Common Core State Standards

Mathematics III

Integrated Pathway

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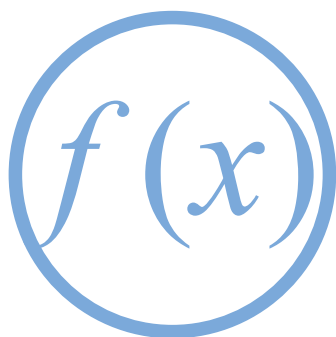
Student Resource
Units 3–4B

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Table of Contents

Unit 4B: Mathematical Modeling and Choosing a Model

Introduction

Welcome to the *CCSS Integrated Pathway: Mathematics III Student Resource Book.* This book will help you learn how to use algebra, geometry, data analysis, and probability to solve problems. Each lesson builds on what you have already learned. As you participate in classroom activities and use this book, you will master important concepts that will help to prepare you for mathematics assessments and other mathematics courses.

This book is your resource as you work your way through the Math III course. It includes explanations of the concepts you will learn in class; math vocabulary and definitions; formulas and rules; and exercises so you can practice the math you are learning. Most of your assignments will come from your teacher, but this book will allow you to review what was covered in class, including terms, formulas, and procedures.

- In **Unit 1: Inferences and Conclusions from Data**, you will learn about using the normal curve, as well as about populations versus random samples and random sampling. This is followed by learning about surveys, experiments, and observational studies—all strategies for collecting data. You will estimate sample proportions and sample means and develop tools for comparing treatments and reading reports. Finally, you will look at making and analyzing decisions with data.

- In **Unit 2A: Polynomial Relationships**, you will begin by exploring polynomial structures and operating with polynomials. Then you will learn how to prove identities. The unit progresses to graphing polynomial functions and solving systems of equations with polynomials. The unit ends with geometric series.

- In **Unit 2B: Rational and Radical Relationships**, you will work with operations with rational expressions. Then you will learn to solve rational and radical equations.

- In **Unit 3: Trigonometry of General Triangles and Trigonometric Functions**, you will start by learning about radians and the unit circle. Then you will explore the trigonometry of general angles, including the Law of Sines and the Law of Cosines. You will move on to graphs of trigonometric functions to model periodic phenomena.

- In **Unit 4A: Mathematical Modeling of Inverse, Logarithmic, and Trigonometric Functions**, you will begin by learning about the inverses of quadratics and other functions. This builds into learning about graphing and interpreting logarithmic functions and models. Then you will learn about modeling trigonometric functions by graphing the sine and cosine functions.

- In **Unit 4B: Mathematical Modeling and Choosing a Model**, you will revisit the process of creating equations in one variable and explore creating constraints and rearranging formulas. You will then learn about transforming models and combining functions. You will review various kinds of functions including linear, exponential, quadratic, piecewise, step, absolute value, square root, and cube root functions, all with an eye to choosing a model for a real-world situation. Finally, you will consider geometric models, including two-dimensional cross sections of three-dimensional shapes.

Each lesson is made up of short sections that explain important concepts, including some completed examples. Each of these sections is followed by a few problems to help you practice what you have learned. The "Words to Know" section at the beginning of each lesson includes important terms introduced in that lesson.

As you move through your Math III course, you will become a more confident and skilled mathematician. We hope this book will serve as a useful resource as you learn.

Unit 3

Trigonometry of General Triangles and Trigonometric Functions

$f(x)$

$\angle ABC$

Q_1

Lesson 1: Radians and the Unit Circle

Common Core State Standards

F–TF.1 Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

F–TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

Essential Questions

1. What is a radian?

2. What is a unit circle and how is it helpful?

3. What is a reference angle and how is it found?

4. How do you find the point at which the terminal side of an angle intersects the unit circle?

5. What are the special angles and how do you find their trigonometric ratios?

WORDS TO KNOW

arc length	the distance between the endpoints of an arc; written as $d(\overset{\frown}{ABC})$ or $m\overset{\frown}{AC}$
central angle	an angle with its vertex at the center of a circle
cosecant	the reciprocal of sine, $\csc\theta = \dfrac{1}{\sin\theta}$; the cosecant of $\theta = $ $\csc\theta = \dfrac{\text{length of hypotenuse}}{\text{length of opposite side}}$

cosine	a trigonometric function of an acute angle in a right triangle that is the ratio of the length of the side adjacent to the length of the hypotenuse; the cosine of $\theta = \cos \theta = \dfrac{\text{length of adjacent side}}{\text{length of hypotenuse}}$
cotangent	the reciprocal of tangent, $\cot\theta = \dfrac{1}{\tan\theta}$; the cotangent of $\theta = \cot \theta = \dfrac{\text{length of adjacent side}}{\text{length of opposite side}}$
coterminal angles	angles that, when drawn in standard position, share the same terminal side
initial side	the stationary ray of an angle from which the measurement of the angle starts
radian	the measure of the central angle that intercepts an arc equal in length to the radius of the circle; π radians $= 180°$
reference angle	the acute angle that the terminal side makes with the x-axis. The sine, cosine, and tangent of the reference angle are the same as that of the original angle (except for the sign, which is based on the quadrant in which the terminal side is located).
secant	the reciprocal of cosine, $\sec\theta = \dfrac{1}{\cos\theta}$; the secant of $\theta = \sec \theta = \dfrac{\text{length of hypotenuse}}{\text{length of adjacent side}}$
sine	a trigonometric function of an acute angle in a right triangle that is the ratio of the length of the opposite side to the length of the hypotenuse; the sine of $\theta = \sin \theta = \dfrac{\text{length of opposite side}}{\text{length of hypotenuse}}$

standard position (of an angle)	a position in which the vertex of the angle is at the origin of the coordinate plane and is the center of the unit circle. The angle's initial side is located along the positive *x*-axis and the terminal side may be in any location.
subtended arc	the section of an arc formed by a central angle that passes through the circle, thus creating the endpoints of the arc
tangent	a trigonometric function of an acute angle in a right triangle that is the ratio of the length of the opposite side to the length of the adjacent side; the tangent of $\theta = \tan\theta = \dfrac{\text{length of opposite side}}{\text{length of adjacent side}}$
terminal side	for an angle in standard position, the movable ray of an angle that can be in any location and which determines the measure of the angle
***theta* (θ)**	a Greek letter commonly used to refer to unknown angle measures
unit circle	a circle with a radius of 1 unit. The center of the circle is located at the origin of the coordinate plane.

Recommended Resources

- BetterExplained.com. "Intuitive Guide to Angles, Degrees, and Radians."

  http://www.walch.com/rr/00218

  This website provides a creative explanation of the radian system of angle measure.

- Khan Academy. "Unit Circle Definition of Trig Functions."

  http://www.walch.com/rr/00219

  This site includes videos and examples of how to use the unit circle to find trigonometric functions.

- MathIsFun.com. "Interactive Unit Circle."

  http://www.walch.com/rr/00220

  This site includes an interactive unit circle and the corresponding points on a coordinate plane.

IXL Links

- Convert between radians and degrees:
  http://www.ixl.com/math/algebra-2/convert-between-radians-and-degrees

- Radians and arc length:
  http://www.ixl.com/math/algebra-2/radians-and-arc-length

- Coterminal angles:
  http://www.ixl.com/math/algebra-2/coterminal-angles

- Reference angles:
  http://www.ixl.com/math/algebra-2/reference-angles

- Find trigonometric ratios using the unit circle:
  http://www.ixl.com/math/algebra-2/find-trigonometric-ratios-using-the-unit-circle

Lesson 3.1.1: Radians

Introduction

The most familiar unit used to measure angles is the degree, where one degree represents $\frac{1}{360}$ of a full rotation. This unit of measurement and its value originated from ancient mathematicians. Some modern theorists propose the number 360 was chosen because the ancient Babylonian calendar had 360 days in the year. Although the number of degrees in a full rotation appears to have been an arbitrary choice, there is another system of angle measurement that is not arbitrary.

Key Concepts

- A **radian** is the measure of the central angle that intercepts an arc equal in length to the radius of the circle.

- A **central angle** is an angle with its vertex at the center of a circle.

- The radian system of measurement compares the length of the arc that the angle subtends to (intersects) the radius.

- The **subtended arc** is the section of an arc formed by a central angle that passes through the circle, thus creating the endpoints of the arc.

- The formula used to represent this relationship is $\theta = \frac{s}{r}$, in which θ is the angle measure in radians, s is the measure of the **arc length** (the distance between the endpoints of an arc), and r is the radius of the circle.

- The lowercase Greek letter ***theta*** (**θ**) is commonly used to refer to an unknown angle measure.

- The measure of an angle described as θ is 1 radian when the arc length equals the radius, as shown in the figure.
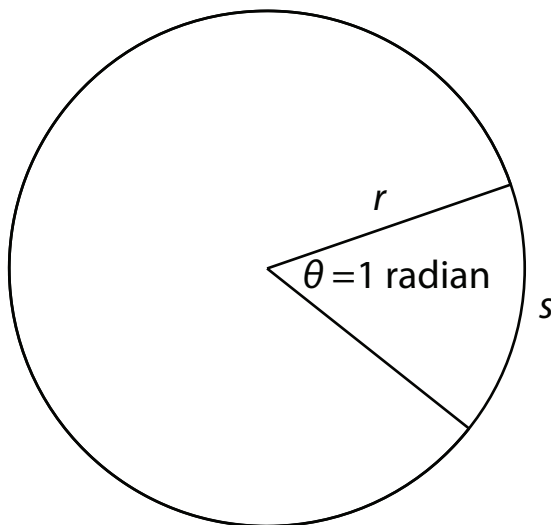
- This can be verified mathematically with the aforementioned formula, $\theta = \dfrac{s}{r}$; for any arc length s or radius r, if $s = r$, then $\dfrac{s}{r} = 1$.

- One radian is approximately equal to 57.3°. While this may appear to be an arbitrary value, recall that the number 360 was an arbitrarily chosen number. Radians are connected mathematically to the properties of a circle in a more easily identifiable way.

- Radians are often expressed in terms of π, which provides an exact measurement instead of a decimal approximation.

- A full rotation (360°) is equal to 2π radians. This is because the arc length of a full rotation is also the circumference of a circle, equal to $2\pi r$. Thus, when $2\pi r$ (the arc length, s) is divided by r, the result is $\theta = \dfrac{s}{r} = \dfrac{2\pi r}{r} = 2\pi \text{ radians}$.

- Therefore, a half rotation (180°) is equal to π radians (since $\dfrac{360°}{2} = 180°$, so by substituting 2π radians for 360°, $\dfrac{2\pi \text{ radians}}{2} = \pi \text{ radians}$), and a 90° rotation is equal to $\dfrac{\pi}{2}$ radians.

- To convert from radians to degrees or vice versa, use an appropriate conversion factor based on the relationship between π radians and 180°. If converting from radians to degrees, multiply by $\dfrac{180°}{\pi \text{ radians}}$ so that the radians cancel and degrees remain. Alternately, if converting from degrees to radians, multiply by $\dfrac{\pi \text{ radians}}{180°}$ so that the degrees cancel and radians remain.

Guided Practice 3.1.1

Example 1

Given the diagram of $\odot B$, find the measure of θ in radians. Round your answer to the nearest ten-thousandth.
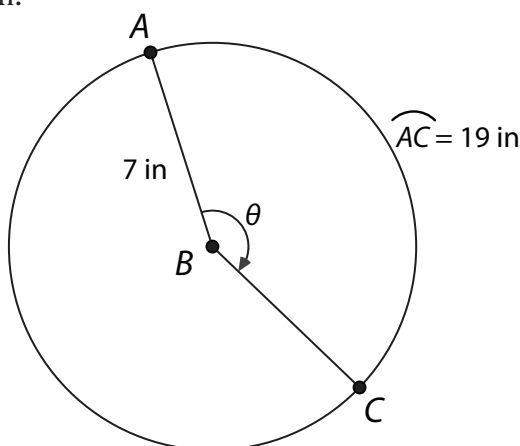
1. Identify the length of the radius and the arc length.

   The length of the radius of the circle is 7 inches and the length of the arc the angle subtends is 19 inches. Thus, $r = 7$ and $s = 19$.

2. Substitute r and s into the formula $\theta = \dfrac{s}{r}$ and solve for θ.

   The formula $\theta = \dfrac{s}{r}$ describes the relationship among an angle measure in radians, an arc length, and the radius of a circle.

   Substitute the known values of r and s into the formula, then solve for θ to determine the measure of the angle in radians.

   $\theta = \dfrac{s}{r}$ Formula for the measure of an angle in radians

   $\theta = \dfrac{(19)}{(7)}$ Substitute 19 for s and 7 for r.

   $\theta \approx 2.7143$ Use a calculator to simplify.

   The measure of the angle is approximately 2.7143 radians.

   Notice that the angle measure in radians shows the ratio of the arc length to the length of the radius. In this instance, the arc length is approximately 2.7143 times the length of the radius.

Example 2

Convert 78° to radians. Give your answer as an exact answer and also as a decimal rounded to the nearest ten-thousandth.

1. Determine which conversion factor to use.

   Since degrees need to be converted to radians, multiply by $\dfrac{\pi \text{ radians}}{180°}$ so that the degrees in the denominator will cancel out and radians will remain.

2. Multiply 78° by the conversion factor.

   Recall that π radians = 180°. Thus, $\dfrac{\pi \text{ radians}}{180°} = 1$, since dividing one quantity by its equivalent is equal to 1. Therefore, multiply 78° by the chosen conversion factor in order to convert the degree measure to its radian equivalent.

   $78° \bullet \dfrac{\pi \text{ radians}}{180°}$ Multiply by the conversion factor, $\dfrac{\pi \text{ radians}}{180°}$.

   $\dfrac{78\pi}{180}$ Multiply, and cancel out the degree symbols.

   $\dfrac{13\pi}{30}$ Reduce the fraction.

   Converted to radians, the exact measure of 78° is $\dfrac{13\pi}{30}$ radians.

3. Use your calculator to find the measure of the angle as a decimal.

   Multiply 13 by π and divide by 30.

   $\dfrac{13\pi}{30} \approx 1.3614$

   Converted to radians, the decimal measure of 78° is approximately 1.3614 radians.

   Recall that radians compare the value of the arc length to the value of the radius; therefore, for this 78° angle, the arc length is approximately 1.3614 times the length of the radius.

Example 3

Convert $\dfrac{2\pi}{3}$ radians to degrees.

1. Determine which conversion factor to use.

   Since radians need to be converted to degrees, multiply by $\dfrac{180°}{\pi \text{ radians}}$ so that the radians in the denominator will cancel out and degrees will remain.

2. Multiply $\dfrac{2\pi}{3}$ radians by the conversion factor.

   $\dfrac{2\pi \text{ radians}}{3} \bullet \dfrac{180°}{\pi \text{ radians}}$ Multiply by the conversion factor, $\dfrac{180°}{\pi \text{ radians}}$.

   $\dfrac{360\pi°}{3\pi}$ Multiply the numerators and denominators, canceling out the radians.

   $120°$ Reduce the fraction, canceling out π.

   Converted to degrees, the measure of $\dfrac{2\pi}{3}$ radians is 120°.

Example 4

Convert 0.5793 radian to degrees. Round your answer to the nearest tenth.

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1. Determine which conversion factor to use.

   Since radians need to be converted to degrees, multiply by $\dfrac{180°}{\pi \text{ radians}}$.

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2. Multiply 0.5793 radian by the conversion factor.

   $0.5793 \text{ radian} \cdot \dfrac{180°}{\pi \text{ radians}}$ Multiply by the conversion factor, $\dfrac{180°}{\pi \text{ radians}}$.

   $\dfrac{180°(0.5793)}{\pi}$ Cancel out the radians.

   $\approx 33.2°$ Use a calculator to simplify.

   Converted to degrees, 0.5793 radian is approximately 33.2°.

UNIT 3 • TRIGONOMETRY OF GENERAL TRIANGLES AND TRIGONOMETRIC FUNCTIONS
Lesson 1: Radians and the Unit Circle

Practice 3.1.1: Radians

For problems 1–3, use the information in the diagrams to find the angle measure of θ in radians. Round your answer to the nearest ten-thousandth, if necessary.

1.
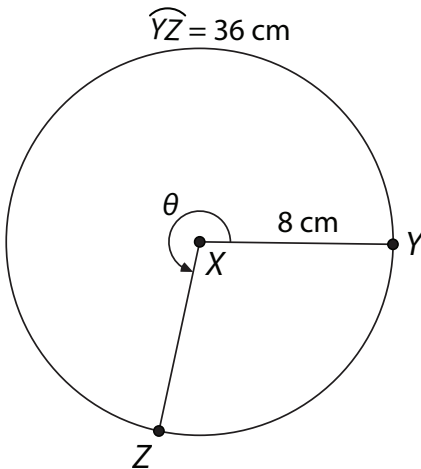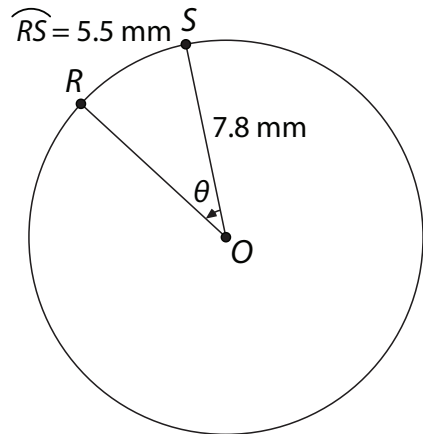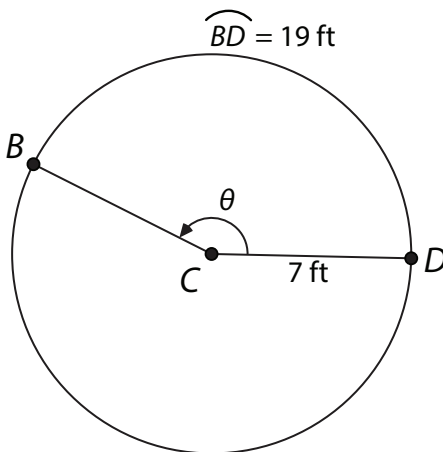
$\overset{\frown}{YZ} = 36$ cm

8 cm

θ

X

Y

Z

3.

$\overset{\frown}{RS} = 5.5$ mm

S

R

7.8 mm

θ

O

2.

$\overset{\frown}{BD} = 19$ ft

B

θ

C 7 ft D

continued

UNIT 3 • TRIGONOMETRY OF GENERAL TRIANGLES AND TRIGONOMETRIC FUNCTIONS
Lesson 1: Radians and the Unit Circle

For problems 4–7, convert each radian measure to degrees. Round your answer to the nearest tenth, if necessary.

4. $\dfrac{13\pi}{9}$ radians

6. 5.2009 radians

5. $\dfrac{5\pi}{7}$ radians

7. 0.7384 radian

Read the following scenario, and use the information in it to complete problems 8–10.

Stephen, Ali, and Easton are taking turns spinning the merry-go-round at the park. They each spin the carousel at a different speed in degrees per second. How fast is each boy's spin in radians per second? Supply an exact answer and also a decimal approximation rounded to the nearest ten-thousandth.

8. Stephen's spin speed: 185° per second

9. Ali's spin speed: 93° per second

10. Easton's spin speed: 122° per second

Lesson 3.1.2: The Unit Circle

Introduction

A **unit circle** is a circle that has a radius of 1 unit, with the center of the circle located at the origin of the coordinate plane. Because $r = 1$ in the unit circle, it can be a useful tool for discussing arc lengths and angles in circles. An angle in a unit circle can be studied in radians or degrees; however, since radians directly relate an angle measure to an arc length, radian measures are more useful in calculations.

Key Concepts

- Angles are typically in **standard position** on a unit circle. This means that the center of the circle is placed at the origin of the coordinate plane, and the vertex of the angle is on the origin at the center of the circle. The **initial side** of the angle (the stationary ray from which the measurement of the angle starts) is located along the positive x-axis. The **terminal side** (the movable ray that determines the measure of the angle) may be in any location.

An angle in standard position on the unit circle
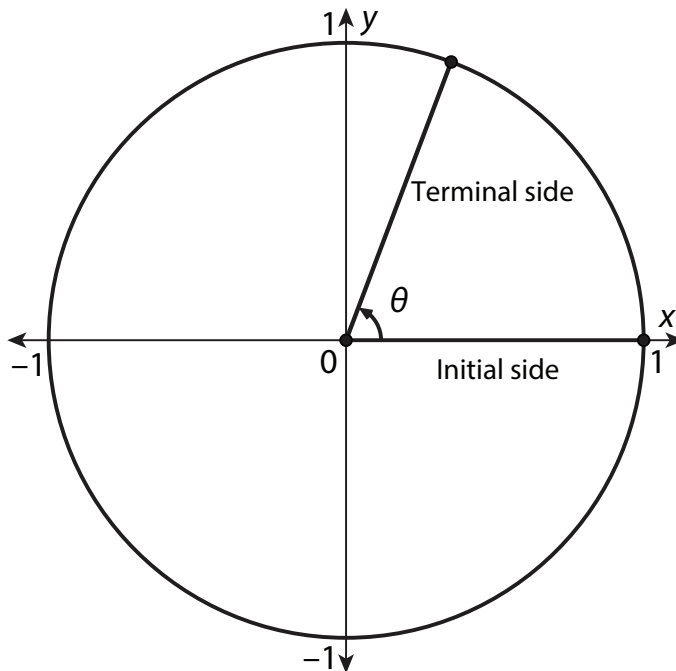
- The terminal side of the angle may be rotated counterclockwise to create a positive angle or clockwise to create a negative angle.

- To sketch an angle in radians on the unit circle, remember that halfway around the circle (180°) is equal to π radians and that a full rotation (360°) is equal to 2π radians. Then use the fraction of π to estimate the angle's location, if it falls somewhere between these measures.

- Within the unit circle, each angle has a **reference angle**. The reference angle is always the acute angle that the terminal side makes with the *x*-axis. The reference angle's sine, cosine, and tangent are the same as that of the original angle except for the sign, which is based on the quadrant in which the terminal side is located.

- Recall that a right triangle has one right angle and two acute angles (less than 90°). Sine, cosine, and tangent are trigonometric functions of an acute angle θ in a right triangle and are determined by the ratios of the lengths of the opposite side, adjacent side, and the hypotenuse of that triangle, summarized as follows.

  - The **sine** of $\theta = \sin \theta = \dfrac{\text{length of opposite side}}{\text{length of hypotenuse}}$.

  - The **cosine** of $\theta = \cos \theta = \dfrac{\text{length of adjacent side}}{\text{length of hypotenuse}}$.

  - The **tangent** of $\theta = \tan \theta = \dfrac{\text{length of opposite side}}{\text{length of adjacent side}}$.

- To find a reference angle, first sketch the original angle to determine which quadrant it lies in. Then, determine the measure of the angle between the terminal side and the *x*-axis. The following table shows the relationships between the reference angle and the original angle (θ) for each quadrant.

Quadrant	Reference angle (degrees)	Reference angle (radians)
I	same as θ	same as θ
II	$180° - \theta$	π radians $- \theta$
III	$\theta - 180°$	$\theta - \pi$ radians
IV	$360° - \theta$	2π radians $- \theta$

- If an angle is larger than 2π radians (360°), subtract a full rotation (2π radians or 360°) until the angle is less than 2π radians (360°). Then, find the reference angle of the resulting angle.

- The coordinates of the point at which the terminal side intersects the unit circle are always given by ($\cos \theta$, $\sin \theta$), where θ is the measure of the angle.

Guided Practice 3.1.2

Example 1

On a unit circle, sketch angles that measure $\dfrac{2\pi}{3}$ radians, $\dfrac{\pi}{4}$ radian, and $\dfrac{9\pi}{7}$ radians.

1. Sketch a unit circle, and then label π radians and 2π radians.

   A half rotation (180°) is π radians and a full rotation (360°) is 2π radians. Notice that 0 radians and 2π radians are in the same location on the unit circle, but represent different angle measures.
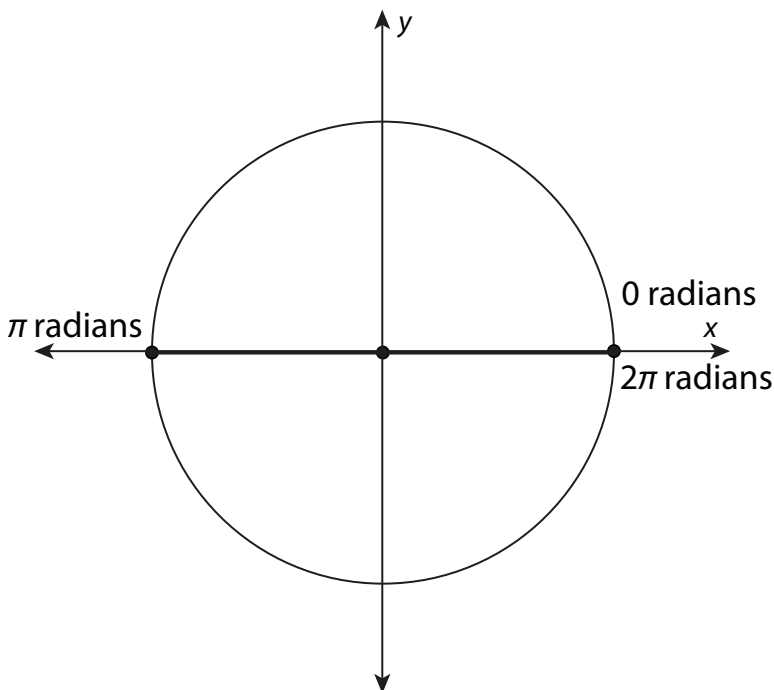
2. Sketch $\dfrac{2\pi}{3}$ radians.

$\dfrac{2\pi}{3}$ is the same as $\dfrac{2}{3}\pi$. In other words, the terminal side is $\dfrac{2}{3}$ of the way between 0 and π. Thus, imagine the semicircle between 0 radians and π radians split into thirds, and then sketch the angle $\dfrac{2}{3}$ of the way around the semicircle.
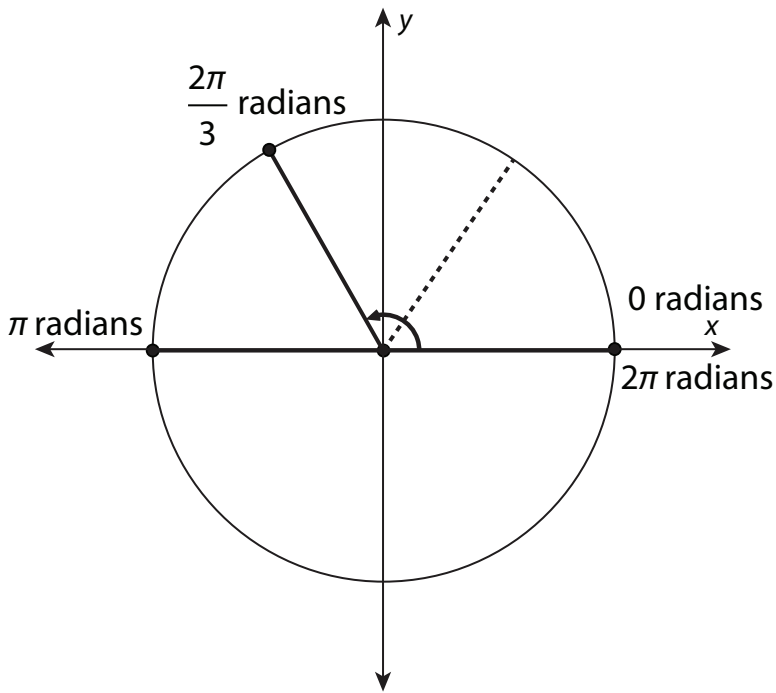
3. Sketch $\dfrac{\pi}{4}$ radian.

$\dfrac{\pi}{4}$ is the same as $\dfrac{1}{4}\pi$. In other words, it is $\dfrac{1}{4}$ of the way to π.
Thus, imagine the semicircle between 0 radians and π radians split
into fourths, and then sketch the angle $\dfrac{1}{4}$ of the way around
the semicircle.
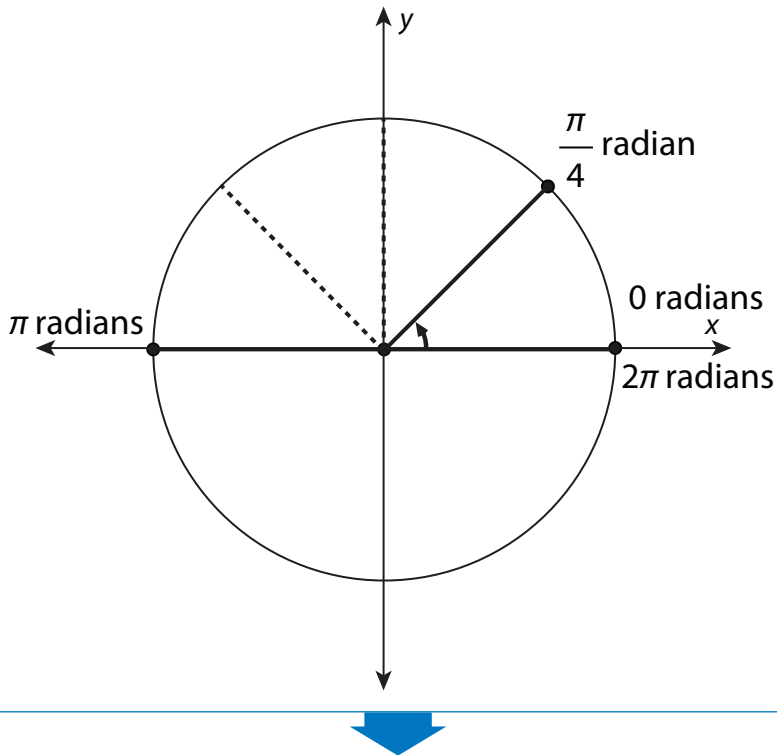
4. Sketch $\dfrac{9\pi}{7}$ radians.

$\dfrac{9\pi}{7}$ is the same as $\dfrac{9}{7}\pi$, which is equal to $1\dfrac{2}{7}\pi$.

Because this value is greater than π, it goes beyond π radians. It is $\dfrac{2}{7}$ of the way past π.

Imagine the semicircle between π radians and 2π radians split into sevenths, and then sketch the angle $\dfrac{2}{7}$ of the way around the semicircle.
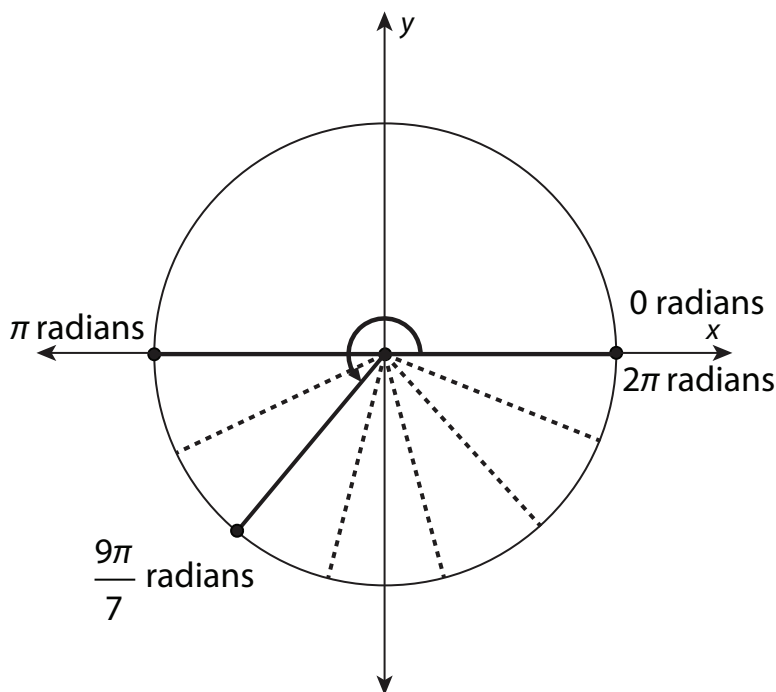
5. Summarize your findings.

The diagram shows the final unit circle with angles that measure $\frac{2\pi}{3}$ radians, $\frac{\pi}{4}$ radian, and $\frac{9\pi}{7}$ radians.
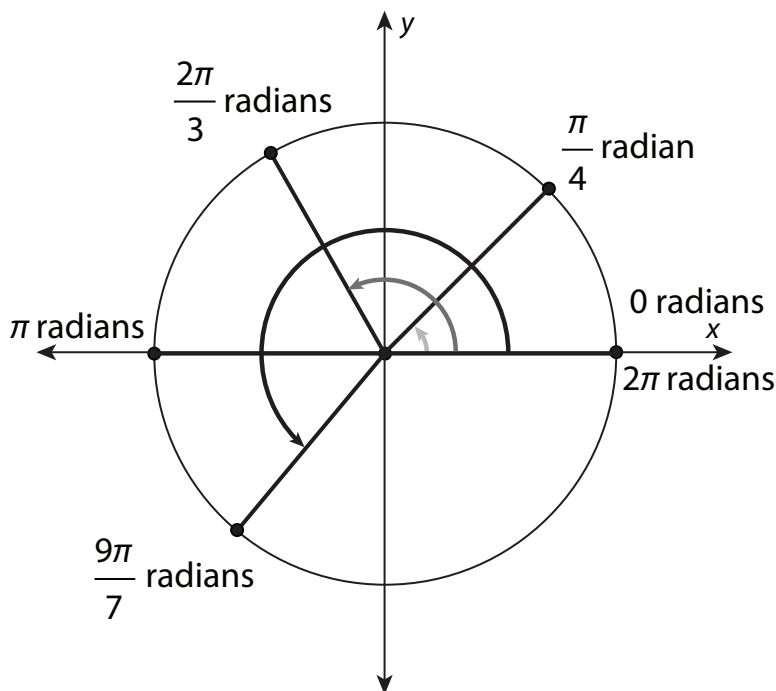
Example 2

Find the reference angles for angles that measure $\dfrac{11\pi}{9}$ radians, $\dfrac{3\pi}{5}$ radians, and 5.895 radians.

1. Sketch an angle with a measure of $\dfrac{11\pi}{9}$ radians on the unit circle.

   $\dfrac{11\pi}{9}$ radians is the same as $1\dfrac{2}{9}\pi$ radians; therefore, this angle will be $\dfrac{2}{9}$ of the way between π radians and 2π radians.
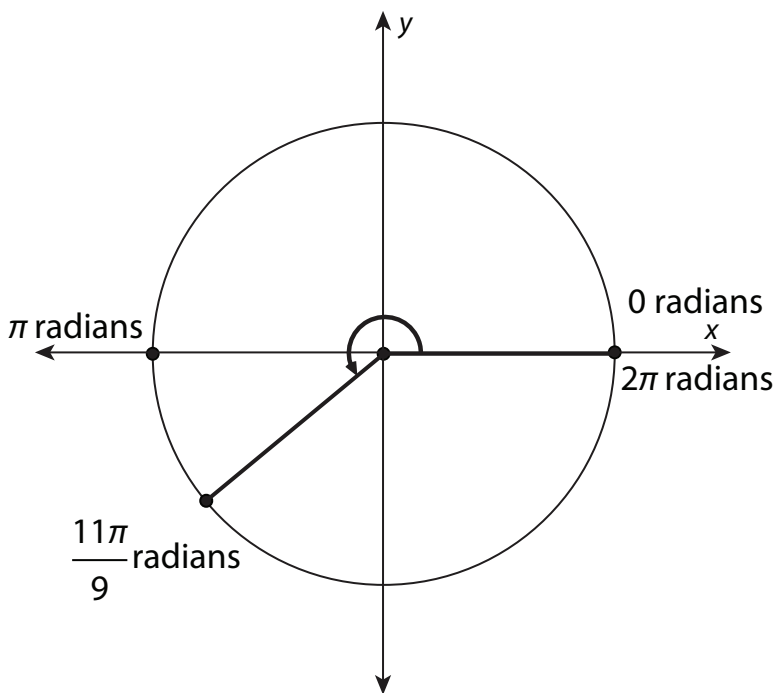
2. Determine the measure of the angle between the terminal side and the *x*-axis.

Since the terminal side falls in Quadrant III, subtract π radians from the original angle measure, $\dfrac{11\pi}{9}$ radians, to find the measure of the reference angle.

$\dfrac{11\pi}{9} - \pi$ Subtract π from the original angle measure.

$= \dfrac{11\pi}{9} - \dfrac{9\pi}{9}$ Rewrite π as a fraction with a common denominator.

$= \dfrac{2\pi}{9}$ Subtract.

The reference angle for $\dfrac{11\pi}{9}$ radians is $\dfrac{2\pi}{9}$ radians.

3. Sketch $\dfrac{3\pi}{5}$ radians.

Sketch $\dfrac{3\pi}{5}$ radians $\dfrac{3}{5}$ of the way between 0 and π radians.
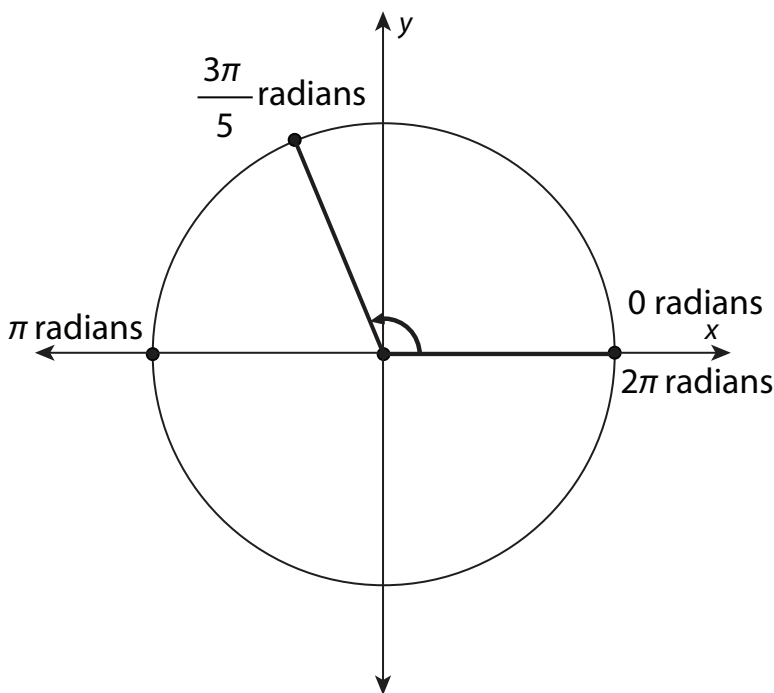
4. Determine the measure of the angle between the terminal side and the *x*-axis.

    Since the terminal side falls in Quadrant II, subtract $\dfrac{3\pi}{5}$ radians from π radians to find the measure of the reference angle.

    $$\pi - \dfrac{3\pi}{5}$$ Subtract the original angle measure from π.

    $$= \dfrac{5\pi}{5} - \dfrac{3\pi}{5}$$ Rewrite π as a fraction with a common denominator.

    $$= \dfrac{2\pi}{5}$$ Subtract.

    The reference angle for $\dfrac{3\pi}{5}$ radians is $\dfrac{2\pi}{5}$ radians.

5. Sketch 5.895 radians.

    Since π is not included in this measurement, we must use decimal approximations. π is approximately 3.14 and 2π is approximately 6.28.

    5.895 is fairly close to 6.28 and thus will fall in Quadrant IV.
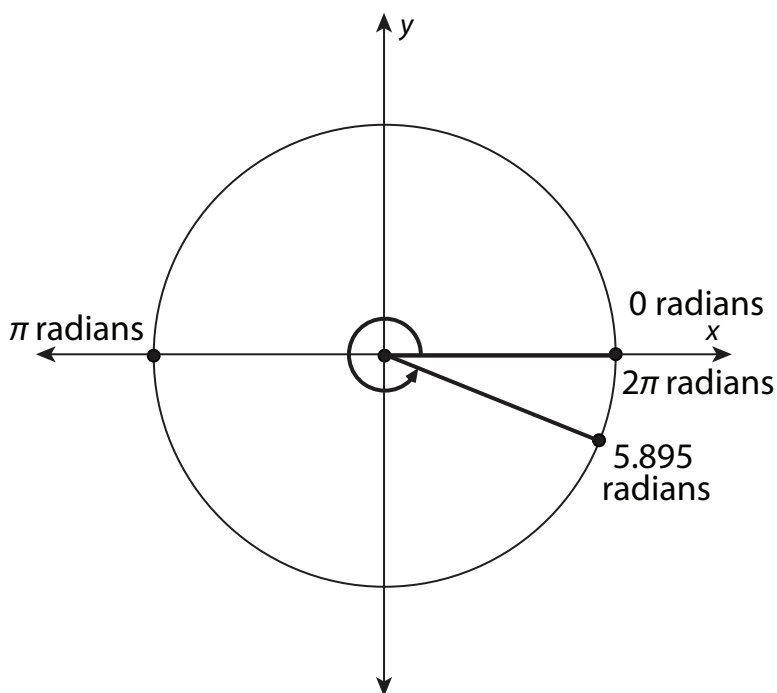
6. Determine the measure of the angle between the terminal side and the *x*-axis.

    Since the terminal side falls in Quadrant IV, subtract 5.895 radians from 2π radians to find a more precise measure of the reference angle.

    $2\pi - 5.895 \approx 0.388$

    The measure of the reference angle for 5.895 radians is approximately 0.388 radian.

Example 3

Use the following diagram of an angle in the unit circle to demonstrate why the point where the terminal side intersects the unit circle is (cos θ, sin θ).
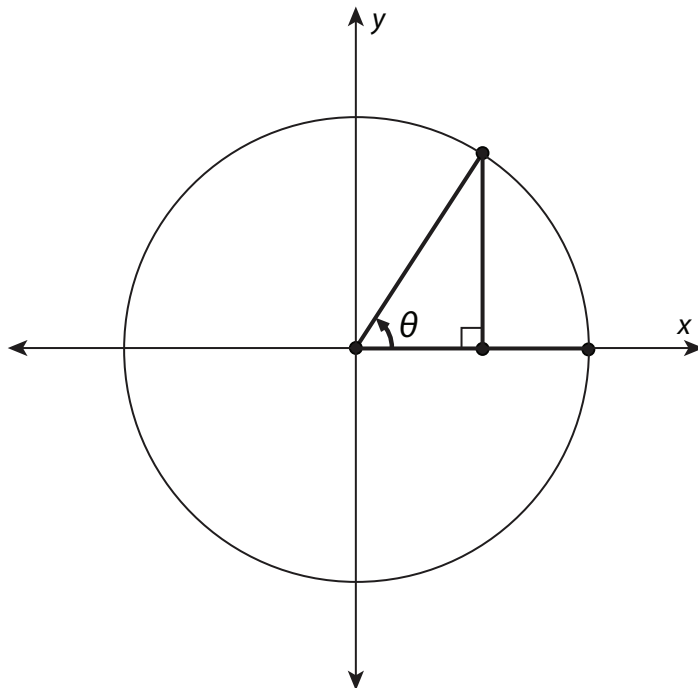
1. Label the three sides of the triangle.

The hypotenuse of the triangle is also the radius of the circle. Since it is a unit circle, the radius is 1.

Label the opposite and adjacent sides.
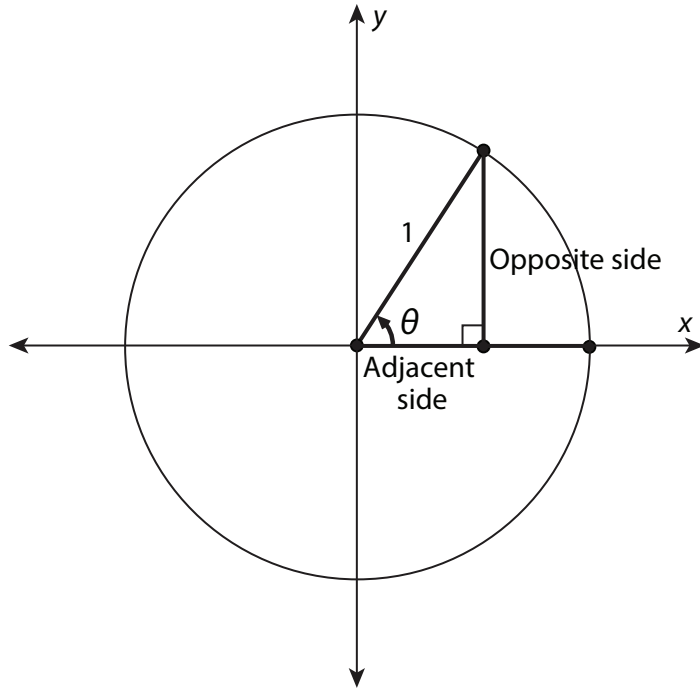
2. Use the cosine ratio to write a statement for the length of the adjacent side.

$$\cos\theta = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}} \qquad \text{Cosine ratio}$$

$$\cos\theta = \frac{\text{length of adjacent side}}{(1)} \qquad \text{Substitute 1 for the length of the hypotenuse.}$$

$$\cos\theta = \text{length of adjacent side} \qquad \text{Divide by 1.}$$

The length of the adjacent side is equal to cos θ.

3. Use the sine ratio to write a statement for the length of the opposite side.

$$\sin\theta = \frac{\text{length of opposite side}}{\text{length of hypotenuse}}$$ Sine ratio

$$\sin\theta = \frac{\text{length of opposite side}}{(1)}$$ Substitute 1 for the length of the hypotenuse.

$$\sin\theta = \text{length of opposite side}$$ Divide by 1.

The length of the opposite side is equal to sin θ.

4. Label the diagram to show the coordinates of the point of intersection of the terminal side and the unit circle.
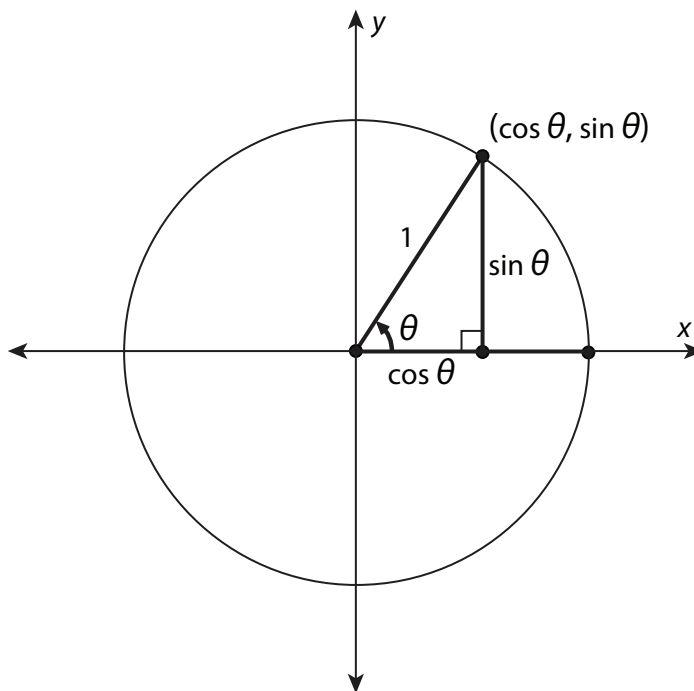
The coordinates of the point where the terminal side intersects the unit circle are (cos θ, sin θ).

Example 4

Find the coordinates of the point where the terminal side intersects the unit circle. Round each coordinate to the nearest hundredth.
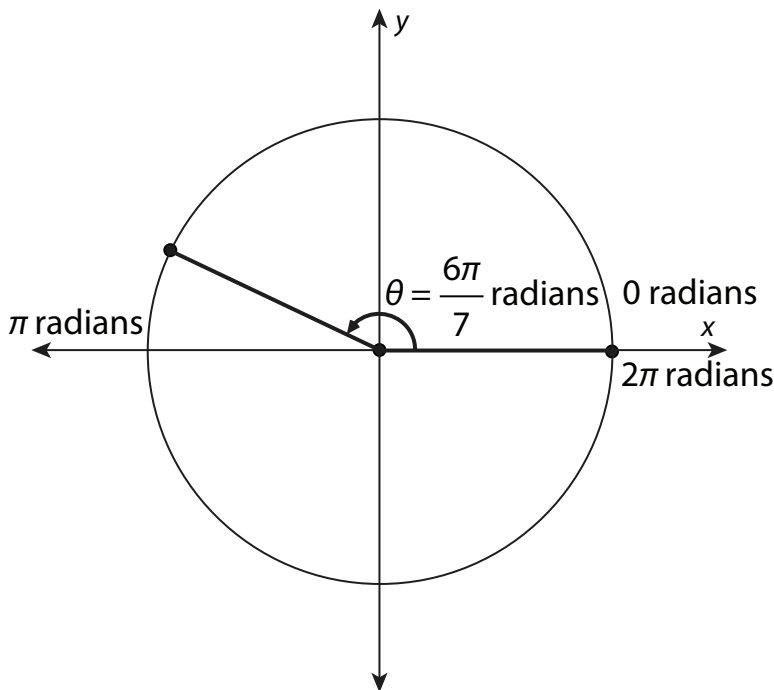
1. Find $\sin \theta$ and $\cos \theta$.

   Since the coordinates of the point where the terminal side intersects the unit circle are given by $(\cos \theta, \sin \theta)$, use this to determine the value of each coordinate.

   We are given in the diagram that $\theta = \dfrac{6\pi}{7}$ radians ; therefore,

   substitute $\dfrac{6\pi}{7}$ for θ: $\cos\theta = \cos\dfrac{6\pi}{7}$ and $\sin\theta = \sin\dfrac{6\pi}{7}$.

   Ensure your calculator is in radian mode, and then calculate $\cos\dfrac{6\pi}{7}$ and $\sin\dfrac{6\pi}{7}$.

   $\cos\dfrac{6\pi}{7} \approx -0.9010$

   $\sin\dfrac{6\pi}{7} \approx 0.4339$

2. Write the coordinates of the point of intersection of the terminal side and the unit circle.

The coordinates of the point of intersection are (cos θ, sin θ).

Substitute the rounded coordinates: –0.90 for cos θ and 0.43 for sin θ.

The approximate coordinates of the point of intersection of the terminal side are (–0.90, 0.43).

UNIT 3 • TRIGONOMETRY OF GENERAL TRIANGLES AND TRIGONOMETRIC FUNCTIONS
Lesson 1: Radians and the Unit Circle

Practice 3.1.2: The Unit Circle

For problems 1–3, sketch each radian measure on the unit circle.

1. $\dfrac{7\pi}{8}$ radians

2. $\dfrac{\pi}{5}$ radian

3. $\dfrac{11\pi}{6}$ radians

For problems 4–7, find the reference angle for each angle measure.

4. 218°

5. $\dfrac{7\pi}{4}$ radians

6. $\dfrac{2\pi}{7}$ radians

7. 2.871 radians

For problems 8–10, find the coordinates of the point where the terminal side of the angle intersects the unit circle. Round each coordinate to the nearest hundredth.

8. $\dfrac{\pi}{7}$ radian

9. $\dfrac{4\pi}{3}$ radians

10. 5.897 radians

Lesson 3.1.3: Special Angles in the Unit Circle

Introduction

Special angles exist within the unit circle. For these special angles, it is possible to calculate the exact coordinates for the point where the terminal side intersects the unit circle. The patterns of the 30°–60°–90° triangle and the 45°–45°–90° triangle can be used to find these points.

Key Concepts

- Recall the pattern for a 45°–45°–90° triangle:
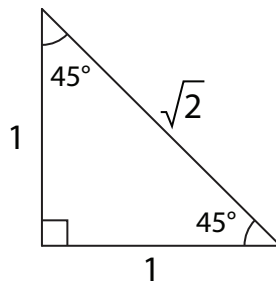
- Notice the two legs are the same length and the hypotenuse is equal to the length of a leg times $\sqrt{2}$.

- Recall the pattern for a 30°–60°–90° triangle:
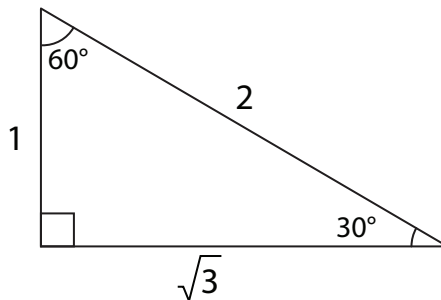
- Notice the hypotenuse is twice as long as the short leg, and the longer leg is equal to the length of the short leg times $\sqrt{3}$.

- Each special angle can be viewed in radians as well as degrees: $30° = \dfrac{\pi}{6}$ radian, $60° = \dfrac{\pi}{3}$ radians, $90° = \dfrac{\pi}{2}$ radians, and $45° = \dfrac{\pi}{4}$ radian.

- Thus, the patterns for the special triangles, noted in radians instead of degrees, are as follows:

45°–45°–90° triangle
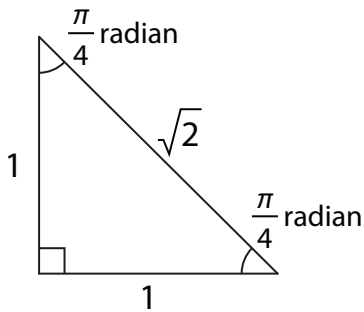
30°–60°–90° triangle
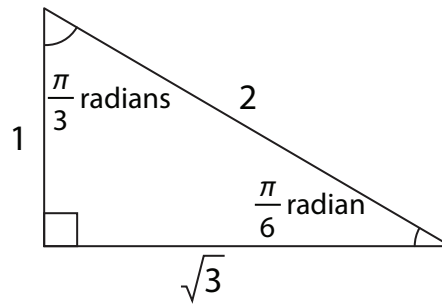
- The special angles continue around the unit circle and can be identified as all angles whose reference angles are special angles.
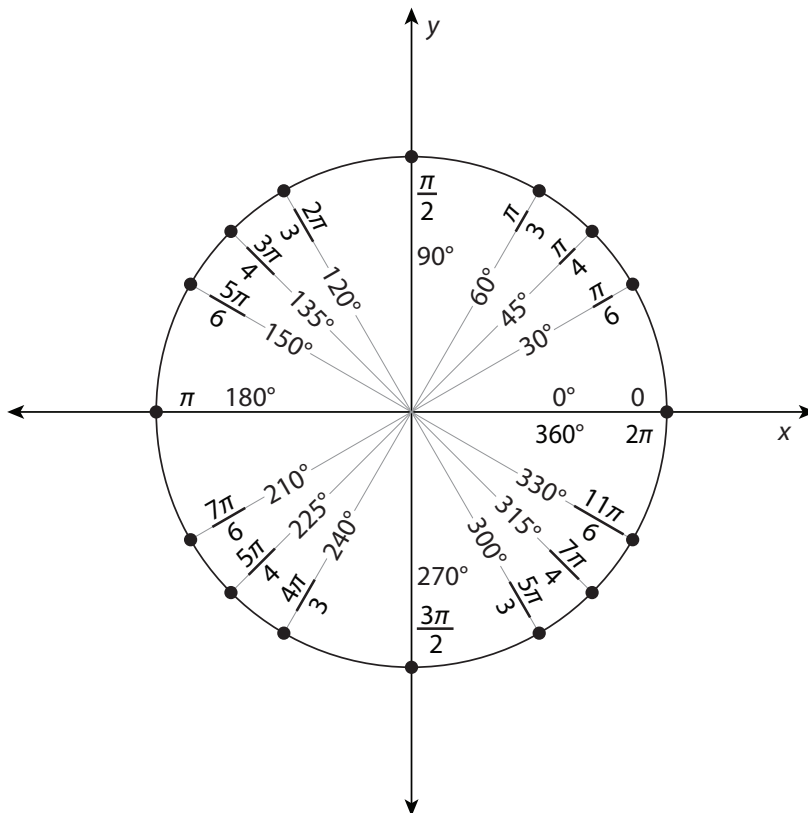
- To find the coordinates of the point where the terminal side of a special angle intersects the unit circle, first identify the reference angle. Then use the pattern to identify cos θ and sin θ. Recall that the coordinates of the point where the terminal side intersects the unit circle are always (cos θ, sin θ). However, since the reference angle was used to find cos θ and sin θ, remember to account for negative coordinates based on which quadrant the point is located in.

- The following illustration gives the coordinates of the points where the terminal side of each special angle intersects the unit circle. While these coordinates can be memorized, it is helpful to understand how to derive them for a given problem.
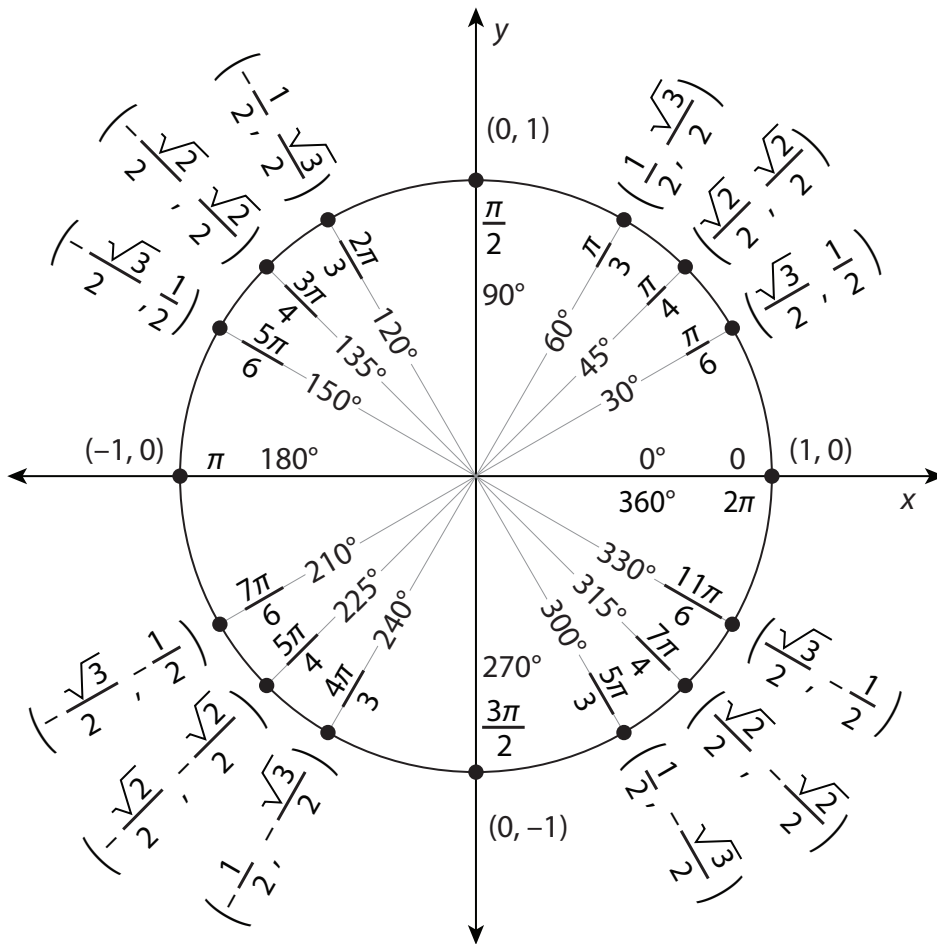
Guided Practice 3.1.3

Example 1

Find the coordinates of the point where the terminal side of a 330° angle intersects the unit circle.

1. Sketch the angle on the unit circle and identify the location of the terminal side.

   A 330° angle is close to a full rotation (360°).
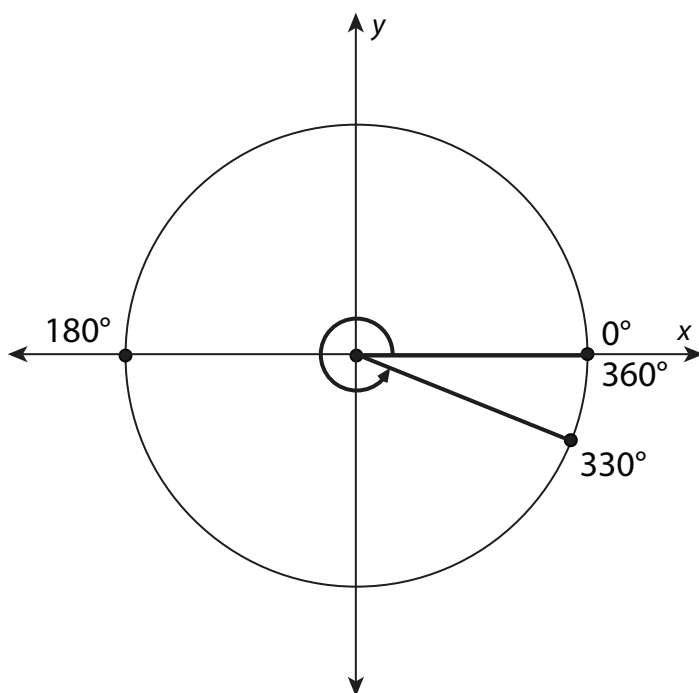
   The terminal side falls in Quadrant IV.

2. Identify the reference angle.

   The reference angle is the angle that the terminal side makes with the x-axis. Since the terminal side is located in Quadrant IV, subtract 330° from 360° to find the reference angle.

   $360 - 330 = 30$

   The reference angle for 330° is 30°.

3. Find the cosine and sine of the reference angle.

Remember the pattern for a 30°–60°–90° triangle:
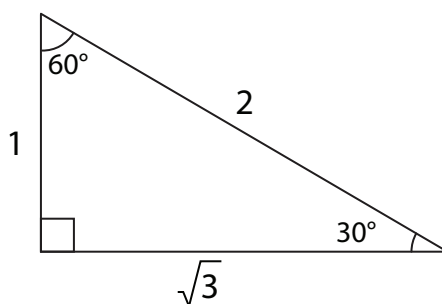
Use the ratios for sine and cosine, substituting in the values from the triangle.

$$\cos\theta = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$$ Cosine ratio

$$\cos(30°) = \frac{\left(\sqrt{3}\right)}{(2)}$$ Substitute $\sqrt{3}$ for the adjacent side, 2 for the hypotenuse, and 30° for θ.

The cosine of the reference angle is $\dfrac{\sqrt{3}}{2}$.

$$\sin\theta = \frac{\text{length of opposite side}}{\text{length of hypotenuse}}$$ Sine ratio

$$\sin(30°) = \frac{(1)}{(2)}$$ Substitute 1 for the opposite side, 2 for the hypotenuse, and 30° for θ.

The sine of the reference angle is $\dfrac{1}{2}$.

4. Determine the coordinates of the point where the terminal side intersects the unit circle.

The coordinates of the point where the terminal side intersects the unit circle are (cos θ, sin θ).

The sine and cosine of the reference angle are the same as the sine and cosine of the original angle except for the sign, which is based on the quadrant in which the terminal side is located.

Since the terminal side is in Quadrant IV, the x-coordinate (cos θ) must be positive and the y-coordinate (sin θ) must be negative.

Therefore, the coordinates of the point at which the terminal side intersects the unit circle are $\left(\dfrac{\sqrt{3}}{2}, -\dfrac{1}{2}\right)$.

Example 2

Find the coordinates of the point where the terminal side of an angle with a measure of $\dfrac{5\pi}{4}$ radians intersects the unit circle.

1. Sketch the angle on the unit circle and identify the location of the terminal side.

   $\dfrac{5\pi}{4}$ is the same as $1\dfrac{1}{4}\pi$ and thus is $\dfrac{1}{4}$ of the way between π radians and 2π radians.
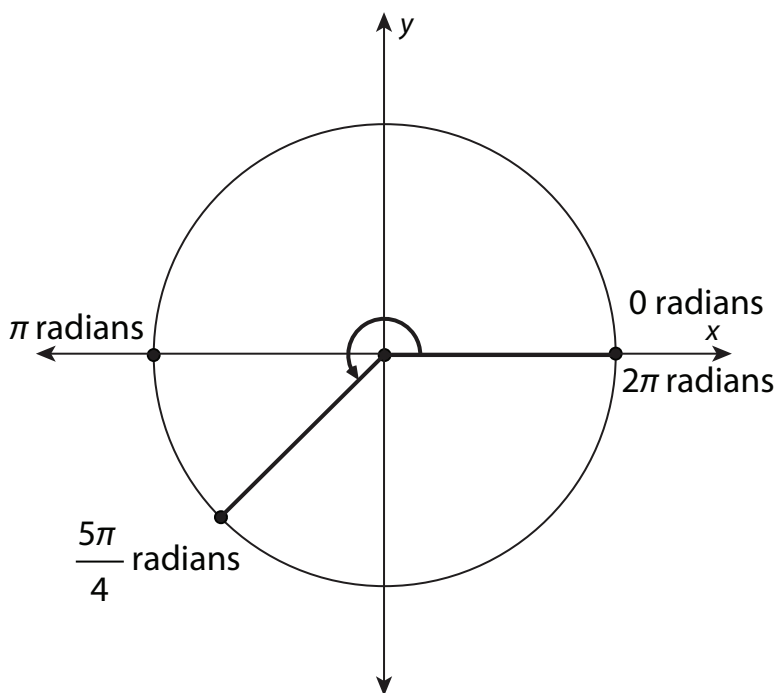
The terminal side falls in Quadrant III.

2. Identify the reference angle.

The reference angle is the angle that the terminal side makes with the *x*-axis. Since it is located in Quadrant III, subtract π radians from $\dfrac{5\pi}{4}$ radians to find the reference angle.

$\dfrac{5\pi}{4} - \pi$ Subtract π from the original angle measure.

$= \dfrac{5\pi}{4} - \dfrac{4\pi}{4}$ Rewrite π as a fraction over a common denominator.

$= \dfrac{\pi}{4}$ Subtract.

The reference angle for $\dfrac{5\pi}{4}$ radians is $\dfrac{\pi}{4}$ radian.

3. Find the cosine and sine of the reference angle.

$\dfrac{\pi}{4}$ radian is the same as 45°. Recall the pattern for a 45°–45°–90° triangle measured in radians:

45°–45°–90° triangle
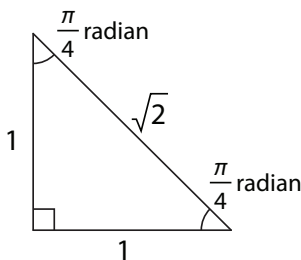
Substitute values from the triangle into the ratios for sine and cosine.

$\cos\theta = \dfrac{\text{length of adjacent side}}{\text{length of hypotenuse}}$ Cosine ratio

$\cos\left(\dfrac{\pi}{4}\right) = \dfrac{(1)}{(\sqrt{2})}$ Substitute 1 for the adjacent side, $\sqrt{2}$ for the hypotenuse, and $\dfrac{\pi}{4}$ radian for θ.

$\cos\dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2}$ Multiply the numerator and denominator by $\sqrt{2}$ to rationalize the denominator.

(continued)

The cosine of the reference angle is $\dfrac{\sqrt{2}}{2}$.

$\sin\theta = \dfrac{\text{length of opposite side}}{\text{length of hypotenuse}}$ 　　Sine ratio

$\sin\left(\dfrac{\pi}{4}\right) = \dfrac{(1)}{(\sqrt{2})}$ 　　Substitute 1 for the opposite side, $\sqrt{2}$ for the hypotenuse, and $\dfrac{\pi}{4}$ radian for θ.

$\sin\dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2}$ 　　Multiply the numerator and denominator by $\sqrt{2}$ to rationalize the denominator.

The sine of the reference angle is $\dfrac{\sqrt{2}}{2}$.

4. Determine the coordinates of the point where the terminal side intersects the unit circle.

The coordinates of the point where the terminal side intersects the unit circle are (cos θ, sin θ).

The sine and cosine of the reference angle are the same as the sine and cosine of the original angle except for the sign, which is based on the quadrant in which the terminal side is located.

Since the terminal side is in Quadrant III, both the *x*-coordinate (cos θ) and the *y*-coordinate (sin θ) must be negative.

Therefore, the coordinates of the point at which the terminal side intersects the unit circle are $\left(-\dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2}\right)$.

Example 3

Find the coordinates of the point where the terminal side of an angle with a measure of $\dfrac{3\pi}{2}$ radians intersects the unit circle.

1. Sketch the angle on the unit circle and identify the location of the terminal side.

   $\dfrac{3\pi}{2}$ is the same as $1\dfrac{1}{2}\pi$ and thus is $\dfrac{1}{2}$ of the way between π radians and 2π radians.
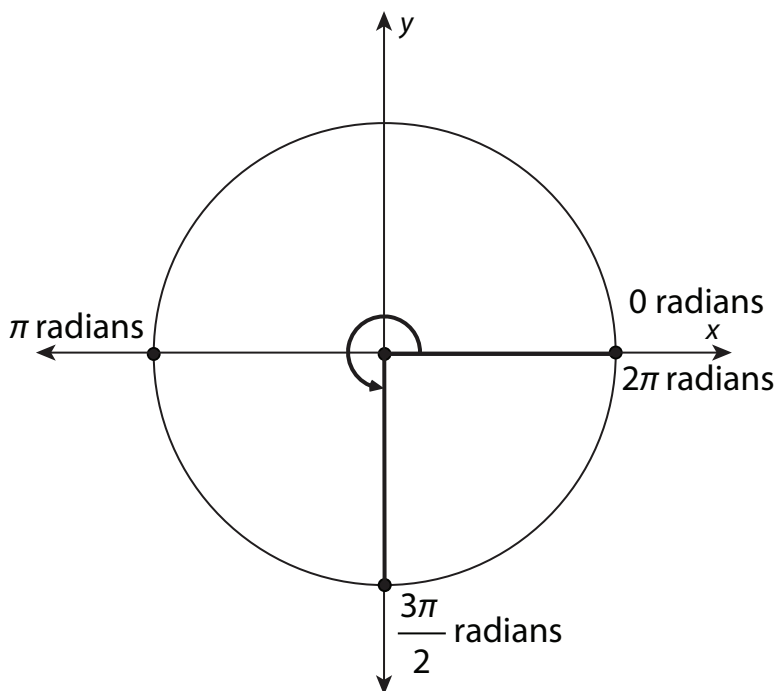
   The terminal side is located along the y-axis.

2. Determine the coordinates of the point where the terminal side intersects the unit circle.

   Since the point is located on the y-axis, the x-coordinate must be 0.

   Since the radius of the unit circle is 1, the y-coordinate must be -1.

   The coordinates of the point where the terminal side intersects the unit circle are $(0, -1)$.

Example 4

Sketch the three special angles that are located in Quadrant II. Label the coordinates of the points where their terminal sides intersect the unit circle. Use degrees.

1. Identify the special angles that are located in Quadrant II.

   The special angles of a unit circle are 30°, 45°, 60°, 90°, and their multiples.

   For the angle to fall in Quadrant II, its measure must be larger than 90° and smaller than 180°.

   The multiples of 30° (up to 180°) are 60°, 90°, 120°, 150°, and 180°. The only multiples of 30° that fall in Quadrant II are 120° and 150°.

   The multiples of 45° (up to 180°) are 90°, 135°, and 180°. The only one of these that falls in Quadrant II is 135°.

   The multiples of 60° and 90° are included in the multiples of 30°.

   Therefore, the special angles that are located in Quadrant II are 120°, 135°, and 150°.

2. Sketch 120°, 135°, and 150° angles on the unit circle.
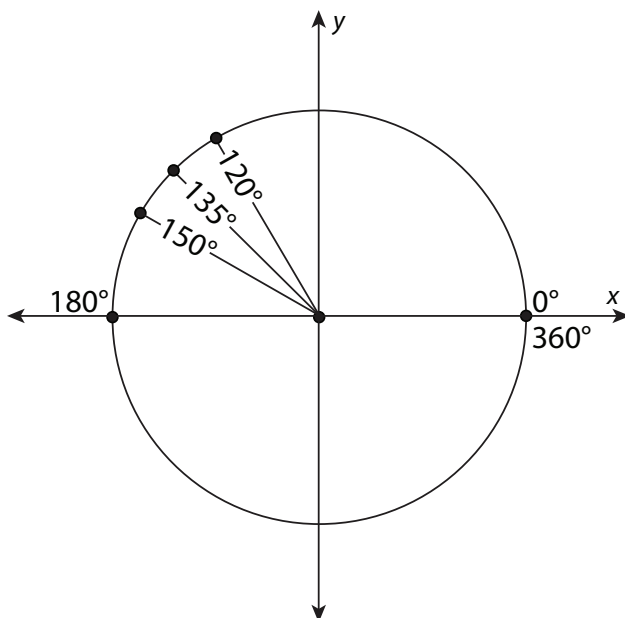
3. Identify the reference angles for the 120°, 135°, and 150° angles.

The reference angle is the angle that the terminal side makes with the *x*-axis. Since these angles are located in Quadrant II, subtract each original angle measure from 180° to find its reference angle.

$180 - 120 = 60$ The reference angle for 120° is 60°.

$180 - 135 = 45$ The reference angle for 135° is 45°.

$180 - 150 = 30$ The reference angle for 150° is 30°.

4. Find the cosine and sine of each reference angle.

Remember the patterns for a 30°–60°–90° triangle and a 45°–45°–90° triangle:
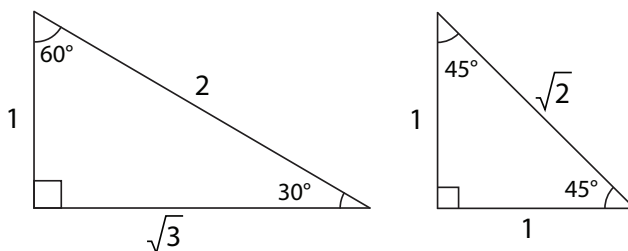
Use the ratios for cosine and sine, substituting in the values from the special right triangles for each angle measure.

Recall that the cosine ratio is $\cos\theta = \dfrac{\text{length of adjacent side}}{\text{length of hypotenuse}}$, and the sine ratio is $\sin\theta = \dfrac{\text{length of opposite side}}{\text{length of hypotenuse}}$.

For a 60° reference angle:

$$\cos 60° = \frac{1}{2} \qquad\qquad \sin 60° = \frac{\sqrt{3}}{2}$$

For a 45° reference angle:

$$\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

For a 30° reference angle:

$$\cos 30° = \frac{\sqrt{3}}{2} \qquad\qquad \sin 30° = \frac{1}{2}$$

5. Determine the coordinates of the point where each terminal side intersects the unit circle and label the coordinates on the sketch.

The coordinates of the point where the terminal side intersects the unit circle are $(\cos \theta, \sin \theta)$.

The sine and cosine of the reference angle are the same as the sine and cosine of the original angle except for the sign, which is based on the quadrant in which the terminal side is located.

Since the terminal sides are in Quadrant II, the x-coordinate ($\cos \theta$) must be negative and the y-coordinate ($\sin \theta$) must be positive.

The terminal side of the 120° angle (whose reference angle is 60°) intersects the unit circle at $\left(-\dfrac{1}{2}, \dfrac{\sqrt{3}}{2} \right)$.

The terminal side of the 135° angle (whose reference angle is 45°) intersects the unit circle at $\left(-\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2} \right)$.

The terminal side of the 150° angle (whose reference angle is 30°) intersects the unit circle at $\left(-\dfrac{\sqrt{3}}{2}, \dfrac{1}{2} \right)$.

Label these coordinates on the sketch.
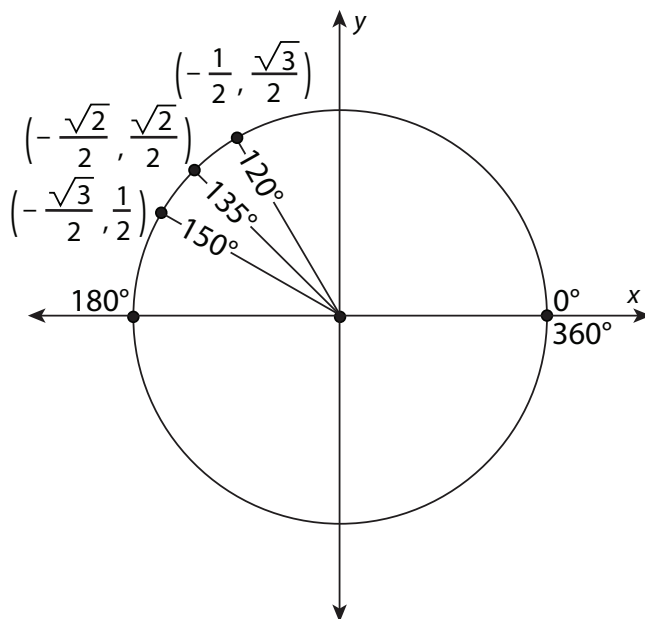
UNIT 3 • TRIGONOMETRY OF GENERAL TRIANGLES AND TRIGONOMETRIC FUNCTIONS
Lesson 1: Radians and the Unit Circle

Practice 3.1.3: Special Angles in the Unit Circle

For problems 1–9, find the coordinates of the point where the terminal side of the angle intersects the unit circle. Give exact answers.

1. 60°

2. 135°

3. 180°

4. 330°

5. $\dfrac{\pi}{6}$ radian

6. $\dfrac{3\pi}{4}$ radians

7. $\dfrac{3\pi}{2}$ radians

8. $\dfrac{5\pi}{3}$ radians

9. $\dfrac{11\pi}{6}$ radians

Use your knowledge of unit circles to complete problem 10.

10. Create a unit circle that contains all the special angles in radians. Label the terminal point of each angle with its coordinates.

Lesson 3.1.4: Evaluating Trigonometric Functions

Introduction

The six trigonometric functions (sine, cosine, tangent, cosecant, secant, and cotangent) can be used to find the length of the sides of a triangle or the measure of an angle if the length of two sides is given. Previously these functions could only be applied to angles up to 90°. However, by using radians and the unit circle, these functions can be applied to any angle.

Key Concepts

- Recall that sine is the ratio of the length of the opposite side to the length of the hypotenuse, cosine is the ratio of the length of the adjacent side to the length of the hypotenuse, and tangent is the ratio of the length of the opposite side to the length of the adjacent side. (You may have used the mnemonic device SOHCAHTOA to help remember these relationships: Sine equals the Opposite side over the Hypotenuse, Cosine equals the Adjacent side over the Hypotenuse, and Tangent equals the Opposite side over the Adjacent side.)

- Three other trigonometric functions, cosecant, secant, and cotangent, are reciprocal functions of the first three. **Cosecant** is the reciprocal of the sine function, **secant** is the reciprocal of the cosine function, and **cotangent** is the reciprocal of the tangent function.

    - The cosecant of $\theta = \csc\theta = \dfrac{\text{length of hypotenuse}}{\text{length of opposite side}}$; $\csc\theta = \dfrac{1}{\sin\theta}$

    - The secant of $\theta = \sec\theta = \dfrac{\text{length of hypotenuse}}{\text{length of adjacent side}}$; $\sec\theta = \dfrac{1}{\cos\theta}$

    - The cotangent of $\theta = \cot\theta = \dfrac{\text{length of adjacent side}}{\text{length of opposite side}}$; $\cot\theta = \dfrac{1}{\tan\theta}$

- The quadrant in which the terminal side is located determines the sign of the trigonometric functions. In Quadrant I, all the trigonometric functions are positive. In Quadrant II, the sine and its reciprocal, the cosecant, are positive and all the other functions are negative. In Quadrant III, the tangent and its reciprocal, the cotangent, are positive, and all other functions are negative. In Quadrant IV, the cosine and its reciprocal, the secant, are positive, and all other functions are negative.

- You can use a mnemonic device to remember in which quadrants the functions are positive: All Students Take Calculus (ASTC).
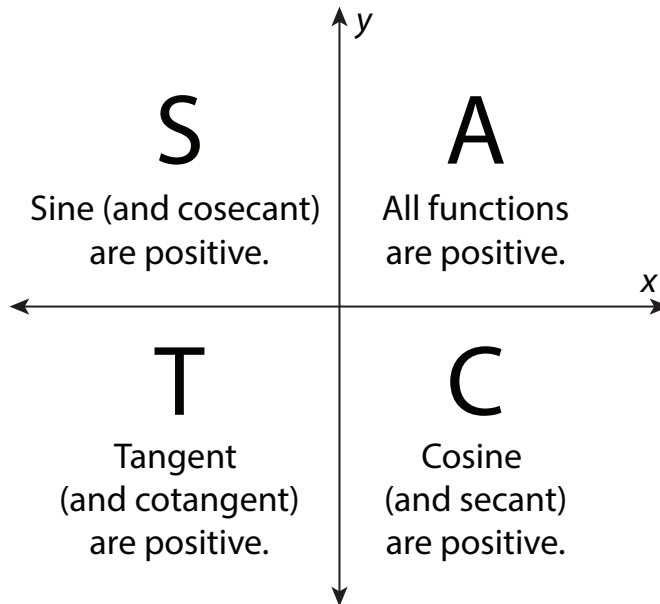
- However, instead of memorizing this, you can also think it through each time, considering whether the opposite and adjacent sides of the reference angle are positive or negative in each quadrant.

- To find a trigonometric function of an angle given a point on its terminal side, first visualize a triangle using the reference angle. The x-coordinate becomes the length of the adjacent side and the y-coordinate becomes the length of the opposite side. The length of the hypotenuse can be found using the Pythagorean Theorem. Determine the sign by remembering the ASTC pattern or by considering the signs of the x- and y-coordinates.

- To find the trigonometric functions of special angles, first find the reference angle and then use the pattern to determine the ratio.

- For angles larger than 2π radians (360°), subtract 2π radians (360°) to find a **coterminal angle**, an angle that shares the same terminal side, that is less than 2π radians (360°). Repeat if necessary.

- For negative angles, find the reference angle and then apply the same method.

Guided Practice 3.1.4

Example 1

What is the sign of each trigonometric ratio for an angle with a measure of $\dfrac{9\pi}{8}$ radians?

1. Sketch the angle to determine in which quadrant it is located.

   $\dfrac{9\pi}{8}$ radians is the same as $1\dfrac{1}{8}\pi$ radians, so the terminal side falls $\dfrac{1}{8}$ of the way between π radians and 2π radians.
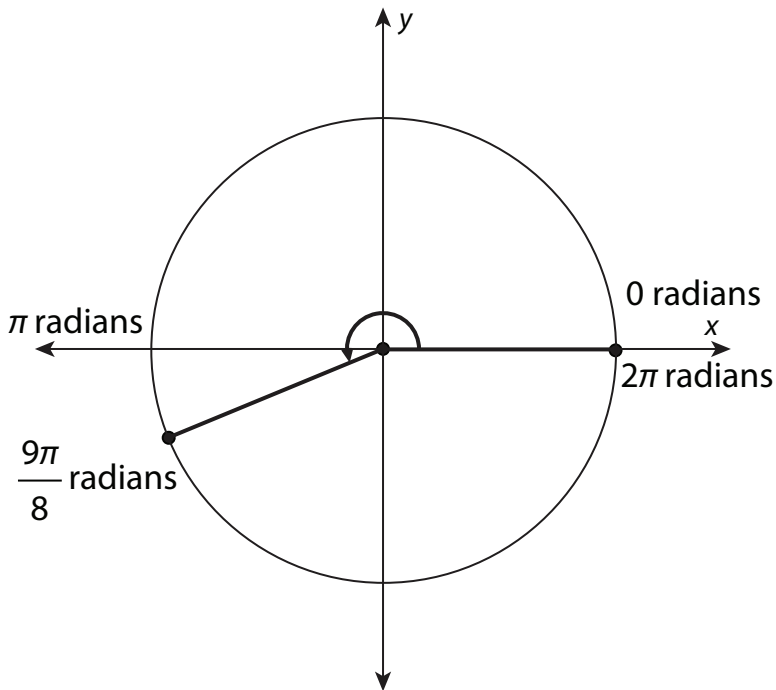
2. Determine the signs of the lengths of the opposite side, adjacent side, and hypotenuse for the reference angle.

The reference angle is the angle the terminal side makes with the *x*-axis. Draw the triangle associated with this reference angle and label its sides.
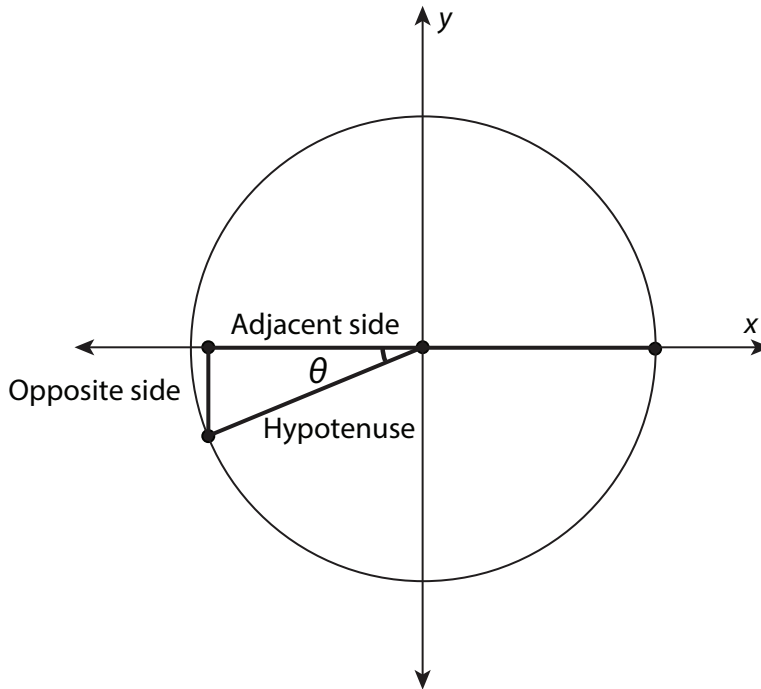
The adjacent side along the *x*-axis is negative since *x* is negative to the left of the origin. The opposite side (which corresponds to the *y*-coordinate of the terminal side) is also negative since *y* is negative below the origin. The hypotenuse is positive since it is the length of the radius.

3. Use the definitions of the trigonometric functions to determine the sign of each and check the results by using the acronym ASTC.

    Organize the information in a table to better see the relationships among the functions.

Trigonometric function	Description	Result
$\sin\theta = \dfrac{\text{length of opposite side}}{\text{length of hypotenuse}}$	For the reference angle, the sign of length of the opposite side is negative and the sign of the length of the hypotenuse is positive.	Negative
$\cos\theta = \dfrac{\text{length of adjacent side}}{\text{length of hypotenuse}}$	For the reference angle, the length of the adjacent side is negative and the length of the hypotenuse is positive.	Negative
$\tan\theta = \dfrac{\text{length of opposite side}}{\text{length of adjacent side}}$	For the reference angle, the length of the opposite side is negative and the length of the adjacent side is negative.	Positive
$\csc\theta = \dfrac{\text{length of hypotenuse}}{\text{length of opposite side}}$	Cosecant is the reciprocal of sine. Since the sine is negative, its reciprocal is also negative.	Negative
$\sec\theta = \dfrac{\text{length of hypotenuse}}{\text{length of adjacent side}}$	Secant is the reciprocal of cosine. Since the cosine is negative, its reciprocal is also negative.	Negative
$\cot\theta = \dfrac{\text{length of adjacent side}}{\text{length of opposite side}}$	Cotangent is the reciprocal of tangent. Since the tangent is positive, its reciprocal is also positive.	Positive

(continued)

Recall the diagram from the Key Concepts.

S	**A**
Sine (and cosecant) are positive.	All functions are positive.
T	**C**
Tangent (and cotangent) are positive.	Cosine (and secant) are positive.
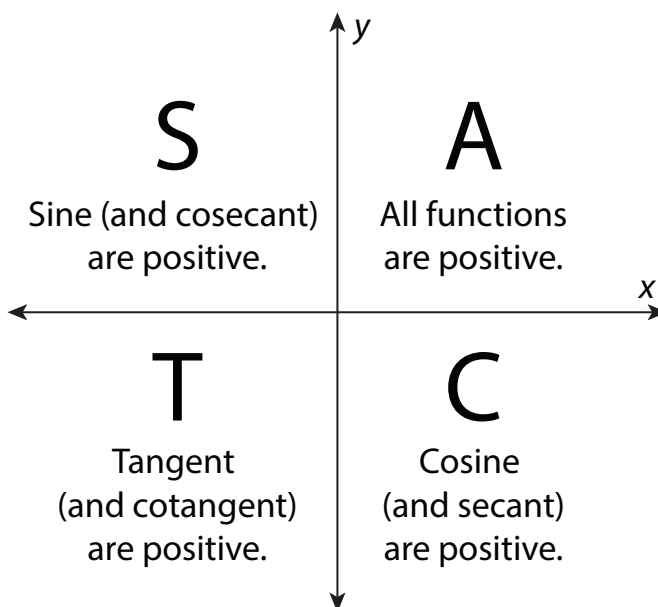
In Quadrant III, where the terminal side of the angle is located, only the tangent and cotangent are positive. This matches the results previously found.

Therefore, for an angle with a measure of $\dfrac{9\pi}{8}$ radians, the tangent and cotangent are positive, and all other trigonometric functions are negative.

Example 2

Find sin θ if θ is a positive angle in standard position with a terminal side that passes through the point (5, –2). Give an exact answer.

1. Sketch the angle and draw in the triangle associated with the reference angle.

   Recall that a positive angle is created by rotating counterclockwise around the origin of the coordinate plane.

   Plot (5, –2) on a coordinate plane and draw the terminal side extending from the origin through that point.

   The reference angle is the angle the terminal side makes with the *x*-axis.
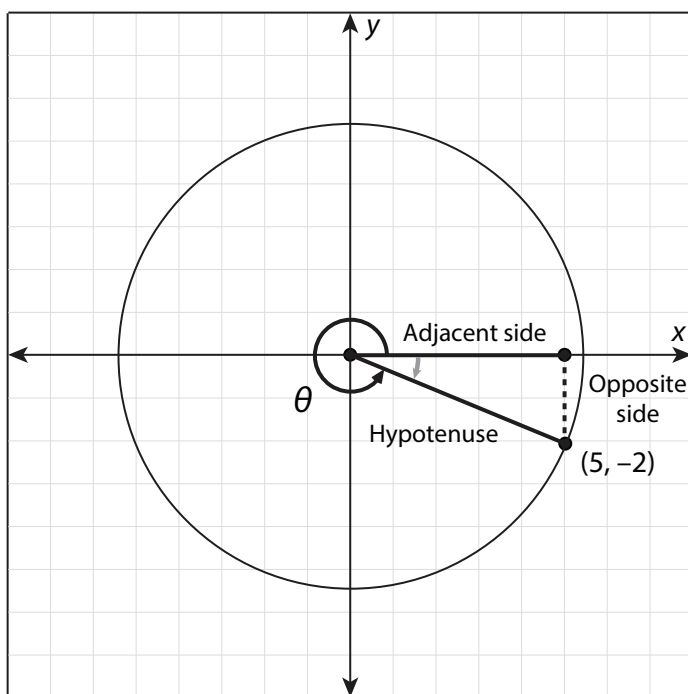
   Notice that θ is nearly 360°, so the reference angle is in the fourth quadrant.

   The magnitude of the *x*-coordinate is the length of the adjacent side and the magnitude of the *y*-coordinate is the length of the opposite side. The hypotenuse can be found using the Pythagorean Theorem. Determine the sign of sin θ by recalling the ASTC pattern or by considering the signs of the *x*- and *y*-coordinates.

2. Find the length of the opposite side and the length of the hypotenuse.

Sine is the ratio of the length of the opposite side to the length of the hypotenuse; therefore, these two lengths must be determined.

The length of the opposite side is the magnitude of the y-coordinate, 2.

Since the opposite side length is known to be 2 and the adjacent side length, 5, can be determined from the sketch, the hypotenuse can be found by using the Pythagorean Theorem.

$c^2 = a^2 + b^2$	Pythagorean Theorem
$c^2 = (2)^2 + (5)^2$	Substitute 2 for a and 5 for b.
$c^2 = 4 + 25$	Simplify the exponents.
$c^2 = 29$	Add.
$c = \sqrt{29}$	Take the square root of both sides.

The length of the hypotenuse is $\sqrt{29}$ units.

3. Find $\sin \theta$.

Now that the lengths of the opposite side and the hypotenuse are known, substitute these values into the sine ratio to determine $\sin \theta$.

$\sin\theta = \dfrac{\text{length of opposite side}}{\text{length of hypotenuse}}$	Sine ratio
$\sin\theta = \dfrac{(2)}{\left(\sqrt{29}\right)}$	Substitute 2 for the opposite side and $\sqrt{29}$ for the hypotenuse.
$\sin\theta = \dfrac{2\sqrt{29}}{29}$	Rationalize the denominator.

According to ASTC, in Quadrant IV only the cosine and secant are positive. The sine is negative.

For a positive angle θ in standard position with a terminal side that passes through the point (5, –2), $\sin\theta = -\dfrac{2\sqrt{29}}{29}$.

Example 3

Find $\csc\dfrac{2\pi}{3}$. Give an exact answer.

1. Sketch the angle and determine its reference angle.

   $\dfrac{2\pi}{3}$ is $\dfrac{2}{3}$ of the way between 0 and π radians.

   The reference angle is the angle the terminal side makes with the *x*-axis.
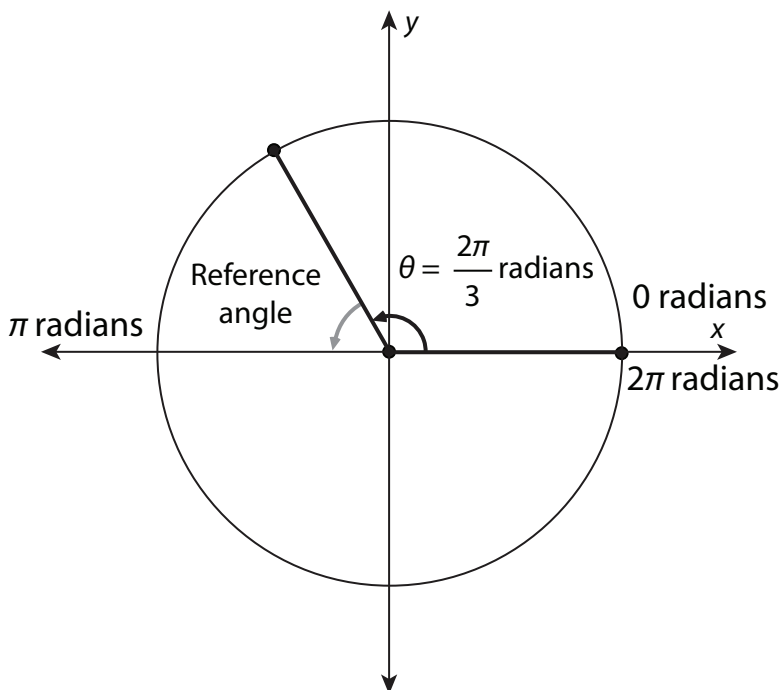
   The reference angle falls in Quadrant II. Therefore, subtract the original angle measure from π radians to find the reference angle.

   $$\pi - \frac{2\pi}{3} = \frac{3\pi}{3} - \frac{2\pi}{3} = \frac{1\pi}{3} = \frac{\pi}{3}$$

   The reference angle for $\dfrac{2\pi}{3}$ radians is $\dfrac{\pi}{3}$ radians.

2. Use the pattern for the special angle to find csc θ.

Recall that $\dfrac{\pi}{3}$ radians is the measure of a special angle in a right triangle. Remember the pattern for the special angle:

30°–60°–90° triangle
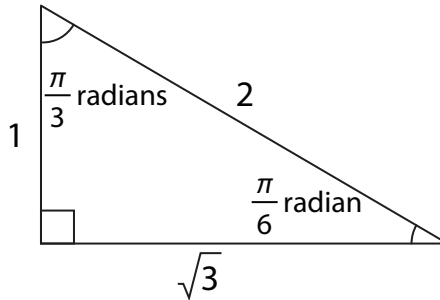
Use the values from this right triangle to determine csc θ.

$$\csc\theta = \frac{\text{length of hypotenuse}}{\text{length of opposite side}}$$
Cosecant ratio

$$\csc\left(\frac{\pi}{3}\right) = \frac{(2)}{\left(\sqrt{3}\right)}$$
Substitute $\dfrac{\pi}{3}$ for θ, 2 for the hypotenuse, and $\sqrt{3}$ for the opposite side.

$$\csc\frac{\pi}{3} = \frac{2\sqrt{3}}{3}$$
Rationalize the denominator.

Recall ASTC: in Quadrant II, only the sine and cosecant are positive. Thus, $\csc\dfrac{\pi}{3}$ must be positive. (Recall that both the opposite side and the hypotenuse are positive in Quadrant II.)

The answer confirms that the cosecant is positive; therefore, $\csc\dfrac{\pi}{3} = \dfrac{2\sqrt{3}}{3}$.

Example 4

Given $\cos\theta = \dfrac{4}{5}$, if θ is in Quadrant I, find $\cot\theta$.

1. Sketch an angle in Quadrant I, draw the associated triangle, and label the sides with the given information.

    Cosine is the ratio of the length of the adjacent side to the length of the hypotenuse. Since $\cos\theta = \dfrac{4}{5}$, 4 is the length of the adjacent side and 5 is the length of the hypotenuse.
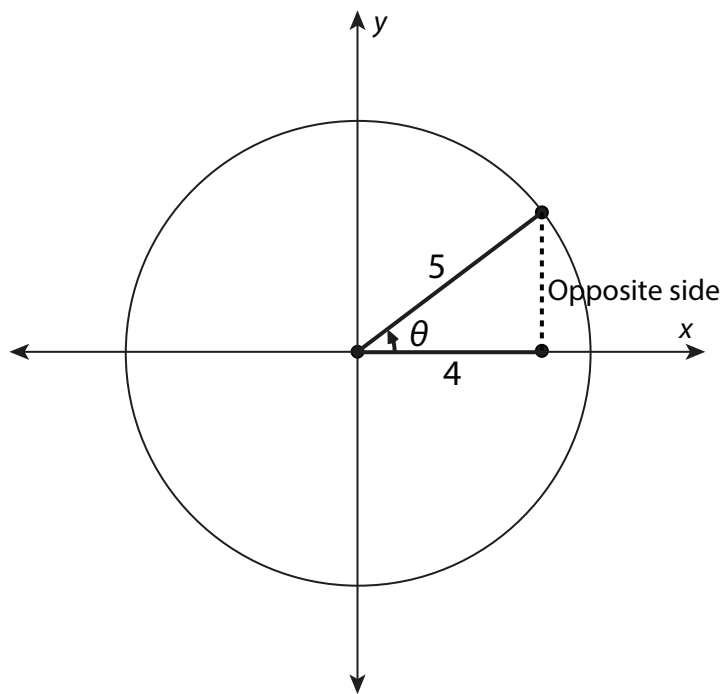
2. Use the Pythagorean Theorem to find the length of the opposite side.

Since the lengths of two sides of the triangle are given, substitute these values into the Pythagorean Theorem and solve for the missing side length.

$c^2 = a^2 + b^2$	Pythagorean Theorem
$(5)^2 = (4)^2 + b^2$	Substitute 5 for c and 4 for a.
$25 = 16 + b^2$	Simplify the exponents.
$9 = b^2$	Subtract 16 from both sides.
$3 = b$	Take the square root of both sides.

The length of the opposite side is 3 units.

3. Find the cotangent.

Use the values from the triangle to determine the cotangent.

$$\cot\theta = \frac{\text{length of adjacent side}}{\text{length of opposite side}}$$ Cotangent ratio

$$\cot\theta = \frac{(4)}{(3)}$$ Substitute 4 for the adjacent side and 3 for the opposite side.

In Quadrant I, all trigonometric ratios are positive, which coincides with the answer found.

Given $\cos\theta = \dfrac{4}{5}$, for an angle θ in Quadrant I, $\cot\theta = \dfrac{4}{3}$.

Example 5

Find cos θ if θ is a positive angle in standard position with a terminal side that passes through the point (–1, 0). Give an exact answer.

1. Sketch the angle.

   Plot the point (–1, 0) and draw the terminal side through it.
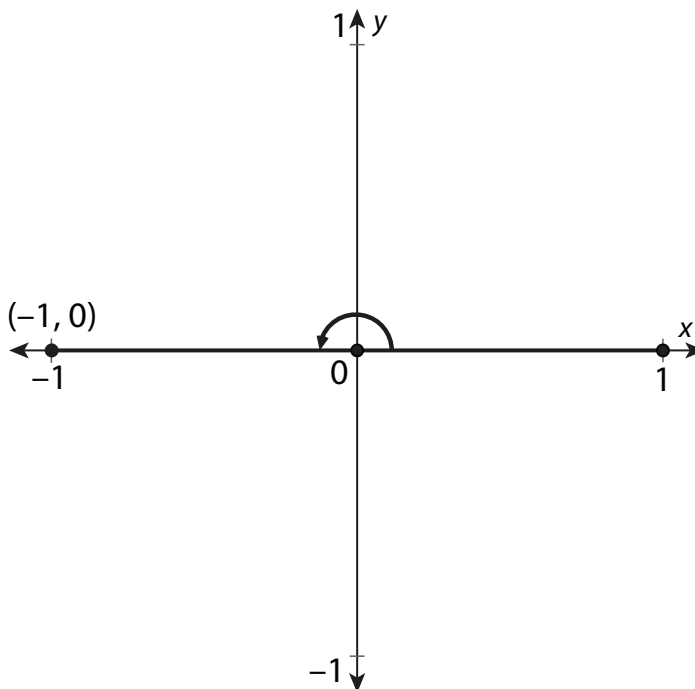
2. Determine the reference angle.

   The reference angle is the angle the terminal side makes with the x-axis. In this case, the reference angle is 0.

3. Determine the lengths of the opposite side, adjacent side, and hypotenuse.

The length of the adjacent side always corresponds with the *x*-coordinate, and the length of the opposite side always corresponds with the *y*-coordinate. (Picture a 1° angle if needed.)

The length of the opposite side is 0 (the *x*-coordinate).

The length of the adjacent side is –1 (the *y*-coordinate).

To find the length of the hypotenuse, use the Pythagorean Theorem.

$c^2 = a^2 + b^2$	Pythagorean Theorem
$c^2 = (-1)^2 + (0)^2$	Substitute –1 for a and 0 for b.
$c^2 = 1 + 0$	Simplify the exponents.
$c^2 = 1$	Add.
$c = 1$	Take the square root of both sides.

The length of the opposite side is 0, the length of the adjacent side is –1, and the length of the hypotenuse is 1.

4. Find cos θ.

Cosine is the ratio of the length of the adjacent side to the length of the hypotenuse.

Substitute the known values into the cosine ratio and solve.

$\cos\theta = \dfrac{\text{length of adjacent side}}{\text{length of hypotenuse}}$	Cosine ratio
$\cos\theta = \dfrac{(-1)}{(1)}$	Substitute –1 for the adjacent side and 1 for the hypotenuse.
$\cos\theta = -1$	Simplify.

In Quadrant II, sine and cosecant are positive, and all other functions are negative, which coincides with the answer found.

For a positive angle θ in standard position with a terminal side that passes through the point (–1, 0), cos $\theta = -1$.

UNIT 3 • TRIGONOMETRY OF GENERAL TRIANGLES AND TRIGONOMETRIC FUNCTIONS
Lesson 1: Radians and the Unit Circle

Practice 3.1.4: Evaluating Trigonometric Functions

For problems 1–4, determine the specified trigonometric ratio for each angle with a terminal side that passes through the given point. Give exact answers.

1. $\sin \theta$; (–8, 6)

2. $\csc \theta$; (2, –1)

3. $\tan \theta$; (0, 1)

4. $\cos \theta$; (–4, –2)

For problems 5–7, determine the specified trigonometric ratio for each special angle. Give exact answers.

5. $\cos\dfrac{11\pi}{6}$ radians

6. $\cot\dfrac{\pi}{4}$ radian

7. $\sec\dfrac{4\pi}{3}$ radians

For problems 8–10, each angle is described by one of its trigonometric ratios and the quadrant in which its terminal side is located. Find the requested trigonometric ratio for the angle. Give an exact answer.

8. Find $\tan \theta$ given $\sin\theta = \dfrac{1}{2}$ with a terminal side in Quadrant II.

9. Find $\cot \theta$ given $\cos\theta = -\dfrac{4}{5}$ with a terminal side in Quadrant III.

10. Find $\csc \theta$ given $\tan\theta = -\dfrac{2}{9}$ with a terminal side in Quadrant IV.

Lesson 2: Trigonometry of General Angles

Common Core State Standards

G–SRT.9 (+) Derive the formula $A = 1/2\ ab \sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.

G–SRT.10 (+) Prove the Laws of Sines and Cosines and use them to solve problems.

G–SRT.11 (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Essential Questions

1. How can you prove the Law of Sines?

2. How can you find the area of a triangle when given two sides and the included angle?

3. How can you prove the Law of Cosines?

4. When should you use the Law of Sines and the Law of Cosines?

WORDS TO KNOW

altitude	the perpendicular line from a vertex of a figure to its opposite side; height
ambiguous case	a situation wherein the Law of Sines produces two possible answers. This only occurs when the lengths of two sides and the measure of the non-included angle are given (SSA).
arccosine	the inverse of the cosine function, written $\cos^{-1}\theta$ or $\mathrm{arccos}\,\theta$
arcsine	the inverse of the sine function, written $\sin^{-1}\theta$ or $\mathrm{arcsin}\,\theta$

cosine	a trigonometric function of an acute angle in a right triangle that is the ratio of the length of the side adjacent to the length of the hypotenuse; the cosine of $\theta = \cos\theta = \dfrac{\text{length of adjacent side}}{\text{length of hypotenuse}}$
included angle	the angle between two sides
Law of Cosines	a formula for any triangle which states $c^2 = a^2 + b^2 - 2ab\cos C$, where C is the included angle in between sides a and b, and c is the nonadjacent side across from $\angle C$
Law of Sines	a formula for any triangle which states $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$, where a represents the measure of the side opposite $\angle A$, b represents the measure of the side opposite $\angle B$, and c represents the measure of the side opposite $\angle C$
oblique triangle	a triangle that does not contain a right angle
sine	a trigonometric function of an acute angle in a right triangle that is the ratio of the length of the opposite side to the length of the hypotenuse; the sine of $\theta = \sin\theta = \dfrac{\text{length of opposite side}}{\text{length of hypotenuse}}$
tangent	a trigonometric function of an acute angle in a right triangle that is the ratio of the length of the opposite side to the length of the adjacent side; the tangent of $\theta = \tan\theta = \dfrac{\text{length of opposite side}}{\text{length of adjacent side}}$
vertex	a point at which two or more lines meet

Recommended Resources

- AlgebraLAB.org. "Law of Sines and Law of Cosines #1."

  http://www.walch.com/rr/00221

  Users can practice solving problems involving the Laws of Sines and Cosines and then check their answers. The multiple-choice practice problems feature "hint" buttons and embedded links to the definitions of terms within problems. Scroll to the bottom of the page and click "Show Related AlgebraLab Documents" to reveal links to additional resources, including lessons, extra practice, study aids, and word problem examples.

- Khan Academy. "Proof of the Law of Cosines."

  http://www.walch.com/rr/00222

  This video tutorial details the proof of the Law of Cosines, a proof of the Law of Sines, and examples of the Law of Cosines. Users can also access practice problems through the sidebar menu.

- MathWarehouse.com. "Law of Sines and Cosines."

  http://www.walch.com/rr/00223

  This site provides a handy chart illustrating which law applies to different situations, as well as a video and practice problems.

IXL Links

- Area of a triangle sine formula:
  http://www.ixl.com/math/algebra-2/area-of-a-triangle-sine-formula

- Area of a triangle law of sines:
  http://www.ixl.com/math/algebra-2/area-of-a-triangle-law-of-sines

- Law of sines:
  http://www.ixl.com/math/geometry/law-of-sines

- Law of cosines:
  http://www.ixl.com/math/geometry/law-of-cosines

- Law of sines:
  http://www.ixl.com/math/algebra-2/law-of-sines

- Law of cosines:
  http://www.ixl.com/math/algebra-2/law-of-cosines

- Area of a triangle law of sines:
  http://www.ixl.com/math/algebra-2/area-of-a-triangle-law-of-sines

- Solve a triangle:
  http://www.ixl.com/math/geometry/solve-a-triangle

- Solve a triangle:
  http://www.ixl.com/math/algebra-2/solve-a-triangle

Lesson 3.2.1: Proving the Law of Sines

Introduction

The trigonometric functions (sine, cosine, and tangent) can be used to solve for unknown angle measures and side lengths of right triangles. However, the Law of Sines, which is derived from the definition of sine, can be used to help solve for unknown angle measures and side lengths of any triangle, even if it is not a right triangle.

The sine ratio can also be used to develop a formula for the area of a triangle when two sides and the **included angle** (the angle between two sides) are known.

Key Concepts

- The trigonometric function sine is the ratio of the length of the opposite side to the length of the hypotenuse; the sine of $\theta = \sin \theta = \dfrac{\text{length of opposite side}}{\text{length of hypotenuse}}$.

- The cosine function is the ratio of the length of the side adjacent to the length of the hypotenuse; the cosine of $\theta = \cos \theta = \dfrac{\text{length of adjacent side}}{\text{length of hypotenuse}}$.

- The tangent function is the ratio of the length of the opposite side to the length of the adjacent side; the tangent of $\theta = \tan \theta = \dfrac{\text{length of opposite side}}{\text{length of adjacent side}}$.

- The Law of Sines can be derived from the definition of the sine function.

- The **Law of Sines** states $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$, where a represents the measure of the side opposite angle A, b represents the measure of the side opposite angle B, and c represents the measure of the side opposite angle C.

- The Law of Sines is true for any triangle, not just a right triangle.

- To use the Law of Sines, the measures of an angle and the side opposite that angle must be known.

- To find an unknown angle or side using the Law of Sines, substitute in the known information; eliminate the fraction that is incomplete and will not help to solve the equation; then, solve the resulting equation.

- The **ambiguous case** of the Law of Sines occurs when the law is used to find an angle measure given two sides and a non-included angle (SSA). The Law of Sines may produce one answer, two answers, or no answer.

- However, in the ambiguous case, if the measure of the angle found using the Law of Sines is greater than the other angle given, there is a second valid answer. To find it, subtract the measure of the first answer from 180° (or π radians). To ensure that an answer is valid, remember that the three angles of a triangle must add up to 180° or π radians. If the sum of two of the angles is greater than 180° or π radians, then the answer is not valid.

- The area of any triangle can be found using the formula $\frac{1}{2}ab\sin C$, in which angle C is the included angle in between the sides with lengths a and b. This formula can be used when two sides and the included angle are known.

- Before finding the area of a triangle, the altitude of the triangle must first be found. Recall that the **altitude** of a triangle is a perpendicular line segment that connects a **vertex,** a point at which two or more lines meet, to a point on the line of its opposite side.

- If you know the sine of an angle and want to find the angle measure, use arcsine. **Arcsine** is the inverse of the sine function, written $\sin^{-1}\theta$ or $\arcsin\theta$.

- To find the sine of an angle, or if you know the sine of an angle and want to find the angle measure, use a calculator.

On a TI-83/84:

Step 1: Make sure your calculator is in the correct mode (degrees or radians).

Step 2: For sine, press [SIN], then input the angle measure. Press [ENTER]. For arcsine, press [2ND][SIN] and then input the sine ratio of an angle measure. Press [ENTER]. The calculator will calculate the angle measure.

On a TI-Nspire:

Step 1: Make sure your calculator is in the correct mode (degrees or radians).

Step 2: From the home screen, arrow down to the Calculate icon and press [enter].

Step 3: For sine, press [trig] and select "sin," then input the angle measure. Press [enter]. For arcsine, press [trig] and select "sin^{-1}," and then input the sine ratio of an angle measure. Press [enter]. The calculator will calculate the angle measure.

Guided Practice 3.2.1

Example 1

Prove the Law of Sines, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$, using the given information for the diagram of $\triangle ABC$ with altitude h and sides a, b, and c.
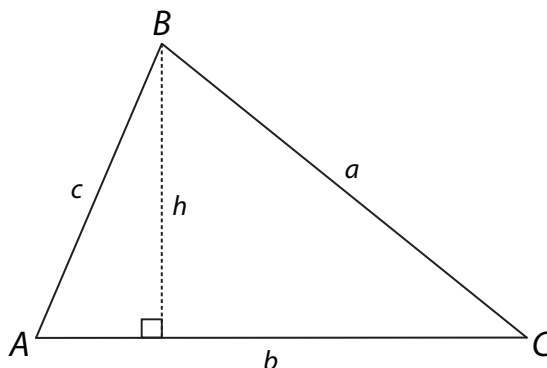
1. Use the definition of sine to create statements for angles A and C.

   Recall that the sine is equal to the ratio of the length of the opposite side to the length of the hypotenuse of a right triangle.

   Therefore, the ratios for $\sin A$ and $\sin C$ are as follows:

   $$\sin A = \frac{h}{c} \qquad \sin C = \frac{h}{a}$$

   Note that a, b, c, and h not only identify the sides and altitude of the triangle, they also represent the lengths of these sides and the altitude. A, B, and C identify the angles and also represent the measures of these angles.

2. Solve $\sin A$ and $\sin C$ for h.

   For $\sin A = \dfrac{h}{c}$, multiply both sides by c to get $c \sin A = h$.

   For $\sin C = \dfrac{h}{a}$, multiply both sides by a to get $a \sin C = h$.

3. Set the equations equal to each other and rearrange.

   Since both expressions equal h, they must also equal each other:

   $c \sin A = a \sin C$

   Divide both sides by sin A and sin C.

   $$\frac{c \sin A}{\sin A} = \frac{a \sin C}{\sin A}$$ Divide both sides by sin A.

   $$c = \frac{a \sin C}{\sin A}$$ Simplify.

   $$\frac{c}{\sin C} = \frac{\dfrac{a \sin C}{\sin A}}{\sin C}$$ Divide both sides by sin C.

   $$\frac{c}{\sin C} = \frac{a}{\sin A}$$ Simplify.

4. Repeat the proof for angles B and C.

   Start by drawing an altitude from angle A to side a.
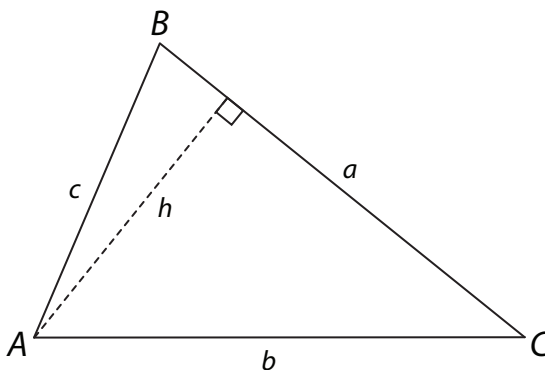
5. Use the definition of sine to create statements for angles B and C.

   Set up ratios for sin B and sin C:

   $$\sin B = \frac{h}{c} \qquad \sin C = \frac{h}{b}$$

6. Solve the equations for sin B and sin C for h.

For $\sin B = \dfrac{h}{c}$, multiply both sides by c to get $c \sin B = h$.

For $\sin C = \dfrac{h}{b}$, multiply both sides by b to get $b \sin C = h$.

7. Set the equations equal to each other and rearrange.

Since both expressions equal h, they must also equal each other.

$c \sin B = b \sin C$

Divide both sides by sin B and sin C.

$\dfrac{c \sin B}{\sin B} = \dfrac{b \sin C}{\sin B}$ Divide both sides by sin B.

$c = \dfrac{b \sin C}{\sin B}$ Simplify.

$\dfrac{c}{\sin C} = \dfrac{\frac{b \sin C}{\sin B}}{\sin C}$ Divide both sides by sin C.

$\dfrac{c}{\sin C} = \dfrac{b}{\sin B}$ Simplify.

8. Combine the two resulting formulas from steps 3 and 7 to complete the proof.

From step 3, $\dfrac{c}{\sin C} = \dfrac{a}{\sin A}$.

From step 7, $\dfrac{c}{\sin C} = \dfrac{b}{\sin B}$.

Since both $\dfrac{a}{\sin A}$ and $\dfrac{b}{\sin B}$ are equal to $\dfrac{c}{\sin C}$,

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$, thus proving the Law of Sines. ✔

Example 2

For $\triangle ABC$, find the length of side b if $m\angle B = \dfrac{\pi}{3}$ radians and $m\angle C = \dfrac{2\pi}{9}$ radians.
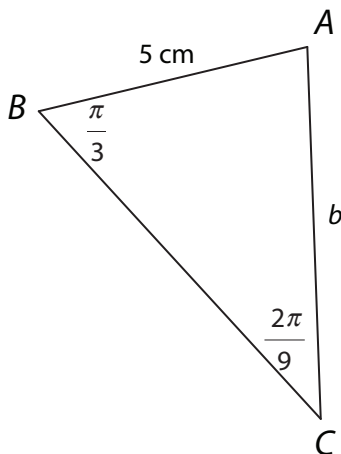
1. Substitute the known values into the Law of Sines.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Formula for the Law of Sines

$$\frac{a}{\sin A} = \frac{b}{\sin\left(\dfrac{\pi}{3}\right)} = \frac{(5)}{\sin\left(\dfrac{2\pi}{9}\right)}$$

Substitute 5 for c, $\dfrac{\pi}{3}$ for $m\angle B$, and $\dfrac{2\pi}{9}$ for $m\angle C$.

Applying the Law of Sines, the given triangle can be represented by

$$\frac{a}{\sin A} = \frac{b}{\sin\dfrac{\pi}{3}} = \frac{5}{\sin\dfrac{2\pi}{9}}.$$

2. Determine which two fractions are necessary for solving the equation and then solve.

Applying the Law of Sines to this example, the following equation results: $\dfrac{a}{\sin A} = \dfrac{b}{\sin\dfrac{\pi}{3}} = \dfrac{5}{\sin\dfrac{2\pi}{9}}$.

Eliminate the first fraction, $\dfrac{a}{\sin A}$, since neither the numerator nor denominator is given.

What remains is $\dfrac{b}{\sin\dfrac{\pi}{3}} = \dfrac{5}{\sin\dfrac{2\pi}{9}}$. Solve this equation for b.

$\dfrac{b}{\sin\dfrac{\pi}{3}} = \dfrac{5}{\sin\dfrac{2\pi}{9}}$ Rewritten equation

$b = \dfrac{5\sin\dfrac{\pi}{3}}{\sin\dfrac{2\pi}{9}}$ Multiply both sides by $\sin\dfrac{\pi}{3}$ to isolate b.

3. Use your calculator to find the value of b.

First, use a calculator to find values for $\sin\dfrac{\pi}{3}$ and $\sin\dfrac{2\pi}{9}$.

$$\sin\frac{\pi}{3} \approx 0.866$$

$$\sin\frac{2\pi}{9} \approx 0.643$$

Substitute these values into the equation from the previous step.

$$b = \frac{5\sin\dfrac{\pi}{3}}{\sin\dfrac{2\pi}{9}}$$ Equation from the previous step

$$b \approx \frac{5(0.866)}{(0.643)}$$ Substitute 0.866 for $\sin\dfrac{\pi}{3}$ and 0.643 for $\sin\dfrac{2\pi}{9}$.

$$b \approx \frac{4.33}{0.643}$$ Multiply 5 • 0.866.

$$b \approx 6.7$$ Divide.

The length of side $b \approx 6.7$ cm.

Example 3

For $\triangle ABC$, find the measure of angle C if $m\angle A = 34°$, $c = 9$ meters, and $a = 7$ meters.

1. Substitute the known values into the Law of Sines.

    $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$ Formula for the Law of Sines

    $$\frac{(7)}{\sin(34°)} = \frac{b}{\sin B} = \frac{(9)}{\sin C}$$ Substitute 7 for a, 34° for $m\angle A$, and 9 for c.

2. Determine which two fractions are necessary for solving the equation and then solve.

    Eliminate the second fraction, $\dfrac{b}{\sin B}$, since neither the numerator nor the denominator is given.

    What remains is $\dfrac{7}{\sin 34°} = \dfrac{9}{\sin C}$. Solve this equation for sin C.

    $$\frac{7}{\sin 34°} = \frac{9}{\sin C}$$ Rewritten equation

    $7 \sin C = 9 \sin 34°$ Eliminate the fractions by multiplying both sides of the equation by sin 34° and sin C.

    $$\sin C = \frac{9 \sin 34°}{7}$$ Divide both sides by 7.

    To solve for $m\angle C$, take the arcsine of both sides.

    $$m\angle C = \arcsin\left(\frac{9 \sin 34°}{7}\right)$$

3. Use your calculator to find $m\angle C$.

    $m\angle C \approx 46.0°$

4. Determine whether the ambiguous case is possible to determine if there is a second valid answer.

The ambiguous case occurs when the Law of Sines is used to find an angle measure using two sides and a non-included angle (SSA). We now know the values for two sides and an angle. Because the angle is not the included angle, there may be two possible answers. Since $m\angle C$ is greater than $m\angle A$, we know there is a second valid answer. Consider the following figure.
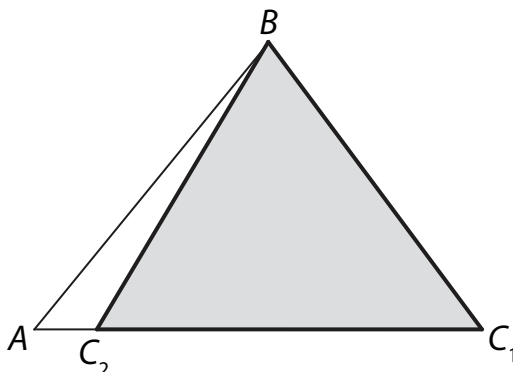
Two possible triangles can result from this information. One would be acute (ABC_1) and the other obtuse (ABC_2). The Law of Sines gave us the acute angle, C_1, but another angle may exist.

Since the measure of the angle found in step 3, $m\angle C \approx 46.0°$, is greater than that of the original angle given ($m\angle A = 34°$), there is a second valid answer.

5. Determine the second valid measure of angle C.

Subtract the first value of angle C from 180°.

$180 - 46 = 134$

This answer is valid because a value of 134° for angle C, when added to the known value of 34° for angle A, still leaves 12° remaining for angle B. (Remember, the sum of the angles of a triangle must equal 180°.)

Therefore, $m\angle C \approx 46.0°$ and $m\angle C \approx 134.0°$.

Example 4

Derive the formula $A = \dfrac{1}{2}ab\sin C$ for the area of $\triangle ABC$ with altitude h.
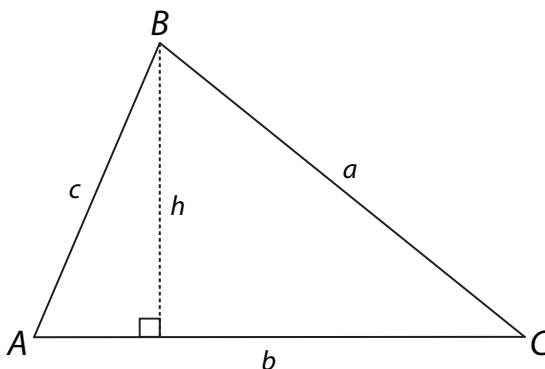
1. Write an expression for the altitude h using $\sin C$.

   The sine of an angle is the ratio of the length of the opposite side to the length of the hypotenuse.

   The angle is C, the opposite side is the altitude h, and the hypotenuse of the triangle formed by angle C and h is a.

   Therefore, $\sin C = \dfrac{h}{a}$.

2. Solve the expression for h.

   Multiply both sides by a.

   $h = a\sin C$

3. Substitute the value for h into the basic formula for the area of a triangle.

   $A = \dfrac{1}{2}bh$ Formula for the area of a triangle

   $A = \dfrac{1}{2}b(a\sin C)$ Substitute $a\sin C$ for h.

   $A = \dfrac{1}{2}ab\sin C$ Apply the Commutative Property of Multiplication.

   Thus it is possible to derive the trigonometric formula

   $A = \dfrac{1}{2}ab\sin C$ for the area of $\triangle ABC$ with altitude h.

Example 5

Find the area of $\triangle DEF$ using the given information.
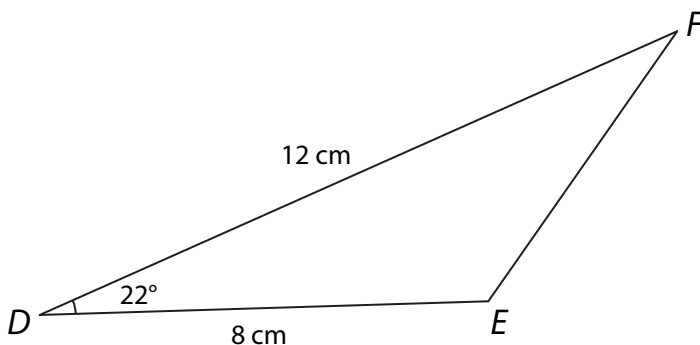
1. Substitute the values into the formula for the area of a triangle.

   Since two sides and the included angle are given, the formula $A = \frac{1}{2}ab \sin C$ can be used. The lengths of the sides are a and b, and C is the measure of the included angle.

   $A = \frac{1}{2}ab \sin C$ \qquad Formula

   $A = \frac{1}{2}(8)(12)\sin 22°$ \qquad Substitute 8 for a, 12 for b, and 22° for C.

   $A = \frac{1}{2}(96)\sin 22°$ \qquad Multiply 8 and 12.

   $A = 48 \sin 22°$ \qquad Simplify.

2. Use your calculator to find the area.

   The area of the triangle is equal to 48 sin 22°.

   Use your calculator to find the area of the triangle. Make sure your calculator is in degree mode.

   $A = 48 \sin 22° \approx 18.0$

   The area of $\triangle DEF$ is approximately 18.0 cm².

UNIT 3 • TRIGONOMETRY OF GENERAL TRIANGLES AND TRIGONOMETRIC FUNCTIONS
Lesson 2: Trigonometry of General Angles

Practice 3.2.1: Proving the Law of Sines

For problems 1–5, use the Law of Sines to find each requested measurement. Round answers to the nearest hundredth.

1. What is the length of $\overline{HF}$?
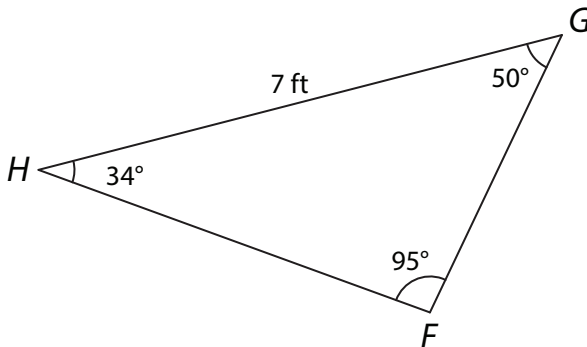
2. If $m\angle A = \dfrac{2\pi}{9}$ radians, what is the measure of angle B?
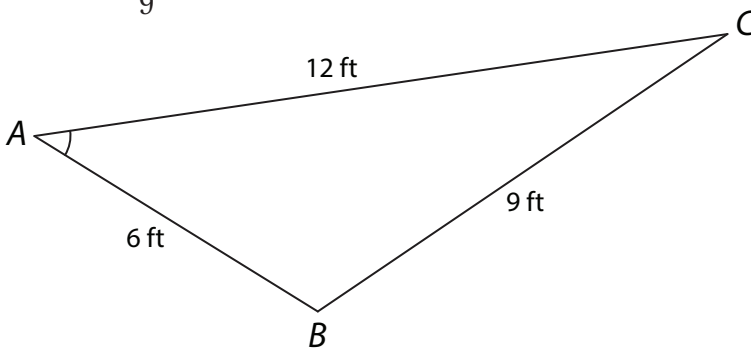
3. For $\triangle JKL$, $JK = 3$ meters, $KL = 4$ meters, and $m\angle L = 29°$. What is the measure of angle J?

4. For $\triangle XYZ$, $YZ = 6$ feet, $m\angle X = 1.342$ radians, and $m\angle Z = 0.431$ radian. What is the length of $\overline{XY}$?

5. For $\triangle ABC$, $AB = 10$ mm, $AC = 8$ mm, and $m\angle B = \dfrac{\pi}{5}$ radians. What is the measure of angle C?

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For problems 6 and 7, find the area of each triangle. Round answers to the nearest hundredth.

6. For $\triangle HIJ$, $HI = 3$ inches, $IJ = 7.5$ inches, $HJ = 7$ inches, and $m\angle H = 87°$.

7. For $\triangle ABC$, $m\angle A = \dfrac{2\pi}{7}$ radians.
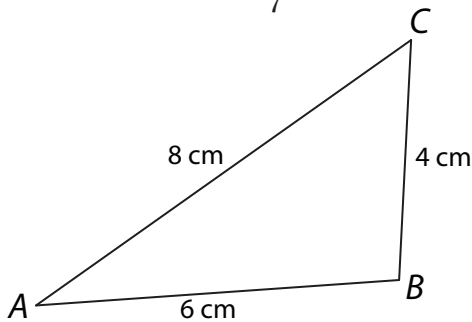
For problems 8–10, find the perimeter of each triangle shown. Round answers to the nearest tenth.
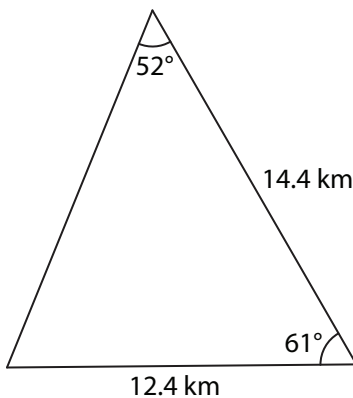
8.

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9.
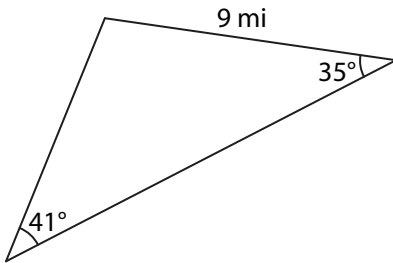
10.
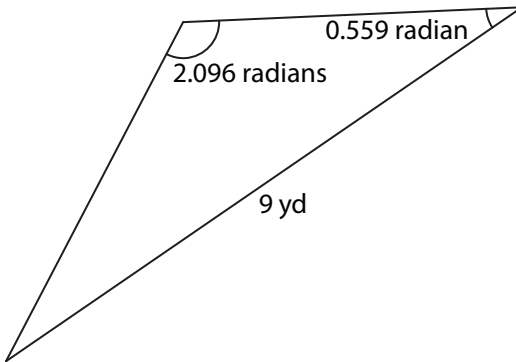
Lesson 3.2.2: Proving the Law of Cosines

Introduction

The sine, cosine, and tangent functions can be used to find unknown lengths of sides or measures of angles in right triangles, and in certain cases the Law of Sines can be used to find missing pieces of **oblique triangles** (triangles that do not contain a right angle). If two angles and a side are known (Angle-Side-Angle or ASA, or Angle-Angle-Side or AAS), or two sides and the non-included angle are known (Side-Side-Angle or SSA), the Law of Sines can be used to solve the triangle. However, sometimes there is not enough information to use the Law of Sines. In this case, one must use the Law of Cosines, which is derived from a variety of other trigonometric formulas.

Key Concepts

- Recall that the cosine function is the ratio of the length of the side adjacent to the length of the hypotenuse; the cosine of $\theta = \cos \theta = \dfrac{\text{length of adjacent side}}{\text{length of hypotenuse}}$.

- The **arccosine** is the inverse of the cosine function, written $\cos^{-1}\theta$ or arccos θ.

- The Law of Cosines can be used to solve any triangle when two sides and the included angle are known (SAS) or when three sides are known (SSS).

- The **Law of Cosines** states $c^2 = a^2 + b^2 - 2ab \cos C$, where C is the included angle in between sides a and b, and c is the (nonadjacent) side across from angle C.

- The Law of Cosines can be derived from the Pythagorean Theorem using the definitions of sine and cosine, along with the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$.

- The Law of Cosines is true for any triangle, not just a right triangle.

- To find the cosine or arccosine of a given angle measure, use a graphing calculator.

> **On a TI-83/84:**
>
> Step 1: Make sure your calculator is in the correct mode (degrees or radians).
>
> Step 2: For cosine, press [COS], then input the angle measure. Press [ENTER]. For arccosine, press [2ND][COS] and then input the angle measure. Press [ENTER].

On a TI-Nspire:

Step 1: Make sure your calculator is in the correct mode (degrees or radians).

Step 2: For cosine, press [trig] and select "cos," then input the angle measure. Press [enter]. For arccosine, press [trig] and select "cos^{-1}," and then input the angle measure. Press [enter].

Guided Practice 3.2.2

Example 1

Given the diagram of $\triangle ABC$ with altitude h, prove the Law of Cosines.
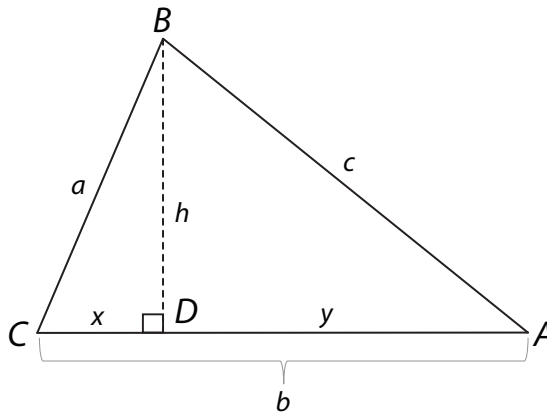
1. Create statements for h, x, and y using the sine and cosine of angle C.

    Recall that in a right triangle, the sine is equal to the ratio of the length of the opposite side to the length of the hypotenuse, and the cosine is the ratio of the length of the adjacent side to the length of the hypotenuse.

    $\sin C = \dfrac{h}{a}$ Set sin C equal to the ratio of sides h and a.

    $h = a \sin C$ Multiply both sides by a and apply the Symmetric Property of Equality.

    $\cos C = \dfrac{x}{a}$ Set cos C equal to the ratio of sides x and a.

    $x = a \cos C$ Multiply both sides by a and apply the Symmetric Property of Equality.

    $x + y = b$ Side b equals the sum of segments x and y.

    $y = b - x$ Subtract x from both sides.

    $y = b - (a \cos C)$ Substitute a cos C for x.

    The expressions for h, x, and y are $h = a \sin C$, $x = a \cos C$, and $y = b - a \cos C$.

The new values for *h*, *x*, and *y* can now be substituted into the diagram, as shown.
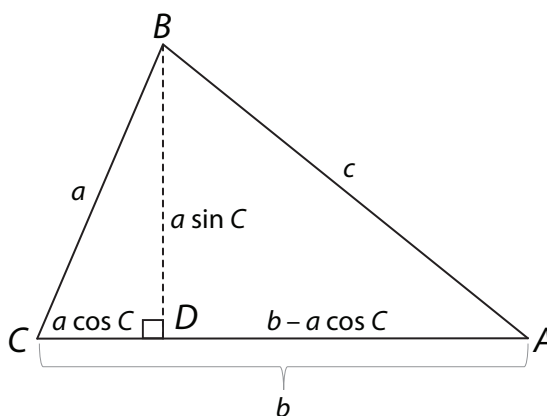
2. Use the Pythagorean Theorem and the known expressions to write an equation for $\triangle ADB$.

From the diagram, we know that $\triangle ADB$ is a right triangle.

Recall that the Pythagorean Theorem states that for a right triangle, $a^2 + b^2 = c^2$, where *a* and *b* are the legs and *c* is the hypotenuse.

$c^2 = a^2 + b^2$	Pythagorean Theorem
$c^2 = (a \sin C)^2 + (b - a \cos C)^2$	Substitute *a* sin *C* for *a* and *b* – *a* cos *C* for *b*.

3. Expand and rearrange the statement to develop the Law of Cosines.

$c^2 = (a \sin C)^2 + (b - a \cos C)^2$	Statement from the previous step
$c^2 = a^2 \sin^2 C + b^2 - 2ab \cos C + a^2 \cos^2 C$	Expand each squared term.
$c^2 = a^2 \sin^2 C + a^2 \cos^2 C + b^2 - 2ab \cos C$	Rearrange the terms.
$c^2 = a^2(\sin^2 C + \cos^2 C) + b^2 - 2ab \cos C$	Factor out the common a^2 from the first two terms on the right side of the equation.
$c^2 = a^2(1) + b^2 - 2ab \cos C$	Substitute 1 for $\sin^2 C + \cos^2 C$ (from the trigonometric identity $\sin^2 \theta + \cos^2 \theta = 1$).
$c^2 = a^2 + b^2 - 2ab \cos C$	Multiply.

The result, $c^2 = a^2 + b^2 - 2ab \cos C$, is the Law of Cosines.

Example 2

Given the diagram of $\triangle DEF$, find the length of side f. Round the length to the nearest tenth.
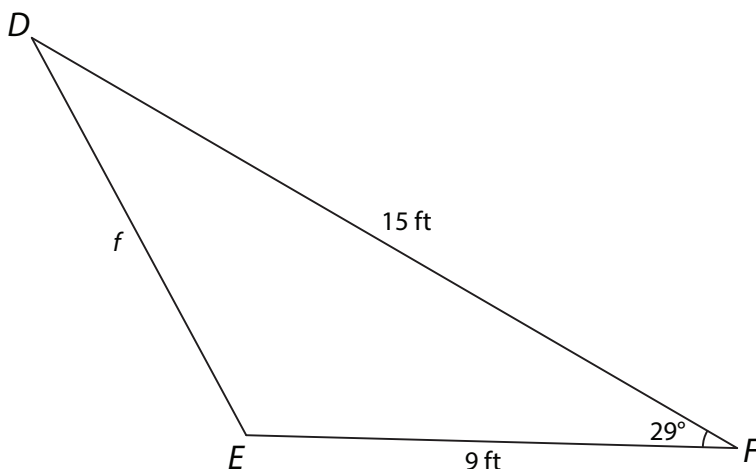
1. Identify which values are equivalent to a, b, and c in the Law of Cosines formula.

   Recall that the Law of Cosines states $c^2 = a^2 + b^2 - 2ab \cos C$, where C is the included angle in between sides a and b, and c is the (nonadjacent) side across from angle C.

   Examine the diagram labels to determine which sides are equivalent to a, b, and c.

   Since sides are named after their opposite angles, the side labeled "9 ft" is side d and the side labeled "15 ft" is side e. Side f is given.

   In this triangle, we can match the angle names and sides: $a = d$, $b = e$, and $c = f$. Therefore, $\angle F$ is the included angle of sides d and e.

   $\angle C$ is the included angle in between sides a and b. Thus $a = 9$, $b = 15$, and $C = 29°$.

   c is the side across from angle C, so $c = f$.

2. Substitute the known values into the Law of Cosines formula.

   Since two sides and the included angle are known (SAS), the Law of Cosines can be used to find the length of the third side.

$c^2 = a^2 + b^2 - 2ab \cos C$	Formula for the Law of Cosines
$(f)^2 = (9)^2 + (15)^2 - 2(9)(15) \cos (29°)$	Substitute 9 for a, 15 for b, f for c, and 29° for C.

3. Solve the resulting equation for f.

$$f^2 = 9^2 + 15^2 - 2(9)(15) \cos 29°$$ Equation from the previous step

$$f^2 = 81 + 225 - 270 \cos 29°$$ Simplify the squares and multiply.

$$f^2 = 306 - 270 \cos 29°$$ Add.

$$f = \sqrt{306 - 270 \, \cos 29°}$$ Take the square root of both sides.

$$f \approx 8.4$$ Use a calculator to find the answer.

The length of side f is approximately 8.4 feet.

Example 3

Given the diagram of $\triangle LMN$, find the measure of angle L in radians.
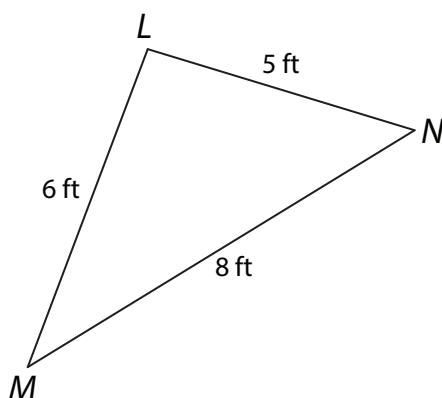
1. Identify which values are equivalent to a, b, and c in the Law of Cosines formula.

    Side c is the side across from angle C. Since we are looking for angle L, let angle L be equivalent to angle C. Thus, $c = 8$.

    a and b are the sides that make up angle C. The sides that make up angle L are 5 feet and 6 feet, so $a = 5$ and $b = 6$.

2. Substitute the known values into the Law of Cosines formula.

    Since three sides are known (SSS), the Law of Cosines can be used to find the measure of an angle.

$c^2 = a^2 + b^2 - 2ab \cos C$	Formula for the Law of Cosines
$(8)^2 = (5)^2 + (6)^2 - 2(5)(6) \cos (L)$	Substitute 8 for c, 5 for a, 6 for b, and L for C.
$64 = 25 + 36 - 60 \cos L$	Simplify the squares and multiply.
$64 = 61 - 60 \cos L$	Add.

3. Solve the resulting equation for angle L.

$64 = 61 - 60 \cos L$	Equation from the previous step
$3 = -60 \cos L$	Subtract 61 from both sides.
$\dfrac{3}{-60} = \cos L$	Divide both sides by -60.
$m\angle L = \arccos\left(-\dfrac{3}{60}\right)$	Take the arccosine of both sides.
$m\angle L \approx 1.621$ radians	Use your calculator in radian mode to find $\arccos\left(-\dfrac{3}{60}\right)$.

    The measure of angle L is approximately 1.621 radians.

Example 4

Given the diagram of $\triangle XYZ$, solve the triangle by finding the missing sides and angles.
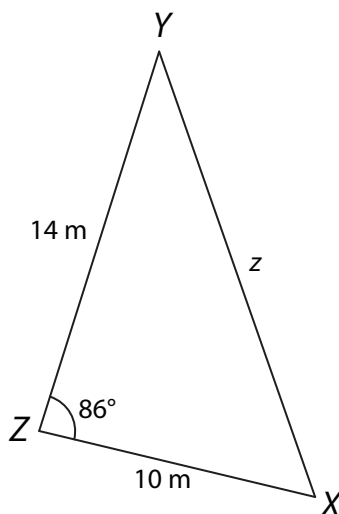
1. Use the Law of Cosines formula to find the length of side z.

    To solve a triangle, you must find the length of all the sides and all the angles.

    Since two sides and the included angle are known (SAS), the Law of Cosines can be used to find the third side.

$c^2 = a^2 + b^2 - 2ab \cos C$	Formula for the Law of Cosines
$(z)^2 = (14)^2 + (10)^2 - 2(14)(10) \cos (86°)$	Substitute z for c, 14 for a, 10 for b, and 86° for C.
$z^2 = 196 + 100 - 280 \cos 86°$	Simplify the squares and multiply.
$z^2 = 296 - 280 \cos 86°$	Add.
$z = \sqrt{296 - 280 \cos 86°}$	Take the square root of both sides.
$z \approx 16.6$	Use your calculator to find the answer.

    The length of side z is approximately 16.6 meters.

2. Find the measure of angle X.

Now that you know the length of side z, you can use either the Law of Sines or the Law of Cosines to find angle X.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$ Formula for the Law of Sines

$$\frac{(z)}{\sin(Z)} = \frac{(x)}{\sin(X)}$$ Substitute z for a, x for b, Z for A, and X for B. The third ratio is unnecessary.

$$\frac{(16.6)}{\sin(86°)} = \frac{(14)}{\sin X}$$ Substitute 16.6 for z, 14 for x, and 86° for Z.

$16.6 \sin X = 14 \sin 86°$ Multiply both sides by sin 86° and sin X.

$$\sin X = \frac{14 \sin 86°}{16.6}$$ Divide both sides by 16.6.

$$m\angle X = \arcsin\left[\frac{14 \sin 86°}{16.6}\right]$$ Take the arcsine of both sides.

$m\angle X \approx 57.3°$

The measure of angle X is approximately 57.3°.

3. Find the measure of angle Y.

The sum of the angles of a triangle is always 180°. Subtract the known values for angles X and Z from 180° to find $m\angle Y$.

$180 - 86 - 57.3 = 36.7$

The measure of angle Y is approximately 36.7°.

4. Summarize your findings.

The missing angle and side measures for $\triangle XYZ$ are $z \approx 16.6$ meters, $m\angle X \approx 57.3°$, and $m\angle Y \approx 36.7°$.

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Practice 3.2.2: Proving the Law of Cosines

For problems 1–4, use the Law of Cosines to find the measure of each requested angle. Round degrees to the nearest tenth and radians to the nearest thousandth.

1. For $\triangle DEF$, find the measure of angle E in degrees if $DE = 7$ in, $DF = 9$ in, and $EF = 4$ in.

2. For $\triangle IJK$, find the measure of angle K in radians if $IJ = 8$ cm, $IK = 10$ cm, and $JK = 11$ cm.

3. Find the measure of angle O in radians.
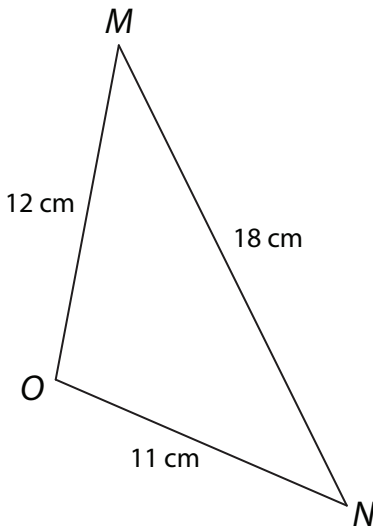
4. Find the measure of angle A in degrees.
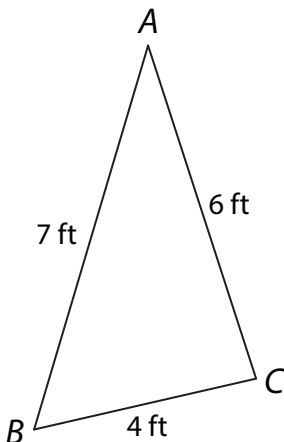
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For problems 5–7, solve each triangle by finding the measure of all the unknown angles and sides. Use degrees for angle measurements and round all answers to the nearest whole number.

5. For $\triangle UVW$, $UV = 9$ mm, $WV = 4$ mm, and $m\angle V = 74°$.

6. For $\triangle BCD$, $BC = 10$ in, $CD = 5$ in, and $BD = 12$ in.

7.
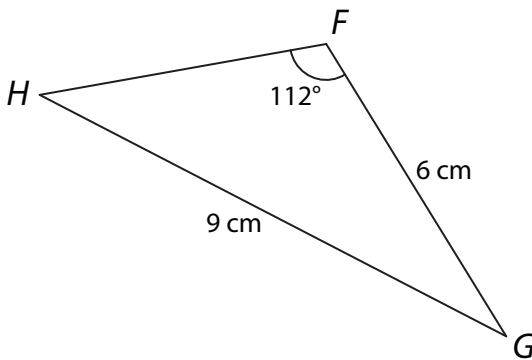
For problems 8–10, find the amount of fencing that would be needed to completely enclose each triangular plot of land. Round answers to the nearest tenth.
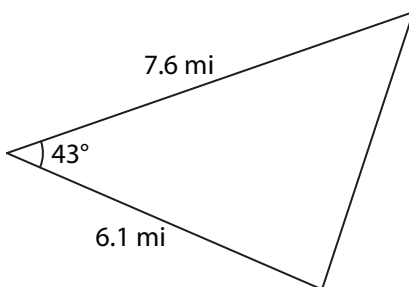
8.

continued

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9.

4.2 km

108°

6.1 km

10.

$\dfrac{3\pi}{7}$ radians

45 yd

65 yd

Lesson 3.2.3: Applying the Laws of Sines and Cosines

Introduction

Because the Law of Sines and the Law of Cosines can be used to solve oblique triangles, they have many real-world applications. Two of the most common applications are in surveying and navigation.

Key Concepts

- In problems describing a real-world situation, if no picture is given, draw and label a diagram.

- Recall that the Law of Sines should be used when two angles and a side are known (ASA or AAS) or two sides and the non-included angle are known (SSA). The Law of Cosines must be used when two sides and the included angle are known (SAS) or when three sides are known (SSS).

- To find the area of a triangle when the height is not known, use the formula $A = \dfrac{1}{2}ab\sin C$.

- The calculated answer to a real-world problem should always be examined to determine if the answer is reasonable given the context.

- When appropriate, answers should include a unit label.

Guided Practice 3.2.3

Example 1

A surveyor needs to measure the width of a building. He identifies two points E and F at ground level on opposite sides of the building. He then walks to a point a distance away from the building: point G. He measures the distance from E to G as 284 feet, the distance from F to G as 322 feet, and the measure of $\angle G$ as 63°. How wide is the building? Round your answer to the nearest foot, then determine whether it is reasonable given the context of the problem.

1. Draw and label a sketch of the situation.

   This will help to organize the known information.
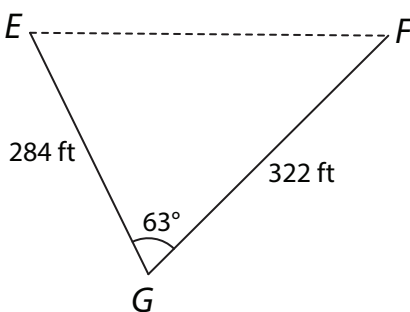
2. Find EF using the Law of Cosines.

   Since two sides and the included angle are known (SAS), the Law of Cosines must be used to find the length of the third side.

   $c^2 = a^2 + b^2 - 2ab \cos C$ — Formula for the Law of Cosines

   $g^2 = (284)^2 + (322)^2 - 2(284)(322) \cos(63°)$ — Substitute g for c, 284 for a, 322 for b, and 63° for C.

   $g = \sqrt{284^2 + 322^2 - 2(284)(322)\cos 63°}$ — Take the square root of both sides.

   $g \approx 318$ — Solve using a calculator.

   The width of the building is approximately 318 feet.

3. Determine if the answer is reasonable.

   Based on the other side lengths, yes, this answer is a reasonable estimate for the width of the building.

Example 2

Hideko is a pilot with a small two-seater plane. She took off from the airport and headed south toward her hometown. After flying for 70 miles, Hideko turned the plane 40° to the west to avoid a large storm. She remained on this course for 95 miles before turning back 65° to the east and continuing on to land near her hometown. How much farther did Hideko fly than she would have if she could have remained on her original course? Determine whether your answer is reasonable given the context of the problem.

1. Draw and label a sketch of the situation.

   Draw a sketch that includes all the known information.

   Each change in the course is a change from the direction in which the plane was previously heading.

   Label the points with letters.
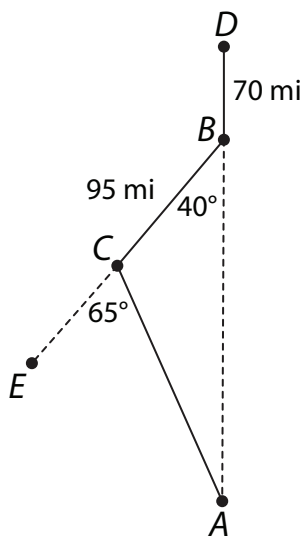
2. Find the measure of $\angle BCA$ and $\angle CAB$.

Since $\angle BCA$ and $\angle ECA$ form a straight line, they are supplementary angles. Supplementary angles sum to 180, so subtract $m\angle ECA$ (65°) from 180° to find $m\angle BCA$: $180 - 65 = 115$.

Since the sum of the angle measures of a triangle is 180°, subtract $m\angle BCA$ (115°) and $m\angle CBA$ (40°) from 180° to find the remaining angle of the triangle, $m\angle CAB$: $180 - 115 - 40 = 25$.

Update the sketch with this new information.
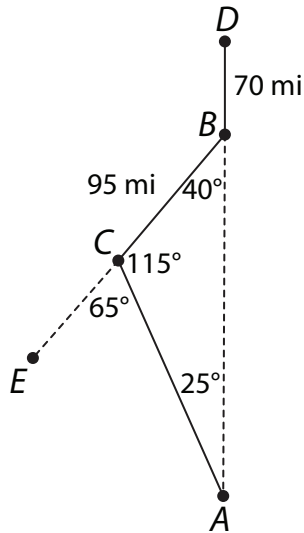
3. Find AC using the Law of Sines.

The Law of Sines must be used since two angles (A and BCA) and one side (BC) are known (AAS).

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

Use the portion of the Law of Sines formula in which a and b are the lengths of the sides across from angles A and B, respectively.

$$\frac{(95)}{\sin(25°)} = \frac{b}{\sin(40°)}$$

Substitute 95 for a, 25° for A, and 40° for B.

$$b = \frac{95\sin 40°}{\sin 25°}$$

Multiply both sides by sin 40°.

$$b \approx 144.5$$

Use your calculator to find the answer. Round to the nearest tenth.

AC is approximately 144.5 miles. Hideko flew approximately 144.5 miles after turning back toward her hometown.

4. Find AB using the Law of Sines.

The Law of Sines is more appropriate for this situation than the Law of Cosines, because the original information can be substituted into the Law of Sines formula, thus reducing the chance for error. (The problem can also be solved using the Law of Cosines, but if a mistake were made in finding AC, AB would also be incorrect.)

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

Use the portion of the Law of Sines formula in which a and c are the measures of the sides across from angles A and C, respectively.

$$\frac{(95)}{\sin(25°)} = \frac{c}{\sin(115°)}$$

Substitute 95 for a, 25° for A, and 115° for C.

$$c = \frac{95\sin 115°}{\sin 25°}$$

Multiply both sides by sin 115°.

$$c \approx 203.7$$

Use your calculator to find the answer. Round to the nearest tenth.

AB is approximately 203.7 miles.

Hideko was 203.7 miles away from her hometown when she had to change course.

5. Determine how much farther Hideko flew as a result of changing course.

The 70 miles Hideko flew before initially changing course are irrelevant, because the distance flown before the detour has no effect on how much distance was added to the trip. Therefore, this figure can be ignored and the difference can be calculated starting at the point at which Hideko turned west away from the storm.

Add *BC* and *AC* to find the total distance Hideko flew between the point at which she turned west and when she landed near her home.

$95 + 144.5 = 239.5$

Hideko flew a total of 239.5 miles from the detour point to her hometown.

If she had stayed on course, she would have flown only 203.7 miles from the detour point to her hometown.

Subtract to find the difference.

$239.5 - 203.7 = 35.8$

Hideko flew an additional 35.8 miles as a result of the detour.

6. Consider whether or not the answer is reasonable.

Yes, 35.8 miles is a reasonable answer. If a much larger answer such as 443 miles had been found, this answer would not make sense because that is a greater distance than the entire trip.

Example 3

The Leaning Tower of Pisa, which is 55.8 meters tall, currently leans to the southeast. When the angle of elevation of the sun is 42° and the tower is leaning toward the sun, the tower's shadow is 57.9 meters long. How many degrees does the tower lean? Round your answer to the nearest degree, then determine whether your answer is reasonable given the context of the problem.

1. Draw and label a sketch of the situation.

   Sketch the situation and label the known information.

   Draw a dotted vertical line to represent where the tower would be if it were perfectly vertical.

   Label the points with letters.
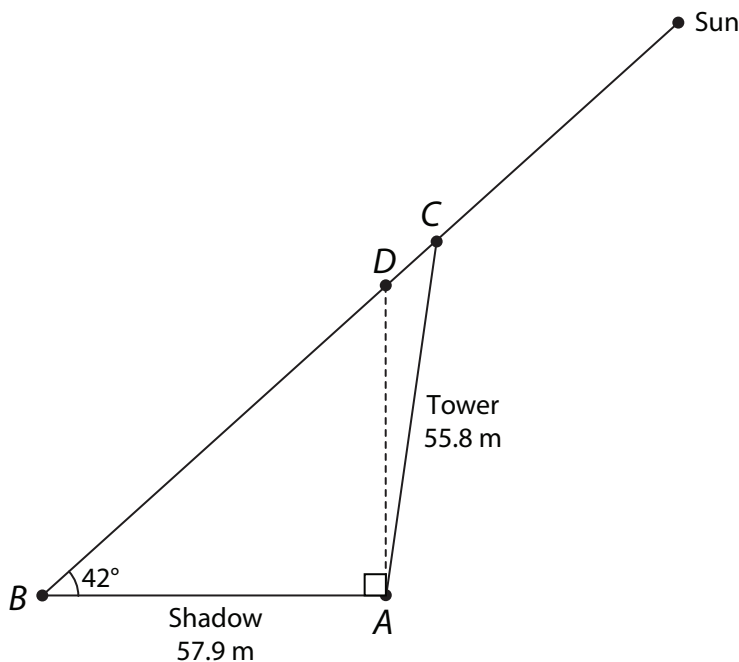
2. Find the measure of $\angle BCA$.

    Since there is not enough information yet to find $m\angle BAC$, $m\angle BCA$ must be found first.

    Two sides and an angle are known (SSA), so use the Law of Sines.

    $$\frac{b}{\sin B} = \frac{c}{\sin C}$$

    Use the portion of the Law of Sines formula in which b and c are the measures of the sides opposite angles B and C, respectively.

    $$\frac{(55.8)}{\sin(42°)} = \frac{(57.9)}{\sin C}$$

    Substitute 55.8 for b, 57.9 for c, and 42° for B.

    $55.8 \sin C = 57.9 \sin 42°$ Cross multiply.

    $$\sin C = \frac{57.9 \sin 42°}{55.8}$$

    Divide both sides by 55.8.

    $$m\angle BCA = \arcsin\left[\frac{57.9 \sin 42°}{55.8}\right]$$

    Take the arcsine of both sides.

    $m\angle BCA \approx 44.0°$

    Use your calculator to find the answer. Round to the nearest tenth.

    The measure of $\angle BCA$ is approximately 44°.

3. Find the measure of $\angle BAC$.

    The angle measures of a triangle sum to 180°. Subtract $m\angle B$ and $m\angle BCA$ from 180 to find $m\angle BAC$.

    $180 - m\angle B - m\angle BCA = m\angle BAC$

    $180 - 42 - 44 = 94$

    The measure of $\angle BAC$ is approximately 94°.

4. Determine how many degrees the tower leans.

$\angle DAC$ represents the angle of the tower's lean.

$\angle DAB$ is a right angle since we drew AD to be a perfectly vertical tower, which would be perpendicular to the ground.

$m\angle DAC = m\angle BAC - m\angle DAB$	Angle Subtraction Property
$m\angle DAC = (94°) - (90°)$	Substitute 94° for $m\angle BAC$ and 90° for $m\angle DAB$.
$m\angle DAC = 4°$	Simplify.

The tower leans approximately 4°.

5. Consider whether or not the answer is reasonable.

Yes, this is a reasonable answer given the height of the tower. A slight lean of such a tall tower would be noticeable without putting the tower in danger of toppling; a larger degree could cause the tower to collapse.

UNIT 3 • TRIGONOMETRY OF GENERAL TRIANGLES AND TRIGONOMETRIC FUNCTIONS
Lesson 2: Trigonometry of General Angles

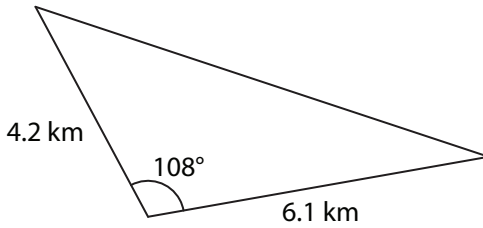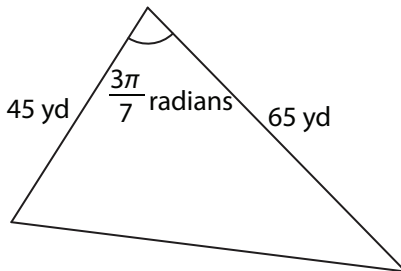## Practice 3.2.3: Applying the Laws of Sines and Cosines

Answer the following questions using the Law of Sines or the Law of Cosines. Round answers to the nearest tenth.

1. Peter and Elizabeth are standing by a barn. Elizabeth heads due north of the barn and Peter heads 25 degrees east of north. When they stop, Peter is 56 meters away from the barn and 63 meters from Elizabeth. How far is Elizabeth from the barn?

2. A plane flew south out of Omaha, Nebraska, on a 781-mile flight to Houston. When the plane was 345 miles away from Houston, it experienced a technical problem and the pilot had to divert to the closest available airport, which was 57 miles away. When the plane landed, it was 312 miles away from Houston. How many degrees did the pilot turn to divert to the closest airport?

3. An archaeologist is trying to measure the width of a recently unearthed housing settlement. She walks to a point far away from the settlement and sets up her instruments. She measures the distance from the left side of the settlement to her location as 212 feet and the distance from the right side of the settlement to her location as 185 feet. She then measures the angle between these two lines of sight as 38°. How wide is the housing settlement?

4. The sun is currently at a 73° angle of elevation. A tree is leaning away the sun at a 3° angle and casts a 12-foot shadow. How tall is the tree?

5. A ship sailed out of New York on its way to London, which is 3,459 miles away. After 2,812 miles of smooth sailing, the captain decided to veer 22° to the left to avoid a fleet of naval ships. After sailing on that bearing for 185 miles, the captain turned the ship back on a direct line to London. How far must the ship travel to reach London?

6. A full moon is currently at a 39° angle of elevation. A 30-foot-tall telephone pole is leaning toward the moon at a 4° angle. How long is the pole's shadow in the moonlight?

7. An environmental scientist identifies two points *E* and *F* on opposite sides of an ice formation. He then walks to a point a distance away from these points: point *G*. He measures the distance from *E* to *G* as 29 meters, the distance from *F* to *G* as 186 meters, and the measure of ∠*G* as 72°. What is the distance across the ice formation?

8. In a parallelogram, opposite angles and opposite sides have the same measure. The sides of a parallelogram are 7 cm and 5 cm long, and the angles are 112° and 68°. What are the lengths of the diagonals?

9. A flagpole that is 35 feet tall is leaning at a 3° angle away from the sun. When the flagpole's shadow is 10 feet long, what is the angle of elevation of the sun?

10. Nicole and Isaiah are both on the same street in New York, headed directly toward One World Trade Center, which is 1,776 feet tall. Nicole, who is closer to the building, measures the angle of elevation to the top of the center from her location as 79°. Isaiah measures the angle of elevation from his to the top of the center to be 54°. How far away is Nicole from Isaiah?

Lesson 3: Graphs of Trigonometric Functions

Common Core State Standard

F–TF.5   Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.★

Essential Questions

1. What is the period of a trigonometric function and how can it be found?

2. What is the amplitude of a trigonometric function and how can it be found?

3. How is the equation of a trigonometric function determined?

WORDS TO KNOW

amplitude	the coefficient a of the sine or cosine term in a function of the form $f(x) = a \sin bx$ or $g(x) = a \cos bx$; on a graph of the cosine or sine function, the vertical distance from the y-coordinate of the maximum point on the graph to the midline of the cosine or sine curve
cosine function	a trigonometric function of the form $f(x) = a \cos [b(x - c)] + d$, in which a, b, c, and d are constants and x is a variable defined in radians over the domain $(-\infty, \infty)$
cycle	the smallest representation of a cosine or sine function graph as defined over a restricted domain that is equal to the period of the function
frequency of a periodic function	the reciprocal of the period for a periodic function; indicates how often the function repeats
maximum	the greatest value or highest point of a function

midline	in a cosine function or sine function of the form $f(x) = \sin x + d$ or $g(x) = \cos x + d$, a horizontal line of the form $y = d$ that bisects the vertical distance on a graph between the minimum and maximum function values
minimum	the least value or lowest point of a function
period	in a cosine or sine function graph, the horizontal distance from a maximum to a maximum or from a minimum to a minimum; one repetition of the period of a function is called a *cycle*
periodic function	a function whose values repeat at regular intervals
periodic phenomena	real-life situations that repeat at regular intervals and can be represented by a periodic function
sine curve	a curve with a constant amplitude and period, which are given by a sine or cosine function; also called a *sine wave* or *sinusoid*
sine function	a trigonometric function of the form $f(x) = a \sin [b(x - c)] + d$, in which a, b, c, and d are constants and x is a variable defined in radians over the domain $(-\infty, \infty)$
sine wave	a curve with a constant amplitude and period given by a sine or cosine function; also called a *sine curve* or *sinusoid*
sinusoid	a curve with a constant amplitude and period given by a sine or cosine function; also called a *sine curve* or *sine wave*

Recommended Resources

- LearnZillion. "Graph Sinusoidal Functions by Plotting Points."

  http://www.walch.com/rr/00224

  This video series explores topics such as how to graph a sine curve by plotting points and how the terminal ray of the unit circle creates the sine curve.

- The Math Page. "Trigonometry."

  http://www.walch.com/rr/00225

  This website provides detailed instructions for how to graph the various trigonometric functions.

- MathIsFun.com. "Graphs of Sine, Cosine, and Tangent."

  http://www.walch.com/rr/00226

  This website shows the various trigonometric functions and how they relate to one another.

Lesson 3.3.1: Periodic Phenomena and Amplitude, Frequency, and Midline

Introduction

The sine, cosine, and tangent functions are **periodic functions** since their values repeat at regular intervals. While the unit circle can be used to study their properties, graphing these functions on a coordinate plane is also useful, since a coordinate plane allows the functions' periodic properties to be examined more closely.

Key Concepts

- The basic sine function, $f(x) = \sin x$, can be graphed using a table of x- and y-values. The graph reveals the basic pattern of the sine curve, as shown.
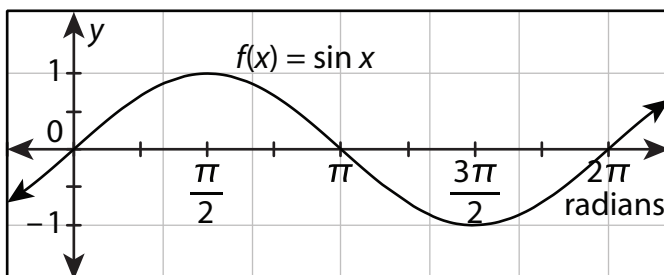
- A **sine curve** has a constant amplitude and period given by a sine or cosine function. It is also called a **sine wave** or **sinusoid**.

- The **amplitude** is the vertical distance from the y-coordinate of the maximum point on the graph to the midline of the curve. The **period** is the horizontal distance from a maximum to a maximum or from a minimum to a minimum. One repetition of the period of a function is called a **cycle**, which is the smallest representation of a cosine or sine function graph as defined over a restricted domain that is equal to the period of the function.

- The basic cosine function, $f(x) = \cos x$, can also be graphed using a table of x- and y-values, which results in the following graph. Notice that the graph of the basic cosine function is also a sine curve, but it has been shifted $\dfrac{\pi}{2}$ radians to the left.
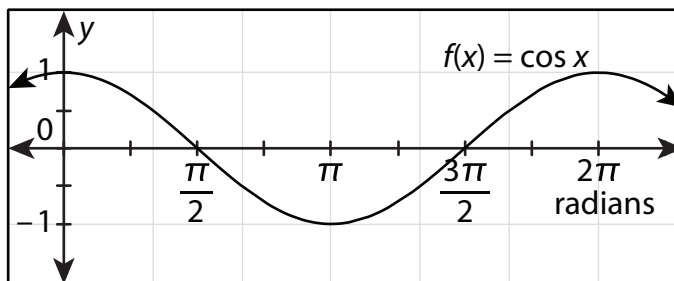
- Likewise, the basic tangent function, $f(x) = \tan x$, can be graphed using a table of x- and y-values. However, its graph is quite different from the graphs of the sine and cosine functions. The tangent function repeats twice as often and approaches both positive and negative infinity.
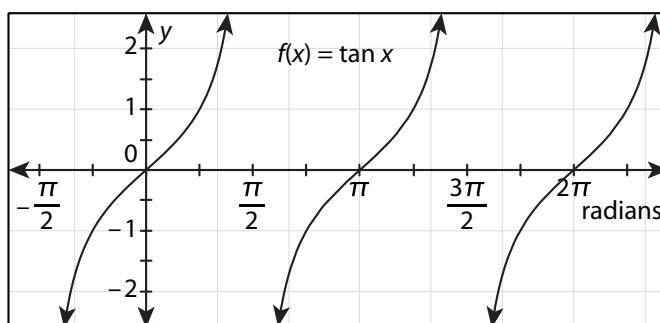
- The **maximum** of a function is the function's greatest value or highest point. The **minimum** of a function is the least value of the function, or its lowest point. A function can have several maxima and minima.

- The period of a sine or cosine function can be determined by finding the horizontal distance between two maxima or between two minima. The period also equals the length of one repetition of the function.

- Note that, in the past, a relative maximum was defined as the greatest value of a function for a particular interval of the function and a relative minimum was defined as the least value of a function for a particular interval of the function. With sine and cosine waves, however, the functions repeat, and the maximum and minimum are the greatest and least values, not for a particular interval, but for the entire function.

- The **midline** of a periodic function is the horizontal line located halfway between a function's minimum and maximum.

- The amplitude of a sine or cosine function can be found by determining how far the function rises above its midline.

- The **frequency of a periodic function** is the reciprocal of its period and indicates how often the function repeats. The higher the frequency, the smaller each wave, thus resulting in more waves appearing in a given portion of the graph.

- The general form of the **sine function** is $f(x) = a \sin [b(x - c)] + d$, and the general form of the **cosine function** is $f(x) = a \cos [b(x - c)] + d$. For both functions, $|a|$ is the amplitude, $\dfrac{2\pi}{b}$ is the period, c is the horizontal shift (also called the phase shift), and d is the vertical shift.

- When calculating the number and locations of these features of a function, viewing the functions on a graphing calculator can be helpful in confirming your answers.

Guided Practice 3.3.1

Example 1

Determine the period, frequency, midline, and amplitude of the graphed function.
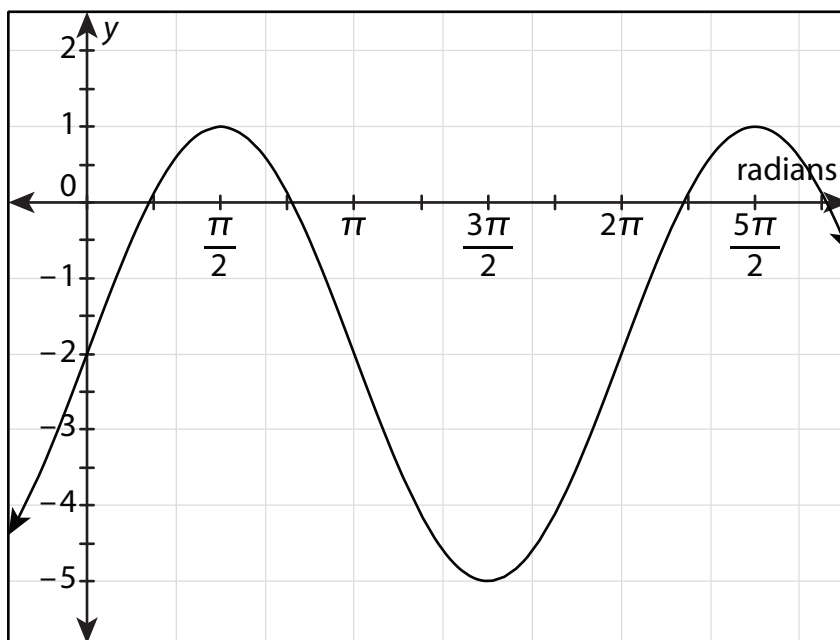
1. Determine the period.

   The period is the length of one cycle of the curve, or the horizontal distance the curve travels before it repeats. It can be found by determining the horizontal distance between two maximum points.

   There are two clearly identifiable maximum points: one at $x = \dfrac{\pi}{2}$ and another at $x = \dfrac{5\pi}{2}$. Subtract these values to find the distance between them.

   $$\frac{5\pi}{2} - \frac{\pi}{2} = \frac{4\pi}{2} = 2\pi$$

   The period of this function is 2π.

   This means the length of one repetition of the curve is 2π.

2. Determine the frequency.

The frequency tells how often a function repeats. It is found by taking the reciprocal of the period.

The period is 2π, so the frequency is $\dfrac{1}{2\pi}$.

This means that the curve repeats $\dfrac{1}{2\pi}$ of a time from the interval 0 to 1. The larger the frequency, the more waves appear in a given portion of the graph.

3. Determine the midline.

The midline of a periodic function is the horizontal line located halfway between the function's minimum and maximum points.

By inspecting the graph, we can see that the maximum point occurs at $y = 1$, and the minimum occurs at $y = -5$.

Find the average of 1 and –5.

$$\frac{1+(-5)}{2} = \frac{-4}{2} = -2$$

The average is –2; therefore, the midline occurs at $y = -2$.

We can verify this by inspecting the graph and confirming that the horizontal line $y = -2$ runs through the middle of the curve.

4. Determine the amplitude.

The amplitude of a sine or cosine function is the height of the curve. It is found by determining how far the function rises above its midline.

The midline is $y = -2$, and the highest the curve rises is $y = 1$.

Subtract to find the distance between them. Or, count the vertical distance from the midline to the maximum point on the graph.

$$1 - (-2) = 3$$

The amplitude of the function is 3.

Example 2

Determine the period, frequency, midline, and amplitude of the graphed function.
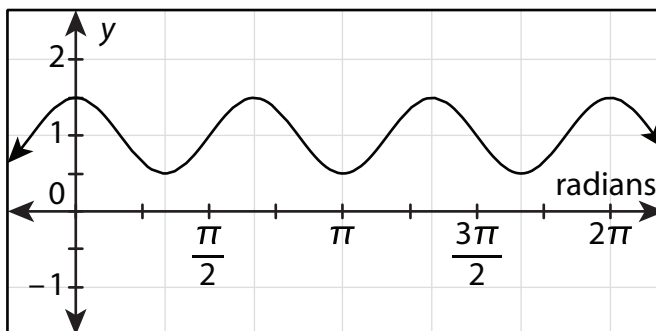
1. Determine the period.

   The period can be found by determining the horizontal distance between two maximum points, but in this case, the exact *x*-value of the maximum points is unclear. Instead, find the period by determining the length of one repetition of the curve.

   The curve repeats 3 times from 0 to 2π. Thus, one repetition is $\dfrac{1}{3}$ of 2π or $\dfrac{2\pi}{3}$.

   The period of this function is $\dfrac{2\pi}{3}$.

2. Determine the frequency.

   The frequency can be found by taking the reciprocal of the period.

   The period, as determined in step 1, is $\dfrac{2\pi}{3}$. The reciprocal of this is $\dfrac{3}{2\pi}$; therefore, the frequency is $\dfrac{3}{2\pi}$.

3. Determine the midline.

Recall that the midline is the horizontal line located halfway between the function's minimum and maximum points.

By inspecting the graph, we can see that the maximum point occurs at $y = 1.5$, and the minimum occurs at $y = 0.5$.

Find the average of 1.5 and 0.5.

$$\frac{1.5 + 0.5}{2} = \frac{2}{2} = 1$$

The average is 1; therefore, the midline occurs at $y = 1$.

We can verify this by inspecting the graph and confirming that the horizontal line $y = 1$ runs through the middle of the curve.

4. Determine the amplitude.

The amplitude is found by determining how far the function rises above its midline.

The midline is $y = 1$, and the highest the curve rises is $y = 1.5$.

Subtract to find the distance between them. (Or, count the vertical distance from the midline to the maximum point on the graph.)

$1.5 - 1 = 0.5$

The amplitude of the function is 0.5.

Example 3

Determine the period, frequency, midline, and amplitude of the function.

$$f(x) = 2 \, \cos\left(\frac{1}{3}x\right) - 4$$

1. Determine the period.

   In the general form of the cosine function, $f(x) = a \, \cos\,[b(x - c)] + d$, the period is $\frac{2\pi}{b}$.

   In $f(x) = 2 \, \cos\left(\frac{1}{3}x\right) - 4$, $b = \frac{1}{3}$ since $\frac{1}{3}$ is the coefficient of x.

   Substitute $\frac{1}{3}$ for b in the formula for the period and then solve.

$\dfrac{2\pi}{b}$	Formula for the period
$= 2\pi \div \left(\dfrac{1}{3}\right)$	Rewrite the fraction as division and substitute $\dfrac{1}{3}$ for b.
$= 2\pi \cdot \dfrac{3}{1}$	To divide, multiply by the reciprocal of the second fraction.
$= 6\pi$	Simplify.

   The period of the function is 6π.

   This means the length of one repetition of the curve is 6π.

2. Determine the frequency.

   The frequency is the reciprocal of the period.

   The reciprocal of 6π is $\frac{1}{6\pi}$. Therefore, the frequency of the function is $\frac{1}{6\pi}$.

3. Determine the midline.

    In the general form of the cosine function, $f(x) = a \cos [b(x - c)] + d$, d is the horizontal shift. In other words, d indicates how many units the midline has moved from the origin.

    In the function $f(x) = 2 \cos \left(\dfrac{1}{3} x \right) - 4$, $d = -4$. This means that the entire function has shifted down 4 units and that the midline is at $y = -4$.

4. Find the amplitude of the function.

    In the general form of the cosine function, $f(x) = a \cos [b(x - c)] + d$, $|a|$ is the amplitude.

    In the function $f(x) = 2 \cos \left(\dfrac{1}{3} x \right) - 4$, $a = 2$.

    Therefore, the amplitude of the function is 2.

    This means that the curve rises 2 units above the midline.

Example 4

Describe the difference between the function $f(x) = 3 \sin (x - 1)$ and the function $g(x) = 3 \sin [2(x - 1)]$.

1. Identify the difference between the two functions and determine the property to which it relates.

    The functions are nearly identical, but for one crucial difference. By inspection, we can see that the second function has a coefficient for the term $x - 1$, whereas the first function does not.

    The general form of the sine function is $f(x) = a \sin [b(x - c)] + d$.

    The difference in the two functions is the b value, the value of the period.

2. Determine the period of each function.

First, find the period of $f(x) = 3 \sin (x - 1)$.

Use the formula for the period, and substitute 1 for b, since the x has no coefficient and the b value is understood to be 1.

$$\frac{2\pi}{b}$$ Formula for the period

$$= \frac{2\pi}{(1)}$$ Substitute 1 for b.

$$= 2\pi$$ Divide.

The period of $f(x) = 3 \sin (x - 1)$ is 2π.

Next, find the period of $g(x) = 3 \sin [2(x - 1)]$.

Use the formula for the period, and substitute 2, the coefficient of the $x - 1$ term, for b.

$$\frac{2\pi}{b}$$ Formula for the period

$$= \frac{2\pi}{(2)}$$ Substitute 2 for b.

$$= \pi$$ Divide.

The period of $g(x) = 3 \sin [2(x - 1)]$ is π.

3. Describe the difference between the functions.

The period of $f(x) = 3 \sin (x - 1)$ is 2π and the period of $g(x) = 3 \sin [2(x - 1)]$ is π. The period of $f(x)$ is twice as long as that of $g(x)$. Therefore, one repetition of the function $f(x)$ will be twice as long as one repetition of the function $g(x)$.

UNIT 3 • TRIGONOMETRY OF GENERAL TRIANGLES AND TRIGONOMETRIC FUNCTIONS
Lesson 3: Graphs of Trigonometric Functions

Practice 3.3.1: Periodic Phenomena and Amplitude, Frequency, and Midline

For problems 1–7, determine the period, frequency, midline, and amplitude of each trigonometric function.

1.

3.

2.

4.

5. $f(x) = 5 \sin [4(x - 1)]$

6. $f(x) = \dfrac{1}{4} \cos (2x) - 3$

7. $f(x) = \sin \left[\dfrac{2}{7}(x - 4) \right] + 5$

For problems 8–10, the formula for the periodic motion of a spring is given. Determine the frequency of the spring, or how many times it oscillates per second.

8. $f(x) = 2 \sin (8\pi t)$

9. $f(x) = \sin (3\pi t)$

10. $f(x) = 4 \cos \left(\dfrac{5}{2} \pi t \right)$

Lesson 3.3.2: Using Trigonometric Functions to Model Periodic Phenomena

Introduction

Each sine or cosine function can be distinguished by its period, frequency, midline, and amplitude. Being able to recognize these aspects of a sine or cosine function can be useful in modeling real-life situations that repeat at regular intervals, or **periodic phenomena**. Once the period, frequency, midline, and amplitude are known, the equation of the function can be determined.

Key Concepts

- Recall that the general form of the sine function is $f(x) = a \sin [b(x - c)] + d$ and the general form of the cosine function is $f(x) = a \cos [b(x - c)] + d$. For both functions, $|a|$ is the amplitude, $\dfrac{2\pi}{b}$ is the period, c is the horizontal shift (also known as the phase shift), and d is the vertical shift.

- To find the equation of a function, determine the a, b, c, and d values, and then substitute them into the general form.

- Since the only difference between the sine and cosine curve is the horizontal shift, these functions can be described both in terms of a sine function and in terms of a cosine function.

- A basic sine function starts at the midline and travels up to its maximum point. Upon reaching the maximum, the function then curves down past the midline to its minimum point, and then returns to the midline.
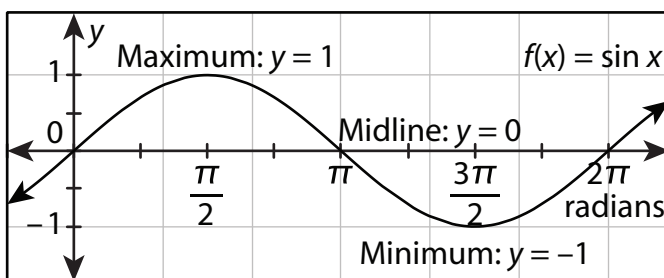
- A basic cosine function, on the other hand, starts at the maximum point and then travels down past the midline to the minimum point before returning up to the maximum point.

$f(x) = \cos x$

Maximum: $y = 1$

Midline: $y = 0$

$\dfrac{\pi}{2}$ π $\dfrac{3\pi}{2}$ 2π radians

Minimum: $y = -1$
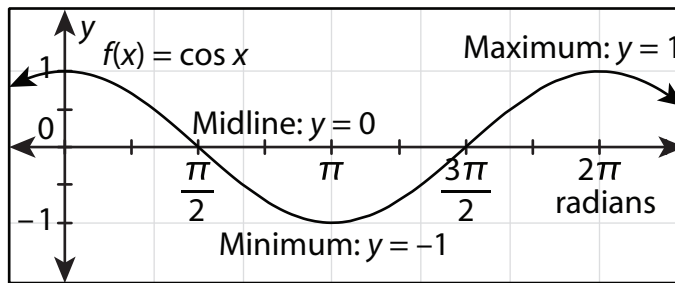
Guided Practice 3.3.2

Example 1

Find the equation of a sine function with no horizontal shift whose frequency is 2. The function rises 3 units above its midline, which is $y = -1$.

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1. Determine the value of a.

   a represents the amplitude of the function, or how far the function rises above its midline.

   Since the function rises 3 units above its midline, its amplitude is 3.

   The value of a, the amplitude, is 3.

---

2. Determine the value of b.

   The period (the length of one cycle of the curve) is equal to $\dfrac{2\pi}{b}$, and the frequency is the reciprocal of the period.

   The reciprocal of the given frequency, 2, is $\dfrac{1}{2}$. Thus, $\dfrac{1}{2}$ is the period.

   Substitute this value into the formula for the period and solve for b.

$\text{period} = \dfrac{2\pi}{b}$	Formula for the period
$\left(\dfrac{1}{2}\right) = \dfrac{2\pi}{b}$	Substitute $\dfrac{1}{2}$ for the period.
$1(b) = 2(2\pi)$	Cross multiply to eliminate the fractions.
$b = 4\pi$	Multiply.

   The value of b is 4π.

---

3. Determine the value of c.

c is the horizontal shift. Since the function has no horizontal shift, $c = 0$.

4. Determine the value of d.

d is the vertical shift.

Since the function's midline is $y = -1$, this means that the entire function has been shifted down one unit (-1), so $d = -1$.

5. Substitute a, b, c, and d into the general form of the sine function.

Substitute the known values into the general form of the sine function and simplify.

$f(x) = a \sin [b(x - c)] + d$	General form of the sine function
$f(x) = (3) \sin \{(4\pi)[x - (0)]\} + (-1)$	Substitute 3 for a, 4π for b, 0 for c, and -1 for d.
$f(x) = 3 \sin (4\pi x) - 1$	Simplify.

The equation of the function is $f(x) = 3 \sin (4\pi x) - 1$.

Example 2

Write an equation to describe the graphed function.
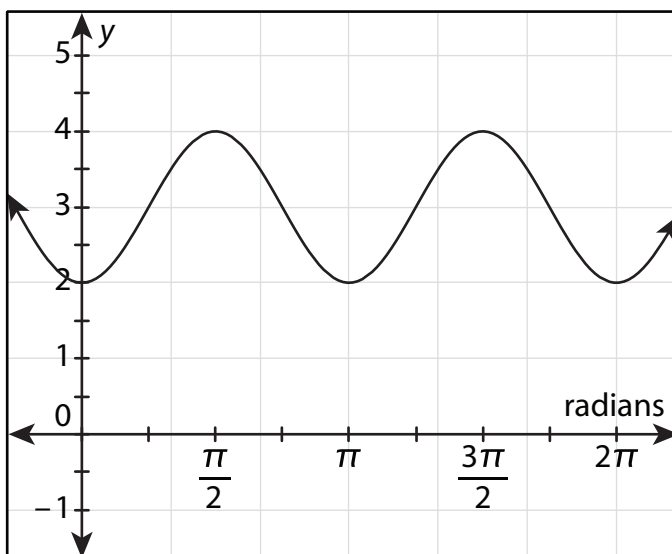
1. Choose to use either the sine or cosine form.

   This periodic function can be described by a sine function or a cosine function. Either function will work; however, since the cosine curve begins at the maximum point, it may allow for more efficient calculations of the features. At the origin, the curve is not at the midline or maximum point, so there is a horizontal shift for both the sine and cosine function.

   There is an identifiable maximum at $x = \dfrac{\pi}{2}$. Thus, this function can be described as a cosine function that has been shifted $\dfrac{\pi}{2}$ units to the right.

   (Note that the curve can also be described as a sine function that has been shifted $\dfrac{\pi}{4}$ units to the right, but this example will focus on finding the equation by using the cosine function.)

2. Determine the value of a.

   a is the amplitude, or how far the function rises above its midline.

   The midline is $y = 3$, and the function rises 1 unit above its midline.

   Therefore, the value of a, the amplitude, is 1.

3. Determine the value of b.

   The period (the length of one cycle of the curve) is equal to $\dfrac{2\pi}{b}$.

   One cycle of this cosine curve starts at $x = \dfrac{\pi}{2}$ and ends at $x = \dfrac{3\pi}{2}$.

   Subtract to find the length of one cycle of the curve.

   $$\frac{3\pi}{2} - \frac{\pi}{2} = \frac{2\pi}{2} = \pi$$

   The period of the function is π.

   Substitute this value into the formula for the period of the function and solve for b.

$\text{period} = \dfrac{2\pi}{b}$	Formula for the period
$(\pi) = \dfrac{2\pi}{b}$	Substitute π for the period.
$\pi b = 2\pi$	Multiply both sides of the equation by b to eliminate the fraction.
$b = \dfrac{2\pi}{\pi}$	Divide both sides of the equation by π.
$b = 2$	Simplify.

   The value of b is 2.

4. Determine the value of c.

    c is the horizontal shift.

    As determined in step 1, the cosine function has been shifted $\dfrac{\pi}{2}$ units to the right. Shifts to the right are designated as positive; shifts to the left are negative.

    The value of c, the horizontal shift, is $\dfrac{\pi}{2}$.

5. Determine the value of d.

    d is the vertical shift.

    The midline is $y = 3$; thus, the function has been shifted up 3 units (+3).

    The value of d, the vertical shift, is 3.

6. Substitute a, b, c, and d into the general form of the cosine function.

    Substitute the known values into the general form of the cosine function and simplify.

    $f(x) = a \cos [b(x - c)] + d$ General form of the cosine function

    $f(x) = (1) \cos \left\{ (2) \left[x - \left(\dfrac{\pi}{2} \right) \right] \right\} + (3)$ Substitute 1 for a, 2 for b, $\dfrac{\pi}{2}$ for c, and 3 for d.

    $f(x) = \cos \left[2 \left(x - \dfrac{\pi}{2} \right) \right] + 3$ Simplify.

    $f(x) = \cos (2x - \pi) + 3$ Distribute to simplify further, if desired.

    The equation of the function is $f(x) = \cos \left[2 \left(x - \dfrac{\pi}{2} \right) \right] + 3$ or $f(x) = \cos (2x - \pi) + 3$.

Example 3

The following graph shows historical average monthly temperatures for the town of Mayorsville starting in January 2000. Write an equation for the graphed function.

Average Monthly Temperature in Mayorsville
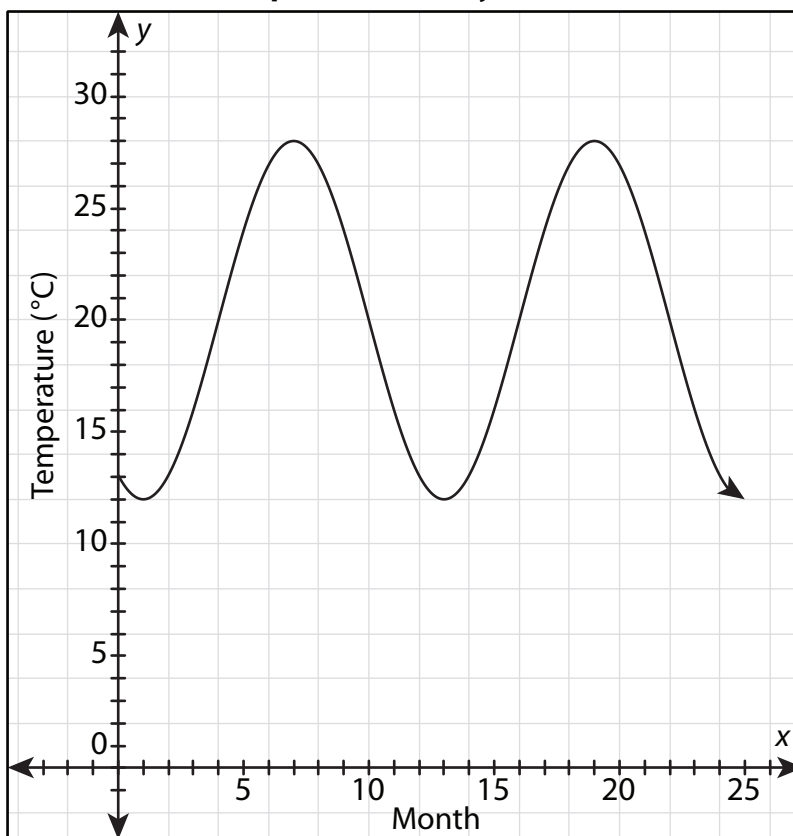
1. Choose to use either the sine or cosine form.

   This periodic function can be described by either a sine function or a cosine function. However, since the cosine curve begins at the maximum point, it may allow for more efficient calculations of the features. At the origin, the curve is not at the midpoint or maximum point, so there is a horizontal shift for both the sine and cosine function.

   There is an identifiable maximum at $x = 7$. Thus, this function can be described as a cosine function that has been shifted 7 units to the right.

   (Similar to the curve in Example 2, this curve can also be described as a sine function that has been shifted 4 units to the right, but this example will focus on finding the equation by using the cosine function.)

2. Determine the value of a.

a is the amplitude, or how far the function rises above its midline.

The function's maximum point is at $y = 28$ and its minimum point is at $y = 12$.

Determine the average of these two values.

$$\frac{28+12}{2} = \frac{40}{2} = 20$$

The average of 28 and 12 is 20, so the midline is $y = 20$.

Subtract to determine how far the function rises above its midline.

$28 - 20 = 8$

The value of a, the amplitude, is 8.

3. Determine the value of b.

The period (the length of one cycle of the curve) is equal to $\dfrac{2\pi}{b}$.

One cycle of this cosine curve starts at $x = 7$ and ends at $x = 19$.

Subtract to find the length of one cycle of the curve.

$19 - 7 = 12$

The period of the function is 12.

Substitute this value into the formula for the period and solve for b.

$$\text{period} = \frac{2\pi}{b} \qquad \text{Formula for the period}$$

$$(12) = \frac{2\pi}{b} \qquad \text{Substitute 12 for the period.}$$

$$12b = 2\pi \qquad \text{Multiply both sides of the equation by } b \text{ to eliminate the fraction.}$$

$$b = \frac{2\pi}{12} \qquad \text{Divide both sides of the equation by 12.}$$

$$b = \frac{\pi}{6} \qquad \text{Simplify.}$$

The value of b is $\dfrac{\pi}{6}$.

4. Determine the value of *c*.

c is the horizontal shift.

As determined in step 1, the cosine function has been shifted 7 units to the right. Shifts to the right are designated as positive; shifts to the left are negative.

The value of *c*, the horizontal shift, is 7.

5. Determine the value of *d*.

d is the vertical shift.

As determined in step 2, the midline is $y = 20$. Thus, the function has been shifted up 20 units (+20).

The value of *d*, the vertical shift, is 20.

6. Substitute *a*, *b*, *c*, and *d* into the general form of the cosine function.

Substitute the known values into the general form of the cosine function and simplify.

$$f(x) = a \cos [b(x - c)] + d$$ General form of the cosine function

$$f(x) = (8) \cos \left\{ \left(\frac{\pi}{6} \right) [x - (7)] \right\} + (20)$$ Substitute 8 for *a*, $\frac{\pi}{6}$ for *b*, 7 for *c*, and 20 for *d*.

$$f(x) = 8 \cos \left[\frac{\pi}{6} (x - 7) \right] + 20$$ Simplify.

The equation of the function is $f(x) = 8 \cos \left[\frac{\pi}{6}(x - 7) \right] + 20$.
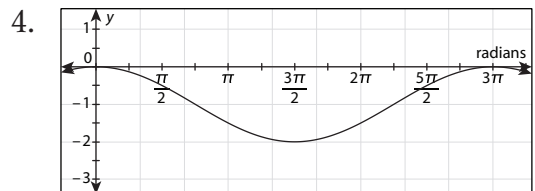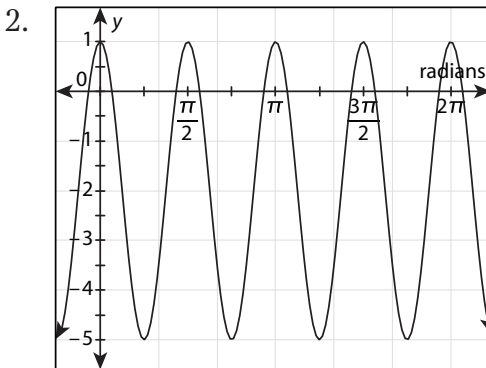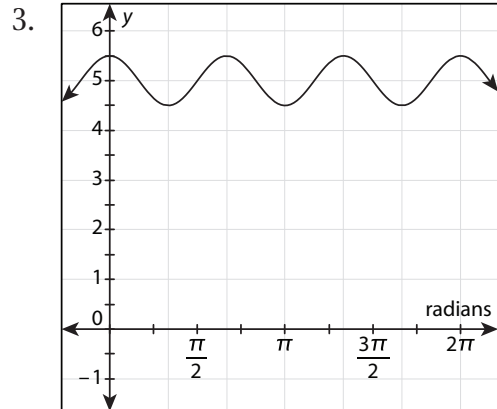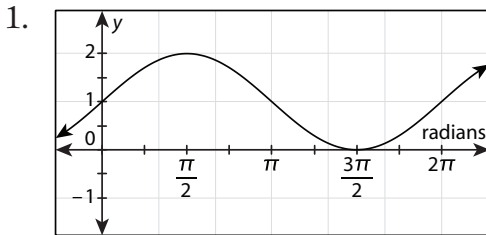
UNIT 3 • TRIGONOMETRY OF GENERAL TRIANGLES AND TRIGONOMETRIC FUNCTIONS
Lesson 3: Graphs of Trigonometric Functions

Practice 3.3.2: Using Trigonometric Functions to Model Periodic Phenomena

For problems 1–7, write a trigonometric equation to describe each function.
Note: Some problems may have more than one correct answer, but only one answer is needed.

1. A sine curve that is shifted $\dfrac{\pi}{5}$ units to the left has a midline at $y = 1$ and rises 2 units above the midline. The length of one cycle of its curve is π.

2. A cosine curve with no horizontal shift has a frequency of $\dfrac{3}{\pi}$ and rises 2 units above its midline, which is at $y = -4$.

3. A cosine curve has two consecutive maximum points at $\left(\dfrac{\pi}{2}, 2 \right)$ and $\left(\dfrac{3\pi}{2}, 2 \right)$, and a minimum point at $(\pi, -4)$.
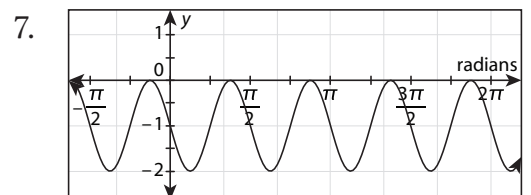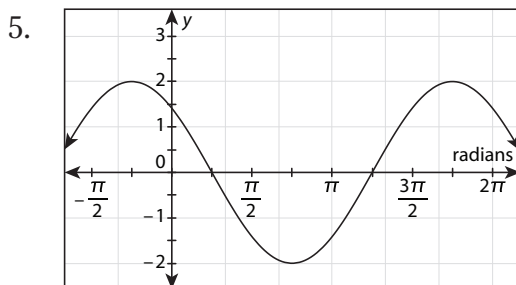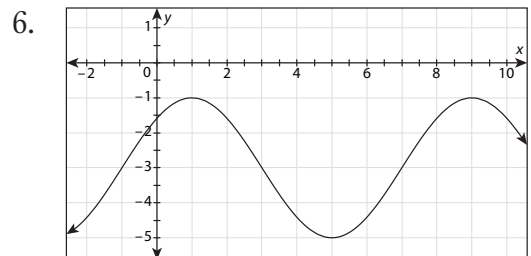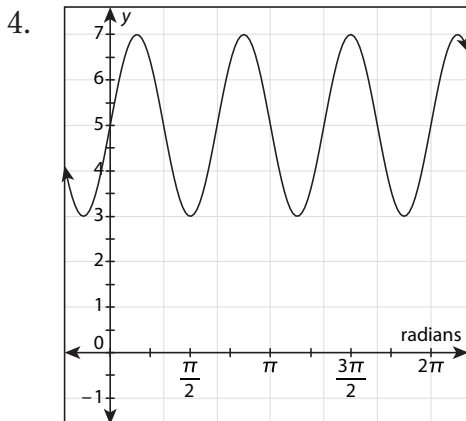
4.

5.

6.

7.

UNIT 3 • TRIGONOMETRY OF GENERAL TRIANGLES AND TRIGONOMETRIC FUNCTIONS
Lesson 3: Graphs of Trigonometric Functions

For problems 8–10, write the trigonometric equation of the function that models each periodic phenomenon.

8. When Isaac released a spring, it completed one oscillation (cycle) every 2 seconds and traveled up to 3 inches from equilibrium (its midline). He knows the oscillation can be described by a cosine function with no horizontal or vertical displacement. Write an equation to describe this function.

9. Elise played a note on her flute that had an amplitude of 3 and a frequency of 1,480 Hz. She knows this note can be described by a sine function with no horizontal or vertical displacement. Write an equation to describe this function.

10. Since January 2010, the average monthly temperatures in Hannonsville have fluctuated in accordance with the function shown in the following graph. Write an equation to describe this function.
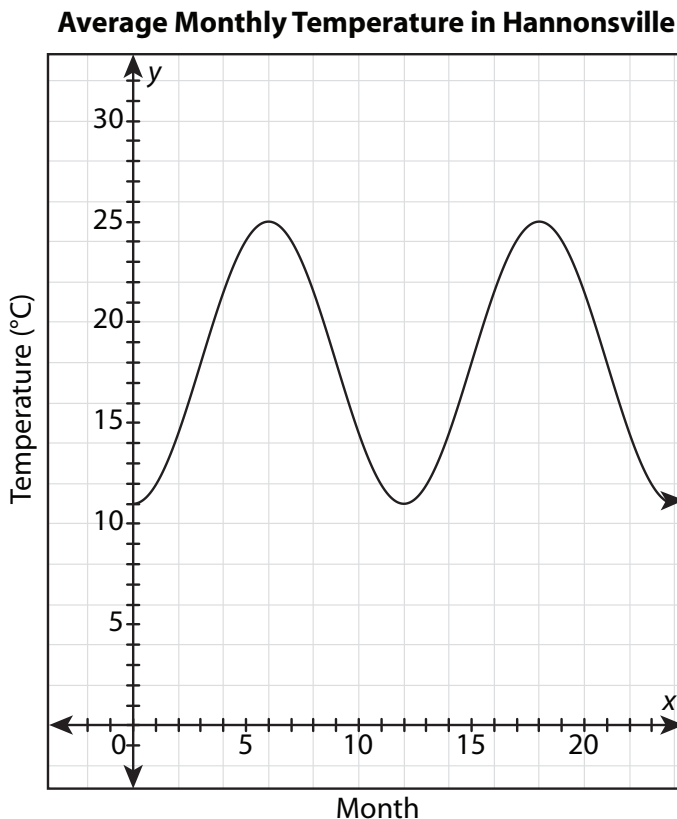
Average Monthly Temperature in Hannonsville

Month

Unit 4A

Mathematical Modeling of Inverse, Logarithmic, and Trigonometric Functions

$f(x)$ $\angle ABC$ Q_1
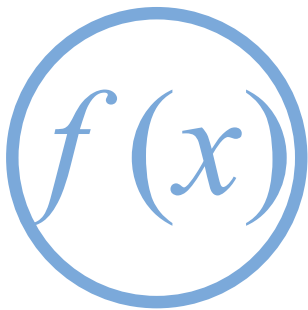
Lesson 1: Inverses of Functions

Common Core State Standard

F–BF.4   Find inverse functions.

　　　　　　a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. *For example, f(x) = 2x³ or f(x) = (x + 1)/(x − 1) for x ≠ 1.*

Essential Questions

1. How can you determine if the functions $f(x)$ and $g(x)$ are inverse functions of each other if $f(a) = b$ and $g(b) = a$?

2. How do the graphs of a function $f(x)$ and its inverse $f^{-1}(x)$ compare?

3. How do the domains and ranges of a function and its inverse compare?

4. How can you identify a function that does not have an inverse?

WORDS TO KNOW

domain	the set of all input values (x-values) that satisfy the given function without restriction
function	a relation in which every element of the domain is paired with exactly one element of the range; that is, for every value of x, there is exactly one value of y
inverse function	the function that may result from switching the x- and y-variables in a given function; the inverse of $f(x)$ is written as $f^{-1}(x)$
inverse relation	a relation $g(x)$ such that $g(f(x)) = x$ and $f(g(y)) = y$ where $f(x)$ is a function
one-to-one correspondence	the feature of a function whereby each value in the domain corresponds to a unique function value; that is, if $x = a$ and $x = b$, the two points would be $(a, f(a))$ and $(b, f(b))$, and if $a \neq b$, then $f(a) \neq f(b)$ for a function to exhibit one-to-one correspondence

quadratic function	a function defined by a second-degree expression of the form $f(x) = ax^2 + bx + c$, where $a \neq 0$ and a, b, and c are constants. The graph of any quadratic function is a parabola.
range	the set of all outputs of a function; the set of y-values that are valid for the function
relation	a relationship between two variables in which at least one value of the domain or independent variable, x, is matched with one or more values of the dependent or range variable, y

Recommended Resources

- Analyzemath.com. "Inverse of Quadratic Functions."

  http://www.walch.com/rr/00227

  This site provides easy-to-follow, simple examples of how to find the inverse of a quadratic function with a restricted domain. *Note*: The shorthand "sqrt(*x*)" is used for $\sqrt{x}$.

- ChiliMath. "Finding the Inverse Function of a Quadratic Function."

  http://www.walch.com/rr/00228

  This tutorial covers how to quickly determine whether a quadratic function has an inverse, and if so, how to find the inverse. Example problems work through the solutions step-by-step.

- MathIsFun.com. "Inverse Functions."

  http://www.walch.com/rr/00229

  This site uses a step-by-step approach to explain and review inverse functions.

IXL Links

- Find inverse functions and relations:
  http://www.ixl.com/math/algebra-2/find-inverse-functions-and-relations

- Solve exponential equations using common logarithms:
  http://www.ixl.com/math/algebra-2/solve-exponential-equations-using-common-logarithms

- Solve exponential equations using natural logarithms:
  http://www.ixl.com/math/algebra-2/solve-exponential-equations-using-natural-logarithms

- Solve logarithmic equations:
  http://www.ixl.com/math/algebra-2/solve-logarithmic-equations

- Solve logarithmic equations ii:
  http://www.ixl.com/math/algebra-2/solve-logarithmic-equations-ii

Lesson 4A.1.1: Determining Inverses of Quadratic Functions

Introduction

Several kinds of relationships exist between variables. For example, a **relation** is a relationship between two variables in which at least one value of the domain or independent variable, x, is matched with one or more values of the dependent or range variable, y. The **domain** refers to the set of all input values (x-values) that satisfy the given function without restriction and the **range** refers to the values of the dependent variable of the function. Sometimes, a relation is said to be a **function**, a relation in which every element of the domain is paired with exactly one element of the range; that is, for every value of x, there is exactly one value of y. Both relations and functions have inverses. In this section, we will explore how to determine the inverse of a quadratic function, as well as how to verify where the inverse exists. A **quadratic function** is a function defined by a second-degree expression of the form $f(x) = ax^2 + bx + c$, where $a \neq 0$ and a, b, and c are constants. The graph of any quadratic function is a parabola.

Key Concepts

- Determining the inverse of a function often requires examining the original function's domain and range.

- An **inverse relation** is a relation $g(x)$ such that $g(f(x)) = x$ and $f(g(y)) = y$ where $f(x)$ is a function.

- If the inverse relation $g(x)$ is also a function, it is said to be an **inverse function**. It is the function that results from switching the x- and y-variables in the given function, $f(x)$. The inverse of $f(x)$ is also written as $f^{-1}(x)$.

- Note that this $f^{-1}(x)$ notation does *not* mean to take the -1 power of $f(x)$. Both notations, $f^{-1}(x)$ and $g(x)$, are used to represent an inverse.

- A function $g(x)$ is the inverse of a function $f(x)$ if $g(f(x)) = x$ and $f(g(y) = y$; or, $g(x)$ is the inverse of $f(x)$ if $f(a) = b$ and $g(b) = a$.

- For example, if $f(x) = x + 1$, then we can switch the domain and range variables (x and $f(x)$) and rename $f(x)$ as $g(x)$ to arrive at its inverse: the inverse of $f(x) = x + 1$ is $x = g(x) + 1$. Or, solved for $g(x)$, $g(x) = x - 1$. Notice that $f(x)$ becomes its inverse $g(x)$ once the variables are switched.

- For a function to have an inverse function, it has to satisfy the condition of **one-to-one correspondence**. This means that each value in the domain corresponds to a unique function value; that is, if $x = a$ and $x = b$, the two points would be $(a, f(a))$ and $(b, f(b))$; if $a \neq b$, then $f(a) \neq f(b)$ for a function to exhibit one-to-one correspondence. In other words, for a function to have an inverse, it cannot have the same range/output value for two different domain values. It is particularly important to check for one-to-one correspondence when finding the inverse of a quadratic function.

- Recall that a quadratic function is a second-degree function of the form $f(x) = ax^2 + bx + c$, where $a \neq 0$ and a, b, and c are constants.

- Since quadratic functions are parabolic, this means there is likely more than one output value for each input value, or more than one input value for each output, depending on the equation. For example, if $f(x) = x^2$, then values of $x = 2$ and $x = -2$ both result in $f(x) = 4$. The resulting points, (2, 4) and (–2, 4), do not have unique output values—they share 4 as the output. Therefore, determining the inverse of a quadratic function requires some additional steps.

- To construct an inverse relation or a possible inverse function for a quadratic function, switch the dependent and independent variables, x and y, and solve for the new function, y.

- For example, take the function $f(x) = 3x^2 - 4$. Switching the variables and renaming $f(x)$ as $g(x)$ results in $x = 3[g(x)]^2 - 4$. Simplifying and solving this new equation for $g(x)$ results in $g(x) = \pm \dfrac{\sqrt{3(x+4)}}{3}$.

- From the graph that follows, notice that the domain of the original function, $f(x) = 3x^2 - 4$, is $(-\infty, \infty)$, whereas the domain of the new inverse relation, $g(x) = \pm \dfrac{\sqrt{3(x+4)}}{3}$, is $[-4, \infty)$. (Also notice that $g(x)$ is called an inverse relation since we haven't determined whether it is a function.) The differing domains suggest that there is no inverse that exists over the entire domain of the original function, but there *is* an inverse over part of the domain: the interval $[-4, \infty)$. Thus for the function $f(x) = 3x^2 - 4$, the inverse function $g(x) = \pm \dfrac{\sqrt{3(x+4)}}{3}$ exists over the restricted domain $[-4, \infty)$. Recall the vertical line test to determine if a relation is a function. If the entire parabola for $g(x)$ were included, the relation $g(x)$ would not be a function.

- Another way to determine whether the inverse of a quadratic function exists is to graph the function and its possible inverse and determine whether the functions show one-to-one correspondence.

- You may recall using the vertical line test to determine if a relation is a function. Once a function passes the *vertical* line test, the *horizontal* line test can then be used to determine if the function is one-to-one; if the horizontal line passes through the function at more than one point, the function fails the test and is not one-to-one.

- To visualize this, look at the graph of $f(x) = 3x^2 - 4$ and its positive inverse, $g(x) = \dfrac{\sqrt{3(x+4)}}{3}$. (Note that the negative inverse, $g(x) = -\dfrac{\sqrt{3(x+4)}}{3}$, has been excluded from the graph.) A horizontal line has been drawn through the graph of $f(x)$ at $y = 1.5$.
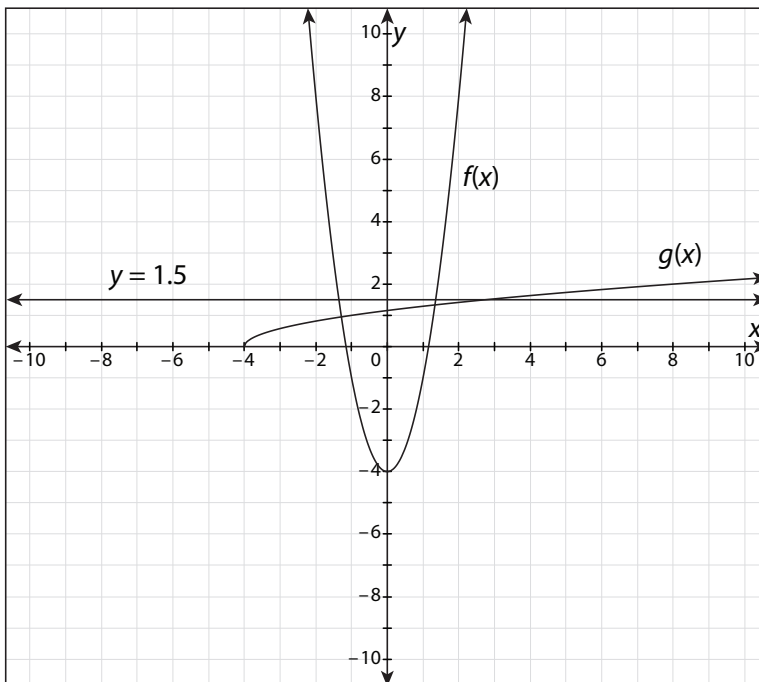
- Notice that the horizontal line $y = 1.5$ intersects the original function at two points, which means that $f(x) = 3x^2 - 4$ does not have an inverse over its entire domain of $(-\infty, \infty)$. In other words, there is not one-to-one correspondence for each point on $f(x)$: the function values of $f(x)$ have *two* possible domain values over some parts of the domain.

- However, if the domain is restricted to the intervals $[0, \infty)$ or $[-4, 0]$, then $f(x)$ does have an inverse $g(x)$ because the domain of $g(x)$, $[-4, \infty)$, includes the restricted domains. Thus, it is possible for a quadratic function to have an inverse, given a restricted domain.

- A graphing calculator can be used to explore a quadratic function and the domain(s) over which its inverse exists.

- Note that a graphing calculator will not provide rigorous proof of the existence of an inverse and its domain. Furthermore, the horizontal line test is not as accurate a method of testing a function and its inverse for one-to-one correspondence. Therefore, when using a graphing calculator, it is necessary to check the results algebraically after plotting the functions.

On a TI-83/84:

Step 1: Press [Y=]. Press [CLEAR] to delete any other functions stored on the screen.

Step 2: At Y_1, enter the original function using your keypad. Press [ENTER].

Step 3: Enter the possible inverse function(s) using your keypad. *Note*:

When both positive and negative inverses are possible, each

function needs to be entered separately to be graphed. For example,

to graph $g(x) = \pm\dfrac{\sqrt{3(x+4)}}{3}$, enter $\dfrac{\sqrt{3(x+4)}}{3}$ at Y_2 and $-\dfrac{\sqrt{3(x+4)}}{3}$

at Y_3. Press [ENTER].

Step 4: To view a graph of $f(x)$ and its inverse $g(x)$, press [GRAPH].

Step 5: To view the table of values, press [2ND][GRAPH]. Take note of any undefined values for the functions.

On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow over to the graphing icon, the second icon from the left, and press [enter].

Step 3: At *f1(x)*, enter the original function using your keypad. Press [enter].

Step 4: To enter the possible inverse function(s), press [tab]. *Note*: When

both positive and negative inverses are possible, each function

needs to be entered separately to be graphed. For example, to graph

$g(x) = \pm \dfrac{\sqrt{3(x+4)}}{3}$, enter $\dfrac{\sqrt{3(x+4)}}{3}$ at *f2(x)* and $-\dfrac{\sqrt{3(x+4)}}{3}$ at

f3(x). Press [enter].

Step 5: To view the graphs of *f(x)* and its inverse *g(x)*, press [tab]. Arrow up to display the equation you wish to graph, and then press [enter]. Repeat for each equation you wish to graph on the same screen.

Step 6: To view the table of values, press [ctrl][T]. Take note of any undefined values for the functions.

Guided Practice 4A.1.1

Example 1

Find the inverse, $g(x)$, of the function $f(x) = 4x^2$ and determine the domain value(s) over which the inverse exists.

1. Switch the domain and function variables, and then rename $f(x)$ as $g(x)$.

    Since the inverse function results from switching the input and output variables of the original function, identify which variable represents the input and which represents the output. For the function $f(x) = 4x^2$, x is the input (independent) variable, and $f(x)$ is the output (dependent) variable. Therefore, swap $f(x)$ and x.

$f(x) = 4x^2$	Original function
$x = 4[f(x)]^2$	Switch $f(x)$ and x.

    Next, rename $f(x)$ as $g(x)$, since this new relation represents the inverse of the original function.

$x = 4[f(x)]^2$	Equation with variables switched
$x = 4[g(x)]^2$	Rename $f(x)$ as $g(x)$.

    The resulting inverse relation, $x = 4[g(x)]^2$, is the *possible* inverse function of the original function. It must be evaluated to determine whether it has one-to-one correspondence with the original function, $f(x) = 4x^2$.

2. Solve the possible inverse for $g(x)$.

$x = 4[g(x)]^2$	Possible inverse
$\dfrac{x}{4} = [g(x)]^2$	Divide both sides by 4.
$g(x)^2 = \dfrac{x}{4}$	Rearrange using the Symmetric Property of Equality.
$g(x) = \pm\sqrt{\dfrac{x}{4}}$	Take the square root of both sides.
$g(x) = \pm\dfrac{\sqrt{x}}{\sqrt{4}}$	Apply the square root to both the numerator and the denominator.
$g(x) = \pm\dfrac{\sqrt{x}}{2}$	Simplify.
$g(x) = \pm\dfrac{1}{2}\sqrt{x}$	Eliminate the denominator by rewriting the division as multiplication by a fraction.

3. Determine the domain of $g(x)$.

We need to know where $g(x)$ is defined in order to know its domain, which we want to compare to the range of $f(x)$.

$g(x)$ is a radical function; specifically, it is a square root function. Recall that the value of the expression inside the radical must be positive in order for the function to be defined. The expression under the square root is x; therefore, the values of x are limited to non-negative numbers.

Thus, the domain of $g(x)$ is $[0, +\infty)$ because x is raised to the $\dfrac{1}{2}$ power (recall that $\sqrt{x} = x^{\frac{1}{2}}$).

4. Determine the range of $g(x)$.

We need to compare the range of $g(x)$ to the domain of $f(x)$.

There are no limitations on the output values; therefore, the range of $g(x)$ is $(-\infty, +\infty)$.

5. Determine the domain of $f(x)$.

The domain of $f(x)$ is $(-\infty, +\infty)$. The domain will need to be restricted to those values over which $f(x)$ shows one-to-one correspondence.

6. Determine whether the function $f(x)$ exhibits one-to-one correspondence.

The function $f(x)$ can only have an inverse over the part of its domain for which it exhibits one-to-one correspondence.

We can determine that the function $f(x) = 4x^2$ does not show one-to-one correspondence for all range values because $x = a$ and $x = -a$ yield equal values for $f(x)$: $f(a) = 4a^2$ and $f(-a) = 4(-a)^2 = 4a^2$. You can apply the horizontal line test and exclude portions of $f(x)$ where this test fails; that is, where the horizontal line crosses $f(x)$ at more than one point.

7. Determine the parts of its domain over which $f(x)$ exhibits one-to-one correspondence.

The restricted domains of $f(x)$ are $(-\infty, 0]$ and $[0, +\infty)$. Only the point (0, 0) is not restricted, because all of the other points do not have one-to-one correspondence. The horizontal line crosses the graph at two points for all other points on the graph.

8. Determine the range of $f(x)$.

Because $f(x)$ is a quadratic function and x^2 can never be negative, the range of $f(x)$ is $[0, +\infty)$.

9. Match domains to ranges to find the inverse(s) of $f(x)$.

In order for a function to have an inverse, it must be one-to-one. Therefore, match the restricted domains of $f(x)$ to the range of $g(x)$.

Restricted domain of $f(x)$:	Inverse function with matching range:
$[0, +\infty)$	$g(x) = \dfrac{1}{2}\sqrt{x}$
$(-\infty, 0]$	$g(x) = -\dfrac{1}{2}\sqrt{x}$

Check that the range of $f(x)$ matches the domain of $g(x)$.

The range of $f(x)$ is $[0, +\infty)$. The domain of $g(x)$ is $[0, +\infty)$.

The inverse of $f(x) = 4x^2$ on the restricted domain $0 \le x < +\infty$ is $g(x) = \dfrac{1}{2}\sqrt{x}$.

The inverse of $f(x) = 4x^2$ on the restricted domain $-\infty < x \le 0$ is $g(x) = -\dfrac{1}{2}\sqrt{x}$.

Example 2

Find the inverse, $f^{-1}(x)$, of the function $f(x) = x(x - 1)$ and determine the domain and range value(s) over which the inverse exists.

1. Determine whether an inverse exists for this function.

   When $f(x)$ is simplified, it will contain x^2, so $f(x)$ is a quadratic function and its graph is a parabola. A parabola does not have an inverse because it does not have one-to-one correspondence. For any y-value in the range of $f(x) = x(x - 1)$, there are two corresponding x-values.

   In order for an inverse function to exist, we must restrict the domain of the given function so that it represents half of the parabola, as determined by the axis of symmetry.

   The axis of symmetry is the line $x = -\dfrac{b}{2a}$.

   Simplify the given equation in order to determine the values of a and b.

   $$f(x) = x(x - 1)$$

   $$f(x) = x^2 - x$$

   a is the coefficient of the x^2 term and b is the coefficient of the x term, so $a = 1$ and $b = -1$. Substitute these values to find the axis of symmetry.

   $$x = -\frac{b}{2a}$$

   $$x = -\frac{(-1)}{2(1)} = \frac{1}{2}$$

   The axis of symmetry is $x = \dfrac{1}{2}$. As long as we restrict the domain of

   $f(x)$ such that either $x \le \dfrac{1}{2}$ or $x \ge \dfrac{1}{2}$, the function will have an inverse.

   Let's use the restricted domain $\left[\dfrac{1}{2}, +\infty\right)$.

2. Determine the range of $f(x)$.

Because $f(x)$ is a parabolic function, its range is limited.

Since the coefficient of the quadratic term is positive, the parabola opens up, and the minimum of the range is the y-coordinate of the minimum of the parabola.

The x-coordinate of the minimum is known; it is the axis of symmetry, $x = \frac{1}{2}$. To determine the y-coordinate, substitute $x = \frac{1}{2}$ into the original function, $f(x) = x(x - 1)$. Then, solve for $f(x)$.

$f(x) = x(x - 1)$ Original function

$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)\left[\left(\frac{1}{2}\right) - 1\right]$ Substitute $\frac{1}{2}$ for x.

$f\left(\frac{1}{2}\right) = -\frac{1}{4}$ Simplify.

The minimum of the function is located at $\left(\frac{1}{2}, -\frac{1}{4}\right)$. Therefore, the range of $f(x)$ is $\left[-\frac{1}{4}, +\infty\right)$.

The restricted domain and corresponding range for $f(x)$ are now known. With this restricted domain, the function now has an inverse.

3. Calculate the inverse by switching the domain and function variables, and then renaming $f(x)$ as $f^{-1}(x)$.

Swap $f(x)$ and x.

$f(x) = x(x-1)$	Original function
$x = f(x)[f(x) - 1]$	Switch $f(x)$ and x.

Next, rename $f(x)$ as $f^{-1}(x)$, since this new relation represents the inverse rather than the original function.

$x = f(x)[f(x) - 1]$	Equation with variables switched
$x = f^{-1}(x)[f^{-1}(x) - 1]$	Rename $f(x)$ as $f^{-1}(x)$.

The inverse for the original function on the restricted domain is $x = f^{-1}(x)[f^{-1}(x) - 1]$.

4. Rewrite the inverse in a form that can be solved for $f^{-1}(x)$.

In this example, $f^{-1}(x)$ can be thought of as the dependent variable in a quadratic function, and x can be thought of as an independent variable. Here, $f^{-1}(x)$ is a single variable. Along with x, it can be manipulated according to the usual rules of algebra.

$x = f^{-1}(x)[f^{-1}(x) - 1]$	Inverse
$0 = f^{-1}(x)[f^{-1}(x) - 1] - x$	Subtract x from both sides.
$0 = [f^{-1}(x)]^2 - f^{-1}(x) - x$	Distribute $f^{-1}(x)$ over $[f^{-1}(x) - 1]$.

Thus, $x = f^{-1}(x)[f^{-1}(x) - 1]$ becomes $0 = [f^{-1}(x)]^2 - f^{-1}(x) - x$.

5. Solve for $f^{-1}(x)$ using the quadratic formula.

The quadratic formula is used for this equation since this function cannot be factored using whole numbers. In this equation, the coefficients in the quadratic formula are $a = 1$, $b = -1$, and $c = -x$. Substitute these values into the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Quadratic formula

$$\left[f^{-1}(x) \right] = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-x)}}{2(1)}$$ Substitute $f^{-1}(x)$ for x, 1 for a, -1 for b, and $-x$ for c.

$$f^{-1}(x) = \frac{1 \pm \sqrt{1 + 4x}}{2}$$ Simplify.

$$f^{-1}(x) = \frac{1}{2} \pm \frac{\sqrt{1 + 4x}}{2}$$ Rewrite as separate terms, with the terms in the numerator each written over the common denominator.

The inverse function is either $f^{-1}(x) = \dfrac{1}{2} + \dfrac{\sqrt{1 + 4x}}{2}$ or $f^{-1}(x) = \dfrac{1}{2} - \dfrac{\sqrt{1 + 4x}}{2}$. These equations represent the lower and upper halves of a "sideways" parabola. To determine the correct inverse function, look at the domain and range of each. Recall that the domain of the original function becomes the range of the inverse function, and the range of the original function becomes the domain of the inverse function.

6. Determine the domain of $f^{-1}(x)$.

The domain of $f^{-1}(x)$ should be the same as the range of the original function, $f(x)$, which is $\left[-\dfrac{1}{4}, +\infty\right)$. Verify this by determining where $f^{-1}(x)$ is defined. The domain of $f^{-1}(x)=\dfrac{1}{2}\pm\dfrac{\sqrt{1+4x}}{2}$ can be found by looking at the binomial under the radical sign, $1 + 4x$. Since the radical represents a real number value, $1 + 4x$ must be non-negative; therefore, $1+4x\geq 0$. Solve this inequality for x.

$1+4x\geq 0$	Inequality
$4x\geq -1$	Subtract 1 from both sides of the inequality.
$x\geq -\dfrac{1}{4}$	Divide both sides of the inequality by 4.

Since x is greater than *or equal to* $-\dfrac{1}{4}$, this value is included in the domain of $f^{-1}(x)$. Therefore, the domain of $f^{-1}(x)$ is $\left[-\dfrac{1}{4}, +\infty\right)$. This is the same as the range of $f(x)$.

7. Determine the range of $f^{-1}(x)$.

The range of $f^{-1}(x)$ should be the same as the restricted domain of the original function, $f(x)$, which is $\left[\dfrac{1}{2}, +\infty\right)$. Remember that we have two options for $f^{-1}(x)$, either $f^{-1}(x) = \dfrac{1}{2} + \dfrac{\sqrt{1+4x}}{2}$ or $f^{-1}(x) = \dfrac{1}{2} - \dfrac{\sqrt{1+4x}}{2}$. Determine which of these is the correct inverse for $f(x)$ on its restricted domain by finding the range of $f^{-1}(x)$ and comparing it to the domain of $f(x)$.

For $f^{-1}(x)$, the lower boundary of the domain is $-\dfrac{1}{4}$. Substitute this x-value into either option for $f^{-1}(x)$ and solve.

$f^{-1}(x) = \dfrac{1}{2} + \dfrac{\sqrt{1+4x}}{2}$ Upper part of $f^{-1}(x)$

$f^{-1}\left(-\dfrac{1}{4}\right) = \dfrac{1}{2} + \dfrac{\sqrt{1+4\left(-\dfrac{1}{4}\right)}}{2}$ Substitute $-\dfrac{1}{4}$ for x.

$f^{-1}\left(-\dfrac{1}{4}\right) = \dfrac{1}{2} + \dfrac{\sqrt{1-1}}{2}$ Multiply.

$f^{-1}\left(-\dfrac{1}{4}\right) = \dfrac{1}{2} + \dfrac{0}{2}$ Subtract.

$f^{-1}\left(-\dfrac{1}{4}\right) = \dfrac{1}{2}$ Add.

(*continued*)

When $x=-\dfrac{1}{4}$, $f^{-1}(x)=\dfrac{1}{2}$. Recall that the axis of symmetry for the original parabola was $x=\dfrac{1}{2}$. Thus, the axis of symmetry for the "sideways" parabola is $f^{-1}(x)=\dfrac{1}{2}$. Therefore, the range is either $\left(-\infty,\dfrac{1}{2}\right]$ or $\left[\dfrac{1}{2},+\infty\right)$. Since we know that the range of $f^{-1}(x)$ must match the domain of $f(x)$, it must be $\left[\dfrac{1}{2},+\infty\right)$, which is the range for the upper part of the inverse function, $f^{-1}(x)=\dfrac{1}{2}+\dfrac{\sqrt{1+4x}}{2}$.

8. Summarize your conclusions.

The inverse of $f(x) = x(x - 1)$ on the restricted domain $\left[\dfrac{1}{2},+\infty\right)$ is

$f^{-1}(x)=\dfrac{1}{2}+\dfrac{\sqrt{1+4x}}{2}$. The domain of $f^{-1}(x)$ is $\left[-\dfrac{1}{4},+\infty\right)$ and

the range of $f^{-1}(x)$ is $\left[\dfrac{1}{2},+\infty\right)$.

Example 3

Use a graphing calculator to graph the quadratic function $f(x) = x^2 - 3x - 1$ and its inverse, $g(x)$. Write an equation for the possible inverse, $g(x)$, using algebraic methods. Then, verify your written equation using points from the graphs of $f(x)$ and $g(x)$.

1. Graph $f(x)$ and its inverse, $g(x)$, using a graphing calculator.

   A graphing calculator does not provide the solid proof that the algebraic methods do for the existence of an inverse and its domain. Therefore, it is necessary to check the results algebraically after the functions are plotted. For this reason, after graphing $f(x)$, you will also need to calculate the table of values for the graph.

   Follow the instructions particular to your calculator model to graph the original function, $f(x) = x^2 - 3x - 1$, and its inverse, $g(x)$.

   On a TI-83/84:

   Step 1: Press [Y=]. Press [CLEAR] to delete any other functions stored on the screen.

   Step 2: Enter the original function on the Y_1 line.

   Step 3: Press [GRAPH] to see a graph of $f(x)$.

   Step 4: Press [2ND][GRAPH] to display a table of x- and y-values.

   Step 5: To see a graph of the inverse of $f(x)$, press [2ND][PRGM] to display the DRAW menu. Select 8: DrawInv.

   Step 6: After "DrawInv" and the cursor appear on the screen, press [VARS]. Arrow over to Y-VARS and select 1: Function.

   Step 7: Highlight 1: Y_1 and press [ENTER]. This will display "DrawInv Y_1." Press [ENTER] to graph the inverse of the function entered as Y_1.

   On a TI-Nspire:

   Step 1: Press the [home] key.

   Step 2: Arrow over to the graphing icon, the second icon from the left, and press [enter].

   Step 3: At $f1(x)$, enter the original function. Press [enter].

   (continued)
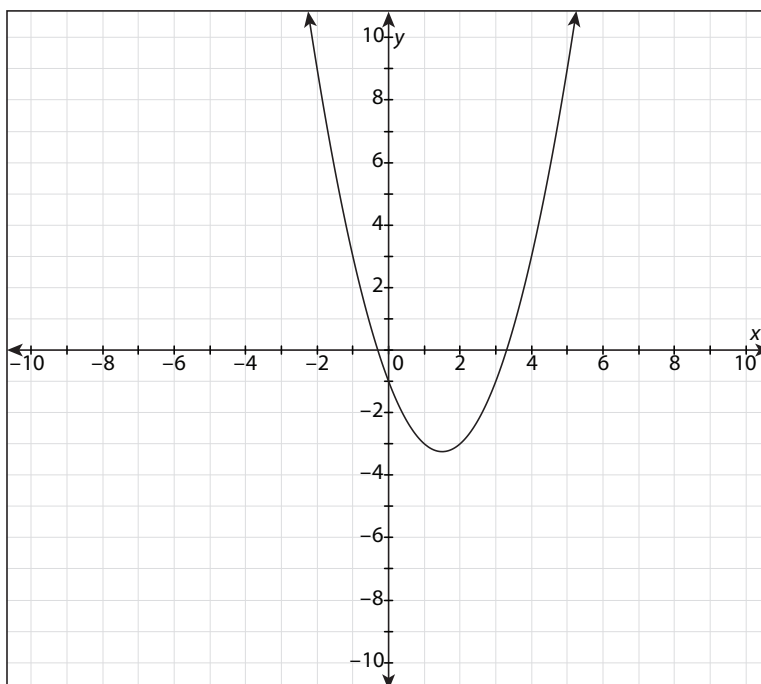
Step 4: To view the table of values, press [ctrl][T]. The function table will be displayed to the right of the graph.

Step 5: To hide the table, press [ctrl][k] to select it (signified by a blinking table border), then press [clear]. *Note*: The table can be displayed again later by pressing [ctrl][T]. After graphing the inverse function, return to this table so that you can observe how the graph of the inverse results from exchanging the *x*- and *y*-values found in the table of values for *f*(*x*).

Step 6: Constructing the inverse of a function requires that the graph of *f*(*x*) be reflected across a constructed line of the form *y* = *x*. First, add a grid to the screen by pressing the [menu] key and selecting 2: View. Select 5: Show Grid.

Step 7: To construct the function *y* = *x*, press [menu] and select 7: Points & Lines. Then select 4: Line. A pencil cursor will be displayed on the graph.

Step 8: Use the NavPad to move the cursor to a point of the form (*a*, *a*) and press the click button. Repeat by moving the cursor to a different point of the form (*b*, *b*) and then pressing the click button. A line of the form *y* = *x* will be displayed.

(*continued*)

Note: A function reflected across the line $y = x$ will produce the inverse because the values of x and y will be exchanged; i.e., point A (a, b) will become point B (b, a).

Step 9: To start the reflection process, press [menu] and select B: Transformation. Then select 2: Reflection. Move the cursor to the function and press the click button to name a point on the function. Then, move the cursor to the line $y = x$ and press the click button. The image of the point on the function will appear. Press [enter] to save the points shown.

Step 10: To see the graph of the inverse, press [menu] and select A: Construction. Then select 6: Locus. Click on the image point and press [enter], then click on the point on the original function to see the inverse relation that goes with the function.

Either calculator will create a graph resembling the one that follows. Observe that this graph of the inverse results from exchanging the x- and y-values found in the table of values created for $f(x)$.
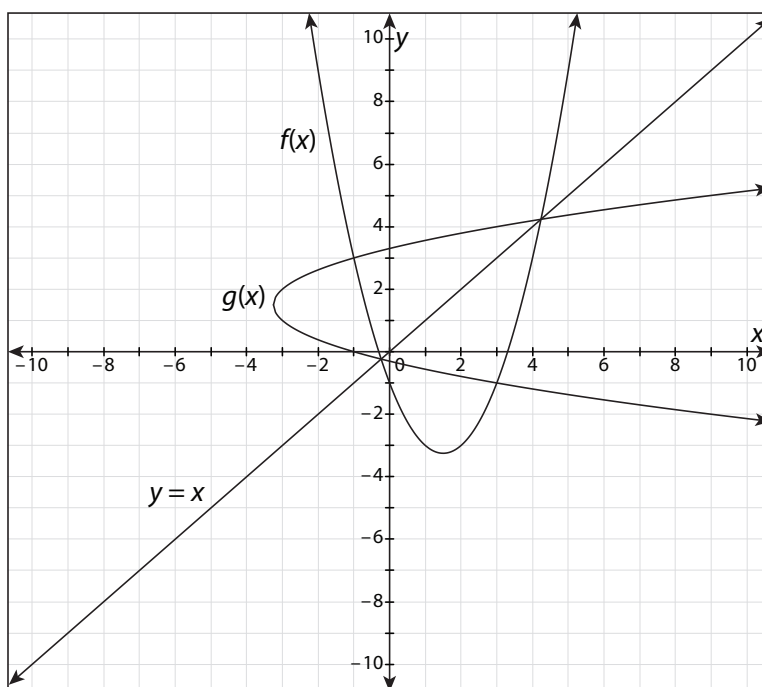
2. Write a possible inverse, $g(x)$, for the function $f(x) = x^2 - 3x - 1$.

Recall that the first step in writing the inverse of a function is to switch the domain and function variables.

$f(x) = x^2 - 3x - 1$ Original function

$x = [f(x)]^2 - 3[f(x)] - 1$ Switch $f(x)$ and x.

Then, rename $f(x)$ as $g(x)$, since this new relation represents the inverse.

$x = [f(x)]^2 - 3[f(x)] - 1$ Equation with variables switched

$x = [g(x)]^2 - 3[g(x)] - 1$ Rename $f(x)$ as $g(x)$.

Rewrite the inverse in standard form.

$x = [g(x)]^2 - 3[g(x)] - 1$ Possible inverse

$0 = [g(x)]^2 - 3[g(x)] - 1 - x$ Subtract x from both sides.

$0 = [g(x)]^2 - 3[g(x)] - (x + 1)$ Distribute -1 over $1 - x$.

Solve this equation for $g(x)$ by using the quadratic formula. Let $a = 1$, $b = -3$, and $c = -(x + 1)$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Quadratic formula

$$[g(x)] = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)\left[-(x+1)\right]}}{2(1)}$$ Substitute 1 for a, -3 for b, $-(x+1)$ for c, and $g(x)$ for x.

$$g(x) = \frac{3 \pm \sqrt{9 + 4x + 4}}{2}$$ Simplify.

$$g(x) = \frac{3}{2} \pm \frac{\sqrt{4x + 13}}{2}$$ Rewrite as separate terms, with the terms in the numerator each written over the common denominator.

The equation $g(x) = \dfrac{3}{2} \pm \dfrac{\sqrt{4x + 13}}{2}$ is a possible inverse for the function $f(x) = x^2 - 3x - 1$.

3. Use points from the graph to verify the possible inverse.

    Refer to the table of values you created in step 1 for the graph of $f(x)$. Substitute an ordered pair from the table of values into the possible inverse equation determined in step 2. This will verify that the inverse is produced by exchanging the domain and function values for the points of the inverse relation graph.

    Let's use the ordered pair $(0, -1)$ from the table of values for $f(x)$.

    Reverse the coordinates: $(-1, 0)$.

    Substitute these reversed coordinates into the equation for $g(x)$, then compare the resulting coordinates to the graph of $g(x)$.

    $$g(x) = \frac{3}{2} \pm \frac{\sqrt{4x + 13}}{2} \qquad \text{Possible inverse of } f(x)$$

    $$g(-1) = \frac{3}{2} \pm \frac{\sqrt{4(-1) + 13}}{2} \qquad \text{Substitute } -1 \text{ for } x.$$

    $$g(-1) = \frac{3}{2} \pm \frac{\sqrt{9}}{2} \qquad \text{Simplify.}$$

    Solve each of these equations for $g(-1)$.

    $$g(-1) = \frac{3}{2} + \frac{\sqrt{9}}{2} \qquad\qquad g(-1) = \frac{3}{2} - \frac{\sqrt{9}}{2}$$

    $$g(-1) = \frac{3}{2} + \frac{3}{2} \qquad\qquad g(-1) = \frac{3}{2} - \frac{3}{2}$$

    $$g(-1) = 3 \qquad\qquad\qquad g(-1) = 0$$

    $g(-1) = 0$ or 3; the corresponding ordered pairs are $(-1, 0)$ and $(-1, 3)$. Compare these points to the graph of $f(x)$.

    The ordered pair $(-1, 3)$ is also a point on the graph of $f(x)$, so the possible inverse relation has been verified.

UNIT 4A • MATHEMATICAL MODELING OF INVERSE, LOGARITHMIC, AND TRIGONOMETRIC FUNCTIONS
Lesson 1: Inverses of Functions

Practice 4A.1.1: Determining Inverses of Quadratic Functions

For problems 1–7, write the inverse for each quadratic function and the restricted domain over which the quadratic function can have an inverse.

1. $f(x) = 5x^2 - 3$

2. $g(x) = x^2 - 7x$

3. $h(x) = (x - 1)(x + 2)$

4. $a(x) = 3x^2 - 8$

5. $b(x) = 5x - x^2$

6. $c(x) = x^2 - 2x - 3$

7. $d(x) = x^2 + 2$

continued

PRACTICE

UNIT 4A • MATHEMATICAL MODELING OF INVERSE, LOGARITHMIC, AND TRIGONOMETRIC FUNCTIONS
Lesson 1: Inverses of Functions

For problems 8 and 9, use what you know about determining inverses of quadratic functions to answer the questions.

8. The area of a rectangular community garden plot is given by the quadratic function $A(x) = x(x + 3)$, where $A(x)$ is the area and x is the width of the garden in meters. What are the restrictions on the domain of the function and its inverse?

9. The distance an inflatable life preserver falls from a rescue helicopter to the surface of a river is given by the quadratic function $h(t) = \frac{1}{2}gt^2$, where g is the life preserver's acceleration due to gravity (equal to about –9.8 meters per second squared, or –9.8 m/s²) and t is time in seconds. What effect does the negative sign on the gravitational acceleration constant have on the restricted domain of the function and its inverse?

Use the given information to complete all parts of problem 10.

10. The growth in renewable power provided by wind turbines from 2001 to 2011 is modeled by the quadratic function $P(y) = 2y^2 - 6y + 3$, where $P(y)$ is the power capacity of installed wind turbines in gigawatts and y is the number of years from 2001 through 2011. Let $y = 1$ represent the year 2001.

    a. Write an inverse for the function.

    b. State the domain of the inverse.

    c. State a realistic restricted domain for the function and its inverse based on the real-world characteristics of the problem.

Lesson 4A.1.2: Determining Inverses of Other Functions

Introduction

Finding the inverse of functions that are not quadratic—such as radical, rational, and cubic functions—is similar to the process for finding the inverse of a quadratic function. As with quadratic functions, the first step in writing the expression for the possible inverse of a function is to switch the dependent and independent values of the original function, then solve for the new function. Then, to verify that the possible inverse actually exists, you must determine whether the function and its possible inverse exhibit one-to-one correspondence. That is, a function $f(x)$ only has an inverse if each value in the domain corresponds to a unique function value. In this lesson, we will determine and verify the inverses for different types of functions.

Key Concepts

- Recall that a function is a relationship between two variables in which each value of the domain or independent variable, x, is matched with one unique value of the dependent or range variable, y.

- An inverse relation, $g(x)$ or $f^{-1}(x)$, is the relation that results from switching the x- and y-variables in the given function, $f(x)$. This relation may be a function or it may be made a function by applying restrictions.

- For example, take the rational function $f(x) = \dfrac{5}{x-1}$. Switching the variables and rewriting with $f^{-1}(x)$ in place of $f(x)$ yields the equation $x = \dfrac{5}{f^{-1}(x)-1}$.

- Remember: $f^{-1}(x)$ notation does *not* mean to take the −1 power of $f(x)$.

- As with quadratic functions, this equation needs to be simplified and solved algebraically for $f^{-1}(x)$. The resulting equation, $f^{-1}(x) = \dfrac{x+5}{x}$, is the possible inverse of the original function.

- Does the possible inverse of the function exist? To visualize this, compare the graphs of the two functions.
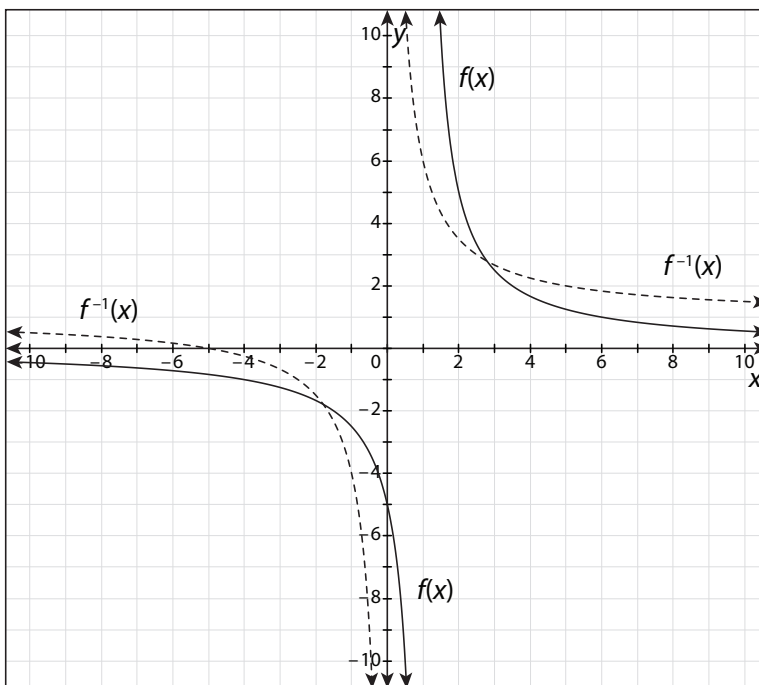
- Notice that the domain of the original function is $(-\infty,1)\cap(1,+\infty)$, whereas the domain of the inverse is $(-\infty,0)\cap(0,+\infty)$. From the graph of the function, it can be seen that the x-values of the domain of the original function extend from negative infinity to 1 and from 1 to positive infinity. This excludes 1 from the domain of the original function. The domain of the inverse extends from negative infinity to 0 and from 0 to positive infinity. The domain of the inverse excludes 0.

- Examine the graph of the function and its inverse and use the horizontal line test. It can be seen that for the function and its inverse, each value in the domain corresponds to a unique function value. Therefore, the function and its inverse exhibit one-to-one correspondence. The original function approaches the horizontal asymptote $y = 0$ as x approaches $\pm\infty$, and the inverse approaches the horizontal asymptote $y = 1$ as x approaches $\pm\infty$.

- To find the domain over which *both* a function and its inverse exist, look at the individual domains of each and where they are defined. In this example, the combined domain is $(-\infty,0)\cap(0,1)\cap(1,+\infty)$ because the function and the inverse have different domain values at which they are undefined.

- As with quadratic functions and their inverses, a graphing calculator can be used to explore a function and the domain(s) over which its inverse exists. Refer to the instructions provided in the previous sub-lesson.

Guided Practice 4A.1.2

Example 1

Find the inverse of the function $f(x) = \dfrac{x}{x-1}$ if it exists, and determine the domain value(s) shared by the inverse and the function.

1. Switch the domain and function variables, and then rewrite using inverse function notation.

   Switch $f(x)$ and x; then, replace $f(x)$ with $f^{-1}(x)$.

   $$f(x) = \frac{x}{x-1} \qquad \text{Original function}$$

   $$x = \frac{f(x)}{\left[f(x)\right] - 1} \qquad \text{Switch the variables.}$$

   $$x = \frac{f^{-1}(x)}{\left[f^{-1}(x)\right] - 1} \qquad \text{Rewrite using } f^{-1}(x) \text{ notation.}$$

   The resulting relation, $x = \dfrac{f^{-1}(x)}{\left[f^{-1}(x)\right] - 1}$, is the possible inverse of the original function. It must be evaluated to determine whether it has one-to-one correspondence with the original function.

2. Solve the possible inverse for $f^{-1}(x)$.

$$x = \frac{f^{-1}(x)}{\left[f^{-1}(x)\right] - 1}$$

Possible inverse

$$x \bullet \left\{\left[f^{-1}(x)\right] - 1\right\} = f^{-1}(x)$$

Multiply both sides of the equation by $\left[f^{-1}(x)\right] - 1$.

$$x\left[f^{-1}(x)\right] - x = f^{-1}(x)$$

Distribute.

$$x\left[f^{-1}(x)\right] = f^{-1}(x) + x$$

Add x to both sides of the equation.

$$x\left[f^{-1}(x)\right] - f^{-1}(x) = x$$

Subtract $f^{-1}(x)$ from both sides of the equation.

$$\left[f^{-1}(x)\right](x - 1) = x$$

Factor out $f^{-1}(x)$ from the left side of the equation.

$$f^{-1}(x) = \frac{x}{x - 1}$$

Divide both sides of the equation by $x - 1$.

3. Determine the domain of $f^{-1}(x)$.

The domain will be needed in order to define where $f^{-1}(x)$ exists in comparison to $f(x)$. The domain of $f^{-1}(x)$ is $(-\infty, 1) \cap (1, +\infty)$ because $f^{-1}(x)$ is undefined at $x = 1$. Therefore, the domain is all real numbers except $x = 1$.

4. Determine the range of $f^{-1}(x)$.

The range of $f^{-1}(x)$ is also $(-\infty, 1) \cap (1, +\infty)$ since $f^{-1}(x)$ has a horizontal asymptote at $y = 1$. Therefore, the range is all real numbers except $y = 1$.

5. Determine whether the function $f(x)$ exhibits one-to-one correspondence.

The function can only have an inverse over the part of its domain for which it exhibits one-to-one correspondence.

Let $x = a$ be a possible domain value of the function so that

$f(a) = \dfrac{a}{a-1}$. Then, let $x = -a$ and evaluate the function:

$f(-a) = \dfrac{-a}{-a-1} = \dfrac{a}{a+1}$.

Since $f(a) \neq f(-a)$, this means that different function values result for the domain values $x = a$ and $x = -a$; therefore, $f(x)$ shows one-to-one correspondence.

This can also be seen on the graphing calculator. Follow the steps appropriate to your calculator model to view the function and its inverse.

On a TI-83/84:

Step 1: Press [Y=]. Press [CLEAR] to delete any other functions stored on the screen.

Step 2: Enter the original function on the Y_1 line.

Step 3: Press [GRAPH] to see a graph of $f(x)$. For better viewing, adjust the domain and function variables by pressing [WINDOW] and changing the values.

Step 4: To view a table of domain and function values, press [2ND] [GRAPH]. This shows that this function exhibits one-to-one correspondence.

Step 5: To see a graph of the inverse of $f(x)$, press [2ND][PRGM] to display the DRAW menu. Select 8: DrawInv.

Step 6: After "DrawInv" and the cursor appear on the screen, press [VARS]. Arrow over to Y-VARS and select 1: Function.

Step 7: Highlight 1: Y_1 and press [ENTER]. This will display "DrawInv Y_1." Press [ENTER] to graph the inverse of the function entered as Y_1.

(*continued*)

On a TI-Nspire:

Step 1: Press [home].

Step 2: Arrow over to the graphing icon, the second icon from the left, and press [enter].

Step 3: At *f1(x)*, enter the original function. Press [enter].

Step 4: Press [tab], and then enter the inverse function at *f2(x)*. Press [enter].

$$f(x) = \frac{x}{x-1}$$

$$f^{-1}(x) = \frac{x}{x-1}$$
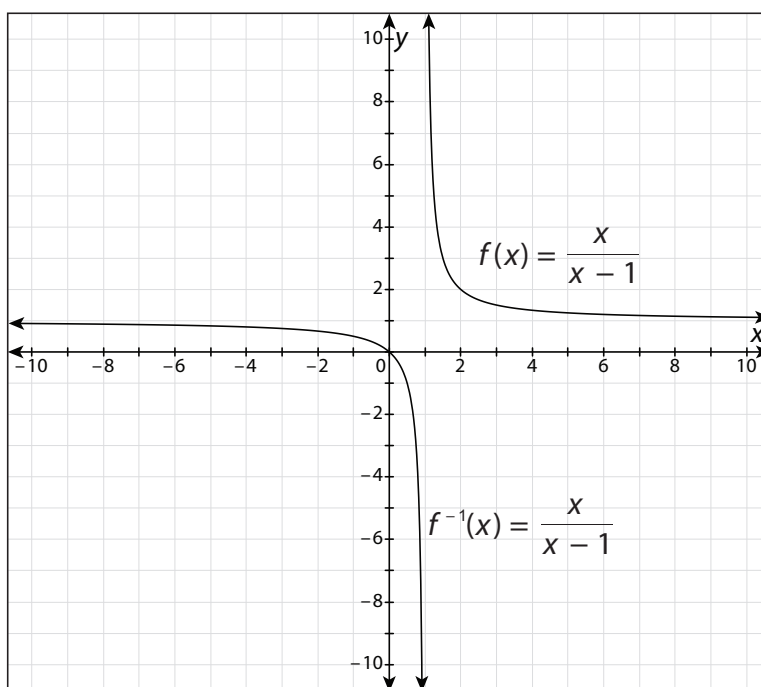
Either calculator will show that the graphs of $f(x)$ and $f^{-1}(x)$ are the same graph.

6. Compare the domains of the function and its inverse to determine where $f(x)$ has an inverse, $f^{-1}(x)$.

   Both the function and its inverse have a domain of $(-\infty,1) \cap (1,+\infty)$. (*Note*: This is a situation in which a function and its inverse have identical equations.)

Example 2

Find the inverse of the radical function $f(x) = \sqrt[3]{x+2}$ if it exists, and determine the domain over which the function and its inverse exist.

1. Switch the domain and function variables, and then rewrite using inverse function notation.

   This is the first step in determining if the function has an inverse.

   Switch $f(x)$ and x; then, replace $f(x)$ with $f^{-1}(x)$.

$f(x) = \sqrt[3]{x+2}$	Original function
$x = \sqrt[3]{\left[f(x)\right]+2}$	Switch the variables.
$x = \sqrt[3]{\left[f^{-1}(x)\right]+2}$	Rewrite using $f^{-1}(x)$ notation.

   Next, solve for $f^{-1}(x)$.

$x = \sqrt[3]{\left[f^{-1}(x)\right]+2}$	Possible inverse
$x^3 = \left[f^{-1}(x)\right]+2$	Cube both sides of the equation.
$x^3 - 2 = f^{-1}(x)$	Subtract 2 from both sides of the equation.
$f^{-1}(x) = x^3 - 2$	Apply the Symmetric Property of Equality.

   The possible inverse of $f(x) = \sqrt[3]{x+2}$ is $f^{-1}(x) = x^3 - 2$.

2. Determine the domain of the function and the domain of the inverse.

   The domain of both the function and its inverse is $(-\infty, +\infty)$ since nonzero cube roots can be positive or negative.

3. Determine whether the function $f(x)$ exhibits one-to-one correspondence with its inverse.

To do this, graph the function and the possible inverse relation. From the graph, visually determine if the possible inverse relation is a function by applying the vertical line test. If both the function and its possible inverse relation pass the horizontal line test, then they have one-to-one correspondence.

Follow the directions appropriate to your calculator model.

On a TI-83/84:

Step 1: Press [Y=]. Press [CLEAR] to delete any other functions stored on the screen.

Step 2: Enter the original function on the Y_1 line.

Step 3: Press [GRAPH] to see a graph of $f(x)$. For better viewing, adjust the domain and function variables by pressing [WINDOW] and changing the values.

Step 4: To view a table of domain and function values, press [2ND] [GRAPH]. This shows that the function exhibits one-to-one correspondence.

Step 5: To see a graph of the inverse of $f(x)$, press [2ND][PRGM] to display the DRAW menu. Select 8: DrawInv.

Step 6: After "DrawInv" and the cursor appear on the screen, press [VARS]. Arrow over to Y-VARS and select 1: Function.

Step 7: Highlight 1: Y_1 and press [ENTER]. This will display "DrawInv Y_1." Press [ENTER] to graph the inverse of the function entered as Y_1.

On a TI-Nspire:

Step 1: Press [home].

Step 2: Arrow over to the graphing icon, the second icon from the left, and press [enter].

Step 3: At $f1(x)$, enter the original function. Press [enter].

(*continued*)

Step 4: Press [tab], and then enter the inverse function at $f2(x)$.
Press [enter].
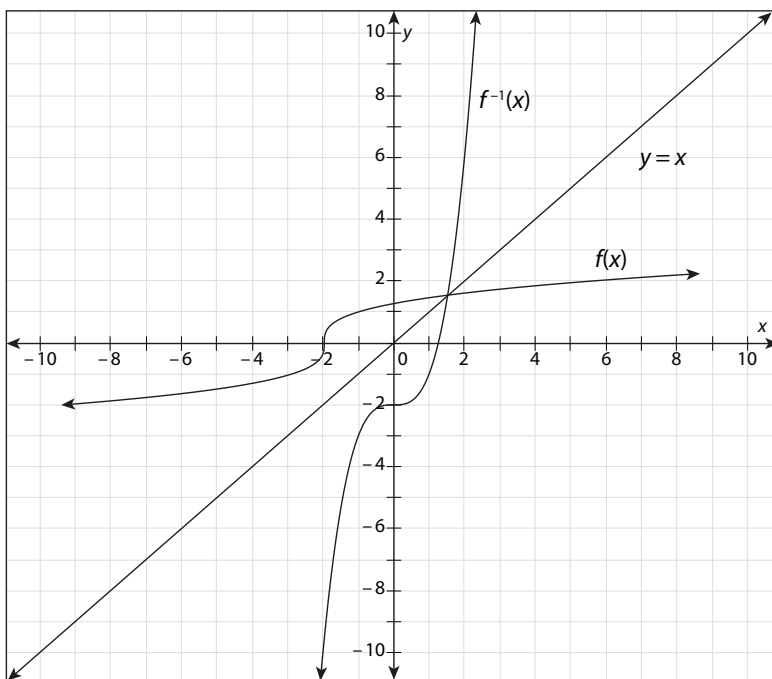
As both graphing calculators show, the function and its inverse are one-to-one over the domain $(-\infty, +\infty)$.

Example 3

Find the inverse of the absolute-value function $f(x) = |x + 2|$ if it exists, and determine the domain and range of the inverse.

1. Write the inverse of the function and simplify it.

    Switch $f(x)$ and x; then, replace $f(x)$ with $f^{-1}(x)$.

    $$f(x) = |x + 2| \qquad \text{Original function}$$
    $$x = |[f(x)] + 2| \qquad \text{Switch the variables.}$$
    $$x = |f^{-1}(x) + 2| \qquad \text{Rewrite using } f^{-1}(x) \text{ notation.}$$

    The absolute-value notation implies that $x = f^{-1}(x) + 2$ or $x = -[f^{-1}(x) + 2]$. Simplify and solve each of these equations for $f^{-1}(x)$.

$x = f^{-1}(x) + 2$	$x = -[f^{-1}(x) + 2]$
$x - 2 = f^{-1}(x)$	$x = -f^{-1}(x) - 2$
$f^{-1}(x) = x - 2$	$f^{-1}(x) = -x - 2$

    The possible inverse can be written as $f^{-1}(x) = x - 2$ or $-x - 2$.

2. Graph the function and its inverse on a graphing calculator.

    On a TI-83/84:

    Step 1: Press [Y=]. Press [CLEAR] to delete any other functions stored on the screen.

    Step 2: Enter the function on the Y_1 line. To enter the absolute value function, select [2ND][CATALOG] and choose "abs(". Press [ENTER].

    Step 3: Enter the possible inverse functions at Y_2 and Y_3. Press [ENTER].

    Step 4: To see the graphs of the functions, press [GRAPH].

    (continued)

On a TI-Nspire:

Step 1: Press the [home] key.

Step 2: Arrow over to the graphing icon, the second icon from the left, and press [enter].

Step 3: At *f1(x)*, enter the original function. To enter the absolute value symbol, press [ctrl][×] to open the math expressions palette, then select the absolute value symbol. Press [enter].

Step 4: Press [tab] to enter the possible inverse function at *f2(x)* and *f3(x)*. Press [enter]. The graphs will be displayed.

The resulting graph should resemble the following.
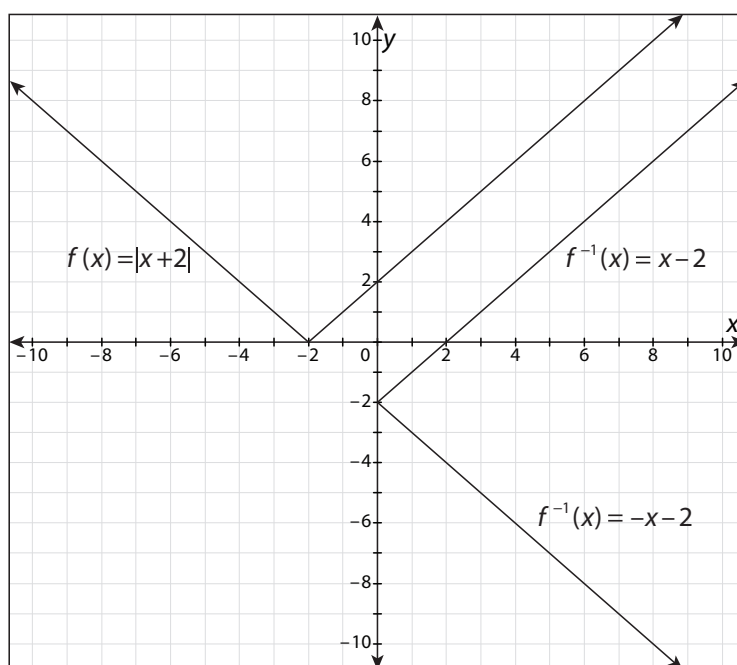
3. Determine the domain and range of the function.

The minimum point on the graph of the function occurs when $|x+2|=0$, which can be represented by the two equalities $x + 2 = 0$ and $-(x + 2) = 0$. The solution to these equations is $x = -2$. Therefore, the domain of the function is $(-\infty, +\infty)$. The range of the function is $[0, +\infty)$ since its minimum value is 0, which occurs at $x = -2$.

4. Identify the restricted domain(s) over which the function can have an inverse.

   The graph shows that the function is not one-to-one over its domain, but it is one-to-one over the restricted domains of $(-\infty, -2]$ and $[-2, +\infty)$. Note that the function can be divided at the point (–2, 0). Therefore, restricted values of the domain are $(-\infty, -2]$ and $[-2, +\infty)$.

5. Find the domain and range of the inverse.

   The inverse can be represented by $f^{-1}(x) = x - 2$ and $f^{-1}(x) = -x - 2$.

   The domain of the inverse is the range of the function. Therefore, the domain of the inverse is $[0, +\infty)$.

   The range of the inverse is the domain of the function. Therefore, the range of the inverse is $(-\infty, -2]$ and $[-2, +\infty)$.

Example 4

Graph the cubic function $f(x) = x^3 - 4x^2 + 3x$ and determine the restricted domain(s) over which its inverse exists.

1. Graph the function using a graphing calculator.

   This is the quickest way to see the function's domain and to specify the restricted domain(s) over which it has an inverse.

   Follow the directions appropriate to your calculator model.

   On a TI-83/84:

   Step 1: Press [Y=]. Press [CLEAR] to delete any other functions stored on the screen.

   Step 2: Enter the function on the Y_1 line. Press [GRAPH].

   On a TI-Nspire:

   Step 1: Press the [home] key.

   Step 2: Arrow over to the graphing icon, the second icon from the left, and press [enter].

   Step 3: At $f1(x)$, enter the original function. Press [enter].

   Keep this view on the calculator for reference in the next example step.

2. Identify the restricted domain(s) over which the function is not one-to-one.

This can be approximated visually, but the graphing calculator's trace features will give a more precise result.

A visual approximation suggests that the restricted domain over which the function is not one-to-one is from $x \approx 0.5$ to $x \approx 3$. This is the portion of the graph that does not pass the horizontal line test.

Confirm this using the method appropriate to your calculator model.

On a TI-83/84:

Step 3: Press [TRACE].

Step 4: Use the left and right arrow keys to move the cursor to the left local maximum at about $x = 0.5$. Notice that the function has a maximum value at the point (0.42553192, 0.62934032).

Step 5: Move the cursor to the right to the point at which the function has approximately the same value as in the previous step, which is (3.0851064, 0.54747021).

On a TI-Nspire:

Step 4: Press [menu]. Select 5: Trace, and then select 1: Graph Trace.

Step 5: Move the cursor to the left local maximum at about $x = 0.5$. Notice that at exactly the maximum function value, "maximum" appears. This point should be (0.45, 0.63).

Step 6: Move the cursor right to the point on the curve that is at approximately the same function value as the maximum found in the previous step. (This may require changing the trace settings: press [menu], select 5: Trace, and then select 3: Trace Step.... Set the trace step to 0.1 and press [enter] to return to the graph.) The coordinates that are close to the function value of 0.63113 found in step 5 are (3.1, 0.65).

(*continued*)

Either calculator will create a graph resembling the following. Keep this view on the calculator for reference in the next example step.

$$f(x) = x^3 - 4x^2 + 3x$$
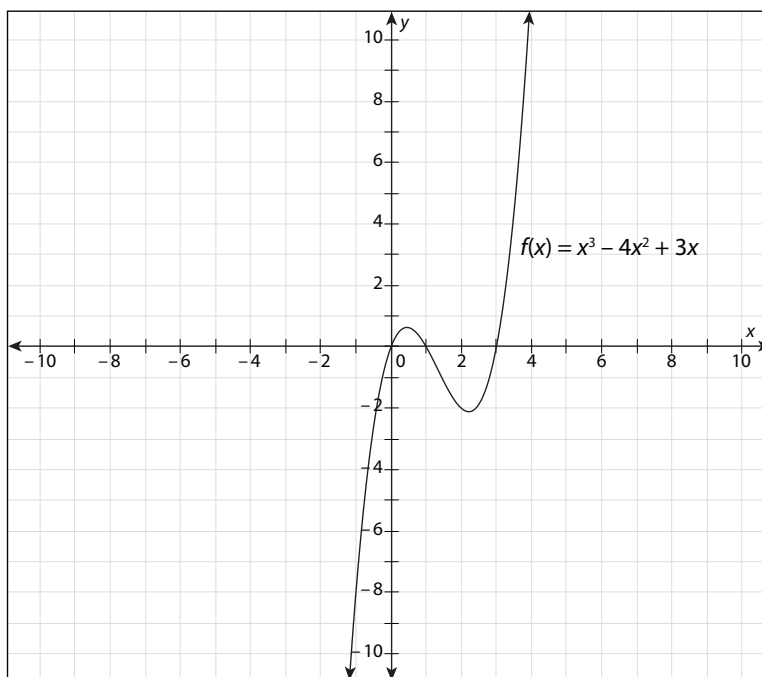
3. Graph the inverse of each function and give the domain value(s) over which the inverse is not one-to-one.

The graphing calculator can be used to display the inverse, which will give an approximation of the restricted domain(s) over which it is one-to-one and how this compares to the restricted domains of the original function.

On a TI-83/84:

Step 6: To see a graph of the inverse of $f(x)$, press [2ND][PRGM] to display the DRAW menu. Select 8: DrawInv.

Step 7: After "DrawInv" and the cursor appear on the screen, press [VARS]. Arrow over to Y-VARS and select 1: Function.

(*continued*)

Step 8: Highlight 1: Y_1 and press [ENTER]. This will display "DrawInv Y_1." Press [ENTER] to graph the inverse of the function entered as Y_1. *Note*: The "DrawInv" function on the TI-83/84 is not interactive, so only an approximation of the restricted domain value(s) is possible by viewing the graph.

Step 9: The inverse does not appear to be one-to-one over the restricted domain from approximately $x = -2$ to approximately $x = 0.5$. To see a more exact value of the locality of $x = 0.5$, press [WINDOW] and reset the end points of the minimum for each axis to -1 and the maximum of each axis to 1. To see the inverse, repeat steps 6–8. By tracing the original function, we can see that the right most point of the inverse is relatively close to the point (0.65957447, 0.52550976).

On a TI-Nspire:

Step 7: Constructing the inverse of a function requires that the graph of $f(x)$ be reflected across a constructed line of the form $y = x$. First, add a grid to the screen by pressing [menu] and selecting 2: View. Select 5: Show Grid.

Step 8: To construct the function $y = x$, press [menu] and select 7: Points & Lines. Then select 4: Line. A pencil cursor will be displayed on the graph.

Step 9: Use the NavPad to move the cursor to a point of the form (a, a) and press the click button. Repeat by moving the cursor to a different point of the form (b, b) and then pressing the click button. A line of the form $y = x$ will be displayed. *Note*: A function reflected across the line $y = x$ will produce the inverse because the values of x and y will be exchanged; i.e., point A (a, b) will become point B (b, a).

Step 10: To start the reflection process, press [menu] and select B: Transformation. Then select 2: Reflection. Move the cursor to the function and press the click button to name a point on the function. Then, move the cursor to the line $y = x$ and press the click button. The image of the point on the function will appear. Press [enter] to save the points shown.

(*continued*)

Step 11: To see the graph of the inverse, press [menu] and select A: Construction. Then select 6: Locus. Click on the image point and press [enter], then click on the point on the original function to see the inverse relation that goes with the function.
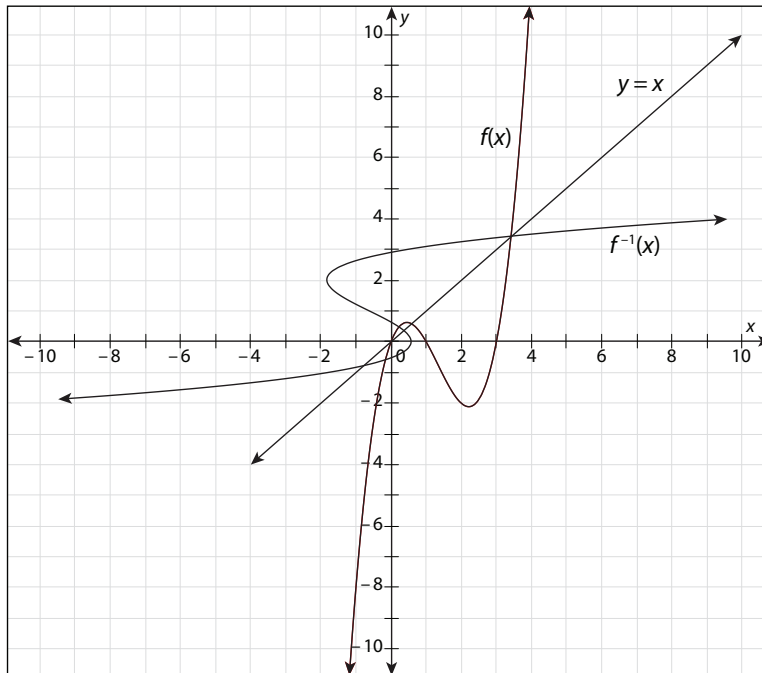
Notice that the function and its inverse are reflected across the line $g(x) = x$.

The restricted domain values appear to be at about $x = -2$ and $x = 0.6$.

4. Compare the domain(s) over which the inverse is one-to-one with the domain(s) over which the function is one-to-one.

This will determine where the function and its inverse are defined, and can be seen on the graph. The function and its inverse exist over the restricted domain intervals $(-\infty, -2) \cap (0.6, +\infty)$. The finite domain values are approximations.

UNIT 4A • MATHEMATICAL MODELING OF INVERSE, LOGARITHMIC, AND TRIGONOMETRIC FUNCTIONS
Lesson 1: Inverses of Functions

Practice 4A.1.2: Determining Inverses of Other Functions

For problems 1–4, write a function for each description, and then determine the inverse of the function.

1. a rational function that has a zero at $x = -1$ and is undefined at $x = \dfrac{1}{2}$

2. a polynomial function that has solutions at $x = 2$ and $x = -2$

3. a radical function that is the cube root of 3 less than the square of a number

4. an absolute-value function that is 2 times the absolute value of the sum of a number and 3

For problems 5–7, determine the shared domain(s) over which each function and its inverse are defined.

5. $f(x) = \dfrac{1}{2x-1}$

6. $g(x) = \sqrt[3]{4x}$

7. $h(x) = \left| 3x^2 - 6 \right|$

PRACTICE

UNIT 4A • MATHEMATICAL MODELING OF INVERSE, LOGARITHMIC, AND TRIGONOMETRIC FUNCTIONS
Lesson 1: Inverses of Functions

For problems 8–10, read the scenarios and use the information given in each to complete the problems.

8. At a coastal fishery, it takes one fisherman t minutes to empty the morning's catch. The time it takes a second fisherman to empty the catch is $t + 2$ minutes. The function that describes how long it takes the two workers working together to empty the catch is given by the function $\dfrac{1}{F(t)} = \dfrac{1}{t} + \dfrac{1}{t+2}$. Find the inverse and describe the restricted domain over which it is defined.

9. A hot-air balloon pilot wants to increase the velocity at which the balloon is rising. She uses the balloon's burner to give the balloon an acceleration of 5 feet per second squared. Before the burn, the balloon's velocity is 10 feet per second. The function $v(x)^2 = v_0^2 + 2ax$ gives the velocity of the balloon as a function of the distance travelled x feet during the time of the acceleration, a. Find the inverse of the function and describe any restrictions on the value of x.

10. The height of a corn stalk when it is ready for harvest is given by the absolute value function $h(x) = |5 - x|$. The upper bound of the function value is 8 feet. Write the inverse of the function and describe the restricted domain of the inverse.

Lesson 2: Modeling Logarithmic Functions

Common Core State Standards

F–IF.4   For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★

F–IF.5   Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★

F–IF.6   Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★

F–IF.7   Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

　　　　　e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

F–IF.8   Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

F–BF.4   Find inverse functions.

　　　　　a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. *For example, f(x) = 2x³ or f(x) = (x + 1)/(x – 1) for x ≠ 1.*

F–LE.4   For exponential models, express as a logarithm the solution to $ab^{ct} = d$, where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.★

Essential Questions

1. If an exponential function is $f(x) = a^x$, what is its inverse function?

2. What exponential function has as its inverse the logarithmic function $\log_a g(x) = x$?

3. What are the domain and range of the logarithmic function that is the inverse of the exponential function $f(x) = a^x$?

4. What function is the inverse of the exponential function $f(x) = ab^{cx}$?

5. How do the values of an exponential function and its inverse logarithmic function vary over a domain and what are the restrictions, if any, on the domain values?

WORDS TO KNOW

argument	the result of raising the base of a logarithm to the power of the logarithm, so that b is the argument of the logarithm $\log_a b = c$
base	the quantity that is being raised to an exponent in an exponential expression; in a^x, a is the base; or, the quantity that is raised to an exponent which is the value of the logarithm, such as 2 in the equation $\log_2 g(x) = 3 - x$
common logarithm	a base-10 logarithm which is usually written without the number 10, such as $\log x = \log_{10} x$
e	an irrational number with an approximate value of 2.71828; e is the base of the natural logarithm ($\ln x$ or $\log_e x$)
exponential function	a function that has a variable in the exponent, such as $f(x) = 5^x$
logarithmic function	the inverse of an exponential function; for the exponential function $f(x) = 5^x$, the inverse logarithmic function is $x = \log_5 f(x)$
natural logarithm	a logarithm whose base is the irrational number e; usually written in the form "ln," which means "$\log_e$"
power	the result of raising a base to an exponent; 32 is a power of 2 since $2^5 = 32$

Recommended Resources

- The Math Page. "Logarithmic and Exponential Functions."

  http://www.walch.com/rr/00230

  This tutorial offers a brief, step-by-step introduction to functions and inverses. The site offers an abundant and diverse selection of examples and problems as well.

- Patrick JMT: Just Math Tutorials. "Properties of Logarithms."

  http://www.walch.com/rr/00231

  This 20-minute instructional video reviews graphing logarithmic functions and the rules of logarithms, and provides detailed instruction on applying these rules.

- S.O.S. Math. "Logarithms and Their Inverses."

  http://www.walch.com/rr/00232

  This comprehensive site gives detailed examples and explanations of how to find the inverses of logarithmic functions.

IXL Links

- Find inverse functions and relations:
  http://www.ixl.com/math/algebra-2/find-inverse-functions-and-relations

- Solve exponential equations using common logarithms:
  http://www.ixl.com/math/algebra-2/solve-exponential-equations-using-common-logarithms

- Solve exponential equations using natural logarithms:
  http://www.ixl.com/math/algebra-2/solve-exponential-equations-using-natural-logarithms

- Solve logarithmic equations:
  http://www.ixl.com/math/algebra-2/solve-logarithmic-equations

- Solve logarithmic equations ii:
  http://www.ixl.com/math/algebra-2/solve-logarithmic-equations-ii

- Solve exponential equations using common logarithms:
  http://www.ixl.com/math/algebra-2/solve-exponential-equations-using-common-logarithms

- Solve exponential equations using natural logarithms:
  http://www.ixl.com/math/algebra-2/solve-exponential-equations-using-natural-logarithms

- Characteristics of quadratic functions:
  http://www.ixl.com/math/algebra-1/characteristics-of-quadratic-functions

- Solve a quadratic equation by factoring:
  http://www.ixl.com/math/algebra-1/solve-a-quadratic-equation-by-factoring

- Complete the square:
  http://www.ixl.com/math/algebra-1/complete-the-square

- Solve a quadratic equation by completing the square:
  http://www.ixl.com/math/algebra-1/solve-a-quadratic-equation-by-completing-the-square

- Characteristics of quadratic functions:
  http://www.ixl.com/math/algebra-2/characteristics-of-quadratic-functions

- Solve a quadratic equation by factoring:
  http://www.ixl.com/math/algebra-2/solve-a-quadratic-equation-by-factoring

- Complete the square:
  http://www.ixl.com/math/algebra-2/complete-the-square

- Convert equations of parabolas from general to vertex form:
  http://www.ixl.com/math/algebra-2/convert-equations-of-parabolas-from-general-to-vertex-form

- Find properties of a parabola from equations in general form:
  http://www.ixl.com/math/algebra-2/find-properties-of-a-parabola-from-equations-in-general-form

- Match exponential functions and graphs:
  http://www.ixl.com/math/algebra-1/match-exponential-functions-and-graphs

- Match exponential functions and graphs:
  http://www.ixl.com/math/algebra-2/match-exponential-functions-and-graphs

- Match exponential functions and graphs:
  http://www.ixl.com/math/algebra-1/match-exponential-functions-and-graphs

- Identify proportional relationships:
  http://www.ixl.com/math/algebra-1/identify-proportional-relationships

- Find the constant of variation:
  http://www.ixl.com/math/algebra-1/find-the-constant-of-variation

- Graph a proportional relationship:
  http://www.ixl.com/math/algebra-1/graph-a-proportional-relationship

- Identify direct variation and inverse variation:
  http://www.ixl.com/math/algebra-1/identify-direct-variation-and-inverse-variation

- Slope intercept form find slope and y intercept:
  http://www.ixl.com/math/algebra-1/slope-intercept-form-find-slope-and-y-intercept

- Standard form find x and y intercepts:
  http://www.ixl.com/math/algebra-1/standard-form-find-x-and-y-intercepts

- Slopes of parallel and perpendicular lines:
  http://www.ixl.com/math/algebra-1/slopes-of-parallel-and-perpendicular-lines

- Characteristics of quadratic functions:
  http://www.ixl.com/math/algebra-1/characteristics-of-quadratic-functions

- Identify linear quadratic and exponential functions from graphs:
  http://www.ixl.com/math/algebra-1/identify-linear-quadratic-and-exponential-functions-from-graphs

- Identify linear quadratic and exponential functions from tables:
  http://www.ixl.com/math/algebra-1/identify-linear-quadratic-and-exponential-functions-from-tables

- Graph an absolute value function:
  http://www.ixl.com/math/algebra-1/graph-an-absolute-value-function

- Rational functions asymptotes and excluded values:
  http://www.ixl.com/math/algebra-1/rational-functions-asymptotes-and-excluded-values

- Slopes of lines:
  http://www.ixl.com/math/geometry/slopes-of-lines

- Characteristics of quadratic functions:
  http://www.ixl.com/math/algebra-2/characteristics-of-quadratic-functions

- Graph a quadratic function:
  http://www.ixl.com/math/algebra-2/graph-a-quadratic-function

- Rational functions asymptotes and excluded values:
  http://www.ixl.com/math/algebra-2/rational-functions-asymptotes-and-excluded-values

- Classify variation:
  http://www.ixl.com/math/algebra-2/classify-variation

- Find the constant of variation:
  http://www.ixl.com/math/algebra-2/find-the-constant-of-variation

- Match exponential functions and graphs:
  http://www.ixl.com/math/algebra-2/match-exponential-functions-and-graphs

- Domain and range of absolute value functions:
  http://www.ixl.com/math/algebra-1/domain-and-range-of-absolute-value-functions

- Domain and range of radical functions:
  http://www.ixl.com/math/algebra-1/domain-and-range-of-radical-functions

- Domain and range:
  http://www.ixl.com/math/algebra-2/domain-and-range

- Domain and range of radical functions:
  http://www.ixl.com/math/algebra-2/domain-and-range-of-radical-functions

- Domain and range of exponential and logarithmic functions:
  http://www.ixl.com/math/algebra-2/domain-and-range-of-exponential-and-logarithmic-functions

- Find the constant of variation:
  http://www.ixl.com/math/algebra-1/find-the-constant-of-variation

- Find the slope of a graph:
  http://www.ixl.com/math/algebra-1/find-the-slope-of-a-graph

- Find slope from two points:
  http://www.ixl.com/math/algebra-1/find-slope-from-two-points

- Slope intercept form find slope and y intercept:
  http://www.ixl.com/math/algebra-1/slope-intercept-form-find-slope-and-y-intercept

- Find the slope of a linear function:
  http://www.ixl.com/math/algebra-2/find-the-slope-of-a-linear-function

Lesson 4A.2.1: Logarithmic Functions as Inverses

Introduction

In this course, you have studied a variety of functions, such as trigonometric functions, quadratic functions, and the inverses of functions. You have worked with exponents in the past and probably realize that exponents are not always whole numbers. You may also recall that sometimes exponents contain variables.

An **exponential function** is a function that has a variable in the exponent, such as $f(x) = 5^x$. The **power** is the result of raising a base to an exponent; 32 is a power of 2 since $2^5 = 32$. The power is also the value of the function's logarithm, such as x in the logarithmic function $x = \log_5 f(x)$ and its exponential function, $f(x) = 5^x$.

Like other functions, exponential functions have inverses, which are called logarithmic functions. A **logarithmic function** is the inverse an exponential function. For example, for the exponential function $f(x) = 5^x$, the inverse logarithmic function is $x = \log_5 f(x)$. If the exponential function is of the form $f(x) = a^x$, then the logarithmic function is of the form $\log_a f(x) = x$. This confirms the relationship between a function $f(x) = y$ and its inverse, $g(y) = x$. This relationship can also be seen from the following graph of an exponential function, $f(x) = 10^x$, and its inverse logarithmic function, $\log_{10} f(x)$.
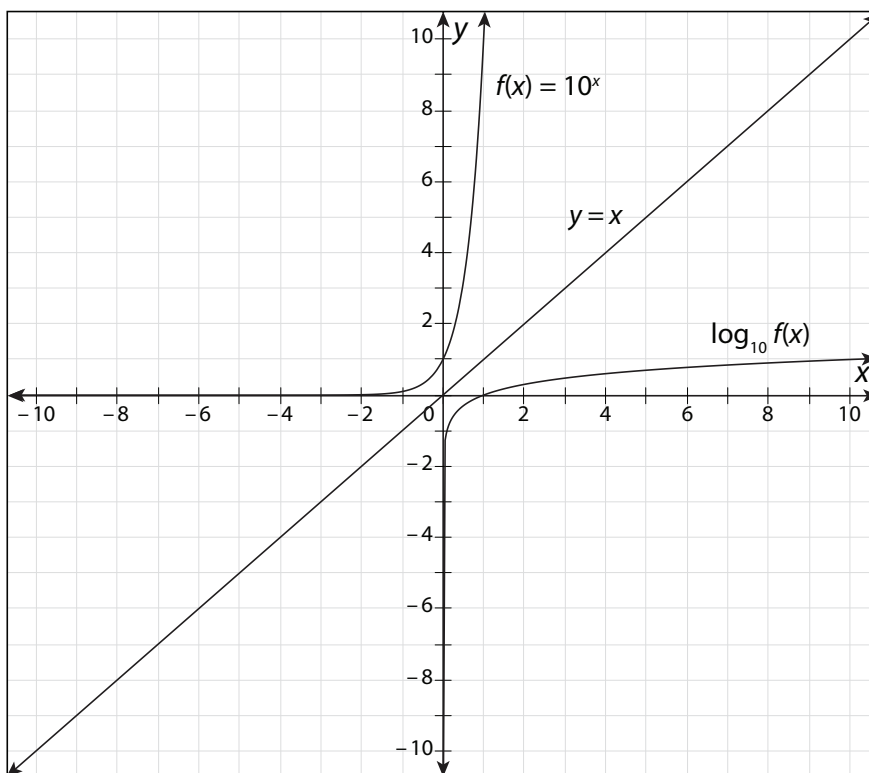
Notice that the exponential function and its inverse logarithmic function are reflected across the line $f(x) = x$ (often written as $y = x$). For example, this means that for the value $x = 3$, the exponential function is given by $f(3) = 10^3$ and its inverse logarithmic function is $\log_{10} f(3) = \log_{10} (10^3) = 3$. In real-world problems, such as the sound-intensity example in the Warm-Up, there will be situations in which the inverse function is more effectively used than the function from which the inverse is derived. Knowledge of the real-world domain of the function can help make the decision about whether the function or its inverse has more meaning. Another factor in deciding which function to work with is how simplified the expressions and numbers are for each function.

Key Concepts

- As the graph in the Introduction shows, the exponential function and its inverse are one-to-one over their domains. The domain of the exponential function is $(-\infty, +\infty)$. However, the domain of the logarithmic function is $(0, +\infty)$.

- The range of the exponential function is $(0, +\infty)$. The range of the logarithmic function is $(-\infty, +\infty)$. This information provides more evidence that the logarithmic function is the inverse of the exponential function.

- In the graphed example, the value of the exponential function is 1 at $x = 0$ because $f(0) = 10^0 = 1$. Correspondingly, the value of the inverse logarithmic function is 0 at $x = 1$ because $\log_{10} (1) = 0$.

- Exponential functions with more constants can be explored using the properties of exponents or by looking at data tables generated by a graphing calculator.

- Use a graphing calculator to explore the domain, range, and other key points of the function $4 \cdot 3^{2x}$ and its inverse logarithmic function by looking at data tables of domain and function values. Follow the directions appropriate to your calculator model.

> **On a TI-83/84:**
>
> Step 1: Press [Y=]. Press [CLEAR] to delete any other functions stored on the screen.
>
> Step 2: At Y1, use your keypad to enter values for the function. Use [X, T, θ, n] for x and [x^2] for any exponents.
>
> Step 3: Press [GRAPH]. Press [WINDOW] to adjust the graph's axes.
>
> Step 4: Press [2ND][GRAPH] to display a table of values. Look at the domain values around $x = 0$.

- Either calculator will show exponential function values that approach 0 as x becomes negative and that increase as x becomes positive.

- To show the corresponding function values for the inverse logarithmic function, switch the x- and y-values, as shown in the following table.

		−2	−1	0	1	2
Exponential function	x	−2	−1	0	1	2
	y	0.05	0.44	4	36	324
Logarithmic function	x	—	—	4	36	324
	y	−2	−1	0	1	2

- Notice that the logarithmic function does not exist for negative domain values.

- The logarithmic function values can be verified with the data table.

- For example, $f(x) = 4 \cdot 3^{2x}$, so $x = \dfrac{1}{2}\log_3\left(\dfrac{f(x)}{4}\right)$. For $x = 0$, $\log_3\left(\dfrac{f(0)}{4}\right) = 0$
  $\rightarrow \log_3\left(\dfrac{4}{4}\right) = \log_3(1) = 0$ or $3^0 = 1$.

- Notice that the coefficient of 4 in the function changes the value of the function to 4 at $x = 0$, and it changes the value of x to 4 when the inverse function is 0.

- Finally, the basic definitions and rules of exponents and logarithms will be needed in order to manipulate and calculate exponential and logarithmic functions, summarized as follows.

Terms and Rules for Logarithms

- In a logarithmic equation, $\log_a b = c$, a is the base, b is the argument, and c is the logarithm of b to the base a.

- The **base** is the quantity that is being raised to an exponent in an exponential expression, such as a in the expression a^x, or the quantity that is raised to an exponent which is the value of the logarithm, such as 2 in the function $\log_2 g(x) = 3 - x$.

- The **argument** is the result of raising the base of a logarithm to the power that is the value of the logarithm, so that b is the argument of the logarithm $\log_a b = c$.

- You may recall the rules for working with exponents; for example, according to the Product of Powers Property, when multiplying two exponents with the same base, keep the base and add the powers: $a^x \cdot a^y = a^{x+y}$. The rules for various operations with logarithms are derived from the rules for exponents. The following table lists some exponent rules, followed by the equation and name of the related logarithmic rule.

Exponent rule	Related logarithm rule	Logarithm rule name
$a^x \cdot a^y = a^{x+y}$	$\log_a (x \cdot y) = \log_a x + \log_a y$	Product rule
$\dfrac{a^x}{a^y} = a^{x-y}$	$\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$	Quotient rule
$(a^x)^y = a^{x \cdot y}$	$\log_a x^y = y \cdot \log_a x$	Power rule

- Another rule, the base change rule, allows for computing with logarithms other than base 10; one form of the equation for this rule is $\log_b a = \dfrac{\log_{10} a}{\log_{10} b}$. (Other forms will be discussed later.)

- This rule is particularly useful when working with calculators that only calculate with logarithms with bases of e (natural logarithms) and 10 (common logarithms).

- The irrational number **e** has a value of approximately 2.71828. A **natural logarithm** is a logarithm with a base of e. Natural logarithms are usually written in the form "ln," which means "$\log_e$." For example, $f(x) = \ln(1-x)$ is understood to be the inverse of the function for the exponential function $g(x) = 1 - e^x$.

- A **common logarithm**, on the other hand, is a logarithm with a base of 10. When writing a common logarithm, the 10 is usually omitted, such that $\log x = \log_{10} x$. For example, the logarithmic function $f(x) = \log(2x-1)$ is understood to be the inverse function for the exponential function $g(x) = 10^{2x-1}$.

Guided Practice 4A.2.1

Example 1

Write the inverse of the exponential function $f(x) = 0.1 \cdot 2^{0.3x}$ in logarithmic form.

---

1. Isolate the exponential term.

   This is necessary in order to use the logarithmic function definition $\log_a f(x) = x$.

   Divide both sides of the equation by 0.1; this is equal to multiplication by 10.

   $f(x) = 0.1 \cdot 2^{0.3x}$ becomes $10 \cdot f(x) = 2^{0.3x}$.

---

2. Rewrite the result as a logarithm.

   $10 \cdot f(x) = 2^{0.3x}$ becomes $\log_2 [10 \cdot f(x)] = 0.3x$.

---

3. Isolate the exponent variable, x.

   $\log_2 [10 \cdot f(x)] = 0.3x$ becomes $x = \dfrac{10}{3} \cdot \log_2 [10 \cdot f(x)]$.

---

4. Use the rules of logarithms to rewrite the result so that the simplest expression possible can be used to evaluate the function numerically.

   Use the product rule for logarithms to rewrite the expression as the sum of two logarithms:

   $$\frac{10}{3} \cdot \log_2 [10 \cdot f(x)] = \frac{10}{3} [\log_2 10] + \frac{10}{3} [\log_2 f(x)] = x$$

   Use the power rule to rewrite the separate logarithms as exponentials:

   $$\frac{10}{3} [\log_2 10] + \frac{10}{3} [\log_2 f(x)] = \log_2 10^{\frac{10}{3}} + \log_2 f(x)^{\frac{10}{3}} = x$$

---

5. Switch the domain and function variables to write the logarithmic inverse as a logarithmic function.

   This step is necessary if the logarithmic function is considered independent of the exponential function from which it was derived.

   $$\log_2 10^{\frac{10}{3}} + \log_2 x^{\frac{10}{3}} = f^{-1}(x)$$

   The inverse of the exponential function $f(x) = 0.1 \bullet 2^{0.3x}$ is

   $$f^{-1}(x) = \log_2 10^{\frac{10}{3}} + \log_2 x^{\frac{10}{3}}.$$

   ✔

Example 2

Find the exponential function on which the logarithmic function $g(x) = \log_6 x^2 - \log_6 25$ is based.

1. Switch the domain and function variables to write the logarithmic function as an exponential function.

   $g(x) = \log_6 x^2 - \log_6 25$ becomes $x = \log_6 [g(x)]^2 - \log_6 25$.

2. Use the rules of logarithms to rewrite the result so that the simplest expression possible can be found for the logarithmic function before it is converted to exponential form.

$x = \log_6 [g(x)]^2 - \log_6 25$	Exponential function from step 1
$x = \log_6 [g(x)]^2 - \log_6 5^2$	Rewrite 25 as 5^2.
$x = 2 \bullet \log_6 g(x) - 2 \bullet \log_6 5$	Apply the power rule to both logarithmic terms.
$x = 2 \bullet [\log_6 g(x) - \log_6 5]$	Factor 2 out from both terms.
$x = 2 \bullet \log_6 \left[\dfrac{g(x)}{5} \right]$	Use the quotient rule to rewrite the logarithmic terms.

3. Solve for the logarithmic term and rewrite the logarithmic function as an exponential function.

The equation may be easier to work with by applying the Symmetric Property of Equality so that x is on the right side of the equation.

$$2 \cdot \log_6 \left[\frac{g(x)}{5} \right] = x$$

$$\log_6 \left[\frac{g(x)}{5} \right] = \frac{x}{2}$$

$$\log_6 [0.2 \cdot g(x)] = 0.5x$$

4. Write the exponential function from the simplified logarithmic function by using the definition of an exponential function and its inverse.

$\log_6 [0.2 \cdot g(x)] = 0.5x$ Logarithmic function

$0.2 \cdot g(x) = 6^{0.5x}$ Rewrite as the inverse.

$g(x) = 5 \cdot 6^{0.5x}$ Divide both sides by 0.2.

The logarithmic function $g(x) = \log_6 x^2 - \log_6 25$ is based on the exponential function $g(x) = 5 \cdot 6^{0.5x}$.

Example 3

Use a logarithmic function to solve the exponential equation $4^{\left(\frac{x-3}{x}\right)} = 5$.

1. Rewrite the exponential function as its inverse logarithmic function.

   This is an alternative method to using the properties of exponents to solve the equation.

   $4^{\left(\frac{x-3}{x}\right)} = 5$ becomes $\log_4 5 = \dfrac{x-3}{x}$.

2. Simplify the result algebraically.

$\log_4 5 = \dfrac{x-3}{x}$	Inverse logarithmic function
$x \bullet \log_4 5 = x - 3$	Multiply both sides by x.
$3 = x - x \bullet \log_4 5$	Add and subtract from both sides.
$3 = x(1 - \log_4 5)$	Factor out x.
$x = \dfrac{3}{1 - \log_4 5}$	Divide both sides by $1 - \log_4 5$.

3. Solve the original exponential equation using the rules of exponents.

This will serve as a check on the logarithmic approach used in steps 1 and 2.

$$4^{\left(\frac{x-3}{x}\right)} = 5$$ Original equation

$$4^{x-3} = 5^x$$ Simplify the fractional exponent.

$$\frac{4^x}{4^3} = 5^x$$ Rewrite subtracted exponents as a fraction.

$$\left(\frac{4}{5}\right)^x = 4^3$$ Cross multiply and simplify.

$$\frac{4}{5} = 4^{\frac{3}{x}}$$ Simplify.

Simplify using the definition of logarithm and the quotient rule.

$$\frac{4}{5} = 4^{\frac{3}{x}}$$ Simplified exponential equation

$$\log_4\left(\frac{4}{5}\right) = \frac{3}{x}$$ Rewrite as a logarithm.

$$\log_4 4 - \log_4 5 = \frac{3}{x}$$ Apply the quotient rule.

The resulting expression can be rearranged to equal x, yielding the result found with the logarithmic function in step 2:

$$\log_4 4 - \log_4 5 = \frac{3}{x} \text{ becomes } x = \frac{3}{1 - \log_4 5}.$$

Example 4

Write the domain and function value of the exponential function $f(x) = 1.23 \cdot 2^{0.7x}$ and its inverse at a domain value of $x = 1.05$. Use a graphing or scientific calculator and the rules of exponents and logarithms to verify your results.

1. Substitute $x = 1.05$ into the function $f(x) = 1.23 \cdot 2^{0.7x}$.

   Evaluate the function using either a graphing calculator or a scientific calculator with exponentiation functionality.

   $$f(1.05) = 1.23 \cdot 2^{0.7(1.05)} \approx 2.05$$

2. Write the domain and function value of the inverse of the exponential function at $x = 1.05$.

   Switch the domain and function values of the exponential function to determine the domain and function values of the inverse logarithmic function at $x = 1.05$.

   The corresponding values for the inverse logarithmic function are given by the ordered pair (2.05, 1.05).

3. Verify that the ordered pair determined in step 2 satisfies the inverse logarithmic function.

   To verify that (2.05, 1.05) satisfies the inverse logarithmic function, first determine the inverse logarithmic function from the original function using the definition of logarithm and the quotient and power rules.

   $f(x) = 1.23 \cdot 2^{0.7x}$ Original function

   $0.7x = \log_2\left[\dfrac{f(x)}{1.23}\right]$ Rewrite as a logarithm.

   $x = \dfrac{10}{7}\log_2 f(x) - \dfrac{10}{7}\log_2 1.23$ Apply the quotient rule.

   $x = \log_2 f(x)^{\frac{10}{7}} - \log_2 1.23^{\frac{10}{7}}$ Apply the power rule.

   (continued)

Switch x and $f(x)$ to write the inverse logarithmic function.

$$f(x) = \log_2 x^{\frac{10}{7}} - \log_2 1.23^{\frac{10}{7}}$$

Substitute the value of x found in step 2.

$$f(2.05) = \log_2 (2.05)^{\frac{10}{7}} - \log_2 1.23^{\frac{10}{7}} \quad \text{Substitute 2.05 for } x.$$

$$f(2.05) = \log_2 \left(\frac{2.05}{1.23} \right)^{\frac{10}{7}} \qquad\qquad \text{Apply the quotient rule.}$$

$$f(2.05) \approx \frac{10}{7} \log_2 1.67 \qquad\qquad \text{Apply the power rule.}$$

4. Use a graphing calculator to estimate the value of the inverse logarithmic function at $x = 2.05$.

The process for estimating the function value will depend on the calculator used. Follow the directions specific to your calculator model.

On a TI-83/84:

Note: The TI-83/84 can only calculate natural (base-e) and common (base-10) logarithms, so the base change rule $\log_b a = \dfrac{\log_{10} a}{\log_{10} b}$ must be used to rewrite the expression found in step 3 before proceeding:

$$\frac{10}{7} \log_2 1.67 = \frac{10}{7} \left(\frac{\log_{10} 1.67}{\log_{10} 2} \right)$$

Step 1: Press [Y=]. Press [CLEAR] to delete any other functions stored on the screen.

Step 2: Use the keypad and [LOG] key to enter the values for the expression. Press [ENTER].

(continued)

On a TI-Nspire:

Note: The TI-Nspire can calculate the logarithm directly without first converting it to a base-10 logarithm.

Step 1: Press [home].

Step 2: Arrow down to the calculator icon, the first icon from the left, and press [enter]. Press [ctrl][clear] to create a new document or to clear any previous calculations on the current document.

Step 3: Press [ctrl][10^x] to bring up the $\log_{\square}\square$ field.

Step 4: Enter the argument of the logarithm into the blank subscript field. Tab to the next blank field to enter the base. Press [enter].

Step 5: To multiply the result by $\dfrac{10}{7}$, press [ctrl][(–)]. Then, enter the numbers and the operations using your keypad.

Press [enter].

Either calculator will return a result of approximately 1.057.

5. Compare the results of steps 2 and 4.

    Both procedures result in an ordered pair of approximately (2.05, 1.06) for the inverse logarithmic function.

UNIT 4A • MATHEMATICAL MODELING OF INVERSE, LOGARITHMIC, AND TRIGONOMETRIC FUNCTIONS
Lesson 2: Modeling Logarithmic Functions

Practice 4A.2.1: Logarithmic Functions as Inverses

For problems 1–3, write the inverse of each logarithmic function as an exponential function.

1. $f(x) = 6 + 4 \cdot \log_{10} (2x - 1)$

2. $g(x) = \dfrac{3 - 4\log_5 x}{6 + \log_5 x}$

3. $h(x) = \dfrac{1}{3}\log_4 \sqrt[3]{x}$

For problems 4–7, state the domain and range of the logarithmic function.

4. $a(x) = \log_9 (8 - x) - 7$

5. $b(x) = \dfrac{4}{\log_3 2x}$

6. $c(x) = 12 - \log_6 (x + 1)$

7. $d(x) = \dfrac{2}{3}\log_4 (5 - x)$

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Use the information given in each scenario to complete problems 8–10.

8. The "loudness" of a whisper is about 20 decibels. The loudness of a school cafeteria is about 60 decibels. The decibel loudness of a sound is given by the function $D(i) = \log_{10}\left(\dfrac{i}{i_0}\right)$, where i is the intensity of a sound in watts per meter squared. The quantity i_0 is 10^{-12} watt per meter squared, which is the threshold of human hearing. How much greater is the intensity i of the cafeteria sound than the intensity of the whisper?

9. The net pH of the stomach acid for a person with indigestion is about 2.75. After taking an antacid, the person's stomach pH changes to 4.15. If the pH is given by the formula pH $= -\log_{10} C$, where C is the concentration of the stomach acid solution, determine the concentration of the solution before and after the antacid was taken.

10. The chance of a fair coin landing heads-up once is 1 in 2. The chance of the coin landing heads-up on each of 3 successive flips of the coin is 1 in 8. What is the chance, $C(n)$, of the coin landing heads-up on each of 12 flips in a row? Write an exponential function for the chance $C(n)$ of the coin landing heads-up n times in a row. Write the inverse, $C^{-1}(n)$, using a logarithm.

Lesson 4A.2.2: Common Logarithms

Introduction

Recall that a base-10 logarithm is called a common logarithm. It is usually written without the base number 10. For example, $f^{-1}(x) = 3 \bullet \log 2x$ is understood to be the inverse of the exponential function $f(x) = 10^{8x^3}$. The newest graphing and scientific calculators can evaluate logarithms with any base, but earlier models may only be capable of evaluating one or two logarithmic function bases. Base 10 is one of those bases. As a result, it is important to know that a logarithm of any base can be evaluated by a quotient of base-10 bases as shown in the Key Concepts that follow. Base-10 or common logarithms are also the basis for a variety of mathematical models used in the sciences and humanities. For example, pH levels and decibel measurements are based on common logarithms.

Key Concepts

- A base-10 or common logarithm is one in which $b = 10$ in the logarithmic function inverse of the exponential function. So, for $f(t) = ab^{ct}$:

$$f^{-1}(t) = \frac{1}{c}\log t - \frac{1}{c}\log a = \log\left(\frac{t}{a}\right)^{\frac{1}{c}}$$

- When written without a base number, the term "log" is understood to be a common logarithm with a base of 10. The base change rule allows a logarithm of any base to be calculated with common logarithms; one form of the base change rule is $\log_a b = \dfrac{\log b}{\log a}$.

- This conversion to common logarithms may not result in a simpler answer, but it will make a much wider variety of logarithmic-calculation tools available because of the widespread use of common logarithms.

Guided Practice 4A.2.2

Example 1

Write the inverse of the exponential function $f(x) = 3 \cdot 10^{2x}$.

1. Identify the base of the exponential function.

    The base is 10 because 10 is raised to the $2x$ power.

2. Switch the variables and use $f^{-1}(x)$ as the inverse function variable.

    $f(x) = 3 \cdot 10^{2x}$ becomes $x = 3 \cdot \left[10^{2f^{-1}(x)}\right]$.

3. Isolate the exponential term.

    This will simplify writing the logarithmic function.

    $$x = 3 \cdot \left[10^{2f^{-1}(x)}\right]$$

    $$\frac{x}{3} = 10^{2f^{-1}(x)}$$

4. Write the inverse logarithmic function.

    Use the inverse logarithmic function definition to rewrite the equation.

    $$\frac{x}{3} = 10^{2f^{-1}(x)}$$

    $$\log\left(\frac{x}{3}\right) = 2 \cdot f^{-1}(x)$$

5. Simplify the result.

   Use algebraic methods and the power rule of logarithms.

   $$\log\left(\frac{x}{3}\right) = 2 \bullet f^{-1}(x) \qquad \text{Inverse logarithmic function}$$

   $$\frac{1}{2}\log\left(\frac{x}{3}\right) = f^{-1}(x) \qquad \text{Divide both sides by 2.}$$

   $$\log\left(\frac{x}{3}\right)^{\frac{1}{2}} = f^{-1}(x) \qquad \text{Apply the power rule.}$$

   The inverse of the exponential function $f(x) = 3 \bullet 10^{2x}$

   is $f^{-1}(x) = \log\left(\frac{x}{3}\right)^{\frac{1}{2}}$.   ✔

Example 2

Write the exponential function that has the inverse $g(x) = 10 - 2 \bullet \log x$.

1. Switch the variables and replace $g(x)$ with $g^{-1}(x)$.

   This can also be done later in the process, but it can be helpful to complete this step first.

   $g(x) = 10 - 2 \bullet \log x$ becomes $x = 10 - 2 \bullet \log [g^{-1}(x)]$.

2. Isolate the logarithmic term.

   The logarithmic term has to be isolated to identify the power of the exponential function.

   $x = 10 - 2 \bullet \log [g^{-1}(x)]$ Equation from step 1

   $x - 10 = -2 \bullet \log [g^{-1}(x)]$ Subtract 10 from both sides.

   $\dfrac{1}{2}(10 - x) = \log\left[g^{-1}(x)\right]$ Divide both sides by –2.

3. Write the exponential function from the logarithmic function.

    Use the definition of logarithm.

    $\frac{1}{2}(10-x)=\log\left[g^{-1}(x)\right]$ becomes $g^{-1}(x)=10^{\frac{1}{2}(10-x)}$.

4. Simplify the exponential function, if needed.

    In this case, the function is already simplified.

    The exponential function that has the inverse $g(x) = 10 - 2 \bullet \log x$

    is $g^{-1}(x)=10^{\frac{1}{2}(10-x)}$.

Example 3

Rewrite the logarithmic function $f(x) = 3 \bullet \log_4 (x - 1) - 2$ using common logarithms.

1. Write the base change rule for changing a logarithm of the form $\log_a b$
    to a common logarithm.

    $$\log_a b=\frac{\log b}{\log a}$$

2. Identify values for a and b in the function $f(x) = 3 \bullet \log_4 (x - 1) - 2$.

    Notice that in the rule $\log_a b=\dfrac{\log b}{\log a}$, the argument of $\log_a b$ is in the
    numerator and the base is in the denominator.

    Look at the logarithmic term of the function $f(x) = 3 \bullet \log_4 (x - 1) - 2$:
    $\log_4 (x - 1)$. The base is 4 and the argument is $x - 1$.

    In $\log_4 (x - 1)$, a is 4 and b is $x - 1$.

3. Write the common logarithmic term for the logarithmic term.

   This can be done separately before substituting in the function in order to avoid possible confusion.

   $$\log_4(x-1)=\frac{\log(x-1)}{\log 4}$$

4. Substitute the common logarithmic term into the original function.

   $$f(x)=3\bullet\log_4(x-1)-2=3\bullet\left[\frac{\log(x-1)}{\log 4}\right]-2$$

5. Simplify as needed.

   Combine the terms on the right side of the equation.

   $$f(x)=3\bullet\left[\frac{\log(x-1)}{\log 4}\right]-2$$

   $$f(x)=\frac{3\bullet\log(x-1)-2\bullet\log 4}{\log 4}$$

6. Replace any constant terms by evaluating the common logarithms.

   This may be necessary for some real-world problems. If this example came from a real-world problem, the logarithm of 4 would be evaluated as shown.

   $$f(x)=\frac{3\bullet\log(x-1)-2\bullet\log 4}{\log 4}$$ Function from step 5

   $$f(x)\approx\frac{3\bullet\log(x-1)-1.2}{0.6}$$ Substitute the calculated values of the logarithmic terms.

   $$f(x)=5\bullet\log(x-1)-2$$ Simplify.

   The logarithmic function $f(x)=3\bullet\log_4(x-1)-2$, rewritten as a common logarithm, is $f(x)=5\bullet\log(x-1)-2$.

Example 4

Write the inverse of the exponential function $g(x) = 2 \cdot 3^{x-4}$ using common logarithms. Evaluate any constants in the logarithmic function.

1. Isolate the exponential term in the function.

   $g(x) = 2 \cdot 3^{x-4}$ becomes $\dfrac{g(x)}{2} = 3^{x-4}$.

2. Write the logarithmic function.

   $\dfrac{g(x)}{2} = 3^{x-4}$ becomes $x - 4 = \log_3 \left[\dfrac{g(x)}{2} \right]$.

3. Simplify the logarithmic function.

   Isolate x. Use the quotient rule on the logarithm's argument.

   $x - 4 = \log_3 \left[\dfrac{g(x)}{2} \right]$ becomes $x = \log_3 g(x) - \log_3 2 + 4$.

4. Switch the variables and use $g^{-1}(x)$ for $g(x)$.

   $x = \log_3 g(x) - \log_3 2 + 4$ becomes $g^{-1}(x) = \log_3 x - \log_3 2 + 4$.

5. Rewrite the logarithmic terms for $g^{-1}(x)$ as common logarithmic fractions using the base change rule.

   Recall that the base change rule is $\log_a b = \dfrac{\log b}{\log a}$.

   The logarithmic terms for $g^{-1}(x) = \log_3 x - \log_3 2 + 4$ are $\log_3 x$ and $\log_3 2$.

   $\log_3 x = \dfrac{\log x}{\log 3}$ and $\log_3 2 = \dfrac{\log 2}{\log 3}$

6. Substitute the common logarithmic fractions into the equation found in step 4 and simplify.

$$g^{-1}(x) = \log_3 x - \log_3 2 + 4$$

$$g^{-1}(x) = \left(\frac{\log x}{\log 3}\right) - \left(\frac{\log 2}{\log 3}\right) + 4$$

$$g^{-1}(x) = \frac{\log x - \log 2 + 4 \cdot \log 3}{\log 3}$$

7. Evaluate the constant terms.

Use a calculator to evaluate log 2 and log 3, and then simplify.

$$g^{-1}(x) = \frac{\log x - \log 2 + 4 \cdot \log 3}{\log 3}$$

$$g^{-1}(x) \approx \frac{\log x - 0.3 + 1.9}{0.5}$$

$$g^{-1}(x) \approx 2 \cdot \log x + 3.2$$

The inverse of the exponential function $g(x) = 2 \cdot 3^{x-4}$, rewritten using common logarithms, is $g^{-1}(x) \approx 2 \cdot \log x + 3.2$.

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Practice 4A.2.2: Common Logarithms

For problems 1–3, write the inverse of the exponential function using common logarithms.

1. $f(x) = 4 \cdot 3^x$

2. $g(x) = 2 \cdot 3^{4x}$

3. $h(x) = 0.2 \cdot 2^{-2x}$

For problems 4–7, rewrite each logarithmic function as a logarithm with a base other than 10.

4. $a(x) = \dfrac{8 \cdot \log x}{\log 2}$

5. $b(x) = \dfrac{\log 7}{7 \cdot \log x}$

6. $c(x) = \dfrac{4 \cdot \log 6 + \log x}{\log 6}$

7. $d(x) = \dfrac{\log 3 \cdot \log(x - 2)}{\log 2 \cdot \log x}$

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Use the information given in each scenario to complete problems 8–10.

8. The growth of smartphone use in a certain country from 2008 to 2011 is given by the function $N(y) = 92 \cdot (1.5)^y$, where $N(y)$ is the number of smartphones in use in millions, and y is the year, with 2008 corresponding to $y = 1$. Use common logs to write the inverse of the function.

9. The decline in cost for electric-vehicle batteries from 2010 through 2020 is projected using the function $C(y) = 1{,}100 \cdot (0.9)^y$, where $C(y)$ is the cost in dollars and y is the year, with 2010 corresponding to $y = 1$. Use common logs to write the inverse of the function.

10. A semi-logarithmic coordinate plane is a coordinate plane on which one axis is labeled in powers of 10 and the other axis is a standard linear axis. The following semi-logarithmic graph shows the number of human genomes sequenced from 2008 to 2011. Use the graph to write a common logarithmic function for the number of human genomes sequenced $G(y)$ as a function of the year y. Let $y = 1$ for 2008.
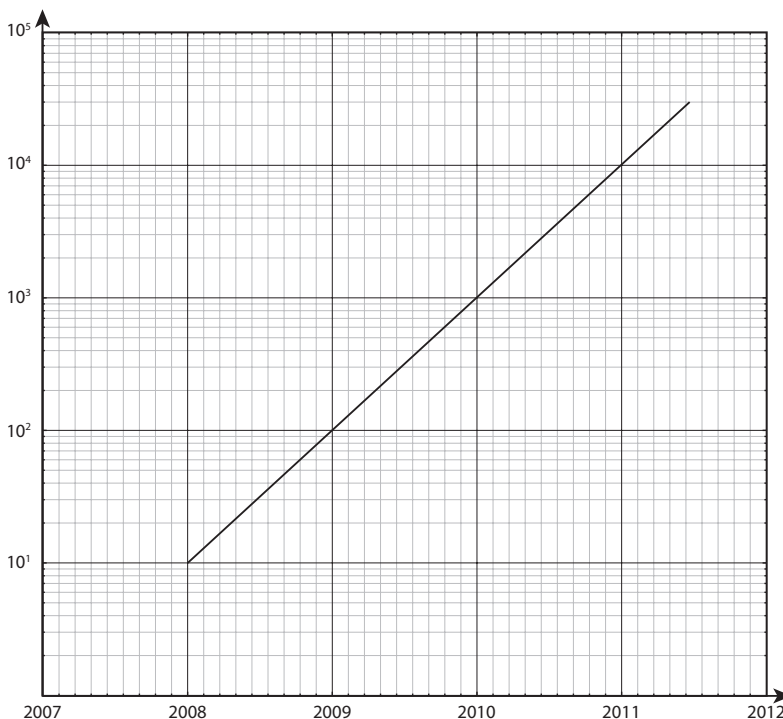
Lesson 4A.2.3: Natural Logarithms

Introduction

Recall that a logarithm with a base of e is called a natural logarithm. A natural logarithm is usually written in the form "ln" to distinguish it from logarithms of other bases. For example, $f^{-1}(x) = \log_e x$ is usually written $f^{-1}(x) = \ln x$ and is the inverse of the exponential function $f(x) = e^x$.

Exponential functions with base e and natural logarithms have been found useful in describing a number of important real-world phenomena, including population growth and extinction, as well as radioactive decay of the isotopes of elements. The latter application has been useful in accurately dating the age of antiquities and fossils of now-extinct living things. The number e, like π, has been found to be useful in describing a wide variety of such real-world phenomena.

Key Concepts

- A natural logarithm is a logarithm with a base of e. The irrational number e is approximately equal to 2.71828. A natural logarithm can be written as $\log_e x$ or, more commonly, $\ln x$.

- Recall the base change rule for logarithms, which allows a logarithm of any base to be rewritten with common logarithms: $\log_a b = \dfrac{\log b}{\log a}$.

- The base change rule can be generalized so that $\log_a b$ can be written as the quotient of two logarithms to base c. This form of the base change rule is written $\log_a b = \dfrac{\log_c b}{\log_c a}$.

- The base change rule also allows any logarithm to be written as the quotient of two natural logarithms: $\log_a b = \dfrac{\ln b}{\ln a}$.

- A graphing calculator can be used to perform calculations and display graphs of natural logarithm functions and the exponential functions to which they are related.

On a TI-83/84:

Step 1: Use the keypad and [LN] to enter the values for the natural logarithm. Press [ENTER].

Step 2: Press [GRAPH] to view the function graph or press [2ND] [GRAPH] to see a table of values for the function.

Step 3: To calculate a specific value of the function for the domain value of $x = 1$, press [2ND][MODE] to return to the calculation screen.

Step 4: Use the keypad and [LN] to enter the values for the natural logarithm, substituting 1 for x. Press [ENTER].

Step 5: Press [2ND][GRAPH] to check the result against a table of values. The result from step 4 should be listed as the result for the function value, Y_1.

On a TI-Nspire:

Step 1: Press [home]. Arrow down to the graphing icon, the second icon from the left, and press [enter].

Step 2: Enter the function to the right of "$f1(x) =$", using [ctrl][e^x] for the natural logarithm. Press [enter].

Step 3: To calculate a specific value of this function for the domain value of $x = 1$, press [home] and select the calculator icon. Press [enter].

Step 4: Enter the function to the right of "$f1(x) =$", using [ctrl][e^x] for the natural logarithm but substituting 1 for x. Press [enter].

Step 5: To check the result against a table of values, press the [ctrl] and arrow keys to return to the screen showing the graph of the original natural logarithm function.

Step 6: Press [menu], then scroll down to 2: View, then 9: Show Table. Press [enter]. The result for $x = 1$ in the "$f1(x):=$" column should match the result from step 4.

Guided Practice 4A.2.3

Example 1

Write the simplified logarithmic function that is the inverse of the exponential function $f(x) = 2 \cdot e^{3x-1}$.

1. Isolate the exponential term.

   In order to write the logarithmic function, the exponential term should be isolated with no coefficients or terms added to it.

   $f(x) = 2 \cdot e^{3x-1}$ can be rewritten as $\dfrac{1}{2} \cdot f(x) = e^{3x-1}$.

2. Rewrite the exponential term to eliminate any other constants that can be removed.

   In this case, e^1 can be removed as shown:

   $$\frac{1}{2} \cdot f(x) = e^{3x-1}$$

   $$\frac{1}{2} \cdot f(x) = \frac{e^{3x}}{e^1}$$

   $$\frac{e}{2} \cdot f(x) = e^{3x}$$

3. Write the inverse logarithmic function.

   To write the inverse, take the natural log of $\dfrac{e}{2} \cdot f(x) = e^{3x}$. This is one of two ways to find the logarithmic function (the other is to take the cube root of both sides of the equation). Simplify algebraically.

   $$\frac{e}{2} \cdot f(x) = e^{3x}$$

   $$\ln\left[\frac{e}{2} \cdot f(x)\right] = 3x$$

   $$x = \frac{1}{3}\ln\left[\frac{e}{2} \cdot f(x)\right]$$

   The inverse function of $\dfrac{e}{2} \cdot f(x) = e^{3x}$ is $x = \dfrac{1}{3}\ln\left[\dfrac{e}{2} \cdot f(x)\right]$.

4. Switch x and $f(x)$ and replace $f(x)$ with $f^{-1}(x)$.

$$f^{-1}(x) = \frac{1}{3}\ln\left[\frac{e}{2} \cdot x\right]$$

5. If possible, simplify the function using any applicable rules of logarithms.

The result in step 4 is complete, but the power rule of logarithms can be used to simplify it further.

$$f^{-1}(x) = \frac{1}{3}\ln\left[\frac{e}{2} \cdot x\right]$$

$$f^{-1}(x) = \ln\left(\frac{e}{2} \cdot x\right)^{\frac{1}{3}}$$

The simplified logarithmic function $f^{-1}(x) = \ln\left(\frac{e}{2} \cdot x\right)^{\frac{1}{3}}$ is the inverse of the exponential function $f(x) = 2 \cdot e^{3x-1}$.

Example 2

Write an exponential function that is equivalent to the natural logarithm function $g(x) = \ln (4 - 5x)^{0.3}$.

---

1. Use the rules of logarithms to simplify the function.

    This is not always needed, but it can make calculations simpler.

$g(x) = \ln (4 - 5x)^{0.3}$	Function
$g(x) = 0.3 \bullet \ln (4 - 5x)$	Apply the power rule.
$\dfrac{10}{3} \bullet g(x) = \ln(4 - 5x)$	Divide both sides by 0.3.

---

2. Switch x and $g(x)$.

    This step can also be done later in the process.

    $\dfrac{10}{3} \bullet g(x) = \ln(4 - 5x)$ becomes $\dfrac{10x}{3} = \ln\{4 - 5[g(x)]\}$.

---

3. Rewrite the resulting expression in exponential form.

    $\dfrac{10x}{3} = \ln\{4 - 5[g(x)]\}$ becomes $4 - 5[g(x)] = e^{\frac{10x}{3}}$.

---

4. Simplify the exponential term, if possible.

    This is needed to isolate the term with the domain variable, x.

    $4 - 5[g(x)] = e^{\frac{10x}{3}}$ is already in simplest form.

5. Isolate $g(x)$.

$$4 - 5[g(x)] = e^{\frac{10x}{3}}$$ Equation from the previous step

$$-5[g(x)] = e^{\frac{10x}{3}} - 4$$ Subtract 4 from both sides.

$$5[g(x)] = 4 - e^{\frac{10x}{3}}$$ Divide both sides by –1 and rearrange the right side.

$$g(x) = \frac{1}{5}\left(4 - e^{\frac{10x}{3}}\right)$$ Divide both sides by 5.

6. Use the rules of logarithms to simplify the exponential term.

The function in step 5 is complete, but it can be simplified further.

$$g(x) = \frac{1}{5}\left(4 - e^{\frac{10x}{3}}\right)$$ Equation from the previous step

$$g(x) = \frac{4}{5} - \frac{1}{5}\left(\sqrt[3]{e^{10x}}\right)$$ Distribute and apply the power rule.

The exponential function $g(x) = \frac{4}{5} - \frac{1}{5}\left(\sqrt[3]{e^{10x}}\right)$ is equivalent to the natural logarithm function $g(x) = \ln(4 - 5x)^{0.3}$.

Example 3

Solve the logarithmic equation $10 + 2 \cdot \ln (x - 4) = 5$ for x. Express the result to three decimal places.

1. Isolate the term containing x.

    This is the same procedure used with solving any equation. In this case, the term to be isolated is $\ln (x - 4)$.

$10 + 2 \cdot \ln (x - 4) = 5$	Logarithmic equation
$2 \cdot \ln (x - 4) = -5$	Subtract 10 from both sides.
$\ln (x - 4) = -2.5$	Divide both sides by 2.

2. Rewrite the result as an exponential equation.

$\ln (x - 4) = -2.5$	Equation from the previous step
$x - 4 = e^{-2.5}$	Rewrite with a base of e.
$x = 4 + e^{-2.5}$	Add 4 to both sides.

3. Use a calculator to determine an approximate value for x to three decimal places.

    Follow the procedures appropriate to your calculator model.

    On a TI-83/84:

    Step 1: Press [Y=]. Press the [CLEAR] key to delete any other functions stored on the screen.

    Step 2: Use the keypad to enter the values for the expression, using [2ND][LN] for the natural log. Press [ENTER].

    On a TI-Nspire:

    Step 1: Press [home]. Arrow down to the calculator icon, the first icon from the left, and press [enter].

    Step 2: Use the keypad to enter the values for the expression, using [ctrl][e^x] for the natural log. Press [enter].

    Either calculator will return a rounded result of $x \approx 4.082$.

Example 4

For the following statement, write the natural logarithm function and its equivalent exponential function:

> The function that is the opposite of four tenths of the natural log of the quantity eight reduced by two times a number.

1. Translate the written information into mathematical expressions.

   Start with one phrase from the statement, and write its mathematical equivalent. Then, build on each phrase and continue writing mathematical equivalents until the entire statement is represented mathematically.

Phrase	Translation
four tenths	0.4
eight reduced by two times a number	$8 - 2x$
the opposite of four tenths of the natural log	$-0.4 \bullet \ln$
the natural log of the quantity eight reduced by two times a number	$\ln (8 - 2x)$
the opposite of four tenths of the natural log of the quantity eight reduced by two times a number	$-0.4 \bullet \ln (8 - 2x)$

2. Write the natural logarithm function using the expressions from step 1.

   $f(x) = -0.4 \bullet \ln (8 - 2x)$

3. Switch x and $f(x)$.

   $f(x) = -0.4 \bullet \ln (8 - 2x)$ becomes $x = -0.4 \bullet \ln \{8 - 2[f(x)]\}$.

4. Isolate the natural logarithm.

$x = -0.4 \bullet \ln\{8 - 2[f(x)]\}$	Equation from the previous step
$-\dfrac{x}{0.4} = \ln\{8 - 2[f(x)]\}$	Divide both sides by –0.4.
$-\dfrac{10x}{4} = \ln\{8 - 2[f(x)]\}$	Rewrite 0.4 as a fraction.

5. Write an exponential function from the logarithmic function in step 4.

$-\dfrac{10x}{4} = \ln\{8 - 2[f(x)]\}$ becomes $8 - 2 \bullet f(x) = e^{-\frac{10x}{4}}$.

6. Simplify the exponent.

$8 - 2 \bullet f(x) = e^{-\frac{10x}{4}}$ becomes $8 - 2 \bullet f(x) = e^{-\frac{5x}{2}}$.

7. Isolate the term containing $f(x)$ and simplify the result as much as possible without calculating any constants.

$8 - 2 \bullet f(x) = e^{-\frac{5x}{2}}$	Equation from the previous step
$-2 \bullet f(x) = e^{-\frac{5x}{2}} - 8$	Subtract 8 from both sides.
$2 \bullet f(x) = 8 - e^{-\frac{5x}{2}}$	Divide both sides by –1 and then rearrange the right side.
$f(x) = 4 - \dfrac{1}{2}e^{-\frac{5x}{2}}$	Divide both sides by 2 and simplify.

8. Summarize your results.

For the given statement, the natural logarithm function is $-\dfrac{10x}{4} = \ln\{8 - 2[f(x)]\}$ and its equivalent simplified exponential function is $f(x) = 4 - \dfrac{1}{2}e^{-\frac{5x}{2}}$.

Example 5

Write the logarithmic function $f(x) = 5 + 2 \cdot \log_5 (x - 2)$ as a natural logarithm function.

1. Rewrite the logarithmic term of the function as the quotient of two natural logarithms.

   Determine and substitute the specific values for a, b, and c from the logarithmic term $f(x) = 5 + 2 \cdot \log_5 (x - 2)$ into the base change rule, $\log_a b = \dfrac{\log_c b}{\log_c a}$. For $\log_5 (x - 2)$, $a = 5$, $b = x - 2$, and $c = e$.

   Recall that the base change rule allows any logarithm to be written as the quotient of two natural logarithms: $\log_a b = \dfrac{\ln b}{\ln a}$. Therefore,

   $\log_5 (x - 2)$ rewritten as the quotient of two natural logarithms is
   $$\log_5(x-2) = \frac{\ln(x-2)}{\ln 5}.$$

2. Substitute the result for the logarithmic term in the original function.

   By substitution, the function $f(x) = 5 + 2 \cdot \log_5 (x - 2)$ becomes
   $$f(x) = 5 + 2 \cdot \left[\frac{\ln(x-2)}{\ln 5} \right].$$

3. Simplify the result.

   The function found in step 2 is simplified, but the result can also be written as shown.
   $$f(x) = 5 + 2 \cdot \left[\frac{\ln(x-2)}{\ln 5} \right]$$
   $$f(x) = \frac{5 \cdot \ln 5 + 2 \cdot \ln(x-2)}{\ln 5}$$

UNIT 4A • MATHEMATICAL MODELING OF INVERSE, LOGARITHMIC, AND TRIGONOMETRIC FUNCTIONS
Lesson 2: Modeling Logarithmic Functions

Practice 4A.2.3: Natural Logarithms

For problems 1–4, use the rule $\log_a b = \dfrac{\ln b}{\ln a}$ to write each function as a natural logarithm function.

1. $f(x) = -3 \cdot \log_4 (2x)$

2. $g(x) = \log (2x - 1) + 1$

3. $f(x) = 6 \cdot \log_5 (x + 4) + \log_3 (2 - x)$

4. $g(x) = [\log_3 (x - 1)]^2$

For problems 5–7, compare the values of the functions $f(x)$ and $g(x)$ at $x = 5$ using >, <, or = signs. Do not use a calculator.

5. $f(x) = \log x$; $g(x) = 0.1 \cdot \log x$

6. $f(x) = 2 \cdot \ln (x - 4)$; $g(x) = 4 \cdot \ln x$

7. $f(x) = \log_2 (x - 4)$; $g(x) = \ln (x - 4)$

UNIT 4A • MATHEMATICAL MODELING OF INVERSE, LOGARITHMIC, AND TRIGONOMETRIC FUNCTIONS
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Use the information given in each scenario to complete problems 8–10.

8. The exponential function $N(x) = N_0(x) \cdot e^{-(\ln 6)x}$ gives the number of fair number cubes labeled 1–6 that come up "6" each time a group of cubes is rolled. After each roll, the cubes that come up 6 are removed. Calculate the number of cubes remaining after 2 rolls if there were 80 number cubes to start.

9. Radon is a naturally occurring radioactive gas that can seep into unsealed basements from the ground. The exponential function $N(t) = N_0(t) \cdot e^{-ct}$, where t represents the number of days, gives the amount of radon found in a home basement at any time after the basement is sealed. If $c = 0.181$, after how many days will half the radon remain once the basement has been sealed?

10. An archaeological team plans to use carbon-14 dating to determine the age of a frozen mastodon. Carbon-14 dating is a way to measure the amount of carbon-14 that remains in a fossil in order to determine the fossil's age. The exponential function $R(t) = R_0(t) \cdot e^{-ct}$ models the amount of radioactivity after the time t has elapsed. The constant c is $1.21 \cdot 10^{-4} \, y^{-1}$ since t is in years (y) and the exponential term must not have any units. Write the exponential function as a natural logarithm function and find t in years if $R(t) = 0.121$ unit of radioactivity and $R_0(t) = 0.231$ unit of radioactivity.

Lesson 4A.2.4: Graphing Logarithmic Functions

Introduction

The functions $f(x) = a \bullet \log_b c$, $g(x) = a + b \bullet \log_c d$, and $h(x) = a \bullet \log_a c$ are "families" of logarithmic functions. The graphs of logarithmic functions exhibit patterns that can help in identifying these functions when the algebraic function is not present.

Key Concepts

- Logarithmic functions or the logarithmic terms within logarithmic functions are equal to 0 when their arguments equal 1. Recall that the argument is the result of raising the base of a logarithm to the power of the logarithm, so that b is the argument of the logarithm $\log_a b = c$.

- For example, let's look at $f(x) = 30 \bullet \ln x$ at $x = 1$: $f(1) = 30 \bullet \ln 1 = 30 \bullet 0 = 0$.

- For another example, let's look at the function $g(x) = -250 + 6 \bullet \log_4 (x + 4)$. Solve for $x = -3$:

  $g(-3) = -250 + 6 \bullet \log_4 [(-3) + 4]$

  $g(-3) = -250 + 6 \bullet \log_4 (1)$

  $g(-3) = -250 + 0 = -250$

- Notice that the argument for each example was equal to 1.

- The values of logarithmic functions or their logarithmic terms approach positive or negative infinity as the argument of the logarithmic term approaches 0. For example, in the function $f(x) = -\ln x$, $f(0.1)$ is about 2.3, but $f(0.001)$ is almost 7. If $x = 10^{-12}$, $f(x)$ is almost 28. The function value increases continuously as the value of x decreases.

- A graphing calculator will also show this function behavior. By looking at a variety of function values of interest (such as very small values of x), it is possible to see trends in the changing values of the function. Examine very small values of x by adjusting the axis scales or using the calculator's trace feature.

- Similarly, the value of logarithmic functions or their logarithmic terms as x becomes very large, positively or negatively, can be seen by substituting values for x or by using a calculator to calculate the values.

- For example, for the function $g(x) = \log x$, the domain is $(0, +\infty)$. As x becomes very large, the value of $g(x)$ increases, too, but at a much slower rate. The value of $g(10^3) = 3$, but the value of $g(10^{200}) = 200$. If you suspect the function value of $g(x)$ has an upper bound, try to find the highest value of the function.

- Follow these basic rules to compare logarithmic functions. However, use caution when defining domains: positive, negative, or zero domain values can result in an undefined function, or change the ordering in a comparison of function values.

Powers, Products, Quotients, Roots, and Sums of Logarithmic Functions

- Families of logarithmic functions are grouped according to the operations shown in the equations of the functions. In real-world problems, you can calculate such combined operations with logarithmic functions best by approximation techniques or with calculators. Each example that follows shows how different operations affect the graph of a logarithmic function.

$f(x) = a \bullet \log_b c$

- Compare the graphs of three functions of the form $f(x) = a \bullet \log_b c$.

On the graph: $h(x) = 3 \bullet \log_{10} x$, $g(x) = 2 \bullet \log_{10} x$, $f(x) = \log_{10} x$
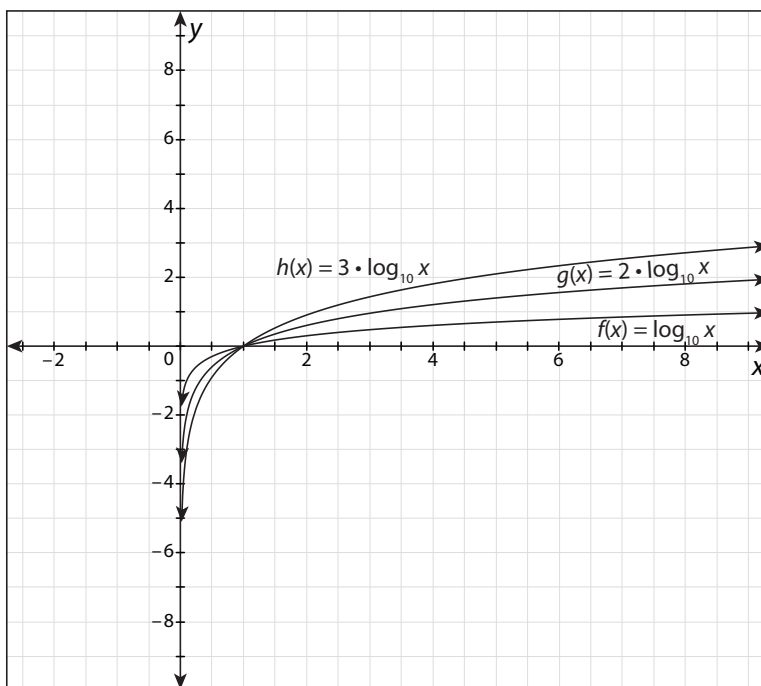
- All three graphs pass through the point (1, 0) because any number raised to the 0 power (the y-value) is equal to 1 (the x-value). The coefficient in front of each logarithm multiplies the logarithmic value at that value of x by the magnitude of the coefficient. This means that $h(x) = 3 \bullet f(x) = 3 \bullet \log x$, and $g(x)$ can be determined as follows.

$$g(x) = \frac{2}{3} \bullet h(x)$$

$$g(x) = \frac{2}{3} \bullet (3 \bullet \log x)$$

$$g(x) = 2 \bullet \log x$$

$f(x) = a + b \bullet \log_c d$

- Observe the graphs of three functions of the form $f(x) = a + b \bullet \log_c d$.
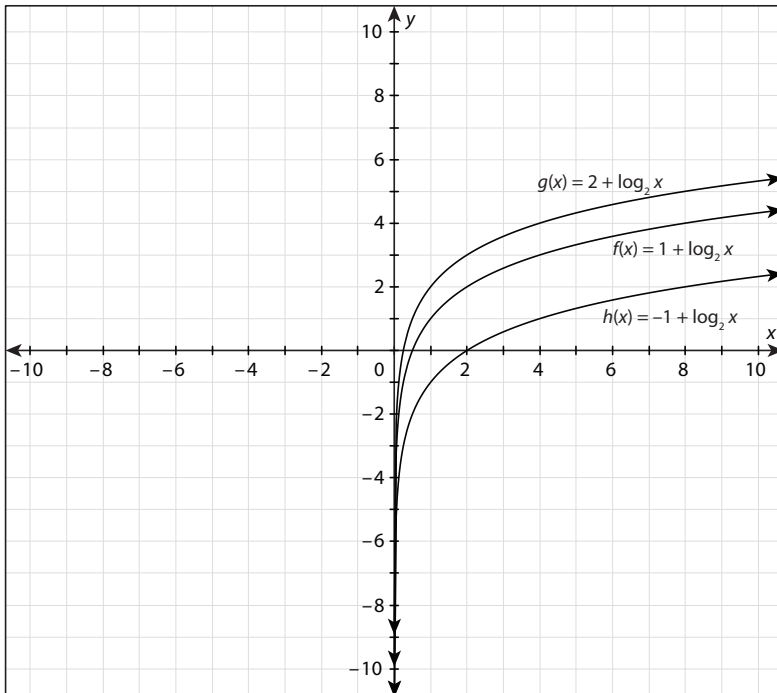
- All three graphs are continuously increasing across the domain $(0, +\infty)$.
  For any value of x in the domain, the y-values are related by the inequality
  $g(x) > f(x) > h(x)$. The x-intercepts are determined by the constant added to $\log_2 x$.

$f(x) = a \bullet \log_b c$ and $g(x) = a \bullet \log_d c$

- Compare the graphs of three functions with different bases.
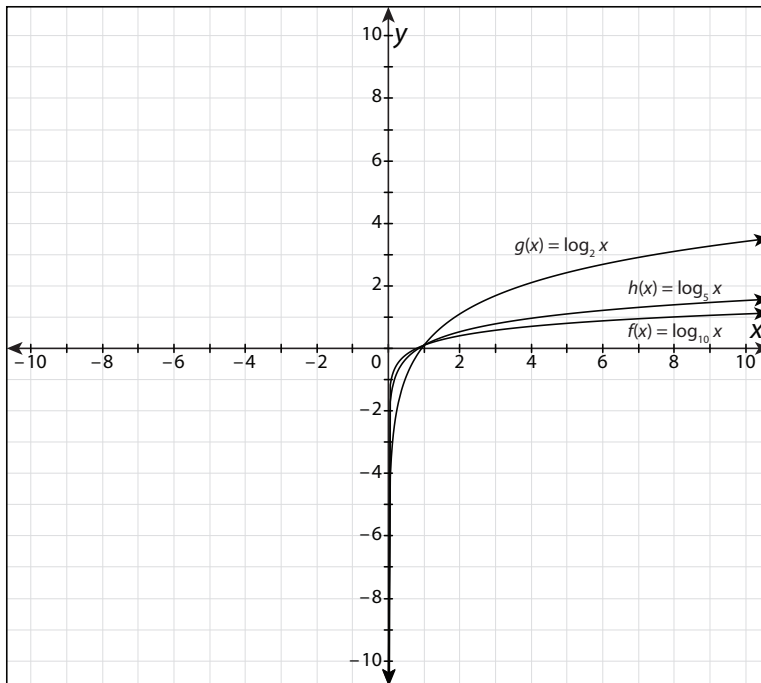
- All three functions contain the point (1, 0), since the logarithm of any base to the power of 0 is equal to 1. As the graphs show, the functions are related by the inequality $g(x) > h(x) > f(x)$ for a specific positive value of x. Comparing the bases of the three functions reveals that they are ordered in the opposite "direction" from the functions: $2 < 5 < 10$ for $g(x) > h(x) > f(x)$.

- Alternately, as x increases, the powers of the smaller bases grow larger in order to remain proportional with larger bases raised to the same power. For example, for $x = 8$, $g(8) = 3$, $h(8) \approx 1.29$, and $f(8) \approx 0.90$.

Guided Practice 4A.2.4

Example 1

Sketch the graphs of the functions $f(x) = 2 \cdot \log_2 x$ and $g(x) = 2 - \log_2 x$ on a coordinate plane. Describe the end behavior of each function.

1. Determine function values for $f(1)$ and $g(1)$ and write the results as ordered pairs.

   Substitute $x = 1$ into each function and solve.

   $f(1) = 2 \cdot \log_2 (1) = 2 \cdot 0 = 0$

   $g(1) = 2 - \log_2 (1) = 2 - 0 = 2$

   Write each point as an ordered pair.

   For $f(x)$: $(1, 0)$

   For $g(x)$: $(1, 2)$

2. Find the value(s) of x at which $f(x) = g(x)$.

   This will reveal any point(s) of intersection, or solutions, of the system of equations.

$2 \cdot \log_2 x = 2 - \log_2 x$	Set the functions equal to each other.
$3 \cdot \log_2 x = 2$	Add $\log_2 x$ to both sides.
$\log_2 x = \dfrac{2}{3}$	Divide both sides by 3.
$2^{\frac{2}{3}} = x$	Simplify.

   Using a calculator, the approximate value of x is 1.6.

3. Find the function value at which the functions are equal.

   Substitute the approximate value of x from step 2 into each function.

   $$f(1.6) \approx g(1.6) \approx \frac{4}{3}$$

4. Write the approximate point at which the functions intersect.

   Since the value of x is rounded, the functions intersect at

   approximately $\left(1.6, \dfrac{4}{3}\right)$ or $\left(1.6, 1.\overline{3}\right)$.

5. Determine additional points for sketching the graph of each function.

   Choose values of x that can be evaluated easily.

   Let $x = 0.25, 0.5, 2,$ and 4.

   Solve $f(x)$ for each given value.

   $f(0.25) = 2 \cdot \log_2 (0.25) = 2 \cdot -2 = -4$

   $f(0.5) = 2 \cdot \log_2 (0.5) = 2 \cdot -1 = -2$

   $f(2) = 2 \cdot \log_2 (2) = 2 \cdot 1 = 2$

   $f(4) = 2 \cdot \log_2 (4) = 2 \cdot 2 = 4$

   Solve $g(x)$ for each given value.

   $g(0.25) = 2 - \log_2 (0.25) = 2 - (-2) = 4$

   $g(0.5) = 2 - \log_2 (0.5) = 2 - (-1) = 3$

   $g(2) = 2 - \log_2 (2) = 2 - 1 = 1$

   $g(4) = 2 - \log_2 (4) = 2 - 2 = 0$

6. Write the additional points.

   Make sure to keep the points with the correct function.

   For $f(x)$: (0.25, –4), (0.5, –2), (2, 2), and (4, 4)

   For $g(x)$: (0.25, 4), (0.5, 3), (2, 1), and (4, 0)

7. Plot and sketch a curve to connect the points for each function.

   The completed curves might suggest that additional points should be plotted to better determine the actual shape of each function's graph.

$f(x) = 2 \cdot \log_2 x$

$g(x) = 2 - \log_2 x$
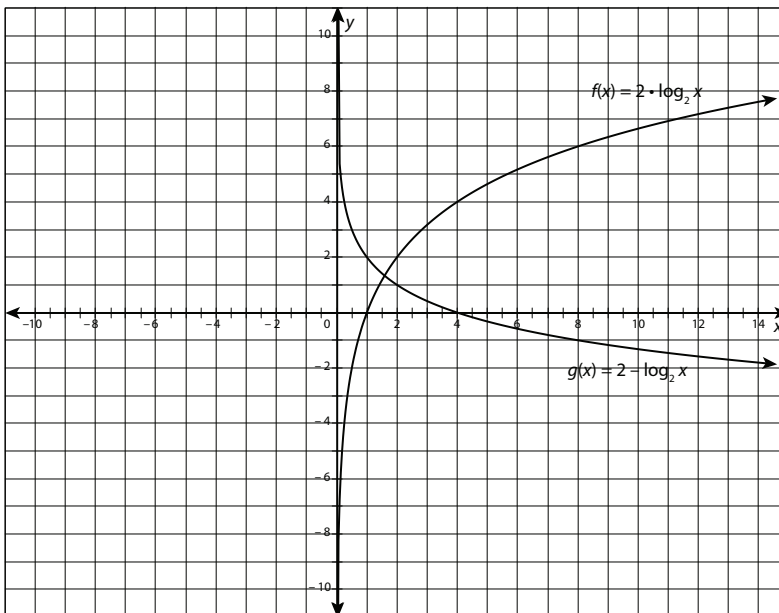
8. Describe what happens to the function values as x becomes very large and as x approaches 0.

It is useful to know the limits of function values as the values go beyond the finite values used for a sketch of a graph.

For $f(x)$: As x approaches positive infinity, $f(x)$ approaches positive infinity. As x approaches 0, $f(x)$ approaches negative infinity.

For $g(x)$: As x approaches positive infinity, $g(x)$ approaches negative infinity. As x approaches 0, $g(x)$ approaches positive infinity.

Example 2

Sketch the graphs of $f(x) = x + \log x$ and $g(x) = \log x$ on a coordinate plane, using a graphing calculator if needed. Write $f(x)$ in terms of $g(x)$ to simplify sketching. Then, describe the end behavior of the functions.

1. Determine function values for $f(1)$ and $g(1)$ and write the results as ordered pairs.

Substitute $x = 1$ into each function and solve.

$f(1) = 1 + \log (1) = 1 + 0 = 1$

$g(1) = \log (1) = 0$

Rewrite the points as ordered pairs.

For $f(x)$: (1, 1)

For $g(x)$: (1, 0)

2. Find the value(s) of x at which $f(x) = g(x)$.

This will reveal any point(s) of intersection, or solutions, of the system of equations.

$f(x) = g(x)$

$x + \log x = \log x$

$x = 0$

The value $x = 0$ is not in the domain of these functions. Therefore, there is no point of intersection or solution for $f(x)$ and $g(x)$.

3. Determine additional points for sketching the graph of each function.

Choose values of x for each function, and then solve the resulting equations.

Let $x = 0.1, 1, 2, 3$, and 4.

The x-values of 2, 3, and 4 will result in common logarithms that cannot be found easily without a calculator that has the capability of computing common logarithms.

Solve $f(x)$ for each given value.

$f(0.1) = (0.1) + \log (0.1) = 0.1 - 1 = -0.9$

$f(1) = (1) + \log (1) = 1$

$f(2) = (2) + \log (2) \approx 2.3$

$f(3) = (3) + \log (3) \approx 3.5$

$f(4) = (4) + \log (4) \approx 4.6$

Solve $g(x)$ for each given value.

$g(0.1) = \log (0.1) = -1$

$g(1) = \log (1) = 0$

$g(2) = \log (2) \approx 0.3$

$g(3) = \log (3) \approx 0.5$

$g(4) = \log (4) \approx 0.6$

4. Write and plot the additional points.

Make sure to keep the points with the correct function.

For $f(x)$: $(0.1, -0.9)$, $(1, 1)$, approximately $(2, 2.3)$, approximately $(3, 3.5)$, and approximately $(4, 4.6)$

For $g(x)$: $(0.1, -1)$, $(1, 0)$, approximately $(2, 0.3)$, approximately $(3, 0.5)$, and approximately $(4, 0.6)$

5. Write $f(x)$ in terms of $g(x)$.

In some problems involving two or more functions graphed on the same coordinate plane, the functions are related, which can simplify graph sketching.

$f(x) = x + \log x = x + g(x)$ or $f(x) - g(x) = x$, which means the difference between the function values at any value of x is x.

6. Plot and sketch a curve to connect the points for each function.

The completed curves might suggest that additional points should be plotted to better determine the actual shape of each function's graph.
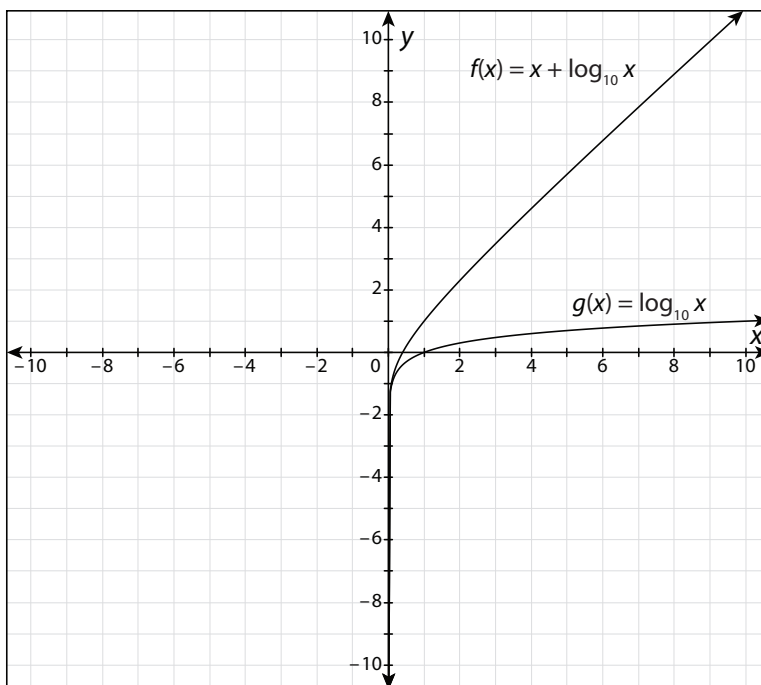
7. Describe what happens to the function values for domain values less than 1.

The graph indicates that the function values diverge.

For $f(x)$: The function values approach negative infinity as x approaches 0. As x approaches 1, the function values converge on 1.

For $g(x)$: The function values approach negative infinity as x approaches 0. As x approaches 1, the function values converge on 0.

8. Describe what happens to the function values for domain values greater than 1.

The graph indicates that the function values increase.

For $f(x)$: The function values approach positive infinity as x increases from $x = 1$ to positive infinity. As x approaches 1 from the right, the function values converge on 1.

For $g(x)$: The function values approach positive infinity as x increases from $x = 1$ to positive infinity. As x approaches 1 from the right, the function values converge on 0.

Example 3

Use the graph of the function $f(x)$ to write the algebraic form of $f(x)$. Assume that $f(x)$ is of the form $a \cdot \log_4 (x + b)$ and includes the point $(15, 4)$. Then, describe how to find the common solution of $f(x)$ and the function $g(x) = 3 \cdot \log_4 x$.
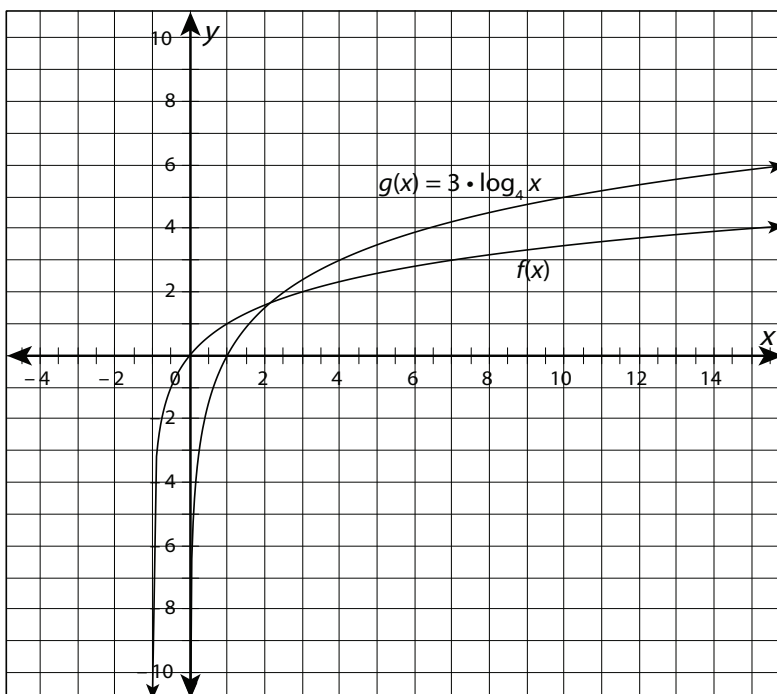
1. Determine the *x*-intercepts and their coordinates for each function.

   Inspect the graph to determine this information.

   The *x*-intercept of $f(x)$ is 0, so the point is (0, 0). The *x*-intercept of $g(x)$ is 1, so the point is (1, 0).

2. Substitute the coordinates of the *x*-intercept of *f*(*x*) into the form $f(x) = a \bullet \log_4 (x + b)$.

This is the first step in writing *f*(*x*).

For $x = 0$, $f(x) = 0$. Substitute and solve.

   $f(x) = a \bullet \log_4 (x + b)$

   $f(0) = a \bullet \log_4 [(0) + b] = 0$

Divide the logarithmic term by *a*.

   $\log_4 b = 0$

$b = 1$ since $4^0 = 1$. Substitute $b = 1$ in $f(x) = a \bullet \log_4 (x + b)$.

   $f(x) = a \bullet \log_4 (x + 1)$

3. Use the other given point on the graph of *f*(*x*) to find *a*.

Substitute the given point (15, 4) into the *f*(*x*) function and then solve.

   $f(x) = a \bullet \log_4 (x + 1)$ Function for *f*(*x*) found in the previous step

   $(4) = a \bullet \log_4 [(15) + 1]$ Substitute 15 for *x* and 4 for *f*(*x*).

   $4 = a \bullet \log_4 (16)$ Add.

   $4 = a \bullet 2$ Simplify.

   $2 = a$ Divide both sides by 2.

4. Substitute the value of *a* into the equation for *f*(*x*) found in step 2.

This is the last step in writing *f*(*x*).

   $f(x) = 2 \bullet \log_4 (x + 1)$

5. Set $f(x) = g(x)$.

This is the first step in finding the common solution of the two functions.

If $f(x) = 2 \bullet \log_4 (x + 1)$ and $g(x) = 3 \bullet \log_4 x$, then $2 \bullet \log_4 (x + 1) = 3 \bullet \log_4 x$ for the value of *x* at which a common solution exists.

6. Simplify the resulting equation.

    Use the rules of logarithms.

    $2 \cdot \log_4 (x + 1) = 3 \cdot \log_4 x$ can be rewritten as $\log_4 (x + 1)^2 = \log_4 x^3$.

7. Rewrite the equation without logarithms.

    When logarithms of the same base are equal, the arguments of those logarithms are equal. Each logarithm has a base of 4; therefore, set the arguments equal to each other.

    $(x + 1)^2 = x^3$

8. Explain how to find x in the resulting equation.

    Since this is a cubic equation, expand the binomial and collect all terms on one side of the equation.

    $(x + 1)^2 = x^3$

    $x^2 + 2x + 1 = x^3$

    $x^3 - x^2 - 2x - 1 = 0$

    There are no small integer solutions to this cubic equation, so x can be found by determining the root of this cubic equation on a graphing calculator. Alternatively, the point of intersection can be found by calculating it with the "calculate intersect" feature on a graphing calculator.

Example 4

The graph shows a logarithmic function of the form $f(x) = a \bullet \log_b (x + c)$, where $(-1, 2)$ is a point on the graph. The exact x-intercept can be determined directly from the graph. Write the x-intercept and estimate the value of the y-intercept from the graph. Then, calculate the actual y-intercept and compare this value with the estimate.
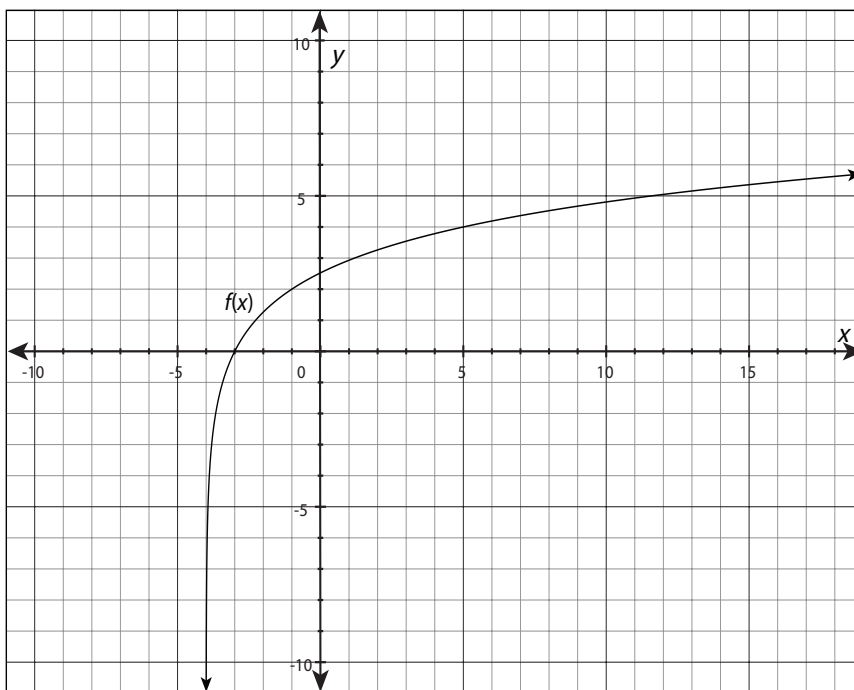
1. Identify the x-intercept from the graph and write its coordinates.

   The x-intercept can be used to help identify the constants in the general form of the function and find the y-intercept.

   The x-intercept is located at the point where the graph crosses the x-axis, or where $y = 0$.

   The x-intercept is -3, so the point containing it is $(-3, 0)$.

2. Estimate the y-intercept from the graph.

   The y-intercept is located at the point where the graph crosses the y-axis, or where $x = 0$. From the graph, the y-intercept appears to be at $y = 2.5$, but this should be verified algebraically.

3. Determine the equation for the y-intercept.

    The y-intercept can be calculated by evaluating the function at $x = 0$. To do this, first determine the equation for the y-intercept. The function is in the form $f(x) = a \bullet \log_b (x + c)$, so substitute 0 for x and simplify.

    $f(0) = a \bullet \log_b (0 + c) = a \bullet \log_b c$

    The equation for the y-intercept is $f(0) = a \bullet \log_b c$.

4. Use the x-intercept and the given point $(-1, 2)$ to find the constants in the function.

    To find one or more of the constants, substitute the coordinates of the x-intercept found in step 1 into the form $f(x) = a \bullet \log_b (x + c)$.

    The x-intercept is located at $(-3, 0)$.

$f(x) = a \bullet \log_b (x + c)$	Given function form
$(0) = a \bullet \log_b [(-3) + c]$	Substitute -3 for x and 0 for $f(x)$.
$0 = \log_b (c - 3)$	Divide both sides by a and simplify.
$b^0 = c - 3$	Apply the power rule of logarithms.
$1 = c - 3$	A quantity raised to a 0 power is equal to 1.
$4 = c$	Add 3 to both sides.

    Therefore, if $c = 4$, then $f(x) = a \bullet \log_b (x + 4)$.

    Now substitute $(-1, 2)$ into $f(x) = a \bullet \log_b (x + 4)$ to find a.

$f(x) = a \bullet \log_b (x + 4)$	Function from the previous step
$2 = a \bullet \log_b [(-1) + 4]$	Substitute -1 for x and 2 for $f(x)$.
$2 = a \bullet \log_b 3$	Add.
$\dfrac{2}{\log_b 3} = a$	Divide both sides by $\log_b 3$.

    The known constants are $a = \dfrac{2}{\log_b 3}$ and $c = 4$.

5. Substitute the value of a into the equation for the y-intercept.

   Use the equation for the y-intercept determined in step 3.

   $f(0) = a \bullet \log_b c$ Equation for the y-intercept

   $f(0) = \left(\dfrac{2}{\log_b 3} \right) \bullet \log_b (4)$ Substitute $\dfrac{2}{\log_b 3}$ for a and 4 for c.

   $f(0) = 2 \bullet \left(\dfrac{\log_b 4}{\log_b 3} \right)$ Rewrite the multiplication.

   $f(0) = (\log_3 4)^2$ Simplify.

6. Use a calculator to evaluate $(\log_3 4)^2$.

   Rewrite $(\log_3 4)^2$ as $2 \bullet \left(\dfrac{\log 4}{\log 3} \right)$ using the power and quotient rules.

   $f(0) = 2 \bullet \left(\dfrac{\log 4}{\log 3} \right) \approx 2.52$

   The y-intercept is approximately 2.52.

7. Compare the estimated y-intercept from the graph with the calculated y-intercept.

   Though the graph suggests that the y-intercept appears to be at (0, 2.5), the y-intercept is actually closer to approximately (0, 2.52), a difference of approximately 0.02.

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Practice 4A.2.4: Graphing Logarithmic Functions

For problems 1–4, sketch $f(x)$ and $g(x)$. Then, calculate the solution to the system of the two functions.

1. $f(x) = -4 + 3 \cdot \log_2 x$
   $g(x) = 3 - 4 \cdot \log_2 x$

2. $f(x) = \log (1 - x)$
   $g(x) = \log (x - 1)$

3. $f(x) = 2 \cdot \ln (x + 2)$
   $g(x) = \ln (x - 2)$

4. $f(x) = 5 \cdot \log_5 x$
   $g(x) = \log_5 (x - 5)$

For problems 5–7, compare the domains and ranges of the three functions in each problem. Then, state the domain(s) over which all three functions are defined.

5. $f(x) = 1 + \log x$
   $g(x) = \log x - 2$
   $h(x) = 1 + 2 \cdot \log x$

6. $f(x) = 3 \cdot \ln (x + 1)$
   $g(x) = -2 \cdot \ln (1 - x)$
   $h(x) = \ln (x - 1)$

7. $f(x) = 1 - 3 \cdot \log_3 (x - 3)$
   $g(x) = 3 - \log_3 (x - 1)$
   $h(x) = -1 + 3 \cdot \log_3 (x + 3)$

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For problems 8–10, use the information in each problem to sketch a graph of the given function on a coordinate plane. Be sure to label the axes so that all the real-world parts of the domain and range are evident. Then, use your graph to solve the problem.

8. A bacterium that is useful in digesting food is found to have a gene that can mutate, resulting in the bacterium's resistance to drugs that treat food poisoning. A researcher found that the number of instances of drug-resistant bacteria populations in a laboratory study can be modeled by the logarithmic function $N(i) = 90 - 65 \cdot \ln i$, where N is the number of instances of food poisoning and i is the number of drug-resistant bacteria populations found in each instance. Describe any conditions placed on the kinds of numbers that can be used for the domain and range. Let the domain be [1, 4] and the range be [0, 90].

9. The management team at a factory that makes smartphone cases has found that the number of defective cases produced can be described by the logarithmic function $D(n) = -80 + 12 \cdot \ln n$, where n is the number of cases produced in a production run and D is the number of defective units produced. The domain of n in a one-time study of the defective units is [1,000, 30,000]. What happens to the *rate* of defective cases produced per 1,000 cases as the size of the production run increases across the domain? Compare two pairs of sequential defective-case numbers to support your answer.

10. A food chemist at a state university studied the effect of pumping extra carbon dioxide into greenhouses where pepper plants grow. She found that the number of peppers produced by these plants can be modeled by the exponential function $N(c) = 20c \cdot e^{-0.4c}$, where N is the number of full-grown peppers and c is the concentration of carbon dioxide. Write a logarithmic function from the exponential function and explain the domain and range of the logarithmic function.

Lesson 4A.2.5: Interpreting Logarithmic Models

Introduction

Expressing or solving logarithmic functions in terms of exponential function models is one technique for solving real-world problems. One factor in determining which type of function is best in a given situation is how the solution to a problem affects a particular audience. For example, environmental scientists may need to present a study of the acidity or alkalinity of a freshwater pond to citizens at a town hall meeting. The citizens might best understand the results if the logarithm-based pH factor is used to describe the chemical condition of the pond rather than the actual concentration of hydronium or hydroxide ions in a sample of the pond water.

Such initial conditions as the upper and lower bound of a domain are essential to the viability of such models. For example, time is nearly always considered to be a positive quantity that moves in an ever-increasing direction. (There are exceptions to this in some of the leading-edge fields of physics, such as cosmology, but such discussions are generally beyond the scope of a mathematics course at this level.)

The ability to move accurately between a function and its inverse is often important in solving real-world problems that employ logarithms. Also, a thorough mastery of the basic rules of exponents and logarithms is essential for such problems.

Finally, be aware of the potential for a graphical or tabular presentation of a problem to aid in the application of logarithmic functions. In fact, visual models based on real data often provide a more accurate picture of a problem than an algebraic model that does not reveal the restricted domain or range of the problem in the way that a graph or table does.

Key Concepts

- Logarithmic functions have a wide range of applications in real-world problems, such as in the fields of biology, chemistry, ecology, and engineering. Logarithmic functions often provide an alternative approach to the use of exponential functions, which might help to increase the understanding of a problem or its solution.

- The logarithmic function is the inverse of an exponential function and vice versa. Recall two basic steps in writing one as the other:

  - A function's inverse switches the values of the domain and range values. For example, if an ordered pair of a function is (3, 5), then the ordered pair (5, 3) solves the inverse of the function.

  - To find the inverse of a function, replace the domain variable (often x) with the range variable ($f(x)$), and change the range variable to $f^{-1}(x)$.

- Apply the basic rules of exponentials and logarithms, described earlier in this lesson.

- The ability to identify the restricted domain and/or range over which a logarithmic function is defined can mean the difference between finding a solution to a problem and misinterpreting the application of that function to the particulars of the problem setting. This is especially true if a function and its inverse have roles to play in formulating and/or solving the problem.

- A graphing calculator can help with the identification of the restraints on real-world problems. The tables that are created when functions are graphed offer approximations of solutions that cannot be easily found using algebraic methods. Adjust the table and window settings as necessary to reflect the particulars of the problem.

Guided Practice 4A.2.5

Example 1

The number of electric vehicles (E) sold in the United States within 3 years of their introduction to the market can be modeled by the logarithmic function $E(y) = 1.67 + 5.74 \cdot \ln y$, where $E(y)$ is the number of vehicles sold in thousands and y is the number of years after introduction to the market. The number of hybrid vehicles (H) sold within 3 years of their introduction to the market can be modeled by the logarithmic function $H(y) = 0.78 + 2.4 \cdot \ln y$. At what value of y was an equal number of each vehicle sold, and which type of vehicle had greater sales over the 3 introductory years? Explain your reasoning with references to the terms in the function and how they compare.

1. Set $E(y) = H(y)$.

   The two functions are equal at a value of y that satisfies the equation $E(y) = H(y)$.

   We are given that $E(y) = 1.67 + 5.74 \cdot \ln y$ and $H(y) = 0.78 + 2.4 \cdot \ln y$.

   Therefore, $1.67 + 5.74 \cdot \ln y = 0.78 + 2.4 \cdot \ln y$.

2. Solve the resulting equation for $\ln y$.

   Isolate the logarithmic terms on one side of the equation and the constants on the other.

$1.67 + 5.74 \cdot \ln y = 0.78 + 2.4 \cdot \ln y$	Set $E(y)$ equal to $H(y)$.
$5.74 \cdot \ln y = 0.78 + 2.4 \cdot \ln y - 1.67$	Subtract 1.67 from both sides.
$5.74 \cdot \ln y - 2.4 \cdot \ln y = 0.78 - 1.67$	Subtract $2.4 \cdot \ln y$ from both sides.
$\ln y\,(5.74 - 2.4) = -0.89$	Factor out $\ln y$ and simplify.
$3.34 \cdot \ln y = -0.89$	Continue to simplify.
$\ln y \approx -0.2665$	Calculate the result.

3. Rewrite the resulting equation using an inverse function.

The exponential function with base e is the inverse of the natural logarithm function in step 2.

If $\ln y \approx -0.2665$, then $y \approx e^{-0.2665}$, which simplifies to $y \approx 0.766$.

4. Interpret the resulting value of y in terms of the conditions of the original problem.

The result also has to be checked for reasonableness as a member of the domain of the real-world problem.

Since y represents time in years, 0.766 would be the equivalent of approximately three-quarters of a year. This is the amount of time after vehicle introduction that the same number of electric vehicles and hybrid vehicles were sold. However, the sales data in the problem are based on 3 full years of sales, which implies a domain of $[1, 3]$. Parts of a year (e.g., 1.5 years, or a year and a half) are defined within this domain, but not for a period of time that falls outside of the domain, such as 0.75 year.

5. Check the value of y found in step 3 by substituting it back into the functions $E(y)$ and $H(y)$.

Values should always be checked to verify that the result is accurate and satisfies the condition(s) of the problem.

Recall that the functions represent the number of vehicles sold in thousands.

$E(0.766) \approx 1.67 + 5.74 \cdot \ln 0.766 \approx 0.140$, or about 140 vehicles

$H(0.766) \approx 0.78 + 2.4 \cdot \ln 0.766 \approx 0.140$, or about 140 vehicles

The results for each function are equal, so the calculated value of y checks out.

6. Show which vehicle type had greater sales over the 3-year period by comparing the characteristics of each function.

Look at the terms in each function and use the values of the domain to support your claim(s).

The constant term of $H(t)$ is less than the constant term of $E(t)$, which means that $E(t)_{\text{constant term}} > H(t)_{\text{constant term}}$. The coefficient of the logarithmic term of $E(t)$ is also greater than the coefficient of the logarithmic term of $H(t)$, which means that $E(t)_{\text{logarithmic term}} > H(t)_{\text{logarithmic term}}$. Furthermore, the logarithmic terms themselves are equal and are only defined for values of y in the range $1 \leq y \leq 3$, which also indicates they are positive. In summary:

$$E(t)_{\text{constant term}} + E(t)_{\text{logarithmic term}} \bullet \ln y > H(t)_{\text{constant term}} + H(t)_{\text{logarithmic term}} \bullet \ln y$$

As the result of step 5 shows, the two vehicle types had the same number of sales during a period of time of about 0.766 of a year that was not included in the functions' domain, namely, before the first sales year was over.

Example 2

A pyramid-shaped token for a board game consists of 4 congruent equilateral faces. Each face is a different color: blue, green, red, or white. Each face is equally likely to end up on the bottom if the token is rolled on the game board. What is the probability of the green face landing on the bottom? The probability of the same event a occurring n times is given by the function $f(n) = a^n$. Write a function for the green face landing on the bottom n times in a row. Then, write the inverse of the function, and explain what the inverse function describes in the context of this problem.

1. State the number of faces on the token.

This is the first step in defining the probability event.

There are 4 faces on the token.

2. State the probability of any of the 4 faces landing on the bottom.

Each face has a 1 in 4, or 0.25, chance of landing on the bottom. Thus, the probability of the green face landing on the bottom is 0.25.

3. Check your answer to step 2 by calculating the chance that *any* of the 4 faces lands on the bottom.

This answer should add up to 1.

The probability of a blue, green, red, *or* white face landing on the bottom is given by $0.25 + 0.25 + 0.25 + 0.25$, which equals 1.

4. Write the function for the green face landing on the bottom n times in a row.

This generalizes the result of step 2 for any number of rolls of the token.

Let $a = \dfrac{1}{4}$ in the function $f(n) = a^n$ and simplify.

$f(n) = a^n$

$f(n) = \left(\dfrac{1}{4}\right)^n$

$f(n) = (4^{-1})^n$

$f(n) = 4^{-n}$

5. Write the inverse of the function determined in step 4.

This will give the logarithm requested in the problem.

The inverse of $f(n) = 4^{-n}$ is $\log_4 f(n) = -n$, which simplifies to $n = -\log_4 f(n)$.

Switch the variables and change $f(n)$ to $f^{-1}(n)$:

$f^{-1}(n) = -\log_4 n = \log_4 n^{-1}$

Therefore, the inverse function is $f^{-1}(n) = \log_4 n^{-1}$.

6. Explain what the inverse function describes in the context of this problem.

Suppose $n = 4$. The value of the function at $n = 4$ is given by the ordered pair $(4, 4^{-4})$, so the corresponding point for the inverse is $(4^{-4}, 4)$. The inverse implies that when a probability of 4^{-4} occurs, the token has been rolled 4 times with the same outcome (4^{-1}) each time.

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Example 3

The owners of a West Virginia pine forest and grasslands preserve introduced a breeding pair of quail to a specific part of the property. The population of quail can be modeled by an exponential function, but a statistics teacher at the local college came up with the logarithmic function $M(n) = 8.33 - \log(50-n)^4$, which models how the number of quail offspring n relates to the number of months M since the introduction of the first pair to the preserve. What is the maximum number of quail (n_{max}) that can be estimated using this model? Explain your answer, and state the domain for the function. What does the constant 8.33 mean in terms of this function model? Write the inverse of the logarithmic function.

1. Name the type of number that has to be used for the variable n.

   The answer to this will also help in describing the domain.

   The variable n refers to individual quail, so n has to be a positive whole number that is equal to at least 2, since the problem specifies a breeding *pair*.

2. State the value of n for which the logarithmic term of the function is undefined.

   This has to be considered since a logarithm is involved.

   The argument of the logarithm, $50 - n$, cannot equal 0.

3. Use the results of steps 1 and 2 to list the domain.

   The variable n is a positive whole number that is at least 2 but less than 50, so the domain is [2, 49].

4. Name the maximum number of quail that can be estimated using the logarithmic model.

   The maximum number of quail is 49, since that is the largest number for n in the domain.

5. Determine the range of the function $M(n)$.

    M is the number of months, so $M(n)$ should be computed using the end points of the domain.

    Evaluate $M(n)$ at the beginning of the domain values, 2.

    $M(2) = 8.33 - \log (50 - 2)^4$

    $M(2) = 8.33 - \log(48)^4$

    $M(2) \approx 1.6$ months

    Evaluate $M(n)$ at the end of the domain values, 49.

    $M(49) = 8.33 - \log (50 - 49)$

    $M(49) = 8.33 - \log 1$

    $M(49) = 8.33$ months

    The time of 8.33 is the maximum time it will take for the quail population to "max out" at 49 birds.

6. Write the inverse of the logarithmic function.

    Simplify the logarithmic function using algebraic methods and the power rule of logarithms in order to write the inverse.

    $M(n) = 8.33 - \log (50 - n)^4$

    $M(n) - 8.33 = -4 \bullet \log (50 - n)$

    $8.33 - M(n) = 4 \bullet \log (50 - n)$

    $\dfrac{8.33 - M(n)}{4} = \log (50 - n)$

    Switch the variables, and change n to m and $M(n)$ to $N(m)$ to indicate that the inverse of the logarithmic function will use the number of months m as the independent variable and the variable $N(m)$ for the number of quail.

    The inverse is $\dfrac{8.33 - m}{4} = \log [50 - N(m)]$, which can be written

    as $50 - N(m) = 10^{\left(\frac{8.33 - m}{4}\right)}$ or $N(m) = 50 - 10^{2.0825 - 0.25m}$.

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Practice 4A.2.5: Interpreting Logarithmic Models

Use the information below to complete problems 1–4.

An air-quality sampling project around the salt marshes along the Massachusetts coast over a 5-year period revealed an increase in the quantities of methane and other gases. This increase implies a decrease in oxygen-producing plants, including grasses and other marsh vegetation. A staff scientist with the state environmental protection department came up with a logarithmic-function model $A(t) = 475 - 85 \cdot \ln t$ that describes the decreasing acreage of such oxygen-producing plants, based on the increased amount of the other gases over the 5-year period with t measured in months.

1. What does the function value $A(1)$ represent in this problem?

2. What is the domain of the function?

3. How many months does the model predict it will take for the plant acreage to be reduced by half of the amount measured from the end of the first year ($t = 1$) of the study?

4. Explain the validity of the model based on the results of problem 3.

Use the given information to complete problem 5.

5. A smelting factory installed "scrubbers" to reduce the amount of acidic particulate matter (dust) being released into the air. A chemist measured the change in rainwater acidity for 6 months following the installation of the scrubbers. Before the scrubbers were installed, the average pH of the rainwater in the area was 6.15. After the scrubbers were installed, the average pH of the rainwater increased to 6.9. By how much did the concentration of acid-producing substances decrease to account for this change in pH? The equation for pH is $\text{pH} = -\log c$, where c is the concentration of acid-producing substances.

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Use the graph and the given information to complete the problems that follow.

In vehicles with air brakes, braking is caused by compressed air in a chamber pressing on a piston that's connected to the brake pad. The following graph shows the work $W(v)$ done by the air brakes on a commuter-rail train coach at two different operating temperatures, $A(v) = \frac{1}{2} \bullet \ln v$ and $B(v) = \frac{1}{3} \bullet \ln v$, where v is the volume of air in the compression chamber.
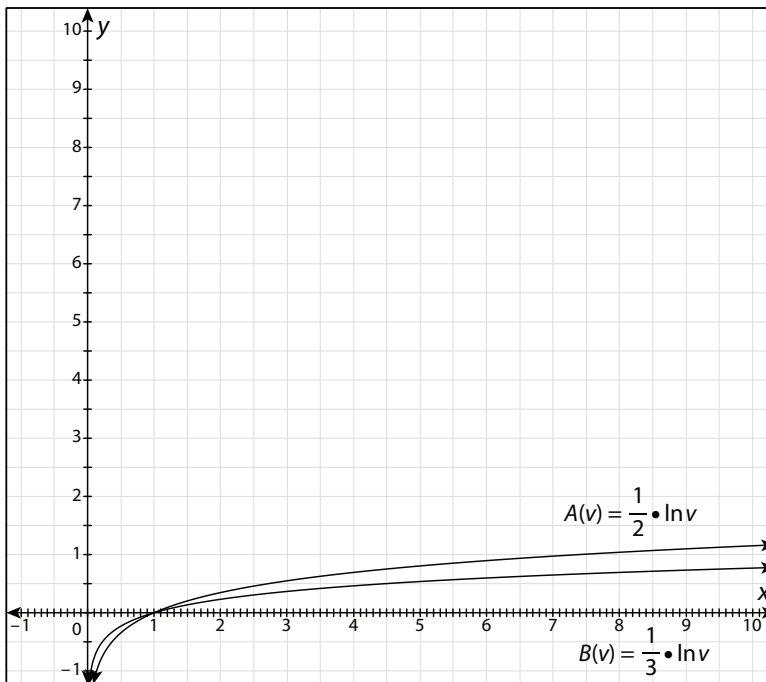
PRACTICE

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6. Draw a horizontal line connecting the graphed curves.

7. What does the horizontal line represent?

8. Without using a calculator, find the volumes of the brake air-compression chamber that correspond to a function value of 0.3 for each function.

9. Write the volume of function $B(v)$ in terms of the volume of function $A(v)$.

10. What do you notice about the exponent of e in the answer to problem 9 in comparison to the value of the work performed?

Lesson 3: Modeling Trigonometric Functions

Common Core State Standard

F–IF.7   Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

e.   Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

Essential Questions

1.   What is the amplitude of a sine function given by $f(x) = a \sin bx$?

2.   What is the period of the sine function given by $f(x) = a \sin bx$?

3.   What is the amplitude of a cosine function given by $g(x) = a \cos bx$?

4.   What is the period of the cosine function given by $g(x) = a \cos bx$?

5.   How do you find the amplitude, domain, period, and range of a trigonometric function?

WORDS TO KNOW

amplitude	the coefficient a of the sine or cosine term in a function of the form $f(x) = a \sin bx$ or $g(x) = a \cos bx$; on a graph of the cosine or sine function, the vertical distance from the y-coordinate of the maximum point on the graph to the midline of the cosine or sine curve
argument	the term $[b(x - c)]$ in a cosine or sine function of the form $f(x) = a \sin [b(x - c)] + d$ or $g(x) = a \cos [b(x - c)] + d$

cosine	a trigonometric function of an acute angle in a right triangle that is the ratio of the length of the side adjacent to the length of the hypotenuse; the cosine of $\theta = \cos \theta = \dfrac{\text{length of adjacent side}}{\text{length of hypotenuse}}$
cosine function	a trigonometric function of the form $f(x) = a \cos bx$, in which a and b are constants and x is a variable defined in radians over the domain $(-\infty, \infty)$
cycle	the smallest representation of a cosine or sine function graph as defined over a restricted domain; equal to one repetition of the period of a function
midline	in a cosine function or sine function of the form $f(x) = \sin x + d$ or $g(x) = \cos x + d$, a horizontal line of the form $y = d$ that bisects the vertical distance on a graph between the minimum and maximum function values
parent function	a function with a simple algebraic rule that represents a family of functions. The graphs of the functions in the family have the same general shape as the parent function. For cosine and sine, the parent functions are $f(x) = \cos x$ and $g(x) = \sin x$, respectively.
period	in a cosine or sine function graph, the horizontal distance from a maximum to a maximum or from a minimum to a minimum; one repetition of the period of a function is called a *cycle*
phase shift	on a cosine or sine function graph, the horizontal distance by which the curve of a parent function is shifted by the addition of a constant or other expression in the argument of the function
radian	the measure of the central angle that intercepts an arc equal in length to the radius of the circle; π radians = 180°

sine	a trigonometric function of an acute angle in a right triangle that is the ratio of the length of the opposite side to the length of the hypotenuse; the sine of $\theta =$ $$\sin\theta = \frac{\text{length of opposite side}}{\text{length of hypotenuse}}$$
sine function	a trigonometric function of the form $f(x) = a \sin bx$, in which a and b are constants and x is a variable defined in radians over the domain $(-\infty, \infty)$
unit circle	a circle with a radius of 1 unit. The center of the circle is located at the origin of the coordinate plane.

Recommended Resources

- Brightstorm, Inc. "Graphs of the Sine and Cosine Functions."

  http://www.walch.com/rr/00233

  This site provides short video tutorials with accompanying transcripts on graphing sine and cosine functions.

- Illuminations. "Trigonometric Graphing."

  http://www.walch.com/rr/00234

  This graphing utility allows users to explore amplitudes, periods, and phase shifts of trigonometric functions.

- Purplemath.com. "Graphing Trigonometric Functions."

  http://www.walch.com/rr/00235

  This website explains how to graph trigonometric functions and includes abundant examples of sine and cosine graphs.

IXL Links

- Match exponential functions and graphs:
  http://www.ixl.com/math/algebra-1/match-exponential-functions-and-graphs

- Match exponential functions and graphs:
  http://www.ixl.com/math/algebra-1/match-exponential-functions-and-graphs

Lesson 4A.3.1: Graphing the Sine Function

Introduction

In previous lessons, you have worked with sine ratios. Recall that **sine** is a trigonometric function of an acute angle in a right triangle. It is the ratio of the length of the opposite side to the length of the hypotenuse, or $\sin \theta = \dfrac{\text{length of opposite side}}{\text{length of hypotenuse}}$. In this lesson, you will explore the graphs of sine functions.

Key Concepts

- A **sine function** is a trigonometric function of the form $f(x) = a \sin bx$, in which a and b are constants and x is a variable defined in radians over the domain $(-\infty, \infty)$.

- Recall that a **radian** is the measure of the central angle that intercepts an arc equal in length to the radius of the circle; π radians = 180°.

- In order to sketch the graph of a sine function, it is necessary to know the amplitude and period of the parent function. A **parent function** is a function with a simple algebraic rule that represents a family of functions. The graphs of the functions in the family have the same general shape as the parent function. For sine, the parent function is $f(x) = \sin x$, as shown in the following graph.

Sine Parent Function, $f(x) = \sin x$
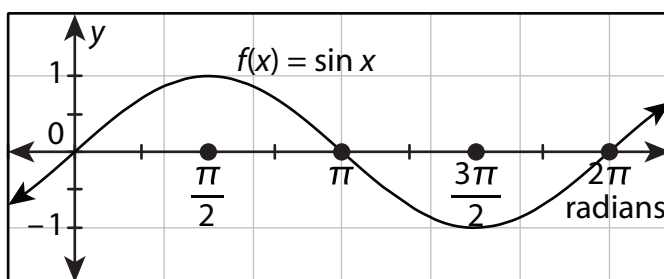
- On this graph, the points shown on the x-axis to the right of the origin correspond to the points $\left(\dfrac{\pi}{2}, 0 \right)$, $(\pi, 0)$, $\left(\dfrac{3\pi}{2}, 0 \right)$, and $(2\pi, 0)$, which are multiples of π in radians. For example, $\dfrac{\pi}{2} \approx 1.57$ and $2\pi \approx 6.28$. (*Note*: Most graphing calculators display the multiples of π as decimal approximations.)

- The **amplitude** of a function is defined as the vertical distance from the y-coordinate of the maximum point on the graph to the midline of the curve. In the graph of the parent function $f(x) = \sin x$, the amplitude is 1, the distance from the midline (x-axis) to the highest point on the curve.

- The **midline** in a function of the form $f(x) = \sin x + d$ is a horizontal line of the form $y = d$ that bisects the vertical distance on a graph between the minimum and maximum function values. In this example, $f(x) = \sin x$ can be written as $f(x) = \sin x + 0$; therefore, the midline is at $y = 0$.

- The **period** of a graphed sine function is the horizontal distance from a maximum to a maximum or from a minimum to a minimum. In other words, it is the beginning of one complete cycle of the graph to the point at which its behavior or shape repeats. A **cycle** is the smallest representation of a sine function graph as defined over a restricted domain that is equal to the period of the function.
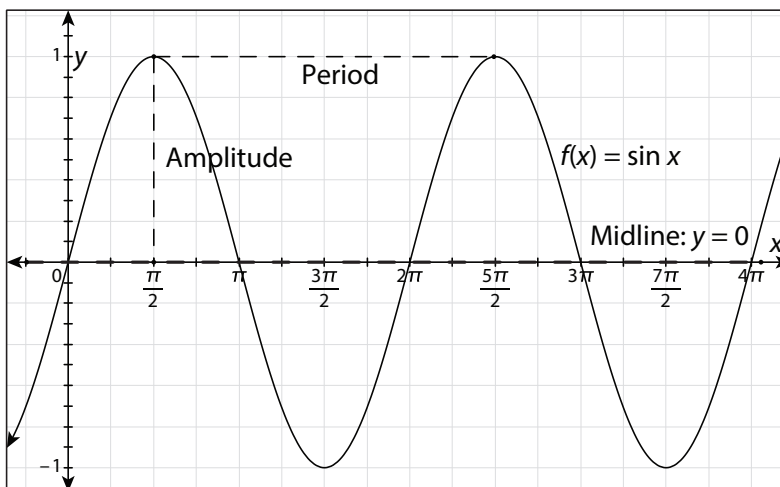
- In this example, the period is 2π or about 6.28, which can also be expressed as 360°. (Recall that a **unit circle** has a radius of 1, which means its circumference is 2π and which creates an angle of 360°.)

- In the general form of the sine function, $f(x) = a \sin bx$, the amplitude is $|a|$ and the period is $\dfrac{2\pi}{b}$. In other words, any change in the value of a has a direct effect on the amplitude of the graph, and any change in the value of b has an inverse effect on the period.

- In an extended form of the general sine function, $g(x) = a \sin [b(x - c)] + d$, the constant d shifts the graph vertically, whereas the constant c shifts the graph horizontally.

- The term $[b(x − c)]$ in a sine function of the form $g(x) = a \sin [b(x − c)] + d$ is the argument of the function.

- The amount of the **phase shift**, or the horizontal distance by which the curve of a parent function is shifted, is found by setting the argument equal to 0 and solving for x. If the function is in the form $f(x) = a \sin [b(x − c)]$, the phase shift is c. If the function is in the form $f(x) = a \sin (bx + c)$, the phase shift is $-\dfrac{c}{b}$.

Expressing the Domain of a Sine Function

- The domain of a sine function is often expressed in radians. However, in some applications and real-world problems, it is useful to express the domain in degrees.

- The conversion between degrees and radians is given by the relationship 2π radians = 360 degrees. Use the following equations to convert between the two units of measurement.

$$1 \text{ radian} = \frac{180}{\pi} \approx 57.3° \text{ or } 1 \text{ degree} = \frac{\pi}{180} \approx 0.02 \text{ radian}$$

- When sketching the graph of a sine function, knowing the values of the function at several critical values of x can be useful.

$$\sin 30° = \sin\left(\frac{\pi}{6} \text{ radian}\right) = 0.5$$

$$\sin 45° = \sin\left(\frac{\pi}{4} \text{ radian}\right) = \frac{\sqrt{2}}{2} \approx 0.71$$

$$\sin 60° = \sin\left(\frac{\pi}{3} \text{ radians}\right) = \frac{\sqrt{3}}{2} \approx 0.87$$

General Characteristics of a Sine Function

- The amplitude of the general sine function $f(x) = a \sin [b(x − c)] + d$ is $|a|$. Any change in a has a *direct* effect on the amplitude of the sine graph. For example, if a increases, the value of the term $a \sin [b(x − c)]$ also increases if the other quantities stay the same.

- The period of the general sine function $f(x) = a \sin [b(x - c)] + d$ is $\dfrac{2\pi}{b}$. Any change in b has an *inverse* effect on the period of the sine graph. For example, if b increases, the value of the period of the function *decreases* if the other quantities stay the same.

- A useful formula for finding the period of any sine function of the form $f(x) = a \sin [b(x - c)] + d$ is to set the argument of the sine function, $[b(x - c)]$, equal to 2π and solve for x. The result is the period of the function.

- In summary, the sine function $f(x) = a \sin [b(x - c)] + d$ has the following characteristics:

Amplitude: a Period: $\dfrac{2\pi}{b}$ Phase shift: c Vertical shift: d

The Unit Circle

- The unit circle is a visual means of viewing sine and cosine values. The unit circle has a radius r of 1 unit and is drawn on a coordinate plane, with the center at $(0, 0)$.
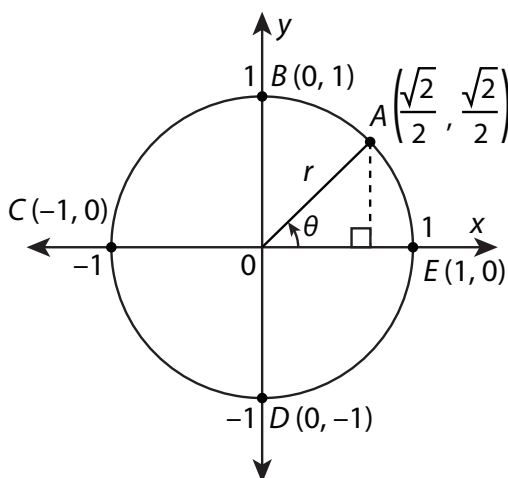
- The coordinates of point A, $\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$, represent the lengths of the sides of a right triangle formed by dropping an altitude to the x-axis. Those x- and y-coordinates also represent the cosine and sine ratios of a 45° (or a $\dfrac{\pi}{4}$-radian) angle θ, respectively, of an isosceles right triangle with a hypotenuse that is 1 unit in length. This originates from the plane geometry definition of the cosine and sine ratios.

- The sine and cosine ratios of any right triangle can be found as follows.
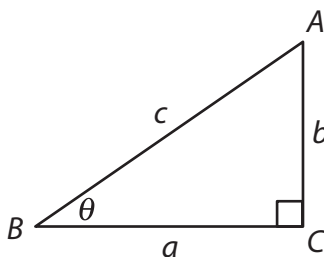
- The cosine of angle θ (abbreviated cos θ) is defined as the ratio of the length of side a (the side that is adjacent to angle θ) to the length of the hypotenuse, side c.

$$\cos\theta = \frac{a}{c} \text{ or } a = c \cos\theta$$

- The sine of angle θ (abbreviated sin θ) is defined as the ratio of the length of side b (the side that is opposite angle θ) to the length of the hypotenuse, side c.

$$\sin\theta = \frac{b}{c} \text{ or } b = c \sin\theta$$

- On the unit circle diagram, the horizontal leg of the right triangle is a and the vertical leg is b. Therefore, the coordinates of a point that lies on the unit circle are (a, b) or $(c \cos\theta, c \sin\theta)$. For a unit circle, $c = 1$, so the coordinates of the point are $(\cos\theta, \sin\theta)$.

- The same constructions for angles of 30° and 60° (equivalent to $\dfrac{\pi}{6}$ and $\dfrac{\pi}{3}$ radians, respectively) can be made for points on the circumference of the circle in the first quadrant. For 30°, the coordinates of the point that lies on the unit circle are $\left(\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$, which is the same as (cos 30°, sin 30°). For 60°, the coordinates of the point that lies on the circle are $\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$, which is the same as (cos 60°, sin 60°).

- The coordinates of the points formed by angles of 30°, 45°, and 60° can be applied to the other three quadrants, but the angle measures will change, and the signs of one or both of the coordinates will change.

- The following diagram shows the angle measures of θ and the related coordinates on the unit circle for the four quadrants:
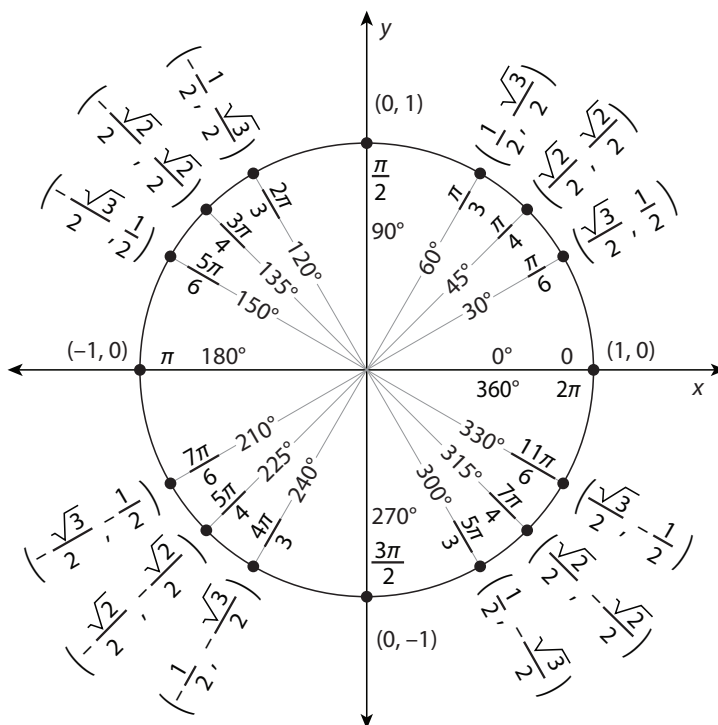
- Angles measured counterclockwise around the unit circle are positive, whereas angles measured clockwise are negative.

- The coordinates of the points (0, 1), (–1, 0), (0, –1), and (1, 0) on the graph of the unit circle represent the cosine and sine ratios of angles that measure 90°, 180°, 270°, and 360° ($\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, and 2π radians, respectively) if the angle θ is measured counterclockwise from the positive x-axis. These points also represent the cosine and sine ratios of angles that measure –270°, –180°, –90°, and 0° ($-\frac{3\pi}{2}$, -2π, $-\frac{\pi}{2}$, and 0 radians) if the angle θ is measured clockwise from the positive x-axis.

- Adding any positive or negative integer multiple of 360° or 2π radians to an angle will produce equivalent coordinates.

Guided Practice 4A.3.1

Example 1

Sketch the graph of $f(x) = 2 \sin x$ over the restricted domain $[-2\pi, 2\pi]$.

1. Identify the amplitude and period of the function.

   This will determine the *x*- and *y*-axis scales.

   The coefficient of the sine term is 2, which is the amplitude. Therefore, the *y*-axis scale should run from at least $y = -2$ to $y = 2$. The coefficient of the sine argument is 1, so $x = 2\pi$, which means the period is 2π. If the graph shows the restricted domain of $[-2\pi, 2\pi]$, it will include two complete "cycles" of the function.

2. Identify any other coefficients or terms that would affect the shape of the graph.

   This is necessary to determine if the graph is translated horizontally or vertically, or if other mathematical terms affect the graph's shape.

   The equation of the function, $f(x) = 2 \sin x$, has no other coefficients or terms that affect the shape or placement of the function's graph.

3. Draw and label the axes for your graph based on steps 1 and 2.

   The x-axis scale will range from -2π to 2π, and the y-axis scale will range from at least -2 to at least 2. Each axis should be divided into sufficient intervals to allow for enough points to be plotted to show the graph. Label the x-axis in increments of $\pm\dfrac{\pi}{2}$ and the y-axis in increments of $\dfrac{1}{2}$.
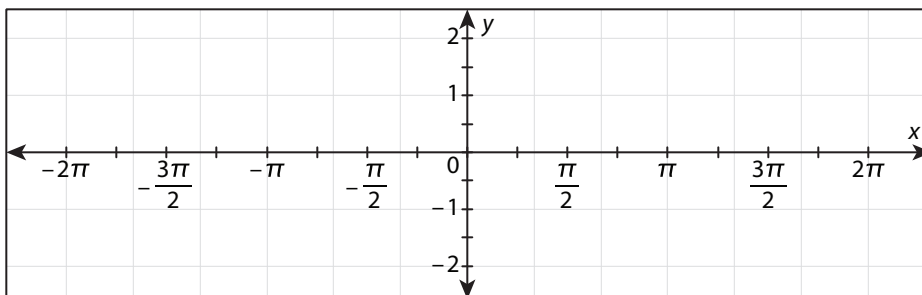
4. Determine the values of the restricted domain for which the function value(s) equal 0. List the points corresponding to these zeros, and plot them on the graph.

   The period of the function $f(x) = 2\sin x$ is 2π. The function is equal to 0 at $x = \pm 2\pi$, $\pm\pi$, and 0. The corresponding points are $(-2\pi, 0)$, $(-\pi, 0)$, $(0, 0)$, $(\pi, 0)$, and $(2\pi, 0)$. Plot the five points as shown.
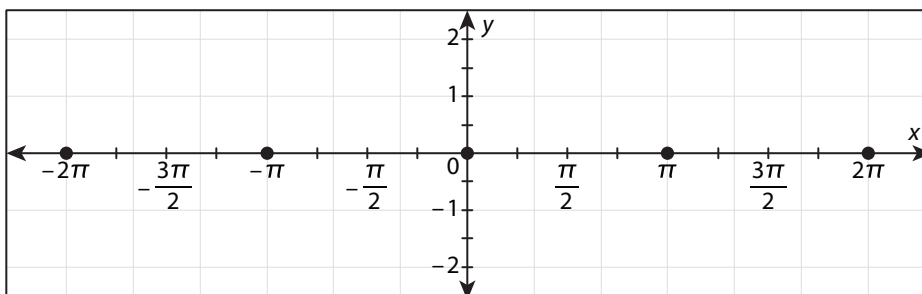
5. Determine what values of the restricted domain are the maximum and minimum of the function value(s). List the points corresponding to these extremes, and plot them on the graph.

The sine function values vary between the two extremes defined by the amplitude. In this case, the amplitude is 2, so the function values for the maximum and minimum values of $f(x) = 2 \sin x$ will vary between -2 and 2. These values occur at $x = \pm\dfrac{\pi}{2}$ and $\pm\dfrac{3\pi}{2}$.

The points for the maximum and minimum values of $f(x) = 2 \sin x$ are $\left(-\dfrac{3\pi}{2}, 2\right)$, $\left(-\dfrac{\pi}{2}, -2\right)$, $\left(\dfrac{\pi}{2}, 2\right)$, and $\left(\dfrac{3\pi}{2}, -2\right)$. Add these points to the graph.
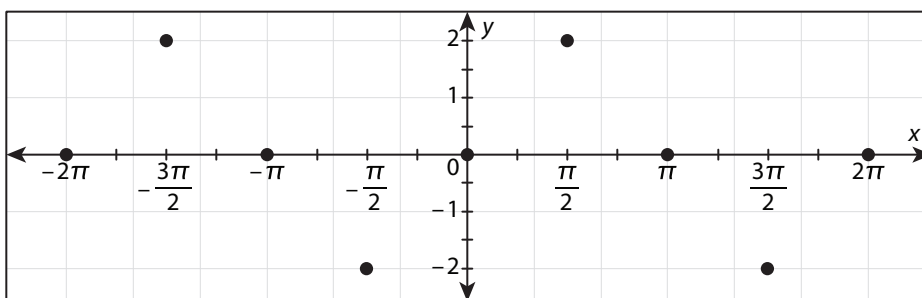
6. Plot points for the sines of $\dfrac{\pi}{6}$, $\dfrac{\pi}{4}$, and $\dfrac{\pi}{3}$ radians.

Refer to the diagram of coordinates for given values of $m\angle\theta$ in the Key Concepts to find the sines, or use a graphing calculator.

These points will fall between the zero at (0, 0) and the function maximum at $\left(\dfrac{\pi}{2}, 2\right)$. Moreover, they will suggest the shape of a quarter-period "piece" of the sine curve, which can be reflected across the y-axis from the maximum point at $\left(\dfrac{\pi}{2}, 2\right)$ to the x-axis to produce the sine curve over the domain of [0, π].

From the diagram of coordinates for $m\angle\theta$, we see that $2\sin\left(\dfrac{\pi}{6}\right)=1$ and $\dfrac{\pi}{6}\approx0.5$, so its ordered pair is (0.5, 1). The coefficient 2 is specific to this function; multiply by the function values $\dfrac{\pi}{4}$ and $\dfrac{\pi}{3}$, respectively, to find the other two function values and points:

$$2\sin\left(\dfrac{\pi}{4}\right)=\dfrac{\sqrt{2}}{2}\approx1.4 \text{ and } \dfrac{\pi}{4}\approx0.8\,;\,(0.8,\ 1.4)$$

$$2\sin\left(\dfrac{\pi}{3}\right)=\dfrac{\sqrt{3}}{2}\approx1.7 \text{ and } \dfrac{\pi}{3}\approx1.1\,;\,(1.1,\ 1.7)$$

Plot (0.5, 1), (0.8, 1.4), and (1.1, 1.7) on the graph between the points (0, 0) and $\left(\dfrac{\pi}{2}, 2\right)$.
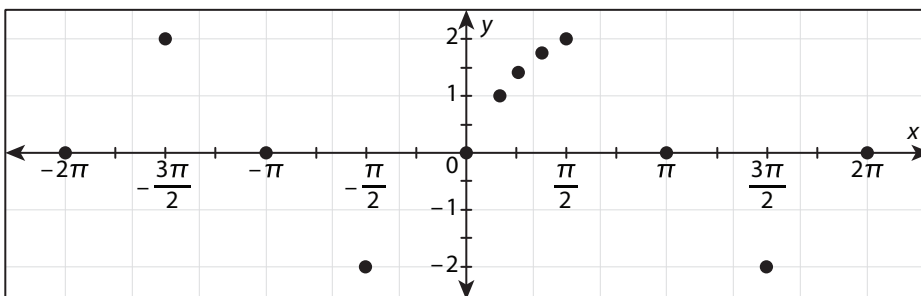
7. Plot additional points.

Plot the points for the x-values $\dfrac{2\pi}{3}$, $\dfrac{3\pi}{4}$, and $\dfrac{5\pi}{6}$, which are between the points $\left(\dfrac{\pi}{2}, 2\right)$ and $(\pi, 0)$.

This will show how the function values are mirrored on either side of the altitude from the maximum point at $\left(\dfrac{\pi}{2}, 2\right)$ to the midline $y = 0$ (the x-axis) for domain values that differ by $\dfrac{\pi}{2}$.

For this function, $f\left(\dfrac{2\pi}{3}\right) = 2\sin\left(\dfrac{2\pi}{3}\right) = 2 \bullet \dfrac{\sqrt{3}}{2} \approx 1.7$, so the approximate point is given by $\left(\dfrac{2\pi}{3}, 1.7\right)$. The other function values can be determined by using a calculator or by finding the appropriate values in the diagram of coordinates for $m\angle\theta$ and multiplying by 2, giving the approximate points as $\left(\dfrac{3\pi}{4}, 1.4\right)$ and $\left(\dfrac{5\pi}{6}, 1\right)$. Plot these points.
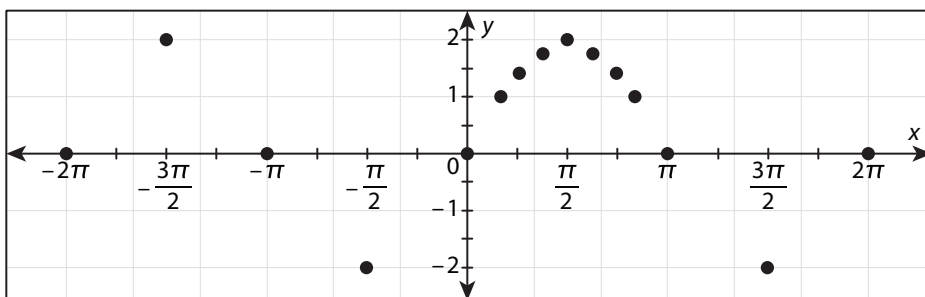
8. Compare the ordered pairs for the graphed points on either side of the *x*-axis.

The points occur where $x = \dfrac{\pi}{6}$ and $\dfrac{5\pi}{6}$, $x = \dfrac{\pi}{4}$ and $\dfrac{3\pi}{4}$, and $x = \dfrac{\pi}{3}$ and $\dfrac{2\pi}{3}$.

The function values for these point pairs are equal, which confirms that the points are mirrored on either side of the altitude from the maximum point at $\left(\dfrac{\pi}{2}, 2 \right)$ to the midline at $y = 0$ (the *x*-axis) over the restricted interval [0, π] for domain values that differ by $\dfrac{\pi}{2}$.

9. Determine the function values for each value of *x* over the restricted domain of [π, 2π]. Plot points on the graph for this part of the domain.

This will show that the function values for values of *x* over the restricted domain [π, 2π] are the opposite of those over the restricted domain [0, π] for domain values that differ by $\dfrac{\pi}{2}$.

Use a graphing calculator or refer to the diagram of coordinates for given values of $m\angle\theta$ in the Key Concepts to find the function values for the *x*-values over the domain [π, 2π].

The *x*-values are $\dfrac{7\pi}{6}$, $\dfrac{5\pi}{4}$, $\dfrac{4\pi}{3}$, $\dfrac{5\pi}{3}$, $\dfrac{7\pi}{4}$, and $\dfrac{11\pi}{6}$.

The calculator will give function values that result in the approximate

points $\left(\dfrac{7\pi}{6}, -1\right)$, $\left(\dfrac{5\pi}{4}, -1.4\right)$, $\left(\dfrac{4\pi}{3}, -1.7\right)$, $\left(\dfrac{5\pi}{3}, -1.7\right)$, $\left(\dfrac{7\pi}{4}, -1.4\right)$, and $\left(\dfrac{11\pi}{6}, -1\right)$.

Plot these points on the graph.
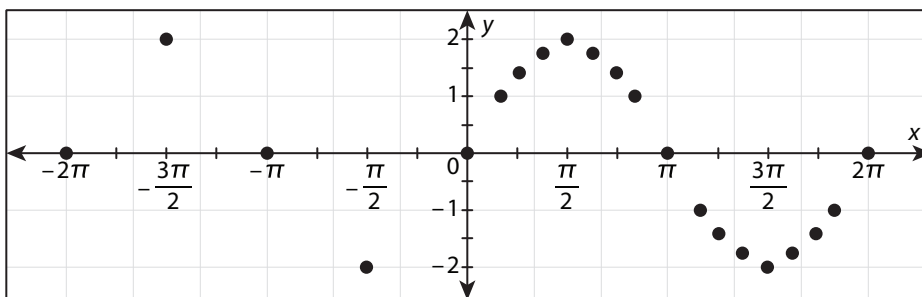
10. Compare the points over the two restricted domains $[0, \pi]$ and $[\pi, 2\pi]$. The points over the restricted domain $[\pi, 2\pi]$ graphed in step 9 are the points previously graphed for the restricted domain $[0, \pi]$, but flipped over the x-axis and reflected over the point $(\pi, 0)$. This reinforces the idea that the function values of the points over the restricted domain $[0, \pi]$ and points over the restricted domain $[\pi, 2\pi]$ are opposites for domain values that differ by $\dfrac{\pi}{2}$.

11. Predict what the shape of the function graph will be over the remainder of the domain, $(-2\pi, 0)$.

Note that the function values change signs for every half period of domain values graphed. Therefore, the shape of the function graph over the domain $(-2\pi, 0)$ should be a reflection of the function graph over the domain $[0, 2\pi]$ across the midline $y = 0$ (the x-axis), followed by a reflection across the y-axis.

12. Plot additional points on the graph to confirm your prediction. Then, draw a curve connecting the points across the domain $[-2\pi, 2\pi]$.

At a minimum, points over the domain $(-\pi, 0)$ should be plotted for x-values that are the opposite of those used for the domain $(\pi, 0)$.
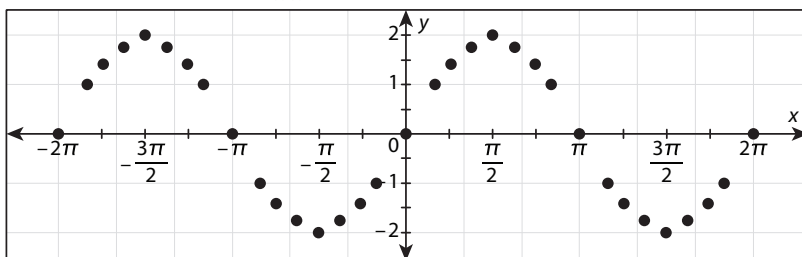
13. Verify the resulting graph using a graphing calculator.

Enter the given function, $f(x) = 2 \sin x$, into your graphing calculator. Regardless of whether you are using the TI-83/84, the TI-Nspire, or a similar calculator, remember to adjust the viewing window values so that the x-axis endpoints include the function's domain and the y-axis endpoints include the amplitude. Use the known x- and y-values to determine a value for the x-axis scale that will provide a comprehensive view of the graphed function. Make sure the mode is set correctly for the given problem (i.e., degrees or radians).

For this function:

- Since the x-values are in increments of π, set the mode to radians.

- The domain is $[-2\pi, 2\pi]$, so set the x-axis endpoints to -2π and 2π.

- The amplitude is 2, so set the y-axis endpoints to at least -2 and 2.

- For the x-axis scale, we know that the smallest interval between x-values is $\dfrac{\pi}{6}$; let this be the x-axis scale.

The resulting graph on the calculator should confirm the accuracy of the sketched function graph.

Example 2

How many complete cycles of the sine function $g(x) = \sin 3x$ are found in the restricted domain $[-180°, 180°]$?

1. Identify the period of $g(x)$.

   This will be needed to calculate the number of complete cycles of $g(x)$ that exist in the restricted domain.

   Set the argument of the function equal to the period of the parent sine function, $\sin x$, and solve for x: $3x = 2\pi$, so $x = \dfrac{2\pi}{3}$.

   The period of the function $g(x) = \sin 3x$ is $\dfrac{2\pi}{3}$.

2. Convert the period to degrees.

   Since the domain is given in degrees, the period must also be in degrees.

   1 radian $= \dfrac{180°}{\pi}$, so $\dfrac{2\pi}{3}$ radians is $\dfrac{2\pi}{3} \cdot \dfrac{180°}{\pi} = \dfrac{360\pi}{3\pi} = 120°$.

   The period of $g(x)$, converted to degrees, is $120°$.

3. Determine the number of complete cycles $g(x)$ makes over the domain $[-180°, 180°]$.

   In order to determine the number of completed cycles, calculate the total number of degrees within the domain of the function, and then divide this number by the period.

   The domain covers $180° - (-180°)$ or $360°$. Therefore, the number of complete cycles of $g(x)$ is $360°$ divided by the period, $120°$: $360 \div 120 = 3$ complete cycles.

4. Verify the result using a graphing calculator.

Enter the given function, $g(x) = \sin 3x$, into your graphing calculator. Regardless of whether you are using the TI-83/84, the TI-Nspire, or a similar calculator, remember to adjust the viewing window values so that the x-axis endpoints include the function's domain and the y-axis endpoints include the amplitude. Use the known x- and y-values to determine a value for the x-axis scale that will provide a comprehensive view of the graphed function. Make sure the mode is set correctly for the given problem (i.e., degrees or radians).

For this function:

- Set the mode to degrees.

- The domain is [−180°, 180°], so set the x-axis endpoints to −180 and 180.

- The amplitude is 1, so set the y-axis endpoints to at least −1 and 1.

- For the x-axis scale, choose a scale value that's a factor of the endpoints. Let the x-axis scale be 30 (a factor of −180 and 180).

Graph the function and count the number of complete cycles.

The resulting graph confirms our calculation that $g(x)$ has 3 complete cycles over the restricted domain [−180°, 180°].
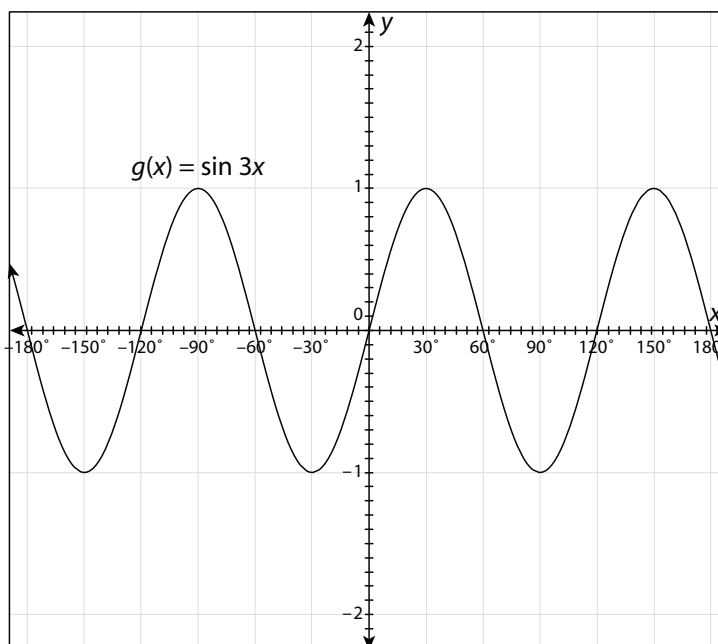
Example 3

Determine the coordinates of the point(s) that represent a maximum positive function value of the function $h(x) = 3 \sin 2x$ over the restricted domain $\left[-\dfrac{3\pi}{4}, \dfrac{5\pi}{3} \right]$.

1. Identify the amplitude and period of the function.

   This information will be needed to determine the location and the magnitude of the maximum function value.

   The amplitude is 3 because 3 is the coefficient of the sine term. The period can be found by setting the argument of the function equal to 2π and solving for x: $2x = 2\pi$, so $x = \pi$.

2. Determine where the function will have maximum values over the domains of its period on either side of the origin.

   This will indicate the value(s) of x at which the function has a maximum.

   At the origin, $x = 0$, so the domains of the function's period on either side of the origin are $[-\pi, 0]$ and $[0, \pi]$.

   The parent sine function $f(x) = \sin x$ has a maximum at $x = \dfrac{\pi}{2}$ and at $x = -\dfrac{3\pi}{2}$, so $h(x) = 3 \sin 2x$ will have a maximum of 3 at $2x = \dfrac{\pi}{2}$ or at $x = \dfrac{\pi}{4}$. It will have another maximum of 3 at $2x = -\dfrac{3\pi}{2}$ or at $x = -\dfrac{3\pi}{4}$.

3. Determine how many maximums $h(x)$ has over the restricted domain $\left[-\dfrac{3\pi}{4}, \dfrac{5\pi}{3}\right]$.

The problem statement gives the restricted domain of $h(x) = 3\sin 2x$ as $\left[-\dfrac{3\pi}{4}, \dfrac{5\pi}{3}\right]$.

One maximum value occurs at $x = -\dfrac{3\pi}{4}$, which is also the lower bound of the restricted domain. The other maximum occurs at $x = \dfrac{\pi}{4}$, which is less than the upper bound of the restricted domain at $x = \dfrac{5\pi}{3}$.

However, it might be useful to find the next maximum on the positive axis and compare it to the upper bound of the restricted domain. The next maximum for $h(x)$ occurs at $x = \dfrac{\pi}{4} + \pi = \dfrac{5\pi}{4}$. This result is less than $x = \dfrac{5\pi}{3}$, so there are two maximums along the positive part of the x-axis that is in the restricted domain. The next maximum for $h(x)$ occurs at $\dfrac{\pi}{4} + 2\pi = \dfrac{9\pi}{4}$, which is greater than $x = \dfrac{5\pi}{3}$.

There are 3 maximum points over the restricted domain: $\left(-\dfrac{3\pi}{4}, 3\right)$, $\left(\dfrac{\pi}{4}, 3\right)$, and $\left(\dfrac{5\pi}{4}, 3\right)$.

4. Check the result using a graphing calculator.

Enter the given function, $h(x) = 3 \sin 2x$, into your graphing calculator. Adjust the viewing window values to include the function's domain and amplitude, with a suitable x-axis scale. Make sure the mode is set correctly for the given problem (i.e., degrees or radians).

For this function:

- Set the mode to radians.

- The domain is $\left[-\dfrac{3\pi}{4}, \dfrac{5\pi}{3} \right]$, so set the x-axis endpoints to $-\dfrac{3\pi}{4}$ and $\dfrac{5\pi}{3}$.

- The amplitude is 3, so set the y-axis endpoints to at least –3 and 3.

- We know that the first function maximum occurs at $x = \dfrac{\pi}{4}$; let $\dfrac{\pi}{4}$ be the x-axis scale.

Graph the function and count the number of maximums.

The resulting graph confirms our calculation that $h(x) = 3 \sin 2x$ has three maximums over the restricted domain $\left[-\dfrac{3\pi}{4}, \dfrac{5\pi}{3} \right]$.

$h(x) = 3 \sin 2x$
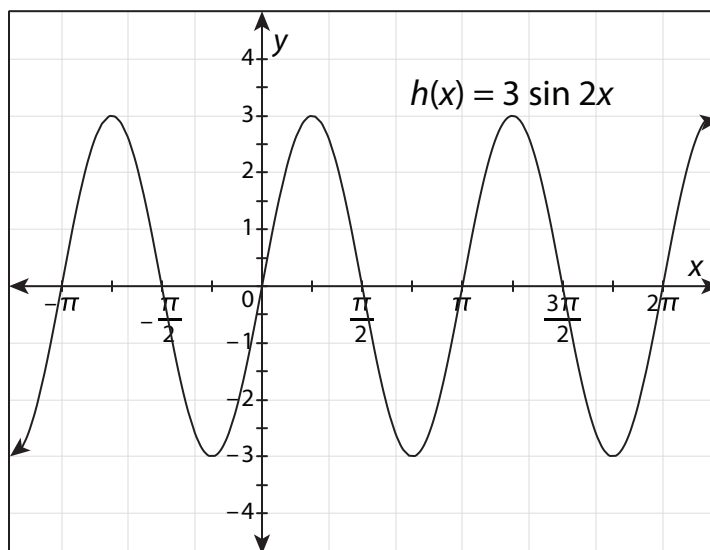
Example 4

Sketch the graph of the function $a(x) = 1 + 2 \sin 3x$ over the restricted domain $[0, 2\pi]$.

1. Identify the amplitude of the function.

   The addition of the constant to the sine term will result in a vertical shift to the graph as well as to the relative locations of the maximum and minimum function values. The amplitude is 2, but the 1 added to the sine term will produce a maximum function value of 3 and a minimum function value of -1. The constant defines the midline, which is $y = 1$.

   Therefore, the maximum function value occurs at $y = 3$ and the minimum function value occurs at $y = -1$.

2. Determine the period of the function.

   Set the argument of the function equal to 2π, and then solve for x to find the period.

   $$3x = 2\pi$$

   $$x = \frac{2\pi}{3}$$

   The period is $\dfrac{2\pi}{3}$.

3. Determine how many cycles of the function can be shown over the domain $[0, 2\pi]$.

   Divide the domain by the period to find the number of cycles.

   $$2\pi \div \frac{2\pi}{3} = \cancel{2\pi} \cdot \frac{3}{\cancel{2\pi}} = 3$$

   There are 3 cycles over the domain $[0, 2\pi]$.

4. Determine the values of x at which the maximum and minimum occur over the domain of the cycle given by $\left[0, \dfrac{2\pi}{3}\right]$.

The maximum function value of the parent sine function occurs at $x = \dfrac{\pi}{2}$, so set $3x = \dfrac{\pi}{2}$ and solve for x. The result is $x = \dfrac{\pi}{6}$.

The minimum function value of the parent sine function occurs at $x = \dfrac{3\pi}{2}$, so set $3x = \dfrac{3\pi}{2}$ and solve for x. The result is $x = \dfrac{\pi}{2}$.

For the domain of the cycle given by $\left[0, \dfrac{2\pi}{3}\right]$, the maximum occurs at $x = \dfrac{\pi}{6}$ and the minimum occurs at $x = \dfrac{\pi}{2}$.

5. Determine the coordinates of the points for the maximum and minimum values.

The values found in the previous step represent the x-values of the coordinates.

Maximum: $x = \dfrac{\pi}{6}$ Minimum: $x = \dfrac{\pi}{2}$

The values found in step 1 represent the y-values of the coordinates.

Maximum: $y = 3$ Minimum: $y = -1$

Write these values as coordinates to determine the points of the maximum and minimum values.

The maximum occurs at the point $\left(\dfrac{\pi}{6}, 3\right)$.

The minimum occurs at the point $\left(\dfrac{\pi}{2}, -1\right)$.

6. Use the period to determine the coordinates of the maximum and minimum points for the remaining two cycles in the restricted domain of the function.

Find the x-values of the coordinates by adding the period, $x = \dfrac{2\pi}{3}$, to the maximum and minimum function values found in step 4. The y-values will be the maximum and minimum function values found in step 1: 3 and −1, respectively.

The maximum function values occur at $x = \dfrac{\pi}{6}$ plus 1 period $\left(x = \dfrac{2\pi}{3} \right)$, and at $\dfrac{\pi}{6}$ plus 2 periods $\left(x = 2 \bullet \dfrac{2\pi}{3} = \dfrac{4\pi}{3} \right)$:

$$\dfrac{\pi}{6} + \dfrac{2\pi}{3} \text{ or } x = \dfrac{5\pi}{6}$$

$$\dfrac{\pi}{6} + \dfrac{4\pi}{3} \text{ or } x = \dfrac{9\pi}{6} = \dfrac{3\pi}{2}$$

The maximum y-value is 3, so the coordinates of these two maximum points are therefore $\left(\dfrac{5\pi}{6}, 3 \right)$ and $\left(\dfrac{3\pi}{2}, 3 \right)$.

Likewise, the minimum function values occur at $x = \dfrac{\pi}{2}$ plus 1 period, and at $\dfrac{\pi}{2}$ plus 2 periods:

$$\dfrac{\pi}{2} + \dfrac{2\pi}{3} \text{ or } x = \dfrac{7\pi}{6}$$

$$\dfrac{\pi}{2} + \dfrac{4\pi}{3} \text{ or } x = \dfrac{11\pi}{6}$$

The minimum y-value is −1; the coordinates of these two minimum points are therefore $\left(\dfrac{7\pi}{6}, -1 \right)$ and $\left(\dfrac{11\pi}{6}, -1 \right)$.

7. Determine the *y*-intercept of the function.

The *y*-intercept is also the equation of the midline of the function.

Substitute $x = 0$ into the function, $a(x) = 1 + 2 \sin 3x$, and then solve.

$a(0) = 1 + 2 \sin 3 \cdot (0) = 1 + 2 \cdot 0 = 1$

The *y*-intercept of the function is at $(0, 1)$.

8. Use the coordinates found in steps 5–7 to sketch the graph of the function.

Sketch the graph of $a(x) = 1 + 2 \sin 3x$ by plotting the points on a coordinate plane and connecting them with a smooth curve, as shown.
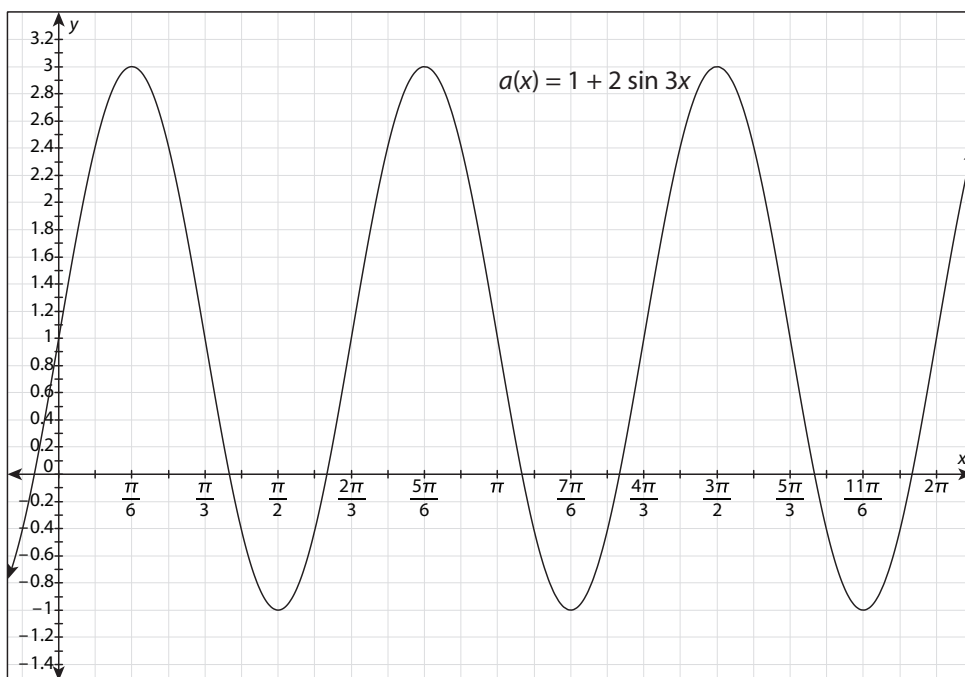
9. Confirm your sketch using a graphing calculator.

Enter the given function, $a(x) = 1 + 2 \sin 3x$, into your graphing calculator. Adjust the viewing window values to include the function's domain and amplitude, with a suitable x-axis scale. Make sure the mode is set correctly for the given problem (i.e., degrees or radians).

For this function:

- Set the mode to radians.

- The domain is $[0, 2\pi]$, so set the x-axis endpoints to 0 and 2π.

- The amplitude of the function is 2 but the midline is $y = 1$, so add 1 to –2 and 2 to determine the y-axis endpoints: –1 and 3.

- The first function maximum occurs at $x = \dfrac{\pi}{6}$; let $\dfrac{\pi}{6}$ be the x-axis scale.

The resulting graph should confirm the accuracy of the sketch.

Example 5

Determine the coordinates of the points at which the first maximum and minimum function values occur for the function $c(x) = \sin\left(x + \dfrac{\pi}{3}\right)$ for values of $x > 0$.

1. Determine the period of the function.

The variable part of the function's argument has no coefficients, so the function has a period of 2π. The presence of the constant $\dfrac{\pi}{3}$ shifts the graph horizontally, but does not affect the amplitude or period of the function.

2. Determine the value of x at which the maximum function value occurs.

The value of x at which the parent function reaches a maximum is $\dfrac{\pi}{2}$. However, the graph of this function is progressing rapidly toward reaching a maximum function value by the amount $\dfrac{\pi}{3}$, which is added to x. Therefore, this function will reach its maximum point at an x-value that is given by the difference $\dfrac{\pi}{2} - \dfrac{\pi}{3}$, or at $x = \dfrac{\pi}{6}$.

3. Determine the value of x at which the minimum function value occurs.

The period of the function is the same as the parent function for sine, $f(x) = \sin x$, so the minimum occurs π units "past" the maximum at $x = \dfrac{\pi}{6}$. Add π to find the minimum:

$$x = \dfrac{\pi}{6} + \pi = \dfrac{7\pi}{6}$$

4. Determine the maximum and minimum function values.

The amplitude of the function is 1, so the maximum function value is 1 and the minimum function value is –1.

5. Write the coordinates of the points for the maximum and minimum function values.

The point with the maximum function value for $x > 0$ is $\left(\dfrac{\pi}{6}, 1 \right)$.

The point with the minimum function value for $x > 0$ is $\left(\dfrac{7\pi}{6}, -1 \right)$.

6. Verify the coordinates using a graphing calculator.

   Enter the given function, $c(x) = \sin\left(x + \dfrac{\pi}{3}\right)$, into your graphing calculator. Adjust the viewing window values to include the function's domain and amplitude, with a suitable x-axis scale. Make sure the mode is set correctly for the given problem (i.e., degrees or radians). For this function:

   - Set the mode to radians.

   - Set the x-axis endpoints to 0 and 2π.

   - The amplitude of the function is 1, so set the y-axis endpoints to at least -1 and 1.

   - The first function maximum occurs at $x = \dfrac{\pi}{6}$; let $\dfrac{\pi}{6}$ be the x-axis scale.

   Your result should resemble the following graph of $c(x) = \sin\left(x + \dfrac{\pi}{3}\right)$ for $x > 0$. Use your calculator's trace feature to confirm that the maximum is at $\left(\dfrac{\pi}{6}, 1\right)$ and the minimum is at $\left(\dfrac{7\pi}{6}, -1\right)$.

$$c(x) = \sin\left(x + \dfrac{\pi}{3}\right)$$

UNIT 4A • MATHEMATICAL MODELING OF INVERSE, LOGARITHMIC, AND TRIGONOMETRIC FUNCTIONS
Lesson 3: Modeling Trigonometric Functions

Practice 4A.3.1: Graphing the Sine Function

For problems 1–4, refer to the provided graph to complete each problem.
(*Note*: Some graphs show only part of a complete cycle. The *x*-axis of each graph is expressed in radians.)

1. Which function has the greater amplitude, *f(x)* or *g(x)*?

2. Which function has the greater period, *f(x)* or *g(x)*?

PRACTICE
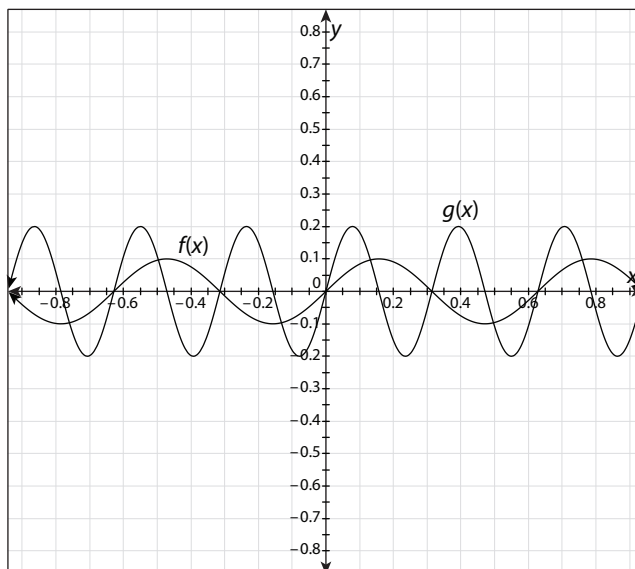
UNIT 4A • MATHEMATICAL MODELING OF INVERSE, LOGARITHMIC, AND TRIGONOMETRIC FUNCTIONS
Lesson 3: Modeling Trigonometric Functions

3. Determine the amount by which the functions $f(x)$ and $g(x)$ are out of phase.

4. Write the simplest form of the sine function shown.

UNIT 4A • MATHEMATICAL MODELING OF INVERSE, LOGARITHMIC, AND TRIGONOMETRIC FUNCTIONS
Lesson 3: Modeling Trigonometric Functions

For problems 5–7, use the given information to find the requested values and coordinates.

5. At what value of $x > 0$ will the first minimum occur for the function $f(x) = 3 \sin 4x$? Determine the coordinates of the point for this value of x.

6. At what value of $x > 0$ will the first zero occur for the function $g(x) = 5 \sin 0.2x$? Determine the coordinates of the point for this value of x.

7. At what value of $x > 0$ will the first maximum occur for $h(x) = \sin (0.5x + 30°)$? Determine the coordinates of the point for this value of x.

Use your knowledge of sine functions to complete problems 8–10.

8. The frequency of a sound is 250 cycles per second. If the sound intensity can be modeled by the sine function $S(t) = 10 \sin 250t$, what is the period of the sound wave?

9. The current in an alternating current circuit can be modeled by the sine function $I(t) = 5 \sin (120\pi t)$. How often does the current reach a peak positive *or* negative value?

10. The original Richter scale for detecting earthquake magnitude was based on calculations involving compressional P-wave amplitudes with a period of about 4 seconds. A newer scale uses calculations based on Rayleigh surface waves that have a different amplitude and a period of about 20 seconds. If both wave intensities can be represented by sine functions of the form $P(t) = A_P \sin c_1 t$ and $R(t) = A_R \sin c_2 t$, how would the arguments of the sine terms be written as a multiple of π? (*Hint*: What is the period of the parent sine function?)

Lesson 4A.3.2: Graphing the Cosine Function

Introduction

Recall that **cosine** is a trigonometric function of an acute angle in a right triangle. It is the ratio of the length of the side adjacent to the length of the hypotenuse; the cosine of $\theta = \cos \theta = \dfrac{\text{length of adjacent side}}{\text{length of hypotenuse}}$. Graphing cosine functions is similar to graphing sine functions. In this lesson, you will explore the graphs of cosine functions.

Key Concepts

- A **cosine function** is a trigonometric function of the form $f(x) = a \cos bx$, in which a and b are constants and x is a variable defined in radians over the domain $(-\infty, \infty)$.

- The following graph shows the parent function $f(x) = \cos x$.

Cosine Parent Function, $f(x) = \cos x$
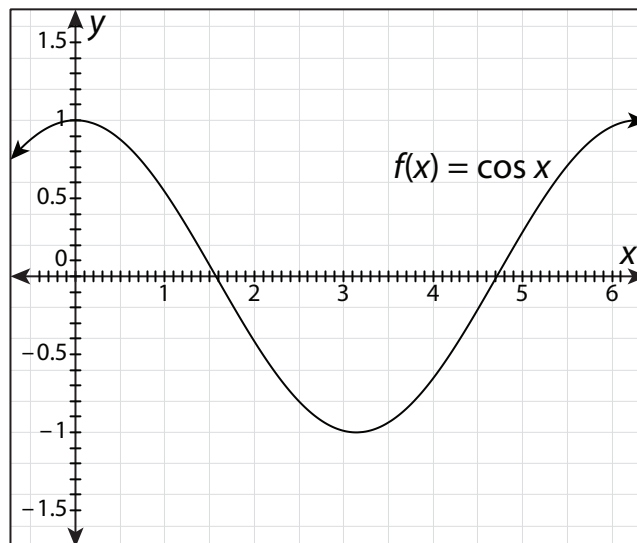
- The general form of a cosine function is $f(x) = a \cos bx$. The extended form is $g(x) = a \cos [b(x - c)] + d$.

- The parts of the equation of a cosine function and the relationships between those parts are nearly identical as those for a sine function:

Comparison of Sine and Cosine Functions

Form	Equation	Amplitude	Period	Argument
Parent	$f(x) = \sin x$	1	2π	x
	$g(x) = \cos x$	1	2π	x
General	$f(x) = a \sin bx$	$\lvert a \rvert$	$\dfrac{2\pi}{b}$	bx
	$g(x) = a \cos bx$	$\lvert a \rvert$	$\dfrac{2\pi}{b}$	bx
Extended	$f(x) = a \sin [b(x - c)] + d$	$\lvert a \rvert$	$\dfrac{2\pi}{b}$	$[b(x - c)]$
	$g(x) = a \cos [b(x - c)] + d$	$\lvert a \rvert$	$\dfrac{2\pi}{b}$	$[b(x - c)]$

- Just as with sines, the domain of a cosine function is generally expressed in radians, but can be converted to degrees to suit a given situation: 2π radians = 360°.

- Furthermore, the methods for determining the phase shift and period of a cosine function are also the same:

  - Phase shift: Set the argument equal to 0 and solve for x.

  - Period: Set the argument equal to 2π and solve for x.

Graphing Cosine Functions

- As with sine functions, in order to sketch the graph of a cosine function, it is necessary to know the amplitude and period of the parent function, $f(x) = \cos x$.

- Recall that the amplitude of a function is defined as the vertical distance from the midline to the highest point of the graph, and the period is the horizontal distance from the beginning of one complete cycle of the graph to the point at which its behavior or shape repeats.

- As with sine functions, any change in the amplitude in the equation of a cosine function has a *direct* effect on the amplitude of the graph, and any change in the period of the equation has an *inverse* effect on the period of the graph:

  - In the general form $g(x) = a \cos bx$, if the amplitude $|a|$ increases, the amplitude of the graph increases. If the value of b increases, the period of the graph decreases.

  - In the extended form $h(x) = a \cos [b(x - c)] + d$, if the amplitude $|a|$ increases, the value of the term $a \cos [b(x - c)]$ also increases if the other quantities stay the same. If the value of b increases, the period of the graph decreases.

  - Furthermore, in the expanded form, $h(x) = a \cos [b(x - c)] + d$, the constant d shifts the graph vertically, whereas the constant c shifts the graph horizontally.

- Recall that on a graph of a cosine or sine function of the form $f(x) = \sin x + d$ or $g(x) = \cos x + d$ the midline is a horizontal line of the form $y = d$ that bisects the vertical distance between the minimum and maximum function values.

- The values of the cosine function at several critical values of x can be useful in sketching the graph of the cosine function. A few of these are listed here:

$$\cos 30° = \cos\left(\frac{\pi}{6} \text{ radian}\right) = \frac{\sqrt{3}}{2} \approx 0.87$$

$$\cos 45° = \cos\left(\frac{\pi}{4} \text{ radian}\right) = \frac{\sqrt{2}}{2} \approx 0.71$$

$$\cos 60° = \cos\left(\frac{\pi}{3} \text{ radians}\right) = \frac{1}{2} = 0.5$$

- Recall that the unit circle can be used to visualize how cosine values are derived.
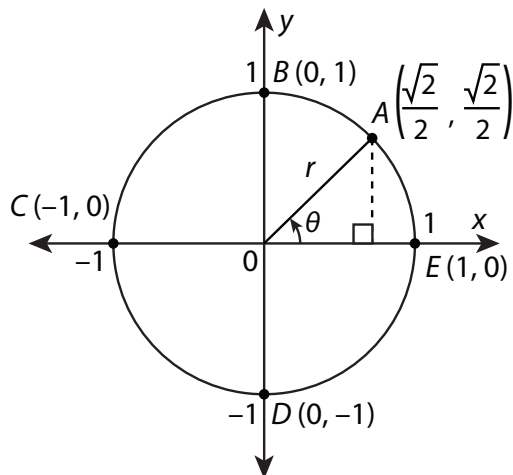
- Notice the right triangle formed by the radius:
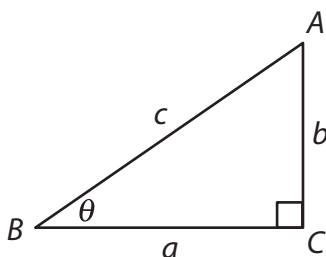
- This triangle can be used to determine the formula for the cosine of angle B, which is defined as the ratio of the length of side a (the side that is adjacent to angle B) to the length of the hypotenuse, side c:

$$\cos B = \frac{a}{c} \text{ or } a = c \cos B$$

- Recall that on the unit circle diagram, the horizontal leg of the right triangle is a and the vertical leg is b. Therefore, the coordinates of a point on the unit circle are (a, b) or $(c \cos \theta, c \sin \theta)$. For a unit circle, $c = 1$, so the coordinates of the point are $(\cos \theta, \sin \theta)$.

- For an angle measuring $30°$ $\left(\frac{\pi}{6} \text{ radian} \right)$ in the first quadrant of the unit circle, the coordinates of the point that lies on the unit circle are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2} \right)$, which is the same as $(\cos 30°, \sin 30°)$.

- Refer to the diagram of coordinates for values of $m\angle\theta$ in the first sub-lesson for the coordinates of the points formed by angles of $30°$, $45°$, and $60°$ in the four quadrants of the coordinate plane.

Guided Practice 4A.3.2

Example 1

Sketch the graph of $f(x) = 3 \cos x$ over the restricted domain $[-2\pi, 2\pi]$.

1. Identify the amplitude and period of the function.

   This will determine the x- and y-axis scales.

   The coefficient of the cosine term is 3, which is the amplitude. Therefore, the y-axis scale should run from at least $y = -3$ to $y = 3$. The coefficient of the cosine argument is 1, so $x = 2\pi$, which means the period is 2π. If the graph shows the restricted domain of $[-2\pi, 2\pi]$, it will include two complete cycles of the function.

2. Identify any other coefficients or terms that would affect the shape of the graph.

   This is necessary to determine if the graph is translated horizontally or vertically, or if other mathematical terms affect the graph's shape.

   The equation of the function, $f(x) = 3 \cos x$, has no other coefficients or terms that affect the shape or placement of the function's graph.

3. Draw and label the axes for your graph based on steps 1 and 2. The *x*-axis scale will range from -2π to 2π, and the *y*-axis scale will range from at least -3 to at least 3. Each axis should be divided into sufficient intervals to allow for enough points to be plotted to show the graph. Label the *x*-axis in increments of $\pm\dfrac{\pi}{2}$ and the *y*-axis in increments of $\dfrac{1}{2}$.
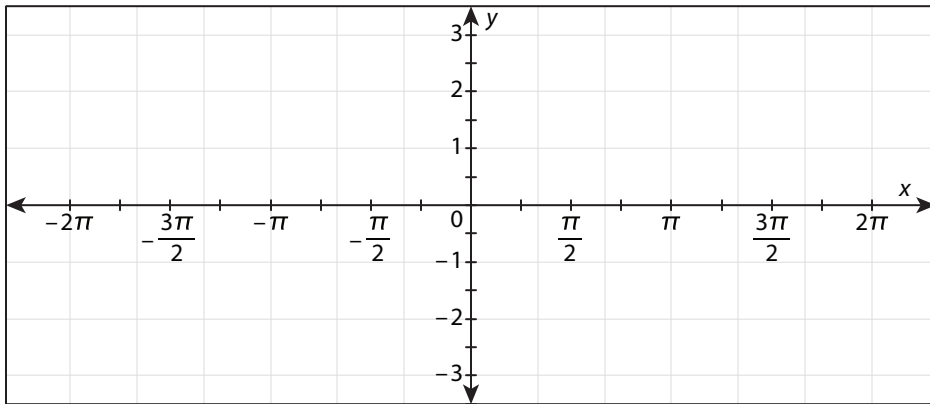
4. Determine the values of the restricted domain for which the function value(s) equal 0. List the points corresponding to these zeros, and plot them on the graph.

This will establish some points on the *x*-axis.

Since the period of the function $f(x) = 3 \cos x$ is 2π, the function is equal to 0 at $x = \pm\dfrac{\pi}{2}$ and $\pm\dfrac{3\pi}{2}$. The corresponding points are $\left(-\dfrac{3\pi}{2}, 0\right)$, $\left(-\dfrac{\pi}{2}, 0\right)$, $\left(\dfrac{\pi}{2}, 0\right)$, and $\left(\dfrac{3\pi}{2}, 0\right)$. Plot the four points as shown.
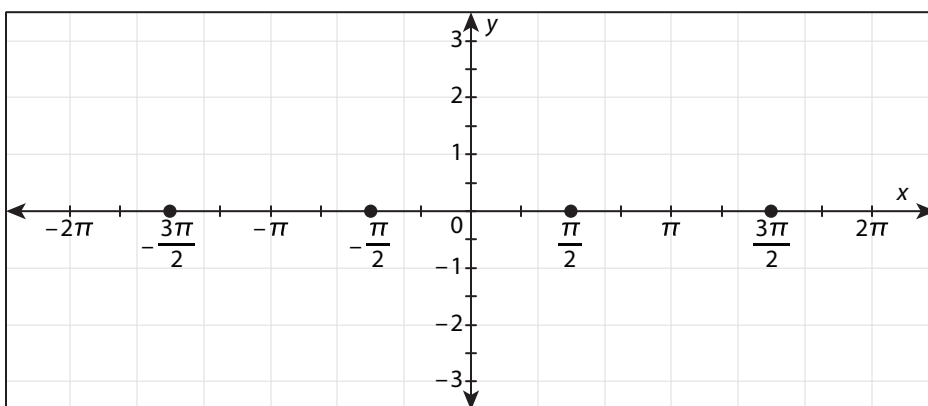
5. Determine what values of the restricted domain are the maximum and minimum of the function value(s). List the points corresponding to these extremes, and plot them on the graph.

The cosine function values vary between the two extremes defined by the amplitude. In this case, the amplitude is 3, so the function values for the maximum and minimum values of $f(x) = 3 \cos x$ will vary between -3 and 3. These values occur at $x = 0$, $x = \pm\pi$, and $x = \pm 2\pi$.

The points for the maximum and minimum values of $f(x) = 3 \cos x$ are $(-2\pi, 3)$, $(-\pi, -3)$, $(0, 3)$, $(\pi, -3)$, and $(2\pi, 3)$. Add these points to the graph.
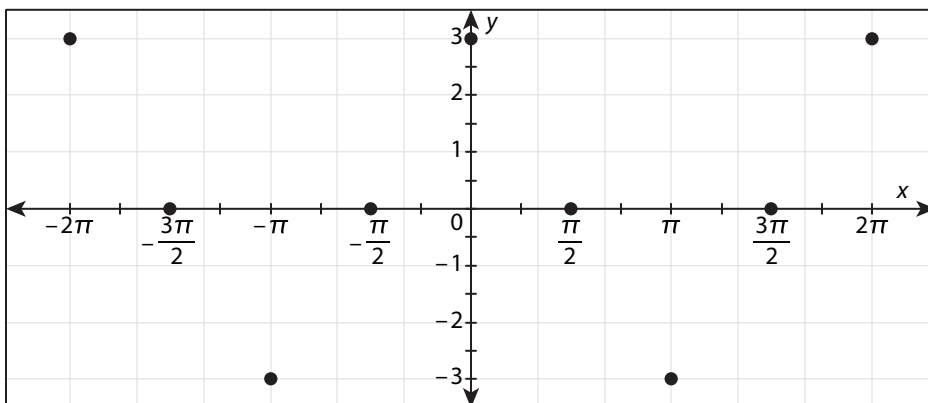
6. Plot points for the cosines of $\dfrac{\pi}{6}$, $\dfrac{\pi}{4}$, and $\dfrac{\pi}{3}$ radians.

Use a calculator to determine the cosines of the given values of $m\angle\theta$.

These points will fall between the function maximum at (0, 3) and the zero at $\left(\dfrac{\pi}{2}, 0\right)$. Moreover, they will suggest the shape of a quarter-period "piece" of the cosine curve, which can be reflected across the y-axis from the zero at $\left(-\dfrac{\pi}{2}, 0\right)$ to the x-axis to produce the cosine curve over the domain of $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

Using your calculator, we see that $3\cos\left(\dfrac{\pi}{6}\right) = \dfrac{3\sqrt{3}}{2} \approx 2.6$ and $\dfrac{\pi}{6} \approx 0.5$, so its ordered pair is (0.5, 2.6). The coefficient 3 is specific to this function; multiply by the function values $\dfrac{\pi}{4}$ and $\dfrac{\pi}{3}$, respectively, to find the other two function values and points:

$$3\cos\left(\dfrac{\pi}{4}\right) = \dfrac{3\sqrt{2}}{2} \approx 2.1 \text{ and } \dfrac{\pi}{4} \approx 0.8 \text{; (0.8, 2.1)}$$

$$3\cos\left(\dfrac{\pi}{3}\right) = 3 \bullet 0.5 = 1.5 \text{ and } \dfrac{\pi}{3} \approx 1.1 \text{; (1.1, 1.5)}$$

Plot these three points on the graph between the points (0, 3) and $\left(\dfrac{\pi}{2}, 0\right)$.
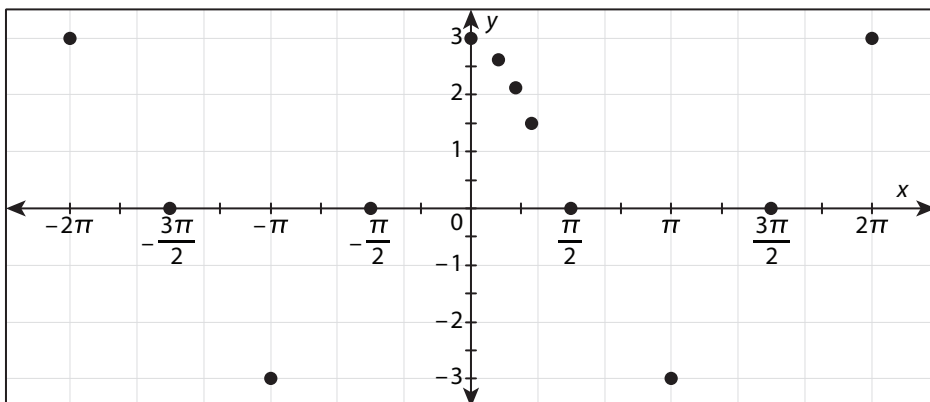
7. Plot additional points.

Plot the points for the *x*-values $\dfrac{2\pi}{3}$, $\dfrac{3\pi}{4}$, and $\dfrac{5\pi}{6}$, which are between the points $\left(\dfrac{\pi}{2},0\right)$ and $\left(\dfrac{3\pi}{2},0\right)$.

This will show how the function values are reflected across the line $x=\dfrac{\pi}{2}$ and then reflected across the midline ($y = 0$) for domain values that differ by $\dfrac{\pi}{2}$.

For this function, $f\left(\dfrac{2\pi}{3}\right)=2\cos\left(\dfrac{2\pi}{3}\right)=3\bullet-0.5=-1.5$, so the point is given by $\left(\dfrac{2\pi}{3},-1.5\right)$. The other function values can be determined by using a calculator. They produce the following two points:

$\left(\dfrac{3\pi}{4},-2.1\right)$ and $\left(\dfrac{5\pi}{6},-2.6\right)$. Plot these three points.
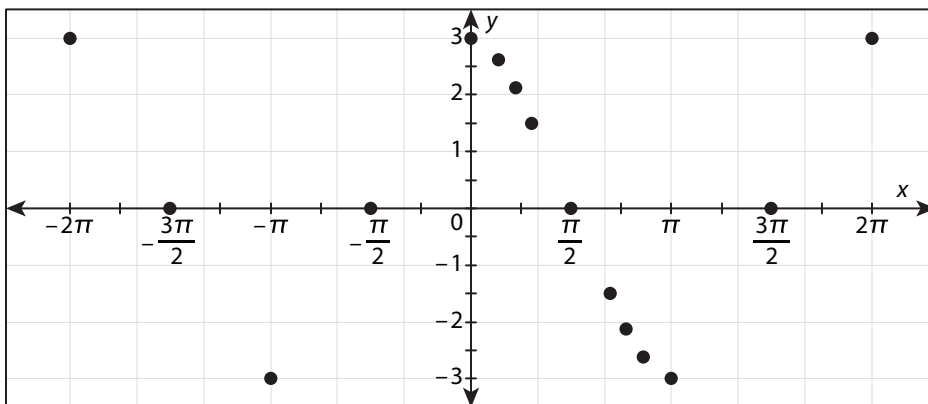
8. Compare the ordered pairs for the graphed points on either side of the *x*-axis.

The points occur where $x = \dfrac{\pi}{6}$ and $\dfrac{5\pi}{6}$, $x = \dfrac{\pi}{4}$ and $\dfrac{3\pi}{4}$, and $x = \dfrac{\pi}{3}$ and $\dfrac{2\pi}{3}$.

The function values for these point pairs are opposites, which confirms that the points are reflected twice across the line $x = \dfrac{\pi}{2}$ and across the function midline (at $y = 0$, the *x*-axis) over the restricted interval $[0, \pi]$.

9. Determine the function values for each value of *x* over the restricted domain of $[\pi, 2\pi]$. Plot points on the graph for this part of the domain.

This will show that the function values for values of *x* over the restricted domain $[\pi, 2\pi]$ are the opposite of those over the restricted domain $[0, \pi]$.

Use a graphing calculator to find the function values for the *x*-values over the domain $[\pi, 2\pi]$.

The *x*-values are $\dfrac{7\pi}{6}$, $\dfrac{5\pi}{4}$, $\dfrac{4\pi}{3}$, $\dfrac{5\pi}{3}$, $\dfrac{7\pi}{4}$, and $\dfrac{11\pi}{6}$.

The calculator will give function values that result in the approximate

points $\left(\dfrac{7\pi}{6}, -2.6\right)$, $\left(\dfrac{5\pi}{4}, -2.1\right)$, $\left(\dfrac{4\pi}{3}, -1.5\right)$, $\left(\dfrac{5\pi}{3}, 1.5\right)$,

$\left(\dfrac{7\pi}{4}, 2.1\right)$, and $\left(\dfrac{11\pi}{6}, 2.6\right)$. Plot these points on the graph.
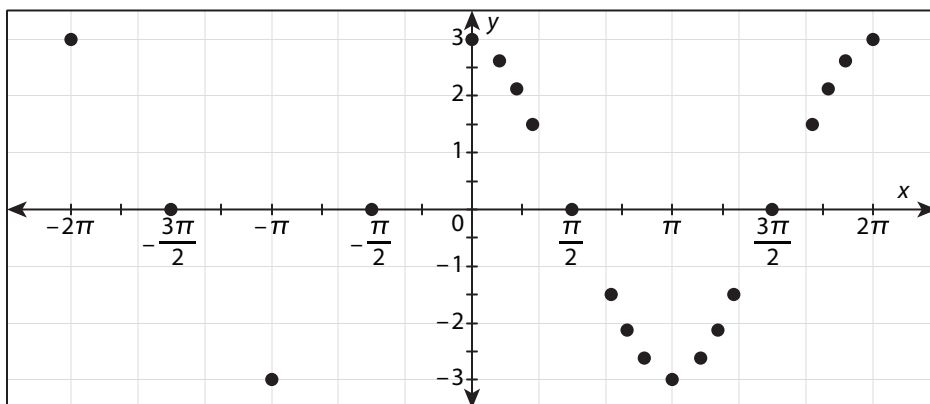
10. Compare the points over the two restricted domains [0, π] and [π, 2π].

The points occur where $y \approx 1.5$ and -1.5, $y \approx 2.1$ and -2.1, and $y = 2.6$ and -2.6.

This confirms that the function values of the points over the restricted domain [0, π] and points over the restricted domain [π, 2π] are opposites for specific domain values.

11. Predict what the shape of the function graph will be over the remainder of the domain, $(-2\pi, 0)$.

Note that the function values change signs every half period of domain values graphed. Therefore, the shape of the function graph over the domain $(-2\pi, 0)$ should be two reflections of the function graph for the domain [0, 2π] across the y-axis ($x = 0$) and across the midline $y = 0$ (the x-axis).

12. Plot additional points on the graph to confirm your prediction. Then, draw a curve connecting the points across the domain $[-2\pi, 2\pi]$.

At a minimum, points over the domain $(-\pi, 0)$ should be plotted for x-values that are the opposite of those used for the domain $(\pi, 0)$.
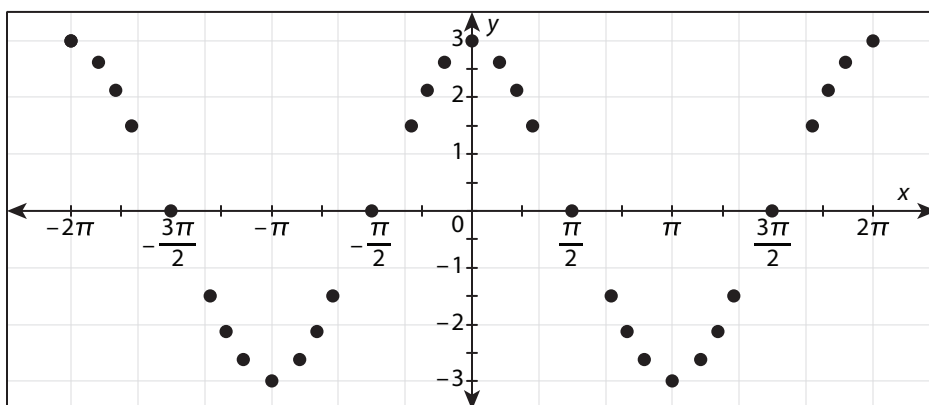
13. Verify the resulting graph using a graphing calculator.

Enter the given function, $f(x) = 3 \cos x$, into your graphing calculator. Regardless of whether you are using the TI-83/84, the TI-Nspire, or a similar calculator, remember to adjust the viewing window values so that the x-axis endpoints include the function's domain and the y-axis endpoints include the amplitude. Use the known x- and y-values to determine a value for the x-axis scale that will provide a comprehensive view of the graphed function. Make sure the mode is set correctly for the given problem (i.e., degrees or radians).

For this function:

- Since the x-values are in increments of π, set the mode to radians.

- The domain is $[-2\pi, 2\pi]$, so set the x-axis endpoints to -2π and 2π.

- The amplitude is 3, so set the y-axis endpoints to at least -3 and 3.

- For the x-axis scale, we know that the smallest interval between x-values is $\dfrac{\pi}{6}$; let this be the x-axis scale.

The resulting graph on the calculator should confirm the accuracy of the sketched function graph.
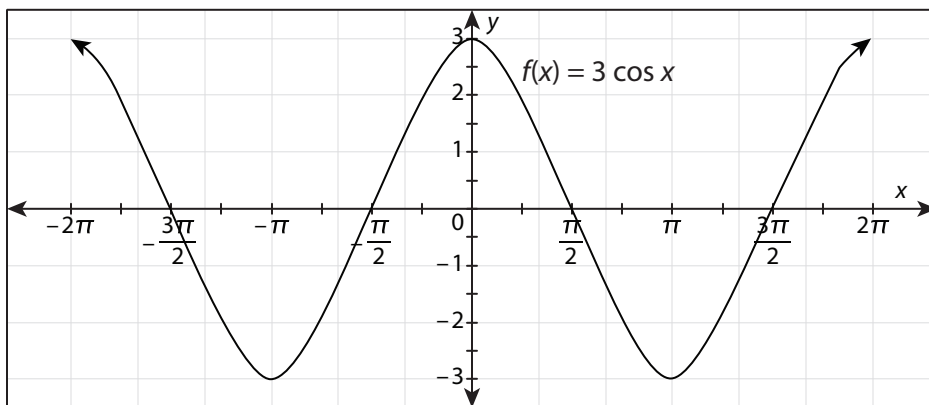
Example 2

How many complete cycles of the cosine function $g(x) = \cos 4x$ are found in the restricted domain $[-270°, 270°]$?

1. Identify the period of $g(x)$.

   This will be needed to calculate the number of complete cycles of $g(x)$ that exist in the restricted domain.

   Set the argument of the function equal to the period of the parent cosine function, $\cos x$, and solve for x: $4x = 2\pi$, so $x = \dfrac{\pi}{2}$.

   The period of the function $g(x) = \cos 4x$ is $\dfrac{\pi}{2}$.

2. Convert the period to degrees.

   Since the domain is given in degrees, the period must also be in degrees.

   $1 \text{ radian} = \dfrac{180°}{\pi}$, so $\dfrac{\pi}{2}$ radians is $\dfrac{\pi}{2} \cdot \dfrac{180°}{\pi} = \dfrac{180\pi}{2\pi} = 90°$.

   The period of $g(x)$, converted to degrees, is $90°$.

3. Determine the number of complete cycles $g(x)$ makes over the domain $[-270°, 270°]$.

   In order to determine the number of completed cycles, calculate the total number of degrees within the domain of the function, and then divide this number by the period.

   The domain covers $270° - (-270°)$ or $540°$. Therefore, the number of complete cycles of $g(x)$ is $540°$ divided by the period, $90°$:
   $540 \div 90 = 6$ complete cycles.

4. Verify the result using a graphing calculator.

Enter the given function, $g(x) = \cos 4x$, into your graphing calculator. Regardless of whether you are using the TI-83/84, the TI-Nspire, or a similar calculator, remember to adjust the viewing window values so that the *x*-axis endpoints include the function's domain and the *y*-axis endpoints include the amplitude. Use the known *x*- and *y*-values to determine a value for the *x*-axis scale that will provide a comprehensive view of the graphed function. Make sure the mode is set correctly for the given problem (i.e., degrees or radians).

For this function:

- Set the mode to degrees.

- The domain is [−270°, 270°], so set the *x*-axis endpoints to −270 and 270.

- The amplitude is 1, so set the *y*-axis endpoints to at least −1 and 1.

- For the *x*-axis scale, choose a scale value that's a factor of the endpoints. Let the *x*-axis scale be 30 (a factor of −270 and 270).

Graph the function and count the number of complete cycles.

The resulting graph confirms our calculation that $g(x)$ has 6 complete cycles over the restricted domain [−270°, 270°].
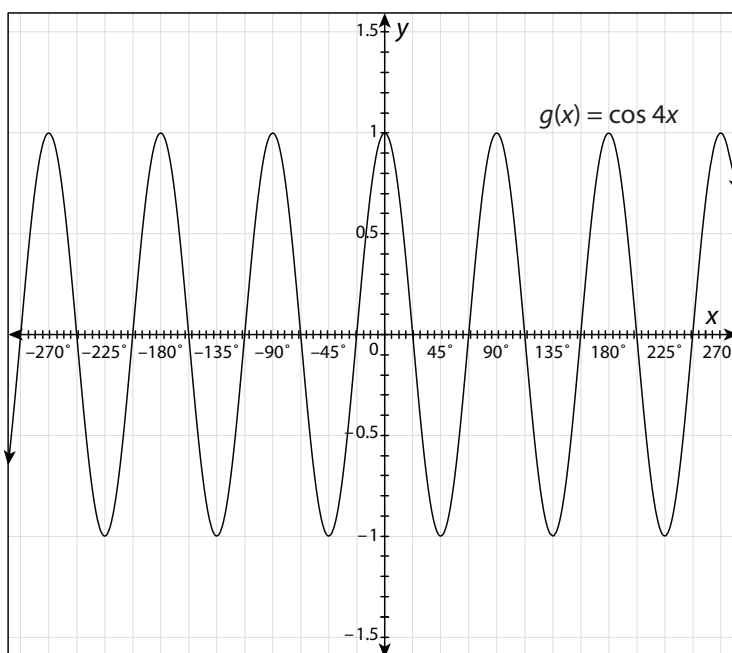
Example 3

Determine the coordinates of the point(s) that represent a maximum positive function value of the function $h(x) = 2 \cos 3x$ over the restricted domain $\left[-\dfrac{3\pi}{2}, \dfrac{2\pi}{3} \right]$.

1. Identify the amplitude and period of the function.

   This information will be needed to determine the location and the magnitude of the maximum function value.

   The amplitude is 2 because 2 is the coefficient of the cosine term. The period can be found by setting the argument of the function equal to 2π and solving for x: $3x = 2\pi$, so $x = \dfrac{2\pi}{3}$.

2. Determine where the function will have maximum values over the domains of its period on either side of the origin.

   This will indicate the value(s) of x at which the function has a maximum.

   At the origin, $x = 0$, so the domains of the function's period on either side of the origin are $\left[-\dfrac{2\pi}{3}, 0 \right]$ and $\left[0, \dfrac{2\pi}{3} \right]$.

   The function $h(x) = 2 \cos 3x$ has a maximum value of 2 at $x = \dfrac{2\pi}{3}$, $x = 0$, and $x = -\dfrac{2\pi}{3}$.

3. Compare the points at which $h(x)$ will have a maximum to the restricted domain of the problem.

The restricted domain of $h(x) = 2 \cos 3x$ is given as $\left[-\dfrac{3\pi}{2}, \dfrac{2\pi}{3} \right]$.

One maximum value occurs at $x = \dfrac{2\pi}{3}$, which is also the upper bound of the restricted domain. However, the lower bound of the restricted domain is $x = -\dfrac{3\pi}{2}$, so the maximum at $x = -\dfrac{2\pi}{3}$ may not be the only maximum greater than the lower bound. Subtract the value of one cycle from $x = -\dfrac{2\pi}{3}$ to find another maximum:

$$x = -\frac{2\pi}{3} - \left(\frac{2\pi}{3} \right) = -\frac{4\pi}{3}$$

The maximum at $x = -\dfrac{4\pi}{3}$ is greater than the lower bound.

Determine whether there is another maximum by subtracting the value of one cycle from $x = -\dfrac{4\pi}{3}$:

$$x = -\frac{4\pi}{3} - \left(\frac{2\pi}{3} \right) = -2\pi$$

There is another maximum at $x = -2\pi$. This value is less than the lower bound, so there are no other maximum values over the restricted domain $\left[-\dfrac{3\pi}{2}, \dfrac{2\pi}{3} \right]$.

4. Determine how many maximum values $h(x)$ has over the restricted domain $\left[-\dfrac{3\pi}{2}, \dfrac{2\pi}{3} \right]$.

Notice that the restricted domain includes the lower and upper bound.

There are 4 maximum points over the restricted domain, at $x = \dfrac{2\pi}{3}$, $x = 0$, $x = -\dfrac{2\pi}{3}$, and $x = -\dfrac{4\pi}{3}$.

5. Check the result using a graphing calculator.

Enter the given function, $h(x) = 2 \cos 3x$, into your graphing calculator. Adjust the viewing window values to include the function's domain and amplitude, with a suitable x-axis scale. Make sure the mode is set correctly for the given problem (i.e., degrees or radians).

For this function:

- Set the mode to radians.

- The domain is $\left[-\dfrac{3\pi}{2}, \dfrac{2\pi}{3} \right]$, so set the x-axis endpoints to $-\dfrac{3\pi}{2}$ and $\dfrac{2\pi}{3}$.

- The amplitude is 2, so set the y-axis endpoints to at least -2 and 2.

- For the x-axis scale, choose a scale value that's a factor of the endpoints. Let the x-axis scale be $\dfrac{\pi}{6}$ (a factor of $-\dfrac{3\pi}{2}$ and $\dfrac{2\pi}{3}$).

Graph the function and count the number of maximums.

(*continued*)

The resulting graph confirms our calculation that $h(x) = 2 \cos 3x$ has four maximums over the restricted domain $\left[-\dfrac{3\pi}{2}, \dfrac{2\pi}{3} \right]$.

$h(x) = 2 \cos 3x$
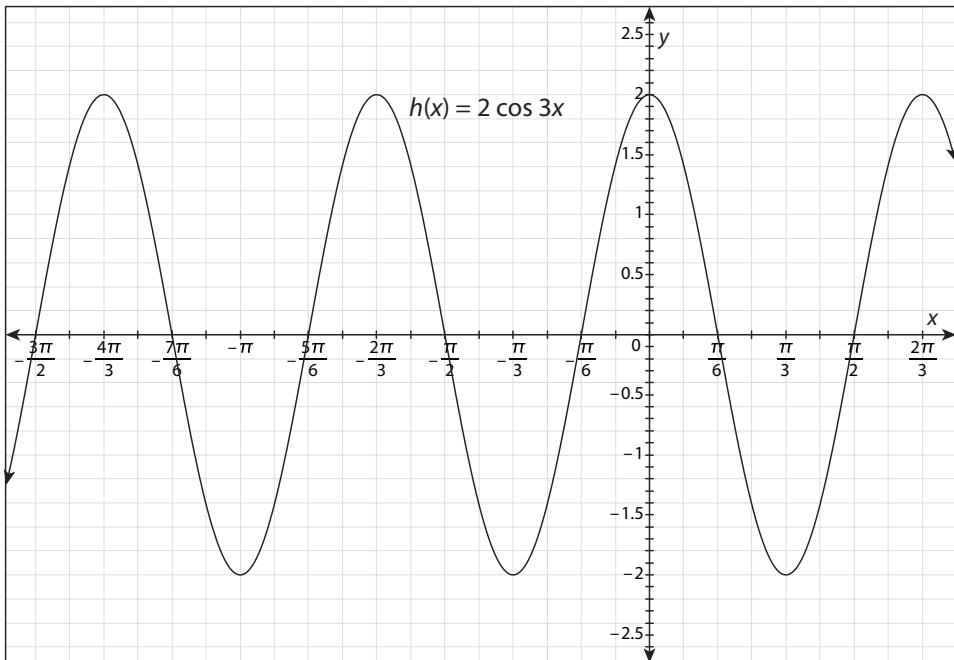
Example 4

Sketch the graph of the function $a(x) = -2 + 4 \cos 2x$ over the restricted domain $[-\pi, \pi]$.

1. Identify the amplitude of the function.

   The addition of the constant to the cosine term will shift the graph and the relative locations of the maximum and minimum values of the function.

   The amplitude is 4, but the -2 added to the cosine term will produce a maximum function value of 2 and a minimum function value of -6. The constant defines the midline, which is $y = -2$.

   Therefore, the maximum function value occurs at $y = 2$ and the minimum function value occurs at $y = -6$.

2. Determine the period of the function.

   Set the argument of the function equal to 2π to find the period.

   $2x = 2\pi$

   $x = \pi$

   The period is equal to π.

3. Determine how many cycles of the function can be shown over the domain $[-\pi, \pi]$.

   First, determine the domain. Subtract the lower bound from the upper bound:

   $\pi - (-\pi) = 2\pi$

   Next, divide the domain by the period, π, to find the number of cycles.

   $$\frac{2\pi}{\pi} = 2$$

   There are 2 cycles over the domain $[-\pi, \pi]$.

4. Determine the values of x at which the maximum and minimum occur over the domain of the cycle given by $[-\pi, \pi]$.

The maximum function values occur at $x = 0$ and $x = \pi$ over the restricted domain $[0, \pi]$, so the minimum function value occurs at half of the distance from $x = 0$ to $x = \pi$, or at $x = \dfrac{\pi}{2}$.

The maximum values over the restricted domain $[-\pi, 0]$ occur at $x = 0$ and $x = -\pi$, so the minimum function value occurs at half of the distance from $x = 0$ to $x = -\pi$, or at $x = -\dfrac{\pi}{2}$.

5. Determine the coordinates of the points for the maximum and minimum values determined in step 4.

The values found in the previous step represent the x-values of the coordinates.

Maximums: $x = -\pi$, 0, and π Minimums: $x = \dfrac{\pi}{2}$ and $-\dfrac{\pi}{2}$

The values found in step 1 represent the y-values of the coordinates.

Maximum: $y = 2$ Minimum: $y = -6$

Write these values as coordinates to determine the points of the maximum and minimum values.

The maximums occur at the points $(-\pi, 2)$, $(0, 2)$, and $(\pi, 2)$.

The minimums occur at the points $\left(-\dfrac{\pi}{2}, -6 \right)$ and $\left(\dfrac{\pi}{2}, -6 \right)$.

6. Use the results from step 5 to sketch the graph of $a(x) = -2 + 4 \cos 2x$ over the restricted domain $[-\pi, \pi]$.

Sketch the graph of $a(x) = -2 + 4 \cos 2x$ by plotting the points on a coordinate plane and connecting them with a smooth curve, as shown.
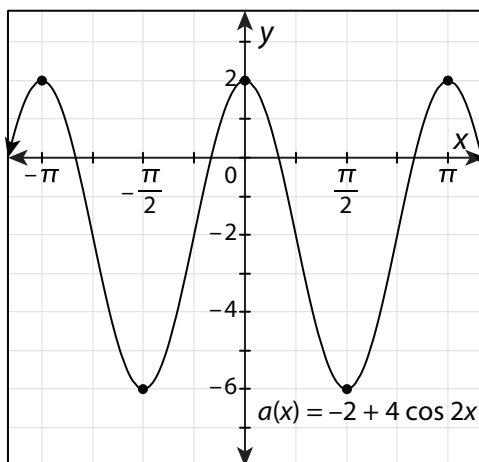
$a(x) = -2 + 4 \cos 2x$

7. Confirm your sketch using a graphing calculator.

Enter the given function, $a(x) = -2 + 4 \cos 2x$, into your graphing calculator. Adjust the viewing window values to include the function's domain and amplitude, with a suitable x-axis scale. Make sure the mode is set correctly for the given problem (i.e., degrees or radians).

For this function:

- Set the mode to radians.

- The domain is $[-\pi, \pi]$, so set the x-axis endpoints to $-\pi$ and π.

- The amplitude of the function is 4 but the midline is $y = -2$, so add -2 to -4 and 4 to determine the y-axis endpoints: -6 and 2.

- We know that the smallest interval between the x-values of the graphed coordinates is $\dfrac{\pi}{2}$; let this be the x-axis scale.

The resulting graph should confirm the accuracy of the sketch.

Example 5

Determine the coordinates of the points at which the first maximum and minimum function values occur for the function $c(x) = \cos\left(x - \dfrac{\pi}{4}\right)$ for values of $x > 0$.

1. Determine the period of the function.

   The variable part of the function's argument has no coefficients, so the function has a period of 2π. The presence of the constant $-\dfrac{\pi}{4}$ shifts the graph horizontally, but does not affect the amplitude or period of the function.

2. Determine the value of x at which the first maximum function value for $x > 0$ occurs.

   The first value of $x > 0$ at which the parent function reaches a maximum is 0. However, with this function, the graph *lags* reaching a maximum function value by the amount of $\dfrac{\pi}{4}$, which is subtracted from x. Therefore, this function will reach its first maximum point at an x-value that is given by the difference $\dfrac{\pi}{4} - \dfrac{\pi}{4} = 0$, so that $\cos\left(\dfrac{\pi}{4} - \dfrac{\pi}{4}\right) = \cos 0 = 1$; therefore, the first maximum is at $x = \dfrac{\pi}{4}$.

   Because of the horizontal shift of $-\dfrac{\pi}{4}$, the function does not reach its first maximum point until $x = \dfrac{\pi}{4}$.

3. Determine the value of x at which the first minimum function value for $x > 0$ occurs.

The period of the function is the same as the parent function for cosine, $f(x) = \cos x$, so the minimum occurs π units "past" the maximum at $x = \dfrac{\pi}{4}$. Add π to find the minimum:

$$x = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$$

4. Determine the maximum and minimum function values.

The amplitude of the function is 1, so the maximum function value is 1 and the minimum function value is –1.

5. Write the coordinates of the points for the maximum and minimum function values.

The point with the first maximum function value for $x > 0$ is $\left(\dfrac{\pi}{4}, 1 \right)$.

The point with the first minimum function value for $x > 0$ is $\left(\dfrac{5\pi}{4}, -1 \right)$.

6. Verify the coordinates using a graphing calculator.

Enter the given function, $c(x) = \cos\left(x - \dfrac{\pi}{4}\right)$, into your graphing calculator. Adjust the viewing window values to include the function's domain and amplitude, with a suitable x-axis scale. Make sure the mode is set correctly for the given problem (i.e., degrees or radians). For this function:

- Set the mode to radians.
- Set the x-axis endpoints to 0 and 2π.
- The amplitude of the function is 1, so set the y-axis endpoints to at least -1 and 1.
- The first function maximum occurs at $x = \dfrac{\pi}{4}$; let $\dfrac{\pi}{4}$ be the x-axis scale.

Your result should resemble the following graph of $c(x) = \cos\left(x - \dfrac{\pi}{4}\right)$ for $x > 0$. Use your calculator's trace feature to confirm that the first maximum is at $\left(\dfrac{\pi}{4}, 1\right)$ and the first minimum is at $\left(\dfrac{5\pi}{4}, -1\right)$.
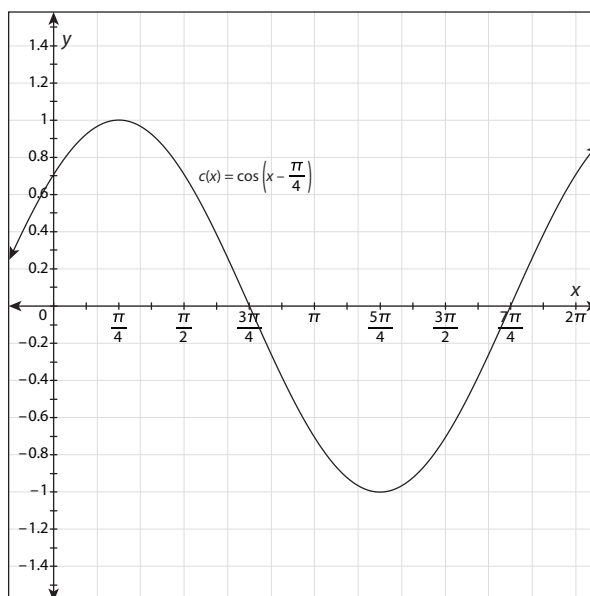
UNIT 4A • MATHEMATICAL MODELING OF INVERSE, LOGARITHMIC, AND TRIGONOMETRIC FUNCTIONS
Lesson 3: Modeling Trigonometric Functions

Practice 4A.3.2: Graphing the Cosine Function

For problems 1–4, refer to the provided graph to complete each problem.
(*Note*: Some graphs show only part of a complete cycle. The *x*-axis of each graph is expressed in radians.)

1. Which function has the greater amplitude, *f(x)* or *g(x)*?
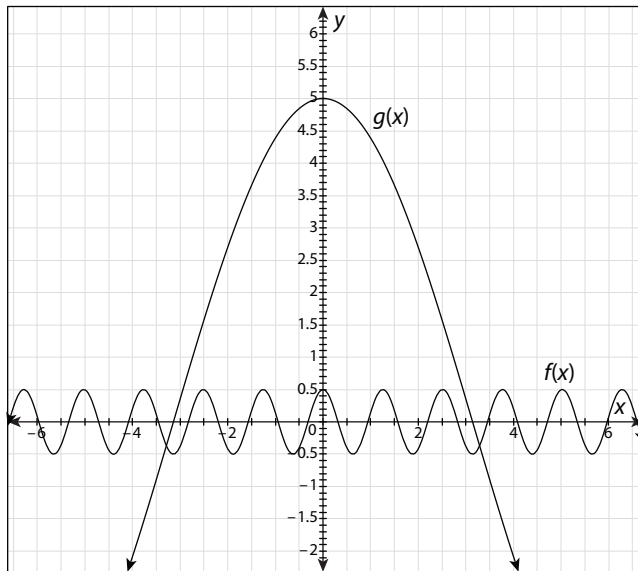
2. Which function has the greater period, *f(x)* or *g(x)*?
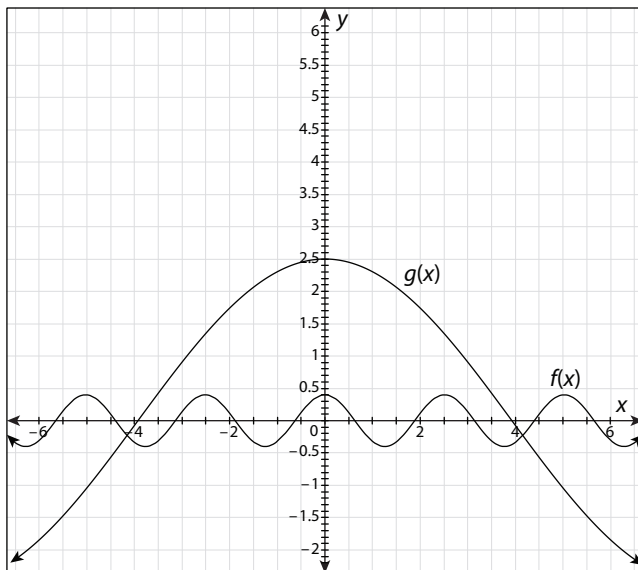
continued

UNIT 4A • MATHEMATICAL MODELING OF INVERSE, LOGARITHMIC, AND TRIGONOMETRIC FUNCTIONS
Lesson 3: Modeling Trigonometric Functions

3. Determine the amount by which the functions $f(x)$ and $g(x)$ are out of phase.
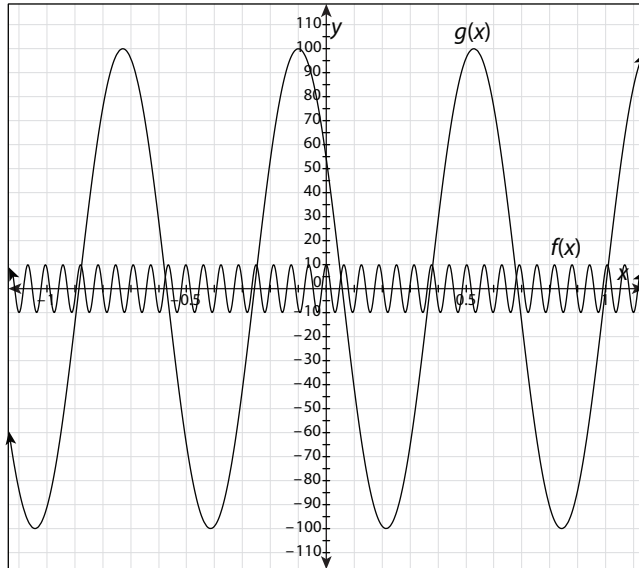
4. Write the simplest form of the cosine function shown.

UNIT 4A • MATHEMATICAL MODELING OF INVERSE, LOGARITHMIC, AND TRIGONOMETRIC FUNCTIONS
Lesson 3: Modeling Trigonometric Functions

For problems 5–7, use the given information to find the requested values and coordinates.

5. At what value of $x > 0$ will the first minimum occur for the function $f(x) = -2 \cos x$? Determine the coordinates of the point for this value of x.

6. At what value of $x > 0$ will the first zero occur for the function $g(x) = 4 \cos 6x$? Determine the coordinates of the point for this value of x.

7. At what value of $x > 0$ will the first maximum occur for the function $h(x) = \cos(60° - x)$? Determine the coordinates of the point for this value of x.

Use your knowledge of cosine functions to complete problems 8–10.

8. A dog whistle produces a high-pitched sound that a dog can hear but humans cannot. The intensity of the sound can be modeled by the function $I(t) = A \cos(6 \cdot 10^4 \cdot \pi \cdot t)$. What are the period and frequency of the sound intensity? The frequency is measured in cycles per second.

9. The horizontal distance a golf ball travels, unaided by gravity, is given by $D(t) = v_0 \cdot t \cos A_0$, in which v_0 is the velocity of the golf ball as it leaves the head of the golf club, t is the golf ball's "hang" time, and A_0 is the measure of the angle at which the golf ball is struck.

   a. If $D(t) = 600$ feet and the hang time is 5 seconds, what is the product $v_0 \cos A_0$ in feet per second?

   b. What is the range of the values of $\cos A_0$?

   c. What is v_0 if $A_0 = 45°$?

10. The average power in an alternating-current utility transmission line can be measured by the function $P_{average} = I_{average} \cdot V_{average} \cdot \cos A$, in which $I_{average}$ is a type of average current and $V_{average}$ is a type of average voltage in the line. The angle A becomes a factor when it is nonzero in certain kinds of circuits that produce phase differences between the current and voltage. (*Hint*: "Household" current delivered by a public utility in the United States at 50–60 cycles per second and at an average of 110–120 volts.)

   a. At what value of A less than 90° will the average power be half of its maximum value?

   b. What does this imply about the phase difference between the current and voltage curves on a graph?

Unit 4B
Mathematical Modeling and Choosing a Model

$f(x)$

$\angle ABC$

Q_1
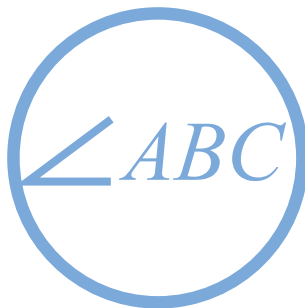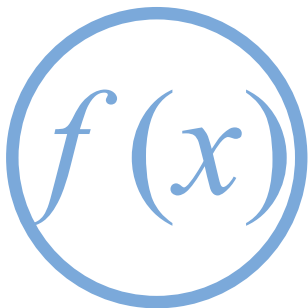
Lesson 1: Creating Equations

Common Core State Standards

A–CED.1   Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*★

A–CED.3   Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.*★

A–CED.4   Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law V = IR to highlight resistance R.*★

Essential Questions

1. How can you tell from looking at a graph or table that an equation in one variable is represented?

2. How can you tell from looking at an equation or equations in one variable how the dependent and independent variables are related?

3. What are the conditions in which an inequality statement in one variable should be written instead of an equality statement?

4. How do you identify the constraints on a function and its variable in a real-world problem?

5. What steps are necessary to isolate a variable in a multi-variable formula?

WORDS TO KNOW

constraint	a limit or restriction on the domain, range, and/or solutions of a mathematical or real-world problem
data fitting	the process of assigning a rule, usually an equation or formula, to a collection of data points as a method of predicting the values of new dependent variables that result from new independent-variable values

data point	a point (x, y) on a two-dimensional coordinate plane that represents the value of an independent variable (x) that results in a specific dependent variable value (y). The term also refers to solutions for an equation or inequality in one variable that originate from a real-world situation. A data point is also called an *ordered pair*.
dependent variable	labeled on the y-axis; the quantity that is based on the input values of the independent variable; the output variable of a function
domain	the set of all input values (x-values) that satisfy the given function without restriction
exponential equation	an equation that has a variable in the exponent
formula	a mathematical statement of the relationship between two or more variables
half plane	a region containing all points on one side of a boundary, which is a line or curve that continues in both directions infinitely. The line or curve may or may not be included in the region. A half plane can be used to represent a solution to an inequality statement.
independent variable	labeled on the x-axis; the quantity that changes based on values chosen; the input variable of a function
inequality	a mathematical statement that compares the value of an expression in one independent variable to the value of a dependent variable using the comparison symbols $>$, $<$, $\geq$, and $\leq$
linear equation	an equation that can be written in the form $ax + by = c$, where a, b, and c are constants; can also be written as $y = mx + b$, in which m is the slope and b is the y-intercept. The graph of a linear equation is a straight line; its solutions are the infinite set of points on the line.
logarithmic equation	an equation that includes a logarithmic expression

ordered pair	a point (x, y) on a two-dimensional coordinate plane that represents the value of an independent variable (x) that results in a specific dependent variable value (y). The term also refers to solutions for an equation or inequality in one variable that originate from a real-world situation. An ordered pair is also called a *data point*.
parent function	a function with a simple algebraic rule that represents a family of functions. The graphs of the functions in the family have the same general shape as the parent function.
quadratic equation	an equation that can be written in the form $y = ax^2 + bx + c$, where x is the independent variable, y is the dependent variable, a, b, and c are constants, and $a \neq 0$
range	the set of all outputs of a function; the set of y-values that are valid for the function
restricted domain	a subset of a function's defined domain
solution set	the set of ordered pairs that represent all of the solutions to an equation or a system of equations

Recommended Resources

- IXL Learning. "Graph a Linear Inequality in One Variable."

  http://www.walch.com/rr/00236

  This site allows users to practice graphing linear inequalities in one variable.

- Khan Academy. "The Quadratic Formula (Quadratic Equation)."

  http://www.walch.com/rr/00237

  This series of videos presents the quadratic equation as well as the quadratic formula. The videos feature various examples and applications.

- Khan Academy. "Solving for a Variable."

  http://www.walch.com/rr/00238

  These videos include topics such as rearranging formulas and solving for an isolated variable, with accompanying examples.

- Monterey Institute. "Writing, Solving, and Graphing Inequalities in One Variable."

  http://www.walch.com/rr/00239

  This website features a comprehensive tutorial on solving inequalities in one variable by applying properties of inequalities. It includes solving one-step and two-step inequalities with examples.

IXL Links

- Write variable equations:
  http://www.ixl.com/math/algebra-1/write-variable-equations

- Model and solve equations using algebra tiles:
  http://www.ixl.com/math/algebra-1/model-and-solve-equations-using-algebra-tiles

- Write and solve equations that represent diagrams:
  http://www.ixl.com/math/algebra-1/write-and-solve-equations-that-represent-diagrams

- Solve linear equations word problems:
  http://www.ixl.com/math/algebra-1/solve-linear-equations-word-problems

- Write inequalities from graphs:
  http://www.ixl.com/math/algebra-1/write-inequalities-from-graphs

- Write compound inequalities from graphs:
  http://www.ixl.com/math/algebra-1/write-compound-inequalities-from-graphs

- Weighted averages word problems:
  http://www.ixl.com/math/algebra-1/weighted-averages-word-problems

- Write variable expressions and equations:
  http://www.ixl.com/math/geometry/write-variable-expressions-and-equations

- Solve linear equations:
  http://www.ixl.com/math/geometry/solve-linear-equations

- Solve linear inequalities:
  http://www.ixl.com/math/geometry/solve-linear-inequalities

- Solve linear equations:
  http://www.ixl.com/math/algebra-2/solve-linear-equations

- Solve linear equations word problems:
  http://www.ixl.com/math/algebra-2/solve-linear-equations-word-problems

- Write inequalities from graphs:
  http://www.ixl.com/math/algebra-2/write-inequalities-from-graphs

- Solve linear inequalities:
  http://www.ixl.com/math/algebra-2/solve-linear-inequalities

- Solve a system of equations by graphing word problems:
  http://www.ixl.com/math/algebra-1/solve-a-system-of-equations-by-graphing-word-problems

- Solve a system of equations using substitution word problems:
  http://www.ixl.com/math/algebra-1/solve-a-system-of-equations-using-substitution-word-problems

- Solve a system of equations using elimination word problems:
  http://www.ixl.com/math/algebra-1/solve-a-system-of-equations-using-elimination-word-problems

- Solve a system of equations using augmented matrices word problems:
  http://www.ixl.com/math/algebra-1/solve-a-system-of-equations-using-augmented-matrices-word-problems

- Solve a system of equations using any method word problems:
  http://www.ixl.com/math/algebra-1/solve-a-system-of-equations-using-any-method-word-problems

- Solve systems of linear equations:
  http://www.ixl.com/math/geometry/solve-systems-of-linear-equations

- Solve a system of equations by graphing word problems:
  http://www.ixl.com/math/algebra-2/solve-a-system-of-equations-by-graphing-word-problems

- Solve a system of equations using substitution word problems:
  http://www.ixl.com/math/algebra-2/solve-a-system-of-equations-using-substitution-word-problems

- Solve a system of equations using elimination word problems:
  http://www.ixl.com/math/algebra-2/solve-a-system-of-equations-using-elimination-word-problems

- Solve a system of equations using any method word problems:
  http://www.ixl.com/math/algebra-2/solve-a-system-of-equations-using-any-method-word-problems

- Find the vertices of a solution set:
  http://www.ixl.com/math/algebra-2/find-the-vertices-of-a-solution-set

- Linear programming:
  http://www.ixl.com/math/algebra-2/linear-programming

- Rate of travel word problems:
  http://www.ixl.com/math/algebra-1/rate-of-travel-word-problems

Lesson 4B.1.1: Creating Equations in One Variable

Introduction

Often, the solutions to equations in real-world situations depend on a single variable. For example, if you are selling lollipops for $1 each for a prom fund-raiser, the amount of money you raise depends only on how many lollipops you sell. Creating equations in one variable to describe a mathematical or real-world situation typically requires analyzing data or viewing a visual display of the data, such as a graph on a coordinate plane. Sometimes you may need to analyze both the data and a visual display of the information.

Key Concepts

- An **independent variable**, labeled on the *x*-axis, is the quantity that changes based on values chosen. It is also referred to as the input variable of an equation or function.

- A **dependent variable**, labeled on the *y*-axis, is the quantity that changes based on the input values of the independent variable. The dependent variable is often referred to as the output variable of an equation or function.

- Mathematically, the number of data points needed to create an equation in one variable depends on the type of function that is being created.

- A **data point** (or **ordered pair**) is a point (*x*, *y*) on a two-dimensional coordinate plane that represents the value of an independent variable (*x*) that results in a specific dependent variable value (*y*).

- Data points also result from solutions for an equation or inequality in one variable that originate from a real-world situation.

- Recall that an **inequality** is a mathematical statement that compares the value of an expression in one independent variable to the value of a dependent variable using the comparison symbols >, <, ≥, and ≤.

- The data points or ordered pairs that make the inequality a true statement are the solutions of the inequality statement.

- Recall that a **linear equation** is an equation that can be written in the form $ax + by = c$, where *a*, *b*, and *c* are constants, or $y = mx + b$, in which *m* is the slope and *b* is the *y*-intercept. The graph of a linear equation is a straight line, and its solutions are the infinite set of points on the line.

- Only two data points are needed to write a linear equation in one variable because the graph of a linear equation is a straight line that is determined by two points. Therefore, the *x*- and *y*-intercepts and the slope of a straight line can be found using only two data points.

- Recollect that a **quadratic equation** is an equation that can be written in the form $y = ax^2 + bx + c$, where x is the independent variable, y is the dependent variable, a, b, and c are constants, and $a \neq 0$. The graph of a quadratic equation is a parabola on a coordinate plane. The values of a, b, and c can be determined if three data points are known. These data points will create three linear equations in three unknowns that can be solved.

- Also recall that an **exponential equation** is an equation that has a variable in the exponent.

- A **logarithmic equation** is an equation that includes a logarithmic expression.

- The following graphs depict several types of one-variable equations.
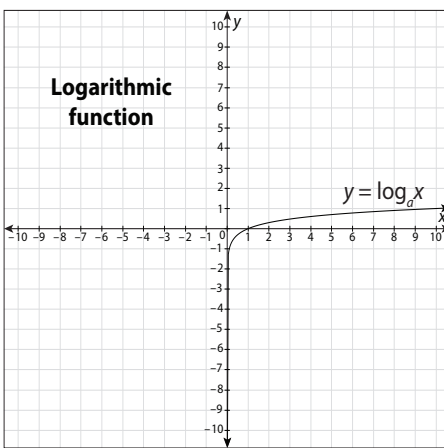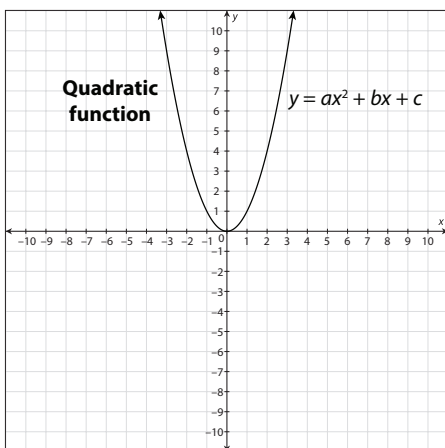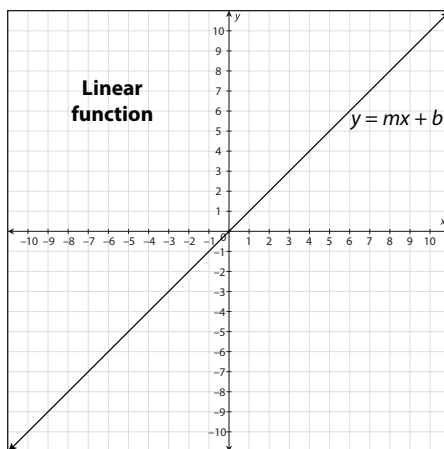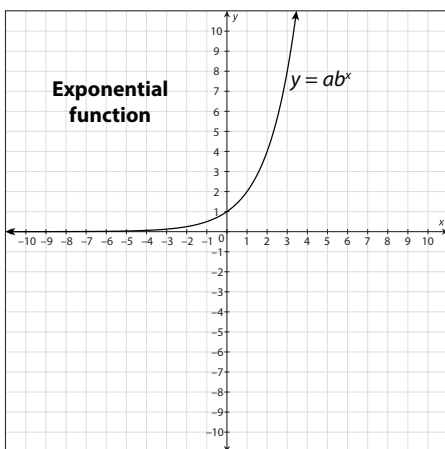
Exponential function: $y = ab^x$

Linear function: $y = mx + b$

Quadratic function: $y = ax^2 + bx + c$

Logarithmic function: $y = \log_a x$

- A **parent function** is a function with a simple algebraic rule that represents a family of functions. The graphs of functions in the family have the same general shape as the parent function.

- When analyzing the graph of an equation or function, it is important to consider the mathematical constraints on the graph. **Constraints** are limits or restrictions on the domain, range, and/or solutions of a mathematical or real-world problem. Constraints can result from mathematical and real-world considerations. Recall that the **domain** is the set of all input values (x-values) that satisfy the given equation or function without restriction, and the **range** is the set of all outputs (y-values) that are valid for the equation or function.

- Later in the lesson, real-world constraints will be applied to one-variable equations and their parent functions.

- Formulas can be useful in determining how variables relate to one another. Formally, a **formula** is a mathematical statement of the relationship between two or more variables. The variables' values are sometimes constrained by mathematical or real-world conditions.

- An inequality in one variable can be written for exponential, linear, logarithmic, and quadratic equations in one variable. The solutions to the inequality are defined as the set of data points or ordered pairs that make the inequality true in the half plane of a coordinate plane.

- A **half plane** is a region containing all points on one side of a boundary, which is a line or curve that continues in both directions infinitely. The line or curve may or may not be included in the region. A half plane can be used to represent a solution to an inequality statement.

- A half plane is used to represent a linear inequality of the form $ax + by > c$ instead of the straight line that represents a linear equation of the form $ax + by = c$. (The linear inequality would be $y > mx + b$ for a linear equation in slope-intercept form, $y = mx + b$.) In fact, the straight line and the half plane together represent the mathematical solutions possible for the inequality. In this case, the straight line is a boundary for the half plane and does not include the points on the line in the solution for the inequality. The inequality is greater than ($>$), not greater than or equal to ($\geq$). The graphs that follow show the one-variable equations from the previous graphs, now changed to inequalities that use the four different types of inequality conditions ($>$, $<$, $\geq$, and $\leq$).

Exponential function — $y > ab^x$

Linear function — $y < mx + b$

Quadratic function — $y \geq ax^2 + bx + c$

Logarithmic function — $y \leq \log_a x$
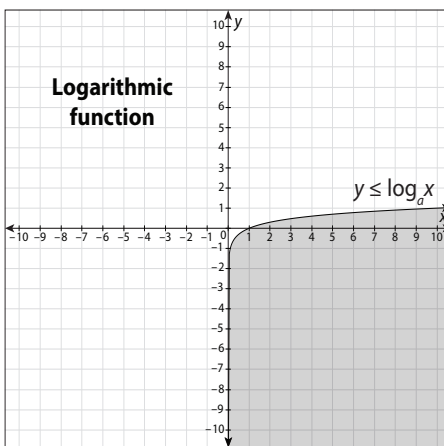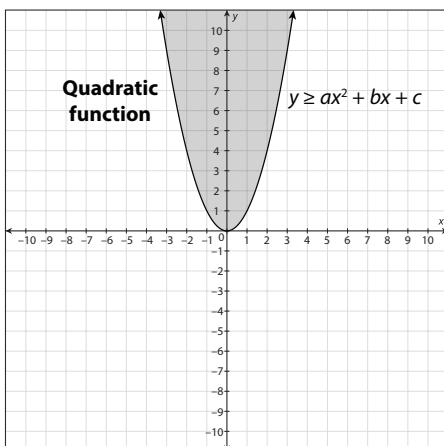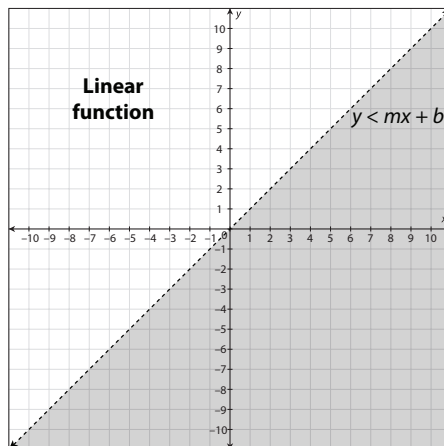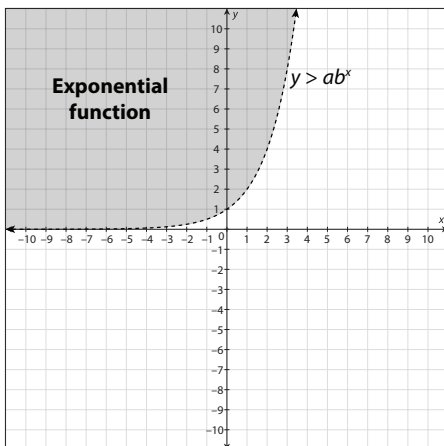
- A graphing calculator can be used to compare the different kinds of equations in one variable that can be written on the basis of a finite number of data points or ordered pairs. This process is sometimes called **data fitting** because an equation is "fitted" to a collection of data points as a way to produce a rule (the equation) that predicts the values of new dependent variables that result from new independent-variable values in the same situation.

- To fit an equation to the graph of a set of data points, follow the directions appropriate to your calculator model. Either calculator will return values for the constants that can be substituted into the equation $y = ax + b$ or $y = ax^2 + bx + c$, depending on the type of equation chosen. These values can be verified by calculating the slope and y-intercept of a line passing through the points.

On a TI-83/84:

Step 1: Press [STAT] to bring up the statistics menu. The first option, 1: Edit, will already be highlighted. Press [ENTER].

Step 2: Arrow up to L1 and press [CLEAR], then [ENTER], to clear the list. Repeat this process to clear L2 as needed.

Step 3: From L1, press the down arrow to move your cursor into the list. Enter the x-value of the first ordered pair. Press [ENTER]. Repeat until all x-values have been entered.

Step 4: Press the right arrow key and enter the y-value of the first ordered pair next to its x-value. Press [ENTER]. Repeat until all y-values have been entered.

Step 5: Press [2ND][Y=] to bring up the STAT PLOTS menu.

Step 6: The first option, Plot 1, will already be highlighted. Press [ENTER].

Step 7: Under Plot 1, select ON if it isn't selected already.

Step 8: Arrow over to Plot 2 and repeat. Check that "Xlist:" is set to "L1" and "Ylist:" is set to "L2." Press [ENTER] to save any changes.

Step 9: Press [ZOOM] and select 6: ZStandard to produce a four-quadrant grid. Notice the graphed data points.

Step 10: To fit an equation to the data points, press [STAT] and arrow over to the CALC menu. Then, select 4: LinReg(ax+b) or 5: QuadReg, depending on the type of graph desired.

Step 11: Press [2ND][1] to type "L1" for Xlist. Arrow down to Ylist and press [2ND][2] to type "L2" for Ylist, if not already shown.

Step 12: Arrow down to "Calculate" and press [ENTER].

On a TI-Nspire:

Step 1: Press the [home] key. Arrow over to the spreadsheet icon, the fourth icon from the left, and press [enter].

Step 2: To clear the lists in your calculator, arrow up to the topmost cell of the table to highlight the entire column, then press [menu]. Choose 3: Data, then 4: Clear Data. Repeat for each column as necessary.

Step 3: Arrow up to the topmost cell of the first column, labeled "A." Press [X][enter] to type *x*. Then, arrow over to the second column, labeled "B." Press [Y][enter] to type *y*.

Step 4: Arrow down to cell A1 and enter the first *x*-value. Press [enter]. Enter the second *x*-value in cell A2 and so on.

Step 5: Move over to cell B1 and enter the first *y*-value. Press [enter]. Enter the second *y*-value in cell B2 and so on.

Step 6: To see a graph of the points, press the [home] key. Arrow over to the graphing icon, the second icon from the left, and press [enter]. Press [menu], then select 3: Graph Type, and 4: Scatter Plot.

Step 7: At the bottom of the screen, use the pop-up menus to enter "*x*" for the *x*-variable and "*y*" for the *y*-variable. Press [enter]. The data points are displayed.

Step 8: If needed, adjust the viewing window. Press [menu], then select 4: Window/Zoom, and then 1: Window Settings. Change the settings as appropriate.

Step 9: To fit an equation to the data points, first press [ctrl] and the up arrow key to display the open windows. Highlight the table of *x*- and *y*-values and press [enter]. Press [menu] and select 4: Statistics, and then 1: Stat Calculations. Select the desired type of equation from the list. The coefficients of the desired equation will be listed in a table.

- *Note*: A word of caution is needed in using a graphing calculator to fit a quadratic equation based on given data. The generated equation is of the "best fit," which means that it may not exactly fit the data points. In real-world problems, such approximations are a result of using inexact or sometimes unreliable measurements or measuring tools.

Guided Practice 4B.1.1

Example 1

Aaron wants to have his company's logo printed on tablet computer cases, so that he can give away the cases as part of a marketing campaign. A specialty printing company will charge Aaron a $750 fee to design and print the personalized cases, plus the cost of the actual cases. The price of each case is $3. How many personalized cases can Aaron purchase if his budget is $1,200?

1. Write an equation in words for determining the total cost to produce the personalized cases.

   Review the problem statement to determine the given information and the information that is needed to solve the problem.

   The *total cost* of buying the cases includes a *fee* plus the *cost of the actual cases*.

   The cost of the cases is determined by the *price of each case* multiplied by the *number of cases*.

   Summarize this information as an equation in words:

   The total cost is the fee plus the price of each case multiplied by the number of cases.

2. Write an equation for the cost for n cases.

   Let C represent the cost of the cases.

   The cost, C, of n cases is equal to $C(n)$.

   From the problem statement, we know that n cases will cost $3 each.

   We also know that the fee is $750.

   In the word equation written for step 1, substitute $C(n)$ for the total cost, $750 for the fee, n for the number of cases, and $3 for the price of each case:

total cost	is	fee	plus	price of each case	multiplied by	number of cases
↓	↓	↓	↓	↓	↓	↓
$C(n)$	=	$750	+	$3	•	n

   The mathematical equation for the cost, C, of n cases is $C(n) = 750 + 3n$.

3. Determine how much money Aaron will have left to spend on cases after paying the fee.

Aaron's budget is $1,200. Subtract the fee of $750 from the amount Aaron has to spend.

$$1200 - 750 = 450$$

After the fee, Aaron has $450 to spend on cases.

4. Determine the number of cases Aaron can buy.

Aaron has $450 to spend on cases and the price of each case is $3. Therefore, divide $450 by $3 to determine the number of cases he can afford.

$$\frac{450}{3} = 150$$

Aaron can buy 150 cases.

5. Use the equation to check the result.

Check the result by substituting 1,200 for $C(n)$ and 150 for n in the equation determined in step 2.

$C(n) = 750 + 3n$	Equation determined in step 2
$(1200) = 750 + 3(150)$	Substitute 1,200 for $C(n)$ and 150 for n.
$1200 = 750 + 450$	Multiply.
$1200 = 1200$	Add.

The answer results in a true statement. Aaron can buy 150 cases with his budget of $1,200.

Example 2

Chloe is driving back home from college for summer vacation. She fuels up her gas tank and then drives for a certain amount of time before passing a roadside attraction. She drives on without stopping, and 3 hours after leaving her college, she has driven 120 miles past the attraction. Seven hours after leaving her college, she has driven 400 miles past the attraction. Write a linear equation in one variable for the distance Chloe covers in t hours, and describe the domain of the linear equation. Assuming that Chloe travels at a constant speed without stopping, use the equation to determine her speed. Then, determine how far she had traveled before she passed the roadside attraction.

1. Name a dependent variable and an independent variable based on the given data.

   Let the independent variable be time in hours, t.

   Let the dependent variable be the distance in miles, d.

2. Write ordered pairs for the data in the problem based on the identified variables.

   The ordered pairs will be of the form (t, d), since the independent variable is on the x-axis and the dependent variable is on the y-axis. Therefore, the ordered pairs for the given data are (3, 120) and (7, 400).

3. Write an equation for the speed in terms of the distance and time.

   This is a distance-rate-time problem in which the rate or speed, r, is the distance divided by the time.

   $$\text{rate} = \frac{\text{distance}}{\text{time}} \text{ or } r = \frac{d}{t}$$

4. Use the two ordered pairs from step 2 and the formula for the slope of a one-variable equation, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, to find the rate.

The slope formula can be used to find the rate because it is of the same form as the formula $r = \dfrac{d}{t}$.

Recall that the two-point slope formula is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ for points (x_1, y_1) and (x_2, y_2). We can think of this as $r = \dfrac{d_2 - d_1}{t_2 - t_1}$ for points (t_1, d_1) and (t_2, d_2). Thus, the slope m would also be the rate r at which Chloe drives.

Let (3, 120) represent (t_1, d_1) and (7, 400) represent (t_2, d_2).

Substitute these values into the formula for slope.

$r = \dfrac{d_2 - d_1}{t_2 - t_1}$	Slope formula written in terms of the rate, r
$r = \dfrac{(400) - (120)}{(7) - (3)}$	Substitute (3, 120) for (t_1, d_1) and (7, 400) for (t_2, d_2).
$r = \dfrac{280}{4}$	Simplify.
$r = 70$	

Chloe's speed during her 7-hour drive was 70 miles per hour.

5. Use the point-slope formula, $y - y_1 = m(x - x_1)$, to write the linear equation in one variable.

   In the point-slope formula, $y - y_1 = m(x - x_1)$, m is the slope and (x_1, y_1) is a point on the line. Use the value $r = 70$ from the previous step for the slope and either of the given ordered pairs for (x_1, y_1) in this formula.

   Let's use (3, 120). Simplify and solve for d.

$y - y_1 = m(x - x_1)$	Point-slope formula
$(d) - (120) = (70)[(t) - (3)]$	Substitute (3, 120) for (x_1, y_1), d for y, r for m, and t for x.
$d - 120 = 70t - 210$	Distribute.
$d = 70t - 90$	Simplify.

6. Find the value of d when $t = 0$.

   $t = 0$ represents the time at which Chloe left college for the drive home.

$d = 70t - 90$	Equation from the previous step
$d = 70(0) - 90$	Substitute 0 for t.
$d = -90$	Simplify.

   When $t = 0$, $d = -90$.

7. Interpret the results based on the information given in the problem to determine how far Chloe drove before passing the roadside attraction.

   If Chloe drove at a constant speed of 70 miles per hour, the ordered pair (3, 120) implies that she had traveled 3 • 70 or 210 miles from her starting point. The distance 120 in the ordered pair implies that Chloe was 210 – 120 or 90 miles past that starting point when she passed the roadside attraction. Therefore, Chloe drove 90 miles before passing the roadside attraction.

Example 3

Write a quadratic equation in one variable that is true for the three data points $(0, 0)$, $(1, 2)$, and $(2, 8)$ by solving a system of three equations based on the standard form of a quadratic equation, $y = ax^2 + bx + c$. Then, use your equation to find the y-value for a fourth point on the same graph that has an x-value of 3.

1. Use the given data points to write three equations using the standard form of a quadratic equation.

   Substitute the x- and y-values from each of the three data points into $y = ax^2 + bx + c$ and simplify.

For (0, 0):	**For (1, 2):**	**For (2, 8):**
$y = ax^2 + bx + c$	$y = ax^2 + bx + c$	$y = ax^2 + bx + c$
$(0) = a(0)^2 + b(0) + c$	$(2) = a(1)^2 + b(1) + c$	$(8) = a(2)^2 + b(2) + c$
$c = 0$	$2 = a + b + c$	$8 = 4a + 2b + c$

   Substitute $c = 0$ into the other two equations to produce two equations in two unknowns.

For (1, 2):	**For (2, 8):**
$2 = a + b + c$	$8 = 4a + 2b + c$
$2 = a + b + (0)$	$8 = 4a + 2b + (0)$
$2 = a + b$	$8 = 4a + 2b$
	$4 = 2a + b$

   We now have two separate equations. We must continue to simplify until only one equation that is true for all three given points remains.

2. Examine the resulting equations for any common traits that can be used to simplify them.

Compare $2 = a + b$ to $4 = 2a + b$.

Notice that the constants in both equations are multiples of 2.

Make the left sides of the two equations equal by multiplying the terms in the first equation by 2.

$2 = a + b$

$2 \cdot 2 = 2 \cdot (a + b)$

$4 = 2a + 2b$

3. Use the results to solve for another variable.

Set the right sides of the two equations equal to each other and solve for any variable possible in order to eliminate another variable from the system of equations.

$4 = 2a + 2b$ Revised first equation

$4 = 2a + b$ Second equation

Set the right sides equal and solve.

$2a + 2b = 2a + b$

$2b - b = 2a - 2a$

$b = 0$

Substitute the value found for b into either of the equations $4 = 2a + 2b$ or $4 = 2a + b$. This will allow the value of the third variable to be found.

Solve for a in $4 = 2a + b$ by substituting 0 for b.

$4 = 2a + b$

$4 = 2a + (0)$

$4 = 2a$

$a = 2$

4. Use the values determined for a, b, and c to write the quadratic equation that is true for the three data points.

Substitute $a = 2$, $b = 0$, and $c = 0$ into the standard form of a quadratic equation and simplify.

$y = ax^2 + bx + c$ Standard form of a quadratic equation

$y = (2)x^2 + (0)x + (0)$ Substitute 2 for a, 0 for b, and 0 for c.

$y = 2x^2$ Simplify.

Check the resulting equation by substituting the x- and y-values from each of the three data points to see if a true statement results. All three data points should produce true statements.

For (0, 0):	**For (1, 2):**	**For (2, 8):**
$y = 2x^2$	$y = 2x^2$	$y = 2x^2$
$(0) = 2(0)^2$	$(2) = 2(1)^2$	$(8) = 2(2)^2 = 2(4)$
$0 = 0$	$2 = 2$	$8 = 8$

All three data points produce true statements for $y = 2x^2$.

5. Use the resulting quadratic equation model for the three given points to predict the location of a fourth point for which $x = 3$.

Since $y = 2x^2$ is true for all three given points, it will be true for all other points on the graph. Therefore, the equation can be used to predict other points.

Substitute 3 for x in the equation and solve for y.

$y = 2x^2$

$y = 2(3)^2 = 2(9) = 18$

The fourth point is (3, 18).

Example 4

The data shows the current i in milliamps (mA) in a cell phone circuit in fractions of a second after a cell-tower signal is received.

Time, t (s)	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Current, i (mA)	2	3	5	8	13	21	34	55	89

Graph these points on a graphing calculator. Use the graph to write an exponential equation of the general form $y = ab^x$ that approximately fits the data. Then, rewrite the resulting equation so that it includes a power of 10.

1. Name an independent variable and a dependent variable.

    In this case, the current increases as a function of the elapsed time.

    Dependent variable: i (current in milliamps)

    Independent variable: t (time in seconds)

2. Use the table of values to determine the data points.

    Follow the format (t, i), using the convention that the independent variable is listed first.

    The data points are (0.1, 2), (0.2, 3), (0.3, 5), (0.4, 8), (0.5, 13), (0.6, 21), (0.7, 34), (0.8, 55), and (0.9, 89).

3. Plot the points on a graphing calculator and use the graph to find the exponential function for the data.

   The graph of the points will allow you to estimate values for the constants *a* and *b* in the general form $y = ab^x$. With these values, you can write an equation that approximately fits the given data.

   On a TI-83/84:

   Step 1: Press [STAT] to bring up the statistics menu. The first option, 1: Edit, will already be highlighted. Press [ENTER].

   Step 2: Arrow up to L1 and press [CLEAR], then [ENTER], to clear the list. Repeat this process to clear L2 as needed.

   Step 3: From L1, press the down arrow to move your cursor into the list. Enter the *t*-value of the first ordered pair. Press [ENTER]. Repeat until all *t*-values have been entered.

   Step 4: Press the right arrow key and enter the *i*-value of the first ordered pair next to its *t*-value. Press [ENTER]. Repeat until all *i*-values have been entered.

   Step 5: Press [2ND][Y=] to bring up the STAT PLOTS menu.

   Step 6: The first option, Plot 1, will already be highlighted. Press [ENTER].

   Step 7: Under Plot 1, select ON if it isn't selected already.

   Step 8: Arrow over to Plot 2 and repeat. Check that "Xlist:" is set to "L1" and "Ylist:" is set to "L2." Press [ENTER] to save any changes.

   Step 9: Press [WINDOW] to set the viewing window to display the data: Xmin = 0, Xmax = 1, Xscl = 0.1, Ymin = 0, Ymax = 90, Yscl = 10.

   Step 10: Press [GRAPH] to see the data graph.

   Step 11: To fit an equation to the data points, press [STAT] and arrow over to the CALC menu. Then, select 0: ExpReg.

   Step 12: Press [2ND][1] to type "L1" for Xlist. Arrow down to Ylist and press [2ND][2] to type "L2" for Ylist, if not already shown.

   Step 13: Arrow down to "Calculate" and press [ENTER]. The resulting equation is of the general form $y = ab^x$, in which *a* and *b* are constants.

   (*continued*)

On a TI-Nspire:

Step 1: Press the [home] key. Arrow over to the spreadsheet icon, the fourth icon from the left, and press [enter].

Step 2: To clear the lists in your calculator, arrow up to the topmost cell of the table to highlight the entire column, then press [menu]. Choose 3: Data, then 4: Clear Data. Repeat for each column as necessary.

Step 3: Arrow up to the topmost cell of the first column, labeled "A." Press [X][enter] to type x. Then, arrow over to the second column, labeled "B." Press [Y][enter] to type y.

Step 4: Arrow down to cell A1 and enter the first t-value. Press [enter]. Enter the second t-value in cell A2 and so on.

Step 5: Move over to cell B1 and enter the first i-value. Press [enter]. Enter the second i-value in cell B2 and so on.

Step 6: To see a graph of the points, press the [home] key. Arrow over to the graphing icon, the second icon from the left, and press [enter]. Press [menu], then select 3: Graph Type, and 4: Scatter Plot.

Step 7: At the bottom of the screen, use the pop-up menus to enter "x" for the t-variable and "y" for the i-variable. Press [enter]. The data points are displayed.

Step 8: If needed, adjust the viewing window. Press [menu], then select 4: Window/Zoom, and then 1: Window Settings. Change the settings as appropriate: XMin = 0, XMax = 1, XScale = 0.1, YMin = 0, YMax = 90, YScale = 10. Select "OK" or press [enter] to view the adjusted graph.

Step 9: To fit an equation to the data points, first press [ctrl] and the up arrow key to display the open windows. Highlight the table of t- and i-values and press [enter]. Press [menu] and select 4: Statistics, and then 1: Stat Calculations. Select A: Exponential Regression.

(continued)

Step 10: At the "Exponential Regression" settings screen, use the pop-up menus to select "x" for X List and "y" for Y List. Press [enter]. The coefficients in the quadratic equation $y = ab^x$ should be listed in the table with the data points.

Either calculator will return approximate results for a and b of $a \approx 1.2$ and $b \approx 120$. Substituting these values into the general form $y = ab^x$ results in the equation $y = 1.2 \cdot 120^x$. Note that this is an equation of best fit, since the constants are approximate.

4. Rewrite the base to include a power of 10.

The base of the exponent variable can be rewritten as a power of 10, and then simplified using the properties of exponents.

$y = 1.2 \cdot 120^x$ Equation from the previous step

$y = 1.2 \cdot (1.2 \cdot 10^2)^x$ Rewrite 120 as a power of 10.

$y = 1.2 \cdot (1.2)^x \cdot (10)^{2x}$ Apply the Power of Powers Property.

$y = 1.2^{x+1} \cdot 10^{2x}$ Rewrite using the Product of Powers Property.

The exponential equation written in base 10 that fits the data in the table is $y = 1.2^{x+1} \cdot 10^{2x}$.

UNIT 4B • MATHEMATICAL MODELING AND CHOOSING A MODEL
Lesson 1: Creating Equations

Practice 4B.1.1: Creating Equations in One Variable

For problems 1–3, write an equation in one variable that fits the data points *exactly* without using a calculator.

1. (1, 3) and (3, 13)

2. (2, 4), (3, 12), and (0, 0)

3. (–4, 5), (3, 12), and (4, 21)

For problems 4–7, write an equation in one variable in the simplest form of the equation type listed, using all three of the graphed data points.
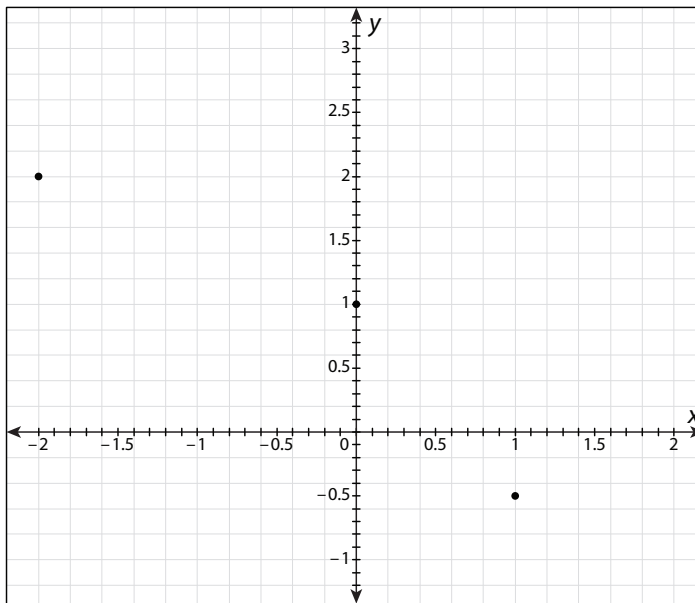
4. an exponential equation

5. a linear equation

6. a logarithmic equation

7. a quadratic equation

UNIT 4B • MATHEMATICAL MODELING AND CHOOSING A MODEL
Lesson 1: Creating Equations

Read the scenario that follows, and use the information in it to complete problems 8–10.

> Darien rode her bike from a starting point (0, 0) at a speed that constantly changed according to the equation $d(t) = -5t^2 + 35t$, in which d is the distance in miles and t is the time in hours. After 2 hours, Darien had traveled 50 miles. Her friend Jaceylo biked from the same starting point (0, 0) for 1 hour at a speed of 30 miles per hour. Then, Jaceylo biked at a different speed for another hour. Jaceylo followed the same route as Darien and ended up at the same destination.

8. Write a linear equation to represent the first hour of Jaceylo's bike ride.

9. Write a linear equation to represent the second hour of Jaceylo's bike ride.

10. At what speed was Jaceylo traveling over the second hour?

Lesson 4B.1.2: Representing and Interpreting Constraints

Introduction

Constraints on mathematical problems can be a function of the mathematical expressions and relationships used. Limitations on the domain of the independent variable in the argument of a logarithm are examples of this. Constraints can also be related to the features of a real-world problem. For example, if time is being measured, it is usually a positive, nonzero quantity. A problem with a real-world context can combine both mathematical and real-world constraints.

Key Concepts

- Recall that a constraint is a limit or restriction on the domain, range, and/or solutions of a mathematical or real-world problem.

- A problem can have constraints that result in a restricted domain for the independent variable in the model or a real-world limitation on the mathematical domain of the model variable(s). A **restricted domain** is a subset of a function's defined domain. The subset is limited by mathematical conditions and/or real-world constraints.

- Constraints can also affect the dependent variable by limiting its range. The range itself might also have constraints regardless of the constraints on the domain of the independent variable used in the model.

- When a constraint is placed on an inequality statement, a subset of the inequality's solution set can result. A **solution set** is the set of ordered pairs that represent all of the solutions to an equation or system of equations. For example, the solution set of the inequality statement $y > 4x - 1$ is all values of x that make the statement true. If x is restricted to values of x that are integers between $113 < x < 137$, a different set of values of y make the inequality true than if x is unrestricted.

- Real-world problems are sometimes modeled with systems of equations or inequalities when there are several different variables, or when it is easier to use different variables than to keep track of a variety of constraints placed on one variable in a model.

- Systems of inequalities have solutions that can be visualized as half planes that overlap with many common solutions, or that share a point in common.

- Sometimes, a system of inequalities that has no solution is nevertheless the solution to a real-world problem.

- Graphing calculators can help you visualize solutions to systems of equations or inequalities. They can also provide the solutions for real-world problems which are sometimes less accessible by manual techniques.

Guided Practice 4B.1.2

Example 1

Meredith runs the souvenir store for a minor-league baseball team. She checked into how much it would cost to have smartphone cases made that feature an image of the team's mascot. One manufacturer charges a $250 fee to design and print the personalized cases, plus the cost of the actual cases. The first 50 cases cost $5 each, and the next 100 cases cost $3 each. Write an equation to determine how many cases Meredith can purchase with a budget of $750. Determine the constraints on the terms of the equation based on the situation, then apply the constraints to solve for the number of cases that can be purchased.

1. Write an equation in words for determining the total cost to produce the personalized cases.

   Review the problem statement to determine the given information and the information that is needed to solve the problem.

   The *total cost* of buying the cases includes a *fee* plus the *cost of the actual cases*.

   The cost of the cases is determined by the *price of each case* multiplied by the *number of cases*.

   Summarize this information as an equation in words:

   The total cost is the fee plus the price of each case multiplied by the number of cases.

2. Write an equation for the cost of *n* cases if *n* is 50 or less.

Let *C* represent the cost of the cases.

The cost, *C*, of *n* cases is equal to $C(n)$.

From the problem statement, we know that *n* cases will cost $5 each if 50 or fewer are purchased.

We also know that the fee is $250.

In the word equation written for step 1, substitute $C(n)$ for the total cost, $250 for the fee, *n* for the number of cases, and $5 for the price of each case when $n \leq 50$:

total cost	is	fee	plus	price of each case	multiplied by	number of cases
↓	↓	↓	↓	↓	↓	↓
$C(n)$	=	$250	+	$5	•	*n*

The mathematical equation for the cost, *C*, of *n* cases when *n* is 50 or less is $C(n) = 250 + 5n$.

3. Write an equation for the cost $C(n, m)$ if *n* is the number of cases that are $5 each and *m* is the number of cases that are $3 each.

Modify the equation for $C(n)$. Let *m* represent the number of cases that are $3 each. Since this quantity represents the cost of additional cases, it is being added to the equation for $C(n)$ to become the equation for $C(n, m)$.

$$C(n, m) = 250 + 5n + 3m$$

4. List any constraints on the value of the expression *m*.

The expression *m* only exists if $n > 50$. Also, since the problem statement specified that the $3 price applies only to the next 100 cases, $m \leq 100$.

5. Write an equation for the amount of money Meredith has available to spend as a function of the cost equation identified in step 3.

Meredith's budget is $750. Substitute this amount for $C(n, m)$ and simplify.

$C(n, m) = 250 + 5n + 3m$	Equation from step 3
$(750) = 250 + 5n + 3m$	Substitute 750 for $C(n, m)$.
$500 = 5n + 3m$	Subtract 250 from both sides.

Based on Meredith's budget of $750, the cost equation is $500 = 5n + 3m$.

6. List any constraints on the terms on the right side of the resulting equation.

For the term $3m$ to exist, $5n$ has to exist and have a maximum value, namely $5 per case • 50 cases, which equals $250.

7. Substitute this constraint into the equation from step 5 and simplify.

Since the maximum value of $5n$ is $250, which is within Meredith's budget, we know that she will have enough money for more than 50 cases. Now we must determine how many more cases she can purchase. Substitute 50 for n to find m, the number of additional cases Meredith can buy for $3.

$500 = 5n + 3m$	Equation from step 5
$500 = 5(50) + 3m$	Substitute 50 for n.
$500 = 250 + 3m$	Simplify.
$250 = 3m$	Subtract 250 from both sides.
$m = 83.\overline{3}$	Divide both sides by 3.

Since Meredith cannot buy a fraction of a case, round down to the nearest whole number.

Meredith can buy up to 83 additional cases at $3 each.

8. Apply any remaining constraints to the value of *m* and determine the total number of cases Meredith can purchase for $750.

Since this scenario has a real-world context, an unstated assumption about *m* and *n* is that both are positive whole numbers. Also, *m* exists when $m \leq 100$ and $n = 50$.

Therefore, Meredith can afford 50 cases for $5 each and 83 additional cases for $3 each.

Meredith can buy a total of 133 personalized cases with her $750 budget.

Example 2

Certain medical tests require that patients be injected with liquids containing trace amounts of radioactive elements in order to track the movement of blood in the circulatory system. The concentration of the radioactive tracer substance diminishes in the human body over time according to the function $C(t) = a\left(\dfrac{2-t}{1+4t-t^2}\right)$, in which *a* is a constant unique to the tracer ($a > 0$), *t* is time in hours, and $C(t)$ is the concentration of the tracer in milligrams per liter. Identify real-world and mathematical constraints on *t*, the time that the tracer is in the body, which allow $C(t)$ to be defined in the context of the situation.

1. Identify a real-world condition that might be placed on the time variable, *t*.

Time cannot be negative, so the values of *t* must be non-negative.

2. Determine any specific mathematical constraints on *t*.

Since the term with *t* is a rational number, $\left(\dfrac{2-t}{1+4t-t^2}\right)$, the denominator will have to be nonzero in order for this rational term of the function to exist.

3. Use the quadratic formula to find value(s) of t for which the denominator is nonzero.

   The denominator of the rational number includes a quadratic expression, $1 + 4t - t^2$. Therefore, the quadratic formula can be used to determine nonzero values of t.

   In the quadratic formula for equations of the form $y = ax^2 + bx + c$, the value of x is $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

   Rearrange the expression $1 + 4t - t^2$ to more readily identify values for a, b, and c: $-t^2 + 4t + 1$.

   Thus, for the denominator of the function $C(t) = a\left(\dfrac{2-t}{1+4t-t^2}\right)$, $a = -1$, $b = 4$, and $c = 1$. Substitute these values for a, b, and c into the quadratic formula, with t in place of x.

   $t = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Quadratic formula with t in place of x

   $t = \dfrac{-(4) \pm \sqrt{(4)^2 - 4(-1)(1)}}{2(-1)}$ Substitute -1 for a, 4 for b, and 1 for c.

   $t = \dfrac{-4 \pm \sqrt{16 + 4}}{-2}$ Simplify.

   $t = \dfrac{4}{2} \pm \dfrac{\sqrt{20}}{2}$ Write the right side as two fractions.

   $t \approx 2 \pm 2.24$ Simplify using a calculator.

   Negative values of t which make the denominator equal to 0 are not important because it has been determined that t cannot be negative.

4. Find the initial concentration of the radioactive tracer at $t = 0$.

This is also a check on the value of the model as a predictor of the concentration having a positive value over the domain of the variable t.

$$C(t) = a\left(\frac{2-t}{1+4t-t^2}\right)$$ Given function

$$C(0) = a\left(\frac{2-(0)}{1+4(0)-(0)^2}\right)$$ Substitute 0 for t.

$$C(0) = a\left(\frac{2}{1}\right)$$ Simplify.

$$C(0) = 2a$$ Continue to simplify.

$2a$ is positive since $a > 0$ in the problem.

5. Combine the results of the previous steps to determine a realistic domain for the time in this problem.

If the tracer is injected at $t = 0$, the domain for t in the tracer-concentration model will be [0, 4.24).

Example 3

A sports shop is holding a sale on a particular brand of tennis racket. The total retail value of the initial stock of rackets was $7,500. The sale price is $120 per racket. Write an equation for the value of the rackets remaining after n rackets are sold. Then, find the number of rackets sold when the total value, $V(n)$, reaches 0, and explain what this reveals about how much "below retail" the sale price is.

1. Use the given information to write a function for the value of the rackets remaining in stock after t hours.

   To find the value of the remaining rackets, subtract the value of sold rackets from $7,500 (the value of the rackets before the sale began).

   value of rackets remaining = 7500 – value of rackets sold

   We are given that n represents the number of rackets sold, and that each racket costs $120.

   Let $V(n)$ represent the value of rackets remaining in stock, V, for n rackets sold.

   Rewrite the word problem using the assigned values and variables.

   value of rackets remaining = 7500 – value of rackets sold

   $V(n) = 7500 – 120n$

   The value function for the number of rackets remaining in stock is $V(n) = 7500 – 120n$.

2. Determine the value of n when the value of the rackets remaining is 0.

   When the value of the rackets remaining in stock is 0, then $V(n) = 0$. Therefore, substitute 0 for $V(n)$ in the value function and solve for n.

$V(n) = 7500 – 120n$	Equation from the previous step
$(0) = 7500 – 120n$	Substitute 0 for $V(n)$.
$120n = 7500$	Add $120n$ to both sides.
$n = 62.5$	Divide both sides by 120.

   The value of n is 62.5 when the value of the rackets remaining is 0.

3. Interpret the result of step 2 and what it means for the equation $V(n) = 0$.

The real-world independent variable n is a positive whole number, so the largest number of rackets that can be sold is 62. The real-world variable $V(n)$ will not have a zero value; its smallest value will be 120 • 0.5 or $60.

4. Interpret these results in terms of the savings per racket.

Multiplying the largest number of rackets that can be sold by the sale price yields $62 \cdot 120 = 7440$. $7,440 is only $60 less than the original retail value of $7,500. Divide this $60 savings among the 62 rackets to determine just how much below retail the sale price really is:

$60 \div 62 \approx 0.97$

The sale price of $120 per racket is only about $0.97 lower than the retail cost per racket.

UNIT 4B • MATHEMATICAL MODELING AND CHOOSING A MODEL
Lesson 1: Creating Equations

Practice 4B.1.2: Representing and Interpreting Constraints

For problems 1–3, determine the restricted domain that corresponds to the range constraint on the dependent variable in the equation.

1. $4x - 3y = 7$ if $y \leq 0$

2. $y = 10 + \log (x + 10)$ if $y > -10$

3. $4x^2 - 9y^3 = 16$ if $y \geq 1$

For problems 4–7, use the given constraints on the variable(s) to determine the solution set(s) of the system.

4. $y < x, x + y > 1$ if $x < 1$

5. $x + 12 \leq 3, 3x - 2y \geq 1$ if $x \leq 0$

6. $4x - y > 3, 3x + 4y < 1$ if $x < 3$ and $y > 4$

7. $|2x - y| > 3$

Use the given information to solve problems 8–10.

8. A race car driver takes off from a standing start and accelerates at a constant rate down a quarter-mile track. The racer crosses the finish line in 2.5 seconds. What are the constraints on the acceleration, distance covered, speed, and time variables for the racer?

9. A rancher fills a 900-gallon water trough at the beginning of a workweek. The rancher's animals drink 60 gallons of water per day, and the trough's automatic-filling control pumps 30 gallons of water into the tank every 2 days. What are the constraints on the capacity of the water trough and the rate at which water is removed from the tank? What is the constraint on the amount of time it takes to empty the tank if the rancher does not refill it weekly?

10. The carrying capacity of an ecosystem is 30 breeding pairs of cottontail rabbits. The growth in the rabbit population can be estimated by the equation $N(t) = N_c(1 - 3^{-0.3t})$, in which N_c is the carrying capacity of breeding pairs of the ecosystem, $N(t)$ is the number of breeding pairs after time t has elapsed, and t is the time in years. What are the constraints on the variables in the equation?

Lesson 4B.1.3: Rearranging Formulas

Introduction

Mathematical formulas that involve exponents, logarithms, powers, and roots have constraints on the values that can be assigned or associated with two or more variables. Real-world formulas involving measurements like area, length, time, and volume have additional restraints, whereas others (such as current, direction, and voltage) can have positive and negative values in addition to the limits imposed by real-world relationships. Rearranging variables in a formula can result in changes to these constraints, so the result of rearranging variables has to be checked and redefined in those cases. Formulas can often be analyzed and visualized in the same way that equations and functions can, and the data resulting from formulas can be presented as data in tables or in graphs.

Key Concepts

- A formula relates two or more variables in a mathematical or a real-world problem context.

- The values of the variables in a formula can be constrained or limited by mathematical or real-world conditions. For example, the formula $A = \frac{1}{2}bh$ for the area of a triangle contains three quantities that are positive. If a triangle is graphed on a coordinate plane and its orientation on the plane results in either b or h being assigned a negative value, then the other variable must also be assigned a negative value so that the area of the triangle is positive.

- Rearranging a formula can leave the constraints on variable values the same or it can change them. For example, in the distance-rate-time formula $d = rt$, time is constrained by the condition $t \geq 0$, with $t = 0$ signifying the time at which motion begins. The formula can be rewritten as $r = \dfrac{d}{t}$; however, $r = \dfrac{d}{t}$ is undefined at $t = 0$, so in this case $t > 0$. This corresponds to the real-world condition that $r \neq 0$ only when $t \neq 0$, since at $t = 0$, there is no distance covered and no motion.

- Formulas can be rearranged to express one or more variables in terms of another variable. A rearranged formula can reduce the number of variables to be calculated.

Guided Practice 4B.1.3

Example 1

The pressure and temperature inside an insulated hot-beverage bottle is related to the volume of the bottle and the amount of beverage in it by a real-world form of the ideal gas law. The ideal gas law is given by the formula $PV = nRT$, in which n is the number of moles (a unit of counting) of the gas in a container, P is the pressure the gas exerts on the container, V is the volume of the container, and T is the temperature in degrees Kelvin. The only constant in the formula is R, which is the ideal, or universal, gas constant. Rearrange the formula to show how the temperature T is affected by doubling each variable n, P, and V. (*Note*: All of the quantities in the formula are nonzero.)

1. Isolate temperature, T, in the given formula, $PV = nRT$.

   Use division to isolate T on one side of the equation.

   $PV = nRT$ Given formula

   $\dfrac{PV}{nR} = \dfrac{\cancel{nR}T}{\cancel{nR}}$ Divide both sides of the formula by nR.

   $\dfrac{PV}{nR} = T$ Simplify.

   $T = \dfrac{PV}{nR}$ Apply the Symmetric Property of Equality.

   The formula $PV = nRT$, isolated for T, is $T = \dfrac{PV}{nR}$.

2. Determine how T is affected in the rearranged formula if n is doubled and P and V stay the same.

We are given that R is a constant, so it will not be affected by changes to the other variables.

Let the original value of n be n_1. If n_1 is doubled, the resulting value of n is $2n_1$.

Write the revised ideal gas law from step 1 for both conditions.

Formula rewritten for n_1 (the original value of n):

$$T_1 = \frac{PV}{n_1 R}$$

Formula rewritten for $2n_1$ (twice the original value of n):

$$T_2 = \frac{PV}{2n_1 R}$$

Factor out $\dfrac{1}{2}$ from $T_2 = \dfrac{PV}{2n_1 R}$:

$$T_2 = \frac{1}{2} \bullet \frac{PV}{n_1 R}$$

Notice that T_2 is half of T_1. In other words, *doubling* the value of n reduces the temperature by *half*. In general, increasing n will decrease T if P and V stay the same.

3. Determine how T is affected in the rearranged formula if P is doubled and n and V stay the same.

Let the original value of P be P_1. If P_1 is doubled, the resulting value of P is $2P_1$.

Write the revised ideal gas law from step 1 for both conditions.

Formula rewritten for P_1 (the original value of P):

$$T_1 = \frac{P_1 V}{nR}$$

Formula rewritten for $2P_1$ (twice the original value of P):

$$T_2 = \frac{2P_1 V}{nR}$$

Factor out 2 from $T_2 = \dfrac{2P_1 V}{nR}$:

$$T_2 = 2 \bullet \frac{P_1 V}{nR}$$

Notice that T_2 is twice T_1. In other words, *doubling* the value of P *doubles* the temperature. In general, increasing P increases T if n and V stay the same.

4. Determine how T is affected in the rearranged formula if V is doubled and n and P stay the same.

Note that V, like P, is a factor of the numerator of the formula $T = \dfrac{PV}{nR}$.

Since both V and P are in the numerator, changes in the volume, V, would have the same effect on temperature, T, as changes in pressure, P. In other words, if we were to double the volume instead of the pressure, the temperature would still be doubled.

Therefore, increasing V increases T if n and P stay the same.

Example 2

The distance, d, an object moves in one direction can be described by the formula $d = vt + \frac{1}{2}at^2$, in which v is the velocity of the object when timing starts, a is the acceleration of the object when timing starts, t is the duration of the motion, and d is the initial distance of the object relative to some arbitrary reference point (e.g., the origin on a coordinate plane) when the timing starts. Rearrange the formula for d so that it can be solved for t using the quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. In this problem, the quantities a, d, and v are constants that can be positive, negative, or 0. Apply the condition $t > 0$ to the resulting values of t and interpret the result(s) for real-world motion if the following values are given: $a = -2$ meters per second squared, $d =$ 10 meters from a reference point, and $v = 5$ meters per second. Describe the motion of the object using these values.

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1. Rewrite the given formula for d so that all of the terms are one side of the equation.

   Rewriting the formula so all terms are on one side of the equation corresponds to having 0 on one side of the equation. Results that are in the form $ax^2 + bx + c = 0$ reveal the values to be substituted into the quadratic formula.

   $$d = vt + \frac{1}{2}at^2 \qquad \text{Given equation}$$

   $$0 = \frac{1}{2}at^2 + vt - d \qquad \text{Subtract } d \text{ from both sides of the equation.}$$

   $$0 = at^2 + 2vt - 2d \qquad \text{Multiply both sides of the equation by 2.}$$

   $$at^2 + 2vt - 2d = 0 \qquad \text{Apply the Symmetric Property of Equality.}$$

   $at^2 + 2vt - 2d = 0$ is a quadratic in the form $ax^2 + bx + c = 0$.

2. Identify the values of the resulting equation that correspond to a, b, c, and x in the quadratic formula.

The quadratic formula is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. For the equation

$at^2 + 2vt - 2d = 0$, $a = a$, $b = 2v$, $c = -2d$, and $x = t$.

3. Use the quadratic formula and the values determined in the previous step to write an equation that can be solved for t.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quadratic formula

$$(t) = \frac{-(2v) \pm \sqrt{(2v)^2 - 4(a)(-2d)}}{2(a)}$$

Substitute a for a, $2v$ for b, $-2d$ for c, and t for x.

$$t = -\frac{v}{a} \pm \frac{\sqrt{4v^2 + 8ad}}{2a}$$

Simplify.

$$t = -\frac{v}{a} \pm \frac{\sqrt{2(2v^2 + 4ad)}}{2a}$$

Factor out 2 from under the radical sign.

$$t = -\frac{v}{a} \pm \frac{\sqrt{v^2 + 2ad}}{a}$$

Divide.

The resulting equation is $t = -\dfrac{v}{a} \pm \dfrac{\sqrt{v^2 + 2ad}}{a}$.

4. Apply the real-world condition $t > 0$ to the "positive" result of the simplified quadratic equation.

Taking the positive result of the radical expression of the inequality

$-\dfrac{v}{a} \pm \dfrac{\sqrt{v^2 + 2ad}}{a} > 0$ yields $-\dfrac{v}{a} + \dfrac{\sqrt{v^2 + 2ad}}{a} > 0$.

$-\dfrac{v}{a} + \dfrac{\sqrt{v^2 + 2ad}}{a} > 0$ simplifies to $\dfrac{\sqrt{v^2 + 2ad}}{a} > \dfrac{v}{a}$.

This result can also be interpreted on the basis of the signs of the quantities a, d, and v. The radical sign also places a mathematical constraint on the relationship of d to v: $v^2 + 2ad \geq 0$ or $v^2 \geq -2ad$. Therefore, $a < 0$, $v > 0$, and $d < 0$.

5. Apply the real-world condition $t > 0$ to the "negative" result of step 3.

Taking the negative result of the radical expression of the inequality

$$-\frac{v}{a} \pm \frac{\sqrt{v^2 + 2ad}}{a} > 0 \text{ yields } -\frac{v}{a} - \frac{\sqrt{v^2 + 2ad}}{a} > 0.$$

$$-\frac{v}{a} - \frac{\sqrt{v^2 + 2ad}}{a} > 0 \text{ simplifies to } \frac{\sqrt{v^2 + 2ad}}{a} < -\frac{v}{a}.$$

This result can also be interpreted on the basis of the signs of the quantities a, d, and v. The radical sign also places a mathematical constraint on the relationship of d to v: $v^2 + 2ad > 0$ or $v^2 \geq -2ad$. Therefore, $a < 0$, $v > 0$, and $d < 0$.

6. Find t for the given values of $a = -2$ meters per second squared, $d = 10$ meters from a reference point, and $v = 5$ meters per second. Describe the motion of the object using these values.

Substitute these values into the simplified quadratic equation for t from step 3.

$$t = -\frac{v}{a} \pm \frac{\sqrt{v^2 + 2d}}{a} \qquad \text{Simplified quadratic equation}$$

$$t = -\frac{(5)}{(-2)} \pm \frac{\sqrt{(5)^2 + 2(10)}}{(-2)} \qquad \text{Substitute } -2 \text{ for } a, 10 \text{ for } d, \text{ and } 5 \text{ for } v.$$

$$t = 2.5 \pm \frac{\sqrt{45}}{2} \qquad \text{Simplify.}$$

$$t = 2.5 \pm 1.5\sqrt{5}$$

$$t \approx -0.85 \text{ or } t \approx 5.85$$

These results give $t \approx -0.85$ or $t \approx 5.85$. Since $t > 0$, only the positive value for t is relevant.

Since only the positive value for t is relevant, the relationships found in step 4 for the positive radical expression should apply: $a < 0$ and $v > 0$.

Note that the given value for a is less than 0 and the given value for v is greater than 0, which follows the pattern of the restrictions on a and v.

7. Use the value of t found in the previous step to calculate d and compare it to the conditions in step 6. Interpret the result for d.

Remember that for the positive radical expression, the value of d for $a < 0$ and $v > 0$ should be given by $d < 0$.

$$d = vt + \frac{1}{2}at^2$$ Original formula

$$d \approx (5)(5.85) + \frac{1}{2}(-2)(5.85)^2$$ Substitute −2 for a, 5.85 for t, and 5 for v.

$$d \approx -4.97$$ Simplify.

This value of d meets the criteria that $d < 0$.

Interpreting these values in the context of the problem, the object started out at 10 meters in a positive direction from the reference point. The acceleration and velocity result in motion that takes the object *back* (a negative distance) toward the reference point by almost 5 meters.

Example 3

The formula for a standard earthquake-body wave scale, m_b, is given by $m_b = \log\left(\dfrac{A}{T}\right) + Q$, in which A is the amplitude of the ground motion in microns (10^{-6} meter), T is the period of the wave, and Q is a correction constant. Determine a formula for the frequency of the earthquake wave if the frequency F is defined as the reciprocal of the wave period. Rearrange the formula to find the range of F values for when the range of T values is [4, 5] seconds. Then, rearrange the formula to find the range of values of A for when the range of values of m_b is [6, 9], the range of T values is [4, 5] seconds, and $Q = 2$.

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1. Substitute an expression for F in place of T in the formula.

   Frequency F can be written as $F = \dfrac{1}{T}$. The argument of the logarithm $\dfrac{A}{T}$ can be written as $A\left(\dfrac{1}{T}\right)$. Therefore, the argument of the logarithm can be written as AF.

   $$m_b = \log\left(\frac{A}{T}\right) + Q \qquad \text{Original formula}$$

   $$m_b = \log(AF) + Q \qquad \text{Substitute } AF \text{ for } \frac{A}{T}.$$

   The formula, using F in place of T, is $m_b = \log(AF) + Q$.

2. Use the definition of the logarithm to rewrite the formula with an exponential term. Then, solve it for F.

   Isolate the logarithm from the rest of the formula, then simplify the result and solve it for F.

   $m_b = \log(AF) + Q$ Modified formula

   $m_b - Q = \log(AF)$ Subtract Q from both sides.

   $10^{m_b - Q} = AF$ Definition of the logarithm

   $\dfrac{10^{m_b}}{10^Q} = AF$ Simplify the exponential term.

   $F = \dfrac{10^{m_b}}{A \bullet 10^Q}$ Divide both sides by A.

   The rewritten formula is $F = \dfrac{10^{m_b}}{A \bullet 10^Q}$.

3. Use the relationship between the period and the frequency to calculate the range of frequency values.

   The period T is related to the frequency F by the relationship $F = \dfrac{1}{T}$.

   If the range of T is [4, 5] seconds, then the range of F values is the

   reciprocal of the period interval values, or $\left[\dfrac{1}{5}, \dfrac{1}{4}\right]$.

4. Write the formula for the earthquake-magnitude scale so that the factors with known values are isolated from A.

To find the range of values of A for when the range of values of m_b is [6, 9] and $Q = 2$, first substitute 2 for Q in the rewritten formula, $F = \dfrac{10^{m_b}}{A \cdot 10^Q}$, and then rearrange the formula so that it's in terms of A.

$F = \dfrac{10^{m_b}}{A \cdot 10^Q}$	Rewritten formula
$F = \dfrac{10^{m_b}}{A \cdot 10^{(2)}}$	Substitute 2 for Q.
$F = \dfrac{10^{m_b - 2}}{A}$	Simplify using the rules of exponents.
$AF = 10^{m_b - 2}$	Multiply both sides by A.
$A = \dfrac{10^{m_b - 2}}{F}$	Divide both sides by F.
$A = \dfrac{10^{m_b - 2}}{\left(\dfrac{1}{T}\right)}$	Substitute $\dfrac{1}{T}$ for F.
$A = T \cdot 10^{m_b - 2}$	Simplify.

The formula, written in terms of A when $Q = 2$, is $A = T \cdot 10^{m_b - 2}$.

5. Use the result of the previous step to write the maximum and minimum values of the range of values for A.

Use the formula, $A = T \cdot 10^{m_b - 2}$, to determine the minimum and maximum values using the given ranges of T and m_b.

The range values for T are given as [4, 5], so the minimum value for T is 4 and the maximum value is 5.

The range values for m_b are given as [6, 9]. Therefore, the minimum value for m_b is 6 and the maximum value is 9.

Substitute the minimum values, $T = 4$ and $m_b = 6$:

$$A_{\text{minimum}} = T \cdot 10^{m_b - 2} \qquad \text{Formula for the minimum value of } A$$

$$A_{\text{minimum}} = (4) \cdot 10^{(6) - 2} \qquad \text{Substitute 4 for } T \text{ and 6 for } m_b.$$

$$A_{\text{minimum}} = 4 \cdot 10^4 \qquad \text{Simplify.}$$

The minimum value of A is $4 \cdot 10^4$, or about 40,000 centimeters.

Substitute the maximum values, $T = 5$ and $m_b = 9$:

$$A_{\text{maximum}} = T \cdot 10^{m_b - 2} \qquad \text{Formula for the maximum value of } A$$

$$A_{\text{maximum}} = (5) \cdot 10^{(9) - 2} \qquad \text{Substitute 5 for } T \text{ and 9 for } m_b.$$

$$A_{\text{maximum}} = 5 \cdot 10^7 \qquad \text{Simplify.}$$

The maximum value of A is $5 \cdot 10^7$, or about 50 meters.

Example 4

The diffusion rate of a gas in the combustion chamber of a diesel engine is directly proportional to the square root of the molecular mass of the gas. This relationship is given by the formula $r = k \cdot \sqrt{m}$, in which k is a constant. Find the diffusion rate for two gases A and B if the molecular mass of gas A is three more than two times the molecular mass of gas B. Describe how the resulting two rates are related.

1. Use the variables m_A, m_B, r_A, and r_B to represent the variables and write the formulas for gas A and gas B.

   Substitute the variables and write the formulas.

   For gas A:

   $r = k \cdot \sqrt{m}$ Original formula

   $r_A = k \cdot \sqrt{m_A}$ Represent r as r_A and m as m_A.

   For gas B:

   $r = k \cdot \sqrt{m}$ Original formula

   $r_B = k \cdot \sqrt{m_B}$ Represent r as r_B and m as m_B.

2. Write an equation for the molecular mass of gas A in terms of the molecular mass of gas B.

   We are given that the mass of gas A is three more than two times the mass of gas B.

   This can be written as $m_A = 2 \cdot m_B + 3$.

3. Substitute the result of step 2 into the formula for gas A.

   $r_A = k \cdot \sqrt{m_A}$ becomes $r_A = k \cdot \sqrt{2 \cdot m_B + 3}$.

4. Rearrange the formulas for r_A and r_B to isolate k.

For gas A:

$$r_A = k \bullet \sqrt{2 \bullet m_B + 3}$$ Rewritten formula for r_A

$$\frac{r_A}{\sqrt{2 \bullet m_B + 3}} = k$$ Divide both sides of the formula by the radical expression.

For gas B:

$$r_B = k \bullet \sqrt{m_B}$$ Formula for r_B

$$\frac{r_B}{\sqrt{m_B}} = k$$ Divide both sides of the formula by the radical expression.

5. Set the resulting formulas equal to each other.

Since both formulas are equivalent to k, they are equivalent to each other.

$$\frac{r_A}{\sqrt{2 \bullet m_B + 3}} = \frac{r_B}{\sqrt{m_B}}$$

6. Rearrange the resulting equation to solve for r_A.

$$\frac{r_A}{\sqrt{2 \bullet m_B + 3}} = \frac{r_B}{\sqrt{m_B}}$$ Equation from the previous step

$$r_A = \frac{r_B \bullet \sqrt{2 \bullet m_B + 3}}{\sqrt{m_B}}$$ Multiply both sides by the denominator on the left side of the equation to isolate r_A.

$$r_A = r_B \bullet \sqrt{\frac{2m_B + 3}{m_B}}$$ Simplify.

$$r_A = r_B \bullet \sqrt{2 + \frac{3}{m_B}}$$ Continue to simplify.

7. Describe how the two rates are related.

The rate of diffusion of gas A is greater than the rate of gas B by a factor of $\sqrt{2+\dfrac{3}{m_B}}$. This is another way of writing the ratio of the square root of the molecular mass of gas A to the square root of the molecular mass of gas B.

✓

Practice 4B.1.3: Rearranging Formulas

For problems 1–4, rearrange the given formulas to complete the problems.

1. If $V = \dfrac{2L}{k}$, $I = k^2L$, and $P = VI$, write k in terms of P and L.

2. If $\log k = \log A - \dfrac{E_a \bullet A}{2.3RT}$, write a formula for E_a.

3. Find $[A^-]$ if $pH = pK_a + \log\left(\dfrac{[A^-]}{[HA]}\right)$, where $[A^-]$ and $[HA]$ are ions.

4. In optics, the focal length of a thin lens, f, is related to the distance of the object being imaged from the lens, o, and the distance of the image formed from the lens, i, by the formula $\dfrac{1}{f} = \dfrac{1}{i} + \dfrac{1}{o}$. Rearrange the formula to find i. Under what condition is $i < 0$ if $f > 0$?

Use the given information to complete problems 5–10. Rearrange the formulas if necessary.

5. The escape velocity of an object moving in Earth's orbit is given by the formula $v_{escape} = \sqrt{\dfrac{2GM_E}{r_E}}$, in which G is a constant, M_E is the approximate mass of Earth, and r_E is the *average* radius of Earth. How does the value of v_{escape} vary if the difference of Earth's actual radius and its average radius is given by the inequality $|r_{actual} - r_E| < 7500$?

continued

6. The impedance, a measure of resistance in an alternating current circuit, is given by the formula $Z = \sqrt{R^2 + (\chi_L - \chi_C)^2}$, in which the three unknowns are resistances of different types of circuit elements. Rewrite the formula to find χ_L if the other two variables are known.

7. In microbiology, the growth rate constant μ of a colony of bacteria related to the time for one generation to grow and mature t_{gen} is given by the formula $\mu = \dfrac{\ln 2}{t_{gen}}$. Rewrite the formula for μ as a base-10 exponential formula.

8. The formula for the ideal gas law, $PV = nRT$, relates the number of moles of a gas, n, to its pressure P, temperature T, and the volume V of the container in which it is held. The quantity R is a constant. All of the quantities are positive. Problems using the ideal gas law are often accompanied by the conditional phrase "at standard temperature and pressure." Rewrite the ideal gas law with P and T on one side of the equation, and describe the meaning of the rewritten formula.

9. How are the three variables [A], [B], and [AB] related if $K_{eq} = \dfrac{[A] \bullet [B]}{[AB]}$ and $K_{eq} < 1$? (*Note*: [A], [B], and [AB] are the concentrations of the elements A and B and the compound AB; e.g., moles per liter, grams per milliliter, etc.)

10. To study hereditary traits, biologists often look closely at alleles, or alternative forms of genes. In population genetics, the frequency of the first allele, p, is related to the frequency of the second allele, q, by the formula $p^2 + 2pq + q^2 = 1$. Write p in terms of q. Then, find p if q is half of p. (*Note*: $p > 0$ and $q > 0$.)

Lesson 2: Transforming a Model and Combining Functions

Common Core State Standards

F–BF.1   Write a function that describes a relationship between two quantities.★

  b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.*

F–BF.3   Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

Essential Questions

1. How can you tell if a parent function, $f(x)$, has been changed by the transformations $g(x) = k \bullet f(x)$ or $g(x) = f(x) + k$?

2. How do the transformations $g(x) = f(kx)$ and $g(x) = f(x + k)$ differ in their effect on a parent function, $f(x)$?

3. How do you distinguish between odd and even functions based on their algebraic expressions and their graphs?

4. What function results when you add, subtract, multiply, or divide two or more functions?

5. What function results when you use a function, $f(x)$, as the independent variable in another function, $g(x)$?

WORDS TO KNOW

combination of functions   the process of combining two or more functions using the operations of addition, subtraction, multiplication, or division to create a new function

composition of functions	the process of substituting one function for the independent variable of another function to create a new function
even function	a function that, when evaluated for $-x$, results in a function that is the same as the original function; $f(-x) = f(x)$
family of functions	a set of functions whose graphs have the same general shape as their parent function. The parent function is the function with a simple algebraic rule that represents the family of functions.
odd function	a function that, when evaluated for $-x$, results in a function that is the opposite of the original function; $f(-x) = -f(x)$
parent function	a function with a simple algebraic rule that represents a family of functions. The graphs of the functions in the family have the same general shape as the parent function.

Recommended Resources

- MathIsFun.com. "Operations with Functions."

  http://www.walch.com/rr/00240

  This site uses a scaffolded, step-by-step approach to review basic operations with functions.

- Purplemath.com. "Function Notation: Even and Odd."

  http://www.walch.com/rr/00241

  This site provides a review of how to determine whether a function is even, odd, or neither using algebraic methods and by evaluating the function's graph. The sample functions are accompanied by illustrations that would be helpful to visual learners.

- SheLovesMath.com. "Parent Functions and Transformations."

  http://www.walch.com/rr/00242

  This comprehensive site reviews basic parent functions and transformations of those functions. The site provides a balanced mix of scaffolded examples and visuals that will help users with a variety of learning-style preferences.

IXL Links

- Transformations of quadratic functions:
  http://www.ixl.com/math/algebra-1/transformations-of-quadratic-functions

- Transformations of absolute value functions:
  http://www.ixl.com/math/algebra-1/transformations-of-absolute-value-functions

- Translations of functions:
  http://www.ixl.com/math/algebra-2/translations-of-functions

- Reflections of functions:
  http://www.ixl.com/math/algebra-2/reflections-of-functions

- Dilations of functions:
  http://www.ixl.com/math/algebra-2/dilations-of-functions

- Transformations of functions:
  http://www.ixl.com/math/algebra-2/transformations-of-functions

- Add and subtract functions:
  http://www.ixl.com/math/algebra-2/add-and-subtract-functions

- Multiply functions:
  http://www.ixl.com/math/algebra-2/multiply-functions

- Divide functions:
  http://www.ixl.com/math/algebra-2/divide-functions

Lesson 4B.2.1: Transformations of Parent Graphs

Introduction

Functions can be transformed, or changed, by performing operations on the dependent and independent variable. For example, adding a constant k to the *independent* variable of a function results in a new function of the form $g(x) = f(x + k)$. On the other hand, adding a constant to the *dependent* variable of a function results in a new function of the form $g(x) = f(x) + k$. Similarly, multiplying the independent variable by a constant yields a new function of the form $g(x) = f(kx)$, while multiplying the dependent variable by the constant yields a new function of the form $g(x) = k \bullet f(x)$. The operation performed, and the variable upon which it is performed, can result in various changes to the graph of the function.

Key Concepts

- Recall that a parent function is a function with a simple algebraic rule that represents a family of functions. The parent function is the simplest form of a function that is changed by the addition or multiplication of constants.

- For linear and quadratic functions, the parent functions can be defined as $f(x) = x$ and $f(x) = x^2$, respectively.

- A **family of functions** is a set of functions whose graphs have the same general shape as their parent function.

- For example, the functions $f(x) = 25x$, $g(x) = 75x$, and $h(x) = 110x$ constitute a family of functions, that are of the general forms $a(x) = k \bullet b(x)$ or $c(x) = b(kx)$.

- Transformations of the dependent or independent variables of a function can shift the location of the function on the coordinate plane. Or, they can make the end behavior of the function more pronounced or less pronounced.

- Transformations can also change the domain and range of the parent function. Solutions to real-world problems involving transformations should include a check of the domain and range values to ensure that the solutions continue to reflect the realities of the problem conditions.

- Graphing a transformed function and its corresponding parent function on a coordinate plane can help show the similarities and differences of the characteristics of each function.

- Three types of transformations can be demonstrated algebraically or graphically.

Case 1: Addition of a Constant

- A constant can be added to the dependent or independent variable.

- The transformation $g(x) = f(x + k)$ adds a constant, k, to the value of the independent variable, x.

- The transformation $g(x) = f(x) + k$ adds a constant, k, to the value of the dependent variable, $f(x)$.

- The functions $g(x) = 5^{x+5}$ and $h(x) = 5^x + 5$ are such transformations of the function $f(x) = 5^x$. All three functions are shown on the following graph. Notice how the transformed functions $g(x)$ and $h(x)$ have been shifted along the axes.
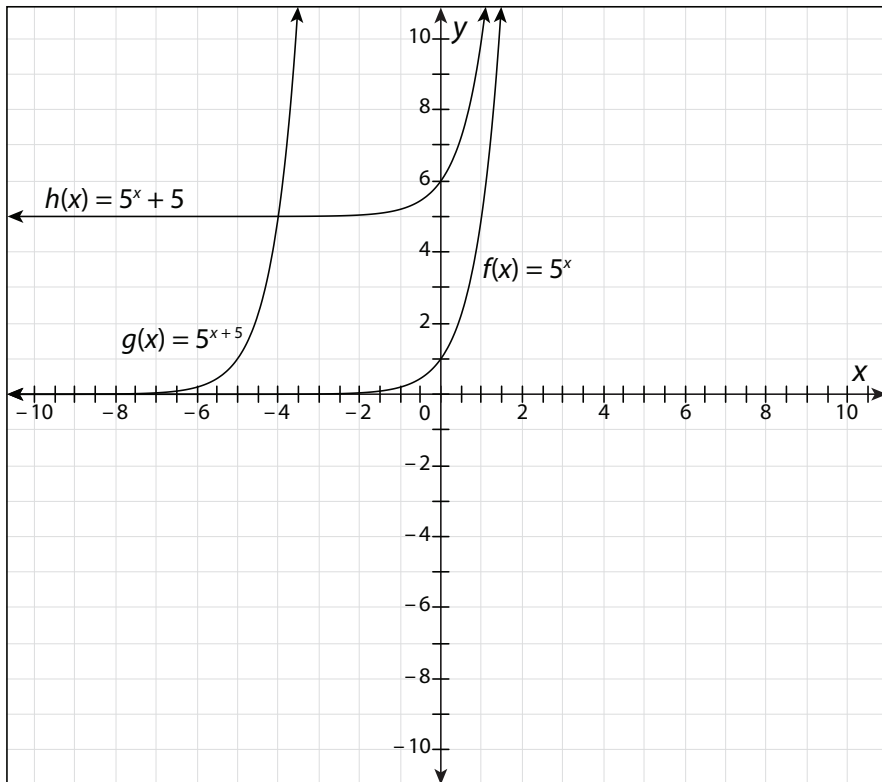
Case 2: Multiplication by a Constant

- The dependent or independent variable can be multiplied by a constant.

- The transformation $g(x) = f(kx)$ multiplies the independent variable, x, by a constant, k.

- The transformation $g(x) = k \bullet f(x)$ multiplies the value of the dependent variable, $f(x)$, by a constant, k.

- The functions $g(x) = 3x$ and $h(x) = -6x$ are such transformations of the function $f(x) = x$. All three functions are shown on the following graph.
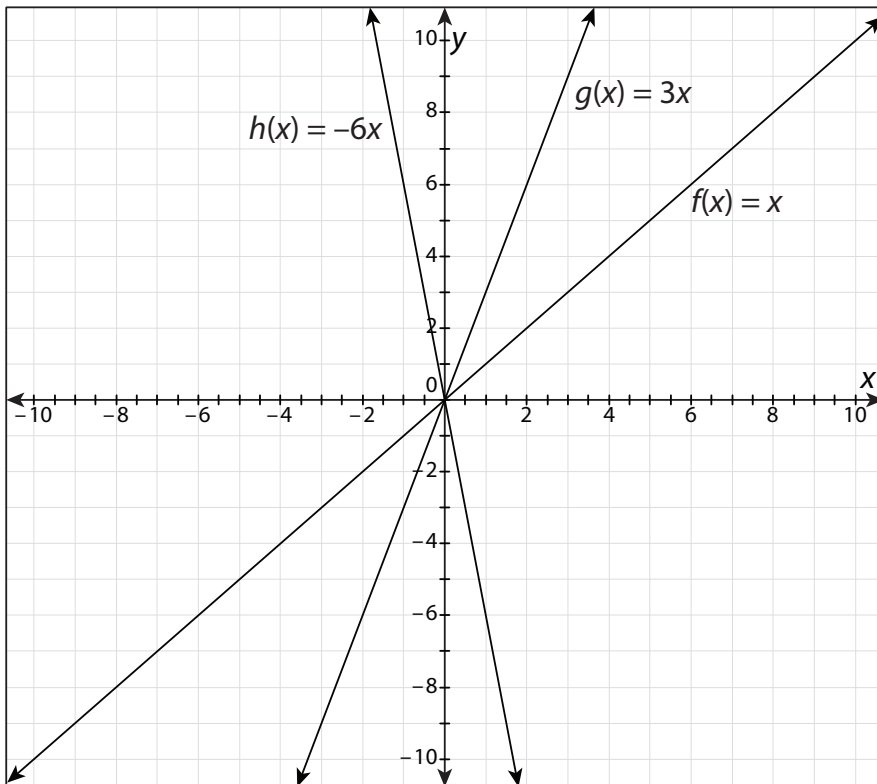
Case 3: Addition of a Constant and Multiplication by a Constant

- The first two cases can be combined to show both addition and multiplication.

- For example, the quadratic function $g(x) = 2(x - 3)^2 + 5$ represents two addition transformations and a multiplication transformation of the parent function $f(x) = x^2$.

  - First, 5 is added to the function:
    $f(x) + 5 = x^2 + 5$

  - Then, 3 is subtracted from x:
    $f(x - 3) + 5 = (x - 3)^2 + 5$

  - Finally, the x-squared term is multiplied by 2:
    $2[f(x - 3)] + 5 = 2[(x - 3)^2] + 5$

  - The resulting function is renamed $g(x)$:
    $g(x) = 2[f(x - 3)] + 5 = 2(x - 3)^2 + 5$

- The two functions $f(x)$ and $g(x)$ are shown on the following graph.
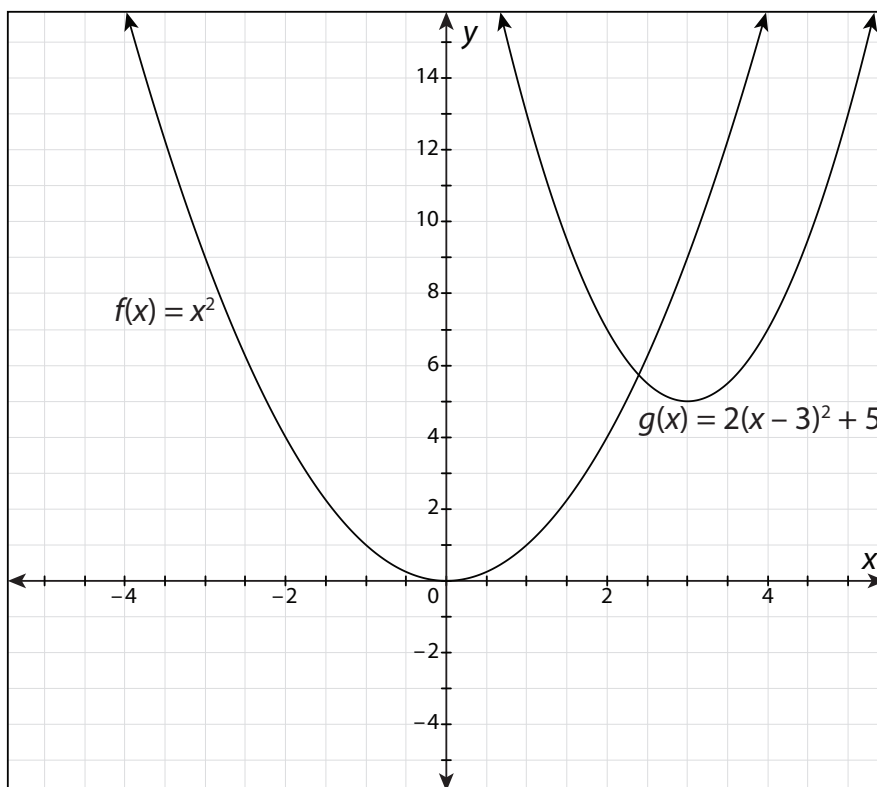
- A graphing calculator also provides an efficient way to change coefficients as well as add terms to functions. The following directions demonstrate how to do this on the TI-83/84 and TI-Nspire calculators.

On a TI-83/84:

Step 1: Graph the family of functions of the form $f(x) = ax + b$ for three values of a ($a = 1, 2,$ and 3) and one value of b ($b = 1$). Then, graph the family of functions for three values of b ($b = 2, 4,$ and 6) and one value of a ($a = 1$).

Step 2: Press [Y=]. Then, enter the *form* of the function using the "list" tool for the coefficient a at the cursor next to "\Y1=" and enter the three values of a in the "list" space in the equation: [2ND][(][1][,][2][,][3][2ND][)][X, T, θ, n][+][1].

Step 3: Press [ENTER] to save the equations. Then, press [GRAPH].

Step 4: Next, press [Y=] and move the cursor to the "\Y2=" line. Enter the *form* of the second family of functions using the "list" tool for the constant that is added to the x term: [X, T, θ, n][+][2ND][(][2][,][4][,][6][2ND][)].

Step 5: Press [ENTER] to save the equations. Then, press [GRAPH].

On a TI-Nspire:

Step 1: Graph the family of functions of the form $f(x) = ax + b$ for three values of a ($a = 1, 2,$ and 3) and one value of b ($b = 1$). Then, graph the family of functions for three values of b ($b = 2, 4,$ and 6) and one value of a ($a = 1$).

Step 2: Press [home]. Arrow down to the graphing icon and press [enter].

Step 3: Enter the *form* of the function using the "list" tool for the coefficient a at the cursor next to "$f1(x) =$" and enter the three values of a in the "list" space in the equation: [ctrl][)][1][,][2][,][3]. Then, press the right arrow key to move your cursor outside of the "list" field and continue entering the equation.

Step 4: Press [enter] to graph the equations.

Step 5: To graph the second family of functions, press [menu] and arrow down to 3: Graph Type, then arrow right to 1: Function. Press [enter].

Step 6: Enter the *form* of the second family of functions at the cursor next to "$f2(x) =$" using the "list" tool for the constant that is added to the x term: [X][+][ctrl][)][2][,][4][,][6]. Then, press the right arrow key to move your cursor outside of the "list" field and continue entering the equation.

Step 7: Press [enter] to graph the equations.

Guided Practice 4B.2.1

Example 1

Compare the graphs of the exponential functions $a(x) = e^{x+2}$, $b(x) = e^x + 2$, $c(x) = e^{2x}$, and $d(x) = 2 \cdot e^x$. Compare each graph's x- and y-intercepts, domain, and range to those of the parent function, $f(x) = e^x$. Finally, determine the value of x at the intersection point of the functions $a(x)$ and $b(x)$.

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1. Graph each of the given functions on a graphing calculator.

   The resulting graph should appear as follows.

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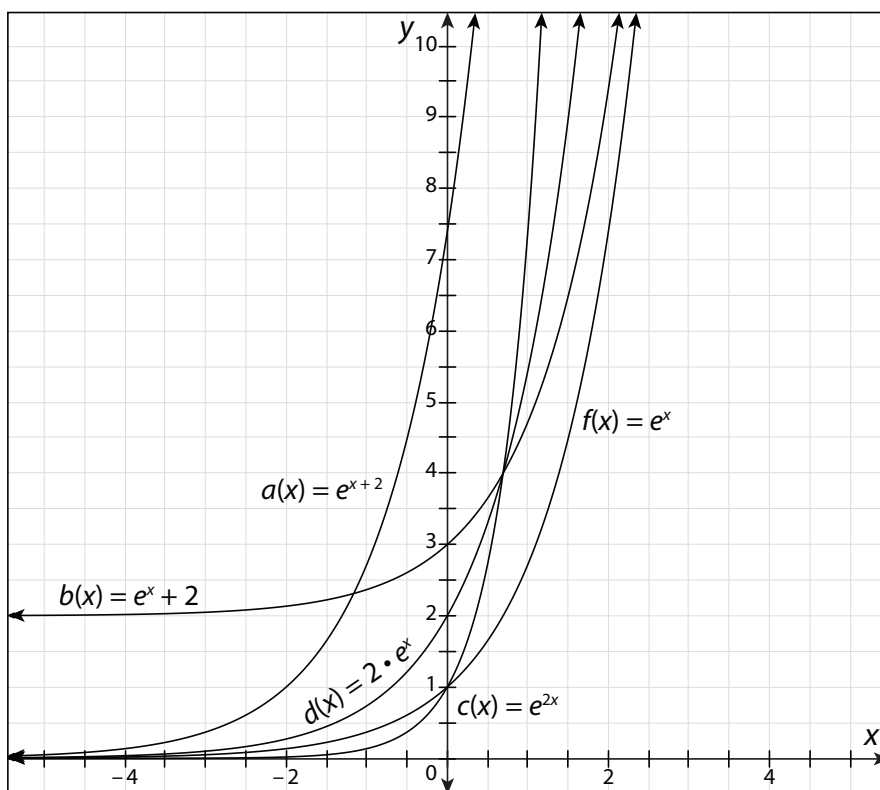
2. Determine and compare the domains and ranges of the functions.

The domain and range values can be inferred from the table of *x*- and *y*-values of the graphing calculator or by looking at the graph of each function.

Function	Domain	Range
$f(x) = e^x$	$(-\infty, \infty)$	$(0, \infty)$
$a(x) = e^{x+2}$	$(-\infty, \infty)$	$(0, \infty)$
$b(x) = e^x + 2$	$(-\infty, \infty)$	$(2, \infty)$
$c(x) = e^{2x}$	$(-\infty, \infty)$	$(0, \infty)$
$d(x) = 2 \cdot e^x$	$(-\infty, \infty)$	$(0, \infty)$

Each transformed function has the same domain as the parent function, $f(x)$. All but $b(x)$ have the same range as the parent function.

3. Determine and compare the *y*-intercepts of the functions.

To determine the *y*-intercept of each function, find the value of each function at $x = 0$.

It can be seen from the graph that there are four different *y*-intercepts, but the values can also be shown algebraically by substituting 0 for *x* in each function.

$f(x) = e^x$ Given function

$f(0) = e^{(0)}$ Substitute 0 for *x*.

$f(0) = 1$ Simplify.

The *y*-intercept of $f(x) = e^x$ is (0, 1).

Follow the same process for the transformed functions.

$a(x) = e^{x+2}$	$b(x) = e^x + 2$	$c(x) = e^{2x}$	$d(x) = 2 \cdot e^x$
$a(0) = e^{[(0)+2]}$	$b(0) = e^{(0)} + 2$	$c(0) = e^{2(0)}$	$d(0) = 2 \cdot e^{(0)}$
$a(0) = e^2$	$b(0) = 1 + 2$	$c(0) = e^0$	$d(0) = 2 \cdot 1$
$a(0) \approx 7.4$	$b(0) = 3$	$c(0) = 1$	$d(0) = 2$

(continued)

For each transformed function, write the *y*-intercept.

- $a(x) = e^{x+2}$: approximately (0, 7.4)

- $b(x) = e^x + 2$: (0, 3)

- $c(x) = e^{2x}$: (0, 1)

- $d(x) = 2 \cdot e^x$: (0, 2)

The *y*-intercept of $c(x)$ is the same as that of the parent function, $f(x)$; all the other transformations have been shifted up along the *y*-axis.

4. Determine and compare the *x*-intercepts of the functions.

   To find the *x*-intercepts, solve each function for $y = 0$.

   None of the functions exist at $y = 0$. The function $b(x)$ does not exist for values of *y* that are less than or equal to 2. It can be seen from the graph that the other functions approach but do not reach $y = 0$.

5. Determine the value of *x* at the intersection point of the functions $a(x)$ and $b(x)$.

   At their intersection point, $a(x) = b(x)$. Therefore, set the right sides of the equations for $a(x)$ and $b(x)$ equal and solve for *x*.

$e^{x+2} = e^x + 2$	Set $a(x)$ equal to $b(x)$.
$e^{x+2} - e^x = 2$	Subtract e^x from both sides.
$e^x(e^2 - 1) = 2$	Rewrite using the Distributive Property.
$e^x = \dfrac{2}{e^2 - 1}$	Divide both sides by $e^2 - 1$.
$x = \ln\left(\dfrac{2}{e^2 - 1}\right)$	Solve for *x* using natural logarithms.
$x = \ln 2 - \ln(e^2 - 1)$	Simplify.
$x \approx -1.16$	

   The value of *x* at the intersection point of the functions $a(x)$ and $b(x)$ is approximately -1.16.

Example 2

The graph shows a parent quadratic function $f(x) = x^2$ and two quadratic functions, $g(x)$ and $h(x)$, derived from it. Use the maximum or minimum point of the quadratic functions to derive their equations, and write the functions in a form that indicates the transformation(s) of the parent function. Then, describe what transformation(s) the parent function underwent to result in each transformed function.
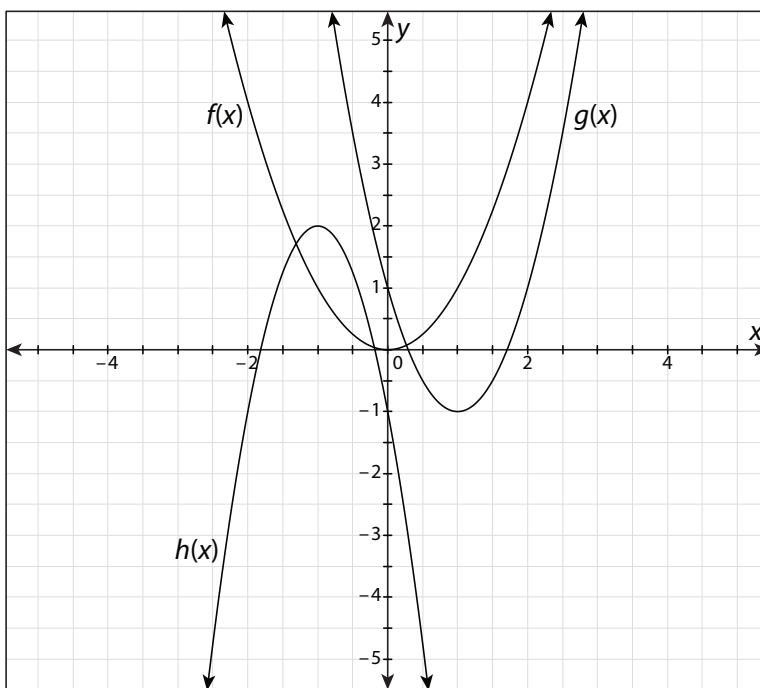
1. Use the vertex form of a quadratic function, $f(x) = a(x - h)^2 + k$, to determine the equation of the transformed function $g(x)$.

   Replace $f(x)$ in the equation of the vertex form with y:

   $y = a(x - h)^2 + k$

   Recall that the vertex of the graph of the quadratic function is also known as the maximum or minimum point of the function and is represented by the coordinate (h, k). (x, y) represents some point on the graph of the function.

   To determine the equation of $g(x)$, first identify the vertex of the graph of $g(x)$.

   It can be seen from the graph that the vertex of $g(x)$ is $(1, -1)$.

   (continued)

Next, identify an additional point on the graph of $g(x)$.

It can be seen from the graph that an additional point on the graph of $g(x)$ is (0, 1).

Use the vertex (1, −1) and the additional point (0, 1) to find the value of a in the standard form of the equation.

$y = a(x - h)^2 + k$ Standard form of a quadratic equation

$(1) = a[(0) - (1)]^2 + (-1)$ Substitute (1, −1) for (h, k) and (0, 1) for (x, y).

$1 = a(-1)^2 - 1$ Simplify.

$1 = a - 1$ Apply the exponent.

$2 = a$ Add 1 to both sides.

The value of a is 2.

Use the value of a and the coordinate of the vertex to write the equation of the transformed function $g(x)$.

$y = a(x - h)^2 + k$ Standard form of a quadratic equation

$[g(x)] = (2)[x - (1)]^2 + (-1)$ Let $a = 2$, $y = g(x)$, and (h, k) be (1, −1).

$g(x) = 2(x - 1)^2 - 1$ Simplify.

The equation of the transformed function $g(x)$ in vertex form is $g(x) = 2(x - 1)^2 - 1$.

2. Use the standard form $y = a(x - h)^2 + k$ to determine the equation of the transformed function $h(x)$.

It can be seen from the graph of $h(x)$ that the vertex is $(-1, 2)$ and an additional point is $(0, -1)$.

Use the vertex $(-1, 2)$ and the additional point $(0, -1)$ to find the value of a in the standard form equation.

$y = a(x - h)^2 + k$	Standard form of a quadratic equation
$(-1) = a[(0) - (-1)]^2 + (2)$	Substitute $(-1, 2)$ for (h, k) and $(0, -1)$ for (x, y).
$-1 = a(1)^2 + 2$	Simplify.
$-1 = a + 2$	Apply the exponent.
$-3 = a$	Subtract 2 from both sides.

The value of a is -3.

Use the value of a and the coordinate of the vertex to write the equation of the transformed function $h(x)$.

$y = a(x - h)^2 + k$	Standard form of a quadratic equation
$[h(x)] = (-3)[x + (1)]^2 + (2)$	Let $a = -3$, $y = h(x)$, and (h, k) be $(-1, 2)$.
$h(x) = -3(x + 1)^2 + 2$	Simplify.

The equation of the transformed function $h(x)$ is $h(x) = -3(x + 1)^2 + 2$.

3. Describe the transformation(s) of the parent function that resulted in the functions $g(x)$ and $h(x)$.

The transformed functions are $g(x) = 2(x - 1)^2 - 1$ and $h(x) = -3(x + 1)^2 + 2$.

To describe the transformation(s) of the parent function $f(x) = x^2$, start inside the parentheses and work outward.

For $g(x)$:

- 1 was subtracted from x to give $x - 1$ in place of x.

- Then, 2 was multiplied by the "new" squared term, $(x - 1)^2$.

- Lastly, 1 was subtracted from the new squared term and its new coefficient to give the final result, $g(x) = 2(x - 1)^2 - 1$.

For $h(x)$:

- 1 was added to x to give $x + 1$ in place of x.

- Then, -3 was multiplied by the "new" squared term, $(x + 1)^2$.

- Finally, 2 was added to the new squared term and its new coefficient to give the final result, $h(x) = -3(x + 1)^2 + 2$.

Example 3

Identify the transformation(s) of the parent function $f(x) = \log x$ that result in the function $g(x) = 1 + 2 \bullet \log (3x + 4)$. Describe the effect of the transformation(s) on the domain and range of the function $g(x)$. Finally, determine the y-intercept(s) of $g(x)$.

1. Identify the transformation(s) of the parent function.

   To determine how the parent function $f(x) = \log x$ was transformed, compare it to $g(x) = 1 + 2 \bullet \log (3x + 4)$.

   - First, note that in $g(x)$, the constant 1 is added to $2 \bullet \log (3x + 4)$. Adding a constant to the function value is the transformation $g(x) = f(x) + k$.

   - Also notice that 2 is multiplied by $\log (3x + 4)$. Multiplying the function value by a constant is the transformation $g(x) = k \bullet f(x)$.

   - The coefficient 3 is in the argument of the logarithm, $\log (3x + 4)$. Multiplying the independent variable by a constant is the transformation $g(x) = f(kx)$.

   - Furthermore, note the addition of 4 in the argument of the logarithm, $\log (3x + 4)$. Adding a constant to the independent variable in the argument of the logarithm is the transformation $g(x) = f(x + k)$.

   Therefore, four transformations—$g(x) = f(x) + k$, $g(x) = k \bullet f(x)$, $g(x) = f(kx)$, and $g(x) = f(x + k)$—have been performed on the parent function $f(x) = \log x$.

2. Compare the domain and range of the parent function to those of the transformed function $g(x)$.

The domain of the parent function is $(0, \infty)$.

Since $g(x)$ is a logarithmic function, the domain of $g(x)$ is defined

by the argument of the logarithmic term, $3x + 4$. The condition

$3x + 4 > 0$ or $x > -\dfrac{4}{3}$ exists on the argument. Therefore, the domain

of $g(x)$ is $\left(-\dfrac{4}{3}, \infty \right)$.

The range of both functions is $(-\infty, \infty)$.

The transformations of $f(x)$ that result in $g(x)$ alter the rate at which the function values change, but not their upper and lower bounds.

3. Calculate the y-intercept of $g(x)$.

The y-intercept of $g(x)$ is defined by the function value $g(0)$. Substitute 0 for x and solve for $g(0)$:

$g(x) = 1 + 2 \bullet \log (3x + 4)$

$g(0) = 1 + 2 \bullet \log [3(0) + 4]$

$g(0) = 1 + 2 \bullet \log 4$

$g(0) \approx 2.2$

The y-intercept of $g(x) = 1 + 2 \bullet \log (3x + 4)$ is approximately 2.2, or (0, 2.2).

UNIT 4B • MATHEMATICAL MODELING AND CHOOSING A MODEL
Lesson 2: Transforming a Model and Combining Functions

Practice 4B.2.1: Transformations of Parent Graphs

For problems 1–3, use the graph of functions $f(x)$ and $g(x)$ to determine the value of the constant for the given general form of both functions. Then, write the equations of $f(x)$ and $g(x)$ in the given form.

1. general form: $a(x + b)$
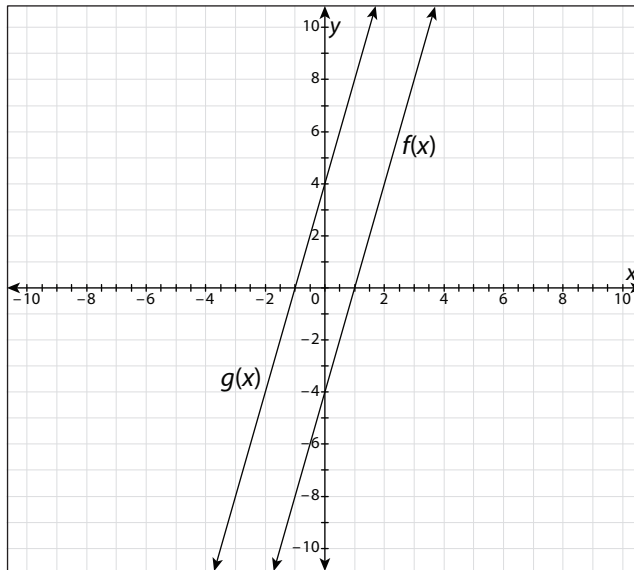
2. general form: ax^2
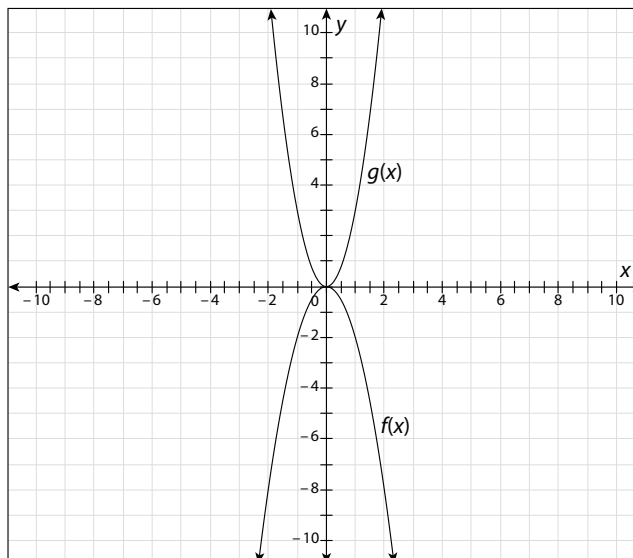
3. general form: $a(b)^x$
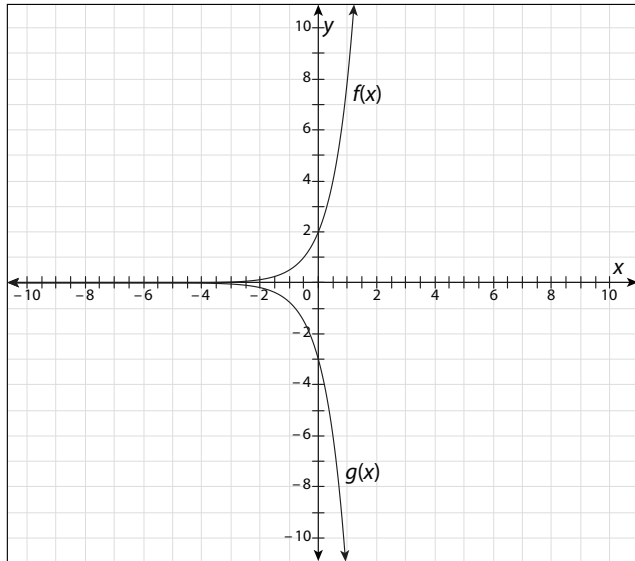
For problems 4–6, use the graph to determine the domain and range for each given function. Write your answers in interval notation.
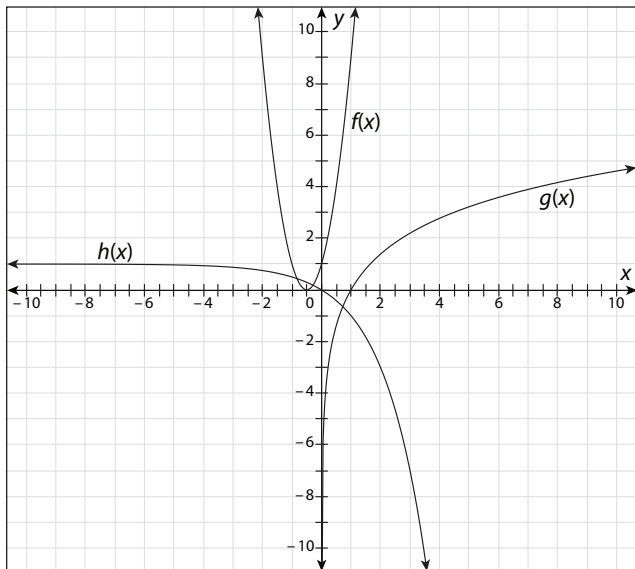
4. $f(x)$

5. $g(x)$

6. $h(x)$

Use the information given in each problem to complete problems 7–10.

7. The area of a circle is given by the function $A(a, b) = \pi(a - b)^2$. What happens to the value of $A(a, b)$ if the value of a is doubled and b is increased by 2?

8. The volume of water flowing into a reservoir is given by the function

   $V(r, t) = r \cdot t$, in which r is the rate in gallons per second and t is the time in

   hours. What happens to the value of t if the volume is doubled and r is

   decreased by a factor of $\dfrac{2}{3}$?

9. The pH of a polluted stream is given by the function $p(c) = -\log c$, in which c is the concentration of acid ions in the stream. The range of p is (0, 14) since the pH scale runs from 0 to 14. How does the domain of c change if the range of p changes to (7, 10)? Recall that on the pH scale, an acidic solution has a pH greater than 7, whereas a basic solution has a pH less than 7.

10. The profit p from selling n units of a product is given by the function $p(n) = 375n - 25{,}000$. What is the domain of n if the range of $p(n)$ is (0, 5000)? How does the domain of p change if the constant term $-25{,}000$ changes to $-40{,}000$ and the coefficient of n increases to 400?

Lesson 4B.2.2: Recognizing Odd and Even Functions

Introduction

When working with functions, it can be useful to recognize when a specific domain value or its opposite changes a function value in a particular way. For some functions, $f(x) = f(-x)$; that is, evaluating a function for $-x$ results in a function equation that is the same as the original function. Such a function is defined as an **even function**. An **odd function** is a function that, when evaluated for $-x$, results in a function equation that is the opposite of the original function; $f(-x) = -f(x)$. Even and odd functions can be changed into other functions using transformations. In some cases, these changed functions are still even or odd functions, but in other cases they no longer meet the criteria for being even or odd.

Key Concepts

- A function $f(x)$ for which $f(x) = f(-x)$ is defined as an even function.

- A function $f(x)$ for which $f(x) = -f(-x)$ is defined as an odd function.

- Whether a function is even, odd, or neither can be determined algebraically by substituting $-x$ for x and comparing the result to the original function.

- The transformations $g(x) = f(kx)$ and $g(x) = k \bullet f(x)$ of an even function result in an even function.

- The transformation $g(x) = f(x) + k$ of an even function results in an even function.

- The transformation $g(x) = f(x) + k$ of an odd function results in a new function that is neither even nor odd.

- A function that contains terms with both odd and even exponents is neither an even nor an odd function.

- To quickly determine if a function is even, odd, or neither, look at its graph:

    - The graph of an even function is symmetric about the y-axis.

    - The graph of an odd function is symmetric about the origin.

    - The graph of a function that is neither odd nor even is not symmetric about either the y-axis or the origin.

- The graphs of trigonometric functions can also be described as even or odd. For example, sine functions are odd, whereas cosine functions are even.

- A graphing calculator can be especially useful for evaluating evenness or oddness for higher-degree polynomials or other non-polynomial functions.

Guided Practice 4B.2.2

Example 1

Show that the function $f(x) = 2x^2 - x$ is neither even nor odd by using two opposite values of x.

1. Evaluate $f(x)$ at a value of x greater than 0.

   For convenience, use a positive single-digit integer, such as $x = 3$.

$f(x) = 2x^2 - x$	Original function
$f(3) = 2(3)^2 - (3)$	Substitute 3 for x.
$f(3) = 18 - 3$	Simplify.
$f(3) = 15$	

   For $x = 3$, the function $f(x) = 2x^2 - x$ is equal to 15.

2. Evaluate $f(x)$ at a value of x that is the opposite of the value of x used in the previous step.

   The value of x used in step 1 is 3; therefore, the opposite value is -3.

$f(x) = 2x^2 - x$	Original function
$f(-3) = 2(-3)^2 - (-3)$	Substitute -3 for x.
$f(-3) = 18 + 3$	Simplify.
$f(-3) = 21$	

   For $x = -3$, the function $f(x) = 2x^2 - x$ is equal to 21.

3. Summarize your findings.

   Use the results from steps 1 and 2 to determine what type of function $f(x)$ is.

   Recall that in an even function, $f(x) = f(-x)$.

   For $f(x) = 2x^2 - x$ to be even, then the result of $f(3)$ must equal the result of $f(-3)$. Since $15 \neq 21$, $f(x) = 2x^2 - x$ is not an even function.

   In an odd function, $f(x) = -f(-x)$.

   For $f(x) = 2x^2 - x$ to be odd, then the result of $f(3)$ must equal the result of $-f(-3)$.

   Since $15 \neq -21$, $f(x) = 2x^2 - x$ is not an odd function.

   Notice that for $f(x) = 2x^2 - x$, $f(x) \neq f(-x)$ and $f(x) \neq -f(-x)$; therefore, the function $f(x)$ is neither even nor odd.

Example 2

Describe how the graph of the function $g(x) = x^3 - 2x$ can be used to determine if the function is even or odd.

1. Graph the function by hand or using a graphing calculator.

   The graph of $g(x) = x^3 - 2x$ is as follows.
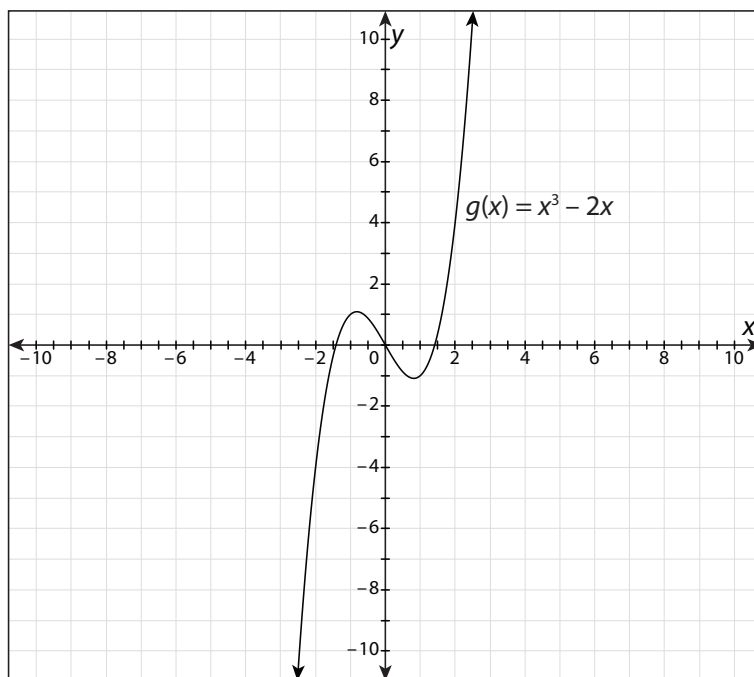
2. Visually compare the parts of the graph to the right and left of the y-axis.

   From the graph, it can be seen that graph is symmetric about the origin.

   Notice that when $x = 1$, $y = -1$, or $(1, -1)$.

   Also notice that when $x = -1$, $y = 1$, or $(-1, 1)$.

   Therefore, it can be hypothesized that opposite values of x result in opposite values of y.

3. Summarize your findings.

   The function $g(x)$ is an odd function since it can be seen from the graph that $g(x) = -g(-x)$.

Example 3

Describe how the graph of the function $h(x) = 6x^6 - 2x^2 - 1$ can be used to determine if the function is even or odd.

1. Graph the function by hand or using a graphing calculator.

   With higher-degree polynomials such as this one, it is more efficient to use a graphing calculator.

   The graph of $h(x) = 6x^6 - 2x^2 - 1$ is as follows.
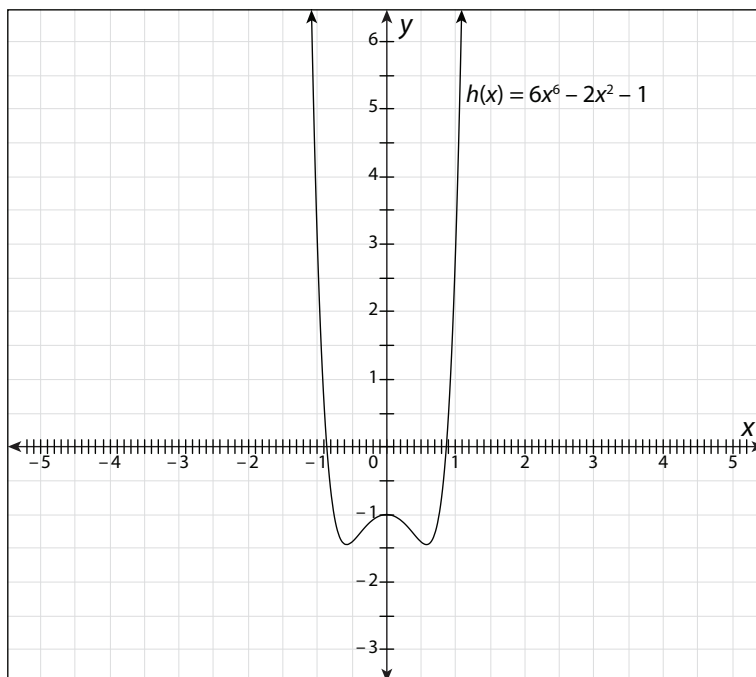
2. Visually compare the parts of the graph to the right and left of the y-axis.

   The graph is symmetric about the y-axis. The values of x on either side of the y-axis are opposites. For example, when $y = 0$, $x = 1$ and $x = -1$.

   It can be hypothesized that opposite values of x result in the same value of y.

3. Describe the function $h(x)$ as an even or an odd function.

   The function $h(x)$ is an even function since it can be seen from the graph that $h(x) = h(-x)$.

Example 4

Compare the functions $f(x) = \log x^2$ and $g(x) = 2 \bullet \log x$ using the definitions of even and odd functions.

1. Determine the domain values for which $f(x)$ exists.

    The domain of $f(x)$ is $(-\infty, 0) \cap (0, \infty)$ since the argument of the logarithm squares the value of x.

2. Describe $f(x)$ using the definitions of even and odd functions.

    The function $f(x) = \log x^2$ can be described as an even function since it can be written as $f(x) = f(-x) = \log x^2$. (Recall that the square of a negative term is positive.)

3. Determine the domain values for which $g(x)$ exists.

    $g(x)$ is a logarithmic function; therefore, the domain of $g(x)$ is $(0, \infty)$.

4. Describe $g(x)$ using the definitions of even and odd functions.

    Notice that the function $f(x) = \log x^2$ can be written as $f(x) = 2 \bullet \log x$, which is the same as $g(x) = 2 \bullet \log x$.

    The function $f(x) = 2 \bullet \log x$ is neither even nor odd since $f(x) \neq f(-x) \neq -f(-x)$; therefore, $g(x)$ is also neither even nor odd.

5. Summarize your findings.

    The function $f(x) = \log x^2$ is an even function, but its rewritten form, $g(x)$, is not.

Practice 4B.2.2: Recognizing Odd and Even Functions

For problems 1–4, change or remove terms to first rewrite each original function as an even function, then as an odd function.

1. $a(x) = x^3 + x^2 + x + 1$

2. $b(x) = 5 - |x - 5|$

3. $c(x) = \dfrac{x^2}{1 - x^2}$

4. $d(x) = \cos x + \sin (90° - x)$

For problems 5–7, use the following graph to determine the equation of each given function, then identify the function as even, odd, or neither.
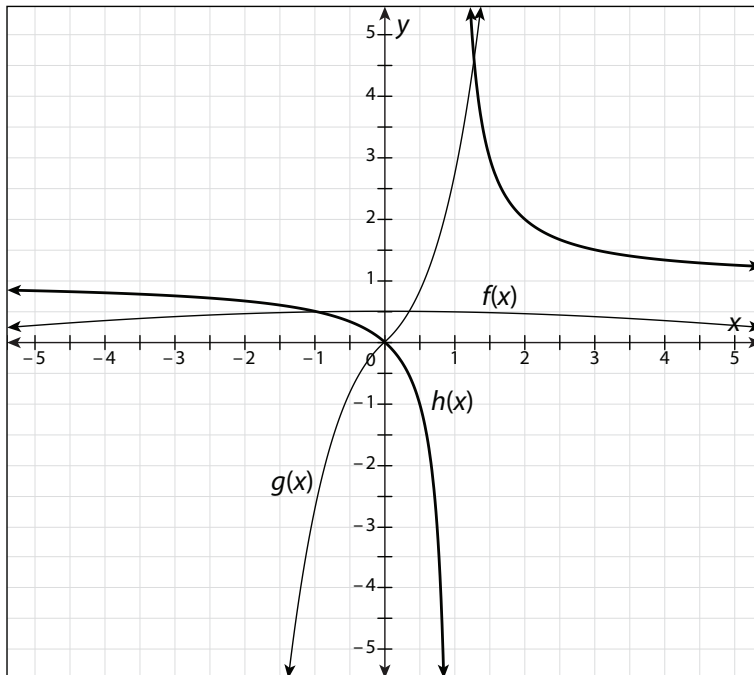
5. $f(x)$

6. $g(x)$

7. $h(x)$

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Use the information and the graph that follows to complete problems 8–10.

Most spiders have 8 eyes—2 "primary" eyes that detect images, and 6 "secondary" eyes that detect shadows and the difference between light and dark. The two types work together to detect the average intensity of light. A primary eye can see polarized light, which is phase shifted from what a secondary eye sees. Mathematically, the light seen by each type of eye can be represented by the functions $I_1(t) = I_{max} \cdot (1 + \cos t)$ and $I_2(t) = I_{max} \cdot (1 - \cos t)$, in which I_{max} is the maximum intensity of each oscillating light wave, and each cosine term represents the varying intensity of the light with time. The graph shows the two cosine terms, $I_1(t)$ and $I_2(t)$, and the average intensity function, $I(t)$.
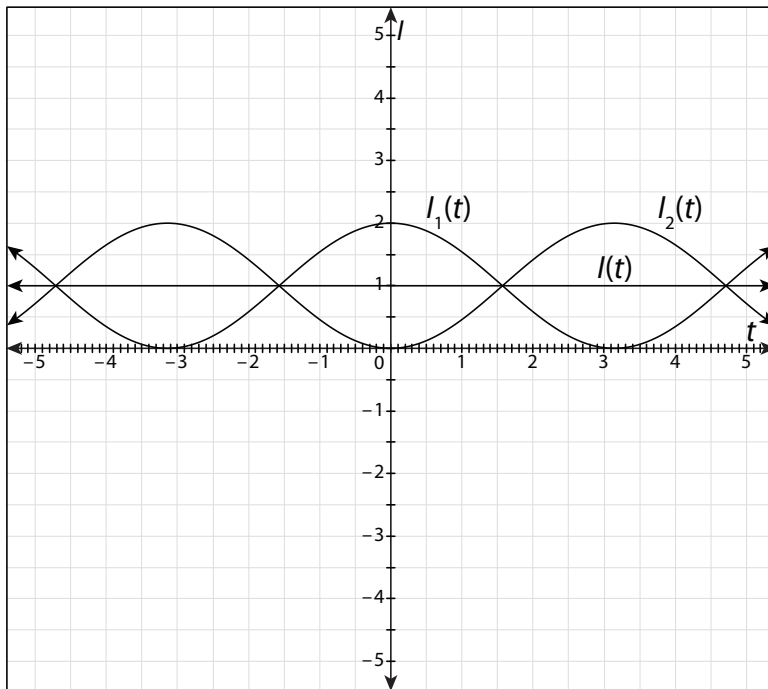
8. Are the functions $I_1(t)$ and $I_2(t)$ even, odd, or neither?

9. Write a function for the average intensity of the light waves. (*Hint*: Recall the definition of an arithmetic mean or "average" of two numbers.)

10. Suppose one light wave is represented as $f(x) = \cos x$ and the other is represented by $g(x) = \cos (90° - x)$. Use the definitions of even and odd functions to determine if $f(x)$ and $g(x)$ are even or odd functions.

Lesson 4B.2.3: Combining Functions

Introduction

Functions can be combined in many ways, such as by being added, subtracted, multiplied, divided, and substituted into each other. Some real-world problems do not lend themselves to modeling by simple functions of one type and/or in one variable with predictable domains and ranges, except as approximations or estimates of solutions. Combinations of functions are useful in modeling such problems and their solutions.

Key Concepts

- The **combination of functions** is the process of adding, subtracting, multiplying, or dividing two or more functions to produce a new function. This table shows how two functions $f(x)$ and $g(x)$ can be combined through addition, subtraction, multiplication, or division to produce a new function $h(x)$.

Combination	Description
$h(x) = (f + g)(x)$	$(f + g)(x) = f(x) + g(x)$; also, $(f + g)(x) = (g + f)(x) = g(x) + f(x)$
$h(x) = (f - g)(x)$	$(f - g)(x) = f(x) - g(x)$; $(f - g)(x) \neq (g - f)(x)$
$h(x) = (g - f)(x)$	$(g - f)(x) = g(x) - f(x)$; $(g - f)(x) \neq (f - g)(x)$
$h(x) = (f \bullet g)(x)$	$(f \bullet g)(x) = f(x) \bullet g(x)$; also, $(f \bullet g)(x) = (g \bullet f)(x) = g(x) \bullet f(x)$
$h(x) = \left(\dfrac{f}{g}\right)(x)$	$\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$; $\left(\dfrac{f}{g}\right)(x) \neq \left(\dfrac{g}{f}\right)(x)$
$h(x) = \left(\dfrac{g}{f}\right)(x)$	$\left(\dfrac{g}{f}\right)(x) = \dfrac{g(x)}{f(x)}$; $\left(\dfrac{g}{f}\right)(x) \neq \left(\dfrac{f}{g}\right)(x)$

- The **composition of functions** is the process of substituting one function for the independent variable of another function to create a new function.

- The notation $g(f(x))$ or $(g \circ f)(x)$ means that the function $f(x)$ is substituted for the independent variable x in the function $g(x)$. Similarly, the notation $f(g(x))$ or $(f \circ g)(x)$ means that the function $g(x)$ is substituted for the independent variable x in the function $f(x)$.

- A special case of function composition is the inverse function. A function $y = g(x)$ is an inverse of $y = f(x)$ if $g(f(x)) = x$ and $f(g(y)) = y$ and the function and its inverse are defined over a common domain.

- A domain and range of a function resulting from the combination or composition of two or more functions should be compared to the domain and range of each of the functions used in the combination or composition; the resulting domain and range of each function may not be the same as they were.

Guided Practice 4B.2.3

Example 1

Find $(f+g)(x)$ if $f(x) = \dfrac{x}{x-1}$ and $g(x) = \dfrac{x-1}{x+2}$. Determine the domains over which $f(x)$, $g(x)$, and $(f+g)(x)$ are defined.

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1. To find $(f+g)(x)$, find $f(x) + g(x)$.

   $(f+g)(x) = f(x) + g(x)$ Rule for finding $(f+g)(x)$

   $(f+g)(x) = \left(\dfrac{x}{x-1}\right) + \left(\dfrac{x-1}{x+2}\right)$ Substitute $\dfrac{x}{x-1}$ for $f(x)$ and $\dfrac{x-1}{x+2}$ for $g(x)$.

   $(f+g)(x) = \dfrac{x(x+2) + (x-1)^2}{(x-1)(x+2)}$ Simplify.

   $(f+g)(x) = \dfrac{2x^2 + 1}{(x-1)(x+2)}$

---

2. Determine the domain of $f(x)$.

   The domain of $f(x)$ is $(-\infty, 1)$ and $(1, \infty)$, or $(-\infty, 1) \cap (1, \infty)$, because $f(x)$ is undefined at $x = 1$.

---

3. Determine the domain of $g(x)$.

   The domain of $g(x)$ is $(-\infty, -2)$ and $(-2, \infty)$, or $(-\infty, -2) \cap (-2, \infty)$, because $g(x)$ is undefined at $x = -2$.

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4. Determine the domain of the combined function $(f+g)(x)$.

   The domain of $(f+g)(x)$ is $(-\infty, -2) \cap (-2, 1) \cap (1, \infty)$ because $(f+g)(x)$ is undefined at $x = -2$ *and* at $x = 1$.

Example 2

Find $(f \cdot g)(x)$ if $f(x) = \sqrt{x-3}$ and $g(x) = \sqrt{4-x}$. Determine the domains over which $f(x)$, $g(x)$, and $(f \cdot g)(x)$ are defined.

1. Find $(f \cdot g)(x)$.

   To find $(f \cdot g)(x)$, find $f(x) \cdot g(x)$.

   $(f \cdot g)(x) = f(x) \cdot g(x)$ Rule for multiplying functions

   $(f \cdot g)(x) = (\sqrt{x-3}) \cdot (\sqrt{4-x})$ Substitute $\sqrt{x-3}$ for $f(x)$ and $\sqrt{4-x}$ for $g(x)$.

   $(f \cdot g)(x) = \sqrt{-x^2 + 7x - 12}$ Simplify.

2. Determine the domain of $f(x)$.

   The domain of $f(x)$ has to meet the condition $x - 3 \geq 0$ or $x \geq 3$, so the domain is $[3, \infty)$.

3. Determine the domain of $g(x)$.

   The domain of $g(x)$ has to meet the condition $4 - x \geq 0$ or $x \leq 4$, so the domain is $(-\infty, 4]$.

4. Determine the domain of $(f \cdot g)(x)$.

   The domain of $(f \cdot g)(x)$ has to meet the condition $-x^2 + 7x - 12 \geq 0$, so the domain is $(-\infty, 4] \cup [3, \infty)$, which reduces to $[3, 4]$ since any value of x less than 3 or greater than 4 results in $(f \cdot g)(x)$ being undefined.

Example 3

Find the function that results from the composition $(f \circ g)(x)$ if $f(x) = \log x$ and $g(x) = \sqrt{x-1}$. Determine the domains of $f(x)$, $g(x)$, and $(f \circ g)(x)$.

1. To find $(f \circ g)(x)$, find $f(g(x))$.

    $(f \circ g)(x) = f(g(x))$ Composition rule

    $(f \circ g)(x) = \log(\sqrt{x-1})$ Substitute $\log(\sqrt{x-1})$ for $f(g(x))$.

    $(f \circ g)(x) = \log(x-1)^{\frac{1}{2}}$ Simplify.

    $(f \circ g)(x) = \dfrac{1}{2} \cdot \log(x-1)$

2. Determine the domain of $f(x)$.

    The domain of $f(x)$ is restricted by the condition $x > 0$. Therefore, the domain of $f(x)$ is $(0, \infty)$.

3. Determine the domain of $g(x)$.

    The domain of $g(x)$ is restricted by the condition $x - 1 \geq 0$ or $x \geq 1$. Therefore, the domain of $g(x)$ is $[1, \infty)$.

4. Determine the domain of the composition $(f \circ g)(x)$.

    The domain of $(f \circ g)(x)$ is restricted by the condition $x - 1 > 0$ or $x > 1$. Therefore, the domain of the composition is $(1, \infty)$.

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Practice 4B.2.3: Combining Functions

For problems 1–3, identify $h(x)$ as a combination, a composition, or both.

1. $h(x) = 4 \cdot g\big[0.4 \cdot f(x)\big]$

2. $h(x) = \dfrac{(f \circ g)(x)}{(g \circ f)(x)}$

3. $h(x) = (2f \cdot 3g)(x)$

For problems 4–7, $f(x) = 2^x$ and $g(x) = \log_2 x$. Write the equation of the combination or composition given and determine its domain.

4. $(f \circ g)(x)$ and $(g \circ f)(x)$

5. $\dfrac{(f \circ g)(x)}{(g \circ f)(x)}$ and $\dfrac{(g \circ f)(x)}{(f \circ g)(x)}$

6. $(f \circ f)(x)$ and $(g \circ g)(x)$

7. $(f^2 \circ g)(x)$ and $(g \circ f^2)(x)$

Use the information given in each problem to complete problems 8–10.

8. The surface area of a sphere with radius r is $S = 4\pi r^2$. Griffin is using a helium tank to inflate a spherical balloon, and the rate at which the radius of the balloon increases is given by $r = \frac{1}{2}t^2$, for which r is in inches and t is the time in seconds. Find a formula for the surface area S of the balloon as a function of time. Use this formula to find the surface area of the balloon after 5 seconds. Round your answer to the nearest square inch.

9. The amount of electrical charge $Q(t)$ transferred in a circuit is related to the current $I(t)$ and the time t by the function $Q(t) = I(t) \bullet t$. The voltage $V(t)$ in the circuit is related to the current $I(t)$ by the function $I(t) = \frac{V(t)}{R}$. Write the voltage $V(t)$ as a function of the charge $Q(t)$.

10. A Doppler weather radar station sends out a radar wave with a frequency of 3 gigahertz. (*Note*: 1 gigahertz is 10^9 hertz or cycles per second.) The radar bounces off of a thunderstorm that is moving toward the radar at a speed of 30 meters per second. The bounce-back wave from the storm is a function of the speed of the storm and therefore has a slightly different frequency, which is given by the function $f_{\text{toward the storm}}(v) = f_{\text{radar}} \cdot \left(1 + \dfrac{v_{\text{storm}}}{c}\right)$, in which c is the speed of light, approximately 3×10^8 meters per second. From the storm, the radar wave returning to the radar station has a different frequency than the wave coming from the radar station. This frequency is given by the function $f_{\text{from the storm}}(v) = f_{\text{toward the storm}} \cdot \left(1 + \dfrac{v_{\text{storm}}}{c}\right)$. Calculate the value of the combination function for this specific thunderstorm speed.

Lesson 3: Comparing Properties Within and Between Functions

Common Core State Standards

F–IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★

F–IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★

F–IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★

F–IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*

Essential Questions

1. What is the significance of a difference in *x*- or *y*-intercepts of different forms of the same function or in comparing two or more functions in real-world situations?

2. In a real-world problem, how does a function model change when different domains result in positive and negative values?

3. When are maximum and minimum function values relevant to solving or visualizing real-world problems?

4. What are the limitations on the domain of a real-world function and how do these limitations affect function values and possible solutions to real-world problems?

5. How do variations in the rate of change of a function value affect the interpretation of real-world problems in which rate of change is a factor?

WORDS TO KNOW

asymptote	an equation that represents sets of points that are not allowed by the conditions in a parent function or model; a line that a function gets closer and closer to as one of the variables increases or decreases without bound
average rate of change	the ratio of the difference of output values to the difference of the corresponding input values: $\dfrac{f(b)-f(a)}{b-a}$; a measure of how a quantity changes over some interval
boundary condition	a constraint or limit on a function or domain value based on real-world conditions or restraints in the problem or its solution
***delta* (Δ)**	a Greek letter commonly used to represent the change in a value
extrema	the minima or maxima of a function
initial condition	a constraint or limit on a function or domain value that exists in the form of a *y*-intercept or other starting-point mathematical restraint in a real-world problem or solution
local maximum	the greatest value of a function for a particular interval of the function; also known as a *relative maximum*
local minimum	the smallest value of a function for a particular interval of the function; also known as a *relative minimum*
periodic function	a function whose values repeat at regular intervals
rate of change	a ratio that describes how much one quantity changes with respect to the change in another quantity; also known as the slope of a line
relative maximum	the greatest value of a function for a particular interval of the function; also known as a *local maximum*
relative minimum	the smallest value of a function for a particular interval of the function; also known as a *local minimum*
symmetry of a function	the property whereby a function exhibits the same behavior (e.g., graph shape, function values, etc.) for specific domain values and their opposites

Recommended Resources

- Interactivate. "Data Flyer."

  http://www.walch.com/rr/00243

  This applet allows users to create graphs by entering data points, then derive the function equation that fits the graph. Graphs can be manipulated to see how changing the constant and/or coefficient affects the shape of the graph.

- OnlineMathLearning.com. "Compare Two Functions."

  http://www.walch.com/rr/00244

  This site offers users access to several videos comparing the properties and features of different types of functions.

- Virtual Nerd. "How Do You Make an Approximate Graph from a Word Problem?"

  http://www.walch.com/rr/00245

  This video tutorial covers how to sketch the graph of a problem using information derived from a word problem.

IXL Links

- Identify proportional relationships:
  http://www.ixl.com/math/algebra-1/identify-proportional-relationships

- Find the constant of variation:
  http://www.ixl.com/math/algebra-1/find-the-constant-of-variation

- Graph a proportional relationship:
  http://www.ixl.com/math/algebra-1/graph-a-proportional-relationship

- Identify direct variation and inverse variation:
  http://www.ixl.com/math/algebra-1/identify-direct-variation-and-inverse-variation

- Slope intercept form find slope and y intercept:
  http://www.ixl.com/math/algebra-1/slope-intercept-form-find-slope-and-y-intercept

- Standard form find x and y intercepts:
  http://www.ixl.com/math/algebra-1/standard-form-find-x-and-y-intercepts

- Slopes of parallel and perpendicular lines:
  http://www.ixl.com/math/algebra-1/slopes-of-parallel-and-perpendicular-lines

- Characteristics of quadratic functions:
  http://www.ixl.com/math/algebra-1/characteristics-of-quadratic-functions

- Identify linear quadratic and exponential functions from graphs:
  http://www.ixl.com/math/algebra-1/identify-linear-quadratic-and-exponential-functions-from-graphs

- Identify linear quadratic and exponential functions from tables:
  http://www.ixl.com/math/algebra-1/identify-linear-quadratic-and-exponential-functions-from-tables

- Graph an absolute value function:
  http://www.ixl.com/math/algebra-1/graph-an-absolute-value-function

- Rational functions asymptotes and excluded values:
  http://www.ixl.com/math/algebra-1/rational-functions-asymptotes-and-excluded-values

- Slopes of lines:
  http://www.ixl.com/math/geometry/slopes-of-lines

- Characteristics of quadratic functions:
  http://www.ixl.com/math/algebra-2/characteristics-of-quadratic-functions

- Graph a quadratic function:
  http://www.ixl.com/math/algebra-2/graph-a-quadratic-function

- Rational functions asymptotes and excluded values:
  http://www.ixl.com/math/algebra-2/rational-functions-asymptotes-and-excluded-values

- Classify variation:
  http://www.ixl.com/math/algebra-2/classify-variation

- Find the constant of variation:
  http://www.ixl.com/math/algebra-2/find-the-constant-of-variation

- Domain and range of absolute value functions:
  http://www.ixl.com/math/algebra-1/domain-and-range-of-absolute-value-functions

- Domain and range of radical functions:
  http://www.ixl.com/math/algebra-1/domain-and-range-of-radical-functions

- Domain and range:
  http://www.ixl.com/math/algebra-2/domain-and-range

- Domain and range of radical functions:
  http://www.ixl.com/math/algebra-2/domain-and-range-of-radical-functions

- Domain and range of exponential and logarithmic functions:
  http://www.ixl.com/math/algebra-2/domain-and-range-of-exponential-and-logarithmic-functions

- Find the constant of variation:
  http://www.ixl.com/math/algebra-1/find-the-constant-of-variation

- Find the slope of a graph:
  http://www.ixl.com/math/algebra-1/find-the-slope-of-a-graph

- Find slope from two points:
  http://www.ixl.com/math/algebra-1/find-slope-from-two-points

- Slope intercept form find slope and y intercept:
  http://www.ixl.com/math/algebra-1/slope-intercept-form-find-slope-and-y-intercept

- Find the slope of a linear function:
  http://www.ixl.com/math/algebra-2/find-the-slope-of-a-linear-function

Lesson 4B.3.1: Reading and Identifying Key Features of Real-World Situation Graphs

Introduction

Mathematical models can be useful in solving real-world problems. However, it is sometimes the case that the features of a real-world mathematical model are limited or specialized forms of the "pure" mathematics counterpart. Visual representations or graphs of the two versions can highlight the differences and refine the model so that it accurately portrays the problem and/or its solution. The graphs and the accompanying data tables produced by graphing calculators can be invaluable in interpreting function features in real-world applications.

Key Concepts

- The domain and range of a mathematical function represented by the intervals $(-\infty, \infty)$ sometimes include values of the domain or the range that do not have meaning in the context of a given problem.

- For example, time and other properties of matter (including area, number of units, and volume) usually have values that are greater than 0. Other physical quantities, such as electrical properties, profit, speed, and temperature, can have both positive and negative values, but they sometimes have interval values that have finite upper and lower values.

- Function values in mathematical and real-world models of problems can be described as increasing, decreasing, or constant. Some real-world situations, such as those involving biological growth or radioactive decay, can be described with only one of these terms in the absence of other factors. For example, natural radioactive decay of a specific isotope of an element is a one-way process (i.e., decreasing) because the reverse process would represent a violation of the accepted physical laws describing the behavior of radioactive elements.

- Mathematical and real-world function models are sometimes restricted to certain unchanging constants. Examples include the values of e and π, the speed of light in a vacuum (often denoted c), and the temperature value of absolute zero.

- Other constants limit function values in a specific problem context. For example, empirical evidence might suggest that only a specific number of polar bears can be supported in an area that has a finite supply of resources the polar bears need to live and multiply.

- The intercepts of a mathematical model have special meanings in real-world problems and/or solutions. At times, the *y*-intercept represents an **initial condition**—that is, a constraint or limit on the domain or function values. An initial condition is a "starting-point" mathematical constraint that dictates function values over other parts of the model of the domain of an independent variable. Typically, the initial condition occurs at a value of 0 for time measurements; however, it might also occur at a value greater than 0. For example, a store may have a certain number of tote bags that need to be sold in a 24-hour period.

- A **boundary condition** is a constraint or limit on a function or domain value based on real-world conditions or restraints in the problem or its solution. Sometimes it represents a physical or quantitative limit. A boundary condition can be represented numerically or with another function.

- Boundary conditions often occur at the asymptotes of exponential, logarithmic, and rational functions. An **asymptote** is an equation that represents sets of points that are not allowed by the conditions in a parent function or model. On a graph, an asymptote is a line that a function gets closer and closer to as one of the variables increases or decreases without bound. Asymptotes represent specific domain values at which mathematical or real-world models do not exist and/or are undefined.

- The **extrema** of a function are the minima or maxima of the function.

- The domain value(s) at which a real-world model has a maximum or a minimum function value are of importance in a wide variety of physical applications.

- With some complicated functions, maximum and minimum function values occur locally; that is, over a restricted domain interval of the function, which may or may not have physical meaning in a real-world model. For example, a function of time given by $f(t) = \dfrac{t}{t+1}$ only exists for values of $t > 0$.

- Such localized values are called the **local maximum** or **relative maximum** (the greatest value of a function for a particular interval of the function) and the **local minimum** or **relative minimum** (the smallest value of a function for a particular interval of the function).

- The **symmetry of a function** is the property whereby a function exhibits the same behavior (such as graph shape and function values) for specific domain values and their opposites.

- Symmetry can be useful in determining if all of the solutions of a model have been identified and if they have relevance for the solution of the real-world context being modeled.

- Symmetries can exist across axes or across lines representing linear equations such as $f(x) = x$, as is the case with functions and their inverses. A special type of symmetry exists for periodic functions. A **periodic function** is a function whose values repeat at regular intervals—that is, it exhibits the same function values or properties over repeating restricted domains.

- The behavior of functions for very large and very small domain values can be important in solving real-world problems and describing restrictions on the domain and range values over which a mathematical model can describe a real-world situation.

- For example, compare the functions $f(t) = \dfrac{t}{t+1}$ and $g(t) = \dfrac{t}{t+2}$. For small values of t such as $t = 1$, the function values can vary by significant amounts: $f(1) = 0.5$ and $g(1) = 0.\overline{3}$. For very large values of t, both functions approach a value of 1; for example, for $t = 100$, $f(100)$ is about 0.9901, whereas $g(100)$ is about 0.98. The difference between these two models for "large" values of t is about 1 percent. On the other hand, for very small values of t, such as $t = 0.01$, $f(0.01)$ is about 0.01 whereas $g(0.01)$ is about 0.005; the difference between the function values for this "small" value of t is about 100 percent.

Guided Practice 4B.3.1

Example 1

The function for finding the amount of time it takes two workers to complete a job is given by $\dfrac{1}{t_{both}} = \dfrac{1}{t_1} + \dfrac{1}{t_2}$. Compare this to the function $\dfrac{1}{f} = \dfrac{1}{d_1} + \dfrac{1}{d_2}$, which relates the focal length f of a lens to the distances between the lens and an object (d_1) and between the lens and the *image* of the object created by the lens (d_2). The lens is two-sided, so if the lens is centered at the origin, the values of f, d_1, and d_2 can be positive or negative, depending on the position of the object. How are the domains and ranges of these two functions alike and different? Solve each function for the function variable on the left side of each equation. How do the variable relationships differ?
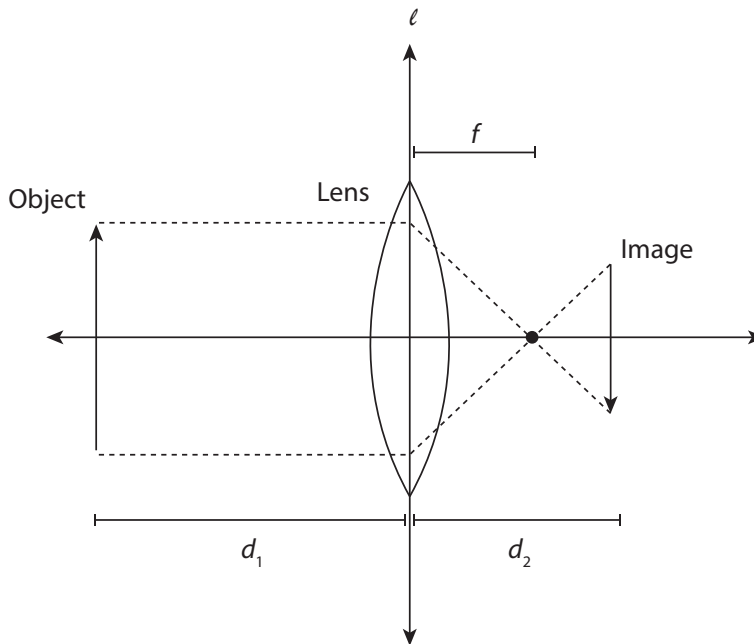
1. State the restrictions, if any, on the domain and range of the first function, t_{both}.

   None of the variables can equal 0 since they are in the denominators of fractions. The variables measure time, so the values of all three variables are assumed to be greater than 0. Therefore, for t_{both}, the restricted domain is $t_1 > 0$ and $t_2 > 0$, and the range is $t_{both} > 0$.

2. State the restrictions, if any, on the domain and range of the second function, f.

None of the variables can equal 0 since they are in the denominators of fractions. According to the diagram, the variables can be positive or negative. The location of the image and/or the object being magnified could be "inside" the focal length f; i.e., between the focal point F and the origin. Therefore, for f, the restricted domain is $d_1 \neq 0$ and $d_2 \neq 0$, and the range is $f \neq 0$.

3. Solve the function $\dfrac{1}{t_{both}} = \dfrac{1}{t_1} + \dfrac{1}{t_2}$ for t_{both}.

$\dfrac{1}{t_{both}} = \dfrac{1}{t_1} + \dfrac{1}{t_2}$ Given function

$1 = \dfrac{1}{t_1} \bullet t_{both} + \dfrac{1}{t_2} \bullet t_{both}$ Multiply both sides by t_{both}.

$1 = \dfrac{t_{both}}{t_1} + \dfrac{t_{both}}{t_2}$ Simplify.

$1 = \dfrac{t_{both} \bullet t_2}{t_1 \bullet t_2} + \dfrac{t_{both} \bullet t_1}{t_2 \bullet t_1}$ Create fractions with common denominators.

$1 = \dfrac{t_{both} \bullet t_2 + t_{both} \bullet t_1}{t_1 \bullet t_2}$ Add.

$1 = \dfrac{t_{both}(t_2 + t_1)}{t_1 \bullet t_2}$ Apply the Distributive Property.

$t_1 \bullet t_2 = t_{both}(t_2 + t_1)$ Multiply both sides by $t_1 \bullet t_2$.

$\dfrac{t_1 \bullet t_2}{t_2 + t_1} = t_{both}$ Divide both sides by $t_2 + t_1$.

$t_{both} = \dfrac{t_1 \bullet t_2}{t_2 + t_1}$ Apply the Symmetric Property of Equality.

The function $\dfrac{1}{t_{both}} = \dfrac{1}{t_1} + \dfrac{1}{t_2}$ solved for t_{both} is $t_{both} = \dfrac{t_1 \bullet t_2}{t_2 + t_1}$.

4. Solve the function $\dfrac{1}{f} = \dfrac{1}{d_1} + \dfrac{1}{d_2}$ for f.

$$\dfrac{1}{f} = \dfrac{1}{d_1} + \dfrac{1}{d_2}$$ Given function

$$1 = \dfrac{1}{d_1} \bullet f + \dfrac{1}{d_2} \bullet f$$ Multiply both sides by f.

$$1 = \dfrac{f}{d_1} + \dfrac{f}{d_2}$$ Simplify.

$$1 = \dfrac{f \bullet d_2}{d_1 \bullet d_2} + \dfrac{f \bullet d_1}{d_2 \bullet d_1}$$ Create fractions with common denominators.

$$1 = \dfrac{f \bullet d_2 + f \bullet d_1}{d_1 \bullet d_2}$$ Add.

$$1 = \dfrac{f(d_2 + d_1)}{d_1 \bullet d_2}$$ Apply the Distributive Property.

$$d_1 \bullet d_2 = f(d_2 + d_1)$$ Multiply both sides by $d_1 \bullet d_2$.

$$\dfrac{d_1 \bullet d_2}{d_2 + d_1} = f$$ Divide both sides by $d_2 + d_1$.

$$f = \dfrac{d_1 \bullet d_2}{d_2 + d_1}$$ Apply the Symmetric Property of Equality.

The function $\dfrac{1}{f} = \dfrac{1}{d_1} + \dfrac{1}{d_2}$ solved for f is $f = \dfrac{d_1 \bullet d_2}{d_2 + d_1}$.

5. Use the restrictions on the domain and range of each function to compare the relationships between the variables in the functions.

For t_{both}, the restricted domain is $t_1 > 0$ and $t_2 > 0$, and the range is $t_{both} > 0$. As a result of these restrictions, the variables in $t_{both} = \dfrac{t_1 \bullet t_2}{t_2 + t_1}$ are all positive, so the function value will always be positive.

For f, the restricted domain is $d_1 \neq 0$ and $d_2 \neq 0$, and the range is $f \neq 0$. As a result of these restrictions, the variables in $f = \dfrac{d_1 \bullet d_2}{d_2 + d_1}$ could be positive or negative, so f could be positive or negative.

Also, the condition $d_2 + d_1 \neq 0$ must be observed in order for the function f to exist.

In summary, the domain and range values for t_{both} will always be positive. Values of the range of the second function, f, can be positive or negative, but the function is only valid for domain values that do not sum to 0 ($d_2 + d_1 \neq 0$).

Example 2

Working together, carpenters Benjamin and Ava can build a table in 4 hours. Use the function $t_{both} = \dfrac{t_1 \bullet t_2}{t_1 + t_2}$ to create a graph and to find the time required for each carpenter to build a table separately if Benjamin takes t hours and Ava takes $t + 2$ hours.

1. Define the variables in the function.

   Using the given information, let $t_1 = t$ hours, $t_2 = t + 2$ hours, and $t_{both} = 4$ hours.

2. Substitute the defined variables into the given function and simplify.

$t_{both} = \dfrac{t_1 \bullet t_2}{t_1 + t_2}$	Given function
$(4) = \dfrac{(t) \bullet (t+2)}{(t) + (t+2)}$	Substitute t for t_1, $t + 2$ for t_2, and 4 for t_{both}.
$4 = \dfrac{t^2 + 2t}{t + (t + 2)}$	Simplify the numerator.
$4 = \dfrac{t^2 + 2t}{2t + 2}$	Simplify the denominator.
$4(2t + 2) = t^2 + 2t$	Multiply both sides by $2t + 2$.
$8t + 8 = t^2 + 2t$	Distribute.
$0 = t^2 - 6t - 8$	Rewrite the equation so that all terms are on one side.

   The simplified function can be written as $0 = t^2 - 6t - 8$.

3. Graph the simplified equation.

   Graph the equation $0 = t^2 - 6t - 8$ either by hand or using a graphing calculator. The equation can be written as $f(t) = t^2 - 6t - 8$.
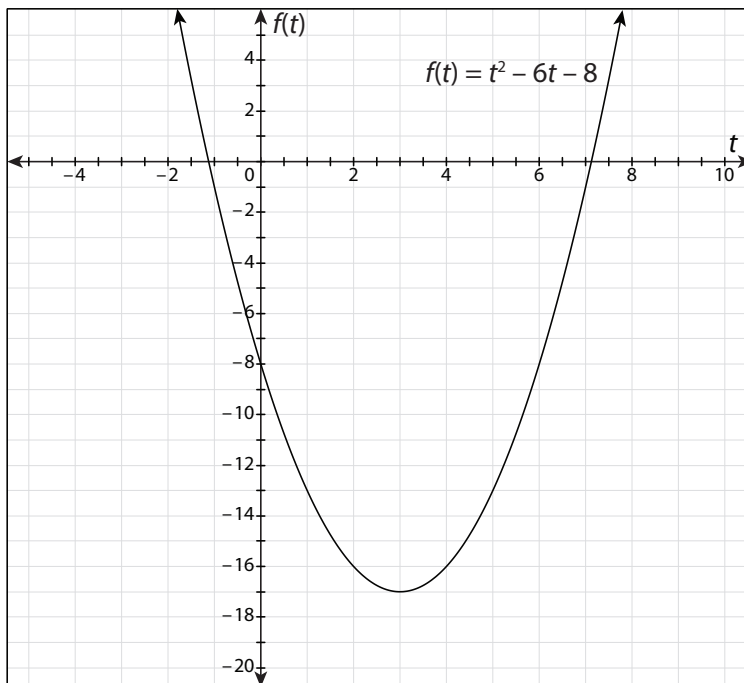
$$f(t) = t^2 - 6t - 8$$

4. Describe the domain of the function and what that implies for the number of possible solutions for t.

   Since t represents time, the domain of t is values of t that are greater than 0. Therefore, only the t-intercept on the positive t-axis (the horizontal axis) gives a real-world solution for t.

5. Solve $0 = t^2 - 6t - 8$ to find the indicated domain value(s) of t and determine the amount of time Benjamin and Ava each take to build a table on their own.

To determine the time required for each carpenter to build

the table when working separately, use the quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, to solve $0 = t^2 - 6t - 8$ for t.

Let $a = 1$, $b = -6$, $c = -8$, and $x = t$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Quadratic formula}$$

$$(t) = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-8)}}{2(1)} \qquad \text{Substitute 1 for } a, -6 \text{ for } b, -8 \text{ for } c, \text{ and } t \text{ for } x.$$

$$t = \frac{6 \pm 2\sqrt{17}}{2} \qquad \text{Simplify.}$$

$$t = 3 \pm \sqrt{17} \qquad \text{Divide.}$$

Only the positive value of the radical, $t = 3 + \sqrt{17}$, will result in a

positive value of t; therefore, $t = 3 + \sqrt{17} \approx 7.1$ hours.

The variable t represents Benjamin's time, so he can build the table by himself in approximately 7.1 hours. Since $t + 2$ represents Ava's time, substitute 7.1 for t and solve to determine that Ava can build the table by herself in approximately 9.1 hours.

Example 3

The diagram shows the distance (d_1) between an object and a lens and the distance (d_2) between the lens and the image it creates of the object. F_1 and F_2 represent the focal points of the two-sided lens. (Note that these points are symmetric about the lens at line ℓ.) Use the diagram and the function $f_1 = \dfrac{d_1 \bullet d_2}{d_1 + d_2}$ to find d_1 and d_2 if the focal length f_1 of the lens is -1 and $d_2 = -d_1 + 2$.
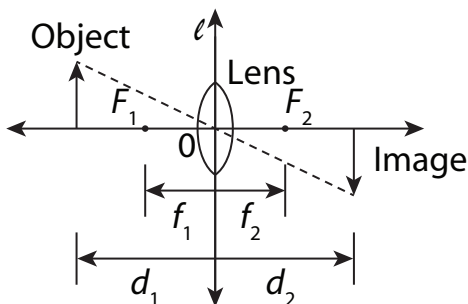
1. Define the variables in the function.

   Using the given information, let $f_1 = -1$ and $d_2 = -d_1 + 2$.

2. Substitute the defined variables into the given function and simplify.

   Substitute each of the defined variables into the given function.

$f_1 = \dfrac{d_1 \bullet d_2}{d_1 + d_2}$	Given function
$(-1) = \dfrac{d_1 \bullet (-d_1 + 2)}{d_1 + (-d_1 + 2)}$	Substitute -1 for f_1 and $-d_1 + 2$ for d_2.
$-1 = \dfrac{-(d_1)^2 + 2 \bullet d_1}{d_1 + (-d_1 + 2)}$	Distribute the numerator.
$-1 = \dfrac{-(d_1)^2 + 2 \bullet d_1}{2}$	Simplify the denominator.
$-2 = -(d_1)^2 + 2 \bullet d_1$	Multiply both sides by 2.
$0 = (d_1)^2 - 2 \bullet d_1 - 2$	Set all terms on one side with a positive leading coefficient.

   The simplified function can be written as $0 = (d_1)^2 - 2d_1 - 2$.

3. Graph the simplified equation.

Graph the equation $0 = (d_1)^2 - 2d_1 - 2$ by hand or using a graphing calculator. This equation can be written as $f(d_1) = (d_1)^2 - 2d_1 - 2$.
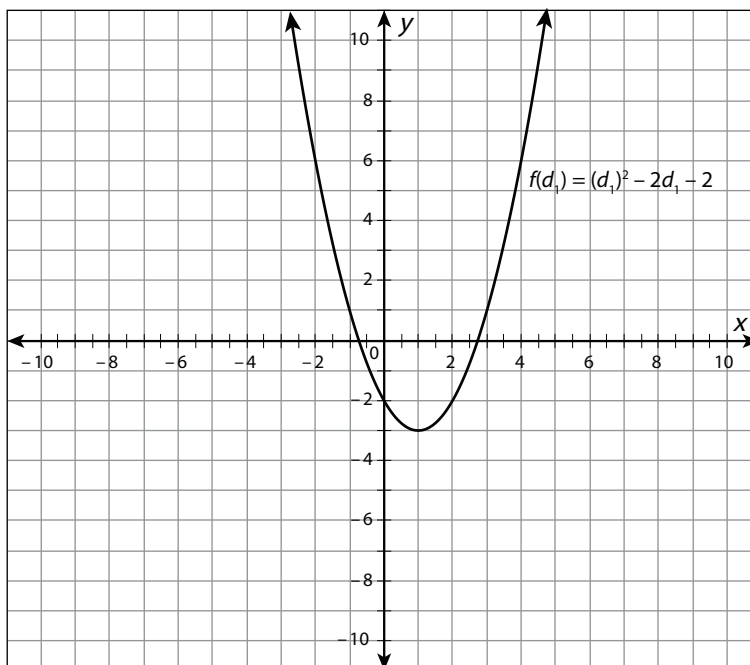
$$f(d_1) = (d_1)^2 - 2d_1 - 2$$

4. Describe the domain of the function and what that implies for the number of possible solutions for d_1 and d_2.

The variables cannot equal 0, but they can be positive or negative. The lens diagram shows that d_1 and d_2 can be positive or negative, depending on where the object is located relative to the lens.

The object could be located between the focal point F_2 and the lens or anywhere to the left of the focal point F_2, as shown in the diagram. This creates the possibility of an infinite number of possible solutions for d_1 and d_2.

5. Solve $0 = (d_1)^2 - 2d_1 - 2$ to find the value(s) of d_1 as described in step 4.

To solve $0 = (d_1)^2 - 2d_1 - 2$, use the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Let $a = 1$, $b = -2$, $c = -2$, and $x = d_1$.

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Quadratic formula

$(d_1) = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)}$ Substitute 1 for a, –2 for b, –2 for c, and d_1 for x.

$d_1 = 1 \pm \dfrac{\sqrt{12}}{2}$ Simplify.

$d_1 = 1 \pm \sqrt{3}$

The solutions are $d_1 = 1 + \sqrt{3} \approx 2.73$ or $d_1 = 1 - \sqrt{3} \approx -0.73$.

6. Find d_2 and interpret the solutions in terms of the locations of the object and image in relation to the lens.

The two distances are related by the formula $d_2 = -d_1 + 2$. Since $d_1 \approx 2.73$ or –0.73, substitute each of these values into the formula and solve for d_2.

$d_2 = -d_1 + 2$ $d_2 = -d_1 + 2$

$d_2 \approx -(2.73) + 2$ $d_2 \approx -(-0.73) + 2$

$d_2 \approx -0.73$ $d_2 \approx 2.73$

The focal length, f_1, was given as –1, which means it is measured to the left of the lens.

A negative value for d_2 means the image is on the left side of the lens. A positive value for d_2 means the image is on the right side of the lens.

Example 4

The table shows the number of earthquakes of six different average magnitudes detected over the course of a year by the Earthquake Hazards Program of the U.S. Geological Survey.

Magnitude, m	6.0	6.5	7.0	7.5	8.0	8.5
Number, N	210	56	15	3	1	0

Use a graphing calculator to find a natural logarithm function that fits this data. Then choose the scales for a graph of the data based on real-world factors. (*Note*: The earthquake magnitude is the power of an exponent, which can be positive, negative, or 0.) Describe the shape of the resulting graph.

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1. Identify the dependent and independent variables.

   The number of earthquakes N depends on the magnitude m, so N is the dependent variable and m is the independent variable.

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2. Use these variables to write the general form of the natural logarithm function.

   The parent function of the natural logarithm function is $f(x) = \ln x$, so this function will be of the form $N(m) = \ln m$.

---

3. Use a graphing calculator to generate a natural logarithm function from the table data.

   You will need to refer to the resulting table of values later, so keep your work on the calculator screen.

   On a TI-83/84:

   Step 1: Press [2ND][Y=] to bring up the STAT PLOTS menu.

   Step 2: The first option, Plot 1, will already be highlighted. Press [ENTER].

   Step 3: Under Plot 1, select ON if it isn't selected already.

   (*continued*)

Step 4: Press [STAT]. Select 1: Edit... to open a table. Arrow up to L1, then press [CLEAR][ENTER] to clear the list. Repeat as needed. Press [ENTER].

Step 5: From L1, press the down arrow to move your cursor into the list. Enter the values from the table for each magnitude, pressing [ENTER] after each number to navigate to the next blank spot in the list.

Step 6: Arrow over to the L2 column. Enter the corresponding values representing the number of earthquakes for each magnitude.

Step 7: To fit an equation to the data points, press [STAT] and arrow over to the CALC menu. Then, select 9: LnReg, "natural logarithm regression."

Step 8: Press [2ND][1] to type "L1" for Xlist. Press [,], then arrow down to Ylist and press [2ND][2] to type "L2" for Ylist, if not already shown.

Step 9: Arrow down to "Calculate" and press [ENTER] to generate a natural logarithm function of the form $y = a + b \cdot \ln x$.

On a TI-Nspire:

Step 1: Press [home]. Arrow over to the spreadsheet icon, the fourth icon from the left, and press [enter].

Step 2: To clear the lists in your calculator, arrow up to the topmost cell of the table to highlight the entire column, then press [menu]. Choose 3: Data, then 4: Clear Data. Repeat for each column as necessary.

Step 3: Arrow over to column A. Name the column "m" and enter the values from the table for each magnitude, pressing [enter] to navigate to the next blank cell.

Step 4: Arrow over to column B. Name the column "n" and enter the corresponding values representing the number of earthquakes for each magnitude.

(*continued*)

Step 5: To generate a natural logarithm function with the data, press [menu]. Select 4: Statistics, then 1: Stat Calculations, and then B: Logarithmic Regression…. Press [enter].

Step 6: Use the pop-up menus to enter "m" in the X List field and "n" in the Y List field. Tab to OK and press [enter]. This will generate a natural logarithm function of the form $y = a + b \cdot \ln x$. The coefficients appear on the right side of the screen in column D.

Either calculator will return approximate results of $a \approx 1075.78$ and $b \approx -520.92$.

4. Write the function and enter it on the graphing calculator.

To write a natural logarithm function of the form $y = a + b \cdot \ln x$, let $y = N(m)$, $a = 1075.78$, $b = -520.92$, and $x = m$.

The function that approximately fits the table data is $N(m) = 1075.78 - 520.92 \cdot \ln m$.

Enter the function on the graphing calculator.

5. Select a scale for the dependent and independent variable axes on a graph of the data.

The m values range from 6.0 to 8.5, so an independent variable scale could be from $m = 0$ to $m = 10$. The N values range from 0 to 210, so a dependent variable scale could be from $N = 0$ to $N = 250$ with a scale of 25 for each interval on the axis. Other scales are acceptable as long as they show all of the data points in the table.

6. Use the results to adjust the scales of the graphing calculator's viewing window, then graph the natural logarithm function and table data.

On a TI-83/84:

Step 1: Press [WINDOW] and change the x and y scales to the values identified earlier: Xmin = 0, Xmax = 10, Xscl = 1, Ymin = 0, Ymax = 250, and Yscl = 25.

Step 2: Press [GRAPH] to view the data plot and function.

(*continued*)

On a TI-Nspire:

Step 1: Press [home]. Arrow over to the graphing icon, the second icon from the left, and press [enter].

Step 2: Press [menu], then select 4: Window/Zoom, and then 1: Window Settings. Change the settings as identified earlier: XMin = 5, XMax = 10, XScale = Auto, YMin = 0, YMax = 250, and YScale = 25. Tab to "OK" and press [enter] to save the settings.

Step 3: Press [menu] when the new window appears and select 3: Graph Type and 4: Scatter Plot.

Step 4: At the bottom of the graph window, beside x type m. Press [tab].

Step 5: Beside y, type n. Press [enter]. The points will appear on the graph.

The result on either calculator includes the data points and the natural logarithm function. (*Note*: Save this data for use with Example 5.)

$$N(m) = 1075.78 - 520.92 \cdot \ln m$$
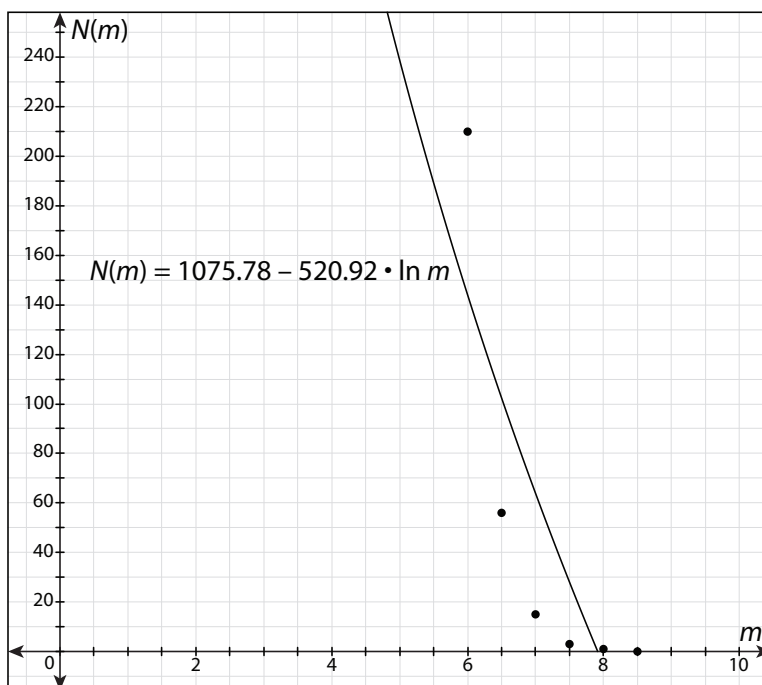
7. Describe the shape of the graph in terms of how the function values change as the earthquake magnitude increases from 6.0 to 8.5 and in terms of changes in N as m changes.

The function value decreases from $m = 6.0$ to $m = 8.5$, and the *rate* of increase also decreases from $m = 6.0$ to $m = 8.5$.

The value of N decreases with each step in the value of m(0.5), but the amount of decrease is constant over each interval. For example, N decreases from 210 to 56 for m from 6.0 to 6.5, and N decreases from 56 to 15 for m from 6.5 to 7.

Example 5

Modify the earthquake data from Example 4 (repeated below) by adding a second dependent variable in the graphing calculator table for the natural logarithm of the number of earthquakes—ln N—of each value of m. Graph this new variable and the earthquake magnitude m and describe the resulting graph. Suggest an explanation for the shape and domain of the new graph given that the earthquake magnitude m is defined as the power of an exponent. What restrictions exist on m?

Magnitude, m	6.0	6.5	7.0	7.5	8.0	8.5
Number, N	210	56	15	3	1	0

1. Modify the graphing calculator data from Example 4 to add values for ln N of each value of m.

On your graphing calculator, return to the table of values for m and N that you created in Example 4. (If you need to enter the values again, refer to the calculator directions in that example.)

On a TI-83/84:

Note: Recall that L1 represents the independent variable m and L2 represents the dependent variable N; therefore, L3 will represent the third variable, ln N.

Step 1: Press [2ND][Y=] to bring up the STAT PLOTS menu.

Step 2: Select 1: Plot 1..., then arrow down to Data List and press [2ND][3] to change the variable to L3. Press [ENTER] to save the change.

(*continued*)

Step 3: Press [STAT]. Select 1: Edit… and press [ENTER].

Step 4: The first *N* value from column L2 is cell L2(1). To enter the natural logarithm of this value, arrow to the first blank cell in L3. Press [LN][2ND][2][(][1][)]. Press [ENTER].

Step 5: Repeat Step 4 for the other nonzero numbers in the L2 column, following the pattern [LN][2ND][2] and changing the number in parentheses depending on the row.

Step 6: Delete the data from cells L1(6) and L2(6) by pressing [DEL] in each of those cells so the calculator can produce the graph for the variables L1 and L3.

On a TI-Nspire:

Step 1: Press [ctrl] and then press up on the NavPad to view all open documents. Select the table document from Example 4.

Step 2: Navigate to the first cell in column C and press up on the NavPad to highlight the column. Press [clear] to delete the data. Repeat for column D.

Step 3: Highlight the top cell in column C. Use your keypad to enter a name for the natural logarithm of *N*, such as "lnn." (*Note*: The "ln" operation key cannot be used for this since it is a variable name.)

Step 4: The first *N* value from column B is cell B1. To enter the natural logarithm of this value, arrow to the first blank cell in column C. Press [=][ctrl][e^x][B][1]. Press [enter].

Step 5: Repeat Step 4 for the other nonzero numbers in the B column, following the pattern [=][ctrl][e^x][B] and changing the last number depending on the row. Do not repeat for the last cell, B6, because the logarithm of 0 is undefined.

Step 6: Clear the data from cells A6 and B6 so the calculator can calculate the graph for the *m* and "lnn" variables.

2. Select a scale for the dependent and independent variable axes on a graph of the revised data.

The settings for the independent variable m can be the same as those used in Example 4, since those did not change. However, the values of the dependent variable "lnn" vary between 0 and 5.3471, so this scale should be reset to values from −1 to 6. Other scales are acceptable as long as they show all of the data points.

On a TI-83/84:

Step 1: Press [WINDOW] and change the x and y scales as needed to reflect the modified values: Xmin = −1, Xmax = 10, Xscl = 1, Ymin = −1, Ymax = 6, and Yscl = 1.

Step 2: Press [GRAPH] to view the data plot.

On a TI-Nspire:

Step 1: Press [home]. Arrow over to the graphing icon, the second icon from the left, and press [enter].

Step 2: Press [menu], then select 4: Window/Zoom, and then 1: Window Settings. Change the settings as needed to reflect the modified values: XMin = −1, XMax = 10, XScale = Auto, YMin = −1, YMax = 6, and YScale = Auto. Tab to "OK" and press [enter] to save the settings.

Step 3: Press [menu] when the new window appears and select 3: Graph Type and 4: Scatter Plot.

Step 4: At the bottom of the graph window, beside x type m. Press [tab].

Step 5: Beside y, type lnn. Press [enter]. The points will appear on the graph.

(continued)

The resulting graph is as follows.
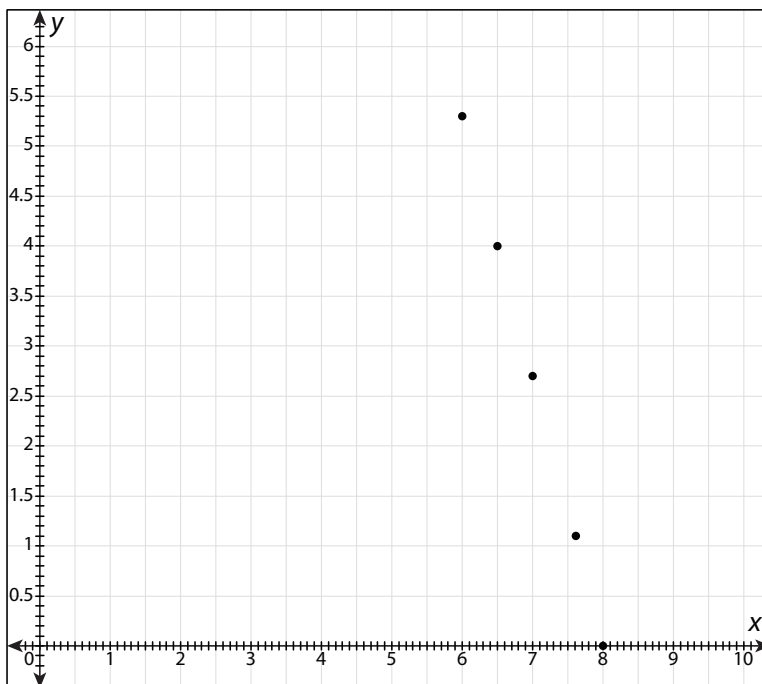
3. Describe the shape of the graph.

   The graph is a straight line with a negative slope.

4. Suggest an explanation for the shape and domain of the new graph based on the definition of m from the problem statement.

   The variable m is described as "the power of an exponent," which is also a definition of a logarithm. This means that in this graph, one logarithm is plotted against another logarithm plus a constant. The average rate of change of one logarithm relative to the other logarithm is constant in this case, which results in a straight line.

   Mathematically, since $\ln N$ is undefined for $m = 8.5$, the domain of the quake magnitudes is restricted to values of m that are less than 8.5. However, since this is a logarithm, recall that the independent variable m will not be less than or equal to 0, as evidenced by the point $(8.5, 0)$. So, since for this real-world situation $\ln N$ cannot be less than 0, the domain of the quake magnitudes is restricted to values of m that are less than or equal to 8.5.

UNIT 4B • MATHEMATICAL MODELING AND CHOOSING A MODEL
Lesson 3: Comparing Properties Within and Between Functions

Practice 4B.3.1: Reading and Identifying Key Features of Real-World Situation Graphs

The following graph shows the time it takes for three different models of microcircuit lithography-fabricator robots to complete the 10 steps in the production of a tablet computer CPU. The functions $f(t)$, $g(t)$, and $h(t)$ model the productivity of the three robots as they produce a finished and ready-to-test CPU for the next stage of the tablet computer assembly. Use the graph to complete problems 1–3.

Hours
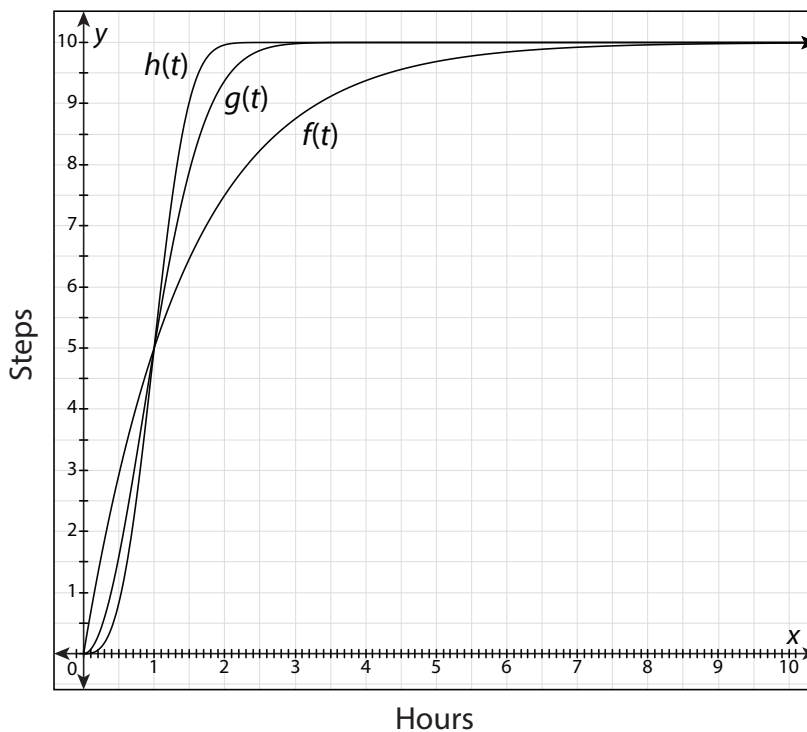
1. Describe the behavior of the function model for each of the three robots as they complete the 10 steps over the time interval shown.

2. What is the significance of the point of intersection of the graphs of the three models?

3. Compare the time at which the three robots complete the 10 steps in the real-world of manufacturing and compare that time to the time at which each of the mathematical models indicate that the 10 steps are completed.

continued

The following graph shows data points representing the increase in the yield of corn $Y(t)$ from 1950 through 2010 in metric tons per hectare (10,000 square meters). The curved line represents a function model of the varying amounts of nitrogen applied to those crops in kilograms per hectare for that time period, which is given by $N(t) = 180(1 - 2^{-t})$. The year 1950 corresponds to $x = 1$, the year 1960 to $x = 2$, and so on. The two different dependent variables are shown on the same graph by using a different y-axis scale for each when analysis or calculations are done. Use the graph to complete problems 4–6.
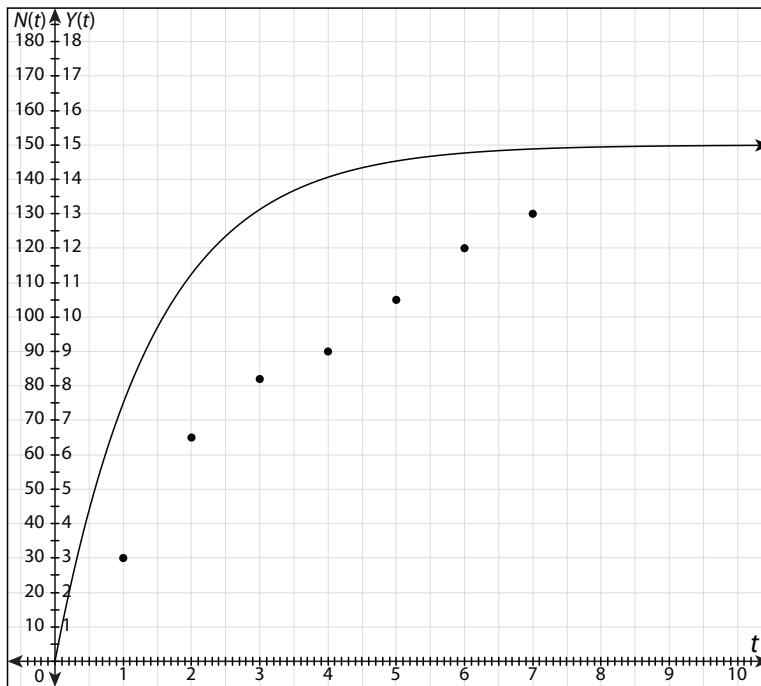
4. The y-axis scale for the corn production runs from 0 to 18. Describe a function model or combination of models for the corn-production data points.

5. Determine an appropriate y-axis scale for the amount of nitrogen applied per hectare. Describe how the amount of nitrogen applied changes over the time period from 1950 to 2010.

6. Compare the function you picked for the rate of corn production with the given function model for the rate of application of nitrogen to the crops over the period from 1950 to 2010. Compare and contrast the behavior of each function.

continued

UNIT 4B • MATHEMATICAL MODELING AND CHOOSING A MODEL
Lesson 3: Comparing Properties Within and Between Functions

The following graph shows two sine-function models for the growth in numbers of hybrid-electric cars, $H(t)$, and battery-electric cars, $B(t)$, from 2007 and projected out to 2020, for which t is the time in years. The year 2007 corresponds to $x = 1$, the year 2008 to $x = 2$, and so on. The y-axis is the historical and projected market share of the two types of vehicles from 0 percent to 6 percent. Use the graph to complete problems 7–10.
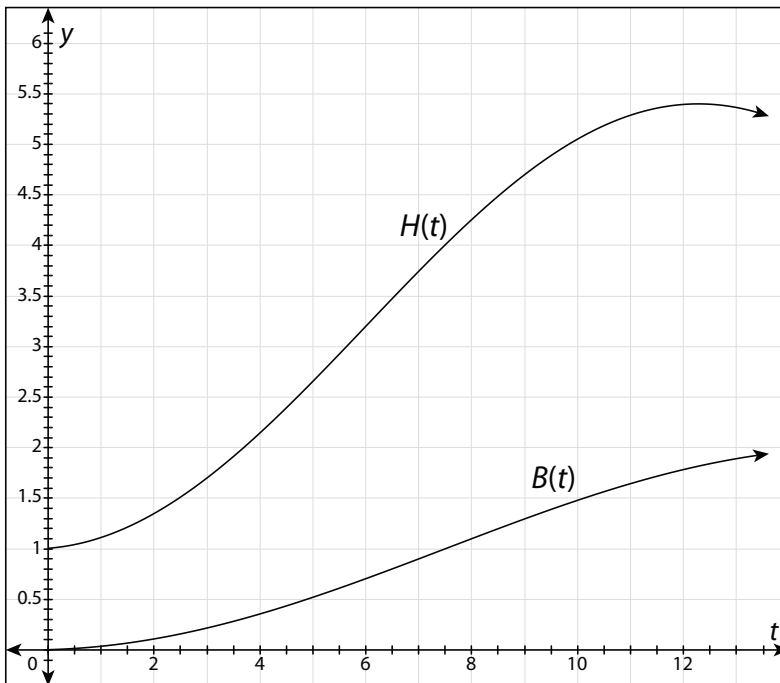
7. Determine the restricted domain of the function $H(t) = 3.2 + 2.2 \sin (0.25 \bullet t - 1.5)$ and its range over that domain.

8. What does the function value $H(0)$ describe?

9. Determine the range of the function $B(t) = 1 + \sin (0.2 \bullet t - 1.5)$ over the restricted domain of $H(t)$.

10. Predict which type of vehicle will have the greater rate of market-share increase based on these models. Explain your answer in terms of the period of the two sine-function models.

Lesson 4B.3.2: Calculating Average Rates of Change
Introduction

The **rate of change** is a ratio that describes how much one quantity changes with respect to the change in another quantity. For a function, the rate of change is the amount by which function values change for each change in domain values. When the rate of change of a function increases by the same amount for every consecutive domain value, the rate of change is said to be constant. Linear functions exhibit this trait; however, not all functions have a constant rate of change.

Key Concepts

- The rate of change of a single function is dependent on the change that occurs between two points (ordered pairs) on the graph of that function. Recall that this is the same as the slope when the function is a line.

- The rate of change of a function is the difference between the values of the function at two different domain values; mathematically, this can be represented by the formula $r = \dfrac{f_1(x) - f_2(x)}{x_1 - x_2} = \dfrac{f_2(x) - f_1(x)}{x_2 - x_1} = \dfrac{\Delta f(x)}{\Delta x}$, in which $f_1(x)$ is the function value at the domain value x_1 and $f_2(x)$ is the function value at the domain value x_2.

- The uppercase Greek letter ***delta*** **(Δ)** is commonly used to represent the "change" in a value; for example, $\Delta f(x)$ can be read as "change in f of x." Therefore, $\Delta f(x)$ and Δx are more concise ways of representing the numerator and denominator, respectively, in the rate of change formula.

- Note that the order in which the function values are compared must be the same as the order in which the domain values are compared. Mixing up the order of the values in the numerator with the order of the values in the denominator will result in an incorrect rate calculation.

- The **average rate of change** is the average of all of the different rates of change of a function value over a restricted domain. Formally, it is the ratio of the difference of output values to the difference of the corresponding input values: $\dfrac{f(b) - f(a)}{b - a}$.

- The average rate of change of a function or a specific variable is useful in those mathematical or real-world problems in which the average rate of change of different function values varies over equal restricted domain intervals.

- The average rate of change is often calculated by substituting the function values that correspond to the endpoints or bounds of a restricted domain interval into the formula for calculating the rate of change.

- The rate of change of a function or variable in a real-world situation can be positive, negative, 0, undefined, or a constant, depending on the situation. In particular, the rate of change of a variable can be negative or 0 even if the variable values are always positive, such as in problems involving area, mass, time, volume, and any number of similar quantities.

- A special case of the constant rate of change can be found with linear functions. For a linear function $ax + by = c$, the rate of change of the function values y is a constant given by the quantity $-\dfrac{a}{b}$. This quantity is the slope of the straight-line graph that represents the linear function. In the slope-intercept form of the linear function $f(x) = mx + b$, the constant m is the rate of change of the function values.

- The average rate of change of a quantity can have special meaning in situations in which the restricted domain over which it is computed is very small or approaches 0. A graphing calculator can be helpful in approximating or visualizing the rate of change in such applications.

Guided Practice 4B.3.2

Example 1

Compare the average rate of change of the function $f(x) = \sin x$ over the restricted domain $[0°, 45°]$ with the average rate of change of the same function over the restricted domain $[45°, 90°]$.

1. Calculate the values of $f(0°)$ and $f(45°)$.

   Substitute $0°$ and $45°$ for x.

   $f(x) = \sin x$ $f(x) = \sin x$

   $f(0°) = \sin (0°)$ $f(45°) = \sin (45°)$

   $f(0°) = 0$ $f(45°) = \dfrac{\sqrt{2}}{2}$

2. Convert degree measures to radian measures.

   In order for the rate of change formula to compare like units, all the measures must be expressed in the same units. $0°$ is the same as 0 radians, and $\dfrac{\sqrt{2}}{2}$ is already expressed in radians. Therefore, the remaining degree measure, $45°$, must be converted to radians. Recall the conversion formula, $1 \text{ degree} = \dfrac{\pi}{180} \text{ radian}$.

   Therefore, $45°$ is $45° \bullet \dfrac{\pi}{180} \text{ radian}$ or $\dfrac{\pi}{4} \text{ radian}$.

3. Calculate the rate of change over the restricted domain [0°, 45°].

Substitute the results from the previous steps into the formula for the rate of change.

$$r = \frac{f_1(x) - f_2(x)}{x_1 - x_2}$$ Rate of change formula

$$r = \frac{\left(\dfrac{\sqrt{2}}{2}\right) - (0)}{\left(\dfrac{\pi}{4}\right) - (0)}$$ Substitute $\dfrac{\sqrt{2}}{2}$ for $f_1(x)$, 0 for $f_2(x)$, $\dfrac{\pi}{4}$ for x_1, and 0 for x_2.

$$r = \frac{4\sqrt{2}}{2\pi}$$ Simplify.

$$r = \frac{2\sqrt{2}}{\pi}$$

$$r \approx 0.9$$

The rate of change over the restricted domain [0°, 45°] is approximately 0.9 radian.

4. Calculate the rate of change over the restricted domain [45°, 90°].

As determined in step 1, $f(45°) = \dfrac{\sqrt{2}}{2}$.

Calculate the value of $f(90°)$.

$f(x) = \sin x$

$f(90°) = \sin(90°)$

$f(90°) = 1$

Convert degree measures to radian measures.

As found in step 2, 45° is equal to $\dfrac{\pi}{4}$ radian.

(continued)

Using the conversion formula, 90° is $90° \cdot \dfrac{\pi}{180}$ radian or $\dfrac{\pi}{2}$ radians.

Substitute the results of the previous steps into the formula for the rate of change.

$r = \dfrac{f_1(x) - f_2(x)}{x_1 - x_2}$ Rate of change formula

$r = \dfrac{\left(\dfrac{\sqrt{2}}{2}\right) - (1)}{\left(\dfrac{\pi}{4}\right) - \left(\dfrac{\pi}{2}\right)}$ Substitute $\dfrac{\sqrt{2}}{2}$ for $f_1(x)$, 1 for $f_2(x)$, $\dfrac{\pi}{4}$ for x_1, and $\dfrac{\pi}{2}$ for x_2.

$r = \dfrac{\dfrac{\sqrt{2} - 2}{2}}{\dfrac{\pi - 2\pi}{4}}$ Simplify.

$r = \dfrac{4\left(\sqrt{2} - 2\right)}{-2\pi}$ Continue to simplify.

$r = \dfrac{2\left(2 - \sqrt{2}\right)}{\pi}$

$r \approx 0.37$

The rate of change over the restricted domain [45°, 90°] is approximately 0.37 radian.

5. Summarize the results of steps 1–4 for the two average rates of change for $f(x) = \sin x$.

The function is increasing over both restricted domains. However, the rate of change of the increase is greater over the interval [0°, 45°] than over the interval [45°, 90°].

Example 2

Two school clubs are selling plants to raise money for a trip. The juggling club started out with 250 plants and had 30 left after selling them for 7 days. The mock trial club started out with 450 plants and had 50 left after selling them for 8 days. Compare the rates at which the two clubs are selling their plants. If the two clubs started selling plants at the same time, which club will sell out of plants first? Explain.

---

1. Determine how many plants the juggling club has sold.

   The juggling club has sold 250 – 30 or 220 plants.

---

2. Calculate the average number of plants the juggling club sold per day.

   The juggling club sold 220 plants over 7 days. Divide to determine the average.

   $$\frac{220}{7} \approx 31$$

   The juggling club sold an average of approximately 31 plants each day.

---

3. Determine how many plants the mock trial club has sold.

   The mock trial club has sold 450 – 50 or 400 plants.

---

4. Calculate the average number of plants the mock trial club sold per day.

   The mock trial club sold 400 plants over 8 days. Divide to determine the average.

   $$\frac{400}{8} = 50$$

   The mock trial club sold an average of 50 plants each day.

5. Describe the rates at which the two clubs sold their plants. Compare the kinds of numbers that represent days, plants, and rates.

The variables "days" and "plants" are positive whole numbers, whereas "rate" can be a negative number. In this situation, the rates for the two clubs are negative, because the rates reflect a reduction in the number of plants each club has in stock. The rate for the juggling club is approximately –31 plants per day because the quantity of plants is being reduced by an average of about 31 per day; similarly, the rate for the mock trial club is –50 plants per day.

6. Calculate how long it will take the juggling club to sell its remaining plants.

The juggling club has 30 plants left and is selling them at a rate of about –31 plants per day, so the juggling club's plants will sell out in 1 day.

7. Calculate how long it will take the mock trial club to sell its remaining plants.

The mock trial club has 50 plants left and is selling them at a rate of –50 plants per day, so the mock trial club's plants will sell out in 1 day.

8. Explain which club will sell out of plants first.

Both clubs will sell off their remaining stock in 1 day. However, assuming each club started selling the plants on the same day, the juggling club will sell out first because it took 1 day less to sell off its initial stock of plants. (Recall that the juggling club sold plants for 7 days and the mock trial club sold plants for 8 days.)

Example 3

Find the average rate of change of the function $f(x) = x^3 + 2x^2 - x - 2$ over the restricted intervals $[-2, -1]$ and $[-1, 1]$. Determine if the rate of change over both intervals is positive, negative, 0, undefined, and/or constant. Use the graph of $f(x)$ to explain how the average rate of change relates to the behavior of the graph across its whole domain.

1. Calculate the average rate of change of $f(x)$ over the interval $[-2, -1]$.

   First find the values of $f(-2)$ and $f(-1)$.

   Calculate $f(-2)$.

$f(x) = x^3 + 2x^2 - x - 2$	Original function
$f(-2) = (-2)^3 + 2(-2)^2 - (-2) - 2$	Substitute -2 for x.
$f(-2) = -8 + 8 + 2 - 2$	Simplify.
$f(-2) = 0$	

   Calculate $f(-1)$.

$f(x) = x^3 + 2x^2 - x - 2$	Original function
$f(-1) = (-1)^3 + 2(-1)^2 - (-1) - 2$	Substitute -1 for x.
$f(-1) = -1 + 2 + 1 - 2$	Simplify.
$f(-1) = 0$	

   Substitute the values for $f(-2)$ and $f(-1)$ into the rate of change formula and solve.

$r = \dfrac{f_1(x) - f_2(x)}{x_1 - x_2}$	Rate of change formula
$r = \dfrac{(0) - (0)}{(-2) - (-1)}$	Substitute 0 for $f_1(x)$, 0 for $f_2(x)$, -2 for x_1, and -1 for x_2.
$r = 0$	Simplify.

   The rate of change of $f(x)$ over the interval $[-2, -1]$ is 0.

2. Calculate the average rate of change of $f(x)$ over the interval $[-1, 1]$.

The value of $f(-1)$ is 0, as found in the previous step.

Find the value of $f(1)$.

Calculate $f(1)$.

$f(x) = x^3 + 2x^2 - x - 2$ Original function

$f(1) = (1)^3 + 2(1)^2 - (1) - 2$ Substitute 1 for x.

$f(1) = 1 + 2 - 1 - 2$ Simplify.

$f(1) = 0$

Substitute the values for $f(-1)$ and $f(1)$ in the rate of change formula.

$$r = \frac{f_1(x) - f_2(x)}{x_1 - x_2}$$ Rate of change formula

$$r = \frac{(0) - (0)}{(-1) - (1)}$$ Substitute 0 for $f_1(x)$, 0 for $f_2(x)$, -1 for x_1, and 1 for x_2.

$r = 0$ Simplify.

The rate of change of $f(x)$ over the interval $[-1, 1]$ is 0.

3. Determine if the rate of change over both intervals is positive, negative, 0, undefined, and/or constant.

The rate of change that occurred over each of the two intervals is 0 and constant.

4. Graph $f(x)$.

Use a graphing calculator to view the graph of $f(x)$.
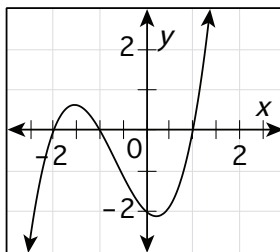
5. Describe what happens to the function values over the restricted domains [–2, –1] and [–1, 1].

The function value goes from 0 to a maximum value and back to 0 over the restricted domain [–2, –1].

The function value goes from 0 to a minimum value and back to 0 over the restricted domain [–1, 1]. For the domain interval $(-\infty, -2)$, the function value range is given by the interval $(-\infty, 0)$; for the domain interval $(1, +\infty)$, the function value range is given by the interval $(0, +\infty)$.

Example 4

Three linear functions—$f(x)$, $g(x)$, and $h(x)$—share a common intersection point of (1, 1) on a coordinate plane. The function $f(x)$ also passes through the point (–2, –3), while the function $g(x)$ passes through the point (–2, 3), and the function $h(x)$ passes through the point (2, 3). Which function has the greatest rate of change? What are the equations of the functions?

1. Calculate the rate of change for $f(x)$.

The rate of change formula can be applied to the two points that satisfy the function $f(x)$: (1, 1) and (–2, –3).

$$r = \frac{f_1(x) - f_2(x)}{x_1 - x_2}$$ Rate of change formula

$$r = \frac{(1) - (-3)}{(1) - (-2)}$$ Substitute 1 for $f_1(x)$, –3 for $f_2(x)$, 1 for x_1, and –2 for x_2.

$$r = \frac{4}{3}$$ Simplify.

The rate of change for $f(x)$ is $\frac{4}{3}$.

2. Calculate the rate of change for $g(x)$.

Substitute the two points that satisfy $g(x)$, (1, 1) and (–2, 3), into the rate of change formula.

$$r = \frac{f_1(x) - f_2(x)}{x_1 - x_2}$$ Rate of change formula

$$r = \frac{(1) - (3)}{(1) - (-2)}$$ Substitute 1 for $f_1(x)$, 3 for $f_2(x)$, 1 for x_1, and –2 for x_2.

$$r = -\frac{2}{3}$$ Simplify.

The rate of change for $g(x)$ is $-\frac{2}{3}$.

3. Calculate the rate of change for $h(x)$.

Substitute the two points that satisfy $h(x)$, (1, 1) and (2, 3), into the rate of change formula.

$$r = \frac{f_1(x) - f_2(x)}{x_1 - x_2}$$ Rate of change formula

$$r = \frac{(1) - (3)}{(1) - (2)}$$ Substitute 1 for $f_1(x)$, 3 for $f_2(x)$, 1 for x_1, and 2 for x_2.

$$r = 2$$ Simplify.

The rate of change for $h(x)$ is 2.

4. Compare the rates of change for the three functions.

The rate of change for $f(x)$ is $\frac{4}{3}$, the rate of change for $g(x)$ is $-\frac{2}{3}$, and the rate of change for $h(x)$ is 2. This can be written symbolically using the inequality $r_{h(x)} > r_{f(x)} > r_{g(x)}$. Therefore, $h(x)$ has the greatest rate of change.

5. Use the point-slope form of a linear equation to write the equation of each function.

The point-slope form of a linear equation is $y - y_1 = m(x - x_1)$, in which m is the rate of change and x_1 and y_1 are coordinates of a point that makes the function a true statement. Use the shared point $(1, 1)$ for the point for each function.

For $f(x)$:

$$y - y_1 = m(x - x_1)$$ Point-slope formula

$$f(x) - (1) = \left(\frac{4}{3}\right)\left[x - (1)\right]$$ Substitute 1 for x_1, 1 for y_1, and $\frac{4}{3}$ for m.

$$f(x) = \frac{4}{3}x - \frac{1}{3}$$ Simplify.

For $g(x)$:

$$y - y_1 = m(x - x_1)$$ Point-slope formula

$$g(x) - (1) = \left(-\frac{2}{3}\right)\left[x - (1)\right]$$ Substitute 1 for x_1, 1 for y_1, and $-\frac{2}{3}$ for m.

$$g(x) = -\frac{2}{3}x + \frac{5}{3}$$ Simplify.

For $h(x)$:

$$y - y_1 = m(x - x_1)$$ Point-slope formula

$$h(x) - (1) = (2)[x - (1)]$$ Substitute 1 for x_1, 1 for y_1, and 2 for m.

$$h(x) = 2x - 1$$ Simplify.

The equations for the three functions are $f(x) = \frac{4}{3}x - \frac{1}{3}$, $g(x) = -\frac{2}{3}x + \frac{5}{3}$, and $h(x) = 2x - 1$.

Example 5

The table gives the function values for $f(x) = \log x$ over the restricted domain [0.001, 1,000]. Use the data to calculate and compare the rates of change of $f(x)$ for the intervals [0.001, 0.01], [0.01, 0.1], [0.1, 1], and [1, 10]. Describe how the rates change as the values of x become larger and smaller.

x	0.001	0.01	0.1	1	10	100	1,000
$f(x)$	−3	−2	−1	0	1	2	3

1. Calculate the rate of change of $f(x)$ over the interval [0.001, 0.01].

   Substitute the table values for x and $f(x)$ into the rate of change formula, $r = \dfrac{\Delta f(x)}{\Delta x}$.

   $r = \dfrac{\Delta f(x)}{\Delta x}$ Rate of change formula

   $r = \dfrac{(-3)-(-2)}{(0.001)-(0.01)}$ Substitute $(-3) - (-2)$ for $\Delta f(x)$ and $(0.001) - (0.01)$ for Δx.

   $r = \dfrac{-1}{-0.009}$ Simplify.

   $r = 111.\overline{1}$

   The rate of change of $f(x)$ over the interval [0.001, 0.01] is $111.\overline{1}$.

2. Calculate the rate of change of $f(x)$ over the interval [0.01, 0.1].

   Substitute the table values for x and $f(x)$ into $r = \dfrac{\Delta f(x)}{\Delta x}$.

   $r = \dfrac{\Delta f(x)}{\Delta x}$ Formula

   $r = \dfrac{(-2)-(-1)}{(0.01)-(0.1)}$ Substitute $(-2) - (-1)$ for $\Delta f(x)$ and $(0.01) - (0.1)$ for Δx.

   $r = \dfrac{-1}{-0.09}$ Simplify.

   $r = 11.\overline{1}$

   The rate of change of $f(x)$ over the interval [0.01, 0.1] is $11.\overline{1}$.

3. Calculate the rate of change of $f(x)$ over the interval $[0.1, 1]$.

Substitute the table values for x and $f(x)$ into $r = \dfrac{\Delta f(x)}{\Delta x}$.

$r = \dfrac{\Delta f(x)}{\Delta x}$ Formula

$r = \dfrac{(-1)-(0)}{(0.1)-(1)}$ Substitute $(-1) - (0)$ for $\Delta f(x)$ and $(0.1) - (1)$ for Δx.

$r = \dfrac{-1}{-0.9}$ Simplify.

$r = 1.\overline{1}$

The rate of change of $f(x)$ over the interval $[0.1, 1]$ is $1.\overline{1}$.

---

4. Calculate the rate of change of $f(x)$ over the interval $[1, 10]$.

Substitute the table values for x and $f(x)$ into $r = \dfrac{\Delta f(x)}{\Delta x}$.

$r = \dfrac{\Delta f(x)}{\Delta x}$ Formula

$r = \dfrac{(0)-(1)}{(1)-(10)}$ Substitute $(0) - (1)$ for $\Delta f(x)$ and $(1) - (10)$ for Δx.

$r = \dfrac{-1}{-9}$ Simplify.

$r = 0.\overline{1}$

The rate of change of $f(x)$ over the interval $[1, 10]$ is $0.\overline{1}$.

---

5. Describe how the rate of change will change as the values of the domain interval increase.

Compare the rate of change r for each interval of interest for $f(x)$:

- For $[0.001, 0.01]$, $r = 111.\overline{1}$.
- For $[0.01, 0.1]$, $r = 11.\overline{1}$.
- For $[0.1, 1]$, $r = 1.\overline{1}$.
- For $[1, 10]$, $r = 0.\overline{1}$.

As the values of x increase, the rate of change decreases.

Practice 4B.3.2: Calculating Average Rates of Change

For problems 1–4, find the average rate of change between $x = 0$ and $x = 1$ for the function given.

1. $a(x) = 2^x - x^2$

2. $b(x) = x^3 - 3^x$

3. $c(x) = \sin(x - 1)$

4. $d(x) = -\cos(1 - x)$

For problems 5–7, determine the average rate of change r over the domain intervals $[-2, -1]$ and $[-1, 0]$ for the function given.

5. $f(x) = \dfrac{x}{x - 1}$

6. $g(x) = \sin 2x$

7. $h(x) = \log(x + 3)$

UNIT 4B • MATHEMATICAL MODELING AND CHOOSING A MODEL
Lesson 3: Comparing Properties Within and Between Functions

The following graph shows the amount of nitrogen fertilizer needed to produce the leaves and stems of two different varieties of corn for use as biofuel stock. The function $f(x)$ represents one variety, and the function $g(x)$ represents the other. Over the "leaf" interval [0, 2.25], the plant uses the nitrogen mostly for growing the leaves. Over the "stem" interval [2.25, 9], the plant uses the nitrogen mostly for growing the stems, which are of the most use in producing biofuel. Use the graph to complete problems 8–10.
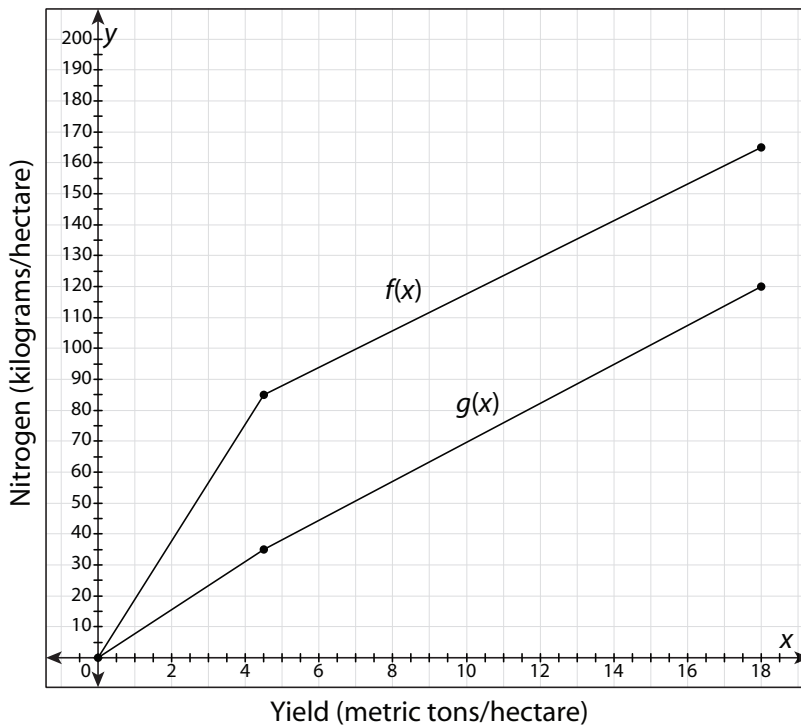
8. Compare the average rates of change of the fertilizer needed for the two different corn varieties represented by the graphs of $f(x)$ and $g(x)$ over the "leaf" interval.

9. Compare the average rates of change of the fertilizer needed for the two different corn varieties represented by the graphs of $f(x)$ and $g(x)$ over the "stem" interval.

10. Which variety of corn would most likely be preferred by biofuel producers? Why?

Lesson 4B.3.3: Comparing Functions

Introduction

Functions can be compared for a variety of purposes. Functions of the same type—for example, quadratic functions—can be compared on the basis of maximum or minimum function values, symmetry about an axis or other vertical line on the coordinate plane, or on the basis of how well the function fits real-world data.

Functions of different types—such as polynomial functions and rational functions—can likewise be compared on the basis of one or more characteristics, such as the number of real solutions or the rate of change over a restricted domain interval.

Key Concepts

- The function characteristics that are used to compare and contrast function types can depend on the purpose of the comparison.

- Recall that the extrema of a function, or the points on a graph of a function at which a maximum or minimum function value occurs, can apply to the function across its entire domain or they can be localized, as in the case of polynomial functions of degree three and higher.

- For example, a cubic equation might be used to model a real-world problem that has a single solution. Graphing a proposed model for the problem can quickly show whether the model meets the criteria of having only one real solution.

- The rate of change of a function value can be calculated to determine which of several different functions (or functions of the same form) best represent a condition of decreasing or increasing function values.

- The symmetry exhibited by a function can be about a vertical line on a coordinate plane or it can be exhibited across a line of symmetry that takes the form of a linear function.

- Symmetry of domain and function values can also be observed in the algebraic form of a function, such as in even or odd functions.

- Domains and ranges can also be used to compare the validity of a function serving as a model for real-world phenomena. In some cases, only a restricted domain or range of a mathematical model is relevant to a specific problem.

- A function's behavior at domain values that are very large or small can be relevant to some real-world problems. Functions might serve as reliable models over a narrowly restricted mathematical domain, but might fail to describe real-world phenomena when domain values increase or decrease by large values.

Guided Practice 4B.3.3

Example 1

Two functions, $f(x) = -2(x - 5)^2 + 3$ and $g(x) = -4x^2 + 8x + 3$, each represent the volume of a rectangular solid. Which solid's function, $f(x)$ or $g(x)$, has the greater maximum volume?

1. Determine the coordinates of the maximum point of $f(x)$.

   To determine the coordinates of the maximum point of a function, first determine the type of function and compare it to the standard form for that function.

   The function $f(x) = -2(x - 5)^2 + 3$ is a quadratic function.

   The vertex form for a quadratic function is $f(x) = a(x - h)^2 + k$, where the coordinates of the maximum ($a < 0$) or minimum ($a > 0$) function value are (h, k).

   Since $f(x) = -2(x - 5)^2 + 3$ is already written in vertex form, it can be seen that for $f(x)$, $h = 5$, $k = 3$ and $a = -2$, which is less than 0; therefore, the maximum point of $f(x)$ is located at $(5, 3)$.

2. Determine the coordinates of the maximum point of $g(x)$.

   $g(x) = -4x^2 + 8x + 3$ is also a quadratic function, but it is written in standard form, $f(x) = ax^2 + bx + c$, where $a = -4$, $b = 8$, and $c = 3$.

   Recall that the x-coordinate of the maximum or minimum of a quadratic in this form is $x = -\dfrac{b}{2a}$.

   $$x = -\frac{b}{2a} \qquad \text{Formula for the } x\text{-coordinate}$$

   $$x = -\frac{(8)}{2(-4)} \qquad \text{Substitute } -4 \text{ for } a \text{ and } 8 \text{ for } b.$$

   $$x = -\frac{8}{-8} \qquad \text{Simplify.}$$

   $$x = 1$$

   (continued)

The x-coordinate of the maximum point of $g(x)$ is 1. To determine the y-coordinate of the maximum, substitute the value of x into the function $g(x)$.

$g(x) = -4x^2 + 8x + 3$	Given function
$g(1) = -4(1)^2 + 8(1) + 3$	Substitute 1 for x.
$g(1) = -4 + 8 + 3$	Simplify.
$g(1) = 7$	

The y-coordinate of the maximum point of $g(x)$ is 7.

The maximum point of $g(x)$ is (1, 7).

3. Determine which function models the greatest maximum volume of its associated rectangular solid.

The function value of the maximum point of $f(x)$ is 3, and the function value of the maximum point of $g(x)$ is 7. Therefore, $g(x)$ represents the function that models the greatest maximum volume for the associated rectangular solid.

Example 2

Sand dune "booming" or singing refers to the audible, tone-specific sounds made by shifting sand dunes under the influence of wind. A team of biophysicists compared the loudness of several sand dunes' booms with the decibel ratings (or sound intensity) of the booms. The table shows some of the team's data, with x representing the loudness x in joules (energy) for each sound and $f(x)$ representing the decibel rating.

x	−0.999	−0.99	−0.9	0	9	99
$f(x)$	−1	0	1	2	3	4

Describe how the domain and function values change over the interval [−0.999, 99]. Then compare the rate of change of the function values over the restricted intervals of [−0.999, −0.99] and [9, 99].

1. Describe how the domain values change over the interval.

Inspect the table to determine how the domain values change. Reading the table values from left to right, the domain values are increasing because each domain value is greater than the one preceding it.

2. Describe how the function values (or range) change over the domain interval.

   The function or range values increase at a constant rate of 1 unit for each of the domain values.

3. Calculate the rate of change for the intervals [–0.999, –0.99] and [9, 99].

   Substitute the table values for x and $f(x)$ into the rate of change formula, $r = \dfrac{\Delta f(x)}{\Delta x}$.

   For the interval [–0.999, –0.99]:

   $$r = \frac{\Delta f(x)}{\Delta x} \qquad \text{Rate of change formula}$$

   $$r = \frac{(-1) - (0)}{(-0.999) - (-0.99)} \qquad \begin{array}{l}\text{Substitute } (-1) - (0) \text{ for } \Delta f(x) \text{ and} \\ (-0.999) - (-0.99) \text{ for } \Delta x.\end{array}$$

   $$r = \frac{-1}{-0.009} \qquad \text{Simplify.}$$

   $$r = 111.\overline{1}$$

   The rate of the change for the interval [–0.999, –0.99] is $111.\overline{1}$.

   For the interval [9, 99]:

   $$r = \frac{\Delta f(x)}{\Delta x} \qquad \text{Rate of change formula}$$

   $$r = \frac{(3) - (4)}{(9) - (99)} \qquad \begin{array}{l}\text{Substitute } (3) - (4) \text{ for } \Delta f(x) \text{ and} \\ (9) - (99) \text{ for } \Delta x.\end{array}$$

   $$r = \frac{-1}{-90} \qquad \text{Simplify.}$$

   $$r = 0.0\overline{1}$$

   The rate of the change for the interval [9, 99] is $0.0\overline{1}$.

4. Compare the rates of change of the function values for the two intervals.

Both rates of change are positive, but the rate of change over the interval [−0.999, −0.99] is 10,000 times greater than the rate of change over the interval [9, 99].

✔

Example 3

Use the data from the table in Example 2 (shown in the table that follows) to find the function that fits the data if the general form of the function is $f(x) = a + \log (x + b)$.

x	−0.999	−0.99	−0.9	0	9	99
$f(x)$	−1	0	1	2	3	4

1. Determine the number of data points that will be needed to find the function.

The general form has four variables—a, b, $f(x)$, and x. However, each data point has both an x and an $f(x)$ value, which leaves two unknowns, a and b. Therefore, two equations in a and b will be needed to find the function.

2. Pick two points from the table and substitute them into the general form of the function to write equations for the points.

Any two points can be chosen. Let the two points be (9, 3) and (99, 4).

For (9, 3):

$f(x) = a + \log (x + b)$ General form

$(3) = a + \log [(9) + b]$ Substitute 9 for x and 3 for $f(x)$.

$3 = a + \log (9 + b)$ Simplify.

(continued)

For (99, 4):

$f(x) = a + \log(x + b)$ — General form

$(4) = a + \log[(99) + b]$ — Substitute 99 for x and 4 for $f(x)$.

$4 = a + \log(99 + b)$ — Simplify.

The equations for the data points (9, 3) and (99, 4) are
$3 = a + \log(9 + b)$ and $4 = a + \log(99 + b)$.

3. Solve the system of resulting equations. Begin by eliminating one of the unknown variables (a or b).

   Either variable can be eliminated; let's eliminate a.

   To eliminate a, subtract one equation from the other.

   $[4 = a + \log(99 + b)] - [3 = a + \log(9 + b)]$ — Set up the subtraction equation.

   $1 = \log(99 + b) - \log(9 + b)$ — Subtract like terms.

   $1 = \log\left(\dfrac{99 + b}{9 + b}\right)$ — Simplify using the quotient rule of logarithms.

   $10^1 = \dfrac{99 + b}{9 + b}$ — Simplify using the definition of a common logarithm.

   $10 \bullet (9 + b) = 99 + b$ — Multiply both sides by $(9 + b)$.

   $90 + 10b = 99 + b$ — Distribute.

   $10b = 9 + b$ — Subtract 90 from both sides.

   $9b = 9$ — Subtract b from both sides.

   $b = 1$ — Divide by 9.

   The value of b is 1.

4. Determine the value of the second unknown variable.

   Substitute the value of b into either equation $4 = a + \log (99 + b)$ or $3 = a + \log (9 + b)$; let's use the latter equation.

$3 = a + \log (9 + b)$	Equation
$3 = a + \log [9 + (1)]$	Substitute 1 for b.
$3 - a = \log 10$	Simplify.
$3 - a = 1$	Simplify the logarithmic term.
$-a = -2$	Subtract 3 from both sides.
$a = 2$	Divide by -1.

   The value of a is 2.

5. Write the function that represents the data in Example 2.

   The general form of the equation is $f(x) = a + \log (x + b)$. Let $a = 2$ and $b = 1$. The resulting function is $f(x) = 2 + \log (x + 1)$.

Example 4

Members of a Florida high school's nature club tracked the estimated number of tree frogs living in a nearby cypress tree for 9 weeks. The estimates are based on the recorded volume of the frogs' singing over a 24-hour period on the same day of each week. The table shows the club's estimates for the number of frogs n per week, with the independent variable t representing time in weeks.

Estimated number of frogs, n	10	14	25	27	29	32	35	36	38
Weeks, t	1	2	3	4	5	6	7	8	9

Use a graphing calculator to graph the data. Then, use the graph to find both an exponential function of the form $f(x) = a \cdot b^x$ and a power function of the form $g(x) = a \cdot x^b$ to fit the data. Compare how closely each function equation models the data on the graph.

1. Create a scatter plot of the table values using a graphing calculator.

   Follow the instructions appropriate to your calculator model.

   On a TI-83/84:

   Step 1: Press [STAT] to bring up the statistics menu. The first option, 1: Edit, will already be highlighted. Press [ENTER].

   Step 2: Arrow up to L1 and press [CLEAR], then [ENTER], to clear the list (or repeatedly hit the DEL key until all old data is deleted). Repeat this process to clear L2 and L3 if needed.

   Step 3: From L1, press the down arrow to move your cursor into the list. Enter each t value from the table into L1, pressing [ENTER] after each number to navigate down to the next blank spot in the list.

   Step 4: Arrow over to L2 and enter the n values as listed in the table.

   Step 5: Press [GRAPH] to see how n varies with t.

   Step 6: To adjust the viewing window, press [WINDOW]. Choose settings as appropriate; for example, the x-axis could run from 1 to 10 and the y-axis could run from 0 to 40.

   On a TI-Nspire:

   Step 1: Press [home]. Arrow over to the spreadsheet icon, the fourth icon from the left, and press [enter].

   Step 2: To clear the lists in your calculator, arrow up to the topmost cell of the table to highlight the entire column, then press [menu]. Use the arrow key to choose 3: Data, then 4: Clear Data, then press [enter]. Repeat for each column as necessary.

   Step 3: Select the top cell in column A and enter "t" since it is the independent variable. Then select the top cell in column B and enter "n" for the dependent variable.

   Step 4: Enter the ordered pairs in the cells A1 through A9 and B1 through B9.

   Step 5: To see the graph, press [menu], and then select 3: Graph Type and 4: Scatter Plot.

   Step 6: Select "t" from the x pop-up menu and "n" from the y pop-up menu to see the points.

   (continued)

Step 7: To adjust the viewing window, press [menu] and then select 4: Window/Zoom and 1: Window Settings. Choose settings as appropriate; for example, the *x*-axis could run from 1 to 10 and the *y*-axis could run from 0 to 40.

The resulting scatter plot should resemble the following.
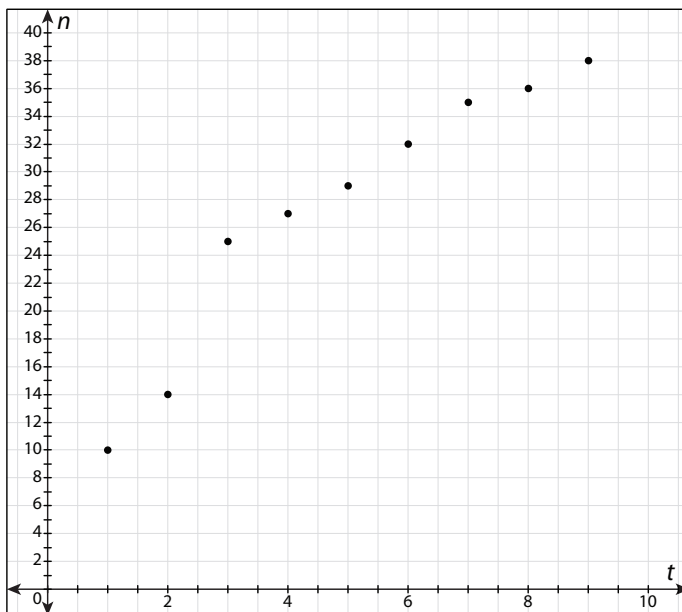
2. Next, use the graphing calculator to fit the data to an exponential function and a power function.

Follow the instructions appropriate to your calculator model. Round the values for *a* and *b* to two significant figures.

On a TI-83/84:

Step 1: Press [STAT] and arrow over to the CALC menu.

Step 2: For an exponential function, arrow down to 0: ExpReg. Press [ENTER]. Enter the L1 variable ([2ND][1]) followed by a comma [,] and the L2 variable ([2ND][2]). Press [ENTER]. The results will be in the form $f(x) = a \bullet b^x$, with $a \approx 12$ and $b \approx 1.2$.

Step 3: For a power function, select A: PwrReg from the CALC menu as described in Step 1. Repeat Step 2. The results will be in the form $g(x) = a \bullet x^b$, with $a \approx 10$ and $b \approx 0.6$.

(*continued*)

On a TI-Nspire:

Step 1: Go back to the data table for n and t. Press [menu] and select 4: Statistics and then 1: Stat Calculations.

Step 2: For an exponential function, select A: Exponential Regression.

Step 3: For X List, select the t variable from the pop-up menu. For Y List, select the n variable from the pop-up menu. Tab to "OK" to see the equation of the function.

Step 4: The results will be in the form $f(x) = a \bullet b^x$, with $a \approx 12$ and $b \approx 1.2$.

Step 5: For a power function, repeat Step 1, then select 9: Power Regression from the Stat Calculations menu.

Step 6: Repeat Step 3.

Step 7: The results will be in the form $g(x) = a \bullet x^b$, with $a \approx 10$ and $b \approx 0.6$.

3. Write the equations of the calculated exponential and power functions from using the variables n and t.

Substitute n, t, and the rounded values for a and b into the general form of each type of function.

	Exponential function $f(x) = a \bullet b^x$	Power function $g(x) = a \bullet x^b$
TI-83/84	$n(t) = 12 \bullet (1.2)^t$	$n(t) = 10 \bullet t^{0.6}$
TI-Nspire	$n(t) = 12 \bullet (1.2)^t$	$n(t) = 10 \bullet t^{0.6}$

4. Plot the exponential and power functions from the previous step on the scatter plots produced by the tree-frog data.

Return to your graph of the scatter plot and enter the equations for the functions $f(x)$ and $g(x)$ from step 3 to add them to the graph. The resulting combined graph should resemble the following.
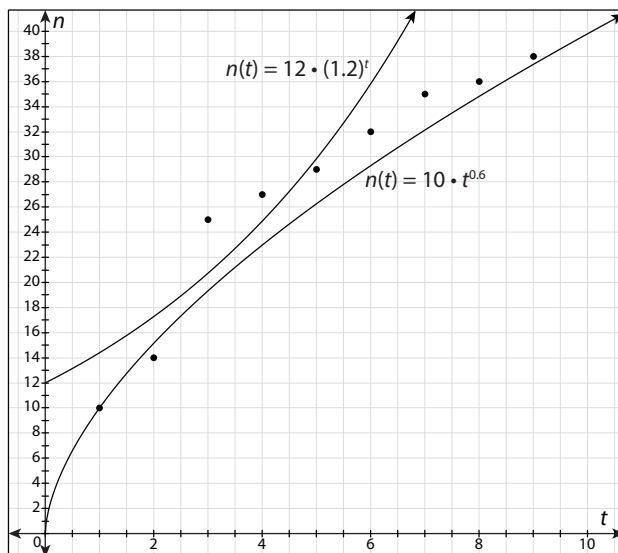
5. Describe how well the exponential function fits the data points.

The exponential function $n(t) = 12 \cdot (1.2)^t$ best fits the data from about $t = 3$ to $t = 6$. Before $t = 3$, the exponential function value is greater than the scatter plot function values. For $t > 6$, the exponential function values diverge from and are greater than the scatter plot function values.

6. Describe how well the power function fits the data points.

The power function $n(t) = 10 \cdot t^{0.6}$ approximately fits the data from $t = 1$ to $t = 9$. The power function values are consistently less than the scatter plot function values from $t = 3$ to $t = 9$, but they follow the trend of the increasing scatter plot function values.

7. Determine which function type, exponential or power, best fits the data.

The function values of the power function $n(t) = 10 \cdot t^{0.6}$ fit the scatter plot data better over this domain interval [1, 9].

Practice 4B.3.3: Comparing Functions

The following graph shows the best times during the day to fish in a neighborhood creek. The horizontal axis represents time and is divided into 24 hours. The vertical axis represents the number of "strikes" $n(t)$ that sport fishers have recorded at different times of the day during 1 fishing season of the year. Use the graph to complete problems 1–3.
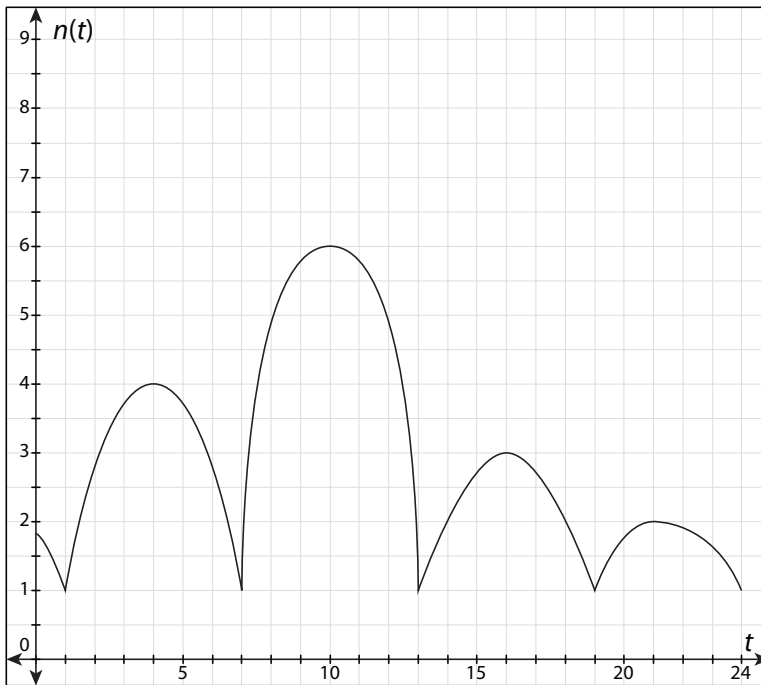
1. Not counting the endpoints, how many different maximum function values occur during the 24-hour period?

2. Not counting the endpoints, how many different minimum function values occur during the 24-hour period?

3. What type of function(s) could be used to model the activeness of the fish?

Use the following information for problems 4–7.

> Bansi, Josh, and Penn are tossing driftwood pieces off of Flat Rock Point, a cliff-top overlook point on the California coast. It takes the driftwood 5 seconds to fall to the rocks in the Pacific Ocean below.

4. Find the average vertical rate of change of Bansi's driftwood over the time interval [1, 1.01] if the function describing the vertical distance Bansi's throw travels is $B(t) = -16t^2 + 5t$.

5. Find the average vertical rate of change of Josh's driftwood over the time interval [1, 1.01] if the function describing the vertical distance Josh's throw travels is $J(t) = -16t^2 + 4t$.

6. Find the average vertical rate of change of Penn's driftwood over the time interval [1, 1.01] if the function describing the vertical distance Penn's throw travels is $P(t) = -16t^2 + 3t$.

7. What do the results of problems 4–6 imply about how to calculate the average vertical rate of change of the driftwood at any time t? Write a function for the average vertical rate of change or speed.

continued

For problems 8–10, find the domain interval over which the rates of change of the two functions are equal. Use the interval $[x, x + k]$ to calculate the rate of change.

8. $a(x) = e^x$; $b(x) = 2 - e^e$

9. $c(x) = \log x$; $d(x) = \log (x - 10)$

10. $e(x) = 2x$; $f(x) = x^2$

Lesson 4: Choosing a Model

Common Core State Standards

A–CED.2   Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★

F–IF.4   For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★

F–IF.5   Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★

F–IF.7   Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

       b.   Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.

F–BF.3   Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

Essential Questions

1. How do you determine the scale to use on the axes of a graph that shows the relationship between two or more variables?

2. How do you interpret the features of a data plot that will determine what kind of function should be used to model the data?

3. How do you decide what restrictions exist on the domain and range variables in a model that represents a real-world problem and/or its solution?

4. How do you sketch a function model so that it represents the relationships in a real-world problem?

5. How do you use graphing calculator technology to explore the effect of constants added to or multiplied by a domain variable in a function that represents a real-world problem?

WORDS TO KNOW

absolute value function	a function of the form $f(x)=|ax+b|+c$, where x is the independent variable and a, b, and c are real numbers
continuous function	a function that does not have a break in its graph across a specified domain
cube root function	a function that contains the cube root of a variable. The general form is $f(x)=a\sqrt[3]{(x-h)-k}$, where a, h, and k are real numbers.
discontinuous function	a function that has a break, hole, or jump in the graph
discrete function	a function in which every element of the domain is individually separate and distinct
exponential function	a function of the form $f(x) = ab^{cx}$, in which a, b, and c are constants; a function that has a variable in the exponent, such as $f(x) = 5^x$
linear function	a function that can be written in the form $ax + by = c$, where a, b, and c are constants; can also be written as $f(x) = mx + b$, in which m is the slope and b is the y-intercept. The graph of a linear function is a straight line; its solutions are the infinite set of points on the line.

piecewise function	a function that is defined by two or more expressions on separate portions of the domain
quadratic function	a function defined by a second-degree expression of the form $f(x) = ax^2 + bx + c$, where $a \neq 0$ and a, b, and c are constants. The graph of any quadratic function is a parabola.
scale	the numbers representing the interval of a variable and the increments into which it is subdivided; usually includes the interval endpoints and the increments of the basic unit of the variable
square root function	a function that contains a square root of a variable. The general form is $f(x) = \sqrt{ax^2 + bx + c}$, where a, b, and c are real numbers.
step function	a function that is a series of disconnected constant functions

Recommended Resources

- Algebra-Class.com. "Step Functions."

  http://www.walch.com/rr/00246

  This site gives a thorough review of the parts of step functions and how to read them.

- MathIsFun.com. "Piecewise Functions."

  http://www.walch.com/rr/00247

  This site offers easy-to-follow models for building an understanding of absolute value functions, floor (step) functions, and general piecewise functions.

- Twining Mathematics, LLC. "Graphs of Cube Roots."

  http://www.walch.com/rr/00248

  This tutorial provides audio-assisted slides and animations that cover how to perform translations of cube root functions.

IXL Links

- Graph a function:
  http://www.ixl.com/math/algebra-1/graph-a-function

- Write a function rule word problems:
  http://www.ixl.com/math/algebra-1/write-a-function-rule-word-problems

- Write a rule for a function table:
  http://www.ixl.com/math/algebra-1/write-a-rule-for-a-function-table

- Write direct variation equations:
  http://www.ixl.com/math/algebra-1/write-direct-variation-equations

- Write inverse variation equations:
  http://www.ixl.com/math/algebra-1/write-inverse-variation-equations

- Write and solve inverse variation equations:
  http://www.ixl.com/math/algebra-1/write-and-solve-inverse-variation-equations

- Slope intercept form graph an equation:
  http://www.ixl.com/math/algebra-1/slope-intercept-form-graph-an-equation

- Slope intercept form write an equation from a graph:
  http://www.ixl.com/math/algebra-1/slope-intercept-form-write-an-equation-from-a-graph

- Slope intercept form write an equation:
  http://www.ixl.com/math/algebra-1/slope-intercept-form-write-an-equation

- Linear function word problems:
  http://www.ixl.com/math/algebra-1/linear-function-word-problems

- Write equations in standard form:
  http://www.ixl.com/math/algebra-1/write-equations-in-standard-form

- Standard form graph an equation:
  http://www.ixl.com/math/algebra-1/standard-form-graph-an-equation

- Point slope form graph an equation:
  http://www.ixl.com/math/algebra-1/point-slope-form-graph-an-equation

- Point slope form write an equation:
  http://www.ixl.com/math/algebra-1/point-slope-form-write-an-equation

- Write linear quadratic and exponential functions:
  http://www.ixl.com/math/algebra-1/write-linear-quadratic-and-exponential-functions

- Graph an absolute value function:
  http://www.ixl.com/math/algebra-1/graph-an-absolute-value-function

- Graph a linear equation:
  http://www.ixl.com/math/geometry/graph-a-linear-equation

- Equations of lines:
  http://www.ixl.com/math/geometry/equations-of-lines

- Graph a linear inequality in two variables:
  http://www.ixl.com/math/algebra-2/graph-a-linear-inequality-in-two-variables

- Graph a quadratic function:
  http://www.ixl.com/math/algebra-2/graph-a-quadratic-function

- Write and solve direct variation equations:
  http://www.ixl.com/math/algebra-2/write-and-solve-direct-variation-equations

- Write and solve inverse variation equations:
  http://www.ixl.com/math/algebra-2/write-and-solve-inverse-variation-equations

- Write joint and combined variation equations:
  http://www.ixl.com/math/algebra-2/write-joint-and-combined-variation-equations

- Write joint and combined variation equations ii:
  http://www.ixl.com/math/algebra-2/write-joint-and-combined-variation-equations-ii

- Solve variation equations:
  http://www.ixl.com/math/algebra-2/solve-variation-equations

- Graph parabolas:
  http://www.ixl.com/math/algebra-2/graph-parabolas

- Graph circles:
  http://www.ixl.com/math/algebra-2/graph-circles

- Identify proportional relationships:
  http://www.ixl.com/math/algebra-1/identify-proportional-relationships

- Find the constant of variation:
  http://www.ixl.com/math/algebra-1/find-the-constant-of-variation

- Graph a proportional relationship:
  http://www.ixl.com/math/algebra-1/graph-a-proportional-relationship

- Identify direct variation and inverse variation:
  http://www.ixl.com/math/algebra-1/identify-direct-variation-and-inverse-variation

- Slope intercept form find slope and y intercept:
  http://www.ixl.com/math/algebra-1/slope-intercept-form-find-slope-and-y-intercept

- Standard form find x and y intercepts:
  http://www.ixl.com/math/algebra-1/standard-form-find-x-and-y-intercepts

- Slopes of parallel and perpendicular lines:
  http://www.ixl.com/math/algebra-1/slopes-of-parallel-and-perpendicular-lines

- Characteristics of quadratic functions:
  http://www.ixl.com/math/algebra-1/characteristics-of-quadratic-functions

- Identify linear quadratic and exponential functions from graphs:
  http://www.ixl.com/math/algebra-1/identify-linear-quadratic-and-exponential-functions-from-graphs

- Identify linear quadratic and exponential functions from tables:
  http://www.ixl.com/math/algebra-1/identify-linear-quadratic-and-exponential-functions-from-tables

- Graph an absolute value function:
  http://www.ixl.com/math/algebra-1/graph-an-absolute-value-function

- Rational functions asymptotes and excluded values:
  http://www.ixl.com/math/algebra-1/rational-functions-asymptotes-and-excluded-values

- Slopes of lines:
  http://www.ixl.com/math/geometry/slopes-of-lines

- Characteristics of quadratic functions:
  http://www.ixl.com/math/algebra-2/characteristics-of-quadratic-functions

- Graph a quadratic function:
  http://www.ixl.com/math/algebra-2/graph-a-quadratic-function

- Rational functions asymptotes and excluded values:
  http://www.ixl.com/math/algebra-2/rational-functions-asymptotes-and-excluded-values

- Classify variation:
  http://www.ixl.com/math/algebra-2/classify-variation

- Find the constant of variation:
  http://www.ixl.com/math/algebra-2/find-the-constant-of-variation

- Domain and range of absolute value functions:
  http://www.ixl.com/math/algebra-1/domain-and-range-of-absolute-value-functions

- Domain and range of radical functions:
  http://www.ixl.com/math/algebra-1/domain-and-range-of-radical-functions

- Domain and range:
  http://www.ixl.com/math/algebra-2/domain-and-range

- Domain and range of radical functions:
  http://www.ixl.com/math/algebra-2/domain-and-range-of-radical-functions

- Domain and range of exponential and logarithmic functions:
  http://www.ixl.com/math/algebra-2/domain-and-range-of-exponential-and-logarithmic-functions

- Transformations of quadratic functions:
  http://www.ixl.com/math/algebra-1/transformations-of-quadratic-functions

- Transformations of absolute value functions:
  http://www.ixl.com/math/algebra-1/transformations-of-absolute-value-functions

- Translations of functions:
  http://www.ixl.com/math/algebra-2/translations-of-functions

- Reflections of functions:
  http://www.ixl.com/math/algebra-2/reflections-of-functions

- Dilations of functions:
  http://www.ixl.com/math/algebra-2/dilations-of-functions

- Transformations of functions:
  http://www.ixl.com/math/algebra-2/transformations-of-functions

- Graph an absolute value function:
  http://www.ixl.com/math/algebra-1/graph-an-absolute-value-function

Lesson 4B.4.1: Linear, Exponential, and Quadratic Functions

Introduction

Exponential, linear, and quadratic functions are used to represent a variety of real-world problems. In some cases, the accuracy and validity of using these function models for a real-world problem is limited to a restricted domain and/or range because of the finite nature of the problem situation; the model and its results apply to a specific set of data points. The key to selecting a function model is an accurate analysis of the data in a problem or its visual representation. Variables such as the domain of the data set and the rate at which the range or function value of a model is changing are critical to selecting a model that best fits a data set. Graphing calculators and other computer-assisted tools can be useful in experimenting with data sets to find the best model to represent that data. Asking "what if" questions about models and their features can help refine those models for best results in depicting a real-world problem and its solution.

Key Concepts

Exponential Functions

- Recall that **exponential functions** are functions that have a variable in the exponent. Exponential functions have the form $f(x) = ab^{cx}$, in which a, b, and c are constants.

- Two special categories of exponential functions are those in which $b = 10$, which is the base of common logarithms, and $b = e$, which is the base of natural logarithms.

- Real-world applications in which $c > 0$ include such subjects as tracking population growth. If $c < 0$, exponential functions can be used to model biological or radioactive decay.

- Combinations of exponential and linear functions can be used to describe the behavior of currents and voltages in electronic circuits, such as those found in cell phones and tablet computers.

Linear Functions

- A **linear function** is a function that can be written in the form $ax + by = c$, where a, b, and c are constants. A linear function can also be written as $f(x) = mx + b$, in which m is the slope and b is the y-intercept. The graph of a linear function is a straight line where the function's solutions are the infinite set of points on the line.

- In large multiple-variable systems of linear equations, the general form of a linear equation might be represented as $ax + by + cz + \ldots = k$, in which the constants are represented by a, b, c, etc., and k; the variables are represented by x, y, z, and so on. Such systems of linear equations are frequently found in large-scale industrial design processes. For example, designers of a new airplane might rely on large systems of linear equations to help account for the complex behavior of air moving over the plane's wing surface.

Quadratic Functions

- Recall that the general form of a **quadratic function** is $f(x) = ax^2 + bx + c$, where $a \neq 0$ and a, b, and c are constants. The graph of any quadratic function is a parabola.

- The vertex form of the quadratic function, $g(x) = a(x - h)^2 + k$, can be used to express the function in terms of the coordinates of its maximum or minimum point, or vertex, (h, k).

- Other types of second-degree equations and relations can be used to represent real-world situations. For example, the motion of planets about a star or the motion of a moon about a planet can be described by a simple second-degree equation of the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. However, the models used in this lesson will all be quadratic functions of the first two types.

Choosing Appropriate Models

- Visual representation of domain and range variables for real-world problems may require calculating conversions between units of distance, mass, time, etc. This will require that an appropriate scale be used for graphs and other visual aids.

- On a graph, **scale** refers to the numbers representing the interval of a variable and the increments into which it is subdivided.

- The scale consists of two components—the interval end points and the increments of the basic unit of the variable. For example, if a time variable is to be shown on the x-axis covering the interval of 0 to 15 seconds in increments of 0.1 second, the scale should be drawn with 0 and 15 as endpoints and 10 markings between each consecutive second (e.g., 1.0, 1.1, 1.2, and so on).

- Sometimes, the available space for such a scale might not be possible, so the interval between the whole units has to be shown as close as possible to variable increments. Using the previously described example, it might only be possible to show half seconds between consecutive whole seconds, in which case the location of a time like 3.7 seconds would have to be approximated.

- Sometimes it is necessary to combine common function types in order to best model the situation. One such combination is the constant-exponential function model, which has the general form $f(x) = a + e^{b-x}$. In the general form of the combination constant-exponential function model, $f(x) = a + e^{b-x}$, a represents the horizontal asymptote and b is a constant.

- The choice of these functions as models for real-world problems can sometimes be justified using graphing calculator techniques. The data-fitting tools of graphing calculators offer a variety of options for comparing function models.

- In other real-world problem contexts, the nature of the variables and the behavior of the data provide the best justification for choosing a model.

- Complicated real-world problems are often difficult to model using purely mathematical functions with well-defined domains and behavior.

- Having the ability to think critically and compare competing function models to describe a real-world problem is often more important than having a thorough knowledge of the functions' characteristics and how they apply to a problem.

Guided Practice 4B.4.1

Example 1

A function commonly used to represent bounded growth of a real-world function value is the exponential-linear combination function of the general form $f(x) = a(1 - b^{-x})$, in which a represents the horizontal asymptote $y = a$ toward which the values of $f(x)$ converge. Use the following graph to determine a and b using the point $\left(1, \dfrac{8}{3}\right)$, then write an exponential-linear combination function that models the graph.
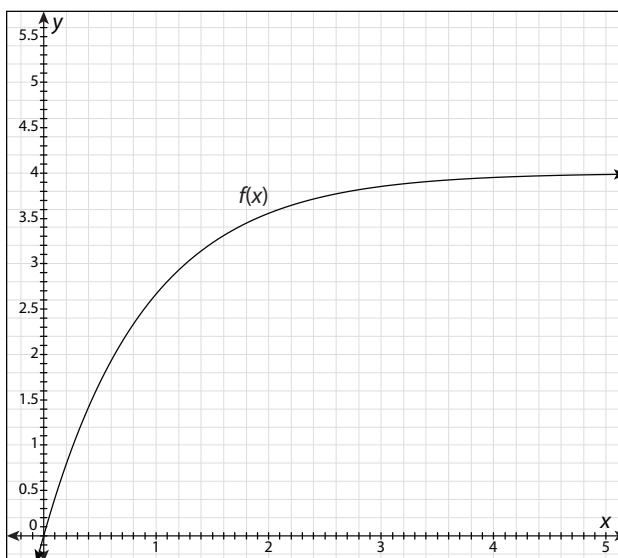
1. Use the graph to identify the asymptote toward which the function values converge as the values of x increase.

   The asymptote shown is a horizontal asymptote that appears to be $y = 4$.

2. Determine the value of a in the general form of the combination function.

   The variable a represents the horizontal asymptote. Therefore, $a = 4$.

3. Update the general form of the combination function with the value of a.

   The general function is $f(x) = a(1 - b^{-x})$. The substitution of 4 for a results in the updated function, $f(x) = 4(1 - b^{-x})$.

4. Find $f(0)$ algebraically and compare it to the graph.

Substitute 0 for x into the equation of the updated function, then solve for $f(0)$.

$f(x) = 4(1 - b^{-x})$	Updated function
$f(0) = 4[1 - b^{-(0)}]$	Substitute 0 for x.
$f(x) = 4\left(1 - \dfrac{1}{b^0}\right)$	Rewrite the negative exponent.
$f(0) = 4(1 - 1)$	Simplify using the Zero Power Property.
$f(0) = 0$	Simplify.

$f(0) = 0$; this matches the point $(0, 0)$ as shown on the graph.

5. Use the point $\left(1, \dfrac{8}{3}\right)$ to find the value of b.

Substitute the x- and y-coordinates of the point $\left(1, \dfrac{8}{3}\right)$ into the equation $f(x) = 4(1 - b^{-x})$.

$f(x) = 4(1 - b^{-x})$	Updated function
$y = 4(1 - b^{-x})$	Let $y = f(x)$.
$\left(\dfrac{8}{3}\right) = 4[1 - b^{-(1)}]$	Substitute 1 for x and $\dfrac{8}{3}$ for y.
$\dfrac{8}{3} = 4\left(1 - \dfrac{1}{b}\right)$	Simplify using the Negative Exponent Property.
$\dfrac{2}{3} = 1 - \dfrac{1}{b}$	Divide both sides by 4.
$-\dfrac{1}{3} = -\dfrac{1}{b}$	Subtract 1 from both sides.
$b = 3$	Solve for b.

The value of b is 3.

6. Write the exponential-linear combination function.

Using the original function, $f(x) = a(1 - b^{-x})$, let $a = 4$ and $b = 3$.

The exponential-linear combination function that models the given graph is $f(x) = 4(1 - 3^{-x})$.

Example 2

The graph models the distance, $s(t)$, traveled by a speeding car moving at a constant rate r of 80 miles per hour, and the distance covered by a state patrol car, $p(t)$, until the patrol car intercepts the speeding car after 5 seconds (t). The general equations for the cars are $s(t) = rt$ for the speeding car and $p(t) = \frac{1}{2}at^2$ for the state patrol car, in which a is the acceleration in miles per second squared. Write the specific equations for the two distances at a time of 5 seconds, and determine appropriate horizontal and vertical axis scales on the graph to reflect the given problem conditions.
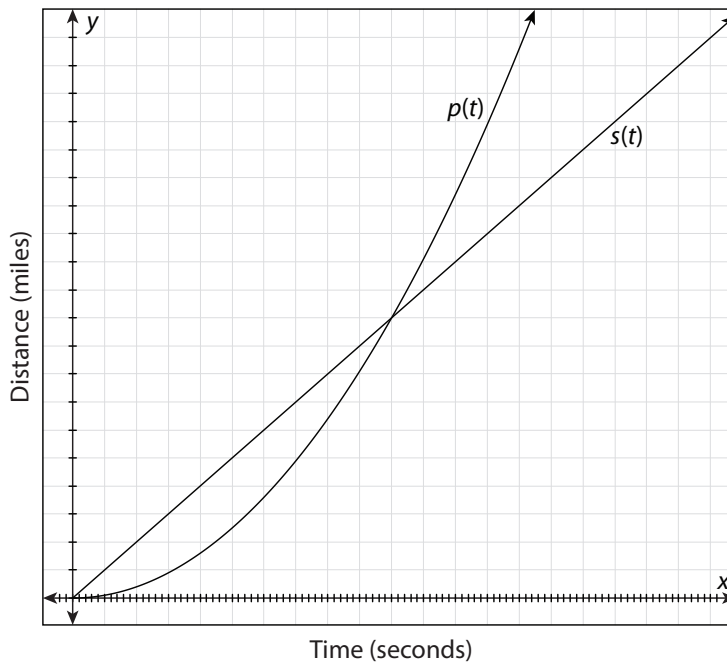
1. Determine how to calculate the distance traveled by the speeding car in 5 seconds.

   The function $s(t) = rt$ is used to calculate the speed, with time t in seconds. However, the speeding car's rate is given as 80 miles per hour. Therefore, this rate has to be converted to seconds.

2. Calculate the speed of the speeding car in miles per second.

   One hour is equal to 60 minutes, which is 3,600 seconds. This means that the conversion between hours and seconds can be represented by the ratios $\dfrac{1 \text{ hour}}{3600 \text{ seconds}}$ and $\dfrac{3600 \text{ seconds}}{1 \text{ hour}}$.

   Use the conversion ratios to determine the speed of the speeding car in miles per second.

   $\text{speed} = \dfrac{80 \text{ miles}}{\text{hour}} \bullet \dfrac{1 \text{ hour}}{3600 \text{ seconds}}$ Multiply the speed of the car by the conversion factor.

   $\text{speed} = \dfrac{80 \text{ miles}}{\cancel{\text{hour}}} \bullet \dfrac{1 \cancel{\text{ hour}}}{3600 \text{ seconds}}$ Cancel like units.

   $\text{speed} = \dfrac{1 \text{ mile}}{45 \text{ seconds}}$ Simplify.

   The speeding car is traveling at 1 mile every 45 seconds, or $\dfrac{1}{45}$ of a mile per second.

3. Use the result of step 2 to calculate the distance covered by the speeding car in 5 seconds.

   Substitute $\dfrac{1}{45}$ for s and 5 for t in the function $s(t) = rt$.

   $s(t) = rt$ Original function

   $s(5) = \left(\dfrac{1}{45}\right)(5)$ Substitute $\dfrac{1}{45}$ for s and 5 for t.

   $s(5) = \dfrac{1}{9}$ Simplify.

   The speeding car travels $\dfrac{1}{9}$ of a mile in 5 seconds.

4. Use the distance covered by the speeding car in 5 seconds to find the acceleration of the state patrol car.

The distance that the patrol car travels to intercept the speeder and the time of 5 seconds are the same as in step 3. Substitute these values into the state patrol car's function, and then solve for the acceleration, a.

$$p(t) = \frac{1}{2}at^2 \qquad \text{State patrol car's function}$$

$$\left(\frac{1}{9}\right) = \frac{1}{2}a(5)^2 \qquad \text{Substitute } \frac{1}{9} \text{ for } p(t) \text{ and 5 for } t.$$

$$\frac{1}{9} = \frac{25a}{2} \qquad \text{Simplify.}$$

$$a = \frac{2}{225} \qquad \text{Solve for } a.$$

The state patrol car's acceleration is $\dfrac{2}{225}$ of a mile per second squared.

5. Write the specific equation for the speeding car's function in terms of the value for r found in step 2.

The rate r found for the speeding car is $\dfrac{1}{45}$ of a mile per second.

Substitute this value into the given function for the speeding car,

$s(t) = rt$.

The equation for the speeding car's function, at any time t, is

$s(t) = \dfrac{1}{45}t$; the distance covered is still in miles, but the speed is in

miles per second, which is compatible with the time units.

6. Write the specific equation for the state patrol car's function in terms of the result of step 4.

The acceleration a found for the state patrol car is $\dfrac{2}{225}$ of a mile per second squared. Substitute this value into the given function for the patrol car, $p(t)=\dfrac{1}{2}at^2$.

The equation for the patrol car is $p(t)=\dfrac{1}{2}\left(\dfrac{2}{225}\right)t^2=\dfrac{1}{225}t^2$. The distance is in miles and the acceleration is in miles per second squared.

7. Determine an appropriate scale for the graph's horizontal axis in order to fit the conditions of the problem.

The horizontal axis t measures time in seconds; the scale is from 0 seconds to 10 seconds in increments of 1 second.

The maximum value of t in the problem is $t = 5$, so the scale does not have to be greater than that value. Therefore, the axis could be limited from 0 to 5. Only whole seconds are used, so the one-second intervals do not need to be subdivided into smaller intervals.

8. Determine an appropriate scale for the vertical axis in order to fit the conditions of the problem.

The vertical axis measures distance in miles; the scale is from 0 miles to 10 miles in increments of 1 mile.

From step 3, the maximum distance represented should be $\dfrac{1}{9}$ of a mile. One possible scale for the vertical axis would be from 0 to $\dfrac{1}{9}$, divided into 10 intervals of $\dfrac{1}{90}$ each. The vertical scale could retain its present scale (from 0 to 10) if the axis title were changed to "Distance $\left(\dfrac{1}{90}\text{ths of a mile}\right)$."

Example 3

Cloud storage refers to keeping data on a remote database instead of your own computer or mobile device in order to save hard drive space. Cloud storage users pay for this service either as part of a data plan subscription or separately. Two market research teams used function models to describe the declining cost of cloud-based computer-storage services. Team A used an exponential function, represented in the graph by $A(t)$; Team B used a quadratic function, represented by $B(t)$. The graphs of the functions are shown over a 10-month period with "the present" represented by the region at which both models reach a minimum function value. The y-axis represents the price in dollars per gigabyte of storage. Domain values that are greater than those represented by the minimum points are projections based on the models. Describe how well the two models fit the data of the changing cost of cloud storage. Use function characteristics to compare the models. Describe the limitations of these models for predicting future changes in the cost of cloud services.
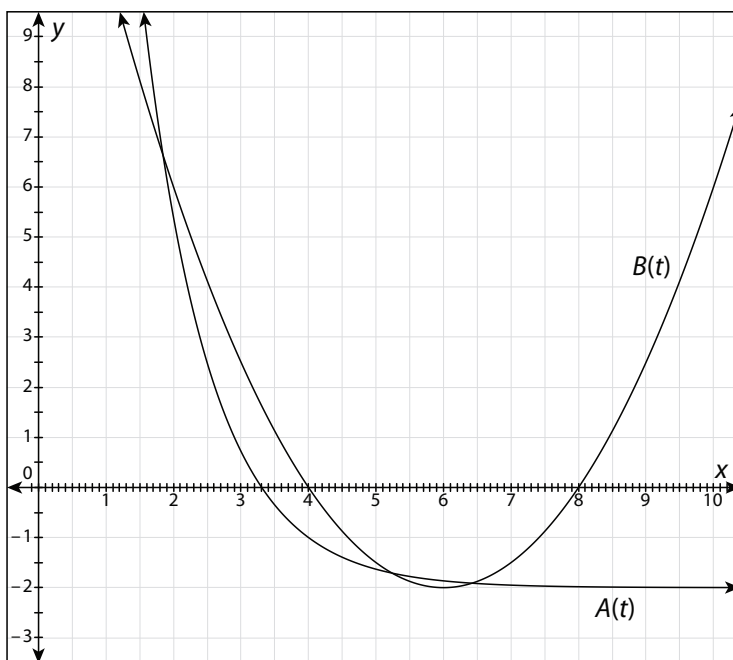
1. State the coordinates of the minimum point of Team B's quadratic function model.

    According to the graph, the coordinates of the minimum appear to be (6, –2).

2. Use the vertex form of a quadratic function, $f(x) = a(x - h)^2 + k$, to write a specific quadratic function $B(t)$ with the identified minimum point for the model used by Team B.

In the form $f(x) = a(x - h)^2 + k$, (h, k) represents the minimum point of the parabola. Therefore, substituting the coordinates of the minimum point at $(6, -2)$, the function can be written as $B(t) = a(t - 6)^2 - 2$.

3. Determine another point on the graph of Team B's function that can be used to find the value of the constant, a, in the function for $B(t)$.

Choose a point from the graph. Let $(t, B(t))$ be $(4, 0)$. Substitute these coordinates into the function for Team B and solve for a.

$B(t) = a(t - 6)^2 - 2$	Function for Team B
$(0) = a[(4) - 6]^2 - 2$	Substitute 4 for t and 0 for $B(t)$.
$0 = a(-2)^2 - 2$	Simplify the expression in the brackets.
$0 = 4a - 2$	Apply the exponent.
$2 = 4a$	Add 2 to both sides.
$a = 0.5$	Divide both sides by 4.

The value of a is 0.5.

4. Write the quadratic function that represents Team B's model.

Using the vertex form of a quadratic function, $f(x) = a(x - h)^2 + k$, the minimum point $(6, -2)$, and the value 0.5 for a, the quadratic function that represents Team B's model is $B(t) = 0.5(t - 6)^2 - 2$.

5. Determine the horizontal asymptote for Team A's exponential function model from the graph.

The horizontal asymptote for Team A's function model appears to be $y = -2$.

6. Use the general form of the combination constant-exponential function model, $f(x) = a + e^{b-x}$, to write a specific function model for $A(t)$.

   In the general form of the combination constant-exponential function model, $f(x) = a + e^{b-x}$, a represents the horizontal asymptote. Therefore, the function can be written as $A(t) = -2 + e^{b-t}$.

7. Find the value of the constant in the resulting function using a point on the graph of Team A's model.

   The constant in the function $A(t) = -2 + e^{b-t}$ is b.

   Choose a point from the graph. Let $(t, A(t))$ be $(4, -1)$.

   $A(t) = -2 + e^{b-t}$ Function for Team A

   $(-1) = = -2 + e^{b-(4)}$ Substitute 4 for t and -1 for $A(t)$.

   $1 = e^{b-4}$ Add 2 to both sides.

   Recall that a number raised to an exponent of 0 will result in 1. Therefore, determine the value for which the exponent of b is 0. When b is 4, the exponent will be $4 - 4$, or 0.

8. Write the combination function that represents Team A's model.

   Using the general function, $f(x) = a + e^{b-x}$, the value -2 for a, and the value 4 for b, the function that represents Team A's model is $A(t) = -2 + e^{4-t}$.

9. Describe what Team A's model shows for the first six months displayed on the graph and what it projects for the remaining four months of the domain interval [0, 10].

   Team A's model describes a declining cost of cloud services for the first six months until "the present," after which the cost is projected to be constant for the remaining four months.

10. Describe what Team B's model shows for the first six months displayed on the graph and what it projects for the remaining four months of the domain interval [0, 10].

Team B's model describes a declining cost of cloud services for the first six months until "the present," after which the cost is projected to increase for the remaining four months.

11. Describe the limitations of these models for predicting future changes in the cost of cloud services.

It is difficult to predict what will happen to the cost of cloud services in the future without additional information. However, both models suggest a declining rate of cost changes over the interval [0, 6]. This favors the exponential model used by Team A, but it is not known if the asymptote $y = -2$ represents a "floor" for the costs or if the costs will decline to a lower-value asymptote with a value of $y < -2$. The future projection of increasing cloud-services cost estimates suggested by Team B's quadratic model does not seem realistic in light of other cost trends in the computer and information sciences field; typically, these costs go down as newer technologies become available and older ones are phased out, or as the technology becomes more widely available and, therefore, more competitively priced.

UNIT 4B • MATHEMATICAL MODELING AND CHOOSING A MODEL
Lesson 4: Choosing a Model

Practice 4B.4.1: Linear, Exponential, and Quadratic Functions

A type of avalanche called a slab avalanche occurs when a densely packed upper layer (or slab) of snow rests on a soft, weaker layer of snow that gives way, causing the slab to move rapidly down the mountain. The data in the following table represent the angle in degrees that a snow slab makes with the sloped snow pack on which it is moving. The speed given is the speed of the leading edge of the avalanche, in meters per second. The first time measurement represents the time at which the avalanche forms, and the last time measurement is the time at which the avalanche wall collapses and the avalanche stops. Use the table to complete problems 1–3.

Angle	30°	28°	27°	26.5°	26.25°	26.125°
Speed (m/s)	10	15	12	10	9	9
Time (s)	10	12	14	16	18	20

1. Describe the average rate of change of the avalanche angle for each of the successive time intervals shown in the table (e.g., 10 seconds to 12 seconds, 12 seconds to 14 seconds).

2. Determine what type of function can model the change in the avalanche angle.

3. Describe the average rate of change of the speed for each of the successive time intervals in the table and determine what type(s) of function(s) could model the speed changes.

For problems 4–10, determine the type of function—exponential, linear, or quadratic—that is most appropriate to model the situation described. Then, use the chosen function model to complete the problem.

4. A public transportation system offers a deferred retirement savings plan that pays participants 5 percent interest compounded annually. What is the domain for an employee who saves with the plan for 30 years?

continued

5. A youth swim club awards an achievement badge to members who swim 100 meters in less than 3 minutes. What is the range for a function that calculates the average speed at which the badge earners swim?

6. A professional golfer's scores for the first three games of an annual golf competition were 78, 76, and 77 strokes, respectively. Compare the rate of change of the first two scores to that of the first and third scores in terms of the appropriate function model.

7. A game token with eight faces numbered 1–8 lands on 8 three times in a row. Given that the game token is equally likely to land on any of its eight faces, what is the asymptote for this outcome?

8. A basketball team scores an average of 75, 77, 80, and 79 points, respectively, in the first four games of tournament season. Calculate the rate of change of the scores for the three consecutive intervals. What would you expect the function model to predict for the next score?

9. The Santa Ana winds in San Dimas Canyon near Glendora, California, increased from an average speed of 30 miles per hour to an average speed of 50 miles per hour over a two-week period. What was the average daily rate of change in the wind speed?

10. A spectrometer, a device for measuring properties of light, is in the shape of a rectangular solid and is being mounted on top of a robotic vehicle designed to explore the surface of Mars. The spectrometer's length and width are $0.4x + 0.3$ meters and $0.3x$ meters, respectively. What is the minimum volume of the spectrometer if its depth is 0.12 meter?

Lesson 4B.4.2: Piecewise, Step, and Absolute Value Functions

Introduction

Many real-world applications of mathematics involve the use of two or more functions to describe relationships among several variables. These functions sometimes apply to the problem over different domains that correspond to different real-world conditions. For example, a quadratic function model might describe a real-world relationship between two variables over the interval $(a, b]$, but an exponential function model might better describe the relationship over the interval (b, c). In other words, portions of the same overall function can be defined by different equations over different domain intervals. The ability to apply a function model to a collection of data points depends, in part, on the ability to identify trends among data values and to associate those with an appropriate function type.

Key Concepts

- A **piecewise function** is a function that is defined by two or more expressions on separate portions of the domain.

- Such functions can be **continuous**, with no break in the graph of the function across a specified domain, or **discontinuous**, in which the graph of the function has a break, hole, or jump.

- A **discrete function** is a function in which every element of the domain is individually separate and distinct.

- Use a brace to show two (or more) pieces of the same function over different restricted domains. When stating the domain of the entire piecewise function, write the domain for each individual piece next to the appropriate expression. For example, for the piecewise function $a(x)$:

$$a(x) = \begin{cases} 4.57 \bullet x; x \leq 10 \\ 3.43 \bullet x; x > 10 \end{cases}$$

- One piece of the function, $a(x) = 4.57 \bullet x$, represents the portion of the function over the restricted domain $x \leq 10$. The other piece, $a(x) = 3.43 \bullet x$, represents the portion of the function for which $x > 10$.

- An **absolute value function** is a function of the form $f(x)=|ax+b|+c$, where x is the independent variable and a, b, and c are real numbers. An absolute value function is a special type of piecewise function because the function values on either side of its maximum or minimum point are equal; for example, if $f(x)=|x-6|$, then $f(3) = f(9)$. In general terms, an absolute value function can be represented by $f(x)=|x+a|+b$, where the maximum or minimum function value occurs at the point $(-a, b)$.

- For a value of $x = m$ in which $m < a$, or a value of $x = m$ in which $m > a$, the function value $f(m)$ is the same because the absolute value of the quantity $a + m$ is the same under both conditions, namely $|a+m|+b$.

- A **step function** (sometimes called a floor function or a postage function) is another special type of piecewise function that is discontinuous. A step function is a combination of one or more functions that are defined over restricted intervals and which may be undefined at other domain points or over other restricted domain intervals. In other words, a step function is a series of disconnected constant functions.

- A common type of step function is one in which the function values are restricted to integer domain values, such as the number of students who join the computer-gaming club—each student must be represented by a whole number (integer) value.

- By contrast, a function that models temperature changes throughout the day would not be restricted to integer domain values, since the temperature fluctuates by fractions of degrees.

- Just as with exponential, linear, and quadratic functions, domain and range variables for real-world problems involving piecewise functions may require adjustments to the axis scales when creating graphs and other visual aids.

Guided Practice 4B.4.2

Example 1

A wholesale store sells paper plates for \$3.27 per package, up to and including 50 packages purchased. If a customer buys more than 50 packages, the price of the entire purchase drops to \$2.73 per package. Write a piecewise function for this pricing structure, and determine the price a deli owner would pay for 75 packages of plates.

1. Write a function for the price paid for 50 or fewer packages of plates.

   Let the total price be t, the price per package be p, and the number of packages be n. Thus, the total price, t, of n packages is $t(n)$.

   The general function $t(n) = p \bullet n$ represents the price paid for 50 or fewer packages of plates.

   We are given that the price per package is \$3.27, up to and including 50 packages purchased; therefore, let $p = 3.27$.

   This function can be written as $t(n) = 3.27 \bullet n$, if $n \le 50$.

2. Write a function for the price paid for more than 50 packages of plates.

   Again, let the total price be t, the price per package be p, the number of packages be n, and the total price, t, of n packages be $t(n)$.

   Thus, the general function from step 1, $t(n) = p \bullet n$, also represents the price paid for more than 50 packages of plates.

   However, the given price per package for more than 50 packages is \$2.73; therefore, let $p = 2.73$.

   Substituting the new price of \$2.73 for p, this function can be written as $t(n) = 2.73 \bullet n$, if $n > 50$.

3. Combine the two functions to show the "pieces" of the piecewise function and the restricted domain for each.

   Write the functions and restrictions for the two different pricing structures using appropriate notation.

   $$t(n) = \begin{cases} 3.27 \bullet n; n \le 50 \\ 2.73 \bullet n; n > 50 \end{cases}$$

> 4. Calculate the price paid for 75 packages of plates.
>
> The part of the piecewise function, $t(n)$, that applies to $n = 75$ is $t(n) = 2.73 \cdot n$. Therefore, the price for 75 packages of plates is $t(75) = 2.73 \cdot 75$ or \$204.75.

Example 2

When heat energy is added to ice that has a temperature less than the freezing point of water, the heat goes into raising the temperature of the ice until it starts to melt. When the ice starts to melt, all of the heat goes into melting the ice, not raising the temperature of the ice-water mixture. After the ice is melted, the heat goes back into raising the temperature of the water that results from the ice melting. The following graph shows these three distinct phenomena.

Heat in joules (in thousands)
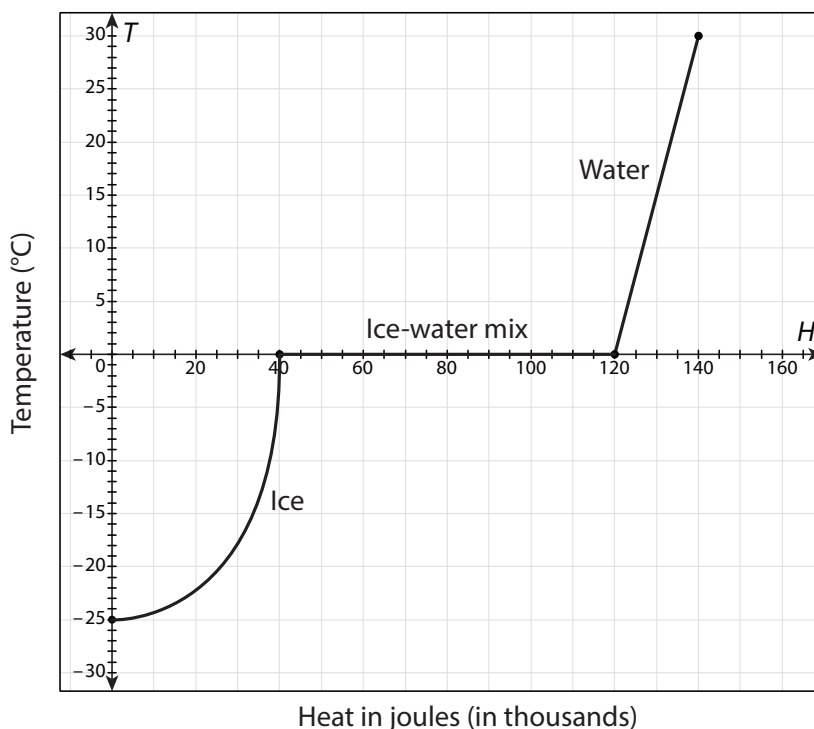
In this situation, the heat energy is added to the ice in such a way that it can be described by a quadratic function model. The "ice-water mix" and "water" parts of the heat-temperature function and graph are linear functions. Use a graphing calculator to derive a function for each of the three parts of this piecewise function. Then, write the complete function and its domain.

1. Determine the values of the three data points for the "ice" part of the function.

   Notice that the *H* or horizontal axis numbers have to be multiplied by 1,000 to yield the "true" values of *x* needed for finding the specific function equation. From the graph, it can be seen that the first point is at (0, –25). The second point appears to have an *H*-coordinate that is located at two thirds of the interval from 0 to 40 multiplied by 1,000. Two thirds of 40,000 is approximately 26,667. The *T*-coordinate of this point is –20; therefore, the second point is at approximately (26,667, –20). The third point has an *H*-coordinate of 40 times 1,000, so the third point's coordinates are (40,000, 0).

   The locations of the three data points can be approximated as (0, –25), (26,667, –20), and (40,000, 0).

2. Use the three data points for the "ice" part of the function to find the equation of the quadratic function.

   Use a graphing calculator to determine the equation of this part of the function. Follow the directions appropriate to your calculator model.

   On a TI-83/84:

   Step 1: Press [STAT] to bring up the statistics menu. The first option, 1: Edit, will already be highlighted. Press [ENTER].

   Step 2: Arrow up to L1 and press [CLEAR], then [ENTER], to clear the list. Repeat this process to clear L2 and L3 if needed.

   Step 3: Enter the three ordered pairs from example step 1 in the L1 and L2 lists. Make sure to enter the *H*-coordinates in L1 and the *T*-coordinates in L2.

   Step 4: Press [STAT]. Arrow to the CALC menu. Select 5: QuadReg.

   Step 5: Press [2ND][1] to type "L1" for Xlist. Arrow down to Ylist and press [2ND][2] to type "L2" for Ylist, if not already shown.

   Step 6: Arrow down to "Calculate" and press [ENTER].

   (*continued*)

On a TI-Nspire:

Step 1: Press [home]. Arrow over to the spreadsheet icon, the fourth icon from the left, and press [enter].

Step 2: To clear the lists in your calculator, arrow up to the topmost cell of the table to highlight the entire column, then press [menu]. Choose 3: Data, then 4: Clear Data. Repeat for each column as necessary.

Step 3: Arrow up to the topmost cell of the first column, labeled "A." Press [X][enter] to type x. Then, arrow over to the second column, labeled "B." Press [Y][enter] to type y.

Step 4: Arrow down to cell A1 and enter the first H-value from the ordered pairs in example step 1. Press [enter]. Enter the second H-value in cell A2 and so on.

Step 5: Move over to cell B1 and enter the first T-value. Press [enter]. Enter the second T-value in cell B2 and so on.

Step 6: To fit an equation to the data points, press [menu] and select 4: Statistics, then 1: Stat Calculations, then 6: Quadratic Regression. Select "x" from the X List pop-up menu. Press [tab] to move to the Y List pop-up menu, then select "y" from the options. Tab to "OK" and press [enter].

Either calculator will return values for the constants that can be substituted into the equation $y = ax^2 + bx + c$.

The result is the quadratic equation $y = (3.28135 \cdot 10^{-8})x^2 + (-6.875 \cdot 10^{-4})x - 25$.

3. Rewrite the resulting equation as a function using the variables H and T, and list the restricted domain over which the function is defined.

The equation generated by the calculator for the "ice" part of the function can be written as shown:

$T(H) = (3.28135 \cdot 10^{-8})H^2 + (-6.875 \cdot 10^{-4})H - 25$

From the graph, it can be seen that the domain for this function is $[0, 40{,}000]$.

4. Write the equation for the "ice-water mix" part of the piecewise function, and list the restricted domain over which the function is defined.

The ice-water mix is a horizontal line with a slope of 0 that would intersect the y-axis at 0. Therefore, the equation for this piece of the function can be written as $T(H) = 0$.

An additional point appears to have an H-coordinate that is located at one-third of the interval from 120 to 160 multiplied by 1,000. One-third of 40,000 is approximately 13,333, so 120,000 + 13,333 is 133,333. The T-coordinate of this point is 0; therefore, the additional data point is at approximately (133,333, 0).

From the graph, it can be seen that the domain for this function is approximately [40,000, 133,333].

5. Write the two points defining the "water" part of the function graph.

From the graph, it can be seen that one point is (120,000, 0) and the other is (140,000, 30).

6. Write the equation for the "water" part of the function.

First, calculate the slope of the line using the formula for slope, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$. Substitute the points determined from the previous step, (140,000, 30) and (120,000, 0), and then solve for the slope, m.

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$ Slope formula

$m = \dfrac{(0) - (30)}{(120,000) - (140,000)}$ Substitute 140,000 for x_1, 120,000 for x_2, 30 for y_1, and 0 for y_2.

$m = \dfrac{-30}{-20,000}$ Simplify.

$m = 1.5 \bullet 10^{-3}$

The slope of the line is $m = 0.0015 \bullet 0.001$.

(*continued*)

Use the point-slope formula, $y - y_1 = m(x - x_1)$, along with the slope and either of the two points from the previous step to write the equation for the "water" part of the function.

Let's use (120,000, 0).

$y - y_1 = m(x - x_1)$ — Point-slope formula

$y - (0) =$ — Substitute 120,000 for x_1, 0 for
$(0.0015 \bullet 0.001)[x - (120,000)]$ — y_1, and 0.0015 $\bullet$ 0.001 for m.

$y = (0.0015 \bullet 0.001)x - 180$ — Simplify.

The equation for the "water" part of the function is
$y = (0.0015 \bullet 0.001)x - 180$.

7. Rewrite the resulting equation as a function using the variables H and T, and list the restricted domain over which the function is defined.

The equation can be written as $T(H) = 0.0015H - 180$.

From the graph, it can be seen that the domain for this function is [120,000, 140,000].

8. Write the complete piecewise function for $T(H)$, and determine its domain.

$$T(H) = \begin{cases} (3.28135 \bullet 10^{-8})H^2 + (-6.875 \bullet 10^{-4})H - 25; 0 \leq H \leq 40,000 \\ 0; 40,000 < H < 120,000 \\ (0.0015)H - 180; 120,000 \leq H \leq 140,000 \end{cases}$$

Comparing the three pieces of the function and their restricted domains, the lowest domain value is 0, and the highest is 140,000. Both are included, as shown by the $\leq$ symbols. Therefore, the domain for the piecewise function $T(H)$ is [0, 140,000].

Example 3

The graph shows a quadratic step function, $f(x) = \dfrac{3}{4}x^2 - 2x - \dfrac{7}{4}$. What domain restriction defines each step? Use the graph to write the equations of the steps of the quadratic function, and determine each step's restricted domain. Then, use this information to determine the domain of the quadratic function.
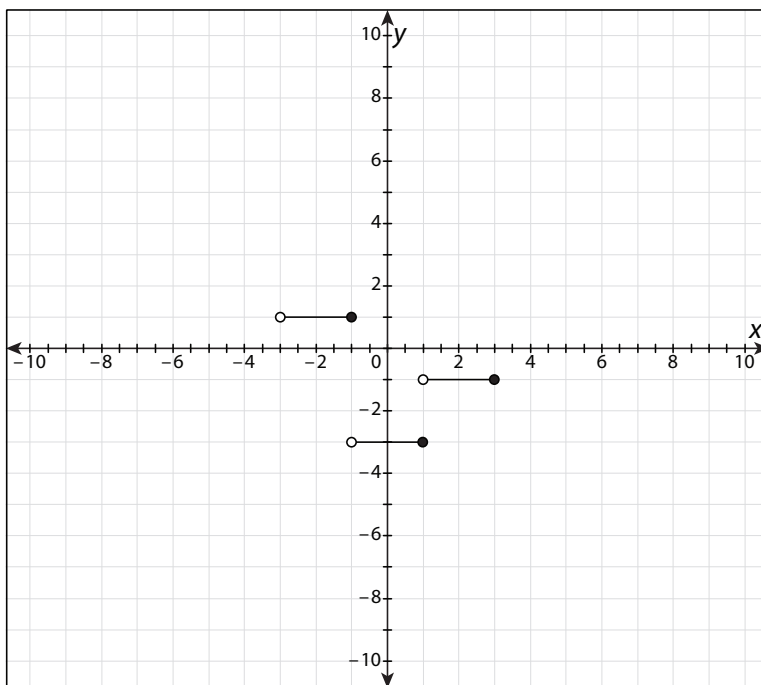
1. Write an equation of the form $y = a$ for each of the three function steps shown.

   The step equations are $y = 1$ (Quadrant II), $y = -3$ (Quadrants III and IV), and $y = -1$ (Quadrant IV).

2. Write the restricted domain for each equation.

   For $y = 1$, there is a closed circle at $x = -1$ and an open circle at $x = -3$. Therefore, the domain of $y = 1$ is $(-3, -1]$.

   For $y = -1$, there is an open circle at $x = 1$ and a closed circle at $x = 3$. Therefore, the domain of $y = -1$ is $(1, 3]$.

   For $y = -3$, there is an open circle at $x = -1$ and a closed circle at $x = 1$. Therefore, the domain of $y = -3$ is $(-1, 1]$.

3. Use the restricted domains of the step functions to determine the rule that defines the domain of the quadratic function $f(x)$.

The value of the quadratic function is given by the upper bound or limit of the interval over which each step is defined. Therefore, the domain of $f(x)$ is defined by the rule $\{x = -1, 1, \text{ and } 3\}$.

4. Write the full function $f(x)$ along with the domain of the quadratic function.

Use the pieces of the function, the given quadratic function, and each of the domains to write the step function $f(x)$.

$$f(x) = \begin{cases} \dfrac{3}{4}x^2 - 2x - \dfrac{7}{4}; x = -1, 1, 3 \\ 1; -3 < x \le -1 \\ -3; -1 < x \le 1 \\ -1; 1 < x \le 3 \end{cases}$$

Example 4

Use a graphing calculator to compare the graphs of the three functions $f(x) = |x| + |x - 3|$, $g(x) = 2x - 3$, and $h(x) = -2x + 3$. How are the functions related? How are they alike and different?

1. Graph the functions using a graphing calculator.

   Enter each function into your graphing calculator. Graph all three functions on the same coordinate plane.

   $f(x) = |x| + |x - 3|$

   $g(x) = 2x - 3$

   $h(x) = -2x + 3$
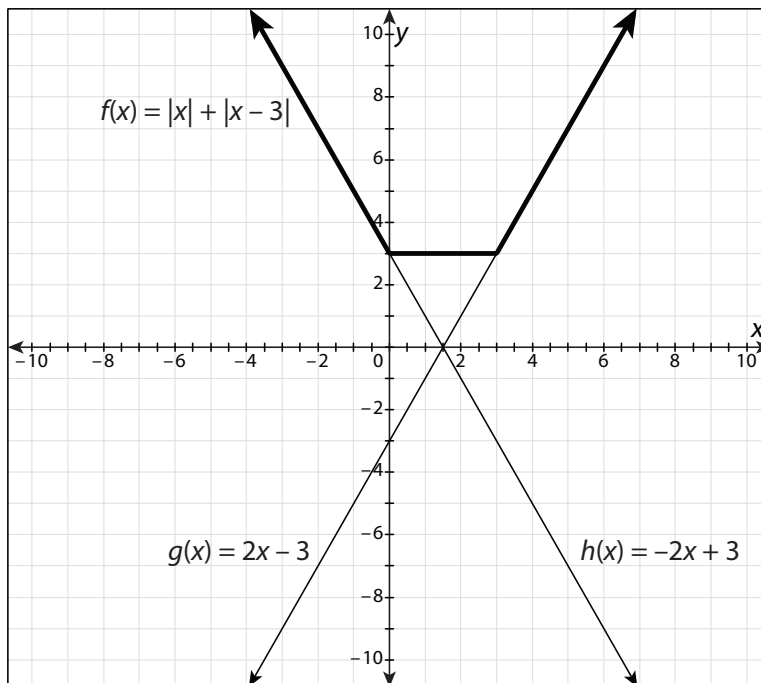
2. Describe the equations and domains of the three pieces of the piecewise function $f(x)$.

   Refer to the graph and the given equations.

   The piecewise function $f(x)$ has one piece that is defined by the rule $f(x) = -2x + 3$ for the restricted domain $(-\infty, 0)$, another that is defined by the rule $f(x) = 3$ for the restricted domain $[0, 3]$, and a third that is defined by the rule $f(x) = 2x - 3$ for the restricted domain $(3, \infty)$. All three rules comprise $f(x)$.

3. Describe what the function $y = 3$ represents for $f(x)$.

   The minimum function value is 3, which occurs over the restricted domain [0, 3]. Therefore, $y = 3$ represents the minimum, or lowest value, of the function $f(x)$.

4. Determine the intersection point of the graphs of $g(x)$ and $h(x)$, and describe what it represents.

   Based on the graph of the functions, the intersection point is (1.5, 0). This point represents the domain value at which the function values are equal.

5. Describe the meaning of the overlapping portions of the graphs of the function pair $f(x)$ and $g(x)$ as well as the function pair $f(x)$ and $h(x)$.

   The overlap means that the function pairs share the same points for those domain values. It also means that the overlapping domains are solutions for the pairs of functions.

6. Determine the solutions for the function pair $f(x)$ and $g(x)$ and the function pair $f(x)$ and $h(x)$.

   Based on the graphs and the overlapping domains, the solution set for $f(x)$ and $g(x)$ is given by the domain interval $[3, \infty)$. The solution set for $f(x)$ and $h(x)$ is given by the domain interval $(-\infty, 0]$.

UNIT 4B • MATHEMATICAL MODELING AND CHOOSING A MODEL
Lesson 4: Choosing a Model

Practice 4B.4.2: Piecewise, Step, and Absolute Value Functions

The following graph is called a phase diagram. It shows the pressure and temperature at which a substance is a gas, a liquid, a solid, or all three phases of matter. The pressure scale, $P(T)$, is in thousands, and the temperature scale T is in degrees Celsius. Use the graph to complete problems 1–4.
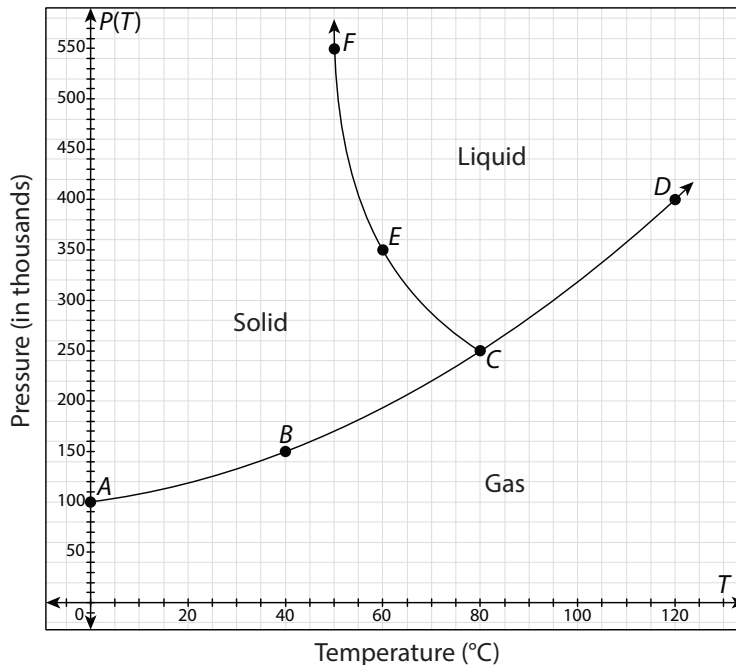
1. Why is the whole graph not a combination or piecewise function?

2. Which points could be used to define individual functions from the data on the graph? Determine the coordinates of those points.

3. What are the coordinates of the point on the graph at which all three phases of the substance exist?

4. What function type(s) can be used to model these data points?

continued

UNIT 4B • MATHEMATICAL MODELING AND CHOOSING A MODEL
Lesson 4: Choosing a Model

Use the table to complete problems 5–7.

x	−3	−2	−1	0	1	2	3
f(x)	10	7	4	3	2	5	8

5. What type of function (linear, exponential, etc.) is represented by the data? Describe the evidence to support your answer, and determine the restricted intervals over which each function piece is defined.

6. The function represented by the data in the table has the general form $f(x)=|x+a|+b\bullet|x-a|$. Find the value of a.

7. Use the result of problem 6 to find the value of b. Be sure to check the values of a and b in each of the three restricted domains of the function.

UNIT 4B • MATHEMATICAL MODELING AND CHOOSING A MODEL
Lesson 4: Choosing a Model

The following graph shows two piecewise functions representing the manufacturing cost (*C*) and the sales (*S*) of a new model of a high-efficiency, long-life electric motor. The time in months over which the motors are introduced and sold is represented by the independent variable *t*. Use the graph to complete problems 8–10.
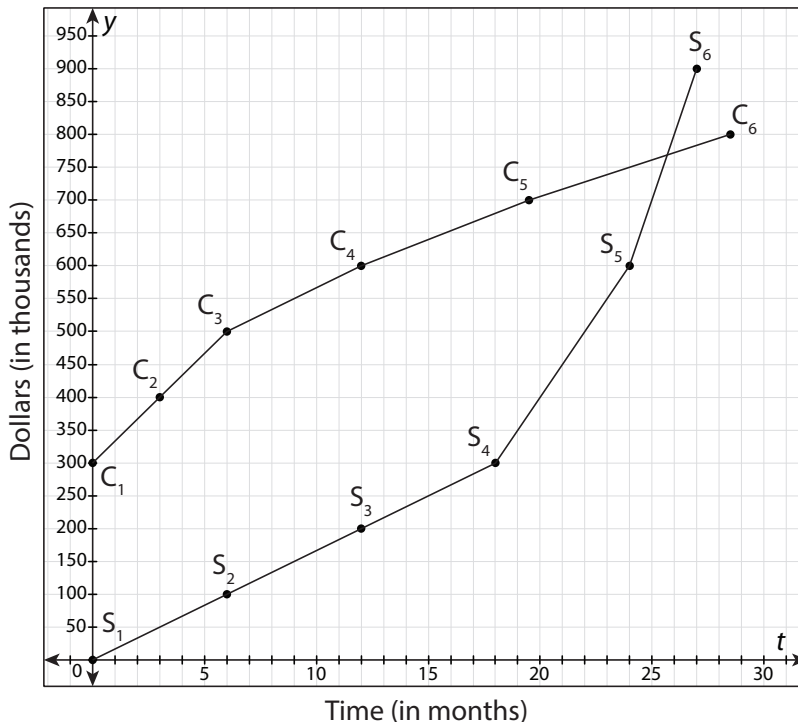
8. Determine the series of points that comprise the unique functions making up the pieces of the cost function $C(t)$. Name the type of function for each letter combination, determine each function's domain, and justify your answers.

9. Determine the letters for the unique functions making up the pieces of the sales function $S(t)$. Name the type of function for each letter combination, determine each function's domain, and justify your answers.

10. Describe at what point the manufacturer starts to make a profit on the motors.

Lesson 4B.4.3: Square Root and Cube Root Functions

Introduction

Cubic functions and quadratic functions have many applications in real-world problems. The inverses of those functions can be just as useful in modeling situations in which the independent variable in a function model is inaccessible, immeasurable, or otherwise not available for use in a calculation. While the domain of a cubic function and its inverse function are often the same, the domain of a quadratic function is often not the same as the domain of its inverse because of restrictions on the value of variables under the radical sign of the inverse. And, as is always the case with real-world problems, the domains of both types of functions must reflect the realities of real-world conditions and factors.

Key Concepts

- A **cube root function** is a function that contains the cube root of a variable. The general form is $f(x) = a\sqrt[3]{(x-h)} - k$, where a, h, and k are real numbers.

- A **square root function** contains a square root of a variable. Square root functions have the general form $f(x) = \sqrt{ax^2 + bx + c}$, where a, b, and c are real numbers.

- Cube root and square root functions, also known as radical functions, are the inverses of cubic and quadratic functions, respectively.

- In both functions, any of the constants a, b, c, and d can be equal to 0, so many such functions are simpler than these general forms.

- The highest power of x does not exceed the reciprocal of the root in this general form because it is assumed that any higher powers of x have been factored out of the terms that are under the radical sign.

- Just as with absolute value, step, and other piecewise functions, domain and range variables for real-world problems involving cube root or square root functions may require adjusting the scale of their graphs to suit the situation.

Guided Practice 4B.4.3

Example 1

A marine biologist measures the volume of a roughly spherical piece of coral by submerging it in a container of seawater of known volume. The coral displaces a volume of $\dfrac{4}{375}\pi$ cubic meters of seawater. What is the radius of the coral? The function for the volume of the coral is given by $V(r)=\dfrac{4}{3}\pi r^3$, where r is the radius of the sphere.

1. Determine the value of r.

   Let $V(r)$ be $\dfrac{4}{375}\pi$ in the formula for the volume of a sphere. Solve for r.

$V(r)=\dfrac{4}{3}\pi r^3$	Formula for the volume of a sphere
$\left(\dfrac{4}{375}\pi\right)=\dfrac{4}{3}\pi r^3$	Substitute $\dfrac{4}{375}\pi$ for $V(r)$.
$\dfrac{4}{125}\pi=4\pi r^3$	Multiply both sides by 3.
$\dfrac{1}{125}=r^3$	Divide both sides by 4π.
$r=\sqrt[3]{\dfrac{1}{125}}$	Take the cube root of both sides.
$r=\dfrac{1}{5}=0.2$	Simplify.

   The radius of a sphere with a volume of $\dfrac{4}{375}\pi$ cubic meters is 0.2 meter.

2. Rewrite the result of step 1 in the context of the problem.

   The radius of the piece of coral is roughly 0.2 meter, or about 20 centimeters.

Example 2

An ecologist drops a data-collection buoy into a river from a bridge. With what velocity does the buoy hit the river surface if the acceleration due to gravity has a magnitude of about 32 feet per second squared and the bridge is 100 feet above the surface of the river? The formula relating an object's acceleration, distance, and velocity is $v(h)^2 = 2ah$, where a is the acceleration, h is the height, and v is the final velocity in feet per second.

1. Rewrite the given formula as a radical function.

$$v(h)^2 = 2ah$$ Original function

$$v(h) = \pm\sqrt{2ah}$$ Take the square root of both sides.

The related radical function is $v(h) = \pm\sqrt{2ah}$.

2. Assign signs to the constants given in the problem, and explain the choice of signs so that the real-world conditions of the problem are met.

If the acceleration due to gravity is defined as a negative value ($a = -32$ feet per second squared) because the object is falling toward Earth, then the height h also has to be negative so that the expression under the radical, $2ah$, will be defined. However, both a and h can be defined as positive as long as all other conditions of the problem (in this case, v, the velocity) are similarly defined.

3. Use the radical function and the defined constants to find the value of v.

Use the negative square root option because the quantities describing the journey of the buoy from the bridge to the river are defined as negative.

$$v(h) = -\sqrt{2ah}$$ Negative radical function

$$v(-100) = -\sqrt{2(-32)(-100)}$$ Substitute −100 for h and −32 for a.

$$v(-100) = -\sqrt{6400}$$ Simplify.

$$v(-100) = -80$$ Take the square root.

The velocity of the buoy when it hits the river is −80 feet per second.

Example 3

A 3-D printer is programmed to produce a plastic game piece. From top to bottom, the game piece has parallel circular bases with radii of r centimeters and $r + 3$ centimeters, respectively, as shown in the diagram. The distance between the circular bases is $3r$ centimeters. The dashed lines on the diagram represent the imaginary tip of the cone that would result if the sides of the game piece continued to that point.

The volume of plastic in cubic centimeters used to produce the game piece is known, but the parts engineer would like to have a function model for the volume that can be solved for r. Use the diagram and the general formula for the volume of a cone, $V_{\text{cone}} = \dfrac{1}{3}\pi r^2 h$, to write a function for the volume of the game piece as a function of r. Determine the domain over which the volume exists. Then, write the volume function of the game piece in such a way that it matches the general form of a cube root function, $f(x) = \sqrt[3]{ax^3 + bx^2 + cx + d}$. Finally, write the cube root function that has to be solved for a game piece volume of 27π cubic centimeters, and use it to calculate the actual height of the game piece and the radii of its bases.
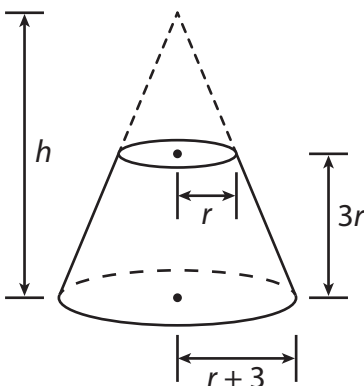
1. Write an expression for the height of the imaginary small cone, shown with dashed lines atop the smaller circular base of the game piece.

   The height of the combined game piece and the small cone is h.
   The height of the game piece is $3r$. Therefore, the height of the small cone is $h - 3r$.

2. Write an equation for finding h in terms of r.

Let $r + 3$ be the radius of the base of the large cone formed by the game piece and the small cone, let r be the radius of the base of the small cone, let h be the height of the large cone, and let $h - 3r$ be the height of the small cone.

Recall that sides of similar triangles are proportional. Use this fact to set up a proportion comparing the large cone to the small cone, and solve this proportion for h: $\dfrac{r}{r+3} = \dfrac{h-3r}{h}$.

$\dfrac{r}{r+3} = \dfrac{h-3r}{h}$	Sides of similar triangles are proportional
$rh = (r+3)(h-3r)$	Cross multiply.
$rh = rh - 3r^2 + 3h - 9r$	Simplify.
$3r^2 + 9r = 3h$	Isolate h.
$r^2 + 3r = h$	Simplify.
$r(r+3) = h$	Factor out r.
$h = r(r+3)$	Apply the Symmetric Property of Equality.

The equation for finding h in terms of r is $h = r(r+3)$.

3. Write an equation for the volume of the game piece in terms of the volumes of both cones in the diagram.

Let V_{GP} represent the volume of the game piece, and V_{LC} and V_{SC} represent the volumes of the large and small cones, respectively.

The volume of the game piece is equal to the difference in volumes of the large and small cones, so the function is given by $V_{GP} = V_{LC} - V_{SC}$.

4. Use the known values for r and h of the large cone and the general equation for the volume of a cone to write an equation for the volume of the large cone.

Recall that the general formula for the volume of a cone is $V_{cone} = \frac{1}{3}\pi r^2 h$.

The diagram shows that the radius of the large cone is $r + 3$ and the height is h.

From step 2, we know that the equation for h in terms of r is $h = r(r + 3)$.

$$V_{cone} = \frac{1}{3}\pi r^2 h \qquad\qquad \text{Formula for the volume of a cone}$$

$$(V_{LC}) = \frac{1}{3}\pi (r+3)^2 h \qquad\qquad \text{Substitute } V_{LC} \text{ for } V_{cone} \text{ and } r + 3 \text{ for } r.$$

$$V_{LC} = \frac{1}{3}\pi (r+3)^2 [r(r+3)] \qquad \text{Substitute } r(r + 3) \text{ for } h.$$

$$V_{LC} = \frac{1}{3}\pi r(r+3)^3 \qquad\qquad \text{Rearrange and combine terms.}$$

The formula for the volume of the large cone in terms of r is $V_{LC} = \frac{1}{3}\pi r(r+3)^3$.

Write an equation for the volume of the small cone.

From step 1, we know that the height of the small cone is $h - 3r$.

Recall that the equation for h in terms of r is $h = r(r + 3)$.

Therefore, the height of the small cone can also be written as $[r(r + 3)] - 3r$, which simplifies to $r^2 + 3r - 3r$.

Substitute this value into the general volume formula and simplify.

$$V_{cone} = \frac{1}{3}\pi r^2 h \qquad\qquad \text{Formula for the volume of a cone}$$

$$(V_{SC}) = \frac{1}{3}\pi r^2 (r^2 + 3r - 3r) \qquad \text{Substitute } V_{SC} \text{ for } V_{cone} \text{ and } r^2 + 3r - 3r \text{ for } h.$$

$$V_{SC} = \frac{1}{3}\pi r^4 \qquad\qquad \text{Simplify and combine terms.}$$

The formula for the volume of the small cone in terms of r is $V_{SC} = \frac{1}{3}\pi r^4$.

5. Write a specific equation for the volume of the game piece.

Use the volume formulas for the large and small cones to write an equation for the volume of the game piece.

Recall the general equation of the game piece from step 3: $V_{GP} = V_{LC} - V_{SC}$.

$V_{GP} = V_{LC} - V_{SC}$	Formula for the volume of the game piece
$(V_{GP}) = \left[\frac{1}{3}\pi r(r+3)^3\right] - \left[\frac{1}{3}\pi r^4\right]$	Substitute values for the large and small cones.
$V_{GP} = \frac{1}{3}\pi r(r^3 + 9r^2 + 18r + 27) - \frac{1}{3}\pi r^4$	Expand the cubic term.
$V_{GP} = \frac{1}{3}\pi r^4 + 3\pi r^3 + 6\pi r^2 + 9\pi r - \frac{1}{3}\pi r^4$	Cancel and combine terms.
$V_{GP} = 3\pi r(r^2 + 2r + 3)$	Simplify.

Therefore, the specific equation for the volume of the game piece is $V_{GP} = 3\pi r(r^2 + 2r + 3)$.

6. Determine the domain for r such that the volume of the game piece exists and is a positive number.

The radius r is a real-world quantity, so it is positive; therefore, $r > 0$. Thus, the minimum volume of the game piece occurs as r approaches 0.

7. Use the specific equation for V_{GP} to write the cubic equation that has to be solved for r if the volume of the game piece is 27π cubic centimeters.

Substitute 27π into the specific equation for V_{GP}, then simplify.

$V_{GP} = 3\pi r(r^2 + 2r + 3)$	Formula for the volume of the game piece
$(27\pi) = 3\pi r(r^2 + 2r + 3)$	Substitute 27π for V_{GP}.
$9 = r(r^2 + 2r + 3)$	Divide both sides by 3π.
$9 = r^3 + 2r^2 + 3r$	Distribute.
$0 = r^3 + 2r^2 + 3r - 9$	Subtract 9 from both sides.

The cubic equation to be solved for r is $0 = r^3 + 2r^2 + 3r - 9$.

8. Use a graphing calculator to find the value(s) of r that can result in a volume of 27π cubic centimeters.

Graph $0 = r^3 + 2r^2 + 3r - 9$, then use your calculator's trace feature to find the values of x at which the function values change from negative to positive (i.e., the zeros of the function). Results may vary depending on your calculator model.

- On a TI-83/84, tracing the graph shows that the solution lies between $x = 1.06$ and $x = 1.28$.

- On a TI-Nspire, tracing the graph yields an x-value of 1.26346 for a y-value of 0.

The value of r that gives a game piece volume of 27π cubic centimeters is about 1.27 centimeters. This means that the game piece has bases with radii of about 1.27 and 4.27 centimeters, respectively, and a height of about 5.4 centimeters.

Example 4

Exoplanets are planets that exist outside of Earth's solar system. As astronomers discover new exoplanets orbiting other stars in our galaxy, they find that some of the same relationships governing the motion of planets in our own solar system apply to those exoplanets, too. One of these is the function $T(r) = \left(\dfrac{2\pi}{\sqrt{GM}} \right) \cdot r^{\frac{3}{2}}$. It relates the length of a "year," $T(r)$, of a planet orbiting a star with an orbit that has an average radius of r, with "year" referring to the amount of time the exoplanet takes to complete one orbit. Compare the average orbital radii of two exoplanets orbiting the same star if one of the exoplanets has a year that is twice as long as the other. (*Note*: The quantities G and M are constants—the universal gravitation constant and the mass of the star about which the planet is orbiting—and do not figure into the calculations for this example.)

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1. Write the relationship of the year of the first exoplanet to that of the exoplanet whose year is twice as long.

   If the year of one exoplanet is T, then the year of the other exoplanet is $2T$.

---

2. Write the function for the exoplanet with the shorter year, and label its average orbital radius r_1.

   Let T be the length of a year. Substitute T for $T(r)$ and r_1 for r in the given function, $T(r) = \left(\dfrac{2\pi}{\sqrt{GM}} \right) \cdot r^{\frac{3}{2}}$:

   $$T = \left(\dfrac{2\pi}{\sqrt{GM}} \right) \cdot r_1^{\frac{3}{2}}$$

---

3. Rearrange the resulting function to solve for r_1.

$$T = \left(\frac{2\pi}{\sqrt{GM}}\right) \bullet r_1^{\frac{3}{2}}$$

Function from the previous step

$$T^2 = \left(\frac{2\pi}{\sqrt{GM}}\right)^2 \bullet r_1^{\left(\frac{3}{2}\right)^2}$$

Square both sides.

$$T^2 = \left(\frac{4\pi^2}{GM}\right) \bullet r_1^3$$

Simplify.

$$\left(\frac{GM}{4\pi^2}\right)T^2 = \left(\frac{\cancel{GM}}{\cancel{4\pi^2}}\right) \bullet \left(\frac{\cancel{4\pi^2}}{\cancel{GM}}\right) \bullet r_1^3$$

Multiply both sides by the reciprocal of the constant term in parentheses; simplify.

$$\frac{GMT^2}{4\pi^2} = r_1^3$$

Multiply the terms on the left side.

$$\sqrt[3]{\frac{GMT^2}{4\pi^2}} = r_1$$

Take the cube root of both sides.

The average orbital radius of the exoplanet with the shorter year is

$$r_1 = \sqrt[3]{\frac{GMT^2}{4\pi^2}}.$$

4. Write the function for the exoplanet with the longer year, and label its average radius r_2.

Let $2T$ be the length of a year. Substitute $2T$ for $T(r)$ and r_2 for r in the given function, $T(r) = \left(\frac{2\pi}{\sqrt{GM}}\right) \bullet r^{\frac{3}{2}}$:

$$2T = \left(\frac{2\pi}{\sqrt{GM}}\right) \bullet r_2^{\frac{3}{2}}$$

5. Rearrange the resulting function to solve for r_2.

$$2T = \left(\frac{2\pi}{\sqrt{GM}} \right) \bullet r_2^{\frac{3}{2}}$$

Function from the previous step

$$4T^2 = \left(\frac{2\pi}{\sqrt{GM}} \right)^2 \bullet r_2^{\left(\frac{3}{2}\right)^2}$$

Square both sides.

$$4T^2 = \left(\frac{4\pi^2}{GM} \right) \bullet r_2^3$$

Simplify.

$$\left(\frac{GM}{4\pi^2} \right) \bullet 4T^2 = \left(\frac{\cancel{GM}}{\cancel{4\pi^2}} \right) \bullet \left(\frac{\cancel{4\pi^2}}{\cancel{GM}} \right) \bullet r_2^3$$

Multiply both sides by the reciprocal of the constant term; simplify.

$$\frac{GMT^2}{\pi^2} = r_2^3$$

Multiply the terms on the left side.

$$\sqrt[3]{\frac{GMT^2}{\pi^2}} = r_2$$

Take the cube root of both sides.

The average orbital radius of the exoplanet with the longer year is

$$r_2 = \sqrt[3]{\frac{GMT^2}{\pi^2}}.$$

6. Compare the average orbital radii of the two exoplanets.

Start by simplifying the formula for r_1.

$$r_1 = \sqrt[3]{\frac{GMT^2}{4\pi^2}}$$
Orbital radius of the exoplanet with the shorter year

$$r_1 = \sqrt[3]{\frac{1}{4} \cdot \frac{GMT^2}{\pi^2}}$$
Factor $\frac{1}{4}$ from the argument.

$$r_1 = \sqrt[3]{\frac{1}{4}} \cdot \sqrt[3]{\frac{GMT^2}{\pi^2}}$$
Rewrite the cube root as a product.

Notice that one of the factors of r_1 is equal to the equation for the larger exoplanet: $r_2 = \sqrt[3]{\frac{GMT^2}{\pi^2}}$. Therefore, the equation for r_1 can be rewritten in terms of r_2 by substituting r_2 into $r_1 = \sqrt[3]{\frac{1}{4}} \cdot \sqrt[3]{\frac{GMT^2}{\pi^2}}$.

$$r_1 = \sqrt[3]{\frac{1}{4}} \cdot \sqrt[3]{\frac{GMT^2}{\pi^2}}$$
Simplified equation for r_1

$$r_1 = \sqrt[3]{\frac{1}{4}} \cdot (r_2)$$
Substitute r_2 for $\sqrt[3]{\frac{GMT^2}{\pi^2}}$.

The resulting equation for r_1 is $r_1 = \sqrt[3]{\frac{1}{4}} \cdot r_2$.

The average orbital radius of the exoplanet with the shorter year is $\sqrt[3]{\frac{1}{4}}$ (or about 0.63) times the average orbital radius of the exoplanet with the longer year.

UNIT 4B • MATHEMATICAL MODELING AND CHOOSING A MODEL
Lesson 4: Choosing a Model

Practice 4B.4.3: Square Root and Cube Root Functions

For problems 1–3, write the equation of each square root function described.

1. the area, $A(x)$, of a rectangle with a diagonal of length x and one side of length $x - 3$

2. the function value, $f(x)$, in an ordered pair, $(x, f(x))$, on an ellipse with its center at the origin of a coordinate system and represented by a function of the form $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$. (*Hint*: The general equation for an ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where (x, y) is a point on the ellipse, a is the radius of the ellipse on the x-axis, and b is the radius on the y-axis.)

3. the length, L, of a simple pendulum with a period of $T(L)$ represented by the function $T(L)^2 = \dfrac{4\pi^2 L}{g}$, in which g is the acceleration due to gravity

For problems 4–6, write the equation of each cube root function described.

4. the length of a side, x, of the square base of a pyramid with a known height of $3x$ and a volume of $V(x)$. (*Hint*: The volume formula for a pyramid is $V = \frac{1}{3}Bh$, in which B is the area of the base.)

5. the number of outcomes, g, of three game events such that the probability, $P(g)$, of getting three specific outcomes is represented by the probability function $P(g) = \dfrac{1}{0.1g \cdot 0.2g \cdot 0.4g}$

6. the time, t, in a function that models the velocity, v, of a hang glider and that includes terms for air resistance, lift, and rotational motion given by $v(t) = -20t + 0.2at^2 - 0.002bt^3$, where a and b are constants

Use the information given in each scenario to complete problems 7–10.

7. Amari and Carson are both trying out for quarterback. Each boy throws a 40-yard pass. Amari's pass is a "lob" pass that reaches a height of 20 yards; Carson's is a "bullet" pass that reaches a height of 10 yards. The distance $d(t)$ traveled by each football is modeled by the function $d(t) = vt + \frac{1}{2}at^2$, where v is the velocity of the football after it leaves the boy's hand, a is the acceleration due to gravity, and t is the time in flight. This function can be used with either the horizontal (40 yards) or vertical (10 or 20 yards high) distance traveled by each football. Assume that the speeds of the two boys' passes are the same at 15 yards per second. How long does it take each boy's football to land?

8. A sphere is submerged in a rectangular tank of water. The length and width of the tank are the same, and are equal to two times the radius of the sphere. The volume of the sphere can be measured by the amount of water displaced when it is submerged in the tank. Different volume levels of the tank can be calculated by measuring the depth, length, and width of the tank. Write a function for r, the radius of the sphere, that gives the difference between the volumes of the tank and the sphere when the depth of the water in the tank is equal to two times the radius of the sphere. Use the volume formulas $V = lwh$ (for a rectangular solid) and $V = \frac{4}{3}\pi r^3$ (for a sphere).

9. The time, $T(r)$, that it takes an exoplanet to complete one orbit around its star (a "year") is given by the function $T(r) = \left(\dfrac{2\pi}{\sqrt{GM}} \right) \bullet r^{\frac{3}{2}}$, where G and M are constants. Compare the length of a year for two different exoplanets if the first exoplanet's sun has a mass M that is half the mass of the second exoplanet's sun. Assume that both exoplanets have the same average orbital radii.

10. The volume of a rectangular solid with dimensions of x, $x + 2$, and $x + 4$ feet is 15 cubic feet. Is it possible for the value of x to result in a volume of 5 cubic feet? If possible, determine an approximate value for x.

Lesson 5: Geometric Modeling

Common Core State Standards

G–GMD.4 Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

G–MG.1 Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).★

G–MG.2 Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).★

G–MG.3 Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).★

Essential Questions

1. What plane geometric figure is formed when a plane intersects a solid figure?

2. What solid figure is formed when a plane figure is rotated a specific number of degrees or radians?

3. How can the density of a quantity be computed for the area or volume in which the quantity is located?

4. How can density be used to compute the total amount of a quantity in an area or a volume?

5. How can the properties of geometric figures and their measurements be used to design solutions to real-world problems?

WORDS TO KNOW

axis of rotation	a line about which a plane figure can be rotated in three-dimensional space to create a solid figure, such as a diameter or a symmetry line
congruent	having the same shape, size, lines, and angles; the symbol for representing congruency between figures is ≅

cross section	the plane figure formed by the intersection of a plane with a solid figure, where the plane is at a right angle to the surface of the solid figure
density	the amount, number, or other quantity per unit of area or volume of some substance or population being studied
plane	a flat, two-dimensional figure without depth that has at least three non-collinear points and extends infinitely in all directions
plane figure	a two-dimensional shape on a plane
polyhedron	a three-dimensional object that has faces made of polygons
regular polyhedron	a polyhedron with faces that are all congruent regular polygons; the angles created by the intersecting faces are congruent, and the cross sections are similar figures
rho (ρ)	a lowercase Greek letter commonly used to represent density
rotation	in three dimensions, a transformation in which a plane figure is moved about one of its sides, a fixed point, or a line that is not located in the plane of the figure, such that a solid figure is produced
similar figures	two figures that are the same shape but not necessarily the same size; the symbol for representing similarity between figures is $\sim$.
solid figure	a three-dimensional object that has length, width, and height (depth)
translation	in three dimensions, the horizontal or vertical movement of a plane figure in a direction that is not in the plane of the figure, such that a solid figure is produced

Recommended Resources

- Annenberg Learner. "Cross Sections."

  http://www.walch.com/rr/00249

  This site provides an overview of cross sections, along with an interactive cube that can be rotated and cut in various locations to show the resulting cross section. Requires Flash.

- LearnZillion.com. "Predict 3D Results of Rotating Simple Figures."

  http://www.walch.com/rr/00250

  This video tutorial reviews how rotations of a plane figure about an axis can be used to create three-dimensional solids.

- MathIsFun.com. "Solid Geometry."

  http://www.walch.com/rr/00251

  This site reviews the geometry of three-dimensional objects; in particular, it covers properties of solids and compares polyhedra to non-polyhedra, with examples of each.

IXL Links

- Front side and top view:
  http://www.ixl.com/math/grade-8/front-side-and-top-view

- Nets of 3 dimensional figures:
  http://www.ixl.com/math/grade-8/nets-of-3-dimensional-figures

- Cross sections of three dimensional figures:
  http://www.ixl.com/math/geometry/cross-sections-of-three-dimensional-figures

- Solids of revolution:
  http://www.ixl.com/math/geometry/solids-of-revolution

- Area and perimeter word problems:
  http://www.ixl.com/math/grade-8/area-and-perimeter-word-problems

- Circles word problems:
  http://www.ixl.com/math/grade-8/circles-word-problems

Lesson 4B.5.1: Two-Dimensional Cross Sections of Three-Dimensional Objects

Introduction

A chef takes a knife and slices a carrot in half. What shape results? Depending on the direction of the cut, the resulting shape may resemble a circle (for a widthwise cut), a triangle (for a lengthwise cut from the tip of the carrot to its base), or an oval (for a diagonal cut). The circle, triangle, and oval are each a type of **plane figure**, a two-dimensional shape on a plane. Recall that a **plane** is a flat, two-dimensional figure without depth that has at least three non-collinear points and extends infinitely in all directions. When planes intersect with **solid figures**, or three-dimensional objects that have length, width, and height (or depth), plane figures are created. In this example, the carrot can be thought of as the solid figure, the knife as the intersecting plane, and the resulting shape as the plane figure.

Plane figures have many applications in real-world problems. Technical drawings of solid figures, such as the parts of a redesigned spacecraft, often rely on back, bottom, front, and rear views of a figure that exists in three dimensions or in three-dimensional coordinate systems. These applications can be as complex as the real-world circumstances in which they are found, but many rely on relatively basic properties of plane and solid geometry for their description and use in calculations.

Similarly, two-dimensional figures can be used to generate solid figures with transformations in three-dimensional space. Solids can be generated by the horizontal or vertical movement of a plane figure through a third dimension, or by the rotation of a plane figure about an axis of rotation or line in a third dimension.

Key Concepts

- In its basic form, a **cross section** is the plane figure formed by the intersection of a plane and a solid figure, where the plane is at a right angle to the surface of the solid figure. The cross section can also be formed by an axis of rotation of the solid figure, given that the axis of rotation is perpendicular to the intersecting plane.

- An **axis of rotation** is a line about which a plane figure can be rotated in three-dimensional space to create a solid figure, such as a diameter or a symmetry line. For example, a line that connects the centers of the circular bases of a cylinder is an axis of rotation.

- The shape of a cross section depends on the solid figure and the location of the cross section.

- A **polyhedron** is a three-dimensional object that has faces made of polygons. Such polygonal plane figures include squares, rectangles, and trapezoids. **Regular polyhedra** are solids that have faces that are all congruent regular polygons. (Recall that **congruent** means having the same shape, size, lines, and angles; the symbol for representing congruency between figures is ≅.) The cross sections of regular polyhedra are similar figures. **Similar figures** are two figures that are the same shape but not necessarily the same size. In other words, corresponding sides have equal ratios. The symbol for representing similarity between figures is $\sim$. One example of a regular polyhedron is a triangular prism with bases consisting of equilateral triangles.

- Cross sections of some polyhedra such as pyramids form similar figures.

- An infinite number of plane figures can be formed by the intersection of a plane with a solid figure, but the cross section is a special case due to its perpendicularity with the surface of the solid.

- A solid figure can be generated by translating a plane figure along one of its two dimensions. In three dimensions, **translation** refers to the horizontal or vertical movement of a plane figure in a direction that is not in the plane of the figure, such that a solid figure is produced. Imagine a pop-up cylindrical clothes hamper—it can be compressed into a flat circle for storage, but when you need to use it you lift up the top part to "open" it up into its cylindrical form.

- A solid figure can also be generated by the rotation of a plane figure in three dimensions. A **rotation** is a transformation in which a plane figure is moved about one of its sides, a fixed point, or a line that isn't in the plane of the figure, such that a solid figure is produced. Similarly, imagine a toothpick with a small paper flag attached. Twirling the toothpick quickly back and forth between your fingers causes the paper flag to form the illusion of a cylinder.

Guided Practice 4B.5.1

Example 1

Compare the areas of two rectangular cross sections of an air-conditioning duct along a length of the duct that shrinks in width from 20 inches to 12 inches and in height from 8 inches to 6 inches. Find the difference of the areas, and determine the percentage that each cross section is of the other. Then, name the solid figure from which the two rectangular cross sections are formed.
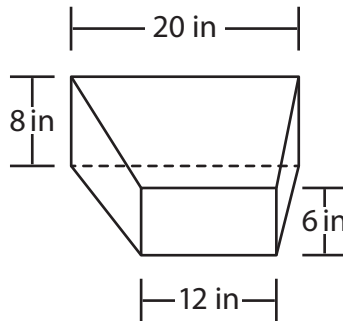
1. Calculate the area of the larger cross section.

   The larger cross section is a rectangle, so use the formula for the area of a rectangle, *area = length • width*. In this case, the "length" is actually the height, so use the formula *area = height • width*, or $A = hw$.

   The larger cross section has a height of 8 inches and a width of 20 inches. Substitute these measurements into the formula $A = hw$ and then solve to determine the area of the larger cross section.

$A = hw$	Rectangular area formula
$A = (8)(20)$	Substitute 8 for h and 20 for w.
$A = 160$	Simplify.

   Recall that area is expressed in square units.

   Thus, the area of the larger cross section is 160 square inches.

2. Calculate the area of the smaller cross section.

The smaller cross section is also a rectangle, so use the formula $A = hw$. However, this cross section has a height of 6 inches and a width of 12 inches. Substitute these measurements into the formula $A = hw$ and then solve to determine the area of the smaller cross section.

$A = hw$	Rectangular area formula
$A = (6)(12)$	Substitute 6 for h and 12 for w.
$A = 72$	Simplify.

The area of the smaller cross section is 72 square inches.

3. Find the difference of the cross section areas.

Subtract the area of the smaller cross section from the area of the larger cross section.

$160 - 72 = 88$

The difference of the areas is 88 square inches.

4. Calculate the percentage that the smaller area is of the larger area.

Divide the smaller area by the larger area and then multiply by 100 to find the percentage.

$\dfrac{\text{smaller area}}{\text{larger area}} \cdot 100$	Percentage formula
$= \dfrac{(72)}{(160)} \cdot 100$	Substitute 72 for the smaller area and 160 for the larger area.
$= 45$	Simplify.

The smaller area is 45% of the size of the larger area.

5. Calculate the percentage that the larger area is of the smaller area.

Divide the larger area by the smaller area and then multiply by 100 to find the percentage.

$$\frac{\text{larger area}}{\text{smaller area}} \cdot 100 \qquad \text{Percentage formula}$$

$$= \frac{(160)}{(72)} \cdot 100 \qquad \begin{array}{l}\text{Substitute 160 for the larger area} \\ \text{and 72 for the smaller area.}\end{array}$$

$$\approx 222.2 \qquad \text{Simplify.}$$

The larger area is about 222% of the size of the smaller area.

6. Determine the solid figure from which the cross sections are formed.

The areas of the rectangular cross sections are unequal, and the rectangular cross sections are not similar. Given this information, the only solid figure from which these cross sections could have been formed is like a pyramid with a rectangular base, except that instead of coming to a point at the top, it comes to a line like the ridge of a roof.

Example 2

The diagram shows the two cross sections, I and II, of a solid figure. What is the geometric solid from which the cross sections are formed?
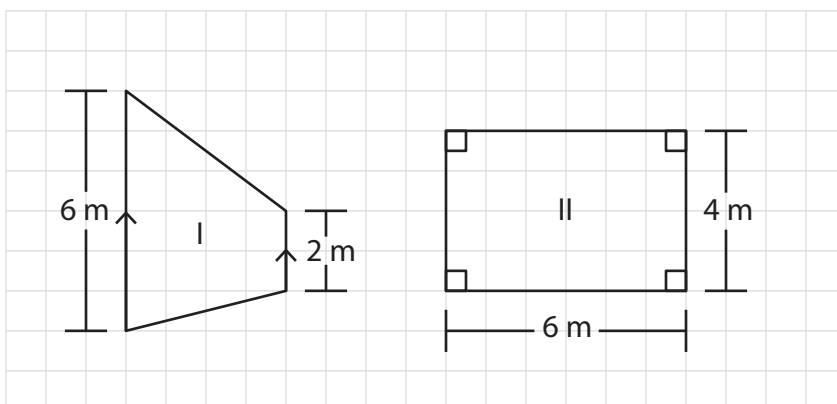
1. Name the plane figure represented by cross section I.

    Cross section I is a trapezoid because it has one pair of parallel sides.

2. Name the plane figure represented by cross section II.

    Cross section II is a rectangle because it has four right angles and two pairs of congruent and parallel sides.

3. Describe how the cross sections are oriented with respect to each other and why.

    The cross sections represent two of the three dimensions of the solid. By definition, a cross section intersects a solid figure at a right angle to the surface of the solid figure. Therefore, since both cross sections must intersect the figure at a right angle and there are two different cross sections, the cross sections are perpendicular to each other.

4. Name the solid figure implied by the cross sections.

    The two cross sections are the only ones for this solid figure. Therefore, the trapezoid is the base of a prism. The rectangle is a cross section of the prism that is parallel to the faces of the prism formed by the parallel sides of the trapezoid, because it has a width that is less than the longer parallel side of the trapezoid and greater than the shorter parallel side of the trapezoid. The diagram that follows shows the solid and its cross sections.
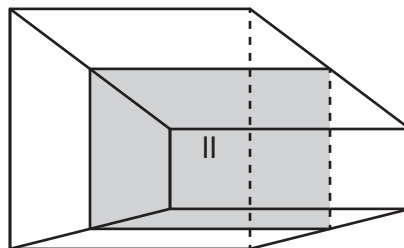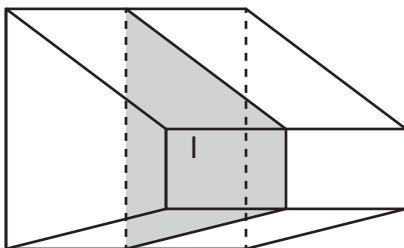
Example 3

A right triangle is rotated 270° about its longer leg, $\overline{AC}$, as shown. Describe the solid figure produced by this rotation as well as what this partial revolution means in terms of the solid figure.

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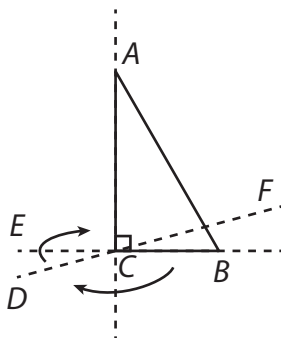
1. Name the sides of the triangle that will move during the rotation.

   Sides $\overline{AB}$ and $\overline{BC}$ will move during the rotation. Side $\overline{AC}$ will remain in place, since it represents an axis of rotation.

---

2. Name the plane figure that the line $\overline{BC}$ will create if it is rotated in space one complete revolution about point C.

   If it is rotated in space one complete revolution, $\overline{BC}$ will create the circumference of a circle with its center at point C; $\overline{BC}$ will be the radius of the circle.

---

3. Describe the solid figure that $\overline{AB}$ will create if it is rotated in space one complete revolution about point A.

   Rotating $\overline{AB}$ one revolution around point A will create a cone that has a slant height of $\overline{AB}$ and a radius that is equal to the circumference of circle C; this circumference will be equal to 2π times the length of $\overline{BC}$, or $2\pi \cdot BC$.

4. Describe the solid figure produced by the rotation.

   The solid figure is a cone with a highest point at point A, a height of $\overline{AC}$, and a circular base with a radius of length $\overline{BC}$.

5. Determine what fraction of a complete revolution 270° represents.

   Recall that 270° is the number of degrees that the lines $\overline{AB}$ and $\overline{BC}$ rotate. Multiply 270° by the conversion factor $\dfrac{1 \text{ revolution}}{360°}$ to convert the degree measure to revolutions.

   $$270° \cdot \frac{1 \text{ revolution}}{360°} = \frac{3}{4} \text{ revolution}$$

   270° is equivalent to $\dfrac{3}{4}$ of a complete revolution.

6. Describe what this partial revolution means in terms of the solid figure.

   A 270° revolution of the right triangle results in a figure that is $\dfrac{3}{4}$ of the cone that would be produced if the right triangle were rotated by one complete revolution.

Example 4

In regular prisms, equilateral polygons form the bases, and the rectangular faces of the prism make up the sides, which are perpendicular to the bases. Planes can intersect regular prisms in such a way as to divide the prism into two congruent halves; these planes are said to bisect the prisms. The diagram shows a regular hexagonal prism (I), a rectangular prism with congruent squares for bases (II), and an octagonal prism (III). Identify the number of such cross sections for each regular prism, and describe how this number relates to the number of faces of the prism.

I   II   III

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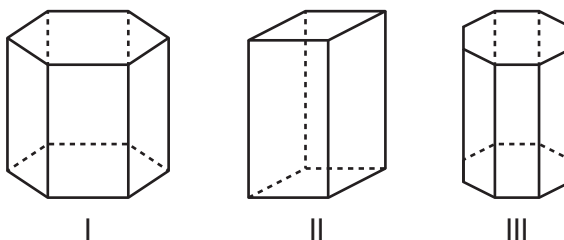
1. Determine the number of faces of each prism.

   Faces include the rectangular sides and the polygonal bases.

   Organize the information in a table.

Figure	Rectangular sides	Bases	Total faces
I	6	2	8
II	4	2	6
III	8	2	10

---

2. Describe the number of ways that each base could be bisected by a plane to produce two congruent halves.

   - Figure I: The 3 pairs of parallel sides of the hexagonal bases could be bisected to produce 3 pairs of congruent prisms, so there are 3 ways.

   - Figure II: The 2 pairs of parallel opposite sides of the square bases could be bisected to produce 2 pairs of congruent prisms, so there are 2 ways.

   - Figure III: The 4 pairs of parallel sides of the hexagonal bases could be bisected to produce 4 pairs of congruent prisms, so there are 4 ways.

3. Describe the number of ways that each prism could be bisected by a plane that is parallel to the bases to produce two congruent halves.

To produce two congruent prisms, each figure could be intersected by 1 plane located halfway between and parallel to the bases. Therefore, there is only 1 way this is possible for each prism.

4. Use the results of steps 2 and 3 to determine the total number of cross sections that produce congruent halves of each prism.

Add the number of ways each prism's base could be bisected (found in step 2) to the number of ways each prism could be bisected (step 3). This gives the total number of cross sections that produce congruent halves.

- Figure I: 3 + 1 = 4 cross sections
- Figure II: 2 + 1 = 3 cross sections
- Figure III: 4 + 1 = 5 cross sections

5. Use the number of faces and cross sections for each figure to suggest a rule relating the total number of faces to the number of cross sections that produce congruent prisms.

Refer to the table from step 1 and the results of step 4 to determine the number of faces and cross sections.

- Figure I has 8 faces and 4 cross sections.
- Figure II has 6 faces and 3 cross sections.
- Figure III has 10 faces and 5 cross sections

Based on these figures, the rule appears to be that the number of cross sections producing congruent prisms is half of the number of faces of the "parent" prism.

6. Use the number of faces and cross sections result for Figure II to predict how many cross sections would be needed to divide a regular decagonal (10-sided) prism into congruent halves.

Based on these results, the number of cross sections would be half the number of faces of the decagonal prism. A decagonal prism has 10 rectangular faces and 2 bases, for a total of 12 faces. Therefore, 6 cross sections would be needed to divide the regular decagonal prism into congruent halves.

Practice 4B.5.1: Two-Dimensional Cross Sections of Three-Dimensional Objects

For problems 1–4, calculate the area of the cross section for each object described.

1. a cross section that is located halfway between the bases of a prism that are congruent triangles with two sides that are 5 inches in length and a third side that is 6 inches in length

2. the two cross sections of unequal areas of a rectangular solid with dimensions of 2 meters by 2 meters by 7 meters

3. a cross section that is located a distance of $\dfrac{1}{\pi}$ yard from the center of a sphere with a volume of 36π cubic yards

4. the cross section of a regular hexagonal prism that is inscribed in a cylinder of radius 20 centimeters

For problems 5–7, use the diagram to describe the object formed by rotating each given figure one complete revolution about line ℓ.

5. arc *FG*

6. sector *FDE*

7. triangle *ABC*

continued

For problems 8–10, calculate the quantity indicated.

8. the cross sections of least and greatest area for a rectangular block of marble with dimensions of 1 meter by 2 meters by 3 meters

9. the maximum diameter of a cross section of a spherical polyester bead that has a volume of 25π cubic microns

10. the dimensions of a rectangular anode "target" in a photoelectric cell that has a cross section of 10^{-18} m^2 if the dimensions of the anode have a 3-to-1 ratio

Lesson 4B.5.2: Density

Introduction

Density is used in a variety of fields to quantify the amount, number, or other quantity per unit of area, volume, or some other basic variable. In some situations, density is used to compare the total number of a quantity with the *average* value of the quantity when it is associated with a second variable. In real-world problems, it is important to carefully define the quantity being averaged, especially with regard to what kind(s) of numbers can be used for quantification. In this sub-lesson, the density of a quantity will be most often compared to a basic unit of area or volume, but these are by no means the only way of averaging a quantity.

Key Concepts

- **Density** is the amount, number, or other quantity per unit of area or volume of some substance or population being studied.

- Density is often symbolized with the lowercase Greek letter ***rho*** (ρ).

- In many real-world problems, density is used to calculate an *average* value of some quantity in terms of a basic unit of area or volume.

- The most basic units of density for these two types of calculations are given by the following formulas:

Equation in words	**Equation in symbols**
$\text{Density}_{\text{Area}} = \dfrac{\text{mass or quantity}}{\text{number of } \textit{square} \text{ units}}$	$\rho_A = \dfrac{m}{A}$
$\text{Density}_{\text{Volume}} = \dfrac{\text{mass or quantity}}{\text{number of } \textit{cubic} \text{ units}}$	$\rho_V = \dfrac{m}{V}$

- Note that these formulas are quite similar, but the area density formula divides by the area, A, (square units) and the volume density formula divides by the volume, V (cubic units). Keeping this distinction in mind can be a useful guide in writing equations for many different density situations.

- Density can vary considerably from one place to another in an area or volume of interest, so it is important to determine if the density calculation will be meaningful for a situation as a whole or if the calculation cannot be reliably used to describe variations in a quantity per unit throughout a problem setting.

- Sometimes the domain of a quantity can determine the most appropriate basic unit to use in order for a density calculation to be meaningful. Fractions of a counting number may be unrealistic for a given situation, such as determining the population density of people living in a certain square mile of a city. Density calculations in such situations make more sense if the contexts in which the quantity is averaged or in which the density is calculated are changed to accommodate the smallest meaningful unit of a quantity of interest.

Guided Practice 4B.5.2

Example 1

A laser-jet printer uses 7,500 droplets of soy-based printer ink to deposit a 6-inch long line of type that is an average of 0.125 of an inch high. How many droplets of ink are deposited per square inch?

1. Calculate the area represented by the line of type.

   The area is the product of the height, 0.125 inch, and the length of the line of type, 6 inches.

   0.125 inch • 6 inches = 0.75 square inch

2. Use the area to calculate the density of ink drops per square inch.

   Recall that density is equal to some quantity divided by the area unit.

   Here, we're dealing with the two-dimensional area of a piece of paper, so use the formula $Density_{Area} = \dfrac{quantity}{number\ of\ square\ units}$.

   The number of drops, or quantity, is given as 7,500.

   The area, or number of square units, is 0.75.

   Substitute these values into the formula to calculate the density.

   $Density_{Area} = \dfrac{quantity}{number\ of\ square\ units}$ — Area density formula

   $Density_{Area} = \dfrac{7500}{0.75}$ — Substitute 7,500 for the quantity and 0.75 for the number of square units.

   $Density_{Area} = 10,000$ — Simplify.

   The ink-jet printer has a print density of 10,000 droplets per square inch. ✓

Example 2

There are 35,000 bats roosting in a pyramid-shaped vaulted cave ceiling that has a height of 30 meters and a rectangular floor that measures 75 by 100 meters. What is the average density of the bat population in the cave? Make sure that the density is a number that reflects the fact that the bat population is measured in counting numbers.

1. Calculate the volume of the cave.

   The cave is pyramid-shaped, with a rectangular floor representing the base of the pyramid.

   Recall that the volume of a pyramid is found using the formula $V = \frac{1}{3}Ah$, where A is the area of the base of the pyramid and h is the pyramid's height.

   The rectangular floor's area can be found using the formula $A = lw$, so use lw for the area, A, substitute the known values, and solve for the volume.

$V = \frac{1}{3}Ah$	Pyramid volume formula
$V = \frac{1}{3}(lw)h$	Substitute lw for A.
$V = \frac{1}{3}(75)(100)(30)$	Substitute 75 for l, 100 for w, and 30 for h.
$V = 75{,}000$	Simplify.

   The pyramid-shaped cave has a volume of 75,000 cubic meters.

2. Calculate the volume density of the bats in the cave.

Use the formula $\text{Density}_{\text{Volume}} = \dfrac{\text{quantity}}{\text{number of cubic units}}$.

The number of bats, or the quantity, is given as 35,000.

The volume of the cave is 75,000 cubic meters.

Substitute these values into the formula to calculate the density.

$$\text{Density}_{\text{Volume}} = \frac{\text{quantity}}{\text{number of cubic units}}$$ Volume density formula

$$\text{Density}_{\text{Volume}} = \frac{(35{,}000)}{(75{,}000)}$$ Substitute 35,000 for the quantity and 75,000 for the number of cubic units.

$$\text{Density}_{\text{Volume}} = 0.4\overline{6}$$ Simplify.

The average density of the bat population in the cave is $0.4\overline{6}$ bats per cubic meter.

3. Determine the domain of the bat population.

The bat population consists of individual bats that are counted using positive integers. Therefore, the domain of the bat population is [1, 35,000].

4. Suggest a different basic volume unit that will result in a bat density measured in positive integers.

One option for the basic volume unit could be a $2 \times 2 \times 2$ meter space within the cave (8 m³) rather than 1 cubic meter. We can write this conversion factor as a ratio, $\dfrac{1 \text{ unit of volume}}{8 \text{ cubic meters}}$.

5. Use the response to step 4 to calculate a more realistic value for the bat density.

Start by converting the total volume of the cave to the new unit of volume measurement.

$$75{,}000 \text{ cubic meters} \cdot \frac{1 \text{ volume unit}}{8 \text{ cubic meters}}$$

Multiply the volume of the cave by the conversion ratio from step 4.

9,375 volume units

Simplify.

Next, substitute the new volume units into the volume formula for the density and recalculate the density. Since the number of bats in the total volume of the cave is unchanged, use 35,000 for the bat quantity.

$$\text{Density}_{\text{Volume}} = \frac{\text{quantity}}{\text{number of cubic units}}$$

Volume density formula

$$\text{Density}_{\text{Volume}} = \frac{(35{,}000)}{(9{,}375)}$$

Substitute 35,000 for the quantity of bats and 9,375 for the volume units.

$$\text{Density}_{\text{Volume}} = 3.7\overline{3}$$

Simplify.

Therefore, there are approximately $3.7\overline{3}$ bats per 1 volume unit, which was defined as 8 m³.

Rounded up to the nearest whole integer, the new calculation gives a density of about 4 bats for each 8 cubic meter space of the cave.

6. Compare the number of bats predicted by the result of step 5 to the number of bats given in the problem.

Multiply the total number of volume units in the cave (9,375 volume units) by the estimated number of bats per volume unit from step 5 (4 bats) to determine the total predicted number of bats in the cave.

$$9375 \text{ volume units} \cdot \frac{4 \text{ bats}}{1 \text{ volume unit}} = 37{,}500$$

Based on our estimate, there are about 37,500 bats in the cave. The original problem gave a bat population of 35,000. Therefore, the estimated population of 37,500 bats is about 7% greater than the given population of 35,000 bats. This is a realistic measurement.

Example 3

Manatees are known to gather in the warmer waters surrounding power plants, which dispose of heated wastewater using discharge canals. The diagram shows the distribution of manatees on a cool winter day in the hot-water discharge basin surrounding a Gulf Coast power plant, as seen on an aerial security photograph. The plant's discharge canal is marked by an X at the end of the positive y-axis. Each data point represents 10 manatees, and each quadrant represents an area measuring 50 yards by 50 yards.
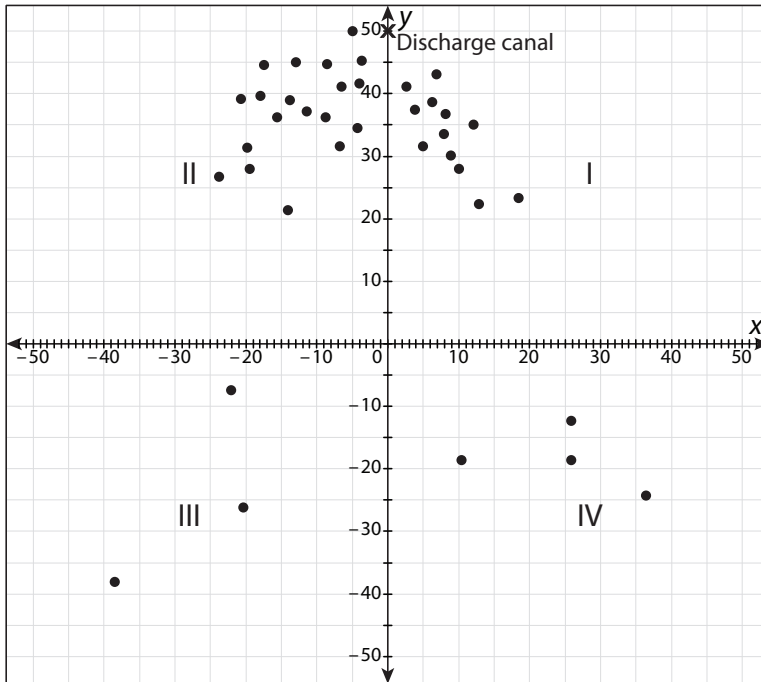
Calculate the density of manatees in each of the four quadrants. Then, calculate the average density of the manatees for all four combined quadrants that make up the discharge basin. Contrast and compare the densities, explain any variations, and make note of any other factors that should be included in relation to the reasonableness of the results and units.

1. Calculate the area of each quadrant.

   Each quadrant has an area of 50 yards • 50 yards, or 2,500 square yards.

2. Write the area density formula using the variable names for the problem.

   The area density formula is $\text{Density}_{\text{Area}} = \dfrac{\text{number of manatees}}{\text{number of square yards}}$.

3. Calculate the density in manatees per square yard for each quadrant.

Divide the number of manatees in each quadrant by the quadrant's area in square yards. Recall that each data point represents 10 manatees.

Organize the calculations in a table.

Quadrant	Manatees	Quadrant area (yd²)	Density formula	Manatee density per yd²
I	120	2,500	$\dfrac{120}{2500}$	0.048
II	190	2,500	$\dfrac{190}{2500}$	0.076
III	30	2,500	$\dfrac{30}{2500}$	0.012
IV	40	2,500	$\dfrac{40}{2500}$	0.016

4. Calculate the total area for all four quadrants of the discharge basin.

The area of the discharge basin is found by multiplying the area of each quadrant (2,500 square yards) by the number of quadrants (4).

$2500 \cdot 4 = 10,000$

The discharge basin has a total area of 10,000 square yards.

5. Calculate the total number of manatees in the discharge basin.

The number of manatees in the discharge basin is found by summing the number of manatees in each quadrant.

$120 + 190 + 30 + 40 = 380$

There are a total of 380 manatees in all four quadrants of the discharge basin.

6. Calculate the average density for all four quadrants.

The total number of manatees is 380 and the area is 10,000 square yards.

Substitute these values into the formula and calculate the area density.

$$\text{Density}_{\text{Area}} = \frac{\text{total number of manatees}}{\text{total square yards}}$$ Density formula

$$\text{Density}_{\text{Area}} = \frac{(380)}{(10,000)}$$ Substitute 380 for the total number of manatees and 10,000 for the total square yards.

$$\text{Density}_{\text{Area}} = 0.038$$ Simplify.

The average density of manatees for the whole discharge basin is approximately 0.038 manatees per square yard.

7. Compare each of the quadrant densities with the average discharge basin density and explain any variations between the individual quadrant densities and the density for the discharge basin as a whole.

Organize the information in a table.

Quadrant(s)	Manatee density
I	0.048
II	0.076
III	0.012
IV	0.016
I–IV	0.038

The large numbers of manatees in quadrants I and II near the hot-water discharge canal and the small numbers of manatees in quadrants III and IV skew the average density for the whole discharge basin.

8. Explain how to change the density formulas so that the densities can be expressed in "whole" manatees.

The area of the discharge basin could be re-scaled so that the basic area unit is small enough to yield manatee densities that are greater than 1. For example, by defining the basic unit of area as a 10-yard by 10-yard area (100 square yards), the resulting density numbers would be greater than 1. This would give a density of about 5 manatees per 100 square yards in Quadrant I, about 8 manatees in Quadrant II, about 1 manatee in Quadrant III, and about 2 manatees in Quadrant IV. The overall density would be about 4 manatees in each 100-square-yard area.

Example 4

The table shows five different volume densities of water taken from an inland salt lake, in kilograms per cubic meter. The samples were all taken at locations that are equidistant and representative of the overall dimensions of the lake. Determine the average saltwater density for the lake.

Density (kg/m³)	Number of samples	Weighted samples (density • samples) (kg/m³)
1,020	12	12,240
1,022	20	20,440
1,025	15	15,375
1,030	22	22,660
1,032	16	16,512

1. Describe how the number of samples for each density affects the average density.

The numbers of samples "weight" the densities they represent, such that a large number of samples for any one density will have a greater effect on the average density. Thus, the 22 samples with a density of 1,030 kilograms per cubic meter will have a greater effect on the average density than the smaller samples collected for the other four densities. However, a combination of any two other densities will have a "weight" greater than that of 1,030 kilograms per cubic meter.

2. Calculate the average saltwater density for the lake.

    First, determine the total number of samples by summing them.

    $$12 + 20 + 15 + 22 + 16 = 85$$

    The total number of samples is 85.

    Next, find the sum of the weighted samples. All samples are in kg/m³.

    $$12{,}240 + 20{,}440 + 15{,}375 + 22{,}660 + 16{,}512 = 87{,}227$$

    The sum of the weighted samples is 87,227 kg/m³.

    Calculate the average by dividing sum of the weighted samples by the total number of samples.

    $$\frac{87{,}227}{85} = 1026.2 \text{ kg/m}^3$$

    The average saltwater density of the lake is 1,026.2 kilograms per cubic meter, or 1,026 kilograms per cubic meter. Interpret the average saltwater density of the samples in terms of the overall saltwater density of the lake.

    Based on these five densities taken in a total of 85 representative samples, the average saltwater density of the lake is about 1,026 kilograms per cubic meter.

UNIT 4B • MATHEMATICAL MODELING AND CHOOSING A MODEL
Lesson 5: Geometric Modeling

Practice 4B.5.2: Density

Use the information given in each scenario to complete problems 1–4.

1. The proverbial "tip of the iceberg" visible above water represents a small part of the volume of the total iceberg; most of it is submerged. Calculate the percentage of an iceberg that is above the water line if the density of ice is 925 kilograms per cubic meter and the density of water is 1,000 kilograms per cubic meter.

2. A car's brakes use fluid to transmit force. In this system, known as a hydraulic system, the hydraulic fluid (brake fluid) has the same density of force, or pressure (measured in pounds per square inch or PSI), everywhere in the system. Every square inch of the system has the same PSI applied to it by the hydraulic fluid, from the master cylinder connected to the brake pedal, to each wheel's brake pistons. The master cylinder transmits the force of the brake pedal to the hydraulic fluid through a piston that moves in a cylinder when the brake pedal is pressed.

   a. What is the density of the force in the brake fluid (in PSI) when the master cylinder piston is applying 50 pounds of force to the surface area of the brake fluid in a 0.75-inch diameter cylinder? Round your answer to the nearest tenth.

   b. That same PSI is present at the wheel end of the braking system, where the brake pad is controlled by two pistons that each have a 1.2-inch diameter. What is the total force (in pounds) applied to the brake pad by the two pistons? Round your answer to the nearest tenth. (*Hint*: Pay attention to the units of each quantity and the desired units in the answer.)

3. The electric scanning "eye" of an early moon lander was in the shape of a trapezoid with its two parallel pairs of sides having dimensions of 3 centimeters and 4 centimeters. Its height was 5 centimeters. The amount of light reaching the eye was reduced by moon dust that was stirred up when the lander reached the moon's surface. Calculate the new effective scanning area of the electric eye if the output of the eye was reduced from 12 volts to 8 volts.

4. A prospector takes a bag of metal nuggets (gold alloyed with mercury) from his claim to the assayer's office to be valued. What percentage of his metal is gold if it assays out at 18 grams per cubic centimeter? (*Note*: The density of pure gold is 19,300 kilograms per cubic meter and the density of pure mercury is 13,600 kilograms per cubic meter.)

Use the information that follows to complete problems 5–10.

> Doughnuts browned in hot vegetable oil are flipped during the cooking process so that each doughnut is evenly cooked throughout. After it is flipped, a very thin line forms on the edge of the doughnut just above the level of the oil. The position of that line depends on the relationship between the density of the doughnut and the density of the oil.
>
> A pastry chef is working on a new product idea: a doughnut bar in the shape of a rectangular prism, with a height of 4 cm, a width of 6 cm, and a length of 10 cm. When one of the bar's 6×10 cm sides is laid flat in cooking oil, the bar floats. When the doughnut is flipped to cook the other side, a line appears x centimeters above the side of the doughnut that is now at the bottom. The density of the vegetable oil is 925 kilograms per cubic meter and the doughnut bar's density is 400 kilograms per cubic meter (for both cooked and uncooked dough).

5. When the doughnut bar is floating in the oil, what quantity is the mass of the oil displaced by the doughnut bar equal to?

continued

6. Use the given information to write and simplify an expression to calculate the mass of the displaced oil based on its density and the height of the line (x) on the doughnut bar. Express your answer in grams.

7. Write and simplify an expression to calculate the mass of the doughnut bar. Express your answer in grams.

8. Combine your answers to problems 5, 6, and 7 to create a single equation, and then solve it for x. Round your answer to the nearest hundredth.

9. What is the ratio of your result for x to the total height of the doughnut bar? Round your answer to the nearest hundredth.

10. What is the ratio of the density of the doughnut dough to the density of the oil? Round your answer to the nearest hundredth. How does this result relate to your answer for problem 9? Explain.

Lesson 4B.5.3: Design

Introduction

In the real world of design of products and services for business, education, government, health care, national security, and other societal institutions, financial and physical constraints regulate the designs that are created and used. Often, the *purpose* of the design dictates the boundaries of these constraints, resulting in a complex interplay among the applied and theoretical factors that go into creating a design. Mathematics can play an essential role in this process.

Key Concepts

- Mathematical concepts that identify the domains, limits, extrema (maximum and minimum values), and solutions of equations and function models are often the first used after the purpose of a design has been identified.

- Concepts in coordinate and plane geometry help designers visualize prototypes. Often, the basic principles of geometry dictate the kinds of mathematical models that are used, too.

- The cost of development of a new design and its ultimate utility value (i.e., value based on expected performance) can also play into the choice of geometric and mathematical models used in the design.

- The number-crunching and visualizing capabilities of calculators and computers can be used to prototype and/or simulate design options.

Guided Practice 4B.5.3

Example 1

Caleb has seven 3-foot fence sections and would like to use them all to create a rectangular garden. How should Caleb use the fence sections to create a garden with the most area? Use the formula for the area of a rectangle, $A = lw$.

1. Assign a variable to the number of fence sections needed to create the maximum area.

   Let x represent the number of sections for the length of the rectangle. There are a total of seven sections, so the width can be represented by $7 - x$.

2. Write a function model for the area of the garden.

   Substitute the values for the length and width into the formula $A = lw$ to calculate the area.

$A = lw$	Area formula for a rectangle
$A = (x)(7 - x)$	Substitute x for l and $7 - x$ for w.
$A = 7x - x^2$	Distribute.

   The area of the garden can be modeled by the function $A = 7x - x^2$.

3. Use a graphing calculator to generate a table of data for different values of x.

   Follow the steps appropriate to your calculator model.

   On a TI-83/84:

   Step 1: Press [Y=]. At Y$_1$, use your keypad to enter the function. Use [X, T, θ, n] for x and [x^2] for any exponents.

   Step 2: Press [GRAPH]. Press [WINDOW] to adjust the graph's axes.

   Step 3: Press [2ND][GRAPH] to display a table of values.

   (continued)

On a TI-Nspire:

Step 1: Press [home]. Arrow down to the graphing icon, the second icon from the left, and press [enter].

Step 2: Enter the function to the right of "$f1(x) =$" and press [enter].

Step 3: To adjust the x- and y-axis scales on the window, press [menu] and select 4: Window and then 1: Window Settings. Enter each setting as needed. Tab to "OK" and press [enter].

Step 4: To see a table of values, press [menu] and scroll down to 2: View and 9: Show Table. (For some models, press [menu] and select 7: Table, then 1: Split-screen Table.)

4. Use the table of values to determine the maximum value(s) of x.

   Either calculator will show that the function has maximum values at $x = 3$ and $x = 4$.

5. Use the results of step 4 to determine the maximum area of the garden.

   $A = 12$ at both $x = 3$ and $x = 4$. Multiply 12 by the squared length of the individual 3-foot fence sections ($3^2 = 9$) to find the maximum area.

   The maximum area at $x = 3$ or $x = 4$ is 12 fence sections squared, or 108 square feet of garden area ($108 = 12 \cdot 9$).

6. Explain how Caleb should use the fence sections to create a garden with the most area. Refer to the domain of x in your explanation.

   The domain of x is whole fence sections, so it can be represented by the interval [0, 7]. Parts of a fence section cannot be used, so either $x = 3$ or $x = 4$ will give the largest area possible given the constraint of the domain. Both values of x give the same dimensions for the garden. That is, either 3 fence sections by 4 fence sections or 4 fence sections by 3 fence sections gives 108 square feet.

   $108 = 9 \cdot 12 = 3^2 \cdot 12 = 3^2 \cdot 4 \cdot 3 = 3^2 \cdot 3 \cdot 4$

Example 2

The city council is comparing two tank options for a new water tower: a cylindrical tank and a spherical tank. Both designs have a diameter of 20 meters. The cylindrical tank has a height of 12 meters. Both designs would cost $300 per square meter of surface area. The average monthly water bill is $100 per cubic meter of water, and either tank would hold enough water for 1 month. The city council decides to go with whichever tank will produce a net profit in its first month, assuming the tank's full volume of water will be emptied completely (i.e., sold to customers) by the end of that month. Which tank will yield the desired profit?

1. Calculate the surface area of the spherical water tank using the formula $A(r) = 4\pi r^2$.

   The radius r of the spherical tank is 10 meters. Substitute this value into the formula to calculate the surface area.

$A(r) = 4\pi r^2$	Formula for the surface area of a sphere
$A(10) = 4\pi(10)^2$	Substitute 10 for r.
$A(10) = 400\pi$	Simplify.

   The surface area of the spherical water tank is 400π or about 1,257 square meters.

2. Use the surface area to calculate the cost of the spherical water tank.

   The cost of the spherical water tank is $300 per square meter of surface area. Therefore, the cost to build the tank is $300 \cdot 400\pi = 120,000\pi$, or about $376,991.

3. Calculate the volume of the spherical water tank using the formula $V(r) = \frac{4}{3}\pi r^3$.

The radius r of the spherical tank is 10 meters. Use this to calculate the volume.

$$V(r) = \frac{4}{3}\pi r^3 \qquad \text{Formula for the volume of a sphere}$$

$$V(10) = \frac{4}{3}\pi (10)^3 \qquad \text{Substitute 10 for } r.$$

$$V(10) = \frac{4000}{3}\pi \qquad \text{Simplify.}$$

The volume of the spherical water tank is $\frac{4000}{3}\pi$ or about 4,189 cubic meters.

4. Use the volume to calculate the expected revenue from emptying the spherical tank.

The average monthly water bill is $100 per cubic meter. The tank holds enough water for 1 month. Therefore, the revenue produced by emptying the spherical water tank is $100 \cdot \frac{4000}{3}\pi = \frac{400,000}{3}\pi$, or about $418,879.

5. Compare the cost of the spherical tank to its expected revenue to determine the net profit or loss for the first month.

The cost of constructing the spherical water tank is about $376,991. The expected revenue from the tank is about $418,879.

$$418,879 - 376,991 = 41,888$$

Therefore, the spherical water tank would produce a net profit of about $41,888 for the first month.

6. Calculate the surface area of the cylindrical water tank using the formula $A(h, r) = 2\pi r^2 + 2\pi rh$.

The height of the tank is 12 meters and the radius is 10 meters. Use these values to calculate the surface area.

$A(h, r) = 2\pi r^2 + 2\pi rh$	Formula for the surface area of a cylinder with bases
$A[(12), (10)] = 2\pi(10)^2 + 2\pi(10)(12)$	Substitute 12 for h and 10 for r.
$A(12, 10) = 440\pi$	Simplify.

The surface area of the cylindrical water tank is 440π or about 1,382 square meters.

7. Use the surface area to calculate the cost of the cylindrical water tank.

The cost of the cylindrical water tank is $300 per square meter. Therefore, the cost to build the tank is $300 \bullet 440\pi = 132,000\pi$, or about $414,690.

8. Calculate the volume of the cylindrical water tank using the formula $V(h, r) = \pi rh$.

The height of the tank is 12 meters and the radius is 10 meters. Use these values to calculate the volume.

$V(h, r) = \pi r^2 h$	Formula for the volume of a cylinder
$V[(12), (10)] = \pi(10)^2(12)$	Substitute 12 for h and 10 for r.
$V(12, 10) = 1200\pi$	Simplify.

The volume of the cylindrical water tank is 1200π or about 3,770 cubic meters.

9. Use the volume to calculate the expected revenue from emptying the cylindrical tank.

   The average water bill is $100 per cubic meter. Therefore, the revenue produced by emptying the cylindrical water tank is $100 \cdot 1200\pi = 120{,}000\pi$, or about $376,991.

10. Compare the cost of building the cylindrical water tank to its expected revenue to determine the net profit or loss for the first month.

    The cost of constructing the cylindrical water tank is about $414,690. The expected revenue is about $376,991.

    $376{,}991 - 414{,}690 = -37{,}699$

    Therefore, the cylindrical water tank would produce a net loss of about $37,699 for the first month.

11. Compare the results for the two water tanks to determine which tank the city council will likely choose.

    Based on the criteria used by the city council to compare the water tanks, the spherical water tank will be a more cost-effective tank to build because the spherical tank is expected to generate a profit in the first month (if completely emptied of water). The cylindrical tank, on the other hand, would generate a net loss.

Example 3

Autumn is an artist who specializes in decorative silk-screen panels. She recently bought several 60-centimeter and 80-centimeter wooden dowels to use as frames for the panels. Autumn decides to make several panels that are in the shape of quadrilaterals with two perpendicular diagonals made from the dowels. One diagonal will be 60 cm, and the other will be 80 cm. Autumn wants these panels to have the largest screen area possible, but she also needs to ensure the design makes the best use of her resources. What configuration of the two sizes of dowels should she use to realize this goal?

1. List all the possible configurations of the dowels in a table.

    The diagonals can be perpendicular bisectors of each other, or one of the dowels can be a perpendicular bisector of the other dowel.

Case	Description
I	The 60-cm dowel is a perpendicular bisector of the 80-cm dowel.
II	Both dowels are perpendicular bisectors of each other.
III	The 80-cm dowel is a perpendicular bisector of the 60-cm dowel.
IV	Neither dowel is a perpendicular bisector of the other.

2. Sketch the four cases from step 1 and find the areas of the resulting figures.

The sketched figures are shown as follows, with each grid box representing 10 cm. Notice that cases I–IV are each comprised of four right triangles. Use this fact to find each area by adding the areas of the triangles contained within each figure. (Recall that the formula for the area of a triangle is $A=\dfrac{1}{2}bh$.)
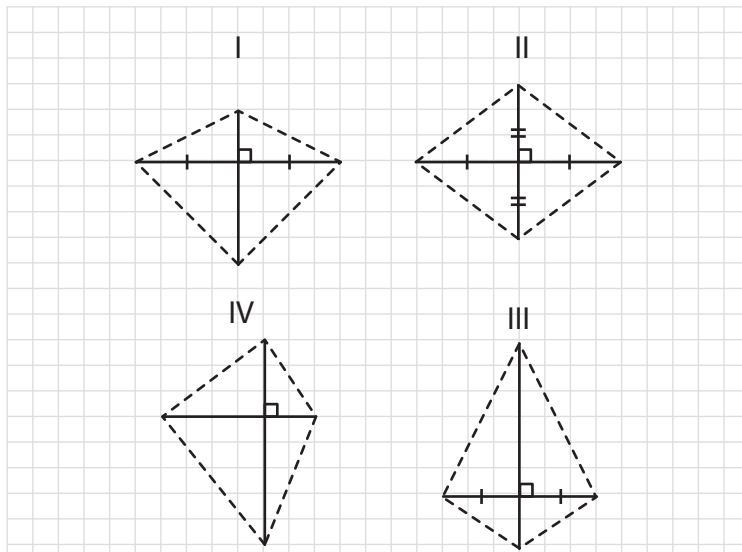
- Case I area: 2,400 cm²
- Case II area: 2,400 cm²
- Case III area: 2,400 cm²
- Case IV area: 2,400 cm²

3. Explain why the figures have equal areas.

   Each panel has diagonal lengths of 60 cm and 80 cm. Visualize sliding the dowels within a panel's frame, while still maintaining perpendicularity. Sliding the dowels changes the sizes of individual triangles within the panel, but does not change the total area enclosed within the panel.

4. Determine which configuration of dowels Autumn should use.

   Since cases I–IV each use the same number of dowels and have the same area, Autumn could use any one of these configurations for her design.

Example 4

A mobile phone manufacturer has to upgrade the circuitry in one of its products. Currently, the voltage in the signal-capture circuit rises according to the exponential function model $V(t) = 8(1 - e^{-at})$, in which a is a positive constant and the coefficient of t, the time in seconds, and V is the circuit voltage in volts. The graph shows how the voltage increases when a cell-phone tower signal is detected for $a = 1$.
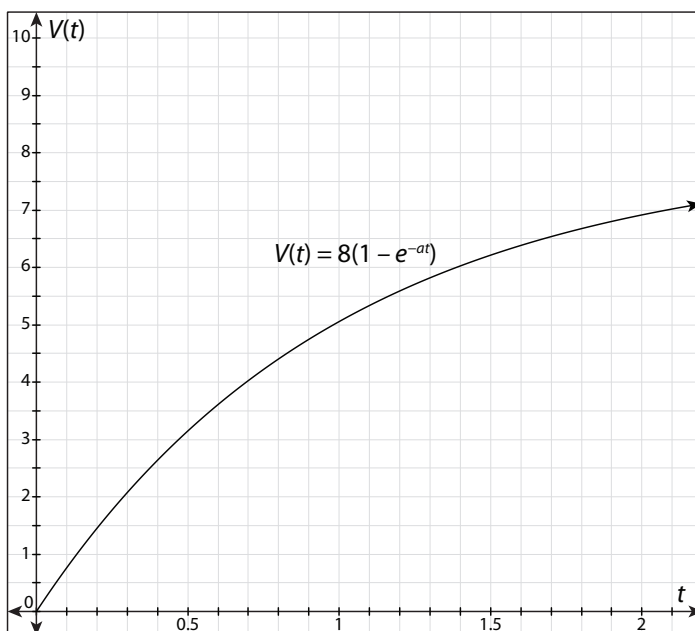
How will the function model have to be modified if the upgraded phone circuit has a voltage "rise time" that results in a circuit voltage of at least 7.9 volts in less than 1 second when a cell-phone tower signal is detected?

1. Write an inequality using the function that reflects the necessary conditions for a faster rise time.

    The voltage has to reach at least 7.9 volts in less than 1 second. Substituting $V = 7.9$ volts and $t = 1$ second in the function model gives $7.9 \geq 8(1 - e^{-a})$.

2. Solve the resulting inequality.

$7.9 \geq 8(1 - e^{-a})$	Inequality from the previous step
$7.9 \geq 8 - 8e^{-a}$	Distribute.
$-0.1 \geq -8e^{-a}$	Isolate the exponential term.
$0.0125 \geq e^{-a}$	Simplify.
$-a \leq \ln(0.0125)$	Rewrite as a natural logarithm.
$a \geq 4.4$	Use a calculator to simplify.

    A value of $a \geq 4.4$ will give a function that models a circuit that will reach a voltage of at least 7.9 volts in a rise time of 1 second.

3. Check the result of the previous step by substituting it into the original exponential function.

$V(t) = 8(1 - e^{-at})$	General exponential function model
$V(1) = 8[1 - e^{-(4.4)(1)}]$	Substitute 1 for t and 4.4 for a.
$V(1) = 8 - 8e^{-4.4}$	Distribute.
$V(1) \approx 8 - 8(0.012)$	Substitute the calculated value of $e^{-4.4}$.
$V(1) \approx 7.9$	Simplify.

    The value $a = 4.4$ meets the conditions of the problem. Therefore, by modifying the function $V(t) = 8(1 - e^{-at})$ so that $a = 4.4$, the upgraded phone circuit will achieve a voltage of at least 7.9 volts in less than 1 second.

UNIT 4B • MATHEMATICAL MODELING AND CHOOSING A MODEL
Lesson 5: Geometric Modeling

Practice 4B.5.3: Design

Use the information given in each problem to answer the questions.

1. Your friend has two chocolate bars and offers you either bar. One chocolate bar is in the shape of a triangular prism, with a triangular base that is 3 cm wide and 2.5 cm high; this bar is 3 cm long. The other chocolate bar is in the shape of a square prism, with a square base with sides of 3 cm; this bar is 1.2 cm long. Find the volume of each bar. Which bar contains more chocolate?

2. Consider a square that measures 4 cm per side. If the length of all sides increases by 1 cm, by what percentage does the perimeter increase? By what percentage does the area increase? Why does one measurement grow so much faster than the other?

3. A toddler's toy features a square with a circle inside it. The circle expands or contracts when the child moves a lever to make the circle bigger or smaller. The square has sides measuring 10 cm, and the circle has radius r. The center of the circle lies at the center of the square. The circle can expand inside the square until its sides touch all four sides of the square. Define a function $A(r)$ that calculates the area between the outside of the circle and the inside of the square as a function of the circle's radius r. Describe the graph of $A(r)$. What type of curve is it, and how does it behave in general? Specify the domain and range of $A(r)$.

4. The Great Pyramid of Giza in Egypt has a square base that measures about 230 m on each side, and is about 147 m high. The average stone used to build the pyramid is in the shape of a rectangular prism measuring about $130 \times 120 \times 70$ cm. An artist has been commissioned to make a full-size replica of the pyramid out of clear plastic blocks with LED lights for a casino in Las Vegas. Assuming that the replica pyramid will be solid (with no chambers or passageways) and the replica will have the same dimensions as the original Great Pyramid, estimate the number of plastic blocks needed to build the replica.

5. A stone slab is being lowered onto a triangular prism by a crane to create a seesaw sculpture. When the slab is in its final position, it will be parallel to the ground, perched on the 2-foot-tall stone triangular prism. However, just before the slab is lowered completely, a supervisor notices the triangular prism is out of place. One side of the slab is in its final position, 2 ft off the ground, at a horizontal distance of 10 ft from the side of a building. The other side is still up in the air, 9.2 ft off the ground and 19.6 ft horizontally from the side of the same building. At what horizontal distance from the side of the building must the peak of the balancing stone be placed so it is exactly in the middle of the stone slab, once lowered?

6. A sign manufacturer creates custom signs that are lit from within. The signs are rectangular boxes to protect the light fixtures inside. The front side of each box is plastic and can be printed with a business name or logo. The other five sides are metal. The boxes are custom built according to preset material sizes. The manufacturer offers standard boxes that are 24 inches tall and 6 inches deep, and orders them in lengths ranging from 2 ft to 40 ft, in multiples of 2 ft. The metal for the box costs $4.20 per square foot. The 24-inch-tall plastic for the front costs $1.75 per square foot, and every 2-foot-long section of sign requires its own light fixture, which costs $27. Regardless of sign length, an additional $250 is charged to cover printing costs for the plastic side. Define and simplify a function for the total cost of a sign (metal box, plastic front, lights, and printing) as a function of the sign's total length, $C(l)$. Specify the domain and range of the function.

7. A new homeowner is repainting the 8-ft-tall walls of a room that measures 12×11 ft. She is considering two different types of paint. Brand A paint hides the existing color with only a single coat of paint, costs $63 per gallon, and covers 320 square feet per gallon. Brand B paint requires two coats to hide the existing color, costs $28 per gallon, and covers 400 square feet per gallon. The painter charges $35 per hour, and paints 100 square feet per hour. Both types of paint must be purchased by the gallon, and extra paint cannot be returned. Determine the total cost of painting the room for each type of paint, and determine which brand of paint she should buy.

continued

8. The landscaping at the front entrance of Morse High School is being redesigned. The principal would like to install some benches in the shape of concrete cylinders (circular prisms) lying on their sides. Each bench would be 2 m long with a diameter of 1 m. In talking over the design with a contractor, the principal learned that each bench would weigh so much that a special (and expensive) crane would have to be rented to move each bench into position. To reduce the weight of the benches and avoid renting the crane, a concrete tube design was suggested. This design keeps the same exterior dimensions but has a hollow center. Concrete weighs 2,400 kilograms per cubic meter. What must be the radius of the hollow tube inside each bench to result in a bench weight of exactly 500 kg? How thick will the remaining concrete be in centimeters? Round your answers to the nearest tenth.

9. Aimee is reframing her grandmother's favorite picture as a gift. She would like to add a band of gold leaf around the edge of the frame. Gold leaf costs about $32 per square foot, and the new frame will measure 32×20 inches. How wide a band of gold leaf could she purchase for $25? Let w represent the width of the gold leaf band in inches.

10. Clive is planning to build a rectangular stone patio measuring 8×12 ft using either large or small stones. The patio will need to be built on top of a bed of sand to support the stones. Large stones average 1.2 square feet and cost $5.40 each, and require a sand bed that is 2 inches deep. Small stones average 0.5 square feet, cost $2.35 each, and need a sand bed that measures 4 inches deep. Sand costs $120 per cubic yard, delivered and installed. How much does it cost to install each type of stone? Which size stone should Clive choose?

Answer Key

Answer Key

Unit 3: Trigonometry of General Triangles and Trigonometric Functions

Lesson 1: Radians and the Unit Circle

Practice 3.1.1: Radians, pp. U3-11–U3-12

1. 4.5 radians
3. 0.6923 radian
5. 128.6°
7. 42.3°
9. Ali's spin speed: $\dfrac{31\pi}{60}$ radians per second; 1.6232 radians per second

Practice 3.1.2: The Unit Circle, p. U3-28

1.

3.
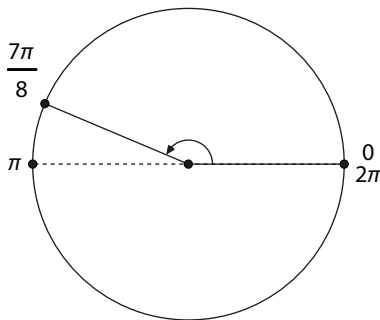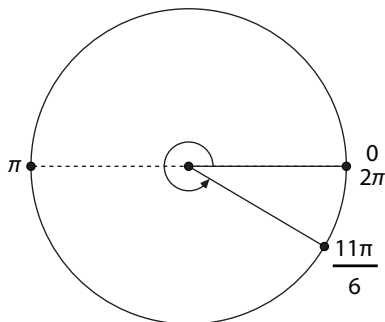
5. $\dfrac{\pi}{4}$ radian
7. 0.271 radian
9. $(-0.5, -0.87)$

Practice 3.1.3: Special Angles in the Unit Circle, p. U3-42

1. $\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$
3. $(-1, 0)$
5. $\left(\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$

7. $(0, -1)$

9. $\left(\dfrac{\sqrt{3}}{2}, -\dfrac{1}{2} \right)$

Practice 3.1.4: Evaluating Trigonometric Functions, p. U3-57

1. $\dfrac{3}{5}$

3. $\dfrac{0}{1}$ or undefined

5. $\dfrac{\sqrt{3}}{2}$

7. $\dfrac{1}{-\dfrac{1}{2}} = -2$

9. $\dfrac{4}{3}$

Lesson 2: Trigonometry of General Angles

Practice 3.2.1: Proving the Law of Sines, pp. U3-73–U3-75

1. 5.38 ft
3. 40.27° and 139.73°
5. 0.83 radian and 2.32 radians
7. 18.76 cm²
9. 30.2 mi

Practice 3.2.2: Proving the Law of Cosines, pp. U3-86–U3-88

1. 106.6°
3. 1.796 radians
5. $UW = 9$ mm, $m\angle U = 26°$, $m\angle W = 80°$
7. $HF = 5$ cm, $m\angle G = 30°$, $m\angle H = 38°$
9. 18.7 km

Practice 3.2.3: Applying the Laws of Sines and Cosines, pp. U3-98–U3-99

1. 109.1 m
3. 131.7 ft
5. 480.5 mi
7. 179.2 m
9. 76.8°

Lesson 3: Graphs of Trigonometric Functions

Practice 3.3.1: Periodic Phenomena and Amplitude, Frequency, and Midline, p. U3-113

1. period: 2π; frequency: $\dfrac{1}{2\pi}$; midline: $y = 1$; amplitude: 1

3. period: $\dfrac{2\pi}{3}$; frequency: $\dfrac{3}{2\pi}$; midline: $y = 5$; amplitude: $\dfrac{1}{2}$

5. period: $\dfrac{\pi}{2}$; frequency: $\dfrac{2}{\pi}$; midline: $y = 0$; amplitude: 5

7. period: 7π; frequency: $\dfrac{1}{7\pi}$; midline: $y = 5$; amplitude: 1

9. $\dfrac{3}{2}$ or 1.5

Practice 3.3.2: Using Trigonometric Functions to Model Periodic Phenomena, pp. U3-124–U3-125

1. $f(x) = 2\sin\left[2\left(x + \dfrac{\pi}{5}\right)\right] + 1$

3. $f(x) = 3\cos\left[2\left(x - \dfrac{\pi}{2}\right)\right] - 1$

5. $f(x) = 2\cos\left(x + \dfrac{\pi}{4}\right)$

7. $f(x) = \sin\left[4\left(x + \dfrac{\pi}{4}\right)\right] - 1$ or $f(x) = \cos\left[4\left(x + \dfrac{\pi}{8}\right)\right] - 1$ or $f(x) = \cos\left[4\left(x - \dfrac{3\pi}{8}\right)\right] - 1$

9. $f(x) = 3\sin(2960\pi x)$

Unit 4A: Mathematical Modeling of Inverse, Logarithmic, and Trigonometric Functions

Lesson 1: Inverses of Functions

Practice 4A.1.1: Determining Inverses of Quadratic Functions, pp. U4A-24–U4A-25

1. $f^{-1}(x) = \pm \dfrac{\sqrt{5(x+3)}}{5}$; $[-3,+\infty)$

3. $h^{-1}(x) = -\dfrac{1}{2} \pm \dfrac{\sqrt{9+4x}}{2}$; $\left[-\dfrac{9}{4}, +\infty \right)$

5. $b^{-1}(x) = \dfrac{-5 \pm \sqrt{25-4x}}{-2}$; $\left(-\infty, 2.5 \right]$ or $\left[2.5, +\infty \right)$

7. $d^{-1}(x) = \pm\sqrt{x-2}$; $\left(-\infty, 0 \right]$ or $\left[0, +\infty \right)$

9. The real-world domain of this function is $[0,+\infty)$ because time is positive. However, the function values have to be negative because the life preserver is falling toward the surface of the river away from the helicopter. This also preserves the positive value of time. The inverse function does not exist in this specific situation in which time is positive, gravity is negative, and the height is negative.

Practice 4A.1.2: Determining Inverses of Other Functions, pp. U4A-43–U4A-44

1. $f(x) = \dfrac{2(x+1)}{2x-1} ; f^{-1}(x) = \dfrac{x+2}{2(x-1)}$

3. $f(x) = \sqrt[3]{x^2-3} ; f^{-1}(x) = \pm\sqrt{x^3+3}$

5. $(-\infty, 0) \cap \left(0, \dfrac{1}{2} \right) \cap \left(\dfrac{1}{2}, +\infty \right)$

7. $\left(-\infty, -\sqrt{2} \right], \left[-\sqrt{2}, 0 \right], \left[0, \sqrt{2} \right]$ and $\left[\sqrt{2}, +\infty \right)$

9. $v^{-1}(x) = \dfrac{x^2}{10} - 10$; $x > 10$

Lesson 2: Modeling Logarithmic Functions

Practice 4A.2.1: Logarithmic Functions as Inverses, pp. U4A-60–U4A-61

1. $f(x) = \dfrac{1}{2}(1 + 10^{\frac{x-6}{4}})$

3. $h(x) = 4^{9x}$

5. domain: $\left[0, \dfrac{1}{2} \right) \cap \left(\dfrac{1}{2}, +\infty \right)$; range: $(-\infty, 0) \cup (0, +\infty)$

7. domain: $(-\infty, 5)$; range: $(-\infty, +\infty)$

9. The concentration of stomach acid before taking the antacid is about 0.00178 units of mass per unit of volume. The concentration after taking the antacid is about 0.00007 units of mass per unit of volume. The stomach acid concentration was reduced by about 25 times.

Practice 4A.2.2: Common Logarithms, pp. U4A-69–U4A-70

1. $f^{-1}(x) = \dfrac{\log\left(\dfrac{x}{4}\right)}{\log 3}$

3. $h^{-1}(x) = -\dfrac{\log 5x}{\log 4}$

5. $b(x) = \dfrac{1}{7 \bullet \log_7 x}$

7. $d(x) = \dfrac{\log_2(x-2)}{\log_3 x}$

9. $C^{-1}(y) = \dfrac{\log\left(\dfrac{y}{1,100}\right)}{\log 9 - 1}$

Practice 4A.2.3: Natural Logarithms, pp. U4A-81–U4A-82

1. $f(x) = -\dfrac{3\ln 2x}{\ln 4}$

3. $f(x) = \dfrac{6 \bullet \ln 3 \bullet \ln(x+4) + \ln 5 \bullet \ln(2-x)}{\ln 3 \bullet \ln 5}$

5. $f(5) = \log 5$; $g(5) = 0.1 \bullet \log 5$; $f(5) > g(5)$

7. $f(5) = \log_2(1) = 0$; $g(5) = \ln(1) = 0$; $f(5) = g(5)$

9. $t \approx 4$ days

Practice 4A.2.4: Graphing Logarithmic Functions, pp. U4A-100–U4A-101

1. solution: $(2, -1)$
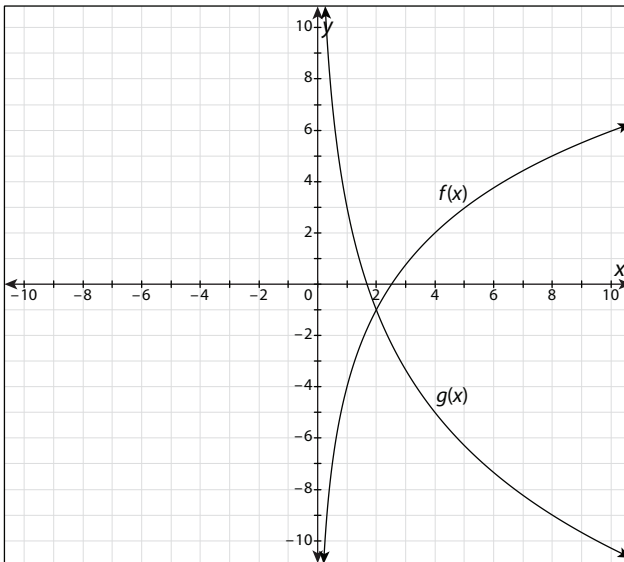
3. no solution
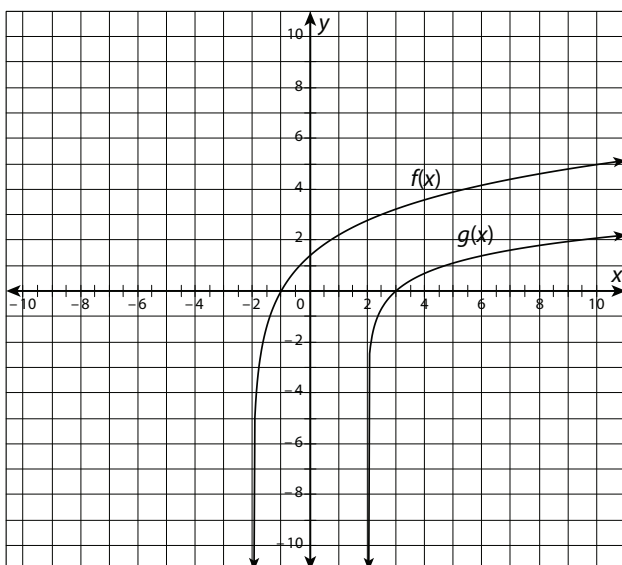
5. The domain of all three functions is $(0,+\infty)$; the range of all three functions is $(-\infty,+\infty)$.

7. The domain of $f(x)$ is $(3,+\infty)$; the domain of $g(x)$ is $(1,+\infty)$; the domain of $h(x)$ is $(-3,+\infty)$. The range of all three functions is $(-\infty,+\infty)$. The common restricted domain for all three functions is $(3,+\infty)$.

9. At $n = 1000$ cases produced, $D \approx 3$ defective cases; at $n = 2000$ cases produced, $D \approx 11$ defective cases. The increase in defects from $n = 1000$ to $n = 2000$ is about 367% or $3\frac{2}{3}$. At $n = 3000$ cases produced, the number of defective cases would about 16. The increase in defects from $n = 2000$ to $n = 3000$ is about 145% or $1\frac{5}{11}$. Other such calculations would indicate that the rate of defective cases decreases as the size of the production run increases.
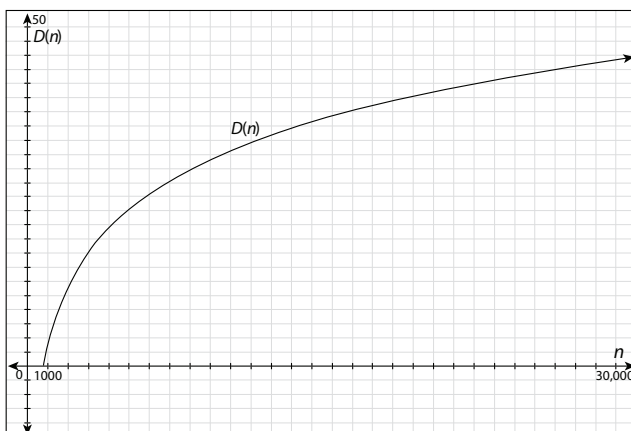
Practice 4A.2.5: Interpreting Logarithmic Models, pp. U4A-110–U4A-112

1. $A(1) = 475 - 85 \bullet \ln 1 = 475$; this is the number of acres of salt marsh after one year of study.

3. Half of 475 is 237.5, so the function becomes $237.5 = 475 - 85 \bullet \ln t$, which simplifies to $\ln t \approx 2.8$ or $t \approx 16.3$ months.

5. $pH_{before} = 6.15 = -\log c_{before}$, which gives $c_{before} = 10^{-6.15}$; $pH_{after} = 6.9 = -\log c_{after}$, which gives $c_{after} = 10^{-6.9}$; the change in concentration can be defined as $c_{before} - c_{after} = 10^{-6.15} - 10^{-6.9}$ or about $5.82 \bullet 10^{-7}$, which means that the acidic substances have been reduced by $5.82 \bullet 10^{-7}$ units of mass.

7. The horizontal line represents the two volumes at which the same amount of work is done at the two different temperatures.

9. $v_B = e^{0.9} = e^{0.6} \bullet e^{0.3} = v_A \bullet e^{0.3}$

Lesson 3: Modeling Trigonometric Functions

Practice 4A.3.1: Graphing the Sine Function, pp. U4A-143–U4A-145

1. $g(x)$

3. The functions appear to be out of phase by 2 radians.

5. The first minimum for $x > 0$ occurs at $x = \dfrac{3\pi}{8}$; the point is $\left(\dfrac{3\pi}{8}, -3 \right)$.

7. The first maximum for $x > 0$ occurs at $x = 120°$; the point is (120°, 1).

9. The period of the current is $\dfrac{1}{60}$ second, so 60 cycles of the current occur per second, which is 60 maximum values and 60 minimum values, or 120 maximum *or* minimum values.

Practice 4A.3.2: Graphing the Cosine Function, pp. U4A-171–U4A-173

1. $g(x)$

3. The functions appear to be out of phase by 0.1.

5. The first minimum for $x > 0$ occurs at $x = 2\pi$; the point is $(2\pi, -2)$.

7. The first maximum for $x > 0$ occurs at $x = 60°$; the point is (60°, 1).

9. a. $v_0 \bullet \cos A_0 = 120$ feet per second

   b. $0 < \cos A < 1$

   c. $D(5) = 5 \bullet v_0 \bullet \cos 60° = 600$; $v_0 = \dfrac{120}{\cos 45°} = 120\sqrt{2}$ feet per second

Unit 4B: Mathematical Modeling and Choosing a Model

Lesson 1: Creating Equations

Practice 4B.1.1: Creating Equations in One Variable, pp. U4B-23–U4B-24

1. $y = 5x - 2$

3. $y = x^2 + 2x - 3$

5. There is no linear equation that fits the three given data points.

7. $y = -\dfrac{1}{4}x^2 - \dfrac{5}{4}x + 1$

9. $d = 20t + 10$

Practice 4B.1.2: Representing and Interpreting Constraints, p. U4B-34

1. $x \le \dfrac{7}{4}$ or $\left(-\infty, \dfrac{7}{4}\right]$

3. $x \ge \dfrac{5}{2}$ or $\left[\dfrac{5}{2}, \infty\right)$

5. $\{x \le 0\}, \left\{y \le -\dfrac{1}{2}\right\}$

7. $\{\varnothing\}$

9. The constraint on the capacity of the trough is given by the interval [0, 900]; the constraint on the rate at which water is removed from the trough is a constant, $r = -60$ gallons per day; however, the automatic refilling of the trough, 30 gallons every two days, is added to the daily removal rate: $r = -60$ gallons per day + 15 gallons per day = -45 gallons per day; the equation $900 - 60d + 15d = 0$ gives the number of days it will take to empty the trough, or $d = 20$ days; if the trough is refilled by the rancher before 20 days is up, it will not go empty.

Practice 4B.1.3: Rearranging Formulas, pp. U4B-51–U4B-52

1. $k = \dfrac{P}{2L^2}$

3. $[A^-] = \dfrac{[HA] \bullet 10^{pH}}{10^{pK_a}}$ or $[A^-] = [HA] \bullet 10^{pH - pK_a}$

5. The escape velocity $v_{escape} = \sqrt{\dfrac{2GM_E}{r_{actual} + 7500}}$ is less than the escape velocity $v_{escape} = \sqrt{\dfrac{2GM_E}{r_{actual} - 7500}}$ because the denominators differ based on the radius inequality.

7. $\mu = \dfrac{\log 2}{t_{gen} \bullet \log e}$

9. [A] $\bullet$ [B] < [AB]; the product of the concentrations of the elements A and B is less than the concentration of the compound AB.

Lesson 2: Transforming a Model and Combining Functions

Practice 4B.2.1: Transformations of Parent Graphs, pp. U4B-69–U4B-71

1. $f(x) = 4(x - 1)$; $g(x) = 4(x + 1)$

3. $f(x) = 2(4^x)$; $g(x) = -3(4^x)$

5. domain: $(0, \infty)$; range: $(-\infty, \infty)$

7. If the value of a is doubled and b is increased by 2, then $A(a, b) = \pi(2a - b - 2)^2$.

9. For a pH range of (0, 14), the domain of c is (10^{-14}); for a pH range of (7, 10), the domain of c is $(10^{-10}, 10^{-7})$.

Practice 4B.2.2: Recognizing Odd and Even Functions, pp. U4B-77–U4B-78

1. For an even function: Remove x^3 and x to give $a(x) = x^2 + 1$.
   For an odd function: Remove x^2 and 1 to give $a(x) = x^3 + x$.

3. For an even function: $c(x) = \dfrac{x^2}{1+x^2}$.

   For an odd function: $c(x) = \dfrac{x}{1+x^2}$.

5. $f(x) = 0.5 \bullet \cos 0.2x$; even

7. $h(x) = \dfrac{x}{x-1}$; neither

9. $I_{average} = \dfrac{1}{2}(I_1 + I_2)$

Practice 4B.2.3: Combining Functions, pp. U4B-83–U4B-85

1. composition

3. combination

5. $\dfrac{(f \circ g)(x)}{(g \circ f)(x)} = \dfrac{2^{\log_2 x}}{\log_2 2^x} = 1$; domain: $(0, \infty)$

   $\dfrac{(g \circ f)(x)}{(f \circ g)(x)} = \dfrac{\log_2 2^x}{2^{\log_2 x}} = 1$; domain: $(0, \infty)$

7. $\left(f^2 \circ g\right)(x) = 2^{2 \bullet \log_2 x} = x^2$; domain: $(0, \infty)$

   $\left(g \circ f^2\right)(x) = \log_2\left(2^{2x}\right) = 2x$; domain: $(-\infty, \infty)$

9. $V(t) = \dfrac{r \bullet Q(t)}{t}$

Lesson 3: Comparing Properties Within and Between Functions

Practice 4B.3.1: Reading and Identifying Key Features of Real-World Situation Graphs, pp. U4B-110–U4B-112

1. The robot represented by $f(t)$ takes longer to complete the initial steps than the robots represented by $g(t)$ and $h(t)$; the robot represented by $h(t)$ completes the initial steps fastest, but its rate of completion of the 10 steps is lower than that of the other two robots.

3. In the real-world of manufacturing, there would be a finite time t at which all three robots would complete the 10 CPU-fabrication steps since a "step" might be assumed to be a positive whole number; mathematically, none of the three robots would ever complete the tenth step if a "step" could be any positive number; all three would get arbitrarily close to 10, depending on the value of t selected.

5. If the function $N(t)$ given in the problem is used, the y-axis scale would have to be 0 to at least 15. The graph represents the exponential function given in the problem, so values of $N(t)$ would increase quickly for initial values of time but would decline over time.

7. The domain for this function model is a restricted domain of 13 years; the domain of the mathematical sine function is unrestricted. The range of $H(t)$ is defined by the constant 3.2 and the amplitude of 2.2, which gives a range of [1, 5.4].

9. approximately [0, 1.9]

Practice 4B.3.2: Calculating Average Rates of Change, pp. U4B-127–U4B-128

1. 0

3. 0.841

5. For [–2, –1], $r = -\dfrac{1}{6}$; for [–1, 0], $r = -\dfrac{1}{2}$.

7. For [–2, –1], $r = \log 2$; for [–1, 0], $r = \log\left(\dfrac{3}{2}\right)$.

9. The amount of fertilizer needed to produce a yield of mostly stems increases at the same rate for both varieties after the leaves are formed, because the lines are parallel, indicating the same rate of nitrogen usage.

Practice 4B.3.3: Comparing Functions, pp. U4B-140–U4B-142

1. There are four maximum function values.

3. Either a quadratic or sine function could be used; in either case, the domain would differ for three or four different functions. If a sine function is used, the function value would be an absolute value of the sine function value.

5. –28.16 feet per second

7. The average change of Josh's function, $J(t) = -16t^2 + 4t$, can be used to find the speed over any interval. For example, the speed between 1 and 2 seconds is $s = \dfrac{J(2) - J(1)}{(2) - (1)} = -44$ feet per second.

9. Two intervals are possible: $\left[x, \dfrac{x + \sqrt{x^2 - 40x}}{2}\right]$ and $\left[x, \dfrac{x - \sqrt{x^2 - 40x}}{2}\right]$.

Lesson 4: Choosing a Model

Practice 4B.4.1: Linear, Exponential, and Quadratic Functions, pp. U4B-159–U4B-160

1. average rates of change in degrees per second (°/s): –1°/s; –0.5°/s; –0.25°/s; –0.125°/s; –0.0625°/s

3. average rates of change in meters per second per second (m/s/s): 2.5 m/s/s; –1.5 m/s/s; –1 m/s/s; –0.5 m/s/s; 0 m/s/s; an exponential model could be used for the speed changes.

5. linear; $\left(0, 33.\overline{3}\right)$

7. exponential; asymptote: $y = 0$

9. linear; $\dfrac{10}{7}$ miles per hour per day

Practice 4B.4.2: Piecewise, Step, and Absolute Value Functions, pp. U4B-173–U4B-175

1. The total piecewise relation does not associate each domain value with a unique function value; e.g., the functions represented by curves using the series of points A, B, and C and C, E, and F do not show this one-to-one correspondence.

3. (80, 250)

5. The data represents a piecewise function because for each interval, [–3, –1], [–1, 1], and [1, 3], the slopes of the linear pieces are different.

7. $a = 1$ and $b = 2$ for all three restricted domains

9. $S_1 S_2 S_3 S_4$ could form a linear function over the domain [0, 18] because of its constant slope; $S_4 S_5 S_6$ could form a quadratic function over the domain [18, 27] because of its increasing rate of change over that interval.

Practice 4B.4.3: Square Root and Cube Root Functions, pp. U4B-188–U4B-191

1. $A(x) = (x - 3)\sqrt{6x - 9}$

3. $L = \dfrac{1}{2\pi}\sqrt{gT}$

5. $g = \dfrac{1}{5} \cdot \sqrt[3]{\dfrac{1}{P}}$

7. Amari's football lands in about 3.4 seconds; Carson's football lands in about 2.8 seconds.

9. $T_{\text{larger sun}} = \dfrac{1}{\sqrt{2}} \cdot T_{\text{smaller sun}} \approx 0.7 \cdot T_{\text{smaller sun}}$

Lesson 5: Geometric Modeling

Practice 4B.5.1: Two-Dimensional Cross Sections of Three-Dimensional Objects, pp. U4B-205–U4B-206

1. 12 in^2

3. $\dfrac{(3\pi - 1)^2}{\pi} \text{ yd}^2$

5. The rotation forms a sphere.

7. The rotation forms a portion of a cylinder reduced by the volume of the cone created by the triangle.

9. $2 \cdot \sqrt[3]{\dfrac{75}{4}} \approx 5.3 \text{ microns}$

Practice 4B.5.2: Density, pp. U4B-218–U4B-220

1. 0.098 or 9.8% of the iceberg is above the water line.

3. The dust-free surface area of the electric eye was reduced to 11.67 square centimeters.

5. $\text{mass}_{\text{displaced oil}} = \text{mass}_{\text{doughnut}}$

7. $\text{mass}_{\text{doughnut}} = \dfrac{400 \text{ kg}}{\text{m}^3} \cdot \dfrac{\text{m}^3}{100^3 \text{ cm}^3} \cdot \dfrac{1000 \text{ g}}{\text{kg}} \cdot 4 \text{ cm} \cdot 6 \text{ cm} \cdot 10 \text{ cm} = 96 \text{ grams}$

9. $\dfrac{1.73}{4} \approx 0.43$

Practice 4B.5.3: Design, pp. U4B-232–U4B-234

1. triangular chocolate: 11.25 cm³; square chocolate: 10.8 cm³. The triangular bar contains more chocolate.

3. $A(r) = 100 - \pi r^2$; the graph is a parabola that opens down with its vertex at (0, 100) and roots at $r \approx \pm 5.64$; domain: (0, 5); range: [21.46, 100)

5. 16 ft

7. Brand A paint: $254.80; Brand B paint: $313.60; the homeowner should buy Brand A.

9. 1.13 in

Glossary

$f(x)$

$\angle ABC$

Q_1

Glossary

English		Español
	#	
68–95–99.7 rule a rule that states percentages of data under the normal curve are as follows: $\mu \pm 1\sigma \approx 68\%$, $\mu \pm 2\sigma \approx 95\%$, and $\mu \pm 3\sigma \approx 99.7\%$; also known as the *Empirical Rule*	U1-1	**regla 68–95–99,7** regla que establece los siguientes porcentajes de datos bajo la curva normal: $\mu \pm 1\sigma \approx 68\%$, $\mu \pm 2\sigma \approx 95\%$ y $\mu \pm 3\sigma \approx 99,7\%$; también se la conoce como *Regla Empírica*
	A	
absolute value equation an equation of the form $y = |ax + b| + c$, where x is the independent variable, y is the dependent variable, and a, b, and c are real numbers	U2A-95	**ecuación de valor absoluto** ecuación de la forma $y = |ax + b| + c$, donde x es la variable independiente, y es la variable dependiente y a, b y c son números reales
absolute value function a function of the form $f(x) = |ax + b| + c$, where x is the independent variable and a, b, and c are real numbers	U4B-144	**función de valor absoluto** función de la forma $f(x) = |ax + b| + c$, donde x es la variable independiente a, b y c son números reales
addition rule for mutually exclusive events If events A and B are mutually exclusive, then the probability that A or B will occur is the sum of the probability of each event; $P(A \text{ or } B) = P(A) + P(B)$.	U1-147	**regla de adición para eventos mutuamente excluyentes** si los eventos A y B son mutuamente excluyentes, la probabilidad de que A o B suceda es la suma de la probabilidad de cada evento; $P(A \text{ or } B) = P(A) + P(B)$.
alternative hypothesis any hypothesis that differs from the null hypothesis; that is, a statement that indicates there is a difference in the data from two treatments; represented by H_a	U1-197	**hipótesis alternativa** toda hipótesis que difiera de la hipótesis nula; es decir, una afirmación que indica que existe una diferencia en los datos de dos tratamientos; representada por H_a

English		Español
altitude the perpendicular line from a vertex of a figure to its opposite side; height	U3-58	**altitud** línea perpendicular desde el vértice de una figura hasta su lado opuesto; altura
ambiguous case a situation wherein the Law of Sines produces two possible answers. This only occurs when the lengths of two sides and the measure of the non-included angle are given (SSA).	U3-58	**caso ambiguo** situación en la cual la Ley de Senos produce dos respuestas posibles. Esto solo ocurre cuando están dadas las longitudes de los dos lados y la medida del ángulo no incluido (SSA).
amplitude the coefficient a of the sine or cosine term in a function of the form $f(x) = a \sin bx$ or $g(x) = a \cos bx$; on a graph of the cosine or sine function, the vertical distance from the y-coordinate of the maximum point on the graph to the midline of the cosine or sine curve	U3-100 U4A-113	**amplitud** el coeficiente a del término de seno o coseno en una función de la forma $f(x) = a \sin bx$ o $g(x) = a \cos bx$; en un gráfico de la función seno o coseno, la distancia vertical desde la coordenada y del punto máximo en la gráfica hasta la línea media de la curva de seno o coseno
arccosine the inverse of the cosine function, written $\cos^{-1}\theta$ or $\arccos\theta$	U3-58	**arcocoseno** inversa de la función coseno; se expresa $\cos^{-1}\theta$ o $\arccos\theta$
arc length the distance between the endpoints of an arc; written as $d(\overset{\frown}{ABC})$ or $m\overset{\frown}{AC}$	U3-1	**longitud de arco** distancia entre los puntos extremos de un arco; se expresa como $d(\overset{\frown}{ABC})$ o $m\overset{\frown}{AC}$
arcsine the inverse of the sine function, written $\sin^{-1}\theta$ or $\arcsin\theta$	U3-58	**arcoseno** inversa de la función seno; se expresa $\text{sen}^{-1}\theta$ o $\arcsin\theta$
argument the result of raising the base of a logarithm to the power of the logarithm, so that b is the argument of the logarithm $\log_a b = c$; the term $[b(x-c)]$ in a cosine or sine function of the form $f(x) = a \sin [b(x-c)] + d$ or $g(x) = a \cos [b(x-c)] + d$	U4A-46 U4A-113	**argumento** el resultado de elevar la base de un logaritmo a la potencia del logaritmo, de manera que b es el argumento del logaritmo $\log_a b = c$; el término $[b(x-c)]$ en una función coseno o seno de la forma $f(x) = a \, \text{sen} \, [b(x-c)] + d$ o $g(x) = a \cos [b(x-c)] + d$

English		Español
asymptote an equation that represents sets of points that are not allowed by the conditions in a parent function or model; a line that a function gets closer and closer to as one of the variables increases or decreases without bound	U4B-87	**asíntota** ecuación que representa conjuntos de puntos que no están permitidos por las condiciones en una función madre o modelo; una línea que una función se acerca cada vez más cerca de una de las variables aumenta o disminuye sin límite
average rate of change the ratio of the difference of output values to the difference of the corresponding input values: $\dfrac{f(b)-f(a)}{b-a}$; a measure of how a quantity changes over some interval	U4B-87	**tasa de cambio promedio** proporción de la diferencia de valores de salida a la diferencia de valores correspondientes de entrada: $\dfrac{f(b)-f(a)}{b-a}$; medida de cuánto cambia una cantidad en cierto intervalo
axis of rotation a line about which a plane figure can be rotated in three-dimensional space to create a solid figure, such as a diameter or a symmetry line	U4B-192	**eje de rotación** línea alrededor de la cual puede girar la figura de un plano en un espacio de tres dimensiones para crear una figura sólida, como un diámetro o una línea de simetría

B

English		Español
base the quantity that is being raised to an exponent in an exponential expression; in a^x, a is the base; or, the quantity that is raised to an exponent which is the value of the logarithm, such as 2 in the equation $\log_2 g(x) = 3 - x$	U4A-46	**base** cantidad que es elevada a un exponente en una expresión exponencial; en a^x, a es la base; o, la cantidad que se eleva a un exponente que es el valor del logaritmo, tal que 2 en la ecuación $\log_2 g(x) = 3 - x$
bias leaning toward one result over another; having a lack of neutrality	U1-63 U1-122 U1-197	**sesgo** inclinación por un resultado sobre otro; carecer de neutralidad

English		Español
biased sample a sample in which some members of the population have a better chance of inclusion in the sample than others	U1-63	**muestra sesgada** muestra en la cual algunos miembros de la población tienen una mayor posibilidad de ser incluidos en la muestra que otros
binomial experiment an experiment in which there are a fixed number of trials, each trial is independent of the others, there are only two possible outcomes (success or failure), and the probability of each outcome is constant from trial to trial	U1-147	**experimento binomial** experimento en el que existe un número fijo de pruebas, cada prueba es independiente de las demás, existen dos resultados posibles (éxito o fracaso) y la probabilidad de cada resultado es constante de prueba a prueba
binomial probability distribution formula the distribution of the probability, P, of exactly x successes out of n trials, if the probability of success is p and the probability of failure is q; given by the formula $P = \begin{pmatrix} n \\ x \end{pmatrix} p^x q^{n-x}$	U1-147	**fórmula de distribución binomial de probabilidad** la distribución de la probabilidad, P, de exactamente x éxitos entre n pruebas, si la probabilidad de éxito es p y la probabilidad de fracaso es q; dada por la fórmula $P = \begin{pmatrix} n \\ x \end{pmatrix} p^x q^{n-x}$

English		Español
Binomial Theorem a theorem stating that a binomial $(a + b)^n$ can be expanded using the formula $$\sum_{k=0}^{n}\frac{n!}{(n-k)!k!}\bullet a^{n-k}b^{k}=1a^{n}b^{0}+\frac{n}{1}\bullet$$ $$a^{n-1}b^{1}+\frac{n(n-1)}{1\bullet2}\bullet a^{n-2}b^{2}+\cdot$$ $$\frac{n(n-1)(n-2)}{1\bullet2\bullet3}\bullet a^{n-3}b^{3}+\cdots+1a^{0}b^{n}$$	U2A-20	**Teorema del Binomio** teorema que expresa que un binomio $(a + b)^n$ puede ampliarse utilizando la fórmula $$\sum_{k=0}^{n}\frac{n!}{(n-k)!k!}\bullet a^{n-k}b^{k}=1a^{n}b^{0}+\frac{n}{1}\bullet$$ $$a^{n-1}b^{1}+\frac{n(n-1)}{1\bullet2}\bullet a^{n-2}b^{2}+\cdot$$ $$\frac{n(n-1)(n-2)}{1\bullet2\bullet3}\bullet a^{n-3}b^{3}+\cdots+1a^{0}b^{n}$$
boundary condition a constraint or limit on a function or domain value based on real-world conditions or restraints in the problem or its solution	U4B-87	**condición de contorno** restricción o límite en un valor de dominio o función basado en condiciones del mundo real o restricciones en el problema o su solución

	C	
central angle an angle with its vertex at the center of a circle	U3-1	**ángulo central** ángulo con su vértice en el centro de un círculo
chance variation a measure showing how precisely a sample reflects the population, with smaller sampling errors resulting from large samples and/or when the data clusters closely around the mean; also called *sampling error*	U1-63	**variación aleatoria** medida que muestra cómo una muestra refleja con precisión la población, con errores de muestreo más pequeños que resultan de muestras grandes y/o cuando los datos se agrupan estrechamente alrededor de la media; también llamada *error de muestreo*
closure a system is closed, or shows closure, under an operation if the result of the operation is within the system	U2A-1	**cierre** un sistema es cerrado, o tiene cierre, en una operación si el resultado de la misma está dentro del sistema

English		Español
cluster sample a sample in which naturally occurring groups of population members are chosen for a sample	U1-64	**muestreo en grupos** muestra en la cual se eligen para una muestra grupos naturalmente ya formados de miembros de la población
coefficient the number multiplied by a variable in an algebraic expression	U2A-1	**coeficiente** número multiplicado por una variable en una expresión algebraica
combination a subset of a group of objects taken from a larger group of objects; the order of the objects does not matter, and objects may be repeated. A combination of size r from a group of n objects can be represented using the notation $_nC_r$, where $_nC_r = \dfrac{n!}{(n-r)!\,r!}$.	U1-64	**combinación** subconjunto de un grupo de objetos tomado de un grupo más grande de objetos; el orden de los objetos no importa y los objetos pueden repetirse. Se puede representar una combinación de tamaño r de un grupo de n objetos con la notación $_nC_r$, donde $_nC_r = \dfrac{n!}{(n-r)!\,r!}$.
combination of functions the process of combining two or more functions using the operations of addition, subtraction, multiplication, or division to create a new function	U4B-53	**combinación de funciones** proceso de combinar dos o más funciones utilizando las operaciones de adición, sustracción, multiplicación o división para crear una nueva función
common denominator a quantity that is a shared multiple of the denominators of two or more fractions	U2B-2 U2B-47	**denominador común** cantidad que es un múltiplo compartido de los denominadores de dos o más fracciones
common logarithm a base-10 logarithm which is usually written without the number 10, such as $\log x = \log_{10} x$	U4A-46	**logaritmo común** logaritmo de base 10 que se escribe normalmente sin el número 10, como $\log x = \log_{10} x$

English		Español
common ratio the ratio of a term in a geometric sequence to the previous term in that sequence; indicated by the variable r and given by the formula $r = \dfrac{a_n}{a_{n-1}}$	U2A-116	**relación común** la relación de un término en una secuencia geométrica con el término anterior en esa secuencia; indicada por la variable r y dada por la fórmula $r = \dfrac{a_n}{a_{n-1}}$
complex conjugate the complex number that when multiplied by another complex number produces a value that is wholly real; the complex conjugate of $a + bi$ is $a - bi$	U2A-21 U2A-47	**conjugado de número complejo** número complejo que cuando se multiplica por otro número complejo produce un valor totalmente real; el conjugado complejo de $a + bi$ es $a - bi$
Complex Conjugate Theorem Let $p(x)$ be a polynomial with real coefficients. If $a + bi$ is a root of the equation $p(x) = 0$, where a and b are real and $b \neq 0$, then $a - bi$ is also a root of the equation.	U2A-48	**Teorema de Complejo Conjugado** suponiendo que $p(x)$ es un polinomio con coeficientes reales, si $a + bi$ es una raíz de la ecuación $p(x) = 0$, donde a y b son reales y $b \neq 0$, entonces $a - bi$ es también una raíz de la ecuación.
complex number a number of the form $a + bi$, where a and b are real numbers and i is the imaginary unit	U2A-21	**número complejo** número en la forma $a + bi$, donde a y b son números reales e i es la unidad imaginaria
composition of functions the process of substituting one function for the independent variable of another function to create a new function	U4B-54	**composición de funciones** proceso de sustituir una función por la variable independiente de otra función para crear una función nueva

English		Español

conditional probability of *B* given *A* the probability that event *B* occurs, given that event *A* has already occurred. If *A* and *B* are two events from a sample space with $P(A) \neq 0$, then the conditional probability of *B* given *A*, denoted $P(B|A)$, has two equivalent expressions:

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)} =$$

$$\frac{\text{number of outcomes of } (A \text{ and } B)}{\text{all outcomes in } A}$$

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probabilidad condicional de *B* dado *A* probabilidad de que el evento *B* se produzca, dado que el evento *A* ya se ha producido. Si *A* y *B* son dos eventos de un espacio de muestra con $P(A) \neq 0$, entonces la probabilidad condicional de *B* dado *A*, indicado $P(B|A)$, tiene dos expresiones equivalentes:

$$P(B|A) = \frac{P(A \text{ y } B)}{P(A)} =$$

$$\frac{\text{cantidad de resultados de } (A \text{ y } B)}{\text{todos los resultados en } A}$$

confidence interval an interval of numbers within which it can be claimed that repeated samples will result in the calculated parameter; generally calculated using the estimate plus or minus the margin of error

U1-147

intervalo de confianza intervalo de números dentro del cual se puede afirmar que las muestras repetidas tendrán como resultado el parámetro calculado; generalmente se calcula usando la estimación más o menos el margen de error

confidence level the probability that a parameter's value can be found in a specified interval; also called *level of confidence*

U1-147
U1-197

nivel de confianza probabilidad de que se pueda encontrar el valor de un parámetro en un intervalo específico; también llamado *grado de confianza*

confounding variable an ignored or unknown variable that influences the result of an experiment, survey, or study

U1-122
U1-197

variable de confusión una variable ignorada o desconocida que influye sobre el resultado de un experimento, encuesta o estudio

congruent having the same shape, size, lines, and angles; the symbol for representing congruency between figures is $\cong$

U4B-192

congruente tener la misma forma, tamaño, rectas y ángulos; el símbolo para representar la congruencia entre números es $\cong$

English		Español
constant term a term whose value does not change	U2A-1	**término constante** término cuyo valor no cambia
constraint a limit or restriction on the domain, range, and/or solutions of a mathematical or real-world problem	U4B-1	**restricción** límite en el dominio, rango y/o soluciones de un problema matemático o del mundo real
continuous data a set of values for which there is at least one value between any two given values	U1-1	**datos continuos** conjunto de valores para el que existe al menos un valor entre dos valores dados
continuous distribution the graphed set of values, a curve, in a continuous data set	U1-1	**distribución continua** conjunto de valores representado gráficamente, una curva, en un conjunto de datos continuos
continuous function a function that does not have a break in its graph across a specified domain	U4B-144	**función continua** función que no tiene una interrupción en su curva a lo largo de un dominio específico
control group the group of participants in a study who are not subjected to the treatment, action, or process being studied in the experiment, in order to form a comparison with participants who are subjected to it	U1-122	**grupo de control** grupo de participantes en un estudio que no están sujetos al tratamiento, acción o proceso que está en estudio en el experimento con el fin de establecer una comparación con participantes que sí lo están.
convenience sample a sample in which members are chosen to minimize the time, effort, or expense involved in sampling	U1-64	**muestreo de conveniencia** muestreo en el cual se eligen los miembros para minimizar el tiempo, esfuerzo o gasto involucrado en este proceso

English		Español
converge to approach a finite limit. If the sequence of the partial sums of a series approaches the value of a given number (the limit), then the entire series converges to that limit; that is, the series has a sum. An infinite series converges when the absolute value of the common ratio r is less than 1 $\left(\lvert r\rvert<1\right)$.	U2A-116	**converger** acercarse a un límite finito. Si la secuencia de las sumas parciales de una serie se acerca al valor de un número dado (el límite), entonces la serie completa converge hacia ese límite; es decir, la serie tiene una suma. Una serie infinita converge cuando el valor absoluto de la relación común r es menor que 1 $\left(\lvert r\rvert<1\right)$.
correlation a measure of the power of the association between exactly two quantifiable variables	U1-197	**correlación** una medida de la potencia de la asociación entre exactamente dos variables cuantificables
cosecant the reciprocal of sine, $\csc\theta=\dfrac{1}{\sin\theta}$; the cosecant of $\theta=$ $\csc\theta=\dfrac{\text{length of hypotenuse}}{\text{length of opposite side}}$	U3-1	**cosecante** la recíproca del seno, $\csc\theta=\dfrac{1}{\sin\theta}$; la cosecante de $\theta=$ $\csc\theta=\dfrac{\text{largo de la hipotenusa}}{\text{largo del lado opuesto}}$
cosine a trigonometric function of an acute angle in a right triangle that is the ratio of the length of the side adjacent to the length of the hypotenuse; the cosine of $\theta=$ $\cos\theta=\dfrac{\text{length of adjacent side}}{\text{length of hypotenuse}}$	U3-2 U3-59 U4A-114	**coseno** función trigonométrica de un ángulo agudo en un triángulo rectángulo que es la proporción de la longitud de lado adyacente a la longitud de la hipotenusa; el coseno de $\theta=\cos\theta=$ $\dfrac{\text{longitud del lado adyacente}}{\text{longitud de la hipotenusa}}$
cosine function a trigonometric function of the form $f(x)=a\cos[b(x-c)]+d$, in which a, b, c, and d are constants and x is a variable defined in radians over the domain $(-\infty,\infty)$	U3-100 U4A-114	**función del coseno** función trigonométrica de la forma $f(x)=a\cos[b(x-c)]+d$, donde a, b, c y x es una variable definida en radianes a lo largo del dominio $(-\infty,\infty)$

English		Español

cotangent the reciprocal

of tangent, $\cot\theta = \dfrac{1}{\tan\theta}$;

the cotangent of $\theta = \cot\theta =$

$\dfrac{\text{length of adjacent side}}{\text{length of opposite side}}$

U3-2

cotangente recíproco de la

tangente, $\cot\theta = \dfrac{1}{\tan\theta}$;

la cotangente de $\theta = \cot\theta =$

$\dfrac{\text{longitud del lado adyacente}}{\text{longitud del lado opuesto}}$

coterminal angles angles that, when drawn in standard position, share the same terminal side

U3-2

ángulos coterminales ángulos que, cuando están trazados en una posición estándar, comparten el mismo lado terminal

critical value a measure of the number of standards of error to be added to or subtracted from the mean in order to achieve the desired confidence level; also known as z_c-*value*

U1-148

valor crítico medida de la cantidad de estándares de error que se suma o se resta de la media para lograr el nivel de confianza deseado; también conocido como *valor z_c*

cross section the plane figure formed by the intersection of a plane with a solid figure, where the plane is at a right angle to the surface of the solid figure

U4B-193

sección transversal figura del plano formada por la intersección de un plano con una figura sólida, donde el plano está en un ángulo recto a la superficie de la figura sólida

cube root function a function that contains the cube root of a variable. The general form is $f(x) = a\sqrt[3]{(x-h)} - k$, where a, h, and k are real numbers.

U4B-144

función raíz cúbica función que contiene la raíz cúbica de una variable. La forma general es $y = a\sqrt[3]{(x-h)} + k$, donde a, h, y k son números reales.

cycle the smallest representation of a cosine or sine function graph as defined over a restricted domain; equal to one repetition of the period of a function

U3-100
U4A-114

ciclo la representación más pequeña de una gráfica de la función coseno o seno definida a través de un dominio restringido; igual a una repetición del período de una función

English		Español
	D	
data numbers in context	U1-122	**datos** números en contexto
data fitting the process of assigning a rule, usually an equation or formula, to a collection of data points as a method of predicting the values of new dependent variables that result from new independent-variable values	U4B-1	**ajuste de datos** proceso de asignar una regla, normalmente una ecuación o fórmula, a una colección de puntos de datos como un método de predecir los valores de nuevas variables dependientes que vienen de nuevos valores de variables independientes
data point a point (x, y) on a two-dimensional coordinate plane that represents the value of an independent variable (x) that results in a specific dependent variable value (y). The term also refers to solutions for an equation or inequality in one variable that originate from a real-world situation. A data point is also called an *ordered pair*.	U4B-2	**punto de datos** un punto (x, y) en un plano de coordenadas de dos dimensiones que representa el valor de una variable independiente (x) que da como resultado un valor específico de variable dependiente (y). El término también se refiere a las soluciones para una ecuación o desigualdad en una variable que proviene de una situación del mundo real. Un punto de datos se denomina también un *par ordenado*.
degree of a one-variable polynomial the greatest exponent of the variable in a polynomial	U2A-1	**grado de un polinomio de una variable** el mayor exponente de la variable en un polinomio
degrees of freedom (*df*) the number of data values that are free to vary in the final calculation of a statistic; that is, values that can change or move without violating the constraints on the data	U1-198	**grados de libertad (*df*)** la cantidad de valores de datos que varían libremente en el cálculo final de una estadística; es decir, los valores que pueden cambiar o moverse sin violar las restricciones en los datos
delta (Δ) a Greek letter commonly used to represent the change in a value	U4B-87	*delta* (Δ) letra griega utilizada comúnmente para representar el cambio en un valor

English		Español
denominator the value located below the line of a rational expression or fraction; the divisor	U2B-2	**denominador** el valor ubicado debajo de la línea de una expresión racional o fracción; el divisor
density the amount, number, or other quantity per unit of area or volume of some substance or population being studied	U4B-193	**densidad** la cantidad, el número u otra cantidad por unidad de área o volumen de alguna sustancia o población que está siendo estudiada
dependent system a system of equations that intersect at every point	U2A-95 U2B-47	**sistema dependiente** sistema de ecuaciones que se cruzan en cada punto
dependent variable labeled on the y-axis; the quantity that is based on the input values of the independent variable; the output variable of a function	U4B-2	**variable dependiente** designada en el eje de y; cantidad que se basa en los valores de entrada de la variable independiente; variable de salida de una función
depressed polynomial the result of dividing a polynomial by one of its binomial factors	U2A-48	**polinomio deprimido** el resultado de dividir un polinomio por uno de sus factores binómicos
descending order polynomials ordered by the power of the variables, with the largest power listed first and the constant last	U2A-1	**orden descendente** polinomios ordenados según la potencia de las variables, con la potencia más grande indicada primero y la constante, última
desirable outcome the data sought or hoped for, represented by p; also known as *favorable outcome* or *success*	U1-148	**resultado deseado** datos buscados o esperados, representado por p; también conocido como *resultado favorable* o *éxito*
discontinuous function a function that has a break, hole, or jump in the graph	U4B-144	**función discontinua** una función que tiene un descanso, agujero, o salto en el gráfico
discrete data a set of values with gaps between successive values	U1-1	**datos discretos** conjunto de valores con interrupciones entre valores sucesivos

English		Español
discrete function a function in which every element of the domain is individually separate and distinct	U2A-116 U4B-144	**función discreta** función en la cual cada elemento del dominio está individualmente separado y distinguible
diverge to *not* approach a finite limit. If a series does not have a sum (that is, the sequence of its partial sums does not approach a given number), then the series diverges. An infinite series diverges when the absolute value of the common ratio r is greater than 1 $\left(|r|>1\right)$.	U2A-116	**divergir** *no* acercarse a un límite finito. Si una serie no tiene una suma (es decir, la secuencia de sus sumas parciales no se acerca a un número dado), entonces la serie diverge. Una serie infinita diverge cuando el valor absoluto de la relación común r es mayor que 1 $\left(|r|>1\right)$.
domain the set of all input values (*x*-values) that satisfy the given function without restriction	U4A-1 U4B-2	**dominio** conjunto de todos los valores de entrada (valores de *x*) que satisfacen la función dada sin restricciones
double-blind study a study in which neither the researcher nor the participants know who has been subjected to the treatment, action, or process being studied, and who is in a control group	U1-122	**estudio doble-ciego** estudio en el cual ni el investigador ni los participantes saben quién se sometió al tratamiento, acción o proceso que está siendo estudiado y quién está en un grupo de control

E

English		Español
e an irrational number with an approximate value of 2.71828; *e* is the base of the natural logarithm ($\ln x$ or $\log_e x$)	U4A-46	*e* número irracional con un valor aproximado de 2,71828; *e* es la base del logaritmo natural (In *x* o $\log_e x$)

English		Español
empirical probability the number of times an event actually occurs divided by the total number of trials, given by the formula $P(E) = \dfrac{\text{number of occurrences of the event}}{\text{total number of trials}}$; also called *experimental probability*	U1-240	**probabilidad empírica** cantidad de veces que se produce un evento dividido por la cantidad total de pruebas, dada por la fórmula $P(E) = \dfrac{\text{cantidad de ocurrencias del evento}}{\text{cantidad total de pruebas}}$; también se denomina *probabilidad experimental*
Empirical Rule a rule that states percentages of data under the normal curve are as follows: $\mu \pm 1\sigma \approx 68\%$, $\mu \pm 2\sigma \approx 95\%$, and $\mu \pm 3\sigma \approx 99.7\%$; also known as the *68–95–99.7 rule*	U1-1	**Regla Empírica** regla que establece los siguientes porcentajes de datos bajo la curva normal: $\mu \pm 1\sigma \approx 68\%$, $\mu \pm 2\sigma \approx 95\%$ y $\mu \pm 3\sigma \approx 99{,}7\%$; también se conoce como *la regla 68–95–99,7*
empty set a set that has no elements, denoted by $\varnothing$; the solution to a system of equations with no intersection points, denoted by { }	U2A-95 U2B-47	**conjunto vacío** un conjunto que no tiene elementos, indicado por $\varnothing$; la solución a un sistema de ecuaciones sin puntos de intersección, indicado por { }
end behavior the behavior of the graph as *x* approaches positive or negative infinity	U2A-48	**comportamiento final** el comportamiento del gráfico a medida que *x* se acerca al infinito positivo o negativo
even-degree polynomial function a polynomial function in which the highest exponent is an even number. Both ends of the graph of an even-degree polynomial function will extend in the same direction, either upward or downward.	U2A-48	**función polinómica de grado par** función polinómica en la cual el exponente mayor es un número par. Ambos extremos del gráfico de una función polinómica de grado par se extenderán en la misma dirección, hacia arriba o hacia abajo.
even function a function that, when evaluated for –*x*, results in a function that is the same as the original function; $f(-x) = f(x)$	U4B-54	**función par** función que, cuando se la evalúa para –*x*, tiene como resultado una función que es igual a la original; $f(-x) = f(x)$

English		Español
expected value an estimate of value that is determined by finding the product of a total value and a probability of a given event; symbolized by $E(X)$	U1-240	**valor esperado** estimación de valor que se determina al encontrar el producto de un valor total y una probabilidad de un evento dado; simbolizado por $E(X)$
experimental probability the number of times an event actually occurs divided by the total number of trials, given by the formula $P(E) = \dfrac{\text{number of occurrences of the event}}{\text{total number of trials}}$; also called *empirical probability*	U1-241	**probabilidad experimental** cantidad de veces que se produce un evento dividido por la cantidad total de pruebas, dada por la fórmula $P(E) = \dfrac{\text{cantidad de ocurrencias del evento}}{\text{cantidad total de pruebas}}$; también se denomina *probabilidad empírica*
experiment a process or action that has observable results	U1-122	**experimento** proceso o acción con resultados observables
explicit formula for a geometric sequence a formula used to find any term in a sequence. The formula is $a_n = a_1 \cdot r^{n-1}$, where n is a positive integer that represents the number of terms in the sequence, r is the common ratio, a_1 is the value of the first term in the sequence, and a_n is the value of the nth term of the sequence.	U2A-117	**fórmula explícita para una secuencia geométrica** fórmula usada para encontrar algún término en una secuencia. La fórmula es $a_n = a_1 \cdot r^{n-1}$, donde n es un número entero positivo que representa la cantidad de términos en la secuencia, r es la relación común, a_1 es el valor del primer término de la secuencia y a_n es el valor del término n de la secuencia.
exponential equation an equation that has a variable in the exponent	U2A-95 U4B-2	**ecuación exponencial** una ecuación que tiene una variable en el exponente
exponential expression an expression that contains a base raised to an exponent	U2A-1	**expresión exponencial** expresión que incluye una base elevada a un exponente

English		Español
exponential function a function of the form $f(x) = ab^{cx}$, in which a, b, and c are constants; a function that has a variable in the exponent, such as $f(x) = 5^x$	U4A-46 U4B-144	**función exponencial** función de la fórmula $f(x) = ab^{cx}$ en la cual a, b y c son constantes; una función que tiene una variable en el exponente, tal como $f(x) = 5^x$
extraneous solution a solution of an equation that arises during the solving process, but which is not a solution of the original equation	U2B-47	**solución extraña** solución de una ecuación que surge durante el proceso de resolución pero que no es una solución de la ecuación original
extrema the minima or maxima of a function	U4B-87	**extremos** los mínimos o máximos de una función

F

English		Español
factor one of two or more numbers or expressions that when multiplied produce a given product	U2A-1	**factor** uno de dos o más números o expresiones que cuando se multiplican generan un producto determinado
factorial the product of an integer and all preceding positive integers, represented using a ! symbol; $n! = n \bullet (n-1) \bullet (n-2) \bullet \cdots \bullet 1$. For example, $5! = 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1$. By definition, $0! = 1$.	U1-64 U1-148 U2A-21	**factorial** producto de un entero y todos los enteros positivos anteriores, que se representa con el símbolo !; $n! = n \bullet (n-1) \bullet (n-2) \bullet ... \bullet 1$. Por ejemplo, $5! = 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1$. Por definición, $0! = 1$.
factor of a polynomial any polynomial that divides evenly into the function $p(x)$	U2A-48	**factor de un polinomio** todo polinomio que divide de manera uniforme en la función $p(x)$
Factor Theorem The binomial $x - a$ is a factor of the polynomial $p(x)$ if and only if $p(a) = 0$.	U2A-48	**Teorema del Factor** el binomio $x - a$ es un factor del polinomio $p(x)$ si y solo si $p(a) = 0$.
failure the occurrence of an event that was not sought out or wanted, represented by q; also known as *undesirable outcome* or *unfavorable outcome*	U1-148	**fracaso** ocurrencia de un evento que no fue buscado ni deseado, representado por q; también conocido como *resultado no deseado* o *resultado desfavorable*

English		Español
fair describes a situation or game in which all of the possible outcomes have an equal chance of occurring	U1-241	**equitativo** describe una situación o juego en el cual todos los resultados posibles tienen igual probabilidad de producirse
false negative result a determination that an experiment has produced an incorrect negative result	U1-241	**resultado falso negativo** determinación de que un experimento ha producido un resultado negativo incorrecto
false positive result a determination that an experiment has produced an incorrect positive result	U1-241	**resultado falso positivo** determinación de que un experimento ha producido un resultado positivo incorrecto
family of functions a set of functions whose graphs have the same general shape as their parent function. The parent function is the function with a simple algebraic rule that represents the family of functions.	U4B-54	**familia de funciones** conjunto de funciones cuyas gráficas tienen la misma forma general que su función raíz. La función raíz es la función con una regla algebraica simple que representa la familia de funciones.
favorable outcome the data sought or hoped for, represented by *p*; also known as *desirable outcome* or *success*	U1-148	**resultado favorable** datos buscados o esperados, representados por *p*; también conocido como *resultado deseado* o *éxito*
finite limited in number	U2A-117	**finito** limitado en número

English		Español
finite geometric series a series that has a limited or definite number of terms; can be written as $\sum_{k=1}^{n} a_1 r^{k-1}$, where n is a positive integer that represents the number of terms in the series, a_1 is the first term, r is the common ratio, and k is the number of the term	U2A-117	**serie geométrica finita** serie que tiene una cantidad limitada o definida de términos; puede ser escrita como $\sum_{k=1}^{n} a_1 r^{k-1}$, donde n es un número entero positivo que representa la cantidad de términos en la serie, a_1 es el primer término, r es la relación común y k es el número del término
formula a mathematical statement of the relationship between two or more variables	U4B-2	**fórmula** afirmación matemática de la relación entre dos o más variables
fraction a ratio of two expressions or quantities	U2B-2	**fracción** relación de dos expresiones o cantidades
frequency of a periodic function the reciprocal of the period for a periodic function; indicates how often the function repeats	U3-100	**frecuencia de una función periódica** recíproca del período para una función periódica; indica con que frecuencia se repite la función
function a relation in which every element of the domain is paired with exactly one element of the range; that is, for every value of x, there is exactly one value of y	U4A-1	**función** relación en la que cada elemento del dominio se empareja con un único elemento del rango; es decir, para cada valor de x, existe exactamente un valor de y
Fundamental Theorem of Algebra If $p(x)$ is a polynomial function of degree $n \geq 1$ with complex coefficients, then the related equation $p(x) = 0$ has at least one complex solution (root).	U2A-48	**Teorema fundamental del álgebra** Si $P(x)$ es una función polinómica de grado $n \geq 1$ con coeficientes complejos, entonces la ecuación relacionada $P(x) = 0$ tiene al menos una solución compleja (raíz).

English		Español
G		
geometric sequence an ordered list of terms in which each new term is the product of the preceding term and a common ratio, r. For a sequence to be geometric, r cannot be equal to 1 ($r \neq 1$).	U2A-117	**secuencia geométrica** lista ordenada de términos en la cual cada término nuevo es el producto del término anterior y una relación común, r. Para que una secuencia sea geométrica, r no puede ser igual a 1 ($r \neq 1$).
geometric series the sum of a specified number of terms from a geometric sequence	U2A-117	**serie geométrica** suma de una cantidad específica de términos de una secuencia geométrica
H		
half plane a region containing all points on one side of a boundary, which is a line or curve that continues in both directions infinitely. The line or curve may or may not be included in the region. A half plane can be used to represent a solution to an inequality statement.	U4B-2	**semiplano** región que contiene todos los puntos en un lado de un límite, que es una línea recta o curva que continúa en ambas direcciones de manera infinita. La línea recta o curva puede o no estar incluida en la región. Un semiplano puede utilizarse para representar una solución a una afirmación de desigualdad.
hypothesis a statement that you are trying to prove or disprove	U1-198	**hipótesis** afirmación que usted intenta probar o desaprobar
hypothesis testing assessing data in order to determine whether the data supports (or fails to support) the hypothesis as it relates to a parameter of the population	U1-198	**prueba de hipótesis** evaluación de datos para determinar si los datos respaldan (o no respaldan) la hipótesis mientras se relaciona con un parámetro de la población
I		
imaginary number any number of the form bi, where b is a real number, $i = \sqrt{-1}$, and $b \neq 0$	U2A-21	**número imaginario** cualquier número de la forma bi, en el que b es un número real, $i = \sqrt{-1}$, y $b \neq 0$
imaginary unit, i the letter i, used to represent the non-real value $i = \sqrt{-1}$	U2A-21	**unidad imaginaria, i** la letra i, utilizada para representar el valor no real $i = \sqrt{-1}$

English		Español
included angle the angle between two sides	U3-59	**ángulo incluido** ángulo entre dos lados
independent system a system of equations with a finite number of points of intersection	U2A-95	**sistema independiente** sistema de ecuaciones con una cantidad finita de puntos de intersección
independent variable labeled on the *x*-axis; the quantity that changes based on values chosen; the input variable of a function	U4B-2	**variable independiente** designada en el eje *x*; cantidad que cambia según los valores seleccionados; variable de entrada de una función
inequality a mathematical statement that compares the value of an expression in one independent variable to the value of a dependent variable using the comparison symbols >, <, ≥, and ≤	U4B-2	**desigualdad** afirmación matemática que compara el valor de una expresión en una variable independiente con el valor de una variable dependiente usando los símbolos de comparación >, <, ≥ y ≤
inference a conclusion reached upon the basis of evidence and reasoning	U1-64	**inferencia** conclusión alcanzada sobre la base de evidencia y razonamiento
infinite having no limit; represented by the infinity symbol, ∞	U2A-117	**infinito** no tiene límite; representado por el símbolo de infinitud ∞
infinite geometric series a series that has an unlimited number of terms $(n=\infty)$; can be written as $\sum_{k=1}^{\infty} a_1 r^{k-1}$, where a_1 is the first term, *r* is the common ratio, and *k* is the number of the term	U2A-117	**serie geométrica infinita** serie que tiene una cantidad ilimitada de términos $(n=\infty)$; se puede escribir como $\sum_{k=1}^{\infty} a_1 r^{k-1}$, donde a_1 es el primer término, *r* es la relación común y *k* es el número del término

English		Español

English		**Español**

English — Español

initial condition a constraint or limit on a function or domain value that exists in the form of a y-intercept or other starting-point mathematical restraint in a real-world problem or solution

U4B-87 · **condición inicial** restricción o límite en el valor de una función o de un dominio que existe en la forma de un intercepto de y u otra restricción matemática del punto de inicio en un problema o solución del mundo real

initial side the stationary ray of an angle from which the measurement of the angle starts

U3-2 · **lado inicial** rayo fijo de un ángulo desde el cual comienza la medición del ángulo

integer a number that is not a fraction or decimal

U2A-48 · **entero** un número que no es una fracción ni un decimal

Integral Zero Theorem If the coefficients of a polynomial function are integers such that $a_n = 1$ and $a_0 \neq 0$, then any rational zeros of the function must be factors of a_0.

U2A-48 · **Teorema Integral Cero** si los coeficientes de una función polinómica son números enteros tales como $a_n = 1$ y $a_0 \neq 0$, entonces todos los ceros racionales de la función deben ser factores de a_0.

interval a set of values between a lower bound and an upper bound

U1-1 · **intervalo** conjunto de valores entre un límite inferior y un límite superior

inverse function the function that may result from switching the x- and y-variables in a given function; the inverse of $f(x)$ is written as $f^{-1}(x)$

U4A-1 · **función inversa** función que se produce como resultado de cambiar las variables x y y en una función determinada; la inversa de $f(x)$ se expresa como $f^{-1}(x)$

inverse relation a relation $g(x)$ such that $g(f(x)) = x$ and $f(g(y)) = y$ where $f(x)$ is a function

U4A-1 · **relación inversa** una relación $g(x)$ tal que $g(f(x)) = x$ y $f(g(y)) = y$ donde $f(x)$ es una función

Irrational Root Theorem If a polynomial $p(x)$ has rational coefficients and $a + b\sqrt{c}$ is a root of the polynomial equation $p(x) = 0$, where a and b are rational and $\sqrt{c}$ is irrational, then $a - b\sqrt{c}$ is also a root of $p(x) = 0$.

U2A-48 · **Teorema de la Raíz Irracional** si un polinomio $p(x)$ tiene coeficientes racionales y $a + b\sqrt{c}$ es una raíz de la ecuación de polinomios $p(x) = 0$, donde a y b son racionales y $\sqrt{c}$ es irracional, entonces $a - b\sqrt{c}$ es también una raíz de $p(x) = 0$.

English		Español
L		
Law of Cosines a formula for any triangle which states $c^2 = a^2 + b^2 - 2ab \cos C$, where C is the included angle in between sides a and b, and c is the nonadjacent side across from $\angle C$	U3-59	**Ley de Cosenos** fórmula para todo triángulo que establece $c^2 = a^2 + b^2 - 2ab \cos C$, donde C es el ángulo incluido entre los lados a y b, y c es el lado no adyacente u opuesto $\angle C$
Law of Sines a formula for any triangle which states $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$, where a represents the measure of the side opposite $\angle A$, b represents the measure of the side opposite $\angle B$, and c represents the measure of the side opposite $\angle C$	U3-59	**Ley de Senos** fórmula para todo triángulo que establece $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$, donde a representa la medida del lado opuesto $\angle A$, b representa la medida del lado opuesto $\angle B$ y c representa la medida del lado opuesto $\angle C$
leading coefficient the coefficient of the term with the highest power	U2A-2	**coeficiente líder** coeficiente del término con la mayor potencia
least common denominator (LCD) the least common multiple of the denominators of two or more fractions	U2B-2 U2B-48	**denominador común mínimo (DCM)** múltiplo común mínimo de los denominadores de dos o más fracciones
level of confidence the probability that a parameter's value can be found in a specified interval; also called *confidence level*	U1-148 U1-198	**grado de confianza** probabilidad de que se pueda encontrar el valor de un parámetro en un intervalo específico; también llamado *nivel de confianza*
like terms terms that contain the same variables raised to the same power	U2A-2	**términos semejantes** términos que contienen las mismas variables elevadas a la misma potencia

English		Español
linear equation an equation that can be written in the form $ax + by = c$, where a, b, and c are constants; can also be written as $y = mx + b$, in which m is the slope and b is the y-intercept. The graph of a linear equation is a straight line; its solutions are the infinite set of points on the line.	U2A-96 U4B-2	**ecuación lineal** ecuación que puede expresarse en la forma $ax + by = c$, donde a, b y c son constantes; también puede escribirse como $y = mx + b$, en donde m es la pendiente y b es el intercepto de y. El gráfico de una ecuación lineal es una línea recta; sus soluciones son el conjunto infinito de puntos en la línea.
linear function a function that can be written in the form $ax + by = c$, where a, b, and c are constants; can also be written as $f(x) = mx + b$, in which m is the slope and b is the y-intercept. The graph of a linear function is a straight line; its solutions are the infinite set of points on the line.	U4B-144	**función lineal** función que puede expresarse en la forma $ax + by = c$, donde a, b y c son constantes; también puede escribirse como $f(x) = mx + b$, donde m es la pendiente y b es el intercepto de y. El gráfico de una función lineal es una línea recta; sus soluciones son el conjunto infinito de puntos en la línea.
local maximum the greatest value of a function for a particular interval of the function; also known as a *relative maximum*	U2A-48 U4B-87	**máximo local** el mayor valor de una función para un intervalo específico de la función; también conocido como *máximo relativo*
local minimum the least value of a function for a particular interval of the function; also known as a *relative minimum*	U2A-48 U4B-87	**mínimo local** el menor valor de una función para un intervalo específico de la función; también conocido como *mínimo relativo*
logarithmic equation an equation that includes a logarithmic expression	U4B-2	**ecuación logarítmica** una ecuación que incluye una expresión logarítmica
logarithmic function the inverse of an exponential function; for the exponential function $f(x) = 5^x$, the inverse logarithmic function is $x = \log_5 f(x)$	U4A-46	**función logarítmica** la inversa de una función exponencial; para la función exponencial $f(x) = 5^x$, la función logarítmica inversa es $x = \log_5 f(x)$

English		Español
	M	
margin of error the quantity that represents the level of confidence in a calculated parameter, abbreviated MOE. The margin of error can be calculated by multiplying the critical value by the standard deviation, if known, or by the SEM.	U1-148	**margen de error** cantidad que representa el nivel de confianza en un parámetro calculado, abreviado MOE. El margen de error puede calcularse multiplicando el valor crítico por la desviación estándar, si se conoce, o por el SEM.
maximum the greatest value or highest point of a function	U3-100	**máximo** el mayor valor o punto más alto de una función
mean a measure of center in a set of numerical data, computed by adding the values in a data set and then dividing the sum by the number of values in the data set; denoted as the Greek lowercase letter *mu*, μ; given by the formula $\mu = \dfrac{x_1 + x_2 + \ldots + x_n}{n}$, where each *x*-value is a data point and *n* is the total number of data points in the set	U1-2	**media** medida del centro en un conjunto de datos numéricos, calculada al sumar los valores en un conjunto de datos y luego al dividir la suma por el número de valores en el conjunto de datos; indicada con la letra griega minúscula *mu*, μ; dada por la fórmula $\mu = \dfrac{x_1 + x_2 + \ldots + x_n}{n}$, donde cada valor de x es un punto de datos y *n* es la cantidad total de puntos de datos en el conjunto
measurement bias bias that occurs when the tool used to measure the data is not accurate, current, or consistent	U1-198	**sesgo de medición** sesgo que se produce cuando la herramienta utilizada para medir los datos no es exacta, actual o constante
median the middle-most value of a data set; 50% of the data is less than this value, and 50% is greater than it	U1-2	**mediana** valor máximo-medio de un conjunto de datos; el 50% de los datos es menor que este valor y el otro 50% es mayor que él

English		Español
midline in a cosine function or sine function of the form $f(x) = \sin x + d$ or $g(x) = \cos x + d$, a horizontal line of the form $y = d$ that bisects the vertical distance on a graph between the minimum and maximum function values	U3-101 U4A-114	**línea media** en una función del coseno o en una función del seno de la forma $f(x) = \operatorname{sen} x + d$ o $g(x) = \cos x + d$, una línea horizontal de la forma $y = d$ que divide en dos la distancia vertical en un gráfico entre los valores de funciones mínimos y máximos
minimum the least value or lowest point of a function	U3-101	**mínimo** el menor valor o el punto más bajo de una función
mu, μ a Greek letter used to represent mean	U1-2	**mu, μ** letra griega usada para representar la media
multiplicity (of a zero) the number of times a zero of a polynomial function occurs	U2A-49	**multiplicidad (de un cero)** la cantidad de veces que sucede cero en una función polinómica
mutually exclusive events events that have no outcomes in common. If A and B are mutually exclusive events, then they cannot both occur.	U1-148	**eventos mutuamente excluyentes** eventos que no tienen resultados en común. Si A y B son eventos mutuamente excluyentes, entonces no pueden producirse ambos.

N

English		Español
natural logarithm a logarithm whose base is the irrational number e; usually written in the form "ln," which means "$\log_e$"	U4A-46	**logaritmo natural** logaritmo cuya base es el número irracional e; escrito normalmente en la forma "ln", que significa "$\log_e$"
negatively skewed a distribution in which there is a "tail" of isolated, spread-out data points to the left of the median. "Tail" describes the visual appearance of the data points in a histogram. Data that is negatively skewed is also called *skewed to the left*.	U1-2	**sesgado negativamente** distribución en la cual existe una "cola" de puntos de datos aislados y esparcidos a la izquierda de la mediana. La "cola" describe la apariencia visual de los puntos de datos en un histograma. Los datos que están sesgados negativamente también se denominan *sesgados a la izquierda*.

English		Español
neutral not biased or skewed toward one side or another; regarding surveys, neutral refers to phrasing questions in a way that does not lead the response toward one particular answer or side of an issue	U1-123	**neutral** no sesgado hacia un lado u otro; respecto de las encuestas, neutral se refiere a la formulación de preguntas de una manera que no conduzca la respuesta hacia una respuesta o lado específico de un tema
nonresponse bias bias that occurs when the respondents to a survey have different characteristics than nonrespondents, causing the population that does not respond to be underrepresented in the survey's results	U1-198	**sesgo sin respuesta** sesgo que se produce cuando los encuestados de una encuesta tienen características diferentes de los no encuestados, dando pie a que la población que no responde sea subrepresentada en los resultados de la encuesta
normal curve a symmetrical curve representing the normal distribution	U1-2	**curva normal** curva simétrica que representa la distribución normal
normal distribution a set of values that are continuous, are symmetric to a mean, and have higher frequencies in intervals close to the mean than equal-sized intervals away from the mean	U1-2	**distribución normal** conjunto de valores que son continuos, simétricos a una media y tienen frecuencias más altas en intervalos cercanos a la media que los intervalos de igual tamaño lejos de la media
null hypothesis the statement or idea that will be tested, represented by H_0; generally characterized by the concept that there is no relationship between the data sets, or that the treatment has no effect on the data	U1-198	**hipótesis nula** afirmación o idea que será probada, representada por H_0; caracterizada generalmente por el concepto de que no hay relación entre los conjuntos de datos o que el tratamiento no tiene efecto en los datos
numerator the value located above the line of a rational expression or fraction; the dividend	U2B-2	**numerador** el valor ubicado por encima de la línea de una expresión racional o fracción; el dividendo

English		Español
	O	
oblique triangle a triangle that does not contain a right angle	U3-59	**triángulo oblicuo** triángulo que no contiene un ángulo recto
observational study a study in which all data, including observations and measurements, are recorded in a way that does not change the subject that is being measured or studied	U1-123	**estudio observacional** estudio en el cual todos los datos, incluyendo las observaciones y las mediciones, están registrados de tal manera que no cambian el objeto que está siendo medido o estudiado
odd-degree polynomial function a polynomial function in which the highest exponent is an odd number. One end of the graph of an odd-degree polynomial function will extend upward and the other end will extend downward.	U2A-49	**función polinómica de grado impar** función polinómica en la cual el exponente mayor es un número impar. Un extremo del gráfico de una función polinómica de grado impar se extenderá hacia arriba y el otro extremo se extenderá hacia abajo.
odd function a function that, when evaluated for $–x$, results in a function that is the opposite of the original function; $f(–x) = –f(x)$	U4B-54	**función impar** función que, cuando se evalúa para $–x$, tiene como resultado una función que es lo opuesto a la función original; $f(–x) = –f(x)$
one-tailed test a t-test performed on a set of data to determine if the data could belong in one of the tails of the bell-shaped distribution curve; with this test, the area under only one tail of the distribution is considered	U1-198	**prueba de una cola o unilateral** una prueba t realizada en un conjunto de datos para determinar si estos podrían pertenecer a una de las colas de la curva de distribución con forma de campana; con esta prueba, solo se considera el área debajo de una cola de la distribución

English		Español
one-to-one correspondence the feature of a function whereby each value in the domain corresponds to a unique function value; that is, if $x = a$ and $x = b$, the two points would be $(a, f(a))$ and $(b, f(b))$, and if $a \neq b$, then $f(a) \neq f(b)$ for a function to exhibit one-to-one correspondence	U4A-1	**correspondencia uno a uno** la característica de una función por la cual cada valor del dominio corresponde a un único valor de función; es decir, si $x = a$ y $x = b$, los dos puntos serían $(a, f(a))$ y $(b, f(b))$, y si $a \neq b$, entonces $f(a) \neq f(b)$ para que una función exhiba una correspondencia uno a uno
ordered pair a point (x, y) on a two-dimensional coordinate plane that represents the value of an independent variable (x) that results in a specific dependent variable value (y). The term also refers to solutions for an equation or inequality in one variable that originate from a real-world situation. An ordered pair is also called a *data point*.	U4B-3	**par ordenado** un punto (x, y) en un plano de coordenadas de dos dimensiones que representa el valor de una variable independiente (x) que da como resultado un valor específico de variable dependiente (y). El término también se refiere a las soluciones para una ecuación o desigualdad en una variable que proviene de una situación del mundo real. Un par ordenado también se denomina un *punto de datos*.
outcome the observable result of an experiment	U1-123	**resultado** producto observable de un experimento
outlier a value far above or below other values of a distribution	U1-2	**valor atípico** valor muy por encima o muy por debajo de otros valores de una distribución

		P		

English		Español
p-value a number between 0 and 1 that determines whether to accept or reject the null hypothesis	U1-198	**valor _p_** número entre 0 y 1 que determina si se acepta o se rechaza la hipótesis nula
parameter numerical value(s) representing the data in a set, including proportion, mean, and variance	U1-64 U1-148 U1-198	**parámetro** valores numéricos que representan los datos en un conjunto, incluyendo la proporción, la media y la varianza

English		Español

parent function a function with a simple algebraic rule that represents a family of functions. The graphs of the functions in the family have the same general shape as the parent function.

U4A-114
U4B-3
U4B-54

función principal función con una regla algebraica simple que representa una familia de funciones. Los gráficos de las funciones en la familia tienen la misma forma general que la función principal.

partial sum the sum of part of a series

U2A-117

suma parcial la suma de la parte de una serie

Pascal's Triangle a triangle displaying a pattern of numbers in which the terms in additional rows are found by adding pairs of terms in previous rows, so that any given term is the sum of the two terms directly above it. The number 1 is the top number of the triangle, and is also the first and last number of each row.

U2A-21

triángulo de Pascal triángulo que muestra un patrón de números en el cual los términos se encuentran en filas adicionales al agregar pares de términos en filas anteriores, de manera que cualquier término dado es la suma de los dos términos directamente por encima de este. El número 1 es el número superior del triángulo y también es el primero y el último de cada fila.

```
          1

      1       1

   1      2      1

 1     3      3      1

1    4    6     4     1
```

```
          1

      1       1

   1      2      1

 1     3      3      1

1    4    6     4     1
```

period in a cosine or sine function graph, the horizontal distance from a maximum to a maximum or from a minimum to a minimum; one repetition of the period of a function is called a *cycle*
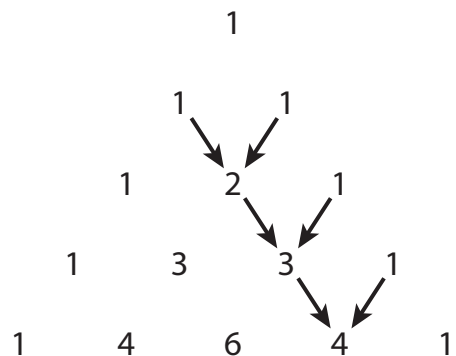
U3-101
U4A-114

período en una curva de la función del seno o coseno, distancia horizontal desde un máximo a un máximo o desde un mínimo a un mínimo; una repetición del período de una función se llama *ciclo*

English		Español
periodic function a function whose values repeat at regular intervals	U3-101 U4B-87	**función periódica** función cuyos valores se repiten a intervalos regulares
periodic phenomena real-life situations that repeat at regular intervals and can be represented by a periodic function	U3-101	**fenómenos periódicos** situaciones de la vida real que se repiten a intervalos regulares y se pueden representar mediante una función periódica
phase shift on a cosine or sine function graph, the horizontal distance by which the curve of a parent function is shifted by the addition of a constant or other expression in the argument of the function	U4A-114	**cambio de fase** en una gráfica de la función del seno o del coseno, distancia horizontal por la cual la curva de una función raíz se cambia por la adición de una constante u otra expresión en el argumento de la función
piecewise function a function that is defined by two or more expressions on separate portions of the domain	U4B-145	**función por partes** función definida por dos o más expresiones en porciones separadas del dominio
placebo a substance that is used as a control in testing new medications; the substance has no medicinal effect on the subject	U1-123	**placebo** sustancia que se utiliza como control en las pruebas de medicamentos nuevos; la sustancia no tiene efecto medicinal sobre el sujeto
plane a flat, two-dimensional figure without depth that has at least three non-collinear points and extends infinitely in all directions	U4B-193	**plano** figura plana, bidimensional, sin profundidad, que tiene al menos tres puntos no colineales y se extiende infinitamente en todas direcciones
plane figure a two-dimensional shape on a plane	U4B-193	**figura plana** forma bidimensional sobre un plano
point(s) of intersection in a graphed system of equations, the ordered pair(s) where graphed functions intersect on a coordinate plane	U2A-96 U2B-48	**punto(s) de intersección** en un sistema de ecuaciones graficado, par(es) ordenado(s) donde las funciones de la gráfica se cruzan sobre un plano de coordenadas

English		Español
polyhedron a three-dimensional object that has faces made of polygons	U4B-193	**poliedro** objeto tridimensional que tiene caras compuestas por polígonos
polynomial an expression that contains variables, numeric quantities, or both, where variables are raised to integer powers greater than or equal to 0	U2A-2	**polinomio** expresión que contiene variables, cantidades numéricas o ambas en donde las variables se elevan a potencias de números enteros mayores o iguales a 0
polynomial equation an equation of the general form $y = a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0$, where a_1 is a rational number, $a_n \neq 0$, and n is a nonnegative integer and the highest degree of the polynomial	U2A-96	**ecuación polinómica** ecuación de la forma general $y = a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0$, donde a_1 es un número racional, $a_n \neq 0$ y n es un número entero no negativo y el grado más alto del polinomio
polynomial function a function of the general form $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0$, where a_1 is a rational number, $a_n \neq 0$, and n is a nonnegative integer and the highest degree of the polynomial	U2A-2 U2A-49	**función polinómica** función de la forma general $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0$, donde a_1 es un número racional, $a_n \neq 0$ y n es un número entero no negativo y el grado más alto del polinomio
polynomial identity a true equation that is often generalized so it can apply to more than one example	U2A-21	**identidad del polinomio** ecuación verdadera que suele generalizarse para que pueda aplicarse a más de un ejemplo
population all of the people, objects, or phenomena of interest in an investigation; the entire data set	U1-148	**población** todas las personas, los objetos o fenómenos de interés en una investigación; el conjunto completo de datos
population all of the people, objects, or phenomena of interest in an investigation	U1-2 U1-64 U1-148	**población** todas las personas, los objetos o fenómenos de interés en una investigación

English		Español
population average the sum of all quantities in a population, divided by the total number of quantities in the population; typically represented by μ; also known as *population mean*	U1-148	**promedio de la población** suma de todas las cantidades de una población, dividida por el número total de cantidades de la población; representada normalmente por μ; también se conoce como *media poblacional*
population mean the sum of all quantities in a population, divided by the total number of quantities in the population; typically represented by μ; also known as *population average*	U1-148	**media poblacional** suma de todas las cantidades de una población, dividida por el número total de cantidades de la población; representada normalmente μ; también se conoce como *promedio de la población*
positively skewed a distribution in which there is a "tail" of isolated, spread-out data points to the right of the median. "Tail" describes the visual appearance of the data points in a histogram. Data that is positively skewed is also called *skewed to the right*.	U1-2	**sesgado positivamente** distribución en la cual existe una "cola" de puntos de datos aislados esparcidos hacia la derecha de la mediana. La "cola" describe la apariencia de los puntos de datos en un histograma. Los datos que están positivamente sesgados también se denominan *sesgados a la derecha*.
power the result of raising a base to an exponent; 32 is a power of 2 since $2^5 = 32$	U4A-46	**potencia** el resultado de elevar una base a un exponente; 32 es una potencia de 2 desde $2^5 = 32$
probability distribution the values of a random variable with associated probabilities	U1-2	**distribución de probabilidad** los valores de una variable aleatoria con probabilidades asociadas
proportion a statement of equality between two ratios	U2B-48	**proporción** afirmación de igualdad entre dos relaciones

English		Español
Q		
quadratic equation an equation that can be written in the form $ax^2 + bx + c = 0$, where x is the variable, a, b, and c are constants, and $a \neq 0$	U2A-96 U2B-2 U4B-3	**ecuación cuadrática** ecuación que se puede expresar en la forma $ax^2 + bx + c = 0$, donde x es la variable, a, b, y c son constantes, y $a \neq 0$
quadratic expression an algebraic expression that can be written in the form $ax^2 + bx + c$, where x is the variable, a, b, and c are constants, and $a \neq 0$	U2B-2	**expresión cuadrática** expresión algebraica que se puede expresar en la forma $ax^2 + bx + c$, donde x es la variable, a, b, y c son constantes, y $a \neq 0$
quadratic formula a formula that states the solutions of a quadratic equation of the form $ax^2 + bx + c = 0$ are given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. A quadratic equation in this form can have no real solutions, one real solution, or two real solutions.	U2B-2	**fórmula cuadrática** fórmula que establece que las soluciones de una ecuación cuadrática de la forma $ax^2 + bx + c = 0$ están dadas por $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Una ecuación cuadrática en esta forma tener ningún solución real, o tener una solución real, o dos soluciones reales.
quadratic function a function defined by a second-degree expression of the form $f(x) = ax^2 + bx + c$, where $a \neq 0$ and a, b, and c are constants. The graph of any quadratic function is a parabola.	U4A-2 U4B-145	**función cuadrática** función definida por una expresión de segundo grado de la forma $f(x) = ax^2 + bx + c$, donde $a \neq 0$ y a, b y c son constantes. La representación gráfica de toda función cuadrática es una parábola.

English		Español
R		
radian the measure of the central angle that intercepts an arc equal in length to the radius of the circle; π radians = 180°	U3-2 U4A-114	**radián** medida del ángulo central que intercepta un arco de longitud igual al radio del círculo; π radianes = 180°
radical equation an algebraic equation in which at least one term includes a radical expression	U2B-48	**ecuación radical** ecuación algebraica en la cual al menos un término incluye una expresión radical
radical expression an expression containing a root	U2B-48	**expresión radical** expresión que contiene una raíz
random the designation of a group or sample that has been formed without following any kind of pattern and without bias. Each group member has been selected without having more of a chance than any other group member of being chosen.	U1-123	**aleatorio** designación de un grupo o muestra que se formó sin seguir ninguna clase de patrón y sin sesgo. Cada miembro del grupo se seleccionó sin tener más probabilidades de ser elegido que cualquier otro miembro del grupo.
randomization the selection of a group, subgroup, or sample without following a pattern, so that the probability of any item in the set being generated is equal; the process used to ensure that a sample best represents the population	U1-123	**aleatorización** selección de un grupo, subgrupo o muestra sin seguir un patrón, de manera que la probabilidad de cualquier elemento en el conjunto que está siendo generado sea igual; proceso utilizado para asegurar que la muestra sea la que mejor represente a la población
random number generator a tool used to select a number without following a pattern, where the probability of generating any number in the set is equal	U1-64 U1-141	**generador de números aleatorios** herramienta utilizada para seleccionar un número sin seguir un patrón, donde la probabilidad de generar cualquier número en el conjunto es igual

English		Español
random sample a subset or portion of a population or set that has been selected without bias, with each item in the population or set having the same chance of being found in the sample	U1-64 U1-149	**muestra aleatoria** subconjunto o porción de población o conjunto que ha sido seleccionado sin sesgo, con cada elemento de la población o conjunto con la misma probabilidad de encontrarse en la muestra
random variable a variable whose numerical value changes depending on each outcome in a sample space; the values of a random variable are associated with chance variation	U1-2	**variable aleatoria** variable cuyo valor numérico cambia según cada resultado en un espacio de muestra; los valores de una variable aleatoria están asociados con una variación al azar
range the set of all outputs of a function; the set of y-values that are valid for the function	U4A-2 U4B-3	**rango** conjunto de todas las salidas de una función; conjunto de valores de y que son válidos para la función
rate of change a ratio that describes how much one quantity changes with respect to the change in another quantity; also known as the slope of a line	U4B-87	**tasa de cambio** proporción que describe cuánto cambia una cantidad con respecto al cambio de otra cantidad; también se la conoce como pendiente de una recta
ratio the relation between two quantities; can be expressed in words, fractions, decimals, or as a percentage	U2B-48	**proporción** relación entre dos cantidades; puede expresarse en palabras, fracciones, decimales o como porcentaje
rational equation an algebraic equation in which at least one term is expressed as a ratio	U2B-48	**ecuación racional** ecuación algebraica en la cual al menos un término está expresado como una relación
rational expression an expression made of the ratio of two polynomials, in which a variable appears in the denominator	U2B-2	**expresión racional** expresión que resulta de la relación de dos polinomios, en la cual una variable aparece en el denominador

	English		Español

English		**Español**
rational number any number that can be written as $\dfrac{m}{n}$, where m and n are integers and $n \neq 0$; any number that can be written as a decimal that ends or repeats	U2B-48	**números racionales** números que pueden expresarse como $\dfrac{m}{n}$, en los que m y n son enteros y $n \neq 0$; cualquier número que puede escribirse como decimal finito o periódico
Rational Root Theorem If the polynomial $p(x)$ has integer coefficients, then every rational root of the polynomial equation $p(x) = 0$ can be written in the form $\dfrac{p}{q}$, where p is a factor of the constant term $p(x)$ and q is a factor of the leading coefficient of $p(x)$.	U2A-49	**Teorema de la Raíz Racional** si el polinomio $p(x)$ tiene coeficientes enteros, entonces toda raíz racional de la ecuación polinómica $p(x) = 0$ puede escribirse en la forma $\dfrac{p}{q}$, donde p es un factor del término constante $p(x)$ y q es un factor del coeficiente principal de $p(x)$.
reciprocal a number that, when multiplied by the original number, has a product of 1	U2B-2	**recíproco** número que multiplicado por el número original tiene producto 1
recursive formula for a geometric sequence a formula used to find the next term in a sequence. The formula is $a_n = a_{n-1} \cdot r$, where n is a positive integer that represents the number of terms in the sequence and r is the common ratio.	U2A-117	**fórmula recursiva para una secuencia geométrica** fórmula utilizada para encontrar el término siguiente en una secuencia. La fórmula es $a_n = a_{n-1} \cdot r$, donde n es un número entero positivo que representa la cantidad de términos en la secuencia y r es la relación común.

English		Español
reference angle the acute angle that the terminal side makes with the *x*-axis. The sine, cosine, and tangent of the reference angle are the same as that of the original angle (except for the sign, which is based on the quadrant in which the terminal side is located).	U3-2	**ángulo de referencia** el ángulo agudo que forma el lado terminal con el eje *x*. El seno, el coseno y la tangente del ángulo de referencia son iguales a los del ángulo original (con excepción del signo, que se basa en el cuadrante en el que se ubica el lado terminal).
regular polyhedron a polyhedron with faces that are all congruent regular polygons; the angles created by the intersecting faces are congruent, and the cross sections are similar figures	U4B-193	**poliedro regular** poliedro cuyas caras son todas polígonos regulares congruentes; los ángulos creados por las caras que se cruzan son congruentes y las secciones transversales son figuras similares
relation a relationship between two variables in which at least one value of the domain or independent variable, *x*, is matched with one or more values of the dependent or range variable, *y*	U4A-2	**relación** relación entre dos variables en la que al menos un valor del dominio o variable independiente, *x*, concuerda con uno o más valores de la variable de rango o dependiente, *y*
relative maximum the greatest value of a function for a particular interval of the function; also known as a *local maximum*	U2A-49 U4B-87	**máximo relativo** el mayor valor de una función para un intervalo en particular de la función; también conocido como *máximo local*
relative minimum the least value of a function for a particular interval of the function; also known as a *local minimum*	U2A-49 U4B-87	**mínimo relativo** el menor valor de una función para un intervalo en particular de la función; también conocido como *mínimo local*
reliability the degree to which a study or experiment performed many times would have similar results	U1-64	**confiabilidad** grado en el cual un estudio o experimento realizado varias veces tendría resultados similares

English		Español
Remainder Theorem For a polynomial $p(x)$ and a number a, dividing $p(x)$ by $x - a$ results in a remainder of $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.	U2A-49	**Teorema del Resto** para un polinomio $p(x)$ y un número a, dividiendo $p(x)$ por $x - a$ resulta un resto de $p(a)$, entonces $p(a) = 0$ si y solo si $(x - a)$ es un factor de $p(x)$.
repeated root a polynomial function with a root that occurs more than once	U2A-49	**raíz repetida** función polinómica con una raíz que aparece más de una vez
representative sample a sample in which the characteristics of the people, objects, or items in the sample are similar to the characteristics of the population	U1-65	**muestra representativa** muestra en la cual las características de las personas, los objetos o elementos en ella son similares a las características de la población
response bias bias that occurs when responses by those surveyed have been influenced in some manner	U1-198	**sesgo de respuesta** sesgo que se produce cuando las respuestas de los encuestados fueron influenciadas de alguna manera
restricted domain a subset of a function's defined domain	U4B-3	**dominio restringido** subconjunto del dominio definido de una función
rho **(ρ)** a lowercase Greek letter commonly used to represent density	U4B-193	*rho* **(ρ)** letra griega minúscula comúnmente utilizada para representar densidad
root the *x*-intercept of a function; also known as *zero*	U2A-49	**raíz** intercepto de *x* de una función; también conocida como *cero*
rotation in three dimensions, a transformation in which a plane figure is moved about one of its sides, a fixed point, or a line that is not located in the plane of the figure, such that a solid figure is produced	U4B-193	**rotación** en tres dimensiones, transformación en la cual una figura plana se mueve sobre uno de sus lados, un punto fijo o una línea que no está ubicada en el plano de la figura, de manera que se produce una figura sólida
S		
sample a subset of the population	U1-3 U1-65	**muestra** subconjunto de la población

English		Español
sample average the sum of all quantities in a sample divided by the total number of quantities in the sample, typically represented by $\overline{x}$; also known as *sample mean*	U1-149	**promedio de la muestra** suma de todas las cantidades en una muestra dividida por el número total de cantidades en la muestra, normalmente representada por $\overline{x}$; también se conoce como *media de la muestra*
sample mean the sum of all quantities in a sample divided by the total number of quantities in the sample, typically represented by $\overline{x}$; also known as *sample average*	U1-149	**media de la muestra** suma de todas las cantidades en una muestra dividida por el número total de cantidades en la muestra, normalmente representada por $\overline{x}$; también se conoce como *promedio de la muestra*
sample population a portion of the population; the number of elements or observations in a sample population is represented by *n*	U1-149	**población de la muestra** porción de la población; la cantidad de elementos u observaciones en una población de muestra se representa por *n*
sample proportion the fraction of favorable results *p* from a sample population *n*; conventionally represented by $\hat{p}$, which is pronounced "p hat." The formula for the sample proportion is $\hat{p}=\dfrac{p}{n}$, where *p* is the number of favorable outcomes and *n* is the number of elements or observations in the sample population.	U1-149	**proporción de la muestra** fracción de los resultados favorables *p* de una población de muestra *n*; convencionalmente representada por $\hat{p}$, que se pronuncia "p hat". La fórmula para la proporción de la muestra es $\hat{p}=\dfrac{p}{n}$, donde *p* es la cantidad de resultados favorables y *n* es la cantidad de elementos u observaciones en la población de la muestra.

English		Español
sample survey a survey carried out using a sampling method so that only a portion of the population is surveyed rather than the whole population	U1-123	**encuesta de muestra** encuesta realizada utilizando un método de muestreo para encuestar solo una porción de la población en lugar de toda la población.
sampling bias errors in estimation caused by flawed (non-representative) sample selection	U1-65	**sesgo de muestreo** errores de cálculo ocasionados por una selección defectuosa (no representativa) de la muestra
sampling error a measure showing how precisely a sample reflects the population, with smaller sampling errors resulting from large samples and/or when the data clusters closely around the mean; also called *chance variation*	U1-65	**error de muestreo** medición que demuestra qué tan precisamente refleja una muestra a una población, con pequeños errores de muestreo ocasionados por muestras grandes y/o cuando los datos se agrupan estrechamente alrededor de la media; también se llama *variación aleatoria*
scale the numbers representing the interval of a variable and the increments into which it is subdivided; usually includes the interval endpoints and the increments of the basic unit of the variable	U4B-145	**escala** números que representan el intervalo de una variable y los incrementos en los cuales esta se subdivide; generalmente incluye los extremos del intervalo y los incrementos de la unidad básica de la variable
secant the reciprocal of cosine, $\sec\theta = \dfrac{1}{\cos\theta}$; the secant of $\theta = $ $\sec\theta = \dfrac{\text{length of hypotenuse}}{\text{length of adjacent side}}$	U3-2	**secante** recíproca del coseno, $\sec\theta = \dfrac{1}{\cos\theta}$; la secante de $\theta = $ $\sec\theta = \dfrac{\text{largo de la hipotenusa}}{\text{largo del lado adyacente}}$
sequence an ordered list of numbers or elements	U2A-117	**secuencia** lista ordenada de números o elementos
series the sum of the terms of a sequence	U2A-117	**serie** suma de los términos de una secuencia

English		Español
***sigma* (lowercase),** σ a Greek letter used to represent standard deviation	U1-3	***sigma* (minúscula) o** σ letra griega utilizada para representar la desviación estándar
***sigma* (uppercase),** Σ a Greek letter used to represent the summation of values	U1-3 U2A-21 U2A-117	***sigma* (mayúscula) o** Σ letra griega utilizada para representar la sumatoria de valores
similar figures two figures that are the same shape but not necessarily the same size; the symbol for representing similarity between figures is $\sim$.	U4B-193	**figuras similares** dos figuras que tienen la misma forma pero no necesariamente el mismo tamaño; el símbolo que representa la similitud entre figuras es $\sim$.
simple random sample a sample in which any combination of a given number of individuals in the population has an equal chance of selection	U1-65	**muestra aleatoria simple** muestra en la cual cualquier combinación de una cantidad dada de individuos de la población tiene iguales posibilidades de selección
simulation a set of data that models an event that could happen in real life	U1-198	**simulación** conjunto de datos que imita un evento que podría suceder en la vida real
sine a trigonometric function of an acute angle in a right triangle that is the ratio of the length of the opposite side to the length of the hypotenuse; the sine of $\theta = \sin \theta =$ $$\frac{\text{length of opposite side}}{\text{length of hypotenuse}}$$	U3-2 U3-59 U4A-115	**seno** función trigonométrica de un ángulo agudo en un triángulo rectángulo que es la proporción de la longitud del lado opuesto a la longitud de la hipotenusa; sen de $\theta =$ sen $\theta =$ $$\frac{\text{longitud del lado opuesto}}{\text{longitud de la hipotenusa}}$$
sine curve a curve with a constant amplitude and period, which are given by a sine or cosine function; also called a *sine wave* or *sinusoid*	U3-101	**curva del seno** curva con amplitud y período constantes que están dados por una función seno o coseno; también se denomina *onda de seno* o *sinusoide*

English		Español

sine function a trigonometric function of the form $f(x) = a \sin [b(x - c)] + d$, in which a, b, c, and d are constants and x is a variable defined in radians over the domain $(-\infty, \infty)$

función seno función trigonométrica de la forma $f(x) = a \sin [b(x - c)] + d$, en la cual a, b, c y d son constantes y x es una variable expresada en radianes sobre el dominio $(-\infty, \infty)$

sine wave a curve with a constant amplitude and period given by a sine or cosine function; also called a *sine curve* or *sinusoid*

onda senoidal curva con amplitud y período constantes dados por una función seno o coseno; también se denomina *curva del seno* o *sinusoide*

sinusoid a curve with a constant amplitude and period given by a sine or cosine function; also called a *sine curve* or *sine wave*

sinusoide curva con amplitud o período constantes dados por una función seno o coseno; también se denomina *curva del seno* u *onda senoidal*

skewed to the left a distribution in which there is a "tail" of isolated, spread-out data points to the left of the median. "Tail" describes the visual appearance of the data points in a histogram. Data that is skewed to the left is also called *negatively skewed*. Example:

sesgado a la izquierda distribución en la cual existe una "cola" de puntos de datos aislados extendidos hacia la izquierda de la mediana. La "cola" describe la apariencia de los puntos de datos en un histograma. Los datos sesgados a la izquierda también se denominan *negativamente sesgados*. Ejemplo:
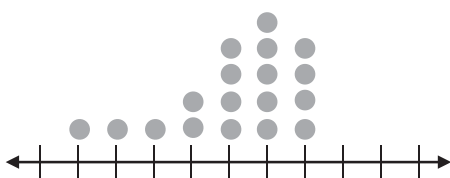
skew to distort or bias, as in data
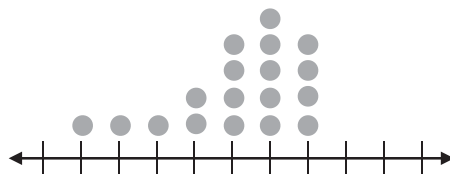
sesgar distorsionar o afectar, como en el caso de los datos

English		Español
skewed to the right a distribution in which there is a "tail" of isolated, spread-out data points to the right of the median. "Tail" describes the visual appearance of the data points in a histogram. Data that is skewed to the right is also called *positively skewed*. Example:	U1-3	**sesgado a la derecha** distribución en la cual existe una "cola" de puntos de datos aislados extendidos hacia la derecha de la mediana. La "cola" describe la apariencia de los puntos de datos en un histograma. Los datos sesgados a la derecha también se denominan *positivamente sesgados*. Ejemplo:
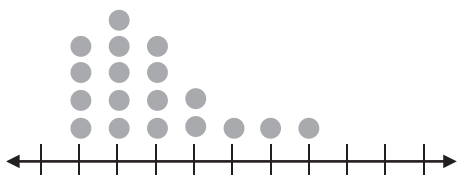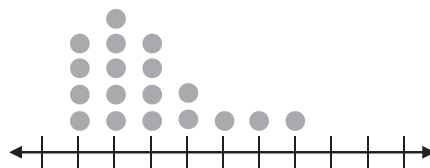
solid figure a three-dimensional object that has length, width, and height (depth)	U4B-193	**figura sólida** objeto tridimensional que tiene largo, ancho y altura (profundidad)
solution set the set of ordered pairs that represent all of the solutions to an equation or a system of equations	U2B-48 U4B-3	**conjunto de soluciones** conjunto de pares ordenados que representa todas las soluciones para una ecuación o sistema de ecuaciones
spread refers to how data is spread out with respect to the mean; sometimes called *variability*	U1-149	**dispersión** forma en que los datos se esparcen con respecto a la media; algunas veces se denomina *variabilidad*
square root function a function that contains a square root of a variable. The general form is $f(x)=\sqrt{ax^2+bx+c}$, where *a*, *b*, and *c* are real numbers.	U4B-145	**función raíz cuadrada** función que contiene la raíz cuadrada de una variable. La forma general es $f(x)=\sqrt{ax^2+bx+c}$, donde *a*, *b* y *c* son números reales.

English		Español
standard deviation how much the data in a given set is spread out, represented by s or σ. The standard deviation of a sample can be found using the following formula: $s = \sqrt{\dfrac{\sum \left(x_i - \overline{x} \right)^2}{n-1}}$. The standard deviation of a population can be found using the following formula: $\sigma = \sqrt{\dfrac{\sum\limits_{i=1}^{n} (x_i - \mu)^2}{n}}$.	U1-3 U1-149	**desviación estándar** cuánto se extienden los datos en un conjunto dado, representada por s o σ. Se puede calcular la desviación estándar de una muestra utilizando la siguiente fórmula: $s = \sqrt{\dfrac{\sum \left(x_i - \overline{x} \right)^2}{n-1}}$. Se puede calcular la desviación estándar de una población utilizando la siguiente fórmula: $\sigma = \sqrt{\dfrac{\sum\limits_{i=1}^{n} (x_i - \mu)^2}{n}}$.
standard error of the mean the variability of the mean of a sample; given by $\text{SEM} = \dfrac{s}{\sqrt{n}}$, where s represents the standard deviation and n is the number of elements or observations in the sample population	U1-149	**error estándar de la media** variabilidad de la media de una muestra; dado por $\text{SEM} = \dfrac{s}{\sqrt{n}}$, donde s representa la desviación estándar y n la cantidad de elementos u observaciones en la población de la muestra

English		Español
standard error of the	U1-150	**error estándar de la proporción**

proportion the variability of the measure of the proportion of a sample, abbreviated SEP. The standard error (*SEP*) of a sample proportion $\hat{p}$ is given by the formula $SEP = \sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$, where $\hat{p}$ is the sample proportion determined by the sample and n is the number of elements or observations in the sample population.

variabilidad de la medida de la proporción de una muestra, abreviada SEP. El error estándar (*SEP*) de una proporción de la muestra $\hat{p}$ está dado por la fórmula $SEP = \sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$, donde $\hat{p}$ es la proporción de la muestra determinada por la muestra y *n* representa la cantidad de elementos u observaciones en la población de la muestra.

standard normal distribution a normal distribution that has a mean of 0 and a standard deviation of 1; data following a standard normal distribution forms a normal curve when graphed   U1-3

distribución normal estándar distribución normal que tiene una media de 0 y una desviación estándar de 1; los datos que siguen una distribución normal estándar forman una curva normal al graficarse

standard position (of an angle) a position in which the vertex of the angle is at the origin of the coordinate plane and is the center of the unit circle. The angle's initial side is located along the positive *x*-axis and the terminal side may be in any location.   U3-3

posición estándar (de un ángulo) posición en la cual el vértice del ángulo está en el origen del plano de coordenadas y es el centro del círculo unitario. El lado inicial del ángulo está ubicado a lo largo del eje positivo *x* y el lado terminal puede estar en cualquier ubicación.

English		Español
statistical significance a measure used to determine whether the outcome of an experiment is a result of the treatment being applied, as opposed to random chance	U1-198	**relevancia estadística** medida utilizada para determinar si el resultado de un experimento es el resultado del tratamiento aplicado, en oposición al resultado producto del azar
statistics a branch of mathematics focusing on how to collect, organize, analyze, and interpret information from data gathered; numbers used to summarize, describe, or represent sets of data	U1-65 U1-123	**estadística** rama de la matemática enfocada en la manera de recabar, organizar, analizar e interpretar la información proveniente de los datos reunidos; números utilizados para resumir, describir o representar conjuntos de datos
step function a function that is a series of disconnected constant functions	U4B-145	**función escalonada** función que es una serie de funciones constantes desconectadas
stratified sample a sample chosen by first dividing a population into subgroups of people or objects that share relevant characteristics, then randomly selecting members of each subgroup for the sample	U1-65	**muestra estratificada** muestra escogida dividiendo primero una población en subgrupos de personas u objetos que comparten características relevantes, luego seleccionando al azar miembros de cada subgrupo para la muestra
subtended arc the section of an arc formed by a central angle that passes through the circle, thus creating the endpoints of the arc	U3-3	**arco subtendido** sección de un arco formada por un ángulo central que pasa por el círculo, creando así los puntos extremos del arco
success the data sought or hoped for, represented by p; also known as *desirable outcome* or *favorable outcome*	U1-150	**éxito** datos buscados o esperados, representados por p; también conocido como *resultado deseado* o *resultado favorable*

English		Español
sum formula for a finite geometric series $S_n = \dfrac{a_1(1-r^n)}{1-r}$, where S_n is the sum, a_1 is the first term, r is the common ratio, and n is the number of terms	U2A-118	**fórmula de suma para una serie geométrica finita** $S_n = \dfrac{a_1(1-r^n)}{1-r}$ donde S_n es la suma, a_1 es el primer término, r es la relación común y n es la cantidad de términos
sum formula for an infinite geometric series $S_n = \dfrac{a_1}{1-r}$, where S_n is the sum, a_1 is the first term, and r is the common ratio	U2A-118	**fórmula de suma para una serie geométrica infinita** $S_n = \dfrac{a_1}{1-r}$, donde S_n es la suma, a_1 es el primer término y r es la relación común
summation notation a symbolic way to represent a series (the sum of a sequence) using the uppercase Greek letter *sigma*, Σ	U1-3 U2A-118	**notación sumatoria** forma simbólica de representar una serie (la suma de una secuencia) utilizando la letra griega mayúscula *sigma*, Σ
survey a study of particular qualities or attributes of items or people of interest to a researcher	U1-123	**encuesta** estudio de las cualidades o atributos particulares de elementos o personas de interés para un investigador
symmetric distribution a data distribution in which a line can be drawn so that the left and right sides are mirror images of each other	U1-4	**distribución simétrica** distribución de datos en la cual se puede trazar una línea de manera que los lados derecho e izquierdo sean imágenes especulares entre sí
symmetry of a function the property whereby a function exhibits the same behavior (e.g., graph shape, function values, etc.) for specific domain values and their opposites	U4B-87	**simetría de una función** propiedad por la cual una función exhibe el mismo comportamiento (por ej., forma de la gráfica, valores de la función, etc.) para valores específicos del dominio y sus opuestos
synthetic division a shorthand way of dividing a polynomial by a linear binomial	U2A-49	**división sintética** forma abreviada de dividir un polinomio por un binomio lineal

English		Español
synthetic substitution the process of using synthetic division to evaluate a function by using only the coefficients	U2A-49	**sustitución sintética** proceso que utiliza la división sintética para evaluar una función utilizando solo los coeficientes
systematic sample a sample drawn by selecting people or objects from a list, chart, or grouping at a uniform interval; for example, selecting every fourth person	U1-65	**muestra sistemática** la muestra se obtiene mediante la selección de personas u objetos a partir de una lista, una tabla o mediante la agrupación a intervalos regulares; por ej., eligiendo una de cada cuatro personas
system of equations a set of equations with the same unknowns	U2A-96 U2B-48	**sistema de ecuaciones** conjunto de ecuaciones con las mismas incógnitas

T

English		Español
***t*-test** a procedure to establish the statistical significance of a set of data using the mean, standard deviation, and degrees of freedom for the sample or population	U1-199	**prueba *t*** procedimiento para establecer la relevancia estadística de un conjunto de datos utilizando la media, la desviación estándar y los grados de libertad para la muestra o población
***t*-value** the result of a *t*-test	U1-199	**valor *t*** resultado de una prueba *t*
tangent a trigonometric function of an acute angle in a right triangle that is the ratio of the length of the opposite side to the length of the adjacent side; the tangent of $\theta = \tan \theta = \dfrac{\text{length of opposite side}}{\text{length of adjacent side}}$	U3-3 U3-59	**tangente** función trigonométrica de un ángulo agudo en un triángulo rectángulo que es la proporción de la longitud del lado opuesto a la longitud del lado adyacente; tangente de $\theta = \tan \theta = \dfrac{\text{longitud del lado opuesto}}{\text{longitud del lado adyacente}}$

English		Español
term an element in a sequence. In the sequence $\{a_1, a_2, a_3, ..., a_n\}$, a_1 is the first term, a_2 is the second term, a_3 is the third term, and a_n is the nth term; a number, a variable, or the product of a number and variable(s)	U2A-2 U2A-118	**término** elemento de una secuencia. En la secuencia $\{a_1, a_2, a_3, ..., a_n\}$, a_1 es el primer término, a_2 es el segundo término, a_3 es el tercer término y a_n es el término n; un número, una variable o el producto de un número y variable(s)
terminal side for an angle in standard position, the movable ray of an angle that can be in any location and which determines the measure of the angle	U3-3	**lado terminal** para un ángulo en posición estándar, el rayo móvil de un ángulo que puede estar en cualquier ubicación y que determina la medida del ángulo
theoretical probability the probability that an outcome will occur as determined through reasoning or calculation, given by the formula $P(E) = \dfrac{\text{number of outcomes in } E}{\text{number of outcomes in the sample space}}$	U1-241	**probabilidad teórica** probabilidad de que un resultado se produzca como se determinó mediante razonamiento o cálculo, dado por la fórmula $P(E) = \dfrac{\text{cantidad de resultados en } E}{\text{cantidad de resultados en el espacio de muestreo}}$
theta (θ) a Greek letter commonly used to refer to unknown angle measures	U3-3	**_theta_ (θ)** letra griega generalmente utilizada para referirse a las medidas de un ángulo desconocido
translation in three dimensions, the horizontal or vertical movement of a plane figure in a direction that is not in the plane of the figure, such that a solid figure is produced	U4B-193	**traslación** en tres dimensiones, movimiento horizontal o vertical de una figura plana en una dirección que no está en el plano de la figura, de manera que se produzca una figura sólida
treatment the process or intervention provided to the population being observed	U1-199	**tratamiento** proceso o intervención efectuada sobre la población que está siendo observada
trial each individual event or selection in an experiment or treatment	U1-150 U1-199 U1-241	**ensayo** cada evento o selección individual en un experimento o tratamiento

English		Español
true negative result a determination that an experiment has produced a correct negative result	U1-241	**resultado negativo verdadero** determinación de que un experimento ha producido un resultado negativo correcto
true positive result a determination that an experiment has produced a correct positive result	U1-241	**resultado positivo verdadero** determinación de que un experimento ha producido un resultado positivo correcto
turning point a point where the graph of the function changes direction, from sloping upward to sloping downward or vice versa	U2A-49	**punto de inflexión** punto en el cual la gráfica de función cambia de dirección, de una inclinación o pendiente ascendente a una descendente o viceversa
two-tailed test a t-test performed on a set of data to determine if the data could belong in either of the tails of the bell-shaped distribution curve; with this test, the area under both tails of the distribution is considered	U1-199	**prueba de dos colas o prueba bilateral** prueba t realizada sobre un conjunto de datos para determinar si esos datos podrían pertenecer a alguna de las colas de una curva de distribución en forma de campana; con esta prueba, se tiene en cuenta el área bajo ambas colas de la distribución

U

English		Español
undesirable outcome the data not sought or hoped for, represented by q; also known as *unfavorable outcome* or *failure*	U1-150	**resultado no deseado** datos no buscados o esperados, representados por q; también conocido como *resultado desfavorable* o *fracaso*
unfavorable outcome the data not sought or hoped for, represented by q; also known as *undesirable outcome* or *failure*	U1-150	**resultado desfavorable** datos no buscados o esperados, representados por q; también conocido como *resultado no deseado* o *fracaso*

English		Español
uniform distribution a set of values that are continuous, are symmetric to a mean, and have equal frequencies corresponding to any two equally sized intervals. In other words, the values are spread out uniformly throughout the distribution.	U1-4	**distribución uniforme** conjunto de valores que son continuos, simétricos respecto de la media y tienen frecuencias iguales que corresponden a cualquiera de dos intervalos del mismo tamaño. En otras palabras, los valores se extienden uniformemente en la distribución.
unit circle a circle with a radius of 1 unit. The center of the circle is located at the origin of the coordinate plane.	U3-3 U4A-115	**círculo unitario** círculo con un radio de una unidad. El centro del círculo está ubicado en el origen del plano de coordenadas.

V

English		Español
validity the degree to which the results obtained from a sample measure what they are intended to measure	U1-65	**validez** el grado en el cual los resultados obtenidos de una muestra miden lo que se pretende que midan
variability refers to how data is spread out with respect to the mean; sometimes called *spread*	U1-150	**variabilidad** hace referencia al modo en que se distribuyen los datos respecto de la media; algunas veces se denomina *dispersión*
vertex a point at which two or more lines meet	U3-59	**vértice** punto en el que se encuentran dos o más líneas
voluntary response bias bias that occurs when the sample is not representative of the population due to the sample having the option of responding to the survey	U1-199	**sesgo de respuesta voluntaria** sesgo que se produce cuando la muestra no es representativa de la población debido a que en la muestra existe la opción de responder a la encuesta (la respuesta es optativa)

| English		Español
	Z	
z_c-value a measure of the number of standards of error to be added or subtracted from the mean in order to achieve the desired confidence level; also known as *critical value*	U1-150	**valor-z_c** medida de la cantidad de estándares de error a sumar o restar de la media para alcanzar el nivel de confianza deseado; conocido también como *valor crítico*
zero the *x*-intercept of a function; also known as *root*	U2A-49	**cero** el intercepto de *x* de una función; también se conoce como *raíz*
z-score the number of standard deviations that a score lies above or below the mean; given by the formula $z = \dfrac{x - \mu}{\sigma}$	U1-4	**puntuación z** cantidad de desviaciones estándar por encima o por debajo de la media que presenta la muestra; dada por la fórmula $z = \dfrac{x - \mu}{\sigma}$

Formulas

$f(x)$ $\angle ABC$ Q_1
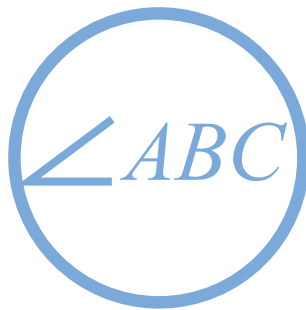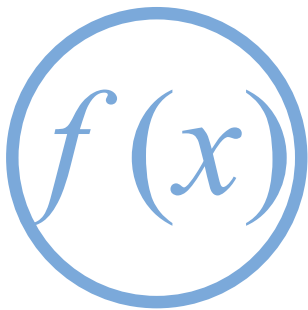
Formulas

ALGEBRA

Symbols

$\approx$	Approximately equal to
$\neq$	Is not equal to
$|a|$	Absolute value of a
$\sqrt{a}$	Square root of a
∞	Infinity
[	Inclusive on the lower bound
]	Inclusive on the upper bound
(	Non-inclusive on the lower bound
)	Non-inclusive on the upper bound
Σ	Sigma
Δ	Delta

General

(x, y)	Ordered pair
$(x, 0)$	x-intercept
$(0, y)$	y-intercept

Linear Equations

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$	Slope
$y = mx + b$	Slope-intercept form
$ax + by = c$	General form
$y - y_1 = m(x - x_1)$	Point-slope form

Exponential Equations

$A = P\left(1 + \dfrac{r}{n}\right)^{nt}$	Compounded interest formula
Compounded...	n (number of times per year)
Yearly/annually	1
Semiannually	2
Quarterly	4
Monthly	12
Weekly	52
Daily	365

Exponential Functions

$1 + r$	Growth factor
$1 - r$	Decay factor
$f(t) = a(1 + r)^t$	Exponential growth function
$f(t) = a(1 - r)^t$	Exponential decay function
$f(x) = ab^x$	Exponential function in general form

Binomial Theorem

$$\sum_{k=0}^{n} \frac{n!}{(n-k)!k!} \bullet a^{n-k}b^k = 1a^n b^0 + \frac{n}{1}a^{n-1}b^1 + \frac{n(n-1)}{1 \bullet 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \bullet 2 \bullet 3}a^{n-3}b^3 + \cdots + 1a^0 b^n$$

Functions	
$f(x)$	Function notation, "f of x"
$f^{-1}(x)$	Inverse function notation
$f(x) = mx + b$	Linear function
$f(x) = b^x + k$	Exponential function
$f(x) = ax^2 + b^x + c$	Quadratic function
$(f + g)(x) = f(x) + g(x)$	Addition
$(f - g)(x) = f(x) - g(x)$	Subtraction
$(f \bullet g)(x) = f(x) \bullet g(x)$	Multiplication
$(f \circ g)(x) = f(g(x))$	Composition
$\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$	Division
$\dfrac{f(b) - f(a)}{b - a}$	Average rate of change
$r = \dfrac{\Delta f(x)}{\Delta g(x)}$	Concise rate of change
$f(-x) = -f(x)$	Odd function
$f(-x) = f(x)$	Even function
$f(x) = \lfloor x \rfloor$	Floor/greatest integer function
$f(x) = \lceil x \rceil$	Ceiling/least integer function
$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$	Polynomial function
$f(x) = a\sqrt[3]{(x - h)} + k$	Cube root function
$f(x) = \sqrt[n]{(x - h)} + k$	Radical function
$f(x) = a|x - h| + k$	Absolute value function
$f(x) = \dfrac{p(x)}{q(x)};\quad q(x) \neq 0$	Rational function
$\log_c bx = c$	Logarithmic function

Formulas

Quadratic Functions and Equations

$x = \dfrac{-b}{2a}$	Axis of symmetry
$x = \dfrac{p+q}{2}$	Axis of symmetry using the midpoint of the x-intercepts
$\left(\dfrac{-b}{2a}, f\left(\dfrac{-b}{2a}\right)\right)$	Vertex
$f(x) = ax^2 + bx + c$	General form
$f(x) = a(x - h)^2 + k$	Vertex form
$f(x) = a(x - p)(x - q)$	Factored/intercept form
$b^2 - 4ac$	Discriminant
$x^2 + bx + \left(\dfrac{b}{2}\right)^2$	Perfect square trinomial
$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Quadratic formula

Properties of Exponents

Property	General rule
Zero Exponent	$a^0 = 1$
Negative Exponent	$b^{-\frac{m}{n}} = \dfrac{1}{b^{\frac{m}{n}}}$
Product of Powers	$a^m \bullet a^n = a^{m+n}$
Quotient of Powers	$\dfrac{a^m}{a^n} = a^{m-n}$
Power of a Power	$\left(b^m\right)^n = b^{mn}$
Power of a Product	$(bc)^n = b^n c^n$
Power of a Quotient	$\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$

Multiplication of Complex Conjugates

$(a + bi)(a - bi) = a^2 + b^2$

Common Polynomial Identities

$(a + b)^2 = a^2 + 2ab + b^2$	Square of Sums
$(a - b)^2 = a^2 - 2ab + b^2$	Square of Differences
$a^2 - b^2 = (a + b)(a - b)$	Difference of Two Squares
$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$	Sum of Two Cubes
$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$	Difference of Two Cubes

Radicals to Rational Exponents

$$\sqrt[n]{a} = a^{\frac{1}{n}}$$

$$\sqrt[n]{x^m} = x^{\frac{m}{n}}$$

Imaginary Numbers

$i = \sqrt{-1}$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

Logarithmic Functions

e	Base of a natural logarithm
$\log_a b = \dfrac{\log b}{\log a}$	Change of base formula
$\dfrac{2\pi}{b}$	Period
$-\dfrac{b}{a}$	Phase shift

Properties of Radicals

$$\sqrt{ab} = \sqrt{a} \bullet \sqrt{b}$$

$$\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$$

Series and Sequences

$r = \dfrac{a_n}{a_{n-1}}$	Common ratio
$a_n = a_1 \bullet r^{n-1}$	Explicit formula for a geometric sequence
$\displaystyle\sum_{k=1}^{n} a_1 r^{k-1}$	Finite geometric series
$\displaystyle\sum_{k=1}^{\infty} a_1 r^{k-1}$	Infinite geometric series
$a_n = a_{n-1} \bullet r$	Recursive formula for a geometric sequence
$S_n = \dfrac{a_1(1-r^n)}{1-r}$	Sum formula for a finite geometric series
$S_n = \dfrac{a_1}{1-r}$	Sum formula for an infinite geometric series
$P = \displaystyle\sum_{k=1}^{n} A\left(\dfrac{1}{1+i}\right)^{k-1}$	Amortization loan formula

Properties of Logarithms

Product property	$\log_a(x \bullet y) = \log_a x + \log_b y$
Quotient property	$\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$
Power property	$\log_a x^y = y \bullet \log_a x$

Formulas

STATISTICS AND DATA ANALYSIS

Symbols	
$\varnothing$	Empty/null set
$\cap$	Intersection, "and"
$\cup$	Union, "or"
$\subset$	Subset
$\overline{A}$	Complement of Set A
!	Factorial
${}_nC_r$	Combination
${}_nP_r$	Permutation
μ	Population mean
$\overline{x}$	Sample mean
σ	Standard deviation of a population
s	Standard deviation of a sample
$\hat{p}$	Sample proportion
SEM	Standard error of the mean
SEP	Standard error of the proportion
MOE	Margin of error
CI	Confidence interval
df	Degrees of freedom

Empirical Rule/68–95–99.7 Rule

$\mu \pm 1\sigma \approx 68\%$

$\mu \pm 2\sigma \approx 95\%$

$\mu \pm 3\sigma \approx 99.7\%$

Common Critical Values

| Confidence level	99%	98%	96%	95%	90%	80%	50%
Critical value (z_c)	2.58	2.33	2.05	1.96	1.645	1.28	0.6745

Formulas

Formulas	
$\mu = \dfrac{x_1 + x_2 + \cdots + x_n}{n}$	Mean of a population
$\bar{x} = \dfrac{x_1 + x_2 + \cdots + x_n}{n}$	Mean of a sample
$\sigma = \sqrt{\dfrac{\sum\limits_{i=1}^{n}(x_i - \mu)^2}{n}}$	Standard deviation of a population
$s = \sqrt{\dfrac{\sum\limits_{i=1}^{n}(x_i - \bar{x})^2}{n}}$	Standard deviation of a sample
$z = \dfrac{x - \mu}{\sigma}$	z-score
$\hat{p} = \dfrac{p}{n}$	Sample proportion
$SEM = \dfrac{s}{\sqrt{n}}$	Standard error of the mean
$SEP = \sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$	Standard error of the proportion
$MOE = \pm z_c \dfrac{s}{\sqrt{n}}$	Margin of error of a sample mean
$MOE = \pm z_c \sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$	Margin of error for a sample proportion
$CI = \hat{p} \pm z_c \sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$	Confidence interval for a sample population with proportion $\hat{p}$
$CI = \bar{x} \pm z_c \dfrac{s}{\sqrt{n}}$	Confidence interval for a sample population with mean $\bar{x}$
$t = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}}$	t-value for two sets of sample data
$t = \dfrac{\bar{x} - \mu_0}{\dfrac{s}{\sqrt{n}}}$	t-value for sample data and population
$df = \dfrac{n_1 - 1 + n_2 - 1}{2}$	Degrees of freedom

Rules and Equations

$P(E) = \dfrac{\text{\# of outcomes in } E}{\text{\# of outcomes in sample space}}$	Probability of event E
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$	Addition rule
$P(\overline{A}) = 1 - P(A)$	Complement rule
$P(B|A) = \dfrac{P(A \cap B)}{P(A)}$	Conditional probability
$E(X) = p_1 P(X_1) + p_2 P(X_2) + p_3 P(X_3)$	Expected value
$P(A \cap B) = P(A) \bullet P(B|A)$	Multiplication rule
$P(A \cap B) = P(A) \bullet P(B)$	Multiplication rule if A and B are independent
$_nC_r = \dfrac{n!}{(n-r)!\,r!}$	Combination
$_nP_r = \dfrac{n!}{(n-r)!}$	Permutation
$n! = n \bullet (n-1) \bullet (n-2) \bullet \ldots \bullet 1$	Factorial
$P = \begin{pmatrix} n \\ x \end{pmatrix} p^x q^{n-x}$	Binomial probability distribution

GEOMETRY

Unit Circle
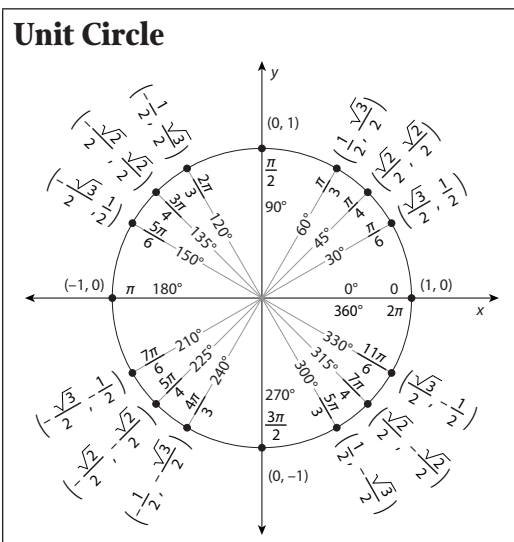
Formulas

Symbols

$\overset{\frown}{ABC}$	Major arc length
$\overset{\frown}{AB}$	Minor arc length
$\angle$	Angle
$\odot$	Circle
$\cong$	Congruent
$\overleftrightarrow{PQ}$	Line
$\overline{PQ}$	Line segment
$\overrightarrow{PQ}$	Ray
$\parallel$	Parallel
$\perp$	Perpendicular
$\bullet$	Point
$\triangle$	Triangle
$\square$	Parallelogram
A'	Prime
$°$	Degrees
θ	Theta
ϕ	Phi
π	Pi
ρ	Rho

Trigonometric Ratios

$\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$	$\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$	$\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$
$\csc\theta = \dfrac{\text{hypotenuse}}{\text{opposite}}$	$\sec\theta = \dfrac{\text{hypotenuse}}{\text{adjacent}}$	$\cot\theta = \dfrac{\text{adjacent}}{\text{opposite}}$

Volume

$V = lwh$	Rectangular prism
$V = Bh$	Prism
$V = \dfrac{1}{3}\pi r^2$	Cone
$V = \dfrac{1}{3}Bh$	Pyramid
$V = \pi r^2 h$	Cylinder
$V = \dfrac{4}{3}\pi r^3$	Sphere

Pythagorean Theorem

$a^2 + b^2 = c^2$

Trigonometric Identities

$$\sin\theta = \cos(90° - \theta)$$

$$\cos\theta = \sin(90° - \theta)$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$

Distance Formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Pi Defined

$$\pi = \frac{\text{circumference}}{\text{diameter}} = \frac{\text{circumference}}{2 \bullet \text{radius}}$$

Area

$A = lw$	Rectangle
$A = \dfrac{1}{2}bh$	Triangle
$A = \dfrac{1}{2}ab\sin C$	Triangle with unknown height
$A = \pi r^2$	Circle
$A = \dfrac{1}{2}(b_1 + b_2)h$	Trapezoid

Formulas

Circumference of a Circle

$C = 2\pi r$	Circumference given the radius
$C = \pi d$	Circumference given the diameter

Inverse Trigonometric Functions

Arcsin $\theta = \sin^{-1}\theta$
Arccos $\theta = \cos^{-1}\theta$
Arctan $\theta = \tan^{-1}\theta$

Converting Between Degrees and Radians

$$\frac{\text{radian measure}}{\pi} = \frac{\text{degree measure}}{180}$$

Arc Length

$s = \theta r$	Arc length (θ in radians)

Laws of Sines and Cosines

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$	Law of Sines
$c^2 = a^2 + b^2 - 2ab\cos C$	Law of Cosines

Density

$\text{Density}_{\text{Area}} = \dfrac{\text{mass or quantity}}{\text{number of } \textit{square} \text{ units}}$	or $\rho_A = \dfrac{m}{A}$	Area density
$\text{Density}_{\text{Volume}} = \dfrac{\text{mass or quantity}}{\text{number of } \textit{cubic} \text{ units}}$	or $\rho = \dfrac{m}{V}$	Volume density

MEASUREMENTS

Length
Metric
1 kilometer (km) = 1000 meters (m)
1 meter (m) = 100 centimeters (cm)
1 centimeter (cm) = 10 millimeters (mm)
Customary
1 mile (mi) = 1760 yards (yd)
1 mile (mi) = 5280 feet (ft)
1 yard (yd) = 3 feet (ft)
1 foot (ft) = 12 inches (in)

Volume and Capacity
Metric
1 liter (L) = 1000 milliliters (mL)
Customary
1 gallon (gal) = 4 quarts (qt)
1 quart (qt) = 2 pints (pt)
1 pint (pt) = 2 cups (c)
1 cup (c) = 8 fluid ounces (fl oz)

Weight and Mass
Metric
1 kilogram (kg) = 1000 grams (g)
1 gram (g) = 1000 milligrams (mg)
1 metric ton (MT) = 1000 kilograms
Customary
1 ton (T) = 2000 pounds (lb)
1 pound (lb) = 16 ounces (oz)