COMBINATORICS: PAST, PRESENT, AND FUTURE

Early Combinatorics in China

ANDREA BRÉARD *

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Abstract

Combinatorial practices in China go back to high Antiquity, when divinatory techniques relied on configurations of broken and unbroken lines. The Book of Change (*Yijing* 易經), compiled under the Zhou dynasty, has transmitted these practices until today, and has been a widely commented and read source. But combinatorial practices in China are not limited to divination and magic squares: a large number of early sources also describe games like Go, chess and games with cards, dominoes and dice that show a combinatorial interest from a more mathematical point of view. The earliest source that systematically discusses permutations and combinations is an 18^{th} century manuscript. Although mathematics had by then been introduced from Europe, the manuscript is clearly based on traditional mathematical concepts and algorithmic modes. In this talk I shall show how early combinatorial practices provide a framework for later mathematical developments in imperial China.

*Université des Sciences et Technologies de Lille, U.F.R de Mathématiques, Laboratoire Paul Painlevé, 59655 Villeneuve d'Ascq Cédex, France.

1 Chen Houyao's Manuscript

Among the now extant Chinese mathematical writings, a single manuscript essay on combinatorics, *The meaning of methods for alternation and combination* (*Cuozong fayi* 錯綜法義) from the late 17th century by Chen Houyao 陳厚耀 (1648-1722) is preserved. It deals systematically with problems of permutation and combination in the case of divination with trigrams, the formation of hexagrams or names with several characters, combinations of the ten heavenly stems (*tiangan* 天干) and the twelve earthly branches (*dizhi* 地支) to form the astronomical sexagesimal cycles, and games of chance such as dice throwing and card games. In the foreword to his treatise, Chen Houyao says:

> The *Nine Chapters*[1] have entirely provided all [mathematical] methods, but they lack of any type of method for alternations and combinations.[2]

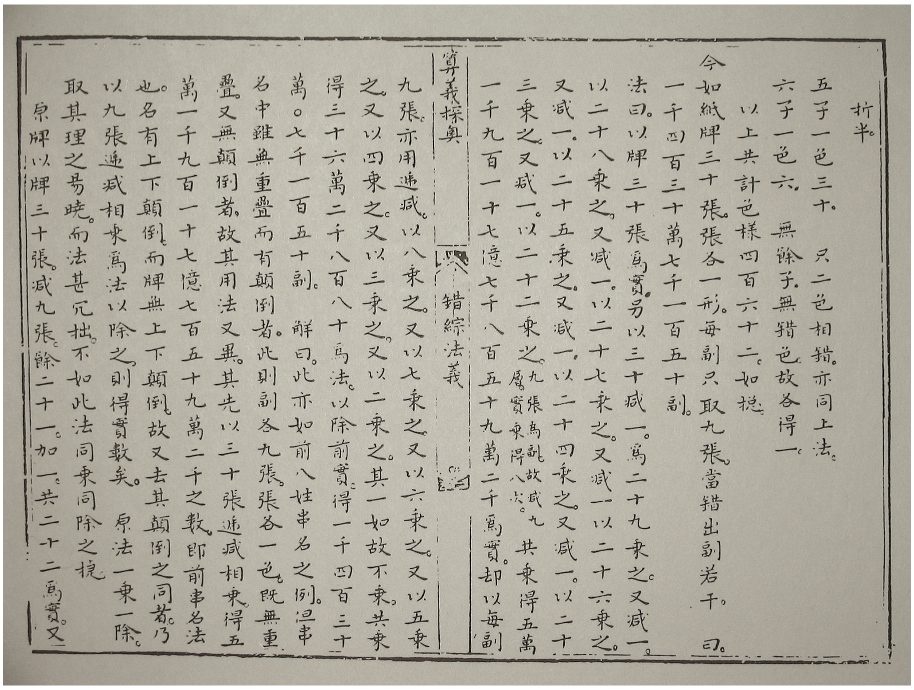
折半。
五子一色三十。只二色相錯。亦同上法。
六子一色六。無餘子。無錯色。故各得一。
以上共計色樣四百六十二。如總
今如紙牌三十張。張各一形。每副只取九張。當錯出副若干。
一千四百三十萬七千一百五十副。　曰。
法曰。以牌三十張爲實。另以三十減一。爲二十九乘之。又減一。
以二十八乘之。又減一。以二十七乘之。又減一以二十六乘之。
又減一。以二十五乘之。又減一。以二十四乘之。又減一。以二十
三乘之。又減一。以二十二乘之。九張爲副故減八次。九屬實乘得 共乘得五萬
一千九百一十七億七千八百五十九萬二千爲實。卻以每副
算義探奧　錯綜法義
九張。亦用遞減。以八乘之。又以七乘之。又以六乘之。又以五乘
之。又以四乘之。又以三乘之。又以二乘之。其一如故不乘。共乘
得三十六萬二千八百八十爲法。以除前實。得一千四百三十
萬。七千一百五十副。　解曰。此亦如前八姓串名之例。但串
名中雖無重疊而有顛倒者。此則副各九張。張各一色。既無重
疊。又無顛倒者。故其用法又異。其先以三十張遞減相乘。得五
萬一千九百一十七億七千八百五十九萬二千之數。即前串名法
也。名有上下顛倒。而牌無上下顛倒。故又去其顛倒之同者。乃
以九張遞減相乘爲法以除之。則得實數矣。　原法一乘一除。
取其理之易曉。而法甚冗拙。不如此法同乘同除之捷。
原牌以牌三十張。減九張。餘二十一。加一。共二十二爲實。又

Figure 1: Chen Houyao (1648-1722), *The meaning of methods for alternation and combination*

[1] Reference to *The Nine Chapters on Mathematical Procedures* (*Jiu zhang suan shu* 九章算術), the foundational work of mathematics in Ancient China.
[2] [Chen Houyao 17th] vol. 4, p. 685.

Unfortunately we do not know whether the author based his methods on knowledge circulating among early Qing dynasty mathematical networks, or whether he learned about combinatorics from the teachings he received from the Kangxi Emperor, who in turn had been instructed by the French Jesuit mathematicians sent by Louis XIV ('Les mathématiciens du Roi'). Although singular in its appearance, the existence of Chen Houyao's short collection of problems is nevertheless an indication that other sources with combinatorial problems might well have circulated during the late imperial period.[3]

Chen Houyao even refers explicitly to his predecessors, trying to improve the efficiency of their algorithms. In the case of the hexagrams he underlines that the calculation of all possible combinations of unbroken and broken lines can either be obtained by successive multiplication of the two possibilities:

$$\begin{aligned}
\text{Number of configurations consisting of 2 lines} &= 2 \cdot 2 = 4 \\
\text{Number of configurations consisting of 3 lines} &= 4 \cdot 2 = 8 \\
&\cdots \\
\text{Number of configurations consisting of 6 lines} &= 32 \cdot 2 = 64
\end{aligned}$$

but also in a much simpler way by squaring the eight possibilities to obtain a trigram (i.e. a configuration made up of three lines):

> If one multiplies by itself the thus obtained number [for trigrams: 8], one economizes half of the mulitplications.[4]

In other problems in which Chen Houyao suggests two alternative algorithms as solution procedures for the stated problem, he not only does name one of them explicitly as the 'original method', but he does also indicate the detailed sequence of operations to follow. In modern mathematical terms the equivalence between Chen's method:

$$\begin{aligned}
C_9^{30} = \binom{30}{9} &= \frac{(30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22)}{(9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2)} \\
&= 5{,}191{,}778{,}592{,}000 \div 362{,}880 = 14{,}307{,}150
\end{aligned}$$

and the 'original' one:

$$C_9^{30} = (\ldots(((22 \cdot 23) \div 2) \cdot 24) \div 3) \cdot \ldots \cdot 30) \div 9.$$

is illustrated in following example:

[3] We can observe similar phenomena of selective textual transmission in China for the development of other mathematical concepts and ideas.

[4] [Chen Houyao 17th] vol. 4, p. 685.

Let us suppose one has thirty playing cards. All cards have a [different] form. Each hand consists of nine cards. How many combinations can one obtain by drawing one hand ? It says: 14,307,150 hands.

The method says: One takes thirty cards as the dividend. Furthermore one subtracts one from thirty. This makes twenty-nine, one multiplies this [the dividend of 30] with it. One then again subtracts one, and multiplies this [the product of 30 and 29] with twenty-eight. [. . .] One multiplies this [the product of 30, 29, 28, 27, 26, 25, 24 and 23] with twenty-two. Nine cards make up one hand. Because one subtracts nine layers, one has to multiply eight times to obtain the dividend. Altogether one obtains by multiplications 5,191,778,592,000 as the dividend. Now one shall also successively subtract each hand of the nine cards. One multiplies this [the nine cards] with eight, further one multiplies this with seven. [. . .] Further one multiplies this [the product of 9, 8, 7, 6, 5, 4 and 3] with two. With one [multiplied] this remains unchanged, therefore one does not multiply. Altogether one obtains by these multiplications 362,880 as divisor. By this one divides the above dividend, one obtains 14,307,150 hands. The explanation (*jie* 觧)[5] says: Here the situation is as in the previous example concerning the combination of eight personal names. Yet although one does not have repetitions in combinations of names, one can still invert their ordering. But here each hand has nine cards, each card has a different color, and there are neither repetitions nor permutations of their ordering. This is the reason why the methods which are used are different. At first, the calculation of 5,191,778,592,000, which one obtains through successive subtraction and multiplication of the thirty cards, equals the previous method concerning the combination of personal names. Names have no above or below, no inversion of ordering. That is why one further has to eliminate the equivalent ones obtained by changing the ordering. Thus one divides this [the dividend 5,191,778,592,000] by the divisor, the successively subtracted and multiplied nine cards. One then obtains the real number.

The original method multiplies and divides alternately once at a time. Its principle is not easy to grasp and the method is very confusing and clumsy (*rongzhuo* 冗拙). It does not equal the efficiency of common multiplication and common division in this method here.[6]

[5]Chen Houyao uses here a graphic variation of the character 解.
[6]Translated from [Chen Houyao 17th] vol. 4, p. 687.

This problem shows that in China games of chance provided a framework for the mathematical treatment of possible outcomes. But it is doubtful that the conceptual step that brings into relation the number of favorable events and the total number of possible events, what in Europe laid the foundations of a mathematical theory of probability, was ever taken in China.[7]

2 Combinatorics and the Arithmetic Triangle

An even earlier prominent candidate for a possible emergence of combinatorial considerations in the Chinese mathematical tradition might be, what in the history of mathematics is known as the 'Pascal Triangle' or the 'Arithmetic Triangle'. It first appeared in China in an early 14$^{\text{th}}$ century treatise, that is in Zhu Shijie's 朱世傑 *Jade Mirror of Four Elements* (*Siyuan yujian* 四元玉鑑, 1303), but it must have been circulating a century earlier. Pascal applies his arithmetic triangle to the theory of combinations, the powers of binomial quantities and the problem of points[8] – the division of stakes in an interrupted game of chance.[9] For early China, we do so far only know of its use in the context of interpolation techniques, solving polynomial equations and constructing finite arithmetical series in the *Jade Mirror of Four Elements*. Zhu's combinatorics played out on another level: the terminology used for the finite series was constructed out of combinations of binomial expressions in the linguistic sense. A mathematical meaning was attached to each pre- or suffix, reflecting the factorization of each term in the arithmetical series. Inverse problems – i.e. how many terms for a given sum – were solved with interpolation techniques using the coefficients from the triangle.[10]

Later, Wang Lai 汪萊 (1768-1813) more systematically relates the finite arithmetical series to combinatorial problems in an essay on *The Mathematical Principles of Sequential Combinations* (*Dijian shuli* 遞兼數理).[11] He illustrates his

[7]In [Bréard 2008] I argue that it is possible that winning schemes in the game of 'ivory tiles' (*yapai* 牙牌), a kind of domino game, were based on combinatorial considerations, but there is no evidence of the existence of a concept of probability. This game is described in popular sources, generally referred to by historians as *Encyclopedias for Daily Use* (*Riyong leishu* 日用類書), which record various popular games and circulated widely among the late Ming dynasty reading public (end of 16$^{\text{th}}$, beginning of 17$^{\text{th}}$ century).

[8]Depicted in the *Traité du triangle arithmétique* "apparently printed in 1654 (though circulated in 1665)." See [Daston 1988] p. 9.

[9][Todhunter 1865] chapter 2 as well as many other historians of probability theory considers the Problem of Points, which prompted the seminal correspondence between Pascal and Fermat in 1654, as the beginning of the theory of probability. See also [Todhunter 1865] chapter 9 on the history of the arithmetic triangle.

[10]See [Bréard 1999a] chapter 4.2. and [Bréard 1999b].

[11]In *juan* 4 of his collected writings *Hengzhai's Mathematics* (*Hengzhai suanxue* 衡齋算學).

procedure with the example of 10 objects, from which sequentially one, two, three, four or five objects are drawn.

> Procedures of sequential combinations had not been discovered in ancient times. Now that I have decided to investigate into them, it is thus appropriate to explain the object of inquiry first. Let us suppose one has all kind of objects. Starting off from one object of which each establishes one configuration, and going up to the all the objects taken together, they form altogether one configuration (*shu* 數). In between lie sequentially: two objects connected to each other form one configuration, we shall discuss how many configurations this can make through exchanging and permuting (*jiao cuo* 交錯); three objects connected to each other form one configuration, we shall discuss how many configurations this can make through exchanging and permuting; four objects, five objects, up to arbitrarily many objects, none doesn't entirely follow that which is the so called procedure of sequential combinations. [12]

The possible outcomes correspond to the sums of higher order series, which Zhu Shijie had already calculated in the *Jade Mirror of Four Elements*, without explicitly referring to problems of combination. Wang Lai links here for the first time in the Chinese tradition combinations to figurate numbers and gives drawings for C_i^{10} (see figure 2 for $i = 1, \ldots 5$). He remarks the symmetry $C_i^{10} = C_{10-i}^{10}$ and calculates the total number of pebbles lined or piled up in triangular or pyramidal shape, the so called 'triangular piles' (*sanjiao dui* 三角堆), using Zhu Shijie's procedures for

Reprint see [Guo Shuchun et al. 1993] vol. 4, p. 1512-1516. For a discussion of its mathematical content, see [Li Zhaohua 1999] p. 52-64.

[12] Translated from [Wang Lai 1854] vol. 4, p. 1512-1513.

calculating the sums of finite arithmetical series of higher order:

$$C_1^{10} = C_9^{10} = 10$$

$$C_2^{10} = C_8^{10} = 1 + 2 + 3 + \ldots + 9 = \sum_{k=1}^{9} k = \frac{9 \cdot 10}{2} = 45$$

$$C_3^{10} = C_7^{10} = 1 + 3 + 6 + 10 + \ldots + 36 = \sum_{k=1}^{8} \frac{k(k+1)}{2} = \frac{8 \cdot 9 \cdot 10}{2 \cdot 3} = 120$$

$$C_4^{10} = C_6^{10} = 1 + 4 + 10 + 20 + \ldots + 84 = \sum_{k=1}^{7} \frac{k(k+1)(k+2)}{6} =$$

$$= \frac{7 \cdot 8 \cdot 9 \cdot 10}{2 \cdot 3 \cdot 4} = 210$$

$$C_5^{10} = 1 + 5 + 15 + 35 + 70 + 126 = \sum_{k=1}^{6} \frac{k(k+1)(k+2)(k+3)}{24} =$$

$$= \frac{6 \cdot 7 \cdot 8 \cdot 9 \cdot 10}{5 \cdot 4 \cdot 3 \cdot 2} = 252$$

The illustrations are suggesting the patterns of formation of every term of the above series, and stem from the Yuan dynasty Chinese tradition of considering piles of discrete objects in different geometric shapes. The series $C_2^{10} = 1 + 2 + 3 + \ldots + 9$ thus becomes a triangle in which pebbles are piled up in rows with 1 to 9 pebbles in each successive row. The next sum is then a regular pyramid, where each layer is composed of one of such triangles, each having (from top to bottom) 1, 3, 6, $\ldots$ or 36 elements. For C_4^{10} for example Wang shows seven pyramids:

$$C_4^{10} = 1 + (1 + 3) + (1 + 3 + 6) + \ldots + (1 + 3 + 6 + 10 + 15 + 21 + 28) = 210$$

Wang Lai's only related mathematical problem given stems from divination. A shaman performing yarrow stalks divination (*shigua* 筮卦) produces a hexagram, i.e. a configuration made up of six lines (*liu yao* 六爻). Wang calculates in two ways the total number of possible configurations of one to six lines that one can produce from this hexagram. A first method proceeds by doubling successively the minimum number of lines in such a configuration and then adding one. Wang Lai remarks that five (i.e. the maximum number of lines that one can obtain minus one) iterations give the total number of possible configurations:

$$
\begin{aligned}
2 \cdot 1 + 1 &= 3 \\
2 \cdot 3 + 1 &= 7 \\
2 \cdot 7 + 1 &= 15 \\
2 \cdot 15 + 1 &= 31 \\
2 \cdot 31 + 1 &= 63
\end{aligned}
$$

or:

$$
\begin{aligned}
63 &= 2 \cdot \Big(2 \cdot \big(2 \cdot (2 \cdot (2 \cdot 1 + 1) + 1) + 1\big) + 1\Big) + 1 \\
&= C_1^6 + C_2^6 + C_3^6 + C_4^6 + C_5^6 + C_6^6
\end{aligned}
$$

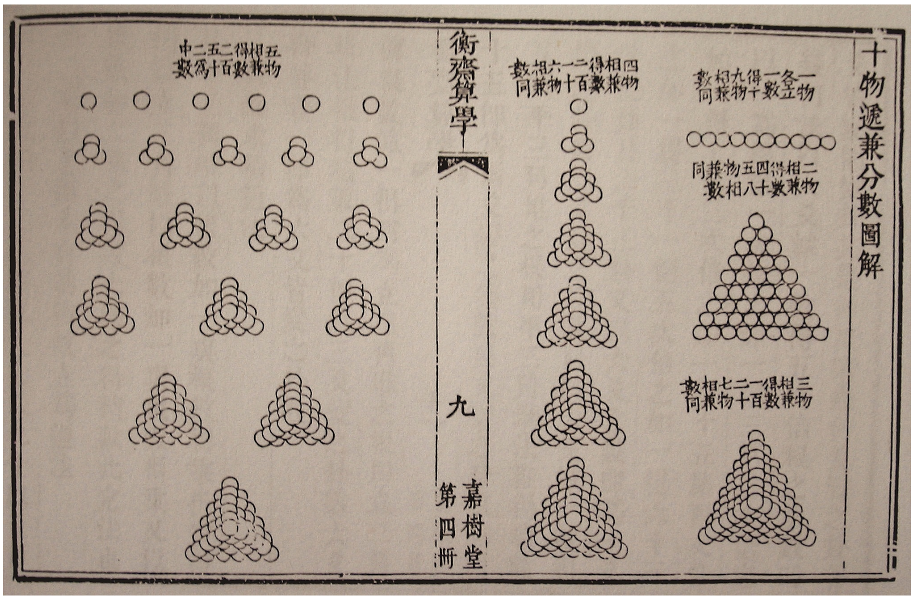

Figure 2: Wang Lai (1768-1813), *Mathematical Principles of Sequential Combinations*

In a second step, Wang Lai calculates the total of 63 configurations by adding up the possibilities for the configurations of 1, 2, … 6 lines. He calculates these using the procedures for 'triangular piles' (i.e. procedures to calculate the sums of triangular numbers). Again, as Chen Houyao did earlier, Wang Lai remarks the symmetry $C_m^n = C_{m-n}^n$:

$$
\begin{aligned}
C_1^6 &= C_5^6 = 6 \\
C_2^6 &= C_4^6 = 15 = \frac{5 \cdot (5+1)}{2} = \sum_{k=1}^{5} k \\
& C_3^6 = 20 = \frac{4 \cdot (4+1) \cdot (4+2)}{6} = \sum_{k=1}^{4} \frac{k(k+1)}{2} = \sum_{k=1}^{4} \sum_{i=1}^{k} i \\
& C_6^6 = 1 \\
\sum_{k=1}^{6} & C_k^6 = 63
\end{aligned}
$$

He finally gives the general procedure for the sum of higher order 'triangular piles' (i.e. finite arithmetic series):

$$\frac{n \cdot (n+1) \cdot (n+2) \cdot \ldots (n+m)}{1 \cdot 2 \cdot \ldots m} = C_m^{n+m}$$

and gives explicitly the procedure and numerical calculations for the 'fourth-order triangular pile' (*si cheng sanjiao dui* 四乘三角堆)[13] with five as the particular 'base number' (*genshu* 根數):

$$\sum_{k=1}^{5} \frac{k(k+1)(k+2)(k+3)}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{5 \cdot (5+1) \cdot (5+2) \cdot (5+3) \cdot (5+4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 126$$

Contrary to other authors,[14] I do not see where Wang Lai should have recognized the general relation:

$$\sum_{k=1}^{n} C_k^n = 2^n - 1$$

As mentioned earlier, Wang Lai does not bring his calculations in connection with the arithmetic triangle. Its seventh line contains precisely the numbers 1, 6, 15, 20, 15, 6 and 1 (C_n^6 for $n = 0, \ldots, 6$), and their sum equals 2^6. But Wang does not refer to the triangle.

[13] Literally the order can be translated as 'multiplication', since as Chen points out the number corresponds to the number of multiplications to be performed to calculate each the dividend and the divisor.

[14] See for example [Li Zhaohua 1986] or Liu Dun's 劉鈍 introduction to [Wang Lai 1854] p. 1479.

3 Li Shanlan

Number theoretic relations in arithmetic triangles will be examined in China more systematically at the end of the 19$^{\text{th}}$ century by Li Shanlan 李善蘭.[15] They led to the now famous 'Li Renshu Identity':[16]

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{m+2n-k}{2n} = \binom{m+n}{n}^2$$

But interpreting Li Shanlan's writings as contributions to combinatorics is problematic.[17] Looking at his work from within the Chinese mathematical tradition, he is rather situated in a specific mathematical domain that developed as early as in Han dynasty. It reflected upon the discretization of continuous solids into a finite number of elements. Ultimately this led Zhu Shijie in 1303 to work out a systematic theory of summation of finite series in relation to the arithmetic triangle. Several Qing dynasty authors, like Li Shanlan, pursued research in this field, but it evolved without any combinatorial interpretation.[18]

Viewing their findings from a purely modern mathematical perspective, their results can be seen as important combinatorial contributions. Yet, as shows a careful reading of Chen Houyao's and Wang Lai's texts against the background of earlier mathematical texts in China, these were the only two that conveyed a combinatorial meaning. Chen's manuscript is particularly original in stating new mathematical problems on games of chance and divination. But his algorithms to calculate combinations and permutations did not seem to inspire further theoretical considerations of chance, at least when we judge from the mathematical sources preserved. Probability theory, an important part of mathematical statistics, seems to have been a mathematical field entirely imported from the West.

[15] [Li Shanlan 1867].
[16] See [Horng Wann-Sheng 1991] p. 206. Renshu was the style name of Li Shanlan.
[17] *Ibid* p. 225 "one of the early masterpieces of combinatorics". See also [Martzloff 1990].
[18] See [Bréard 1999a] chapter 5.

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Résumé

Andrea Bréard is Associate Professor in mathematics at the Université des Sciences et Technologies de Lille. She holds PhDs in Philosophy from the Technical University Berlin and in Epistemology from the Université Paris 7. Her book [Bréard 1999a] on the development of concepts of finite arithmetic series in China from the first to the 19th century was awarded the 'Prix des Jeunes Historiens' by the Académie Internationale d'Histoire des Sciences and the 'Prix de Thèse' by the Fondation Sino-Française pour l'Éducation et la Culture. She has recently been working on the history of statistics and probability theory in late imperial China and prepared a book-length manuscript on *Reform, Bureaucratic Expansion and Production of Numbers : Statistics in Early 20*th *Century China*.