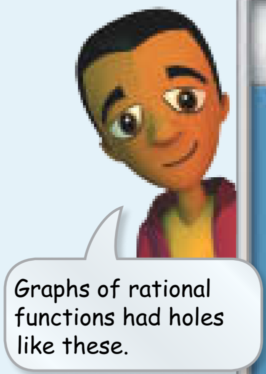
CC-32 Trigonometric Identities

Common Core State Standards

MACC.912.F-TF.3.8 Prove the Pythagorean identity $\sin^2(x) + \cos^2(x) = 1$ and use it to find $\sin(x)$, $\cos(x)$, or $\tan(x)$, given $\sin(x)$, $\cos(x)$, or $\tan(x)$, and the quadrant of the angle.

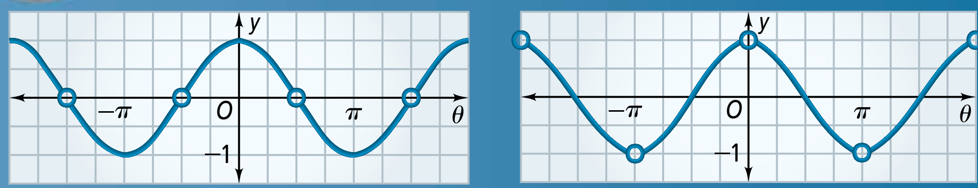
MP 1, MP 2, MP 3, MP 4

Objective To verify trigonometric identities



MATHEMATICAL PRACTICES

SOLVE IT! **Getting Ready!**



What could be the function for each graph? Explain your reasoning.

You may recognize $x^2 = 5x - 6$ as an equation that you are to solve to find the few, if any, values of x that make the equation true. On the other hand, you may recognize $\frac{x^5}{x^3} = x^2$, as an *identity*, an equation that is true for all values of x for which the expressions in the equation are defined. (Here, $\frac{x^5}{x^3}$ is not defined for $x = 0$.)

Lesson Vocabulary
- trigonometric identity

A **trigonometric identity** in one variable is a trigonometric equation that is true for all values of the variable for which all expressions in the equation are defined.

Essential Understanding The interrelationships among the six basic trigonometric functions make it possible to write trigonometric expressions in various equivalent forms, some of which can be significantly easier to work with than others in mathematical applications.

Some trigonometric identities are definitions or follow immediately from definitions.

take note

Key Concept Basic Identities

Reciprocal Identities

$\csc\theta = \frac{1}{\sin\theta}$ $\sec\theta = \frac{1}{\cos\theta}$ $\tan\theta = \frac{1}{\cot\theta}$

$\sin\theta = \frac{1}{\csc\theta}$ $\cos\theta = \frac{1}{\sec\theta}$ $\cot\theta = \frac{1}{\tan\theta}$

Tangent Identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$

Cotangent Identity $\cot\theta = \frac{\cos\theta}{\sin\theta}$

The *domain of validity* of an identity is the set of values of the variable for which all expressions in the equation are defined.

Problem 1 Finding the Domain of Validity

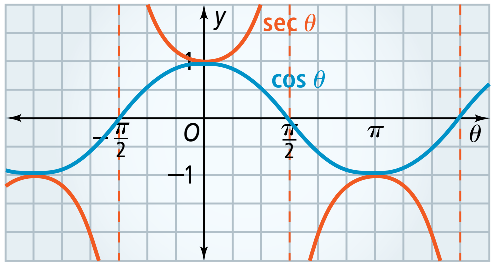
Plan

How can an expression be undefined?
An expression could contain a denominator that could be zero or it could contain an expression that is itself undefined for some values.

What is the domain of validity of each trigonometric identity?

A $\cos\theta = \frac{1}{\sec\theta}$.

The domain of $\cos\theta$ is all real numbers. The domain of $\frac{1}{\sec\theta}$ excludes all zeros of $\sec\theta$ (of which there are none) and all values θ for which $\sec\theta$ is undefined (odd multiples of $\frac{\pi}{2}$).



Therefore the domain of validity of $\cos\theta = \frac{1}{\sec\theta}$ is the set of real numbers except for the odd multiples of $\frac{\pi}{2}$.

B $\sec\theta = \frac{1}{\cos\theta}$.

The domain of validity is the same as part (a), because $\sec\theta$ is not defined for odd multiples of $\frac{\pi}{2}$, and the odd multiples of $\frac{\pi}{2}$ are the zeros of $\cos\theta$.

Got It? **1.** What is the domain of validity of the trigonometric identity $\sin\theta = \frac{1}{\csc\theta}$?

You can use known identities to verify other identities. To verify an identity, you can use previously known identities to transform one side of the equation to look like the other side.

Problem 2 Verifying an Identity Using Basic Identities

Plan

What identity do you know that you can use?
Look for a way to write the expression on the left in terms of $\sin\theta$ and $\cos\theta$. The identity $\sec\theta = \frac{1}{\cos\theta}$ does the job.

Verify the identity. What is the domain of validity?

A $(\sin\theta)(\sec\theta) = \tan\theta$

$(\sin\theta)(\sec\theta) = \sin\theta \cdot \frac{1}{\cos\theta}$ Reciprocal Identity

$= \frac{\sin\theta}{\cos\theta}$ Simplify.

$= \tan\theta$ Tangent Identity

The domain of $\sin\theta$ is all real numbers. The domains of $\sec\theta$ and $\tan\theta$ exclude all zeros of $\cos\theta$. These are the odd multiples of $\frac{\pi}{2}$. The domain of validity is the set of real numbers except for the odd multiples of $\frac{\pi}{2}$.

B $\frac{1}{\cot\theta} = \tan\theta$

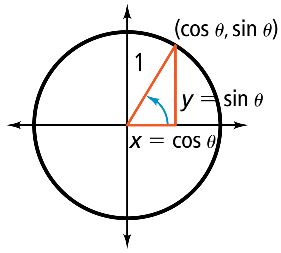
$\frac{1}{\cot\theta} = \frac{1}{\frac{1}{\tan\theta}}$ Definition of cotangent

$= \tan\theta$ Simplify.

The domain of $\cot\theta$ excludes multiples of π. Also, $\cot\theta = 0$ at the odd multiples of $\frac{\pi}{2}$. The domain of validity is the set of real numbers except *all* multiples of $\frac{\pi}{2}$.

Got It? **2.** Verify the identity $\frac{\csc\theta}{\sec\theta} = \cot\theta$. What is the domain of validity?

You can use the unit circle and the Pythagorean Theorem to verify another identity. The circle with its center at the origin with a radius of 1 is called the unit circle, and has an equation $x^2 + y^2 = 1$.



Every angle θ determines a unique point on the unit circle with x- and y-coordinates $(x, y) = (\cos\theta, \sin\theta)$.

Therefore, for every angle θ,

$$(\cos\theta)^2 + (\sin\theta)^2 = 1 \qquad \text{or} \qquad \cos^2\theta + \sin^2\theta = 1.$$

This form allows you to write the identity without using parentheses.

This is a Pythagorean identity. You will verify two others in Problem 3.

You can use the basic and Pythagorean identities to verify other identities. To prove identities, transform the expression on one side of the equation to the expression on the other side. It often helps to write everything in terms of sines and cosines.

Problem 3 Verifying a Pythagorean Identity

Plan

With which side should you work? It usually is easier to begin with the more complicated-looking side.

Verify the Pythagorean identity $1 + \tan^2\theta = \sec^2\theta$.

$1 + \tan^2\theta = 1 + \left(\frac{\sin\theta}{\cos\theta}\right)^2$	Tangent Identity
$= 1 + \frac{\sin^2\theta}{\cos^2\theta}$	Simplify.
$= \frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta}$	Find a common denominator.
$= \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}$	Add.
$= \frac{1}{\cos^2\theta}$	Pythagorean identity
$= \sec^2\theta$	Reciprocal identity

You have transformed the expression on the left side of the equation to become the expression on the right side. The equation is an identity.

Got It? **3. a.** Verify the third Pythagorean identity, $1 + \cot^2\theta = \csc^2\theta$.

b. Reasoning Explain why the domain of validity is not the same for all three Pythagorean identities.

You have now seen all three Pythagorean identities.

take note

Key Concept Pythagorean Identities

$\cos^2\theta + \sin^2\theta = 1$ $\qquad$ $1 + \tan^2\theta = \sec^2\theta$ $\qquad$ $1 + \cot^2\theta = \csc^2\theta$

There are many trigonometric identities. Most do not have specific names.

Problem 4 Verifying an Identity

Plan

How do you begin when both sides look complicated?
It often is easier to collapse a difference (or sum) into a product than to expand a product into a difference.

Verify the identity $\tan^2\theta - \sin^2\theta = \tan^2\theta\sin^2\theta$.

$\tan^2\theta - \sin^2\theta = \frac{\sin^2\theta}{\cos^2\theta} - \sin^2\theta$	Tangent identity
$= \frac{\sin^2\theta}{\cos^2\theta} - \frac{\sin^2\theta\cos^2\theta}{\cos^2\theta}$	Use a common denominator.
$= \frac{\sin^2\theta - \sin^2\theta\cos^2\theta}{\cos^2\theta}$	Simplify.
$= \frac{\sin^2\theta(1-\cos^2\theta)}{\cos^2\theta}$	Factor.
$= \frac{\sin^2\theta(\sin^2\theta)}{\cos^2\theta}$	Pythagorean identity
$= \frac{\sin^2\theta}{\cos^2\theta}\sin^2\theta$	Rewrite the fraction.
$= \tan^2\theta\sin^2\theta$	Tangent Identity

Got It? **4.** Verify the identity $\sec^2\theta - \sec^2\theta\cos^2\theta = \tan^2\theta$.

You can use trigonometric identities to simplify trigonometric expressions.

Problem 5 Simplifying an Expression

What is a simplified trigonometric expression for $\csc\theta\tan\theta$?

Think	Write
Write the expression. Then replace $\csc\theta$ with $\frac{1}{\sin\theta}$.	$\csc\theta\tan\theta = \frac{1}{\sin\theta}\cdot\tan\theta$
Replace $\tan\theta$ with $\frac{\sin\theta}{\cos\theta}$.	$= \frac{1}{\sin\theta}\cdot\frac{\sin\theta}{\cos\theta}$
	$= \frac{\sin\theta}{\sin\theta\cos\theta}$
Simplify.	$= \frac{1}{\cos\theta}$
$\frac{1}{\cos\theta} = \sec\theta$.	$= \sec\theta$

Got It? **5.** What is a simplified trigonometric expression for $\sec\theta\cot\theta$?

Lesson Check

Do you know HOW?

Verify each identity.

1. $\tan\theta \csc\theta = \sec\theta$
2. $\csc^2\theta - \cot^2\theta = 1$
3. $\sin\theta \tan\theta = \sec\theta - \cos\theta$
4. Simplify $\tan\theta \cot\theta - \sin^2\theta$.

Do you UNDERSTAND? MATHEMATICAL PRACTICES

5. **Vocabulary** How does the identity $\cos^2\theta + \sin^2\theta = 1$ relate to the Pythagorean Theorem?
6. **Error Analysis** A student simplified the expression $2 - \cos^2\theta$ to $1 - \sin^2\theta$. What error did the student make? What is the correct simplified expression?

Practice and Problem-Solving Exercises MATHEMATICAL PRACTICES

A Practice

Verify each identity. Give the domain of validity for each identity. See Problems 1–4.

7. $\cos\theta \cot\theta = \frac{1}{\sin\theta} - \sin\theta$
8. $\sin\theta \cot\theta = \cos\theta$
9. $\cos\theta \tan\theta = \sin\theta$
10. $\sin\theta \sec\theta = \tan\theta$
11. $\cos\theta \sec\theta = 1$
12. $\tan\theta \cot\theta = 1$
13. $\sin\theta \csc\theta = 1$
14. $\cot\theta = \csc\theta \cos\theta$
15. $\csc\theta - \sin\theta = \cot\theta \cos\theta$

Simplify each trigonometric expression. See Problem 5.

16. $\tan\theta \cot\theta$
17. $1 - \cos^2\theta$
18. $\sec^2\theta - 1$
19. $1 - \csc^2\theta$
20. $\sec^2\theta \cot^2\theta$
21. $\cos\theta \tan\theta$
22. $\sin\theta \cot\theta$
23. $\sin\theta \csc\theta$
24. $\sec\theta \cos\theta \sin\theta$
25. $\sin\theta \sec\theta \cot\theta$
26. $\sec^2\theta - \tan^2\theta$
27. $\frac{\sin\theta}{\cos\theta \tan\theta}$

B Apply

28. **Think About a Plan** Simplify the expression $\frac{\tan\theta}{\sec\theta - \cos\theta}$.
    - Can you write everything in terms of $\sin\theta$, $\cos\theta$, or both?
    - Are there any trigonometric identities that can help you simplify the expression?

Simplify each trigonometric expression.

29. $\cos\theta + \sin\theta \tan\theta$
30. $\csc\theta \cos\theta \tan\theta$
31. $\tan\theta(\cot\theta + \tan\theta)$
32. $\sin^2\theta + \cos^2\theta + \tan^2\theta$
33. $\sin\theta(1 + \cot^2\theta)$
34. $\sin^2\theta \csc\theta \sec\theta$
35. $\sec\theta \cos\theta - \cos^2\theta$
36. $\csc\theta - \cos\theta \cot\theta$
37. $\csc^2\theta(1 - \cos^2\theta)$
38. $\frac{\csc\theta}{\sin\theta + \cos\theta \cot\theta}$
39. $\frac{\cos\theta \csc\theta}{\cot\theta}$
40. $\frac{\sin^2\theta \csc\theta \sec\theta}{\tan\theta}$

Express the first trigonometric function in terms of the second.

41. $\sin\theta$, $\cos\theta$

42. $\tan\theta$, $\cos\theta$

43. $\cot\theta$, $\sin\theta$

44. $\csc\theta$, $\cot\theta$

45. $\cot\theta$, $\csc\theta$

46. $\sec\theta$, $\tan\theta$

Verify each identity.

47. $\sin^2\theta\tan^2\theta = \tan^2\theta - \sin^2\theta$

48. $\sec\theta - \sin\theta\tan\theta = \cos\theta$

49. $\sin\theta\cos\theta(\tan\theta + \cot\theta) = 1$

50. $\dfrac{1-\sin\theta}{\cos\theta} = \dfrac{\cos\theta}{1+\sin\theta}$

51. $\dfrac{\sec\theta}{\cot\theta + \tan\theta} = \sin\theta$

52. $(\cot\theta + 1)^2 = \csc^2\theta + 2\cot\theta$

53. Express $\cos\theta\csc\theta\cot\theta$ in terms of $\sin\theta$.

54. Express $\frac{\cos\theta}{\sec\theta + \tan\theta}$ in terms of $\sin\theta$.

Use the identity $\sin^2\theta + \cos^2\theta = 1$ and the basic identities to answer the following questions. Show all your work.

55. Given that $\sin\theta = 0.5$ and θ is in the first quadrant, what are $\cos\theta$ and $\tan\theta$?

56. Given that $\sin\theta = 0.5$ and θ is in the second quadrant, what are $\cos\theta$ and $\tan\theta$?

57. Given that $\cos\theta = -0.6$ and θ is in the third quadrant, what are $\sin\theta$ and $\tan\theta$?

58. Given that $\sin\theta = 0.48$ and θ is in the second quadrant, what are $\cos\theta$ and $\tan\theta$?

59. Given that $\tan\theta = 1.2$ and θ is in the first quadrant, what are $\sin\theta$ and $\cos\theta$?

60. Given that $\tan\theta = 3.6$ and θ is in the third quadrant, what are $\sin\theta$ and $\cos\theta$?

61. Given that $\sin\theta = 0.2$ and $\tan\theta < 0$, what is $\cos\theta$?

Challenge

62. The unit circle is a useful tool for verifying identities. Use the diagram at the right to verify the identity $\sin(\theta + \pi) = -\sin\theta$.

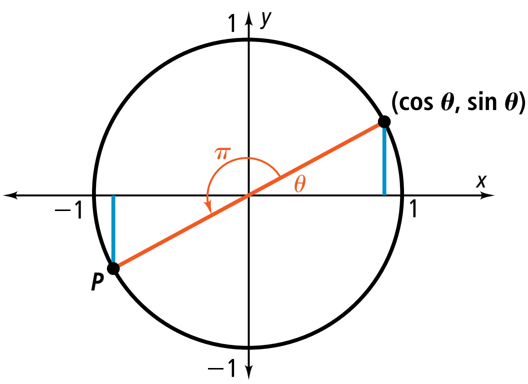
a. Explain why the y-coordinate of point P is $\sin(\theta + \pi)$.

b. Prove that the two triangles shown are congruent.

c. Use part (b) to show that the two blue segments are congruent.

d. Use part (c) to show that the y-coordinate of P is $-\sin\theta$.

e. Use parts (a) and (d) to conclude that $\sin(\theta + \pi) = -\sin\theta$.



Use the diagram in Exercise 62 to verify each identity.

63. $\cos(\theta + \pi) = -\cos\theta$

64. $\tan(\theta + \pi) = \tan\theta$

Simplify each trigonometric expression.

65. $\dfrac{\cot^2\theta - \csc^2\theta}{\tan^2\theta - \sec^2\theta}$

66. $(1 - \sin\theta)(1 + \sin\theta)\csc^2\theta + 1$

STEM **67. Physics** When a ray of light passes from one medium into a second, the angle of incidence θ_1 and the angle of refraction θ_2 are related by Snell's law: $n_1 \sin \theta_1 = n_2 \sin \theta_2$, where n_1 is the index of refraction of the first medium and n_2 is the index of refraction of the second medium. How are θ_1 and θ_2 related if $n_2 > n_1$? If $n_2 < n_1$? If $n_2 = n_1$?