 - \sqrt{36} = 0$, so $t = 33$ will cause problems in the denominator. At last we can find the domain of r: we need $t \geq -3$, but $t \neq 33$. Our final answer is $[-3, 33) \cup (33, \infty)$.

6. It's tempting to simplify $I(x) = \frac{3x^2}{x} = 3x$, and, since there are no longer any denominators, claim that there are no longer any restrictions. However, in simplifying $I(x)$, we are assuming $x \neq 0$, since $\frac{0}{0}$ is undefined.[6] Proceeding as before, we find the domain of I to be all real numbers except 0: $(-\infty, 0) \cup (0, \infty)$. □

It is worth reiterating the importance of finding the domain of a function *before* simplifying, as evidenced by the function I in the previous example. Even though the formula $I(x)$ simplifies to

[5] Do you remember why? Consider squaring both sides to 'solve' $\sqrt{t+1} = -2$.

[6] More precisely, the fraction $\frac{0}{0}$ is an 'indeterminate form'. Calculus is required to tame such beasts.

$3x$, it would be inaccurate to write $I(x) = 3x$ without adding the stipulation that $x \neq 0$. It would be analogous to not reporting taxable income or some other sin of omission.

1.4.1 Modeling with Functions

The importance of Mathematics to our society lies in its value to approximate, or **model** real-world phenomenon. Whether it be used to predict the high temperature on a given day, determine the hours of daylight on a given day, or predict population trends of various and sundry real and mythical beasts, Mathematics is second only to literacy in the importance of humanity's development.[7]

It is important to keep in mind that anytime Mathematics is used to approximate reality, there are always limitations to the model. For example, suppose grapes are on sale at the local market for \$1.50 per pound. Then one pound of grapes costs \$1.50, two pounds of grapes cost \$3.00, and so forth. Suppose we want to develop a formula which relates the cost of buying grapes to the amount of grapes being purchased. Since these two quantities vary from situation to situation, we assign them variables. Let c denote the cost of the grapes and let g denote the amount of grapes purchased. To find the cost c of the grapes, we multiply the amount of grapes g by the price \$1.50 dollars per pound to get

$$c = 1.5g$$

In order for the units to be correct in the formula, g must be measured in *pounds* of grapes in which case the computed value of c is measured in *dollars*. Since we're interested in finding the cost c given an amount g, we think of g as the independent variable and c as the dependent variable. Using the language of function notation, we write

$$c(g) = 1.5g$$

where g is the amount of grapes purchased (in pounds) and $c(g)$ is the cost (in dollars). For example, $c(5)$ represents the cost, in dollars, to purchase 5 pounds of grapes. In this case, $c(5) = 1.5(5) = 7.5$, so it would cost \$7.50. If, on the other hand, we wanted to find the *amount* of grapes we can purchase for \$5, we would need to set $c(g) = 5$ and solve for g. In this case, $c(g) = 1.5g$, so solving $c(g) = 5$ is equivalent to solving $1.5g = 5$ Doing so gives $g = \frac{5}{1.5} = 3.\overline{3}$. This means we can purchase exactly $3.\overline{3}$ pounds of grapes for \$5. Of course, you would be hard-pressed to buy exactly $3.\overline{3}$ pounds of grapes,[8] and this leads us to our next topic of discussion, the **applied domain**[9] of a function.

Even though, mathematically, $c(g) = 1.5g$ has no domain restrictions (there are no denominators and no even-indexed radicals), there are certain values of g that don't make any physical sense. For example, $g = -1$ corresponds to 'purchasing' -1 pounds of grapes.[10] Also, unless the 'local market' mentioned is the State of California (or some other exporter of grapes), it also doesn't

[7] In Carl's humble opinion, of course . . .

[8] You could get close... within a certain specified margin of error, perhaps.

[9] or, 'explicit domain'

[10] Maybe this means *returning* a pound of grapes?

make much sense for $g = 500{,}000{,}000$, either. So the reality of the situation limits what g can be, and these limits determine the applied domain of g. Typically, an applied domain is stated explicitly. In this case, it would be common to see something like $c(g) = 1.5g$, $0 \leq g \leq 100$, meaning the number of pounds of grapes purchased is limited from 0 up to 100. The upper bound here, 100 may represent the inventory of the market, or some other limit as set by local policy or law. Even with this restriction, our model has its limitations. As we saw above, it is virtually impossible to buy exactly $3.\overline{3}$ pounds of grapes so that our cost is exactly \$5. In this case, being sensible shoppers, we would most likely 'round down' and purchase 3 pounds of grapes or however close the market scale can read to $3.\overline{3}$ without being over. It is time for a more sophisticated example.

Example 1.4.4. The height h in feet of a model rocket above the ground t seconds after lift-off is given by

$$h(t) = \begin{cases} -5t^2 + 100t, & \text{if } 0 \leq t \leq 20 \\ 0, & \text{if } t > 20 \end{cases}$$

1. Find and interpret $h(10)$ and $h(60)$.

2. Solve $h(t) = 375$ and interpret your answers.

Solution.

1. We first note that the independent variable here is t, chosen because it represents time. Secondly, the function is broken up into two rules: one formula for values of t between 0 and 20 inclusive, and another for values of t greater than 20. Since $t = 10$ satisfies the inequality $0 \leq t \leq 20$, we use the first formula listed, $h(t) = -5t^2 + 100t$, to find $h(10)$. We get $h(10) = -5(10)^2 + 100(10) = 500$. Since t represents the number of seconds since lift-off and $h(t)$ is the height above the ground in feet, the equation $h(10) = 500$ means that 10 seconds after lift-off, the model rocket is 500 feet above the ground. To find $h(60)$, we note that $t = 60$ satisfies $t > 20$, so we use the rule $h(t) = 0$. This function returns a value of 0 regardless of what value is substituted in for t, so $h(60) = 0$. This means that 60 seconds after lift-off, the rocket is 0 feet above the ground; in other words, a minute after lift-off, the rocket has already returned to Earth.

2. Since the function h is defined in pieces, we need to solve $h(t) = 375$ in pieces. For $0 \leq t \leq 20$, $h(t) = -5t^2 + 100t$, so for these values of t, we solve $-5t^2 + 100t = 375$. Rearranging terms, we get $5t^2 - 100t + 375 = 0$, and factoring gives $5(t - 5)(t - 15) = 0$. Our answers are $t = 5$ and $t = 15$, and since both of these values of t lie between 0 and 20, we keep both solutions. For $t > 20$, $h(t) = 0$, and in this case, there are no solutions to $0 = 375$. In terms of the model rocket, solving $h(t) = 375$ corresponds to finding when, if ever, the rocket reaches 375 feet above the ground. Our two answers, $t = 5$ and $t = 15$ correspond to the rocket reaching this altitude *twice* – once 5 seconds after launch, and again 15 seconds after launch.[11] □

[11] What goes up . . .

The type of function in the previous example is called a **piecewise-defined** function, or 'piecewise' function for short. Many real-world phenomena (e.g. postal rates,[12] income tax formulas[13]) are modeled by such functions.

By the way, if we wanted to avoid using a piecewise function in Example 1.4.4, we could have used $h(t) = -5t^2 + 100t$ on the explicit domain $0 \leq t \leq 20$ because after 20 seconds, the rocket is on the ground and stops moving. In many cases, though, piecewise functions are your only choice, so it's best to understand them well.

Mathematical modeling is not a one-section topic. It's not even a one-*course* topic as is evidenced by undergraduate and graduate courses in mathematical modeling being offered at many universities. Thus our goal in this section cannot possibly be to tell you the whole story. What we can do is get you started. As we study new classes of functions, we will see what phenomena they can be used to model. In that respect, mathematical modeling cannot be a topic in a book, but rather, must be a theme of the book. For now, we have you explore some very basic models in the Exercises because you need to crawl to walk to run. As we learn more about functions, we'll help you build your own models and get you on your way to applying Mathematics to your world.

[12]See the United States Postal Service website http://www.usps.com/prices/first-class-mail-prices.htm

[13]See the Internal Revenue Service's website http://www.irs.gov/pub/irs-pdf/i1040tt.pdf

1.4.2 Exercises

In Exercises 1 - 10, find an expression for $f(x)$ and state its domain.

1. f is a function that takes a real number x and performs the following three steps in the order given: (1) multiply 2; (2) add 3; (3) divide by 4.
2. f is a function that takes a real number x and performs the following three steps in the order given: (1) add 3; (2) multiply by 2; (3) divide by 4.
3. f is a function that takes a real number x and performs the following three steps in the order given: (1) divide by 4; (2) add 3; (3) multiply by 2.
4. f is a function that takes a real number x and performs the following three steps in the order given: (1) multiply 2; (2) add 3; (3) take the square root.
5. f is a function that takes a real number x and performs the following three steps in the order given: (1) add 3; (2) multiply 2; (3) take the square root.
6. f is a function that takes a real number x and performs the following three steps in the order given: (1) add 3; (2) take the square root; (3) multiply by 2.
7. f is a function that takes a real number x and performs the following three steps in the order given: (1) take the square root; (2) subtract 13; (3) make the quantity the denominator of a fraction with numerator 4.
8. f is a function that takes a real number x and performs the following three steps in the order given: (1) subtract 13; (2) take the square root; (3) make the quantity the denominator of a fraction with numerator 4.
9. f is a function that takes a real number x and performs the following three steps in the order given: (1) take the square root; (2) make the quantity the denominator of a fraction with numerator 4; (3) subtract 13.
10. f is a function that takes a real number x and performs the following three steps in the order given: (1) make the quantity the denominator of a fraction with numerator 4; (2) take the square root; (3) subtract 13.

In Exercises 11 - 18, use the given function f to find and simplify the following:

- $f(3)$
- $f(-1)$
- $f\left(\frac{3}{2}\right)$
- $f(4x)$
- $4f(x)$
- $f(-x)$
- $f(x-4)$
- $f(x)-4$
- $f\left(x^2\right)$

11. $f(x) = 2x + 1$

12. $f(x) = 3 - 4x$

13. $f(x) = 2 - x^2$

14. $f(x) = x^2 - 3x + 2$

15. $f(x) = \dfrac{x}{x-1}$

16. $f(x) = \dfrac{2}{x^3}$

17. $f(x) = 6$

18. $f(x) = 0$

In Exercises 19 - 26, use the given function f to find and simplify the following:

- $f(2)$
- $f(-2)$
- $f(2a)$
- $2f(a)$
- $f(a+2)$
- $f(a) + f(2)$
- $f\left(\frac{2}{a}\right)$
- $\frac{f(a)}{2}$
- $f(a+h)$

19. $f(x) = 2x - 5$

20. $f(x) = 5 - 2x$

21. $f(x) = 2x^2 - 1$

22. $f(x) = 3x^2 + 3x - 2$

23. $f(x) = \sqrt{2x+1}$

24. $f(x) = 117$

25. $f(x) = \dfrac{x}{2}$

26. $f(x) = \dfrac{2}{x}$

In Exercises 27 - 34, use the given function f to find $f(0)$ and solve $f(x) = 0$

27. $f(x) = 2x - 1$

28. $f(x) = 3 - \frac{2}{5}x$

29. $f(x) = 2x^2 - 6$

30. $f(x) = x^2 - x - 12$

31. $f(x) = \sqrt{x+4}$

32. $f(x) = \sqrt{1-2x}$

33. $f(x) = \dfrac{3}{4-x}$

34. $f(x) = \dfrac{3x^2 - 12x}{4 - x^2}$

35. Let $f(x) = \begin{cases} x+5, & x \leq -3 \\ \sqrt{9-x^2}, & -3 < x \leq 3 \\ -x+5, & x > 3 \end{cases}$ Compute the following function values.

(a) $f(-4)$

(b) $f(-3)$

(c) $f(3)$

(d) $f(3.001)$

(e) $f(-3.001)$

(f) $f(2)$

36. Let $f(x) = \begin{cases} x^2 & \text{if } x \le -1 \\ \sqrt{1-x^2} & \text{if } -1 < x \le 1 \\ x & \text{if } x > 1 \end{cases}$ Compute the following function values.

(a) $f(4)$ (b) $f(-3)$ (c) $f(1)$

(d) $f(0)$ (e) $f(-1)$ (f) $f(-0.999)$

In Exercises 37 - 62, find the (implied) domain of the function.

37. $f(x) = x^4 - 13x^3 + 56x^2 - 19$

38. $f(x) = x^2 + 4$

39. $f(x) = \dfrac{x-2}{x+1}$

40. $f(x) = \dfrac{3x}{x^2+x-2}$

41. $f(x) = \dfrac{2x}{x^2+3}$

42. $f(x) = \dfrac{2x}{x^2-3}$

43. $f(x) = \dfrac{x+4}{x^2-36}$

44. $f(x) = \dfrac{x-2}{x-2}$

45. $f(x) = \sqrt{3-x}$

46. $f(x) = \sqrt{2x+5}$

47. $f(x) = 9x\sqrt{x+3}$

48. $f(x) = \dfrac{\sqrt{7-x}}{x^2+1}$

49. $f(x) = \sqrt{6x-2}$

50. $f(x) = \dfrac{6}{\sqrt{6x-2}}$

51. $f(x) = \sqrt[3]{6x-2}$

52. $f(x) = \dfrac{6}{4-\sqrt{6x-2}}$

53. $f(x) = \dfrac{\sqrt{6x-2}}{x^2-36}$

54. $f(x) = \dfrac{\sqrt[3]{6x-2}}{x^2+36}$

55. $s(t) = \dfrac{t}{t-8}$

56. $Q(r) = \dfrac{\sqrt{r}}{r-8}$

57. $b(\theta) = \dfrac{\theta}{\sqrt{\theta-8}}$

58. $A(x) = \sqrt{x-7} + \sqrt{9-x}$

59. $\alpha(y) = \sqrt[3]{\dfrac{y}{y-8}}$

60. $g(v) = \dfrac{1}{4-\dfrac{1}{v^2}}$

61. $T(t) = \dfrac{\sqrt{t}-8}{5-t}$

62. $u(w) = \dfrac{w-8}{5-\sqrt{w}}$

63. The area A enclosed by a square, in square inches, is a function of the length of one of its sides x, when measured in inches. This relation is expressed by the formula $A(x) = x^2$ for $x > 0$. Find $A(3)$ and solve $A(x) = 36$. Interpret your answers to each. Why is x restricted to $x > 0$?

64. The area A enclosed by a circle, in square meters, is a function of its radius r, when measured in meters. This relation is expressed by the formula $A(r) = \pi r^2$ for $r > 0$. Find $A(2)$ and solve $A(r) = 16\pi$. Interpret your answers to each. Why is r restricted to $r > 0$?

65. The volume V enclosed by a cube, in cubic centimeters, is a function of the length of one of its sides x, when measured in centimeters. This relation is expressed by the formula $V(x) = x^3$ for $x > 0$. Find $V(5)$ and solve $V(x) = 27$. Interpret your answers to each. Why is x restricted to $x > 0$?

66. The volume V enclosed by a sphere, in cubic feet, is a function of the radius of the sphere r, when measured in feet. This relation is expressed by the formula $V(r) = \frac{4\pi}{3}r^3$ for $r > 0$. Find $V(3)$ and solve $V(r) = \frac{32\pi}{3}$. Interpret your answers to each. Why is r restricted to $r > 0$?

67. The height of an object dropped from the roof of an eight story building is modeled by: $h(t) = -16t^2 + 64$, $0 \leq t \leq 2$. Here, h is the height of the object off the ground, in feet, t seconds after the object is dropped. Find $h(0)$ and solve $h(t) = 0$. Interpret your answers to each. Why is t restricted to $0 \leq t \leq 2$?

68. The temperature T in degrees Fahrenheit t hours after 6 AM is given by $T(t) = -\frac{1}{2}t^2 + 8t + 3$ for $0 \leq t \leq 12$. Find and interpret $T(0)$, $T(6)$ and $T(12)$.

69. The function $C(x) = x^2 - 10x + 27$ models the cost, in *hundreds* of dollars, to produce x *thousand* pens. Find and interpret $C(0)$, $C(2)$ and $C(5)$.

70. Using data from the Bureau of Transportation Statistics, the average fuel economy F in miles per gallon for passenger cars in the US can be modeled by $F(t) = -0.0076t^2 + 0.45t + 16$, $0 \leq t \leq 28$, where t is the number of years since 1980. Use your calculator to find $F(0)$, $F(14)$ and $F(28)$. Round your answers to two decimal places and interpret your answers to each.

71. The population of Sasquatch in Portage County can be modeled by the function $P(t) = \frac{150t}{t+15}$, where t represents the number of years since 1803. Find and interpret $P(0)$ and $P(205)$. Discuss with your classmates what the applied domain and range of P should be.

72. For n copies of the book *Me and my Sasquatch*, a print on-demand company charges $C(n)$ dollars, where $C(n)$ is determined by the formula

$$C(n) = \begin{cases} 15n & \text{if} \quad 1 \leq n \leq 25 \\ 13.50n & \text{if} \quad 25 < n \leq 50 \\ 12n & \text{if} \quad n > 50 \end{cases}$$

(a) Find and interpret $C(20)$.

(b) How much does it cost to order 50 copies of the book? What about 51 copies?

(c) Your answer to 72b should get you thinking. Suppose a bookstore estimates it will sell 50 copies of the book. How many books can, in fact, be ordered for the same price as those 50 copies? (Round your answer to a whole number of books.)

73. An on-line comic book retailer charges shipping costs according to the following formula

$$S(n) = \begin{cases} 1.5n + 2.5 & \text{if} \quad 1 \leq n \leq 14 \\ 0 & \text{if} \quad n \geq 15 \end{cases}$$

where n is the number of comic books purchased and $S(n)$ is the shipping cost in dollars.

(a) What is the cost to ship 10 comic books?

(b) What is the significance of the formula $S(n) = 0$ for $n \geq 15$?

74. The cost C (in dollars) to talk m minutes a month on a mobile phone plan is modeled by

$$C(m) = \begin{cases} 25 & \text{if} \quad 0 \leq m \leq 1000 \\ 25 + 0.1(m - 1000) & \text{if} \quad m > 1000 \end{cases}$$

(a) How much does it cost to talk 750 minutes per month with this plan?

(b) How much does it cost to talk 20 hours a month with this plan?

(c) Explain the terms of the plan verbally.

75. We have through our examples tried to convince you that, in general, $f(a+b) \neq f(a) + f(b)$. It has been our experience that students refuse to believe us so we'll try again with a different approach. With the help of your classmates, find a function f for which the following properties are always true.

(a) $f(0) = f(-1+1) = f(-1) + f(1)$

(b) $f(5) = f(2+3) = f(2) + f(3)$

(c) $f(-6) = f(0-6) = f(0) - f(6)$

(d) $f(a+b) = f(a) + f(b)$ regardless of what two numbers we give you for a and b.

How many functions did you find that failed to satisfy the conditions above? Did $f(x) = x^2$ work? What about $f(x) = \sqrt{x}$ or $f(x) = 3x + 7$ or $f(x) = \dfrac{1}{x}$? Did you find an attribute common to those functions that did succeed? You should have, because there is only one extremely special family of functions that actually works here. Thus we return to our previous statement, **in general**, $f(a+b) \neq f(a) + f(b)$.

1.4.3 Answers

1. $f(x) = \frac{2x+3}{4}$
Domain: $(-\infty, \infty)$

2. $f(x) = \frac{2(x+3)}{4} = \frac{x+3}{2}$
Domain: $(-\infty, \infty)$

3. $f(x) = 2\left(\frac{x}{4} + 3\right) = \frac{1}{2}x + 6$
Domain: $(-\infty, \infty)$

4. $f(x) = \sqrt{2x+3}$
Domain: $\left[-\frac{3}{2}, \infty\right)$

5. $f(x) = \sqrt{2(x+3)} = \sqrt{2x+6}$
Domain: $[-3, \infty)$

6. $f(x) = 2\sqrt{x+3}$
Domain: $[-3, \infty)$

7. $f(x) = \frac{4}{\sqrt{x}-13}$
Domain: $[0, 169) \cup (169, \infty)$

8. $f(x) = \frac{4}{\sqrt{x-13}}$
Domain: $(13, \infty)$

9. $f(x) = \frac{4}{\sqrt{x}} - 13$
Domain: $(0, \infty)$

10. $f(x) = \sqrt{\frac{4}{x}} - 13 = \frac{2}{\sqrt{x}} - 13$
Domain: $(0, \infty)$

11. For $f(x) = 2x + 1$

- $f(3) = 7$
- $f(-1) = -1$
- $f\left(\frac{3}{2}\right) = 4$
- $f(4x) = 8x + 1$
- $4f(x) = 8x + 4$
- $f(-x) = -2x + 1$
- $f(x-4) = 2x - 7$
- $f(x) - 4 = 2x - 3$
- $f\left(x^2\right) = 2x^2 + 1$

12. For $f(x) = 3 - 4x$

- $f(3) = -9$
- $f(-1) = 7$
- $f\left(\frac{3}{2}\right) = -3$
- $f(4x) = 3 - 16x$
- $4f(x) = 12 - 16x$
- $f(-x) = 4x + 3$
- $f(x-4) = 19 - 4x$
- $f(x) - 4 = -4x - 1$
- $f\left(x^2\right) = 3 - 4x^2$

13. For $f(x) = 2 - x^2$

- $f(3) = -7$
- $f(-1) = 1$
- $f\left(\frac{3}{2}\right) = -\frac{1}{4}$
- $f(4x) = 2 - 16x^2$
- $4f(x) = 8 - 4x^2$
- $f(-x) = 2 - x^2$
- $f(x-4) = -x^2 + 8x - 14$
- $f(x) - 4 = -x^2 - 2$
- $f\left(x^2\right) = 2 - x^4$

14. For $f(x) = x^2 - 3x + 2$

- $f(3) = 2$
- $f(-1) = 6$
- $f\left(\frac{3}{2}\right) = -\frac{1}{4}$
- $f(4x) = 16x^2 - 12x + 2$
- $4f(x) = 4x^2 - 12x + 8$
- $f(-x) = x^2 + 3x + 2$
- $f(x-4) = x^2 - 11x + 30$
- $f(x) - 4 = x^2 - 3x - 2$
- $f\left(x^2\right) = x^4 - 3x^2 + 2$

15. For $f(x) = \frac{x}{x-1}$

- $f(3) = \frac{3}{2}$
- $f(-1) = \frac{1}{2}$
- $f\left(\frac{3}{2}\right) = 3$
- $f(4x) = \frac{4x}{4x-1}$
- $4f(x) = \frac{4x}{x-1}$
- $f(-x) = \frac{x}{x+1}$
- $f(x-4) = \frac{x-4}{x-5}$
- $f(x) - 4 = \frac{x}{x-1} - 4 = \frac{4-3x}{x-1}$
- $f\left(x^2\right) = \frac{x^2}{x^2-1}$

16. For $f(x) = \frac{2}{x^3}$

- $f(3) = \frac{2}{27}$
- $f(-1) = -2$
- $f\left(\frac{3}{2}\right) = \frac{16}{27}$
- $f(4x) = \frac{1}{32x^3}$
- $4f(x) = \frac{8}{x^3}$
- $f(-x) = -\frac{2}{x^3}$
- $f(x-4) = \frac{2}{(x-4)^3} = \frac{2}{x^3-12x^2+48x-64}$
- $f(x) - 4 = \frac{2}{x^3} - 4 = \frac{2-4x^3}{x^3}$
- $f\left(x^2\right) = \frac{2}{x^6}$

17. For $f(x) = 6$

- $f(3) = 6$
- $f(-1) = 6$
- $f\left(\frac{3}{2}\right) = 6$
- $f(4x) = 6$
- $4f(x) = 24$
- $f(-x) = 6$
- $f(x-4) = 6$
- $f(x) - 4 = 2$
- $f\left(x^2\right) = 6$

18. For $f(x) = 0$

- $f(3) = 0$
- $f(-1) = 0$
- $f\left(\frac{3}{2}\right) = 0$
- $f(4x) = 0$
- $4f(x) = 0$
- $f(-x) = 0$
- $f(x-4) = 0$
- $f(x) - 4 = -4$
- $f\left(x^2\right) = 0$

19. For $f(x) = 2x - 5$

- $f(2) = -1$
- $f(-2) = -9$
- $f(2a) = 4a - 5$
- $2f(a) = 4a - 10$
- $f(a+2) = 2a - 1$
- $f(a) + f(2) = 2a - 6$
- $f\left(\frac{2}{a}\right) = \frac{4}{a} - 5 = \frac{4-5a}{a}$
- $\frac{f(a)}{2} = \frac{2a-5}{2}$
- $f(a+h) = 2a + 2h - 5$

20. For $f(x) = 5 - 2x$

- $f(2) = 1$
- $f(-2) = 9$
- $f(2a) = 5 - 4a$
- $2f(a) = 10 - 4a$
- $f(a+2) = 1 - 2a$
- $f(a) + f(2) = 6 - 2a$
- $f\left(\frac{2}{a}\right) = 5 - \frac{4}{a} = \frac{5a-4}{a}$
- $\frac{f(a)}{2} = \frac{5-2a}{2}$
- $f(a+h) = 5 - 2a - 2h$

21. For $f(x) = 2x^2 - 1$

- $f(2) = 7$
- $f(-2) = 7$
- $f(2a) = 8a^2 - 1$
- $2f(a) = 4a^2 - 2$
- $f(a+2) = 2a^2 + 8a + 7$
- $f(a) + f(2) = 2a^2 + 6$
- $f\left(\frac{2}{a}\right) = \frac{8}{a^2} - 1 = \frac{8-a^2}{a^2}$
- $\frac{f(a)}{2} = \frac{2a^2-1}{2}$
- $f(a+h) = 2a^2 + 4ah + 2h^2 - 1$

22. For $f(x) = 3x^2 + 3x - 2$

- $f(2) = 16$
- $f(-2) = 4$
- $f(2a) = 12a^2 + 6a - 2$
- $2f(a) = 6a^2 + 6a - 4$
- $f(a+2) = 3a^2 + 15a + 16$
- $f(a) + f(2) = 3a^2 + 3a + 14$
- $f\left(\frac{2}{a}\right) = \frac{12}{a^2} + \frac{6}{a} - 2 = \frac{12+6a-2a^2}{a^2}$
- $\frac{f(a)}{2} = \frac{3a^2+3a-2}{2}$
- $f(a+h) = 3a^2+6ah+3h^2+3a + 3h - 2$

23. For $f(x) = \sqrt{2x+1}$

- $f(2) = \sqrt{5}$
- $f(-2)$ is not real
- $f(2a) = \sqrt{4a+1}$
- $2f(a) = 2\sqrt{2a+1}$
- $f(a+2) = \sqrt{2a+5}$
- $f(a) + f(2) = \sqrt{2a+1} + \sqrt{5}$
- $f\left(\frac{2}{a}\right) = \sqrt{\frac{4}{a}+1} = \sqrt{\frac{a+4}{a}}$
- $\frac{f(a)}{2} = \frac{\sqrt{2a+1}}{2}$
- $f(a+h) = \sqrt{2a+2h+1}$

24. For $f(x) = 117$

- $f(2) = 117$
- $f(-2) = 117$
- $f(2a) = 117$
- $2f(a) = 234$
- $f(a+2) = 117$
- $f(a) + f(2) = 234$
- $f\left(\frac{2}{a}\right) = 117$
- $\frac{f(a)}{2} = \frac{117}{2}$
- $f(a+h) = 117$

25. For $f(x) = \frac{x}{2}$

- $f(2) = 1$
- $f(-2) = -1$
- $f(2a) = a$
- $2f(a) = a$
- $f(a+2) = \frac{a+2}{2}$
- $f(a) + f(2) = \frac{a}{2} + 1 = \frac{a+2}{2}$
- $f\left(\frac{2}{a}\right) = \frac{1}{a}$
- $\frac{f(a)}{2} = \frac{a}{4}$
- $f(a+h) = \frac{a+h}{2}$

26. For $f(x) = \frac{2}{x}$

- $f(2) = 1$
- $f(-2) = -1$
- $f(2a) = \frac{1}{a}$
- $2f(a) = \frac{4}{a}$
- $f(a+2) = \frac{2}{a+2}$
- $f(a) + f(2) = \frac{2}{a} + 1 = \frac{a+2}{2}$
- $f\left(\frac{2}{a}\right) = a$
- $\frac{f(a)}{2} = \frac{1}{a}$
- $f(a+h) = \frac{2}{a+h}$

27. For $f(x) = 2x - 1$, $f(0) = -1$ and $f(x) = 0$ when $x = \frac{1}{2}$

28. For $f(x) = 3 - \frac{2}{5}x$, $f(0) = 3$ and $f(x) = 0$ when $x = \frac{15}{2}$

29. For $f(x) = 2x^2 - 6$, $f(0) = -6$ and $f(x) = 0$ when $x = \pm\sqrt{3}$

30. For $f(x) = x^2 - x - 12$, $f(0) = -12$ and $f(x) = 0$ when $x = -3$ or $x = 4$

31. For $f(x) = \sqrt{x+4}$, $f(0) = 2$ and $f(x) = 0$ when $x = -4$

32. For $f(x) = \sqrt{1-2x}$, $f(0) = 1$ and $f(x) = 0$ when $x = \frac{1}{2}$

33. For $f(x) = \frac{3}{4-x}$, $f(0) = \frac{3}{4}$ and $f(x)$ is never equal to 0

34. For $f(x) = \frac{3x^2-12x}{4-x^2}$, $f(0) = 0$ and $f(x) = 0$ when $x = 0$ or $x = 4$

35. (a) $f(-4) = 1$ (b) $f(-3) = 2$ (c) $f(3) = 0$
(d) $f(3.001) = 1.999$ (e) $f(-3.001) = 1.999$ (f) $f(2) = \sqrt{5}$

36. (a) $f(4) = 4$ (b) $f(-3) = 9$ (c) $f(1) = 0$
(d) $f(0) = 1$ (e) $f(-1) = 1$ (f) $f(-0.999) \approx 0.0447$

37. $(-\infty, \infty)$

38. $(-\infty, \infty)$

39. $(-\infty, -1) \cup (-1, \infty)$

40. $(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$

41. $(-\infty, \infty)$

42. $(-\infty, -\sqrt{3}) \cup (-\sqrt{3}, \sqrt{3}) \cup (\sqrt{3}, \infty)$

43. $(-\infty, -6) \cup (-6, 6) \cup (6, \infty)$

44. $(-\infty, 2) \cup (2, \infty)$

45. $(-\infty, 3]$

46. $\left[-\frac{5}{2}, \infty\right)$

47. $[-3, \infty)$

48. $(-\infty, 7]$

49. $\left[\frac{1}{3}, \infty\right)$

50. $\left(\frac{1}{3}, \infty\right)$

51. $(-\infty, \infty)$

52. $\left[\frac{1}{3}, 3\right) \cup (3, \infty)$

53. $\left[\frac{1}{3}, 6\right) \cup (6, \infty)$

54. $(-\infty, \infty)$

55. $(-\infty, 8) \cup (8, \infty)$

56. $[0, 8) \cup (8, \infty)$

57. $(8, \infty)$

58. $[7, 9]$

59. $(-\infty, 8) \cup (8, \infty)$

60. $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 0\right) \cup \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$

61. $[0, 5) \cup (5, \infty)$

62. $[0, 25) \cup (25, \infty)$

63. $A(3) = 9$, so the area enclosed by a square with a side of length 3 inches is 9 square inches. The solutions to $A(x) = 36$ are $x = \pm 6$. Since x is restricted to $x > 0$, we only keep $x = 6$. This means for the area enclosed by the square to be 36 square inches, the length of the side needs to be 6 inches. Since x represents a length, $x > 0$.

64. $A(2) = 4\pi$, so the area enclosed by a circle with radius 2 meters is 4π square meters. The solutions to $A(r) = 16\pi$ are $r = \pm 4$. Since r is restricted to $r > 0$, we only keep $r = 4$. This means for the area enclosed by the circle to be 16π square meters, the radius needs to be 4 meters. Since r represents a radius (length), $r > 0$.

65. $V(5) = 125$, so the volume enclosed by a cube with a side of length 5 centimeters is 125 cubic centimeters. The solution to $V(x) = 27$ is $x = 3$. This means for the volume enclosed by the cube to be 27 cubic centimeters, the length of the side needs to 3 centimeters. Since x represents a length, $x > 0$.

66. $V(3) = 36\pi$, so the volume enclosed by a sphere with radius 3 feet is 36π cubic feet. The solution to $V(r) = \frac{32\pi}{3}$ is $r = 2$. This means for the volume enclosed by the sphere to be $\frac{32\pi}{3}$ cubic feet, the radius needs to 2 feet. Since r represents a radius (length), $r > 0$.

67. $h(0) = 64$, so at the moment the object is dropped off the building, the object is 64 feet off of the ground. The solutions to $h(t) = 0$ are $t = \pm 2$. Since we restrict $0 \leq t \leq 2$, we only keep $t = 2$. This means 2 seconds after the object is dropped off the building, it is 0 feet off the ground. Said differently, the object hits the ground after 2 seconds. The restriction $0 \leq t \leq 2$ restricts the time to be between the moment the object is released and the moment it hits the ground.

68. $T(0) = 3$, so at 6 AM (0 hours after 6 AM), it is 3° Fahrenheit. $T(6) = 33$, so at noon (6 hours after 6 AM), the temperature is 33° Fahrenheit. $T(12) = 27$, so at 6 PM (12 hours after 6 AM), it is 27° Fahrenheit.

69. $C(0) = 27$, so to make 0 pens, it costs[14] \$2700. $C(2) = 11$, so to make 2000 pens, it costs \$1100. $C(5) = 2$, so to make 5000 pens, it costs \$2000.

70. $F(0) = 16.00$, so in 1980 (0 years after 1980), the average fuel economy of passenger cars in the US was 16.00 miles per gallon. $F(14) = 20.81$, so in 1994 (14 years after 1980), the average fuel economy of passenger cars in the US was 20.81 miles per gallon. $F(28) = 22.64$, so in 2008 (28 years after 1980), the average fuel economy of passenger cars in the US was 22.64 miles per gallon.

71. $P(0) = 0$ which means in 1803 (0 years after 1803), there are no Sasquatch in Portage County. $P(205) = \frac{3075}{22} \approx 139.77$, so in 2008 (205 years after 1803), there were between 139 and 140 Sasquatch in Portage County.

72. (a) $C(20) = 300$. It costs \$300 for 20 copies of the book.
    (b) $C(50) = 675$, so it costs \$675 for 50 copies of the book. $C(51) = 612$, so it costs \$612 for 51 copies of the book.
    (c) 56 books.

73. (a) $S(10) = 17.5$, so it costs \$17.50 to ship 10 comic books.
    (b) There is free shipping on orders of 15 or more comic books.

74. (a) $C(750) = 25$, so it costs \$25 to talk 750 minutes per month with this plan.
    (b) Since 20 hours = 1200 minutes, we substitute $m = 1200$ and get $C(1200) = 45$. It costs \$45 to talk 20 hours per month with this plan.
    (c) It costs \$25 for up to 1000 minutes and 10 cents per minute for each minute over 1000 minutes.

[14] This is called the 'fixed' or 'start-up' cost. We'll revisit this concept on page 199.

1.5 Function Arithmetic

In the previous section we used the newly defined function notation to make sense of expressions such as '$f(x)+2$' and '$2f(x)$' for a given function f. It would seem natural, then, that functions should have their own arithmetic which is consistent with the arithmetic of real numbers. The following definitions allow us to add, subtract, multiply and divide functions using the arithmetic we already know for real numbers.

Function Arithmetic

Suppose f and g are functions and x is in both the domain of f and the domain of g.[a]

- The **sum** of f and g, denoted $f+g$, is the function defined by the formula

$$(f+g)(x) = f(x) + g(x)$$

- The **difference** of f and g, denoted $f-g$, is the function defined by the formula

$$(f-g)(x) = f(x) - g(x)$$

- The **product** of f and g, denoted fg, is the function defined by the formula

$$(fg)(x) = f(x)g(x)$$

- The **quotient** of f and g, denoted $\dfrac{f}{g}$, is the function defined by the formula

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},$$

provided $g(x) \neq 0$.

[a] Thus x is an element of the intersection of the two domains.

In other words, to add two functions, we add their outputs; to subtract two functions, we subtract their outputs, and so on. Note that while the formula $(f+g)(x) = f(x) + g(x)$ looks suspiciously like some kind of distributive property, it is nothing of the sort; the addition on the left hand side of the equation is *function* addition, and we are using this equation to *define* the output of the new function $f+g$ as the sum of the real number outputs from f and g.

Example 1.5.1. Let $f(x) = 6x^2 - 2x$ and $g(x) = 3 - \dfrac{1}{x}$.

1. Find $(f+g)(-1)$
2. Find $(fg)(2)$
3. Find the domain of $g-f$ then find and simplify a formula for $(g-f)(x)$.

4. Find the domain of $\left(\frac{g}{f}\right)$ then find and simplify a formula for $\left(\frac{g}{f}\right)(x)$.

Solution.

1. To find $(f+g)(-1)$ we first find $f(-1) = 8$ and $g(-1) = 4$. By definition, we have that $(f+g)(-1) = f(-1) + g(-1) = 8 + 4 = 12$.

2. To find $(fg)(2)$, we first need $f(2)$ and $g(2)$. Since $f(2) = 20$ and $g(2) = \frac{5}{2}$, our formula yields $(fg)(2) = f(2)g(2) = (20)\left(\frac{5}{2}\right) = 50$.

3. One method to find the domain of $g-f$ is to find the domain of g and of f separately, then find the intersection of these two sets. Owing to the denominator in the expression $g(x) = 3 - \frac{1}{x}$, we get that the domain of g is $(-\infty, 0) \cup (0, \infty)$. Since $f(x) = 6x^2 - 2x$ is valid for all real numbers, we have no further restrictions. Thus the domain of $g - f$ matches the domain of g, namely, $(-\infty, 0) \cup (0, \infty)$.

   A second method is to analyze the formula for $(g-f)(x)$ *before simplifying* and look for the usual domain issues. In this case,

   $$(g-f)(x) = g(x) - f(x) = \left(3 - \frac{1}{x}\right) - \left(6x^2 - 2x\right),$$

   so we find, as before, the domain is $(-\infty, 0) \cup (0, \infty)$.

   Moving along, we need to simplify a formula for $(g-f)(x)$. In this case, we get common denominators and attempt to reduce the resulting fraction. Doing so, we get

   $$\begin{aligned}
   (g-f)(x) &= g(x) - f(x) \\
   &= \left(3 - \frac{1}{x}\right) - \left(6x^2 - 2x\right) \\
   &= 3 - \frac{1}{x} - 6x^2 + 2x \\
   &= \frac{3x}{x} - \frac{1}{x} - \frac{6x^3}{x} + \frac{2x^2}{x} \qquad \text{get common denominators} \\
   &= \frac{3x - 1 - 6x^3 - 2x^2}{x} \\
   &= \frac{-6x^3 - 2x^2 + 3x - 1}{x}
   \end{aligned}$$

4. As in the previous example, we have two ways to approach finding the domain of $\frac{g}{f}$. First, we can find the domain of g and f separately, and find the intersection of these two sets. In addition, since $\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)}$, we are introducing a new denominator, namely $f(x)$, so we need to guard against this being 0 as well. Our previous work tells us that the domain of g is $(-\infty, 0) \cup (0, \infty)$ and the domain of f is $(-\infty, \infty)$. Setting $f(x) = 0$ gives $6x^2 - 2x = 0$

or $x = 0, \frac{1}{3}$. As a result, the domain of $\frac{g}{f}$ is all real numbers except $x = 0$ and $x = \frac{1}{3}$, or $(-\infty, 0) \cup \left(0, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$.

Alternatively, we may proceed as above and analyze the expression $\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)}$ *before* simplifying. In this case,

$$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{3 - \frac{1}{x}}{6x^2 - 2x}$$

We see immediately from the 'little' denominator that $x \neq 0$. To keep the 'big' denominator away from 0, we solve $6x^2 - 2x = 0$ and get $x = 0$ or $x = \frac{1}{3}$. Hence, as before, we find the domain of $\frac{g}{f}$ to be $(-\infty, 0) \cup \left(0, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$.

Next, we find and simplify a formula for $\left(\frac{g}{f}\right)(x)$.

$$\begin{aligned}
\left(\frac{g}{f}\right)(x) &= \frac{g(x)}{f(x)} \\
&= \frac{3 - \frac{1}{x}}{6x^2 - 2x} \\
&= \frac{3 - \frac{1}{x}}{6x^2 - 2x} \cdot \frac{x}{x} && \text{simplify compound fractions} \\
&= \frac{\left(3 - \frac{1}{x}\right) x}{\left(6x^2 - 2x\right) x} \\
&= \frac{3x - 1}{\left(6x^2 - 2x\right) x} \\
&= \frac{3x - 1}{2x^2(3x - 1)} && \text{factor} \\
&= \frac{\cancel{(3x - 1)}^{\,1}}{2x^2\cancel{(3x - 1)}} && \text{cancel} \\
&= \frac{1}{2x^2}
\end{aligned}$$

□

Please note the importance of finding the domain of a function *before* simplifying its expression. In number 4 in Example 1.5.1 above, had we waited to find the domain of $\frac{g}{f}$ until after simplifying, we'd just have the formula $\frac{1}{2x^2}$ to go by, and we would (incorrectly!) state the domain as $(-\infty, 0) \cup (0, \infty)$, since the other troublesome number, $x = \frac{1}{3}$, was canceled away.

Next, we turn our attention to the **difference quotient** of a function.

Definition 1.6. Given a function f, the **difference quotient** of f is the expression

$$\frac{f(x+h)-f(x)}{h}$$

We will revisit this concept in Section 2.1, but for now, we use it as a way to practice function notation and function arithmetic. For reasons which will become clear in Calculus, 'simplifying' a difference quotient means rewriting it in a form where the 'h' in the definition of the difference quotient cancels from the denominator. Once that happens, we consider our work to be done.

Example 1.5.2. Find and simplify the difference quotients for the following functions

1. $f(x) = x^2 - x - 2$
2. $g(x) = \dfrac{3}{2x+1}$
3. $r(x) = \sqrt{x}$

Solution.

1. To find $f(x+h)$, we replace every occurrence of x in the formula $f(x) = x^2 - x - 2$ with the quantity $(x+h)$ to get

$$f(x+h) = (x+h)^2 - (x+h) - 2 = x^2 + 2xh + h^2 - x - h - 2.$$

So the difference quotient is

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{\left(x^2+2xh+h^2-x-h-2\right)-\left(x^2-x-2\right)}{h} \\
&= \frac{x^2+2xh+h^2-x-h-2-x^2+x+2}{h} \\
&= \frac{2xh+h^2-h}{h} \\
&= \frac{h(2x+h-1)}{h} && \text{factor} \\
&= \frac{\cancel{h}(2x+h-1)}{\cancel{h}} && \text{cancel} \\
&= 2x+h-1.
\end{aligned}$$

2. To find $g(x+h)$, we replace every occurrence of x in the formula $g(x) = \frac{3}{2x+1}$ with the quantity $(x+h)$ to get

$$g(x+h) = \frac{3}{2(x+h)+1} = \frac{3}{2x+2h+1},$$

which yields

$$\begin{aligned}
\frac{g(x+h)-g(x)}{h} &= \frac{\dfrac{3}{2x+2h+1}-\dfrac{3}{2x+1}}{h} \\
&= \frac{1}{h}\cdot\left[\frac{3}{2x+2h+1}-\frac{3}{2x+1}\right] \\
&= \frac{1}{h}\cdot\left[\frac{3(2x+1)-3(2x+2h+1)}{(2x+2h+1)(2x+1)}\right] \\
&= \frac{6x+3-6x-6h-3}{h(2x+2h+1)(2x+1)} \\
&= \frac{-6h}{h(2x+2h+1)(2x+1)} \\
&= \frac{-6\cancel{h}}{\cancel{h}(2x+2h+1)(2x+1)} = \frac{-6}{(2x+2h+1)(2x+1)}
\end{aligned}$$

Since we have managed to cancel the original 'h' from the denominator, we are done.

3. For $r(x)=\sqrt{x}$, we get $r(x+h)=\sqrt{x+h}$ so the difference quotient is

$$\frac{r(x+h)-r(x)}{h}=\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

In order to cancel the 'h' from the denominator, we rationalize the *numerator* by multiplying by its conjugate.[1]

$$\begin{aligned}
\frac{r(x+h)-r(x)}{h} &= \frac{\sqrt{x+h}-\sqrt{x}}{h} \\
&= \frac{\left(\sqrt{x+h}-\sqrt{x}\right)}{h}\cdot\frac{\left(\sqrt{x+h}+\sqrt{x}\right)}{\left(\sqrt{x+h}+\sqrt{x}\right)} && \text{Multiply by the conjugate.} \\
&= \frac{\left(\sqrt{x+h}\right)^2-\left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h}+\sqrt{x}\right)} && \text{Difference of Squares.} \\
&= \frac{(x+h)-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\
&= \frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)} = \frac{1}{\sqrt{x+h}+\sqrt{x}}
\end{aligned}$$

Since we have removed the original 'h' from the denominator, we are done. □

[1] Rationalizing the *numerator*!? How's that for a twist!

As mentioned before, we will revisit difference quotients in Section 2.1 where we will explain them geometrically. For now, we want to move on to some classic applications of function arithmetic from Economics and for that, we need to think like an entrepreneur.[2]

Suppose you are a manufacturer making a certain product.[3] Let x be the **production level**, that is, the number of items produced in a given time period. It is customary to let $C(x)$ denote the function which calculates the total **cost** of producing the x items. The quantity $C(0)$, which represents the cost of producing no items, is called the **fixed** cost, and represents the amount of money required to begin production. Associated with the total cost $C(x)$ is cost per item, or **average cost**, denoted $\overline{C}(x)$ and read 'C-bar' of x. To compute $\overline{C}(x)$, we take the total cost $C(x)$ and divide by the number of items produced x to get

$$\overline{C}(x) = \frac{C(x)}{x}$$

On the retail end, we have the **price** p charged per item. To simplify the dialog and computations in this text, we assume that *the number of items sold equals the number of items produced*. From a retail perspective, it seems natural to think of the number of items sold, x, as a function of the price charged, p. After all, the retailer can easily adjust the price to sell more product. In the language of functions, x would be the *dependent* variable and p would be the *independent* variable or, using function notation, we have a function $x(p)$. While we will adopt this convention later in the text, we will hold with tradition at this point and consider the price p as a function of the number of items sold, x. That is, we regard x as the independent variable and p as the dependent variable and speak of the **price-demand** function, $p(x)$. Hence, $p(x)$ returns the price charged per item when x items are produced and sold. Our next function to consider is the **revenue** function, $R(x)$. The function $R(x)$ computes the amount of money collected as a result of selling x items. Since $p(x)$ is the price charged per item, we have $R(x) = xp(x)$. Finally, the **profit** function, $P(x)$ calculates how much money is earned after the costs are paid. That is, $P(x) = (R - C)(x) = R(x) - C(x)$.

It is high time for an example.

Example 1.5.3. Let x represent the number of dOpi media players ('dOpis'[4]) produced and sold in a typical week. Suppose the cost, in dollars, to produce x dOpis is given by $C(x) = 100x + 2000$, for $x \geq 0$, and the price, in dollars per dOpi, is given by $p(x) = 450 - 15x$ for $0 \leq x \leq 30$.

1. Find and interpret $C(0)$.
2. Find and interpret $\overline{C}(10)$.
3. Find and interpret $p(0)$ and $p(20)$.
4. Solve $p(x) = 0$ and interpret the result.
5. Find and simplify expressions for the revenue function $R(x)$ and the profit function $P(x)$.
6. Find and interpret $R(0)$ and $P(0)$.
7. Solve $P(x) = 0$ and interpret the result.

[2]Not really, but "entrepreneur" is the buzzword of the day and we're trying to be trendy.
[3]Poorly designed resin Sasquatch statues, for example. Feel free to choose your own entrepreneurial fantasy.
[4]Pronounced 'dopeys' . . .

Solution.

1. We substitute $x = 0$ into the formula for $C(x)$ and get $C(0) = 100(0) + 2000 = 2000$. This means to produce 0 dOpis, it costs \$2000. In other words, the fixed (or start-up) costs are \$2000. The reader is encouraged to contemplate what sorts of expenses these might be.

2. Since $\overline{C}(x) = \frac{C(x)}{x}$, $\overline{C}(10) = \frac{C(10)}{10} = \frac{3000}{10} = 300$. This means when 10 dOpis are produced, the cost to manufacture them amounts to \$300 per dOpi.

3. Plugging $x = 0$ into the expression for $p(x)$ gives $p(0) = 450 - 15(0) = 450$. This means no dOpis are sold if the price is \$450 per dOpi. On the other hand, $p(20) = 450 - 15(20) = 150$ which means to sell 20 dOpis in a typical week, the price should be set at \$150 per dOpi.

4. Setting $p(x) = 0$ gives $450 - 15x = 0$. Solving gives $x = 30$. This means in order to sell 30 dOpis in a typical week, the price needs to be set to \$0. What's more, this means that even if dOpis were given away for free, the retailer would only be able to move 30 of them.[5]

5. To find the revenue, we compute $R(x) = xp(x) = x(450 - 15x) = 450x - 15x^2$. Since the formula for $p(x)$ is valid only for $0 \leq x \leq 30$, our formula $R(x)$ is also restricted to $0 \leq x \leq 30$. For the profit, $P(x) = (R - C)(x) = R(x) - C(x)$. Using the given formula for $C(x)$ and the derived formula for $R(x)$, we get $P(x) = \left(450x - 15x^2\right) - (100x + 2000) = -15x^2 + 350x - 2000$. As before, the validity of this formula is for $0 \leq x \leq 30$ only.

6. We find $R(0) = 0$ which means if no dOpis are sold, we have no revenue, which makes sense. Turning to profit, $P(0) = -2000$ since $P(x) = R(x) - C(x)$ and $P(0) = R(0) - C(0) = -2000$. This means that if no dOpis are sold, more money (\$2000 to be exact!) was put into producing the dOpis than was recouped in sales. In number 1, we found the fixed costs to be \$2000, so it makes sense that if we sell no dOpis, we are out those start-up costs.

7. Setting $P(x) = 0$ gives $-15x^2 + 350x - 2000 = 0$. Factoring gives $-5(x - 10)(3x - 40) = 0$ so $x = 10$ or $x = \frac{40}{3}$. What do these values mean in the context of the problem? Since $P(x) = R(x) - C(x)$, solving $P(x) = 0$ is the same as solving $R(x) = C(x)$. This means that the solutions to $P(x) = 0$ are the production (and sales) figures for which the sales revenue exactly balances the total production costs. These are the so-called '**break even**' points. The solution $x = 10$ means 10 dOpis should be produced (and sold) during the week to recoup the cost of production. For $x = \frac{40}{3} = 13.\overline{3}$, things are a bit more complicated. Even though $x = 13.\overline{3}$ satisfies $0 \leq x \leq 30$, and hence is in the domain of P, it doesn't make sense in the context of this problem to produce a fractional part of a dOpi.[6] Evaluating $P(13) = 15$ and $P(14) = -40$, we see that producing and selling 13 dOpis per week makes a (slight) profit, whereas producing just one more puts us back into the red. While breaking even is nice, we ultimately would like to find what production level (and price) will result in the largest profit, and we'll do just that . . . in Section 2.4. □

[5] Imagine that! Giving something away for free and hardly anyone taking advantage of it . . .

[6] We've seen this sort of thing before in Section 1.4.1.

1.5.1 Exercises

In Exercises 1 - 10, use the pair of functions f and g to find the following values if they exist.

- $(f+g)(2)$
- $(f-g)(-1)$
- $(g-f)(1)$
- $(fg)\left(\frac{1}{2}\right)$
- $\left(\frac{f}{g}\right)(0)$
- $\left(\frac{g}{f}\right)(-2)$

1. $f(x) = 3x+1$ and $g(x) = 4-x$
2. $f(x) = x^2$ and $g(x) = -2x+1$
3. $f(x) = x^2-x$ and $g(x) = 12-x^2$
4. $f(x) = 2x^3$ and $g(x) = -x^2-2x-3$
5. $f(x) = \sqrt{x+3}$ and $g(x) = 2x-1$
6. $f(x) = \sqrt{4-x}$ and $g(x) = \sqrt{x+2}$
7. $f(x) = 2x$ and $g(x) = \dfrac{1}{2x+1}$
8. $f(x) = x^2$ and $g(x) = \dfrac{3}{2x-3}$
9. $f(x) = x^2$ and $g(x) = \dfrac{1}{x^2}$
10. $f(x) = x^2+1$ and $g(x) = \dfrac{1}{x^2+1}$

In Exercises 11 - 20, use the pair of functions f and g to find the domain of the indicated function then find and simplify an expression for it.

- $(f+g)(x)$
- $(f-g)(x)$
- $(fg)(x)$
- $\left(\frac{f}{g}\right)(x)$

11. $f(x) = 2x+1$ and $g(x) = x-2$
12. $f(x) = 1-4x$ and $g(x) = 2x-1$
13. $f(x) = x^2$ and $g(x) = 3x-1$
14. $f(x) = x^2-x$ and $g(x) = 7x$
15. $f(x) = x^2-4$ and $g(x) = 3x+6$
16. $f(x) = -x^2+x+6$ and $g(x) = x^2-9$
17. $f(x) = \dfrac{x}{2}$ and $g(x) = \dfrac{2}{x}$
18. $f(x) = x-1$ and $g(x) = \dfrac{1}{x-1}$
19. $f(x) = x$ and $g(x) = \sqrt{x+1}$
20. $f(x) = \sqrt{x-5}$ and $g(x) = f(x) = \sqrt{x-5}$

In Exercises 21 - 45, find and simplify the difference quotient $\dfrac{f(x+h)-f(x)}{h}$ for the given function.

21. $f(x) = 2x-5$
22. $f(x) = -3x+5$
23. $f(x) = 6$
24. $f(x) = 3x^2-x$
25. $f(x) = -x^2+2x-1$
26. $f(x) = 4x^2$

27. $f(x) = x - x^2$

28. $f(x) = x^3 + 1$

29. $f(x) = mx + b$ where $m \neq 0$

30. $f(x) = ax^2 + bx + c$ where $a \neq 0$

31. $f(x) = \dfrac{2}{x}$

32. $f(x) = \dfrac{3}{1 - x}$

33. $f(x) = \dfrac{1}{x^2}$

34. $f(x) = \dfrac{2}{x + 5}$

35. $f(x) = \dfrac{1}{4x - 3}$

36. $f(x) = \dfrac{3x}{x + 1}$

37. $f(x) = \dfrac{x}{x - 9}$

38. $f(x) = \dfrac{x^2}{2x + 1}$

39. $f(x) = \sqrt{x - 9}$

40. $f(x) = \sqrt{2x + 1}$

41. $f(x) = \sqrt{-4x + 5}$

42. $f(x) = \sqrt{4 - x}$

43. $f(x) = \sqrt{ax + b}$, where $a \neq 0$.

44. $f(x) = x\sqrt{x}$

45. $f(x) = \sqrt[3]{x}$. **HINT:** $(a - b)\left(a^2 + ab + b^2\right) = a^3 - b^3$

In Exercises 46 - 50, $C(x)$ denotes the cost to produce x items and $p(x)$ denotes the price-demand function in the given economic scenario. In each Exercise, do the following:

- Find and interpret $C(0)$.
- Find and interpret $\overline{C}(10)$.
- Find and interpret $p(5)$
- Find and simplify $R(x)$.
- Find and simplify $P(x)$.
- Solve $P(x) = 0$ and interpret.

46. The cost, in dollars, to produce x "I'd rather be a Sasquatch" T-Shirts is $C(x) = 2x + 26$, $x \geq 0$ and the price-demand function, in dollars per shirt, is $p(x) = 30 - 2x$, $0 \leq x \leq 15$.

47. The cost, in dollars, to produce x bottles of 100% All-Natural Certified Free-Trade Organic Sasquatch Tonic is $C(x) = 10x + 100$, $x \geq 0$ and the price-demand function, in dollars per bottle, is $p(x) = 35 - x$, $0 \leq x \leq 35$.

48. The cost, in cents, to produce x cups of Mountain Thunder Lemonade at Junior's Lemonade Stand is $C(x) = 18x + 240$, $x \geq 0$ and the price-demand function, in cents per cup, is $p(x) = 90 - 3x$, $0 \leq x \leq 30$.

49. The daily cost, in dollars, to produce x Sasquatch Berry Pies $C(x) = 3x + 36$, $x \geq 0$ and the price-demand function, in dollars per pie, is $p(x) = 12 - 0.5x$, $0 \leq x \leq 24$.

50. The monthly cost, in hundreds of dollars, to produce x custom built electric scooters is $C(x) = 20x + 1000$, $x \geq 0$ and the price-demand function, in hundreds of dollars per scooter, is $p(x) = 140 - 2x$, $0 \leq x \leq 70$.

In Exercises 51 - 62, let f be the function defined by

$$f = \{(-3,4),(-2,2),(-1,0),(0,1),(1,3),(2,4),(3,-1)\}$$

and let g be the function defined

$$g = \{(-3,-2),(-2,0),(-1,-4),(0,0),(1,-3),(2,1),(3,2)\}$$

. Compute the indicated value if it exists.

51. $(f+g)(-3)$
52. $(f-g)(2)$
53. $(fg)(-1)$
54. $(g+f)(1)$
55. $(g-f)(3)$
56. $(gf)(-3)$
57. $\left(\frac{f}{g}\right)(-2)$
58. $\left(\frac{f}{g}\right)(-1)$
59. $\left(\frac{f}{g}\right)(2)$
60. $\left(\frac{g}{f}\right)(-1)$
61. $\left(\frac{g}{f}\right)(3)$
62. $\left(\frac{g}{f}\right)(-3)$

Summary of Common Economic Functions

Suppose x represents the quantity of items produced and sold.

- The price-demand function $p(x)$ calculates the price per item.
- The revenue function $R(x)$ calculates the total money collected by selling x items at a price $p(x)$, $R(x) = x\,p(x)$.
- The cost function $C(x)$ calculates the cost to produce x items. The value $C(0)$ is called the fixed cost or start-up cost.
- The average cost function $\overline{C}(x) = \frac{C(x)}{x}$ calculates the cost per item when making x items. Here, we necessarily assume $x > 0$.
- The profit function $P(x)$ calculates the money earned after costs are paid when x items are produced and sold, $P(x) = (R - C)(x) = R(x) - C(x)$.

1.5.2 Answers

1. For $f(x) = 3x + 1$ and $g(x) = 4 - x$

   - $(f + g)(2) = 9$
   - $(f - g)(-1) = -7$
   - $(g - f)(1) = -1$
   - $(fg)\left(\frac{1}{2}\right) = \frac{35}{4}$
   - $\left(\frac{f}{g}\right)(0) = \frac{1}{4}$
   - $\left(\frac{g}{f}\right)(-2) = -\frac{6}{5}$

2. For $f(x) = x^2$ and $g(x) = -2x + 1$

   - $(f + g)(2) = 1$
   - $(f - g)(-1) = -2$
   - $(g - f)(1) = -2$
   - $(fg)\left(\frac{1}{2}\right) = 0$
   - $\left(\frac{f}{g}\right)(0) = 0$
   - $\left(\frac{g}{f}\right)(-2) = \frac{5}{4}$

3. For $f(x) = x^2 - x$ and $g(x) = 12 - x^2$

   - $(f + g)(2) = 10$
   - $(f - g)(-1) = -9$
   - $(g - f)(1) = 11$
   - $(fg)\left(\frac{1}{2}\right) = -\frac{47}{16}$
   - $\left(\frac{f}{g}\right)(0) = 0$
   - $\left(\frac{g}{f}\right)(-2) = \frac{4}{3}$

4. For $f(x) = 2x^3$ and $g(x) = -x^2 - 2x - 3$

   - $(f + g)(2) = 5$
   - $(f - g)(-1) = 0$
   - $(g - f)(1) = -8$
   - $(fg)\left(\frac{1}{2}\right) = -\frac{17}{16}$
   - $\left(\frac{f}{g}\right)(0) = 0$
   - $\left(\frac{g}{f}\right)(-2) = \frac{3}{16}$

5. For $f(x) = \sqrt{x + 3}$ and $g(x) = 2x - 1$

   - $(f + g)(2) = 3 + \sqrt{5}$
   - $(f - g)(-1) = 3 + \sqrt{2}$
   - $(g - f)(1) = -1$
   - $(fg)\left(\frac{1}{2}\right) = 0$
   - $\left(\frac{f}{g}\right)(0) = -\sqrt{3}$
   - $\left(\frac{g}{f}\right)(-2) = -5$

6. For $f(x) = \sqrt{4 - x}$ and $g(x) = \sqrt{x + 2}$

   - $(f + g)(2) = 2 + \sqrt{2}$
   - $(f - g)(-1) = -1 + \sqrt{5}$
   - $(g - f)(1) = 0$
   - $(fg)\left(\frac{1}{2}\right) = \frac{\sqrt{35}}{2}$
   - $\left(\frac{f}{g}\right)(0) = \sqrt{2}$
   - $\left(\frac{g}{f}\right)(-2) = 0$

7. For $f(x) = 2x$ and $g(x) = \frac{1}{2x+1}$

- $(f+g)(2) = \frac{21}{5}$
- $(f-g)(-1) = -1$
- $(g-f)(1) = -\frac{5}{3}$
- $(fg)\left(\frac{1}{2}\right) = \frac{1}{2}$
- $\left(\frac{f}{g}\right)(0) = 0$
- $\left(\frac{g}{f}\right)(-2) = \frac{1}{12}$

8. For $f(x) = x^2$ and $g(x) = \frac{3}{2x-3}$

- $(f+g)(2) = 7$
- $(f-g)(-1) = \frac{8}{5}$
- $(g-f)(1) = -4$
- $(fg)\left(\frac{1}{2}\right) = -\frac{3}{8}$
- $\left(\frac{f}{g}\right)(0) = 0$
- $\left(\frac{g}{f}\right)(-2) = -\frac{3}{28}$

9. For $f(x) = x^2$ and $g(x) = \frac{1}{x^2}$

- $(f+g)(2) = \frac{17}{4}$
- $(f-g)(-1) = 0$
- $(g-f)(1) = 0$
- $(fg)\left(\frac{1}{2}\right) = 1$
- $\left(\frac{f}{g}\right)(0)$ is undefined.
- $\left(\frac{g}{f}\right)(-2) = \frac{1}{16}$

10. For $f(x) = x^2 + 1$ and $g(x) = \frac{1}{x^2+1}$

- $(f+g)(2) = \frac{26}{5}$
- $(f-g)(-1) = \frac{3}{2}$
- $(g-f)(1) = -\frac{3}{2}$
- $(fg)\left(\frac{1}{2}\right) = 1$
- $\left(\frac{f}{g}\right)(0) = 1$
- $\left(\frac{g}{f}\right)(-2) = \frac{1}{25}$

11. For $f(x) = 2x + 1$ and $g(x) = x - 2$

- $(f+g)(x) = 3x - 1$
  Domain: $(-\infty, \infty)$
- $(f-g)(x) = x + 3$
  Domain: $(-\infty, \infty)$
- $(fg)(x) = 2x^2 - 3x - 2$
  Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{2x+1}{x-2}$
  Domain: $(-\infty, 2) \cup (2, \infty)$

12. For $f(x) = 1 - 4x$ and $g(x) = 2x - 1$

- $(f+g)(x) = -2x$
  Domain: $(-\infty, \infty)$
- $(f-g)(x) = 2 - 6x$
  Domain: $(-\infty, \infty)$
- $(fg)(x) = -8x^2 + 6x - 1$
  Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{1-4x}{2x-1}$
  Domain: $\left(-\infty, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$

13. For $f(x) = x^2$ and $g(x) = 3x - 1$

- $(f + g)(x) = x^2 + 3x - 1$
  Domain: $(-\infty, \infty)$
- $(f - g)(x) = x^2 - 3x + 1$
  Domain: $(-\infty, \infty)$
- $(fg)(x) = 3x^3 - x^2$
  Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{x^2}{3x-1}$
  Domain: $\left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$

14. For $f(x) = x^2 - x$ and $g(x) = 7x$

- $(f + g)(x) = x^2 + 6x$
  Domain: $(-\infty, \infty)$
- $(f - g)(x) = x^2 - 8x$
  Domain: $(-\infty, \infty)$
- $(fg)(x) = 7x^3 - 7x^2$
  Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{x-1}{7}$
  Domain: $(-\infty, 0) \cup (0, \infty)$

15. For $f(x) = x^2 - 4$ and $g(x) = 3x + 6$

- $(f + g)(x) = x^2 + 3x + 2$
  Domain: $(-\infty, \infty)$
- $(f - g)(x) = x^2 - 3x - 10$
  Domain: $(-\infty, \infty)$
- $(fg)(x) = 3x^3 + 6x^2 - 12x - 24$
  Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{x-2}{3}$
  Domain: $(-\infty, -2) \cup (-2, \infty)$

16. For $f(x) = -x^2 + x + 6$ and $g(x) = x^2 - 9$

- $(f + g)(x) = x - 3$
  Domain: $(-\infty, \infty)$
- $(f - g)(x) = -2x^2 + x + 15$
  Domain: $(-\infty, \infty)$
- $(fg)(x) = -x^4 + x^3 + 15x^2 - 9x - 54$
  Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = -\frac{x+2}{x+3}$
  Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

17. For $f(x) = \frac{x}{2}$ and $g(x) = \frac{2}{x}$

- $(f + g)(x) = \frac{x^2+4}{2x}$
  Domain: $(-\infty, 0) \cup (0, \infty)$
- $(f - g)(x) = \frac{x^2-4}{2x}$
  Domain: $(-\infty, 0) \cup (0, \infty)$
- $(fg)(x) = 1$
  Domain: $(-\infty, 0) \cup (0, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{x^2}{4}$
  Domain: $(-\infty, 0) \cup (0, \infty)$

18. For $f(x) = x - 1$ and $g(x) = \frac{1}{x-1}$

- $(f+g)(x) = \frac{x^2-2x+2}{x-1}$
Domain: $(-\infty, 1) \cup (1, \infty)$
- $(f-g)(x) = \frac{x^2-2x}{x-1}$
Domain: $(-\infty, 1) \cup (1, \infty)$
- $(fg)(x) = 1$
Domain: $(-\infty, 1) \cup (1, \infty)$
- $\left(\frac{f}{g}\right)(x) = x^2 - 2x + 1$
Domain: $(-\infty, 1) \cup (1, \infty)$

19. For $f(x) = x$ and $g(x) = \sqrt{x+1}$

- $(f+g)(x) = x + \sqrt{x+1}$
Domain: $[-1, \infty)$
- $(f-g)(x) = x - \sqrt{x+1}$
Domain: $[-1, \infty)$
- $(fg)(x) = x\sqrt{x+1}$
Domain: $[-1, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{x}{\sqrt{x+1}}$
Domain: $(-1, \infty)$

20. For $f(x) = \sqrt{x-5}$ and $g(x) = f(x) = \sqrt{x-5}$

- $(f+g)(x) = 2\sqrt{x-5}$
Domain: $[5, \infty)$
- $(f-g)(x) = 0$
Domain: $[5, \infty)$
- $(fg)(x) = x - 5$
Domain: $[5, \infty)$
- $\left(\frac{f}{g}\right)(x) = 1$
Domain: $(5, \infty)$

21. 2
22. -3
23. 0
24. $6x + 3h - 1$
25. $-2x - h + 2$
26. $8x + 4h$
27. $-2x - h + 1$
28. $3x^2 + 3xh + h^2$
29. m
30. $2ax + ah + b$
31. $\dfrac{-2}{x(x+h)}$
32. $\dfrac{3}{(1-x-h)(1-x)}$
33. $\dfrac{-(2x+h)}{x^2(x+h)^2}$
34. $\dfrac{-2}{(x+5)(x+h+5)}$
35. $\dfrac{-4}{(4x-3)(4x+4h-3)}$
36. $\dfrac{3}{(x+1)(x+h+1)}$

37. $\dfrac{-9}{(x-9)(x+h-9)}$

38. $\dfrac{2x^2+2xh+2x+h}{(2x+1)(2x+2h+1)}$

39. $\dfrac{1}{\sqrt{x+h-9}+\sqrt{x-9}}$

40. $\dfrac{2}{\sqrt{2x+2h+1}+\sqrt{2x+1}}$

41. $\dfrac{-4}{\sqrt{-4x-4h+5}+\sqrt{-4x+5}}$

42. $\dfrac{-1}{\sqrt{4-x-h}+\sqrt{4-x}}$

43. $\dfrac{a}{\sqrt{ax+ah+b}+\sqrt{ax+b}}$

44. $\dfrac{3x^2+3xh+h^2}{(x+h)^{3/2}+x^{3/2}}$

45. $\dfrac{1}{(x+h)^{2/3}+(x+h)^{1/3}x^{1/3}+x^{2/3}}$

46.
- $C(0) = 26$, so the fixed costs are \$26.
- $\overline{C}(10) = 4.6$, so when 10 shirts are produced, the cost per shirt is \$4.60.
- $p(5) = 20$, so to sell 5 shirts, set the price at \$20 per shirt.
- $R(x) = -2x^2 + 30x$, $0 \leq x \leq 15$
- $P(x) = -2x^2 + 28x - 26$, $0 \leq x \leq 15$
- $P(x) = 0$ when $x = 1$ and $x = 13$. These are the 'break even' points, so selling 1 shirt or 13 shirts will guarantee the revenue earned exactly recoups the cost of production.

47.
- $C(0) = 100$, so the fixed costs are \$100.
- $\overline{C}(10) = 20$, so when 10 bottles of tonic are produced, the cost per bottle is \$20.
- $p(5) = 30$, so to sell 5 bottles of tonic, set the price at \$30 per bottle.
- $R(x) = -x^2 + 35x$, $0 \leq x \leq 35$
- $P(x) = -x^2 + 25x - 100$, $0 \leq x \leq 35$
- $P(x) = 0$ when $x = 5$ and $x = 20$. These are the 'break even' points, so selling 5 bottles of tonic or 20 bottles of tonic will guarantee the revenue earned exactly recoups the cost of production.

48.
- $C(0) = 240$, so the fixed costs are 240¢ or \$2.40.
- $\overline{C}(10) = 42$, so when 10 cups of lemonade are made, the cost per cup is 42¢.
- $p(5) = 75$, so to sell 5 cups of lemonade, set the price at 75¢ per cup.
- $R(x) = -3x^2 + 90x$, $0 \leq x \leq 30$
- $P(x) = -3x^2 + 72x - 240$, $0 \leq x \leq 30$
- $P(x) = 0$ when $x = 4$ and $x = 20$. These are the 'break even' points, so selling 4 cups of lemonade or 20 cups of lemonade will guarantee the revenue earned exactly recoups the cost of production.

49.
- $C(0) = 36$, so the daily fixed costs are \$36.
- $\overline{C}(10) = 6.6$, so when 10 pies are made, the cost per pie is \$6.60.
- $p(5) = 9.5$, so to sell 5 pies a day, set the price at \$9.50 per pie.
- $R(x) = -0.5x^2 + 12x$, $0 \leq x \leq 24$
- $P(x) = -0.5x^2 + 9x - 36$, $0 \leq x \leq 24$
- $P(x) = 0$ when $x = 6$ and $x = 12$. These are the 'break even' points, so selling 6 pies or 12 pies a day will guarantee the revenue earned exactly recoups the cost of production.

50.
- $C(0) = 1000$, so the monthly fixed costs are 1000 *hundred* dollars, or \$100,000.
- $\overline{C}(10) = 120$, so when 10 scooters are made, the cost per scooter is 120 hundred dollars, or \$12,000.
- $p(5) = 130$, so to sell 5 scooters a month, set the price at 130 hundred dollars, or \$13,000 per scooter.
- $R(x) = -2x^2 + 140x$, $0 \leq x \leq 70$
- $P(x) = -2x^2 + 120x - 1000$, $0 \leq x \leq 70$
- $P(x) = 0$ when $x = 10$ and $x = 50$. These are the 'break even' points, so selling 10 scooters or 50 scooters a month will guarantee the revenue earned exactly recoups the cost of production.

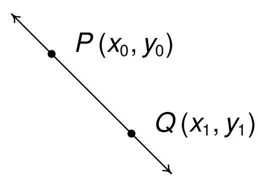
51. $(f + g)(-3) = 2$
52. $(f - g)(2) = 3$
53. $(fg)(-1) = 0$
54. $(g + f)(1) = 0$
55. $(g - f)(3) = 3$
56. $(gf)(-3) = -8$
57. $\left(\frac{f}{g}\right)(-2)$ does not exist
58. $\left(\frac{f}{g}\right)(-1) = 0$
59. $\left(\frac{f}{g}\right)(2) = 4$
60. $\left(\frac{g}{f}\right)(-1)$ does not exist
61. $\left(\frac{g}{f}\right)(3) = -2$
62. $\left(\frac{g}{f}\right)(-3) = -\frac{1}{2}$

Chapter 2

Linear and Quadratic Functions

2.1 Linear Functions

We now begin the study of families of functions. Our first family, linear functions, are old friends as we shall soon see. Recall from Geometry that two distinct points in the plane determine a unique line containing those points, as indicated below.



To give a sense of the 'steepness' of the line, we recall that we can compute the **slope** of the line using the formula below.

Equation 2.1. The **slope** m of the line containing the points $P\,(x_0, y_0)$ and $Q\,(x_1, y_1)$ is:

$$m = \frac{y_1 - y_0}{x_1 - x_0},$$

provided $x_1 \neq x_0$.

A couple of notes about Equation 2.1 are in order. First, don't ask why we use the letter 'm' to represent slope. There are many explanations out there, but apparently no one really knows for sure.[1] Secondly, the stipulation $x_1 \neq x_0$ ensures that we aren't trying to divide by zero. The reader is invited to pause to think about what is happening geometrically; the anxious reader can skip along to the next example.

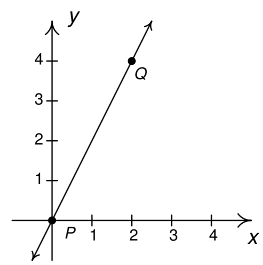
Example 2.1.1. Find the slope of the line containing the following pairs of points, if it exists. Plot each pair of points and the line containing them.

[1]See www.mathforum.org or www.mathworld.wolfram.com for discussions on this topic.

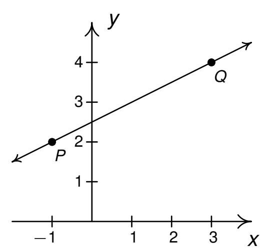
1. $P(0,0)$, $Q(2,4)$
2. $P(-1,2)$, $Q(3,4)$
3. $P(-2,3)$, $Q(2,-3)$
4. $P(-3,2)$, $Q(4,2)$
5. $P(2,3)$, $Q(2,-1)$
6. $P(2,3)$, $Q(2.1,-1)$

Solution. In each of these examples, we apply the slope formula, Equation 2.1.

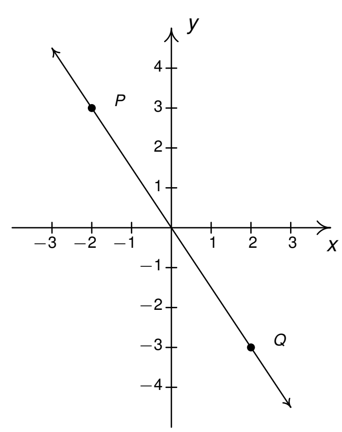
1. $m = \dfrac{4-0}{2-0} = \dfrac{4}{2} = 2$



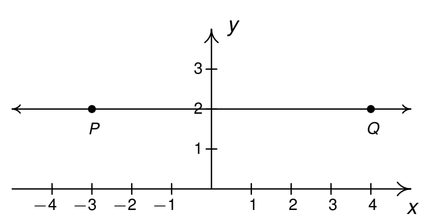
2. $m = \dfrac{4-2}{3-(-1)} = \dfrac{2}{4} = \dfrac{1}{2}$



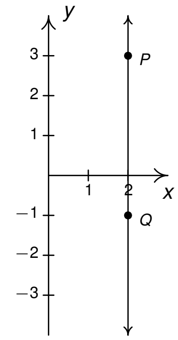
3. $m = \dfrac{-3-3}{2-(-2)} = \dfrac{-6}{4} = -\dfrac{3}{2}$



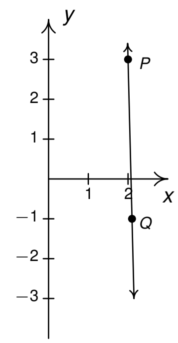
4. $m = \dfrac{2-2}{4-(-3)} = \dfrac{0}{7} = 0$



5. $m = \dfrac{-1-3}{2-2} = \dfrac{-4}{0}$, which is undefined

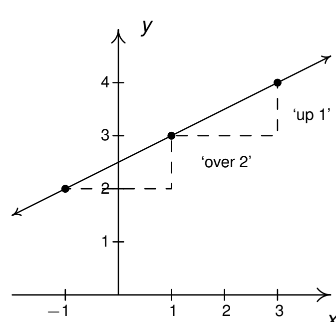


6. $m = \dfrac{-1-3}{2.1-2} = \dfrac{-4}{0.1} = -40$



□

A few comments about Example 2.1.1 are in order. First, for reasons which will be made clear soon, if the slope is positive then the resulting line is said to be increasing. If it is negative, we say the line is decreasing. A slope of 0 results in a horizontal line which we say is constant, and an undefined slope results in a vertical line.[2] Second, the larger the slope is in absolute value, the steeper the line. You may recall from Intermediate Algebra that slope can be described as the ratio '$\frac{\text{rise}}{\text{run}}$'. For example, in the second part of Example 2.1.1, we found the slope to be $\frac{1}{2}$. We can interpret this as a rise of 1 unit upward for every 2 units to the right we travel along the line, as shown below.



[2]Some authors use the unfortunate moniker 'no slope' when a slope is undefined. It's easy to confuse the notions of 'no slope' with 'slope of 0'. For this reason, we will describe slopes of vertical lines as 'undefined'.

Using more formal notation, given points (x_0, y_0) and (x_1, y_1), we use the Greek letter delta 'Δ' to write $\Delta y = y_1 - y_0$ and $\Delta x = x_1 - x_0$. In most scientific circles, the symbol Δ means 'change in'. Hence, we may write

$$m = \frac{\Delta y}{\Delta x},$$

which describes the slope as the **rate of change** of y with respect to x. Rates of change abound in the 'real world', as the next example illustrates.

Example 2.1.2. Suppose that two separate temperature readings were taken at the ranger station on the top of Mt. Sasquatch: at 6 AM the temperature was 24°F and at 10 AM it was 32°F.

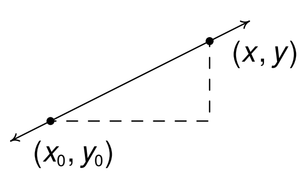
1. Find the slope of the line containing the points $(6, 24)$ and $(10, 32)$.
2. Interpret your answer to the first part in terms of temperature and time.
3. Predict the temperature at noon.

Solution.

1. For the slope, we have $m = \frac{32-24}{10-6} = \frac{8}{4} = 2$.
2. Since the values in the numerator correspond to the temperatures in °F, and the values in the denominator correspond to time in hours, we can interpret the slope as $2 = \dfrac{2}{1} = \dfrac{2^\circ \text{ F}}{1 \text{ hour}}$, or 2°F per hour. Since the slope is positive, we know this corresponds to an increasing line. Hence, the temperature is increasing at a rate of 2°F per hour.
3. Noon is two hours after 10 AM. Assuming a temperature increase of 2°F per hour, in two hours the temperature should rise 4°F. Since the temperature at 10 AM is 32°F, we would expect the temperature at noon to be $32 + 4 = 36$°F. □

Now it may well happen that in the previous scenario, at noon the temperature is only 33°F. This doesn't mean our calculations are incorrect, rather, it means that the temperature change throughout the day isn't a constant 2°F per hour. As discussed in Section 1.4.1, mathematical models are just that: models. The predictions we get out of the models may be mathematically accurate, but may not resemble what happens in the real world.

In Section 1.2, we discussed the equations of vertical and horizontal lines. Using the concept of slope, we can develop equations for the other varieties of lines. Suppose a line has a slope of m and contains the point (x_0, y_0). Suppose (x, y) is another point on the line, as indicated below.



Equation 2.1 yields

$$\begin{aligned} m &= \frac{y - y_0}{x - x_0} \\ m(x - x_0) &= y - y_0 \\ y - y_0 &= m(x - x_0) \end{aligned}$$

We have just derived the **point-slope form** of a line.[3]

Equation 2.2. The **point-slope form** of the line with slope m containing the point (x_0, y_0) is the equation $y - y_0 = m(x - x_0)$.

Example 2.1.3. Write the equation of the line containing the points $(-1, 3)$ and $(2, 1)$.

Solution. In order to use Equation 2.2 we need to find the slope of the line in question so we use Equation 2.1 to get $m = \frac{\Delta y}{\Delta x} = \frac{1-3}{2-(-1)} = -\frac{2}{3}$. We are spoiled for choice for a point (x_0, y_0). We'll use $(-1, 3)$ and leave it to the reader to check that using $(2, 1)$ results in the same equation. Substituting into the point-slope form of the line, we get

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 3 &= -\frac{2}{3}(x - (-1)) \\ y - 3 &= -\frac{2}{3}(x + 1) \\ y - 3 &= -\frac{2}{3}x - \frac{2}{3} \\ y &= -\frac{2}{3}x + \frac{7}{3}. \end{aligned}$$

We can check our answer by showing that both $(-1, 3)$ and $(2, 1)$ are on the graph of $y = -\frac{2}{3}x + \frac{7}{3}$ algebraically, as we did in Section 1.2.1. □

In simplifying the equation of the line in the previous example, we produced another form of a line, the **slope-intercept form**. This is the familiar $y = mx + b$ form you have probably seen in Intermediate Algebra. The 'intercept' in 'slope-intercept' comes from the fact that if we set $x = 0$, we get $y = b$. In other words, the y-intercept of the line $y = mx + b$ is $(0, b)$.

Equation 2.3. The **slope-intercept form** of the line with slope m and y-intercept $(0, b)$ is the equation $y = mx + b$.

Note that if we have slope $m = 0$, we get the equation $y = b$ which matches our formula for a horizontal line given in Section 1.2. The formula given in Equation 2.3 can be used to describe all lines except vertical lines. All lines except vertical lines are functions (Why is this?) so we have finally reached a good point to introduce **linear functions**.

[3] We can also understand this equation in terms of applying transformations to the function $I(x) = x$. See the Exercises.

Definition 2.1. A **linear function** is a function of the form

$$f(x) = mx + b,$$

where m and b are real numbers with $m \neq 0$. The domain of a linear function is $(-\infty, \infty)$.

For the case $m = 0$, we get $f(x) = b$. These are given their own classification.

Definition 2.2. A **constant function** is a function of the form

$$f(x) = b,$$

where b is real number. The domain of a constant function is $(-\infty, \infty)$.

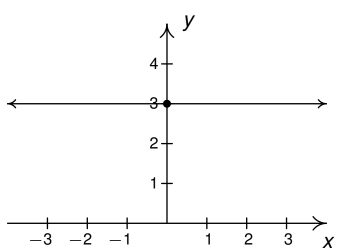
Recall that to graph a function, f, we graph the equation $y = f(x)$. Hence, the graph of a linear function is a line with slope m and y-intercept $(0, b)$; the graph of a constant function is a horizontal line (a line with slope $m = 0$) and a y-intercept of $(0, b)$.

Example 2.1.4. Graph the following functions. Identify the slope and y-intercept.

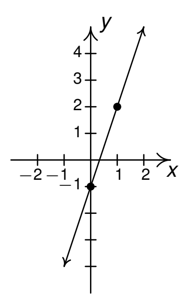
1. $f(x) = 3$
2. $f(x) = 3x - 1$
3. $f(x) = \dfrac{3 - 2x}{4}$
4. $f(x) = \dfrac{x^2 - 4}{x - 2}$

Solution.

1. To graph $f(x) = 3$, we graph $y = 3$. This is a horizontal line ($m = 0$) through $(0, 3)$.
2. The graph of $f(x) = 3x - 1$ is the graph of the line $y = 3x - 1$. Comparison of this equation with Equation 2.3 yields $m = 3$ and $b = -1$. Hence, our slope is 3 and our y-intercept is $(0, -1)$. To get another point on the line, we can plot $(1, f(1)) = (1, 2)$.



$f(x) = 3$



$f(x) = 3x - 1$

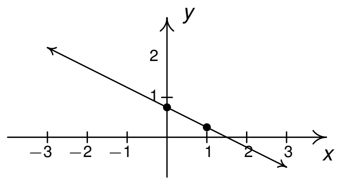
3. At first glance, the function $f(x) = \frac{3-2x}{4}$ does not fit the form in Definition 2.1 but after some rearranging we get $f(x) = \frac{3-2x}{4} = \frac{3}{4} - \frac{2x}{4} = -\frac{1}{2}x + \frac{3}{4}$. We identify $m = -\frac{1}{2}$ and $b = \frac{3}{4}$. Hence,

our graph is a line with a slope of $-\frac{1}{2}$ and a y-intercept of $\left(0, \frac{3}{4}\right)$. Plotting an additional point, we can choose $(1, f(1))$ to get $\left(1, \frac{1}{4}\right)$.

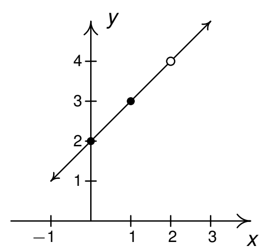
4. If we simplify the expression for f, we get

$$f(x) = \frac{x^2 - 4}{x - 2} = \frac{\cancel{(x - 2)}(x + 2)}{\cancel{(x - 2)}} = x + 2.$$

If we were to state $f(x) = x + 2$, we would be committing a sin of omission. Remember, to find the domain of a function, we do so **before** we simplify! In this case, f has big problems when $x = 2$, and as such, the domain of f is $(-\infty, 2) \cup (2, \infty)$. To indicate this, we write $f(x) = x + 2$, $x \neq 2$. So, except at $x = 2$, we graph the line $y = x + 2$. The slope $m = 1$ and the y-intercept is $(0, 2)$. A second point on the graph is $(1, f(1)) = (1, 3)$. Since our function f is not defined at $x = 2$, we put an open circle at the point that would be on the line $y = x + 2$ when $x = 2$, namely $(2, 4)$.



$$f(x) = \frac{3 - 2x}{4}$$



$$f(x) = \frac{x^2 - 4}{x - 2}$$

□

The last two functions in the previous example showcase some of the difficulty in defining a linear function using the phrase 'of the form' as in Definition 2.1, since some algebraic manipulations may be needed to rewrite a given function to match 'the form'. Keep in mind that the domains of linear and constant functions are all real numbers $(-\infty, \infty)$, so while $f(x) = \frac{x^2-4}{x-2}$ simplified to a formula $f(x) = x + 2$, f is not considered a linear function since its domain excludes $x = 2$. However, we would consider

$$f(x) = \frac{2x^2 + 2}{x^2 + 1}$$

to be a constant function since its domain is all real numbers (Can you tell us why?) and

$$f(x) = \frac{2x^2 + 2}{x^2 + 1} = \frac{2\cancel{(x^2 + 1)}}{\cancel{(x^2 + 1)}} = 2$$

The following example uses linear functions to model some basic economic relationships.

Example 2.1.5. The cost C, in dollars, to produce x PortaBoy[4] game systems for a local retailer is given by $C(x) = 80x + 150$ for $x \geq 0$.

1. Find and interpret $C(10)$.
2. How many PortaBoys can be produced for \$15,000?
3. Explain the significance of the restriction on the domain, $x \geq 0$.
4. Find and interpret $C(0)$.
5. Find and interpret the slope of the graph of $y = C(x)$.

Solution.

1. To find $C(10)$, we replace every occurrence of x with 10 in the formula for $C(x)$ to get $C(10) = 80(10) + 150 = 950$. Since x represents the number of PortaBoys produced, and $C(x)$ represents the cost, in dollars, $C(10) = 950$ means it costs \$950 to produce 10 PortaBoys for the local retailer.
2. To find how many PortaBoys can be produced for \$15,000, we solve $C(x) = 15000$, or $80x + 150 = 15000$. Solving, we get $x = \frac{14850}{80} = 185.625$. Since we can only produce a whole number amount of PortaBoys, we can produce 185 PortaBoys for \$15,000.
3. The restriction $x \geq 0$ is the applied domain, as discussed in Section 1.4.1. In this context, x represents the number of PortaBoys produced. It makes no sense to produce a negative quantity of game systems.[5]
4. We find $C(0) = 80(0) + 150 = 150$. This means it costs \$150 to produce 0 PortaBoys. As mentioned on page 199, this is the fixed, or start-up cost of this venture.
5. If we were to graph $y = C(x)$, we would be graphing the portion of the line $y = 80x + 150$ for $x \geq 0$. We recognize the slope, $m = 80$. Like any slope, we can interpret this as a rate of change. Here, $C(x)$ is the cost in dollars, while x measures the number of PortaBoys so

$$m = \frac{\Delta y}{\Delta x} = \frac{\Delta C}{\Delta x} = 80 = \frac{80}{1} = \frac{\$80}{1 \text{ PortaBoy}}.$$

   In other words, the cost is increasing at a rate of \$80 per PortaBoy produced. This is often called the **variable cost** for this venture. □

The next example asks us to find a linear function to model a related economic problem.

[4]The similarity of this name to PortaJohn is deliberate.

[5]Actually, it makes no sense to produce a fractional part of a game system, either, as we saw in the previous part of this example. This absurdity, however, seems quite forgivable in some textbooks but not to us.

Example 2.1.6. The local retailer in Example 2.1.5 has determined that the number x of PortaBoy game systems sold in a week is related to the price p in dollars of each system. When the price was \$220, 20 game systems were sold in a week. When the systems went on sale the following week, 40 systems were sold at \$190 a piece.

1. Find a linear function which fits this data. Use the weekly sales x as the independent variable and the price p as the dependent variable.

2. Find a suitable applied domain.

3. Interpret the slope.

4. If the retailer wants to sell 150 PortaBoys next week, what should the price be?

5. What would the weekly sales be if the price were set at \$150 per system?

Solution.

1. We recall from Section 1.4 the meaning of 'independent' and 'dependent' variable. Since x is to be the independent variable, and p the dependent variable, we treat x as the input variable and p as the output variable. Hence, we are looking for a function of the form $p(x) = mx + b$. To determine m and b, we use the fact that 20 PortaBoys were sold during the week when the price was 220 dollars and 40 units were sold when the price was 190 dollars. Using function notation, these two facts can be translated as $p(20) = 220$ and $p(40) = 190$. Since m represents the rate of change of p with respect to x, we have

$$m = \frac{\Delta p}{\Delta x} = \frac{190 - 220}{40 - 20} = \frac{-30}{20} = -1.5.$$

   We now have determined $p(x) = -1.5x+b$. To determine b, we can use our given data again. Using $p(20) = 220$, we substitute $x = 20$ into $p(x) = 1.5x + b$ and set the result equal to 220: $-1.5(20) + b = 220$. Solving, we get $b = 250$. Hence, we get $p(x) = -1.5x + 250$. We can check our formula by computing $p(20)$ and $p(40)$ to see if we get 220 and 190, respectively. You may recall from page 199 that the function $p(x)$ is called the price-demand (or simply demand) function for this venture.

2. To determine the applied domain, we look at the physical constraints of the problem. Certainly, we can't sell a negative number of PortaBoys, so $x \geq 0$. However, we also note that the slope of this linear function is negative, and as such, the price is decreasing as more units are sold. Thus another constraint on the price is $p(x) \geq 0$. Solving $-1.5x + 250 \geq 0$ results in $-1.5x \geq -250$ or $x \leq \frac{500}{3} = 166.\overline{6}$. Since x represents the number of PortaBoys sold in a week, we round down to 166. As a result, a reasonable applied domain for p is $[0, 166]$.

3. The slope $m = -1.5$, once again, represents the rate of change of the price of a system with respect to weekly sales of PortaBoys. Since the slope is negative, we have that the price is decreasing at a rate of \$1.50 per PortaBoy sold. (Said differently, you can sell one more PortaBoy for every \$1.50 drop in price.)

4. To determine the price which will move 150 PortaBoys, we find $p(150) = -1.5(150) + 250 = 25$. That is, the price would have to be \$25.

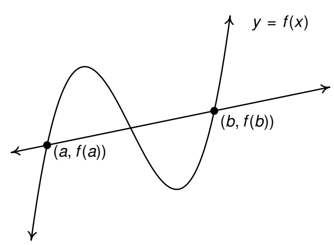
5. If the price of a PortaBoy were set at \$150, we have $p(x) = 150$, or, $-1.5x + 250 = 150$. Solving, we get $-1.5x = -100$ or $x = 66.\overline{6}$. This means you would be able to sell 66 PortaBoys a week if the price were \$150 per system. □

Not all real-world phenomena can be modeled using linear functions. Nevertheless, it is possible to use the concept of slope to help analyze non-linear functions using the following.

Definition 2.3. Let f be a function defined on the interval $[a, b]$. The **average rate of change** of f over $[a, b]$ is defined as:

$$\frac{\Delta f}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

Geometrically, if we have the graph of $y = f(x)$, the average rate of change over $[a, b]$ is the slope of the line which connects $(a, f(a))$ and $(b, f(b))$. This is called the **secant line** through these points. For that reason, some textbooks use the notation m_{sec} for the average rate of change of a function. Note that for a linear function $m = m_{\text{sec}}$, or in other words, its rate of change over an interval is the same as its average rate of change.



The graph of $y = f(x)$ and its secant line through $(a, f(a))$ and $(b, f(b))$

The interested reader may question the adjective 'average' in the phrase 'average rate of change'. In the figure above, we can see that the function changes wildly on $[a, b]$, yet the slope of the secant line only captures a snapshot of the action at a and b. This situation is entirely analogous to the average speed on a trip. Suppose it takes you 2 hours to travel 100 miles. Your average speed is $\frac{100 \text{ miles}}{2 \text{ hours}} = 50$ miles per hour. However, it is entirely possible that at the start of your journey, you traveled 25 miles per hour, then sped up to 65 miles per hour, and so forth. The average rate of change is akin to your average speed on the trip. Your speedometer measures your speed at any one instant along the trip, your **instantaneous rate of change**, and this is one of the central themes of Calculus.[6]

[6]Here we go again...

When interpreting rates of change, we interpret them the same way we did slopes. In the context of functions, it may be helpful to think of the average rate of change as:

$$\frac{\text{change in outputs}}{\text{change in inputs}}$$

Example 2.1.7. Recall from page 199, the revenue from selling x units at a price p per unit is given by the formula $R = xp$. Suppose we are in the scenario of Examples 2.1.5 and 2.1.6.

1. Find and simplify an expression for the weekly revenue $R(x)$ as a function of weekly sales x.
2. Find and interpret the average rate of change of $R(x)$ over the interval $[0, 50]$.
3. Find and interpret the average rate of change of $R(x)$ as x changes from 50 to 100 and compare that to your result in part 2.
4. Find and interpret the average rate of change of weekly revenue as weekly sales increase from 100 PortaBoys to 150 PortaBoys.

Solution.

1. Since $R = xp$, we substitute $p(x) = -1.5x + 250$ from Example 2.1.6 to get $R(x) = x(-1.5x + 250) = -1.5x^2 + 250x$. Since we determined the price-demand function $p(x)$ is restricted to $0 \leq x \leq 166$, $R(x)$ is restricted to these values of x as well.
2. Using Definition 2.3, we get that the average rate of change is

$$\frac{\Delta R}{\Delta x} = \frac{R(50) - R(0)}{50 - 0} = \frac{8750 - 0}{50 - 0} = 175.$$

   Interpreting this slope as we have in similar situations, we conclude that for every additional PortaBoy sold during a given week, the weekly revenue increases \$175.

3. The wording of this part is slightly different than that in Definition 2.3, but its meaning is to find the average rate of change of R over the interval $[50, 100]$. To find this rate of change, we compute

$$\frac{\Delta R}{\Delta x} = \frac{R(100) - R(50)}{100 - 50} = \frac{10000 - 8750}{50} = 25.$$

   In other words, for each additional PortaBoy sold, the revenue increases by \$25. Note that while the revenue is still increasing by selling more game systems, we aren't getting as much of an increase as we did in part 2 of this example. (Can you think of why this would happen?)

4. Translating the English to the mathematics, we are being asked to find the average rate of change of R over the interval $[100, 150]$. We find

$$\frac{\Delta R}{\Delta x} = \frac{R(150) - R(100)}{150 - 100} = \frac{3750 - 10000}{50} = -125.$$

This means that we are losing \$125 dollars of weekly revenue for each additional PortaBoy sold. (Can you think why this is possible?) □

We close this section with a new look at difference quotients which were first introduced in Section 1.4. If we wish to compute the average rate of change of a function f over the interval $[x, x+h]$, then we would have

$$\frac{\Delta f}{\Delta x} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

As we have indicated, the rate of change of a function (average or otherwise) is of great importance in Calculus.[7] Also, we have the geometric interpretation of difference quotients which was promised to you back on page 199 – a difference quotient yields the slope of a secant line.

[7]So we are not torturing you with these for nothing.

2.1.1 Exercises

In Exercises 1 - 10, find both the point-slope form and the slope-intercept form of the line with the given slope which passes through the given point.

1. $m = 3, \quad P(3, -1)$
2. $m = -2, \quad P(-5, 8)$
3. $m = -1, \quad P(-7, -1)$
4. $m = \frac{2}{3}, \quad P(-2, 1)$
5. $m = -\frac{1}{5}, \quad P(10, 4)$
6. $m = \frac{1}{7}, \quad P(-1, 4)$
7. $m = 0, \quad P(3, 117)$
8. $m = -\sqrt{2}, \quad P(0, -3)$
9. $m = -5, \quad P(\sqrt{3}, 2\sqrt{3})$
10. $m = 678, \quad P(-1, -12)$

In Exercises 11 - 20, find the slope-intercept form of the line which passes through the given points.

11. $P(0, 0), \; Q(-3, 5)$
12. $P(-1, -2), \; Q(3, -2)$
13. $P(5, 0), \; Q(0, -8)$
14. $P(3, -5), \; Q(7, 4)$
15. $P(-1, 5), \; Q(7, 5)$
16. $P(4, -8), \; Q(5, -8)$
17. $P\left(\frac{1}{2}, \frac{3}{4}\right), \; Q\left(\frac{5}{2}, -\frac{7}{4}\right)$
18. $P\left(\frac{2}{3}, \frac{7}{2}\right), \; Q\left(-\frac{1}{3}, \frac{3}{2}\right)$
19. $P\left(\sqrt{2}, -\sqrt{2}\right), \; Q\left(-\sqrt{2}, \sqrt{2}\right)$
20. $P\left(-\sqrt{3}, -1\right), \; Q\left(\sqrt{3}, 1\right)$

In Exercises 21 - 26, graph the function. Find the slope, y-intercept and x-intercept, if any exist.

21. $f(x) = 2x - 1$
22. $f(x) = 3 - x$
23. $f(x) = 3$
24. $f(x) = 0$
25. $f(x) = \frac{2}{3}x + \frac{1}{3}$
26. $f(x) = \dfrac{1 - x}{2}$
27. Find all of the points on the line $y = 2x + 1$ which are 4 units from the point $(-1, 3)$.
28. Jeff can walk comfortably at 3 miles per hour. Find a linear function d that represents the total distance Jeff can walk in t hours, assuming he doesn't take any breaks.
29. Carl can stuff 6 envelopes per *minute*. Find a linear function E that represents the total number of envelopes Carl can stuff after t *hours*, assuming he doesn't take any breaks.
30. A landscaping company charges \$45 per cubic yard of mulch plus a delivery charge of \$20. Find a linear function which computes the total cost C (in dollars) to deliver x cubic yards of mulch.

31. A plumber charges \$50 for a service call plus \$80 per hour. If she spends no longer than 8 hours a day at any one site, find a linear function that represents her total daily charges C (in dollars) as a function of time t (in hours) spent at any one given location.

32. A salesperson is paid \$200 per week plus 5% commission on her weekly sales of x dollars. Find a linear function that represents her total weekly pay, W (in dollars) in terms of x. What must her weekly sales be in order for her to earn \$475.00 for the week?

33. An on-demand publisher charges \$22.50 to print a 600 page book and \$15.50 to print a 400 page book. Find a linear function which models the cost of a book C as a function of the number of pages p. Interpret the slope of the linear function and find and interpret $C(0)$.

34. The Topology Taxi Company charges \$2.50 for the first fifth of a mile and \$0.45 for each additional fifth of a mile. Find a linear function which models the taxi fare F as a function of the number of miles driven, m. Interpret the slope of the linear function and find and interpret $F(0)$.

35. Water freezes at 0° Celsius and 32° Fahrenheit and it boils at 100°C and 212°F.

    (a) Find a linear function F that expresses temperature in the Fahrenheit scale in terms of degrees Celsius. Use this function to convert 20°C into Fahrenheit.

    (b) Find a linear function C that expresses temperature in the Celsius scale in terms of degrees Fahrenheit. Use this function to convert 110°F into Celsius.

    (c) Is there a temperature n such that $F(n) = C(n)$?

36. Legend has it that a bull Sasquatch in rut will howl approximately 9 times per hour when it is $40^\circ F$ outside and only 5 times per hour if it's $70^\circ F$. Assuming that the number of howls per hour, N, can be represented by a linear function of temperature Fahrenheit, find the number of howls per hour he'll make when it's only $20^\circ F$ outside. What is the applied domain of this function? Why?

37. Economic forces beyond anyone's control have changed the cost function for PortaBoys to $C(x) = 105x + 175$. Rework Example 2.1.5 with this new cost function.

38. In response to the economic forces in Exercise 37 above, the local retailer sets the selling price of a PortaBoy at \$250. Remarkably, 30 units were sold each week. When the systems went on sale for \$220, 40 units per week were sold. Rework Examples 2.1.6 and 2.1.7 with this new data. What difficulties do you encounter?

39. A local pizza store offers medium two-topping pizzas delivered for \$6.00 per pizza plus a \$1.50 delivery charge per order. On weekends, the store runs a 'game day' special: if six or more medium two-topping pizzas are ordered, they are \$5.50 each with no delivery charge. Write a piecewise-defined linear function which calculates the cost C (in dollars) of p medium two-topping pizzas delivered during a weekend.

40. A restaurant offers a buffet which costs \$15 per person. For parties of 10 or more people, a group discount applies, and the cost is \$12.50 per person. Write a piecewise-defined linear function which calculates the total bill T of a party of n people who all choose the buffet.

41. A mobile plan charges a base monthly rate of \$10 for the first 500 minutes of air time plus a charge of 15¢ for each additional minute. Write a piecewise-defined linear function which calculates the monthly cost C (in dollars) for using m minutes of air time.

    HINT: You may want to revisit Exercise 74 in Section 1.4

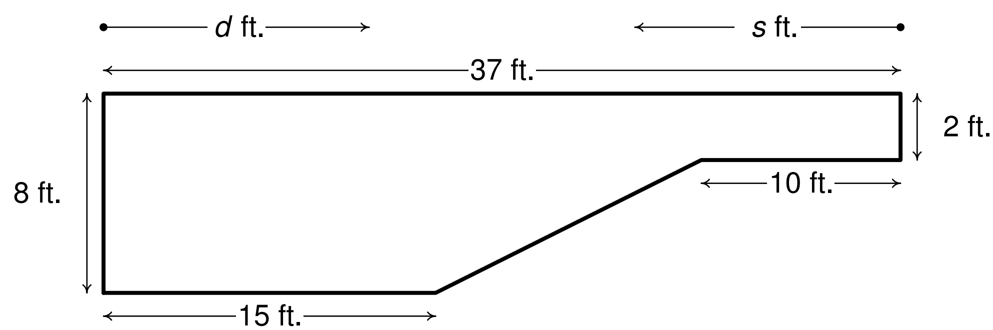
42. The local pet shop charges 12¢ per cricket up to 100 crickets, and 10¢ per cricket thereafter. Write a piecewise-defined linear function which calculates the price P, in dollars, of purchasing c crickets.

43. The cross-section of a swimming pool is below. Write a piecewise-defined linear function which describes the depth of the pool, D (in feet) as a function of:

    (a) the distance (in feet) from the edge of the shallow end of the pool, d.

    (b) the distance (in feet) from the edge of the deep end of the pool, s.

    (c) Graph each of the functions in (a) and (b). Discuss with your classmates how to transform one into the other and how they relate to the diagram of the pool.



In Exercises 44 - 49, compute the average rate of change of the function over the specified interval.

44. $f(x) = x^3$, $[-1, 2]$

45. $f(x) = \dfrac{1}{x}$, $[1, 5]$

46. $f(x) = \sqrt{x}$, $[0, 16]$

47. $f(x) = x^2$, $[-3, 3]$

48. $f(x) = \dfrac{x+4}{x-3}$, $[5, 7]$

49. $f(x) = 3x^2 + 2x - 7$, $[-4, 2]$

In Exercises 50 - 53, compute the average rate of change of the given function over the interval $[x, x+h]$. Here we assume $[x, x+h]$ is in the domain of the function.

50. $f(x) = x^3$

51. $f(x) = \dfrac{1}{x}$

52. $f(x) = \dfrac{x+4}{x-3}$

53. $f(x) = 3x^2 + 2x - 7$

54. The height of an object dropped from the roof of an eight story building is modeled by: $h(t) = -16t^2 + 64$, $0 \leq t \leq 2$. Here, h is the height of the object off the ground in feet, t seconds after the object is dropped. Find and interpret the average rate of change of h over the interval $[0, 2]$.

55. Using data from Bureau of Transportation Statistics, the average fuel economy F in miles per gallon for passenger cars in the US can be modeled by $F(t) = -0.0076t^2 + 0.45t + 16$, $0 \leq t \leq 28$, where t is the number of years since 1980. Find and interpret the average rate of change of F over the interval $[0, 28]$.

56. The temperature T in degrees Fahrenheit t hours after 6 AM is given by:

$$T(t) = -\frac{1}{2}t^2 + 8t + 32, \quad 0 \leq t \leq 12$$

(a) Find and interpret $T(4)$, $T(8)$ and $T(12)$.

(b) Find and interpret the average rate of change of T over the interval $[4, 8]$.

(c) Find and interpret the average rate of change of T from $t = 8$ to $t = 12$.

(d) Find and interpret the average rate of temperature change between 10 AM and 6 PM.

57. Suppose $C(x) = x^2 - 10x + 27$ represents the costs, in *hundreds*, to produce x *thousand* pens. Find and interpret the average rate of change as production is increased from making 3000 to 5000 pens.

58. With the help of your classmates find several other “real-world” examples of rates of change that are used to describe non-linear phenomena.

(Parallel Lines) Recall from Intermediate Algebra that parallel lines have the same slope. (Please note that two vertical lines are also parallel to one another even though they have an undefined slope.) In Exercises 59 - 64, you are given a line and a point which is not on that line. Find the line parallel to the given line which passes through the given point.

59. $y = 3x + 2$, $P(0, 0)$

60. $y = -6x + 5$, $P(3, 2)$

61. $y = \frac{2}{3}x - 7$, $P(6,0)$

62. $y = \dfrac{4-x}{3}$, $P(1,-1)$

63. $y = 6$, $P(3,-2)$

64. $x = 1$, $P(-5,0)$

(Perpendicular Lines) Recall from Intermediate Algebra that two non-vertical lines are perpendicular if and only if they have negative reciprocal slopes. That is to say, if one line has slope m_1 and the other has slope m_2 then $m_1 \cdot m_2 = -1$. (You will be guided through a proof of this result in Exercise 71.) Please note that a horizontal line is perpendicular to a vertical line and vice versa, so we assume $m_1 \neq 0$ and $m_2 \neq 0$. In Exercises 65 - 70, you are given a line and a point which is not on that line. Find the line perpendicular to the given line which passes through the given point.

65. $y = \frac{1}{3}x + 2$, $P(0,0)$

66. $y = -6x + 5$, $P(3,2)$

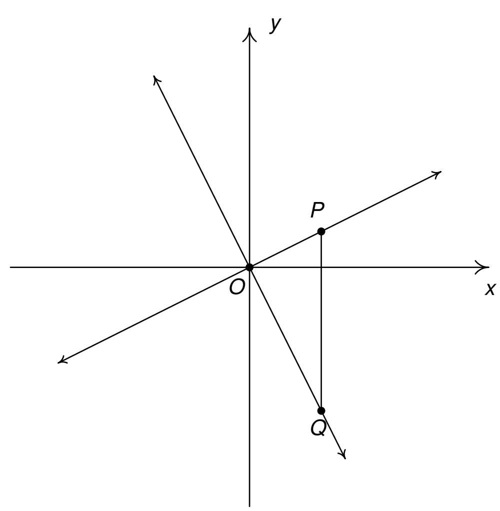
67. $y = \frac{2}{3}x - 7$, $P(6,0)$

68. $y = \dfrac{4-x}{3}$, $P(1,-1)$

69. $y = 6$, $P(3,-2)$

70. $x = 1$, $P(-5,0)$

71. We shall now prove that $y = m_1x + b_1$ is perpendicular to $y = m_2x + b_2$ if and only if $m_1 \cdot m_2 = -1$. To make our lives easier we shall assume that $m_1 > 0$ and $m_2 < 0$. We can also "move" the lines so that their point of intersection is the origin without messing things up, so we'll assume $b_1 = b_2 = 0$. (Take a moment with your classmates to discuss why this is okay.) Graphing the lines and plotting the points $O(0,0)$, $P(1,m_1)$ and $Q(1,m_2)$ gives us the following set up.



The line $y = m_1x$ will be perpendicular to the line $y = m_2x$ if and only if $\triangle OPQ$ is a right triangle. Let d_1 be the distance from O to P, let d_2 be the distance from O to Q and let d_3 be the distance from P to Q. Use the Pythagorean Theorem to show that $\triangle OPQ$ is a right triangle if and only if $m_1 \cdot m_2 = -1$ by showing $d_1^2 + d_2^2 = d_3^2$ if and only if $m_1 \cdot m_2 = -1$.

72. Show that if $a \neq b$, the line containing the points (a, b) and (b, a) is perpendicular to the line $y = x$. (Coupled with the result from Example 1.1.6 on page 134, we have now shown that the line $y = x$ is a *perpendicular* bisector of the line segment connecting (a, b) and (b, a). This means the points (a, b) and (b, a) are symmetric about the line $y = x$.

73. The function defined by $I(x) = x$ is called the Identity Function. Discuss with your classmates why this name makes sense.

2.1.2 Answers

1. $y + 1 = 3(x - 3)$
$y = 3x - 10$

2. $y - 8 = -2(x + 5)$
$y = -2x - 2$

3. $y + 1 = -(x + 7)$
$y = -x - 8$

4. $y - 1 = \frac{2}{3}(x + 2)$
$y = \frac{2}{3}x + \frac{7}{3}$

5. $y - 4 = -\frac{1}{5}(x - 10)$
$y = -\frac{1}{5}x + 6$

6. $y - 4 = \frac{1}{7}(x + 1)$
$y = \frac{1}{7}x + \frac{29}{7}$

7. $y - 117 = 0$
$y = 117$

8. $y + 3 = -\sqrt{2}(x - 0)$
$y = -\sqrt{2}x - 3$

9. $y - 2\sqrt{3} = -5(x - \sqrt{3})$
$y = -5x + 7\sqrt{3}$

10. $y + 12 = 678(x + 1)$
$y = 678x + 666$

11. $y = -\frac{5}{3}x$

12. $y = -2$

13. $y = \frac{8}{5}x - 8$

14. $y = \frac{9}{4}x - \frac{47}{4}$

15. $y = 5$

16. $y = -8$

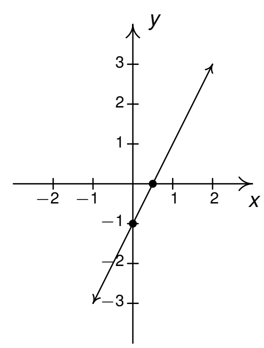
17. $y = -\frac{5}{4}x + \frac{11}{8}$

18. $y = 2x + \frac{13}{6}$

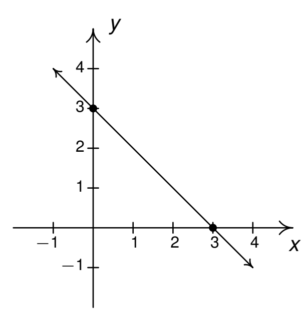
19. $y = -x$

20. $y = \frac{\sqrt{3}}{3}x$

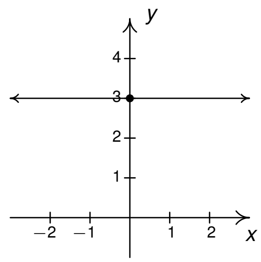
21. $f(x) = 2x - 1$
slope: $m = 2$
y-intercept: $(0, -1)$
x-intercept: $\left(\frac{1}{2}, 0\right)$



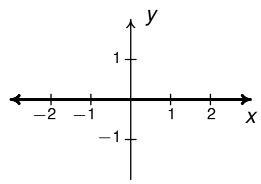
22. $f(x) = 3 - x$
slope: $m = -1$
y-intercept: $(0, 3)$
x-intercept: $(3, 0)$



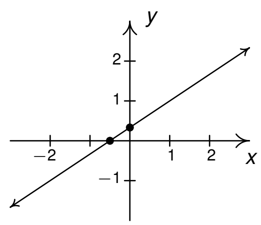
23. $f(x) = 3$
slope: $m = 0$
y-intercept: $(0, 3)$
x-intercept: none



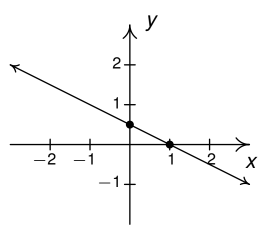
24. $f(x) = 0$
slope: $m = 0$
y-intercept: $(0, 0)$
x-intercept: $\{(x, 0) \mid x \text{ is a real number}\}$



25. $f(x) = \frac{2}{3}x + \frac{1}{3}$
slope: $m = \frac{2}{3}$
y-intercept: $\left(0, \frac{1}{3}\right)$
x-intercept: $\left(-\frac{1}{2}, 0\right)$



26. $f(x) = \dfrac{1 - x}{2}$
slope: $m = -\frac{1}{2}$
y-intercept: $\left(0, \frac{1}{2}\right)$
x-intercept: $(1, 0)$



27. $(-1, -1)$ and $\left(\frac{11}{5}, \frac{27}{5}\right)$

28. $d(t) = 3t$, $t \geq 0$.

29. $E(t) = 360t$, $t \geq 0$.

30. $C(x) = 45x + 20$, $x \geq 0$.

31. $C(t) = 80t + 50$, $0 \leq t \leq 8$.

32. $W(x) = 200 + .05x$, $x \geq 0$ She must make \$5500 in weekly sales.

33. $C(p) = 0.035p + 1.5$ The slope 0.035 means it costs 3.5¢ per page. $C(0) = 1.5$ means there is a fixed, or start-up, cost of \$1.50 to make each book.

34. $F(m) = 2.25m + 2.05$ The slope 2.25 means it costs an additional \$2.25 for each mile beyond the first 0.2 miles. $F(0) = 2.05$, so according to the model, it would cost \$2.05 for a trip of 0 miles. Would this ever really happen? Depends on the driver and the passenger, we suppose.

35. (a) $F(C) = \frac{9}{5}C + 32$ (b) $C(F) = \frac{5}{9}(F - 32) = \frac{5}{9}F - \frac{160}{9}$

(c) $F(-40) = -40 = C(-40)$.

36. $N(T) = -\frac{2}{15}T + \frac{43}{3}$

Having a negative number of howls makes no sense and since $N(107.5) = 0$ we can put an upper bound of $107.5^\circ F$ on the domain. The lower bound is trickier because there's nothing other than common sense to go on. As it gets colder, he howls more often. At some point it will either be so cold that he freezes to death or he's howling non-stop. So we're going to say that he can withstand temperatures no lower than $-60^\circ F$ so that the applied domain is $[-60, 107.5]$.

39. $C(p) = \begin{cases} 6p + 1.5 & \text{if } 1 \leq p \leq 5 \\ 5.5p & \text{if } p \geq 6 \end{cases}$

40. $T(n) = \begin{cases} 15n & \text{if } 1 \leq n \leq 9 \\ 12.5n & \text{if } n \geq 10 \end{cases}$

41. $C(m) = \begin{cases} 10 & \text{if } 0 \leq m \leq 500 \\ 10 + 0.15(m - 500) & \text{if } m > 500 \end{cases}$

42. $P(c) = \begin{cases} 0.12c & \text{if } 1 \leq c \leq 100 \\ 12 + 0.1(c - 100) & \text{if } c > 100 \end{cases}$

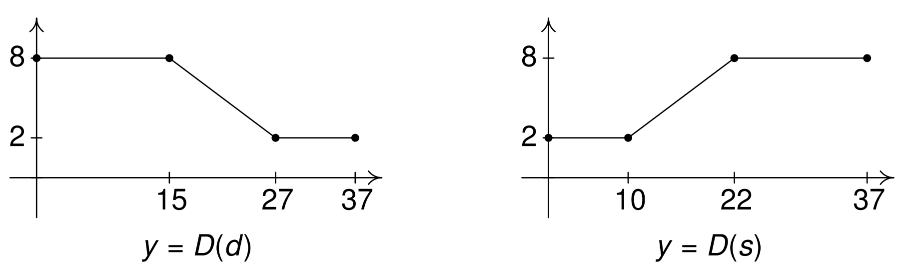
43. (a)

$$D(d) = \begin{cases} 8 & \text{if } 0 \leq d \leq 15 \\ -\frac{1}{2}d + \frac{31}{2} & \text{if } 15 \leq d \leq 27 \\ 2 & \text{if } 27 \leq d \leq 37 \end{cases}$$

(b)

$$D(s) = \begin{cases} 2 & \text{if } 0 \leq s \leq 10 \\ \frac{1}{2}s - 3 & \text{if } 10 \leq s \leq 22 \\ 8 & \text{if } 22 \leq s \leq 37 \end{cases}$$

(c)



44. $\dfrac{2^3-(-1)^3}{2-(-1)} = 3$

45. $\dfrac{\frac{1}{5}-\frac{1}{1}}{5-1} = -\dfrac{1}{5}$

46. $\dfrac{\sqrt{16}-\sqrt{0}}{16-0} = \dfrac{1}{4}$

47. $\dfrac{3^2-(-3)^2}{3-(-3)} = 0$

48. $\dfrac{\frac{7+4}{7-3}-\frac{5+4}{5-3}}{7-5} = -\dfrac{7}{8}$

49. $\dfrac{(3(2)^2+2(2)-7)-(3(-4)^2+2(-4)-7)}{2-(-4)} = -4$

50. $3x^2+3xh+h^2$

51. $\dfrac{-1}{x(x+h)}$

52. $\dfrac{-7}{(x-3)(x+h-3)}$

53. $6x+3h+2$

54. The average rate of change is $\frac{h(2)-h(0)}{2-0} = -32$. During the first two seconds after it is dropped, the object has fallen at an average rate of 32 feet per second. (This is called the *average velocity* of the object.)

55. The average rate of change is $\frac{F(28)-F(0)}{28-0} = 0.2372$. During the years from 1980 to 2008, the average fuel economy of passenger cars in the US increased, on average, at a rate of 0.2372 miles per gallon per year.

56. (a) $T(4) = 56$, so at 10 AM (4 hours after 6 AM), it is 56°F. $T(8) = 64$, so at 2 PM (8 hours after 6 AM), it is 64°F. $T(12) = 56$, so at 6 PM (12 hours after 6 AM), it is 56°F.
    (b) The average rate of change is $\frac{T(8)-T(4)}{8-4} = 2$. Between 10 AM and 2 PM, the temperature increases, on average, at a rate of 2°F per hour.
    (c) The average rate of change is $\frac{T(12)-T(8)}{12-8} = -2$. Between 2 PM and 6 PM, the temperature decreases, on average, at a rate of 2°F per hour.
    (d) The average rate of change is $\frac{T(12)-T(4)}{12-4} = 0$. Between 10 AM and 6 PM, the temperature, on average, remains constant.

57. The average rate of change is $\frac{C(5)-C(3)}{5-3} = -2$. As production is increased from 3000 to 5000 pens, the cost decreases at an average rate of $200 per 1000 pens produced (20¢ per pen.)

59. $y = 3x$

60. $y = -6x+20$

61. $y = \frac{2}{3}x-4$

62. $y = -\frac{1}{3}x-\frac{2}{3}$

63. $y = -2$

64. $x = -5$

65. $y = -3x$

66. $y = \frac{1}{6}x+\frac{3}{2}$

67. $y = -\frac{3}{2}x+9$

68. $y = 3x-4$

69. $x = 3$

70. $y = 0$

2.2 Systems of Linear Equations

Up until now, when we concerned ourselves with solving different types of equations there was only one equation to solve at a time. Given an equation $f(x) = g(x)$, we could check our solutions geometrically by finding where the graphs of $y = f(x)$ and $y = g(x)$ intersect. The x-coordinates of these intersection points correspond to the solutions to the equation $f(x) = g(x)$, and the y-coordinates were largely ignored. If we modify the problem and ask for the intersection points of the graphs of $y = f(x)$ and $y = g(x)$, where both the solution to x and y are of interest, we have what is known as a **system of equations**, usually written as

$$\left\{ \begin{array}{rcl} y & = & f(x) \\ y & = & g(x) \end{array} \right.$$

The 'curly bracket' notation means we are to find all **pairs** of points (x, y) which satisfy **both** equations. In this section we will focus on systems of linear equations in two variables.

Definition 2.4. A **linear equation in two variables** is an equation of the form $a_1 x + a_2 y = c$ where a_1, a_2 and c are real numbers and at least one of a_1 and a_2 is nonzero.

We are using subscripts in Definition 2.4 to indicate different, but fixed, real numbers. For example, $3x - \frac{y}{2} = 0.1$ is a linear equation in two variables with $a_1 = 3$, $a_2 = -\frac{1}{2}$ and $c = 0.1$. We can also consider $x = 5$ to be a linear equation in two variables by identifying $a_1 = 1$, $a_2 = 0$, and $c = 5$. If a_1 and a_2 are both 0, then depending on c, we get either an equation which is *always* true, called an **identity**, or an equation which is *never* true, called a **contradiction**. (If $c = 0$, then we get $0 = 0$, which is always true. If $c \neq 0$, then we'd have $0 \neq 0$, which is never true.) Even though identities and contradictions have a large role to play in the upcoming sections, we do not consider them linear equations. The key to identifying linear equations is to note that the variables involved are to the first power and that the coefficients of the variables are numbers. Some examples of equations which are non-linear are $x^2 + y = 1$, $xy = 5$ and $e^{2x} + \ln(y) = 1$. We leave it to the reader to explain why these do not satisfy Definition 2.4. From what we know from Sections 1.2 and 2.1, the graphs of linear equations are lines. If we couple two or more linear equations together, in effect to find the points of intersection of two or more lines, we obtain a **system of linear equations in two variables**. Our first example discusses some of the basic techniques to solve such equations.

Example 2.2.1. Solve the following systems of equations.

1. $\left\{ \begin{array}{rcl} 2x - y & = & 1 \\ y & = & 3 \end{array} \right.$

2. $\left\{ \begin{array}{rcr} 3x + 4y & = & -2 \\ -3x - y & = & 5 \end{array} \right.$

3. $\left\{ \begin{array}{rcl} \frac{x}{3} - \frac{4y}{5} & = & \frac{7}{5} \\ \frac{2x}{9} + \frac{y}{3} & = & \frac{1}{2} \end{array} \right.$

4. $\left\{ \begin{array}{rcl} 2x - 4y & = & 6 \\ 3x - 6y & = & 9 \end{array} \right.$

5. $\left\{ \begin{array}{rcr} 6x + 3y & = & 9 \\ 4x + 2y & = & 12 \end{array} \right.$

6. $\left\{ \begin{array}{rcr} x - y & = & 0 \\ x + y & = & 2 \\ -2x + y & = & -2 \end{array} \right.$

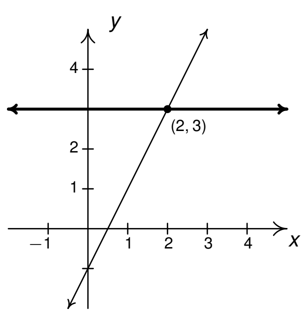
Solution.

1. Our first system is nearly solved for us. The second equation tells us that $y = 3$. To find the corresponding value of x, we **substitute** this value for y into the the first equation to obtain $2x - 3 = 1$, so that $x = 2$. Our solution to the system is $(2, 3)$. To check this algebraically, we substitute $x = 2$ and $y = 3$ into each equation and see that they are satisfied. We see $2(2) - 3 = 1$, and $3 = 3$, as required. To check our answer graphically, we graph the lines $2x - y = 1$ and $y = 3$ and verify that they intersect at $(2, 3)$.

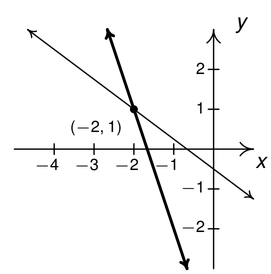
2. To solve the second system, we use the **addition** method to **eliminate** the variable x. We take the two equations as given and 'add equals to equals' to obtain

$$\begin{array}{rrcr} & 3x + 4y & = & -2 \\ + & (-3x - y & = & 5) \\ \hline & 3y & = & 3 \end{array}$$

   This gives us $y = 1$. We now substitute $y = 1$ into either of the two equations, say $-3x - y = 5$, to get $-3x - 1 = 5$ so that $x = -2$. Our solution is $(-2, 1)$. Substituting $x = -2$ and $y = 1$ into the first equation gives $3(-2) + 4(1) = -2$, which is true, and, likewise, when we check $(-2, 1)$ in the second equation, we get $-3(-2) - 1 = 5$, which is also true. Geometrically, the lines $3x + 4y = -2$ and $-3x - y = 5$ intersect at $(-2, 1)$.



$2x - y = 1$
$y = 3$



$3x + 4y = -2$
$-3x - y = 5$

3. The equations in the third system are more approachable if we clear denominators. We multiply both sides of the first equation by 15 and both sides of the second equation by 18 to obtain the kinder, gentler system

$$\left\{ \begin{array}{rcr} 5x - 12y & = & 21 \\ 4x + 6y & = & 9 \end{array} \right.$$

   Adding these two equations directly fails to eliminate either of the variables, but we note that if we multiply the first equation by 4 and the second by -5, we will be in a position to eliminate the x term

$$\begin{array}{rrcr} & 20x-48y & = & 84 \\ + & (-20x-30y & = & -45) \\ \hline & -78y & = & 39 \end{array}$$

From this we get $y=-\frac{1}{2}$. We can temporarily avoid too much unpleasantness by choosing to substitute $y=-\frac{1}{2}$ into one of the equivalent equations we found by clearing denominators, say into $5x-12y=21$. We get $5x+6=21$ which gives $x=3$. Our answer is $\left(3,-\frac{1}{2}\right)$. At this point, we have no choice − in order to check an answer algebraically, we must see if the answer satisfies both of the *original* equations, so we substitute $x=3$ and $y=-\frac{1}{2}$ into both $\frac{x}{3}-\frac{4y}{5}=\frac{7}{5}$ and $\frac{2x}{9}+\frac{y}{3}=\frac{1}{2}$. We leave it to the reader to verify that the solution is correct. Graphing both of the lines involved with considerable care yields an intersection point of $\left(3,-\frac{1}{2}\right)$.

4. An eerie calm settles over us as we cautiously approach our fourth system. Do its friendly integer coefficients belie something more sinister? We note that if we multiply both sides of the first equation by 3 and the both sides of the second equation by −2, we are ready to eliminate the x

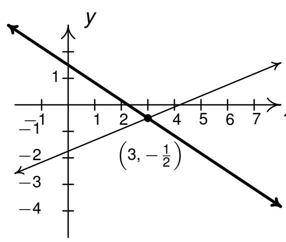
$$\begin{array}{rrcr} & 6x-12y & = & 18 \\ + & (-6x+12y & = & -18) \\ \hline & 0 & = & 0 \end{array}$$

We eliminated not only the x, but the y as well and we are left with the identity $0=0$. This means that these two different linear equations are, in fact, equivalent. In other words, if an ordered pair (x,y) satisfies the equation $2x-4y=6$, it *automatically* satisfies the equation $3x-6y=9$. One way to describe the solution set to this system is to use the roster method[1] and write $\{(x,y)\,|\,2x-4y=6\}$. While this is correct (and corresponds exactly to what's happening graphically, as we shall see shortly), we take this opportunity to introduce the notion of a **parametric solution to a system.** Our first step is to solve $2x-4y=6$ for one of the variables, say $y=\frac{1}{2}x-\frac{3}{2}$. For each value of x, the formula $y=\frac{1}{2}x-\frac{3}{2}$ determines the corresponding y-value of a solution. Since we have no restriction on x, it is called a **free variable**. We let $x=t$, a so-called 'parameter', and get $y=\frac{1}{2}t-\frac{3}{2}$. Our set of solutions can then be described as $\left\{\left(t,\frac{1}{2}t-\frac{3}{2}\right)\,\middle|\,-\infty<t<\infty\right\}$.[2] For specific values of t, we can generate solutions. For example, $t=0$ gives us the solution $\left(0,-\frac{3}{2}\right)$; $t=117$ gives us $(117,57)$, and while we can readily check each of these particular solutions satisfy both equations, the question is how do we check our general answer algebraically? Same as

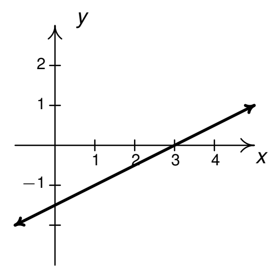
[1] See Section 1.2 for a review of this.

[2] Note that we could have just as easily chosen to solve $2x-4y=6$ for x to obtain $x=2y+3$. Letting y be the parameter t, we have that for any value of t, $x=2t+3$, which gives $\{(2t+3,t)\,|\,-\infty<t<\infty\}$. There is no one correct way to parameterize the solution set, which is why it is always best to check your answer.

always. We claim that for any real number t, the pair $\left(t, \frac{1}{2}t - \frac{3}{2}\right)$ satisfies both equations. Substituting $x = t$ and $y = \frac{1}{2}t - \frac{3}{2}$ into $2x - 4y = 6$ gives $2t - 4\left(\frac{1}{2}t - \frac{3}{2}\right) = 6$. Simplifying, we get $2t - 2t + 6 = 6$, which is always true. Similarly, when we make these substitutions in the equation $3x - 6y = 9$, we get $3t - 6\left(\frac{1}{2}t - \frac{3}{2}\right) = 9$ which reduces to $3t - 3t + 9 = 9$, so it checks out, too. Geometrically, $2x - 4y = 6$ and $3x - 6y = 9$ are the same line, which means that they intersect at every point on their graphs. The reader is encouraged to think about how our parametric solution says exactly that.



$\frac{x}{3} - \frac{4y}{5} = \frac{7}{5}$
$\mathbf{\frac{2x}{9} + \frac{y}{3} = \frac{1}{2}}$



$2x - 4y = 6$
$\mathbf{3x - 6y = 9}$
(Same line.)

5. Multiplying both sides of the first equation by 2 and the both sides of the second equation by -3, we set the stage to eliminate x

$$\begin{array}{rrcr} & 12x + 6y & = & 18 \\ + & (-12x - 6y & = & -36) \\ \hline & 0 & = & -18 \end{array}$$

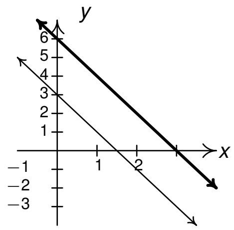
As in the previous example, both x and y dropped out of the equation, but we are left with an irrevocable contradiction, $0 = -18$. This tells us that it is impossible to find a pair (x, y) which satisfies both equations; in other words, the system has no solution. Graphically, the lines $6x + 3y = 9$ and $4x + 2y = 12$ are distinct and parallel, so they do not intersect.

6. We can begin to solve our last system by adding the first two equations

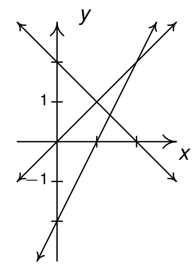
$$\begin{array}{rrcr} & x - y & = & 0 \\ + & (x + y & = & 2) \\ \hline & 2x & = & 2 \end{array}$$

which gives $x = 1$. Substituting this into the first equation gives $1 - y = 0$ so that $y = 1$. We seem to have determined a solution to our system, $(1, 1)$. While this checks in the first two equations, when we substitute $x = 1$ and $y = 1$ into the third equation, we get $-2(1) + (1) = -2$

which simplifies to the contradiction $-1 = -2$. Graphing the lines $x - y = 0$, $x + y = 2$, and $-2x + y = -2$, we see that the first two lines do, in fact, intersect at $(1, 1)$, however, all three lines never intersect at the same point simultaneously, which is what is required if a solution to the system is to be found.



$6x + 3y = 9$
$\mathbf{4x + 2y = 12}$



$y - x = 0$
$y + x = 2$
$-2x + y = -2$

□

A few remarks about Example 2.2.1 are in order. It is clear that some systems of equations have solutions, and some do not. Those which have solutions are called **consistent**, those with no solution are called **inconsistent**. We also distinguish the two different types of behavior among consistent systems. Those which admit free variables are called **dependent**; those with no free variables are called **independent**.[3] Using this new vocabulary, we classify numbers 1, 2 and 3 in Example 2.2.1 as consistent independent systems, number 4 is consistent dependent, and numbers 5 and 6 are inconsistent.[4] The system in 6 above is called **overdetermined**, since we have more equations than variables.[5] Not surprisingly, a system with more variables than equations is called **underdetermined**. While the system in number 6 above is overdetermined and inconsistent, there exist overdetermined consistent systems (both dependent and independent) and we leave it to the reader to think about what is happening algebraically and geometrically in these cases.

[3]In the case of systems of linear equations, regardless of the number of equations or variables, consistent independent systems have exactly one solution. The reader is encouraged to think about why this is the case for linear equations in two variables. Hint: think geometrically.

[4]The adjectives 'dependent' and 'independent' apply only to *consistent* systems – they describe the *type* of solutions. Is there a free variable (dependent) or not (independent)?

[5]If we think if each variable being an unknown quantity, then ostensibly, to recover two unknown quantities, we need two pieces of information - i.e., two equations. Having more than two equations suggests we have more information than necessary to determine the values of the unknowns. While this is not necessarily the case, it does explain the choice of terminology 'overdetermined'.

We close this section with a standard 'mixture' type application of systems of linear equations.

Example 2.2.2. Lucas needs to create 500 milliliters (mL) of a 50% acid solution. He has stock solutions of 30% and 80% acid. Set-up and solve a system of linear equations which determines the amount of the stock solutions which would produce the required solution.

Solution. We are after two unknowns, the amount (in mL) of the 30% stock solution (which we'll call x) and the amount (in mL) of the 80% stock solution (which we'll call y). We now need to determine some relationships between the two variables. Our goal is to produce 500 milliliters of a 50% acid solution. This product has two defining characteristics. First, it must be 500 mL; second, it must be 50% acid. We take each of these qualities in turn. First, the total volume of 500 mL must be the sum of the contributed volumes of the two stock solutions. That is

$$\text{amount of 30\% stock solution} + \text{amount of 80\% stock solution} = 500\,\text{mL}$$

Using our defined variables, this reduces to $x + y = 500$. Next, we need to make sure the final solution is 50% acid. Since 50% of 500 mL is 250 mL, the final solution must contain 250 mL of acid. We have

$$\text{amount of acid in 30\% stock solution} + \text{amount of acid 80\% stock solution} = 250\,\text{mL}$$

The amount of acid in x mL of 30% stock is $0.30x$ and the amount of acid in y mL of 80% solution is $0.80y$. We have $0.30x+0.80y = 250$. Converting to fractions,[6] our system of equations becomes

$$\left\{ \begin{array}{rcl} x + y & = & 500 \\ \frac{3}{10}x + \frac{4}{5}y & = & 250 \end{array} \right.$$

which we can solve by elimination or substitution. We choose to eliminate y and multiply the first equation by $-\frac{4}{5}$ to obtain

$$\begin{array}{rrcl} & -\frac{4}{5}x - \frac{4}{5}y & = & -400 \\ + & \left(\frac{3}{10}x + \frac{4}{5}y\right. & = & \left.250\right) \\ \hline & -\frac{1}{2}x & = & -150 \end{array}$$

which gives $x = 300$ and $y = 200$. This means that to produce 500 mL of a 50% acid solution, we need 300 mL of the 30% stock solution and 200 mL of the 80% stock solution.

[6] We do this only because we believe students can use all of the practice with fractions they can get!

2.2.1 Exercises

In Exercises 1 - 14, solve the given system using substitution and/or elimination. Check your answers both algebraically and graphically.

1. $\begin{cases} x + 2y = 5 \\ x = 6 \end{cases}$

2. $\begin{cases} 2y - 3x = 1 \\ y = -3 \end{cases}$

3. $\begin{cases} \frac{x+2y}{4} = -5 \\ \frac{3x-y}{2} = 1 \end{cases}$

4. $\begin{cases} \frac{2}{3}x - \frac{1}{5}y = 3 \\ \frac{1}{2}x + \frac{3}{4}y = 1 \end{cases}$

5. $\begin{cases} \frac{1}{2}x - \frac{1}{3}y = -1 \\ 2y - 3x = 6 \end{cases}$

6. $\begin{cases} x + 4y = 6 \\ \frac{1}{12}x + \frac{1}{3}y = \frac{1}{2} \end{cases}$

7. $\begin{cases} 3y - \frac{3}{2}x = -\frac{15}{2} \\ \frac{1}{2}x - y = \frac{3}{2} \end{cases}$

8. $\begin{cases} \frac{5}{6}x + \frac{5}{3}y = -\frac{7}{3} \\ -\frac{10}{3}x - \frac{20}{3}y = 10 \end{cases}$

9. $\begin{cases} -5x + y = 17 \\ x + y = 5 \end{cases}$

10. $\begin{cases} 2x - 3y = -3.2 - 0.2x + 0.1y \\ x = 0.6x - y + 8.8 \end{cases}$

11. $\begin{cases} \frac{2}{13}x + 2y = -2(y + 1) \\ -\frac{3}{13}x = -5(6 - y) \end{cases}$

12. $\begin{cases} 2x + 5y = 25 + 4.5x \\ 27 + 10y = 3.75x + 4y \end{cases}$

13. $\begin{cases} \pi x + \pi y = 62 \\ y = x + 10 \end{cases}$

14. $\begin{cases} \sqrt{3}(x - y) + 1 = \sqrt{3}(x + y) \\ 2x + \frac{6}{\sqrt{3}}y = \sqrt{3} + 1 \end{cases}$

15. A football game was played between Penn State and Ohio State. The two teams scored a total of 45 points, and Penn State won by a margin of 19 points. How many points did Ohio State score?

16. In Section 2.1, Exercise 35, we learned that the temperature C in Celcius degrees can be expressed in terms of the temperature F in Farenheit degrees: $C = \frac{5}{9}(F - 32)$. For what temperature is the value in both scales the same?

17. The admission fee at a movie theater is \$8 for children and \$12 for adults. On a certain showing, there are 280 visitors and the theater collects \$2520 through admission fees. How many adults visited the movie theater during that showing?

18. If five pencils and two notebooks together cost \$7.20, and one pencil and four notebooks together cost \$9, how much does a bundle of 3 pencils and 3 notebooks cost?

19. Two customers buy the same kind of bread and milk at the same store. The first customer is charged \$11 for a loaf of bread and two gallons of milk, while the second customer is charged \$12.50 for three loaves of bread and one gallon of milk. Find the price for a loaf of bread and that for a gallon of milk at this store.

20. A local buffet charges \$7.50 per person for the basic buffet and \$9.25 for the deluxe buffet (which includes crab legs.) If 27 diners went out to eat and the total bill was \$227.00 before taxes, how many chose the basic buffet and how many chose the deluxe buffet?

21. A restaurant sells two sizes of orders of french fries. On Monday, they sold 5 large orders of fries and 3 small orders of fries for a sales total of \$33. On Tuesday, they sold 4 large orders of fries and 6 small orders of fries for a sales total of \$39. How much would a customer pay for one small order and one large order of fries?

22. At The Old Home Fill'er Up and Keep on a-Truckin' Cafe, Mavis mixes two different types of coffee beans to produce a house blend. The first type costs \$3 per pound and the second costs \$8 per pound. How much of each type does Mavis use to make 50 pounds of a blend which costs \$6 per pound?

23. Skippy has a total of $\$10,000$ to split between two investments. One account offers 3% simple interest, and the other account offers 8% simple interest. For tax reasons, he can only earn \$500 in interest the entire year. How much money should Skippy invest in each account to earn \$500 in interest for the year?

24. A 10% salt solution is to be mixed with pure water to produce 75 gallons of a 3% salt solution. How much of each is needed?

25. You are trying to make 100 mL of a 62% acid solution using stock solutions at 50% and 80%, respectively. How much of each solution is needed?

26. A 60% orange juice drink is to be mixed with a 10% orange juice drink to obtain 30 gallons of a mixture that is 40% orange juice. How much of each drink is needed?

27. Twelve years ago my father was twice as old as I was, and two years ago our combined age was 110. How old is my father now?

28. The length of a rectangle is $10\sqrt{2}$ inches more than its width. The perimeter of the rectangle is $56\sqrt{2}$ inches. Find the area of the rectangle.

29. A piece of wire 124 cm long is cut into two pieces, and each piece is then bent into a circle. The radius of one of the two circles is 10 cm greater than the radius of the other. Find the radius of the smaller circle. (Recall that the circumference of a circle of radius r is $2\pi r$.)

30. In a certain piggy bank there are only nickels and quarters. Their combined value is \$9.15 and their combined weight is one pound. Ninety nickels weigh one pound. Eighty quarters weigh one pound. How many nickels are there in the piggy bank?

2.2.2 Answers

1. Solution $\left(6, -\frac{1}{2}\right)$
2. Solution $\left(-\frac{7}{3}, -3\right)$
3. Solution $\left(-\frac{16}{7}, -\frac{62}{7}\right)$
4. Solution $\left(\frac{49}{12}, -\frac{25}{18}\right)$
5. Solution $\left(t, \frac{3}{2}t + 3\right)$ for all real numbers t
6. Solution $(6 - 4t, t)$ for all real numbers t
7. No solution
8. No solution
9. Solution $(-2, 7)$
10. Solution $(7, 6)$
11. Solution $(845, -33)$
12. Solution $(76, 43)$
13. Solution $(\frac{31}{\pi} - 5, \frac{31}{\pi} + 5)$
14. Solution $(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{6})$
15. Ohio State scored 13 points.
16. $-40\ C^\circ = -40\ F^\circ$.
17. 70 adults.
18. A bundle of 3 pencils and 3 notebooks costs \$8.10.
19. A loaf of bread costs \$2.80 and a gallon of milk costs \$4.10.
20. 13 chose the basic buffet and 14 chose the deluxe buffet.
21. A customer would pay \$8.
22. Mavis needs 20 pounds of \$3 per pound coffee and 30 pounds of \$8 per pound coffee.
23. Skippy needs to invest \$6000 in the 3% account and \$4000 in the 8% account.
24. 22.5 gallons of the 10% solution and 52.5 gallons of pure water.
25. 60 mL of the 50% stock solution and 40 mL of the 80% stock solution.
26. 18 gallons of the 60% orange juice drink and 12 gallons of the 10% orange juice drink.
27. My father is 72 years old now.
28. 342 square inches.
29. $\frac{31-5\pi}{\pi}$ cm.
30. There are 63 nickels in the piggy bank.

2.4 Quadratic Functions

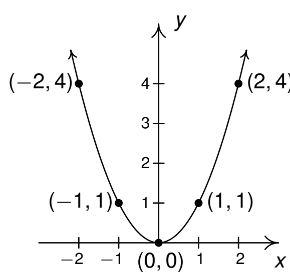
You may recall studying quadratic equations in Intermediate Algebra. In this section, we review those equations in the context of our next family of functions: the quadratic functions.

Definition 2.5. A **quadratic function** is a function of the form

$$f(x) = ax^2 + bx + c,$$

where a, b and c are real numbers with $a \neq 0$. The domain of a quadratic function is $(-\infty, \infty)$.

The most basic quadratic function is $f(x) = x^2$, whose graph appears below. Its shape should look familiar from Intermediate Algebra – it is called a **parabola**. The point $(0, 0)$ is called the **vertex** of the parabola. In this case, the vertex is a relative minimum and is also where the absolute minimum value of f can be found.



$f(x) = x^2$

Much like many of the absolute value functions in Section 2.3, knowing the graph of $f(x) = x^2$ enables us to graph an entire family of quadratic functions using transformations.

Example 2.4.1. Graph the following functions starting with the graph of $f(x) = x^2$ and using transformations. Find the vertex, state the range and find the x- and y-intercepts, if any exist.

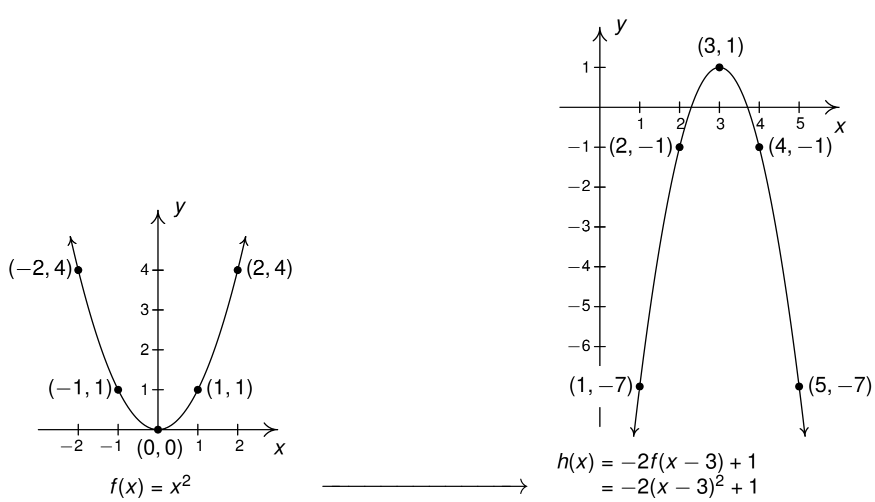
1. $g(x) = (x + 2)^2 - 3$
2. $h(x) = -2(x - 3)^2 + 1$

Solution.

1. Since $g(x) = (x+2)^2 - 3 = f(x+2) - 3$, Theorem 1.7 instructs us to first *subtract* 2 from each of the x-values of the points on $y = f(x)$. This shifts the graph of $y = f(x)$ to the *left* 2 units and moves $(-2, 4)$ to $(-4, 4)$, $(-1, 1)$ to $(-3, 1)$, $(0, 0)$ to $(-2, 0)$, $(1, 1)$ to $(-1, 1)$ and $(2, 4)$ to $(0, 4)$. Next, we *subtract* 3 from each of the y-values of these new points. This moves the graph *down* 3 units and moves $(-4, 4)$ to $(-4, 1)$, $(-3, 1)$ to $(-3, -2)$, $(-2, 0)$ to $(-2, 3)$, $(-1, 1)$ to $(-1, -2)$ and $(0, 4)$ to $(0, 1)$. We connect the dots in parabolic fashion to get

From the graph, we see that the vertex has moved from $(0,0)$ on the graph of $y = f(x)$ to $(-2,-3)$ on the graph of $y = g(x)$. This sets $[-3,\infty)$ as the range of g. We see that the graph of $y = g(x)$ crosses the x-axis twice, so we expect two x-intercepts. To find these, we set $y = g(x) = 0$ and solve. Doing so yields the equation $(x+2)^2 - 3 = 0$, or $(x+2)^2 = 3$. Extracting square roots gives $x + 2 = \pm\sqrt{3}$, or $x = -2 \pm \sqrt{3}$. Our x-intercepts are $(-2-\sqrt{3},0) \approx (-3.73,0)$ and $(-2+\sqrt{3},0) \approx (-0.27,0)$. The y-intercept of the graph, $(0,1)$ was one of the points we originally plotted, so we are done.

2. Following Theorem 1.7 once more, to graph $h(x) = -2(x-3)^2 + 1 = -2f(x-3) + 1$, we first start by *adding* 3 to each of the x-values of the points on the graph of $y = f(x)$. This effects a horizontal shift *right* 3 units and moves $(-2,4)$ to $(1,4)$, $(-1,1)$ to $(2,1)$, $(0,0)$ to $(3,0)$, $(1,1)$ to $(4,1)$ and $(2,4)$ to $(5,4)$. Next, we *multiply* each of our y-values first by -2 and then *add* 1 to that result. Geometrically, this is a vertical *stretch* by a factor of 2, followed by a reflection about the x-axis, followed by a vertical shift *up* 1 unit. This moves $(1,4)$ to $(1,-7)$, $(2,1)$ to $(2,-1)$, $(3,0)$ to $(3,1)$, $(4,1)$ to $(4,-1)$ and $(5,4)$ to $(5,-7)$.



The vertex is $(3,1)$ which makes the range of h $(-\infty,1]$. From our graph, we know that there are two x-intercepts, so we set $y = h(x) = 0$ and solve. We get $-2(x-3)^2 + 1 = 0$

which gives $(x-3)^2 = \frac{1}{2}$. Extracting square roots[1] gives $x - 3 = \pm\frac{\sqrt{2}}{2}$, so that when we add 3 to each side,[2] we get $x = \frac{6\pm\sqrt{2}}{2}$. Hence, our x-intercepts are $\left(\frac{6-\sqrt{2}}{2}, 0\right) \approx (2.29, 0)$ and $\left(\frac{6+\sqrt{2}}{2}, 0\right) \approx (3.71, 0)$. Although our graph doesn't show it, there is a y-intercept which can be found by setting $x = 0$. With $h(0) = -2(0-3)^2 + 1 = -17$, we have that our y-intercept is $(0, -17)$. □

A few remarks about Example 2.4.1 are in order. First note that neither the formula given for $g(x)$ nor the one given for $h(x)$ match the form given in Definition 2.5. We could, of course, convert both $g(x)$ and $h(x)$ into that form by expanding and collecting like terms. Doing so, we find $g(x) = (x+2)^2 - 3 = x^2 + 4x + 1$ and $h(x) = -2(x-3)^2 + 1 = -2x^2 + 12x - 17$. While these 'simplified' formulas for $g(x)$ and $h(x)$ satisfy Definition 2.5, they do not lend themselves to graphing easily. For that reason, the form of g and h presented in Example 2.4.2 is given a special name, which we list below, along with the form presented in Definition 2.5.

Definition 2.6. Standard and General Form of Quadratic Functions: Suppose f is a quadratic function.

- The **general form** of the quadratic function f is $f(x) = ax^2 + bx + c$, where a, b and c are real numbers with $a \neq 0$.
- The **standard form** of the quadratic function f is $f(x) = a(x-h)^2 + k$, where a, h and k are real numbers with $a \neq 0$.

It is important to note at this stage that we have no guarantees that *every* quadratic function can be written in standard form. This is actually true, and we prove this later in the exposition, but for now we celebrate the advantages of the standard form, starting with the following theorem.

Theorem 2.1. Vertex Formula for Quadratics in Standard Form: For the quadratic function $f(x) = a(x-h)^2 + k$, where a, h and k are real numbers with $a \neq 0$, the vertex of the graph of $y = f(x)$ is (h, k).

We can readily verify the formula given Theorem 2.1 with the two functions given in Example 2.4.1. After a (slight) rewrite, $g(x) = (x+2)^2 - 3 = (x-(-2))^2 + (-3)$, and we identify $h = -2$ and $k = -3$. Sure enough, we found the vertex of the graph of $y = g(x)$ to be $(-2, -3)$. For $h(x) = -2(x-3) + 1$, no rewrite is needed. We can directly identify $h = 3$ and $k = 1$ and, sure enough, we found the vertex of the graph of $y = h(x)$ to be $(3, 1)$.

To see why the formula in Theorem 2.1 produces the vertex, consider the graph of the equation $y = a(x-h)^2 + k$. When we substitute $x = h$, we get $y = k$, so (h, k) is on the graph. If $x \neq h$, then $x - h \neq 0$ so $(x-h)^2$ is a positive number. If $a > 0$, then $a(x-h)^2$ is positive, thus $y = a(x-h)^2 + k$ is always a number larger than k. This means that when $a > 0$, (h, k) is the lowest point on the graph and thus the parabola must open upwards, making (h, k) the vertex. A similar argument

[1] and rationalizing denominators!
[2] and get common denominators!

shows that if $a < 0$, (h, k) is the highest point on the graph, so the parabola opens downwards, and (h, k) is also the vertex in this case.

Alternatively, we can apply the machinery in Section 1.7. Since the vertex of $y = x^2$ is $(0, 0)$, we can determine the vertex of $y = a(x - h)^2 + k$ by determining the final destination of $(0, 0)$ as it is moved through each transformation. To obtain the formula $f(x) = a(x - h)^2 + k$, we start with $g(x) = x^2$ and first define $g_1(x) = ag(x) = ax^2$. This is results in a vertical scaling and/or reflection.[3] Since we multiply the output by a, we multiply the y-coordinates on the graph of g by a, so the point $(0, 0)$ remains $(0, 0)$ and remains the vertex. Next, we define $g_2(x) = g_1(x - h) = a(x - h)^2$. This induces a horizontal shift right or left h units[4] moves the vertex, in either case, to $(h, 0)$. Finally, $f(x) = g_2(x) + k = a(x - h)^2 + k$ which effects a vertical shift up or down k units[5] resulting in the vertex moving from $(h, 0)$ to (h, k).

In addition to verifying Theorem 2.1, the arguments in the two preceding paragraphs have also shown us the role of the number a in the graphs of quadratic functions. The graph of $y = a(x - h)^2 + k$ is a parabola 'opening upwards' if $a > 0$, and 'opening downwards' if $a < 0$. Moreover, the symmetry enjoyed by the graph of $y = x^2$ about the y-axis is translated to a symmetry about the vertical line $x = h$ which is the vertical line through the vertex.[6] This line is called the **axis of symmetry** of the parabola and is dashed in the figures below.

Graphs of $y = a(x - h)^2 + k$.

Without a doubt, the standard form of the function, coupled with the machinery in Section 1.7, allows us to list the attributes of the graphs of such functions quickly and elegantly. What remains to be shown, however, is the fact that every quadratic function *can be written* in standard form. To convert a quadratic function given in general form into standard form, we employ the ancient rite of 'Completing the Square'. We remind the reader how this is done in our next example.

Example 2.4.2. Convert the functions below from general form to standard form. Find the vertex, axis of symmetry and any x- or y-intercepts. Graph each function and determine its range.

1. $f(x) = x^2 - 4x + 3$.
2. $g(x) = 6 - x - x^2$

[3] Just a scaling if $a > 0$. If $a < 0$, there is a reflection involved.
[4] Right if $h > 0$, left if $h < 0$.
[5] Up if $k > 0$, down if $k < 0$
[6] You should use transformations to verify this!

Solution.

1. To convert from general form to standard form, we complete the square.[7] First, we verify that the coefficient of x^2 is 1. Next, we find the coefficient of x, in this case -4, and take half of it to get $\frac{1}{2}(-4) = -2$. This tells us that our target perfect square quantity is $(x-2)^2$. To get an expression equivalent to $(x-2)^2$, we need to add $(-2)^2 = 4$ to the $x^2 - 4x$ to create a perfect square trinomial, but to keep the balance, we must also subtract it. We collect the terms which create the perfect square and gather the remaining constant terms. Putting it all together, we get

$$\begin{aligned} f(x) &= x^2 - 4x + 3 && \text{(Compute } \tfrac{1}{2}(-4) = -2.) \\ &= \left(x^2 - 4x + \underline{4} - \underline{4}\right) + 3 && \text{(Add and subtract } (-2)^2 = 4 \text{ to } (x^2 + 4x).) \\ &= \left(x^2 - 4x + 4\right) - 4 + 3 && \text{(Group the perfect square trinomial.)} \\ &= (x-2)^2 - 1 && \text{(Factor the perfect square trinomial.)} \end{aligned}$$

   Of course, we can always check our answer by multiplying out $f(x) = (x-2)^2 - 1$ to see that it simplifies to $f(x) = x^2 - 4x - 1$. In the form $f(x) = (x-2)^2 - 1$, we readily find the vertex to be $(2, -1)$ which makes the axis of symmetry $x = 2$. To find the x-intercepts, we set $y = f(x) = 0$. We are spoiled for choice, since we have *two* formulas for $f(x)$. Since we recognize $f(x) = x^2 - 4x + 3$ to be easily factorable,[8] we proceed to solve $x^2 - 4x + 3 = 0$. Factoring gives $(x-3)(x-1) = 0$ so that $x = 3$ or $x = 1$. The x-intercepts are then $(1, 0)$ and $(3, 0)$. To find the y-intercept, we set $x = 0$. Once again, the general form $f(x) = x^2 - 4x + 3$ is easiest to work with here, and we find $y = f(0) = 3$. Hence, the y-intercept is $(0, 3)$. With the vertex, axis of symmetry and the intercepts, we get a pretty good graph without the need to plot additional points. We see that the range of f is $[-1, \infty)$ and we are done.

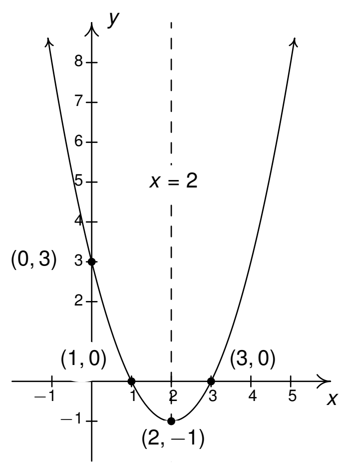
2. To get started, we rewrite $g(x) = 6 - x - x^2 = -x^2 - x + 6$ and note that the coefficient of x^2 is -1, not 1. This means our first step is to factor out the (-1) from both the x^2 and x terms. We then follow the completing the square recipe as above.

$$\begin{aligned} g(x) &= -x^2 - x + 6 \\ &= -\left[x^2 + x\right] + 6 \\ &= -\left[x^2 + x + \underline{\frac{1}{4}} - \underline{\frac{1}{4}}\right] + 6 \\ &= -\left(x^2 + x + \frac{1}{4}\right) + \frac{1}{4} + 6 \\ &= -\left(x + \frac{1}{2}\right)^2 + \frac{25}{4} \end{aligned}$$

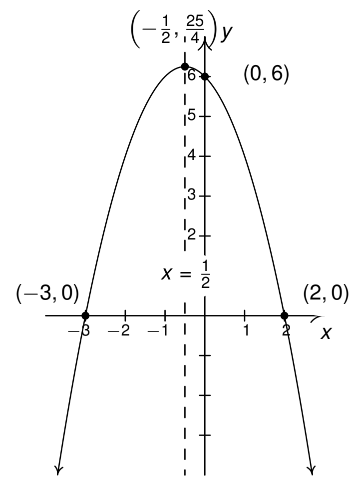
[7] If you forget why we do what we do to complete the square, start with $a(x-h)^2 + k$, multiply it out, step by step, and then reverse the process.

[8] Experience pays off, here!

From $g(x) = -\left(x+\frac{1}{2}\right)^2 + \frac{25}{4}$, we get the vertex to be $\left(-\frac{1}{2}, \frac{25}{4}\right)$ and the axis of symmetry to be $x = -\frac{1}{2}$. To get the x-intercepts, we opt to set the given formula $g(x) = 6 - x - x^2 = 0$. Solving, we get $x = -3$ and $x = 2$, so the x-intercepts are $(-3, 0)$ and $(2, 0)$. Setting $x = 0$, we find $g(0) = 6$, so the y-intercept is $(0, 6)$. Plotting these points gives us the graph below. We see that the range of g is $\left(-\infty, \frac{25}{4}\right]$.



$f(x) = x^2 - 4x + 3$



$g(x) = 6 - x - x^2$

□

With Example 2.4.2 fresh in our minds, we are now in a position to show that every quadratic function can be written in standard form. We begin with $f(x) = ax^2 + bx + c$, assume $a \neq 0$, and complete the square in *complete* generality.

$$\begin{aligned}
f(x) &= ax^2 + bx + c \\
&= a\left[x^2 + \frac{b}{a}x\right] + c && \text{(Factor out coefficient of } x^2 \text{ from } x^2 \text{ and } x.) \\
&= a\left[x^2 + \frac{b}{a}x + \underline{\frac{b^2}{4a^2}} - \underline{\frac{b^2}{4a^2}}\right] + c \\
&= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - a\left(\frac{b^2}{4a^2}\right) + c && \text{(Group the perfect square trinomial.)} \\
&= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} && \text{(Factor and get a common denominator.)}
\end{aligned}$$

Comparing this last expression with the standard form, we identify $(x - h)$ with $\left(x + \frac{b}{2a}\right)$ so that $h = -\frac{b}{2a}$. Instead of memorizing the value $k = \frac{4ac-b^2}{4a}$, we see that $f\left(-\frac{b}{2a}\right) = \frac{4ac-b^2}{4a}$. As such, we have derived a vertex formula for the general form. We summarize both vertex formulas in the box at the top of the next page.

Equation 2.4. Vertex Formulas for Quadratic Functions: Suppose a, b, c, h and k are real numbers with $a \neq 0$.

- If $f(x) = a(x-h)^2 + k$, the vertex of the graph of $y = f(x)$ is the point (h, k).
- If $f(x) = ax^2 + bx + c$, the vertex of the graph of $y = f(x)$ is the point $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

There are two more results which can be gleaned from the completed-square form of the general form of a quadratic function,

$$f(x) = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

We have seen that the number a in the standard form of a quadratic function determines whether the parabola opens upwards (if $a > 0$) or downwards (if $a < 0$). We see here that this number a is none other than the coefficient of x^2 in the general form of the quadratic function. In other words, it is the coefficient of x^2 alone which determines this behavior. The second treasure is a re-discovery of the **quadratic formula**.

Equation 2.5. The Quadratic Formula: If a, b and c are real numbers with $a \neq 0$, then the solutions to $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Assuming the conditions of Equation 2.5, the solutions to $ax^2 + bx + c = 0$ are precisely the zeros of $f(x) = ax^2 + bx + c$. Since

$$f(x) = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

the equation $ax^2 + bx + c = 0$ is equivalent to

$$a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} = 0.$$

Solving gives

$$\begin{aligned} a\left(x + \frac{b}{2a}\right)^2 &= -\frac{4ac - b^2}{4a} \\ \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \end{aligned}$$

$$\begin{aligned}
x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \\
x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

In our discussions of domain, we were warned against having negative numbers underneath the square root. Given that $\sqrt{b^2 - 4ac}$ is part of the Quadratic Formula, we will need to pay special attention to the radicand $b^2 - 4ac$. It turns out that the quantity $b^2 - 4ac$ plays a critical role in determining the nature of the solutions to a quadratic equation. It is given a special name.

Definition 2.7. If a, b and c are real numbers with $a \neq 0$, then the **discriminant** of the quadratic equation $ax^2 + bx + c = 0$ is the quantity $b^2 - 4ac$.

The discriminant 'discriminates' between the kinds of solutions we get from a quadratic equation. These cases, and their relation to the discriminant, are summarized below.

Theorem 2.2. Discriminant Trichotomy: Let a, b and c be real numbers with $a \neq 0$.

- If $b^2 - 4ac < 0$, the equation $ax^2 + bx + c = 0$ has no real solutions.
- If $b^2 - 4ac = 0$, the equation $ax^2 + bx + c = 0$ has exactly one real solution.
- If $b^2 - 4ac > 0$, the equation $ax^2 + bx + c = 0$ has exactly two real solutions.

The proof of Theorem 2.2 stems from the position of the discriminant in the quadratic equation, and is left as a good mental exercise for the reader. The next example exploits the fruits of all of our labor in this section thus far.

Example 2.4.3. Recall that the profit (defined on page 199) for a product is defined by the equation Profit = Revenue − Cost, or $P(x) = R(x) - C(x)$. In Example 2.1.7 the weekly revenue, in dollars, made by selling x PortaBoy Game Systems was found to be $R(x) = -1.5x^2 + 250x$ with the restriction (carried over from the price-demand function) that $0 \leq x \leq 166$. The cost, in dollars, to produce x PortaBoy Game Systems is given in Example 2.1.5 as $C(x) = 80x + 150$ for $x \geq 0$.

1. Determine the weekly profit function $P(x)$.
2. Graph $y = P(x)$. Include the x- and y-intercepts as well as the vertex and axis of symmetry.
3. Interpret the zeros of P.
4. Interpret the vertex of the graph of $y = P(x)$.

5. Recall that the weekly price-demand equation for PortaBoys is $p(x) = -1.5x + 250$, where $p(x)$ is the price per PortaBoy, in dollars, and x is the weekly sales. What should the price per system be in order to maximize profit?

Solution.

1. To find the profit function $P(x)$, we subtract

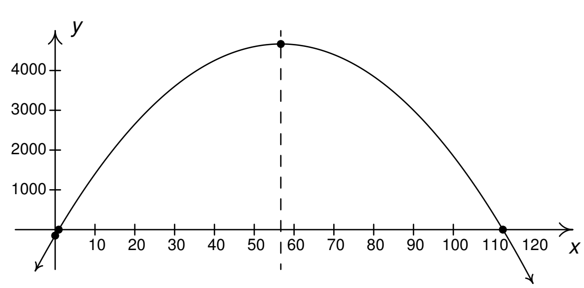
$$P(x) = R(x) - C(x) = \left(-1.5x^2 + 250x\right) - (80x + 150) = -1.5x^2 + 170x - 150.$$

Since the revenue function is valid when $0 \leq x \leq 166$, P is also restricted to these values.

2. To find the x-intercepts, we set $P(x) = 0$ and solve $-1.5x^2 + 170x - 150 = 0$. The mere thought of trying to factor the left hand side of this equation could do serious psychological damage, so we resort to the quadratic formula, Equation 2.5. Identifying $a = -1.5$, $b = 170$, and $c = -150$, we obtain

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-170 \pm \sqrt{170^2 - 4(-1.5)(-150)}}{2(-1.5)} \\ &= \frac{-170 \pm \sqrt{28000}}{-3} = \frac{170 \pm 20\sqrt{70}}{3} \end{aligned}$$

We get two x-intercepts: $\left(\frac{170-20\sqrt{70}}{3}, 0\right)$ and $\left(\frac{170+20\sqrt{70}}{3}, 0\right)$. To find the y-intercept, we set $x = 0$ and find $y = P(0) = -150$ for a y-intercept of $(0, -150)$. To find the vertex, we use the fact that $P(x) = -1.5x^2 + 170x - 150$ is in the general form of a quadratic function and appeal to Equation 2.4. Substituting $a = -1.5$ and $b = 170$, we get $x = -\frac{170}{2(-1.5)} = \frac{170}{3}$. To find the y-coordinate of the vertex, we compute $P\left(\frac{170}{3}\right) = \frac{14000}{3}$ and find that our vertex is $\left(\frac{170}{3}, \frac{14000}{3}\right)$. The axis of symmetry is the vertical line passing through the vertex so it is the line $x = \frac{170}{3}$. To sketch a reasonable graph, we approximate the x-intercepts, $(0.89, 0)$ and $(112.44, 0)$, and the vertex, $(56.67, 4666.67)$. (Note that in order to get the x-intercepts and the vertex to show up in the same picture, we had to scale the x-axis differently than the y-axis. This results in the left-hand x-intercept and the y-intercept being uncomfortably close to each other and to the origin in the picture.)



3. The zeros of P are the solutions to $P(x) = 0$, which we have found to be approximately 0.89 and 112.44. As we saw in Example 1.5.3, these are the 'break-even' points of the profit function, where enough product is sold to recover the cost spent to make the product. More importantly, we see from the graph that as long as x is between 0.89 and 112.44, the graph $y = P(x)$ is above the x-axis, meaning $y = P(x) > 0$ there. This means that for these values of x, a profit is being made. Since x represents the weekly sales of PortaBoy Game Systems, we round the zeros to positive integers and have that as long as 1, but no more than 112 game systems are sold weekly, the retailer will make a profit.

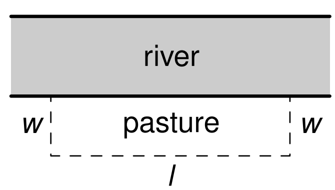
4. From the graph, we see that the maximum value of P occurs at the vertex, which is approximately $(56.67, 4666.67)$. As above, x represents the weekly sales of PortaBoy systems, so we can't sell 56.67 game systems. Comparing $P(56) = 4666$ and $P(57) = 4666.5$, we conclude that we will make a maximum profit of \$4666.50 if we sell 57 game systems.

5. In the previous part, we found that we need to sell 57 PortaBoys per week to maximize profit. To find the price per PortaBoy, we substitute $x = 57$ into the price-demand function to get $p(57) = -1.5(57) + 250 = 164.5$. The price should be set at \$164.50. □

Our next example is another classic application of quadratic functions.

Example 2.4.4. Much to Donnie's surprise and delight, he inherits a large parcel of land in Ashtabula County from one of his (e)strange(d) relatives. The time is finally right for him to pursue his dream of farming alpaca. He wishes to build a rectangular pasture, and estimates that he has enough money for 200 linear feet of fencing material. If he makes the pasture adjacent to a stream (so no fencing is required on that side), what are the dimensions of the pasture which maximize the area? What is the maximum area? If an average alpaca needs 25 square feet of grazing area, how many alpaca can Donnie keep in his pasture?

Solution. It is always helpful to sketch the problem situation, so we do so below.



We are tasked to find the dimensions of the pasture which would give a maximum area. We let w denote the width of the pasture and we let l denote the length of the pasture. Since the units given to us in the statement of the problem are feet, we assume w and l are measured in feet. The area of the pasture, which we'll call A, is related to w and l by the equation $A = wl$. Since w and l are both measured in feet, A has units of feet^2, or square feet. We are given the total amount of fencing available is 200 feet, which means $w + l + w = 200$, or, $l + 2w = 200$. We now have two equations, $A = wl$ and $l + 2w = 200$. In order to use the tools given to us in this section to *maximize* A, we need to use the information given to write A as a function of just *one* variable, either w or l. This is where we use the equation $l + 2w = 200$. Solving for l, we find $l = 200 - 2w$, and we substitute this into our equation for A. We get $A = wl = w(200 - 2w) = 200w - 2w^2$. We now have A as a function of w, $A(w) = 200w - 2w^2 = -2w^2 + 200w$.

Before we go any further, we need to find the applied domain of A so that we know what values of w make sense in this problem situation.[9] Since w represents the width of the pasture, $w > 0$. Likewise, l represents the length of the pasture, so $l = 200 - 2w > 0$. Solving this latter inequality, we find $w < 100$. Hence, the function we wish to maximize is $A(w) = -2w^2 + 200w$ for $0 < w < 100$. Since A is a quadratic function (of w), we know that the graph of $y = A(w)$ is a parabola. Since the coefficient of w^2 is -2, we know that this parabola opens downwards. This means that there is a maximum value to be found, and we know it occurs at the vertex. Using the vertex formula, we find $w = -\frac{200}{2(-2)} = 50$, and $A(50) = -2(50)^2 + 200(50) = 5000$. Since $w = 50$ lies in the applied domain, $0 < w < 100$, we have that the area of the pasture is maximized when the width is 50 feet. To find the length, we use $l = 200 - 2w$ and find $l = 200 - 2(50) = 100$, so the length of the pasture is 100 feet. The maximum area is $A(50) = 5000$, or 5000 square feet. If an average alpaca requires 25 square feet of pasture, Donnie can raise $\frac{5000}{25} = 200$ average alpaca. $\square$

[9]Donnie would be very upset if, for example, we told him the width of the pasture needs to be -50 feet.

2.4.1 Exercises

In Exercises 1 - 15, graph the quadratic function. Find the x- and y-intercepts, if any exist. If the function is given in general form, convert it into standard form $f(x) = a(x-h)^2 + k$. Find the range. Identify the vertex and the axis of symmetry, and determine whether the vertex yields a maximum or minimum.

1. $f(x) = x^2 + 2$
2. $f(x) = -(x+2)^2$
3. $f(x) = x^2 - 2x - 8$
4. $f(x) = -2(x+1)^2 + 4$
5. $f(x) = 2x^2 - 4x - 1$
6. $f(x) = -3x^2 + 4x - 7$
7. $f(x) = x^2 - 4x$
8. $f(x) = x^2 + 2x$
9. $f(x) = 2x^2 - 6x + 4$
10. $f(x) = x^2 - 8x + 16$
11. $f(x) = x^2 + x + 1$
12. $f(x) = -3x^2 + 5x + 4$
13. $f(x) = \frac{1}{2}x^2 + 3x + \frac{5}{2}$
14. $f(x) = -\frac{1}{2}x^2 + x - 2$
15. $f(x) = \frac{1}{3}x^2 + 2x + 3$

In Exercises 16 - 20, the cost and price-demand functions are given for different scenarios. For each scenario,

- Find the profit function $P(x)$.
- Find the number of items which need to be sold in order to maximize profit.
- Find the maximum profit.
- Find the price to charge per item in order to maximize profit.
- Find and interpret break-even points.

16. The cost, in dollars, to produce x "I'd rather be a Sasquatch" T-Shirts is $C(x) = 2x+26$, $x \geq 0$ and the price-demand function, in dollars per shirt, is $p(x) = 30 - 2x$, $0 \leq x \leq 15$.
17. The cost, in dollars, to produce x bottles of 100% All-Natural Certified Free-Trade Organic Sasquatch Tonic is $C(x) = 10x + 100$, $x \geq 0$ and the price-demand function, in dollars per bottle, is $p(x) = 35 - x$, $0 \leq x \leq 35$.
18. The cost, in cents, to produce x cups of Mountain Thunder Lemonade at Junior's Lemonade Stand is $C(x) = 18x + 240$, $x \geq 0$ and the price-demand function, in cents per cup, is $p(x) = 90 - 3x$, $0 \leq x \leq 30$.
19. The daily cost, in dollars, to produce x Sasquatch Berry Pies is $C(x) = 3x + 36$, $x \geq 0$ and the price-demand function, in dollars per pie, is $p(x) = 12 - 0.5x$, $0 \leq x \leq 24$.
20. The monthly cost, in *hundreds* of dollars, to produce x custom built electric scooters is $C(x) = 20x + 1000$, $x \geq 0$ and the price-demand function, in *hundreds* of dollars per scooter, is $p(x) = 140 - 2x$, $0 \leq x \leq 70$.

21. The International Silver Strings Submarine Band holds a bake sale each year to fund their trip to the National Sasquatch Convention. It has been determined that the cost in dollars of baking x cookies is $C(x) = 0.1x + 25$ and that the demand function for their cookies is $p = 10 - .01x$. How many cookies should they bake in order to maximize their profit?

22. Using data from Bureau of Transportation Statistics, the average fuel economy F in miles per gallon for passenger cars in the US can be modeled by $F(t) = -0.0076t^2 + 0.45t + 16$, $0 \leq t \leq 28$, where t is the number of years since 1980. Find and interpret the coordinates of the vertex of the graph of $y = F(t)$.

23. The temperature T, in degrees Fahrenheit, t hours after 6 AM is given by:

$$T(t) = -\frac{1}{2}t^2 + 8t + 32, \quad 0 \leq t \leq 12$$

What is the warmest temperature of the day? When does this happen?

24. Suppose $C(x) = x^2 - 10x + 27$ represents the costs, in *hundreds*, to produce x *thousand* pens. How many pens should be produced to minimize the cost? What is this minimum cost?

25. Skippy wishes to plant a vegetable garden along one side of his house. In his garage, he found 32 linear feet of fencing. Since one side of the garden will border the house, Skippy doesn't need fencing along that side. What are the dimensions of the garden which will maximize the area of the garden? What is the maximum area of the garden?

26. In the situation of Example 2.4.4, Donnie has a nightmare that one of his alpaca herd fell into the river and drowned. To avoid this, he wants to move his rectangular pasture *away* from the river. This means that all four sides of the pasture require fencing. If the total amount of fencing available is still 200 linear feet, what dimensions maximize the area of the pasture now? What is the maximum area? Assuming an average alpaca requires 25 square feet of pasture, how many alpaca can he raise now?

27. What is the largest rectangular area one can enclose with 14 inches of string?

28. The height of an object dropped from the roof of an eight story building is modeled by $h(t) = -16t^2 + 64$, $0 \leq t \leq 2$. Here, h is the height of the object off the ground, in feet, t seconds after the object is dropped. How long before the object hits the ground?

29. The height h in feet of a model rocket above the ground t seconds after lift-off is given by $h(t) = -5t^2 + 100t$, for $0 \leq t \leq 20$. When does the rocket reach its maximum height above the ground? What is its maximum height?

30. Carl's friend Jason participates in the Highland Games. In one event, the hammer throw, the height h in feet of the hammer above the ground t seconds after Jason lets it go is modeled by $h(t) = -16t^2 + 22.08t + 6$. What is the hammer's maximum height? What is the hammer's total time in the air? Round your answers to two decimal places.

31. Assuming no air resistance or forces other than the Earth's gravity, the height above the ground at time t of a falling object is given by $s(t) = -4.9t^2 + v_0 t + s_0$ where s is in meters, t is in seconds, v_0 is the object's initial velocity in meters per second and s_0 is its initial position in meters.

    (a) What is the applied domain of this function?

    (b) Discuss with your classmates what each of $v_0 > 0$, $v_0 = 0$ and $v_0 < 0$ would mean.

    (c) Come up with a scenario in which $s_0 < 0$.

    (d) Let's say a slingshot is used to shoot a marble straight up from the ground ($s_0 = 0$) with an initial velocity of 15 meters per second. What is the marble's maximum height above the ground? At what time will it hit the ground?

    (e) Now shoot the marble from the top of a tower which is 25 meters tall. When does it hit the ground?

    (f) What would the height function be if instead of shooting the marble up off of the tower, you were to shoot it straight DOWN from the top of the tower?

32. The two towers of a suspension bridge are 400 feet apart. The parabolic cable[10] attached to the tops of the towers is 10 feet above the point on the bridge deck that is midway between the towers. If the towers are 100 feet tall, find the height of the cable directly above a point of the bridge deck that is 50 feet to the right of the left-hand tower.

33. Find all of the points on the line $y = 1 - x$ which are 2 units from $(1, -1)$.

34. Let L be the line $y = 2x + 1$. Find a function $D(x)$ which measures the distance *squared* from a point on L to $(0, 0)$. Use this to find the point on L closest to $(0, 0)$.

35. With the help of your classmates, show that if a quadratic function $f(x) = ax^2 + bx + c$ has two real zeros then the x-coordinate of the vertex is the midpoint of the zeros.

In Exercises 36 - 41, solve the quadratic equation for the indicated variable.

36. $x^2 - 10y^2 = 0$ for x

37. $y^2 - 4y = x^2 - 4$ for x

38. $x^2 - mx = 1$ for x

39. $y^2 - 3y = 4x$ for y

40. $y^2 - 4y = x^2 - 4$ for y

41. $-gt^2 + v_0 t + s_0 = 0$ for t (Assume $g \neq 0$.)

[10] The weight of the bridge deck forces the bridge cable into a parabola and a free hanging cable such as a power line does not form a parabola. We shall see in Exercise 35 in Section 6.5 what shape a free hanging cable makes.

2.4.2 Answers

1. $f(x) = x^2 + 2$
   No x-intercepts
   y-intercept $(0, 2)$
   Range: $[2, \infty)$
   Vertex $(0, 2)$ is a minimum
   Axis of symmetry $x = 0$

2. $f(x) = -(x + 2)^2$
   x-intercept $(-2, 0)$
   y-intercept $(0, -4)$
   Range: $(-\infty, 0]$
   Vertex $(-2, 0)$ is a maximum
   Axis of symmetry $x = -2$

3. $f(x) = (x - 1)^2 - 9$
   x-intercepts $(-2, 0)$ and $(4, 0)$
   y-intercept $(0, -8)$
   Range: $[-9, \infty)$
   Vertex $(1, -9)$ is a minimum
   Axis of symmetry $x = 1$

4. $f(x) = -2(x + 1)^2 + 4$
   x-intercepts $(-1 - \sqrt{2}, 0)$ & $(-1 + \sqrt{2}, 0)$
   y-intercept $(0, 2)$
   Range: $(-\infty, 4]$
   Vertex $(-1, 4)$ is a maximum
   Axis of symmetry $x = -1$

5. $f(x) = 2(x - 1)^2 - 3$
   x-intercepts $\left(\frac{2-\sqrt{6}}{2}, 0\right)$ & $\left(\frac{2+\sqrt{6}}{2}, 0\right)$
   y-intercept $(0, -1)$
   Range: $[-3, \infty)$
   Vertex $(1, -3)$ is a minimum
   Axis of symmetry $x = 1$

6. $f(x) = -3\left(x - \frac{2}{3}\right)^2 - \frac{17}{3}$
   No x-intercepts
   y-intercept $(0, -7)$
   Range: $\left(-\infty, -\frac{17}{3}\right]$
   Vertex $\left(\frac{2}{3}, -\frac{17}{3}\right)$ is a maximum
   Axis of symmetry $x = \frac{2}{3}$

7. $f(x) = (x - 2)^2 - 4$
   x-intercepts $(0, 0)$ & $(4, 0)$
   y-intercept $(0, 0)$
   Range: $[-4, \infty)$
   Vertex $(2, -4)$ is a minimum
   Axis of symmetry $x = 2$

8. $f(x) = (x + 1)^2 - 1$
   x-intercepts $(-2, 0)$ & $(0, 0)$
   y-intercept $(0, 0)$
   Range: $[-1, \infty)$
   Vertex $(-1, -1)$ is a minimum
   Axis of symmetry $x = -1$

9. $f(x) = 2(x - \frac{3}{2})^2 - \frac{1}{2}$
   x-intercepts $(1, 0)$ & $(2, 0)$
   y-intercept $(0, 4)$
   Range: $[-\frac{1}{2}, \infty)$
   Vertex $\left(\frac{3}{2}, -\frac{1}{2}\right)$ is a minimum
   Axis of symmetry $x = \frac{3}{2}$

10. $f(x) = (x - 4)^2$
    x-intercept $(4, 0)$
    y-intercept $(0, 16)$
    Range: $[0, \infty)$
    Vertex $(4, 0)$ is a minimum
    Axis of symmetry $x = 4$

11. $f(x) = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$
No x-intercepts
y-intercept $(0, 1)$
Range: $\left[\frac{3}{4}, \infty\right)$
Vertex $\left(-\frac{1}{2}, \frac{3}{4}\right)$ is a minimum
Axis of symmetry $x = -\frac{1}{2}$

12. $f(x) = -3\left(x - \frac{5}{6}\right)^2 + \frac{73}{12}$
x-intercepts $\left(\frac{5-\sqrt{73}}{6}, 0\right)$ & $\left(\frac{5+\sqrt{73}}{6}, 0\right)$
y-intercept $(0, 4)$
Range: $\left(-\infty, \frac{73}{12}\right]$
Vertex $\left(\frac{5}{6}, \frac{73}{12}\right)$ is a maximum
Axis of symmetry $x = \frac{5}{6}$

13. $f(x) = \frac{1}{2}(x + 3)^2 - 2$
x-intercepts $(-5, 0)$ & $(-1, 0)$
y-intercept $(0, \frac{5}{2})$
Range: $[-2, \infty)$
Vertex $(-3, -2)$ is a minimum
Axis of symmetry $x = -3$

14. $f(x) = -\frac{1}{2}(x - 1)^2 - \frac{3}{2}$
No x-intercepts
y-intercept $(0, -2)$
Range: $(-\infty, -\frac{3}{2}]$
Vertex $\left(1, -\frac{3}{2}\right)$ is a maximum
Axis of symmetry $x = 1$

15. $f(x) = \frac{1}{3}(x + 3)^2$
x-intercept $(-3, 0)$
y-intercept $(0, 3)$
Range: $[0, \infty)$
Vertex $(-3, 0)$ is a minimum
Axis of symmetry $x = -3$

16.
- $P(x) = -2x^2 + 28x - 26$, for $0 \leq x \leq 15$.
- 7 T-shirts should be made and sold to maximize profit.
- The maximum profit is \$72.
- The price per T-shirt should be set at \$16 to maximize profit.
- The break even points are $x = 1$ and $x = 13$, so to make a profit, between 1 and 13 T-shirts need to be made and sold.

17.
- $P(x) = -x^2 + 25x - 100$, for $0 \leq x \leq 35$
- Since the vertex occurs at $x = 12.5$, and it is impossible to make or sell 12.5 bottles of tonic, maximum profit occurs when either 12 or 13 bottles of tonic are made and sold.
- The maximum profit is \$56.
- The price per bottle can be either \$23 (to sell 12 bottles) or \$22 (to sell 13 bottles.) Both will result in the maximum profit.
- The break even points are $x = 5$ and $x = 20$, so to make a profit, between 5 and 20 bottles of tonic need to be made and sold.

18.
- $P(x) = -3x^2 + 72x - 240$, for $0 \leq x \leq 30$
- 12 cups of lemonade need to be made and sold to maximize profit.
- The maximum profit is 192¢ or \$1.92.
- The price per cup should be set at 54¢ per cup to maximize profit.

- The break even points are $x = 4$ and $x = 20$, so to make a profit, between 4 and 20 cups of lemonade need to be made and sold.

19.
- $P(x) = -0.5x^2 + 9x - 36$, for $0 \leq x \leq 24$
- 9 pies should be made and sold to maximize the daily profit.
- The maximum daily profit is \$4.50.
- The price per pie should be set at \$7.50 to maximize profit.
- The break even points are $x = 6$ and $x = 12$, so to make a profit, between 6 and 12 pies need to be made and sold daily.

20.
- $P(x) = -2x^2 + 120x - 1000$, for $0 \leq x \leq 70$
- 30 scooters need to be made and sold to maximize profit.
- The maximum monthly profit is 800 hundred dollars, or \$80,000.
- The price per scooter should be set at 80 hundred dollars, or \$8000 per scooter.
- The break even points are $x = 10$ and $x = 50$, so to make a profit, between 10 and 50 scooters need to be made and sold monthly.

21. 495 cookies

22. The vertex is (approximately) $(29.60, 22.66)$, which corresponds to a maximum fuel economy of 22.66 miles per gallon, reached sometime between 2009 and 2010 (29 – 30 years after 1980.) Unfortunately, the model is only valid up until 2008 (28 years after 1908.) So, at this point, we are using the model to *predict* the maximum fuel economy.

23. 64° at 2 PM (8 hours after 6 AM.)

24. 5000 pens should be produced for a cost of \$200.

25. 8 feet by 16 feet; maximum area is 128 square feet.

26. 50 feet by 50 feet; maximum area is 2500 feet; he can raise 100 average alpacas.

27. The largest rectangle has area 12.25 square inches.

28. 2 seconds.

29. The rocket reaches its maximum height of 500 feet 10 seconds after lift-off.

30. The hammer reaches a maximum height of approximately 13.62 feet. The hammer is in the air approximately 1.61 seconds.

31. (a) The applied domain is $[0, \infty)$.

    (d) The height function is this case is $s(t) = -4.9t^2 + 15t$. The vertex of this parabola is approximately $(1.53, 11.48)$ so the maximum height reached by the marble is 11.48 meters. It hits the ground again when $t \approx 3.06$ seconds.

(e) The revised height function is $s(t) = -4.9t^2 + 15t + 25$ which has zeros at $t \approx -1.20$ and $t \approx 4.26$. We ignore the negative value and claim that the marble will hit the ground after 4.26 seconds.

(f) Shooting down means the initial velocity is negative so the height functions becomes $s(t) = -4.9t^2 - 15t + 25$.

32. Make the vertex of the parabola $(0, 10)$ so that the point on the top of the left-hand tower where the cable connects is $(-200, 100)$ and the point on the top of the right-hand tower is $(200, 100)$. Then the parabola is given by $p(x) = \frac{9}{4000}x^2 + 10$. Standing 50 feet to the right of the left-hand tower means you're standing at $x = -150$ and $p(-150) = 60.625$. So the cable is 60.625 feet above the bridge deck there.

33. $\left(\dfrac{3-\sqrt{7}}{2}, \dfrac{-1+\sqrt{7}}{2}\right), \left(\dfrac{3+\sqrt{7}}{2}, \dfrac{-1-\sqrt{7}}{2}\right)$

34. $D(x) = x^2 + (2x+1)^2 = 5x^2 + 4x + 1$, D is minimized when $x = -\frac{2}{5}$, so the point on $y = 2x + 1$ closest to $(0, 0)$ is $\left(-\frac{2}{5}, \frac{1}{5}\right)$

36. $x = \pm y\sqrt{10}$

37. $x = \pm(y - 2)$

38. $x = \dfrac{m \pm \sqrt{m^2 + 4}}{2}$

39. $y = \dfrac{3 \pm \sqrt{16x + 9}}{2}$

40. $y = 2 \pm x$

41. $t = \dfrac{v_0 \pm \sqrt{v_0^2 + 4gs_0}}{2g}$