COMMUTATIVE ALGEBRA

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Contents

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1. Introduction

00AP Basic commutative algebra will be explained in this document. A reference is [Mat70].

2. Conventions

00AQ A ring is commutative with 1. The zero ring is a ring. In fact it is the only ring that does not have a prime ideal. The Kronecker symbol δ_{ij} will be used. If $R \to S$ is a ring map and $\mathfrak{q}$ a prime of S, then we use the notation "$\mathfrak{p} = R \cap \mathfrak{q}$" to indicate the prime which is the inverse image of $\mathfrak{q}$ under $R \to S$ even if R is not a subring of S and even if $R \to S$ is not injective.

3. Basic notions

00AR The following is a list of basic notions in commutative algebra. Some of these notions are discussed in more detail in the text that follows and some are defined in the list, but others are considered basic and will not be defined. If you are not familiar with most of the italicized concepts, then we suggest looking at an introductory text on algebra before continuing.

00AS (1) R is a *ring*,
00AT (2) $x \in R$ is *nilpotent*,
00AU (3) $x \in R$ is a *zerodivisor*,
00AV (4) $x \in R$ is a *unit*,
00AW (5) $e \in R$ is an *idempotent*,
00AX (6) an idempotent $e \in R$ is called *trivial* if $e = 1$ or $e = 0$,
00AY (7) $\varphi : R_1 \to R_2$ is a *ring homomorphism*,
00AZ (8) $\varphi : R_1 \to R_2$ is *of finite presentation*, or R_2 *is a finitely presented* R_1*-algebra*, see Definition 6.1,
00B0 (9) $\varphi : R_1 \to R_2$ is *of finite type*, or R_2 *is a finite type* R_1*-algebra*, see Definition 6.1,
00B1 (10) $\varphi : R_1 \to R_2$ is *finite*, or R_2 *is a finite* R_1*-algebra*,
00B2 (11) R is a *(integral) domain*,
00B3 (12) R is *reduced*,
00B4 (13) R is *Noetherian*,
00B5 (14) R is a *principal ideal domain* or a *PID*,
00B6 (15) R is a *Euclidean domain*,
00B7 (16) R is a *unique factorization domain* or a *UFD*,
00B8 (17) R is a *discrete valuation ring* or a *dvr*,
00B9 (18) K is a *field*,
00BA (19) L/K is a *field extension*,
00BB (20) L/K is an *algebraic field extension*,
00BC (21) $\{t_i\}_{i \in I}$ is a *transcendence basis* for L over K,
00BD (22) the *transcendence degree* $\text{trdeg}(L/K)$ of L over K,
00BE (23) the field k is *algebraically closed*,
00BF (24) if L/K is algebraic, and Ω/K an extension with Ω algebraically closed, then there exists a ring map $L \to \Omega$ extending the map on K,
00BG (25) $I \subset R$ is an *ideal*,
00BH (26) $I \subset R$ is *radical*,
00BI (27) if I is an ideal then we have its *radical* $\sqrt{I}$,
00BJ (28) $I \subset R$ is *nilpotent* means that $I^n = 0$ for some $n \in \mathbf{N}$,
0543 (29) $I \subset R$ is *locally nilpotent* means that every element of I is nilpotent,
00BK (30) $\mathfrak{p} \subset R$ is a *prime ideal*,
00BL (31) if $\mathfrak{p} \subset R$ is prime and if $I, J \subset R$ are ideal, and if $IJ \subset \mathfrak{p}$, then $I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$.
00BM (32) $\mathfrak{m} \subset R$ is a *maximal ideal*,
00BN (33) any nonzero ring has a maximal ideal,

00BO (34) the *Jacobson radical* of R is $\text{rad}(R) = \bigcap_{\mathfrak m \subset R} \mathfrak m$ the intersection of all the maximal ideals of R,
00BP (35) the ideal (T) *generated* by a subset $T \subset R$,
00BQ (36) the *quotient ring* R/I,
00BR (37) an ideal I in the ring R is prime if and only if R/I is a domain,
00BS (38) an ideal I in the ring R is maximal if and only if the ring R/I is a field,
00BT (39) if $\varphi : R_1 \to R_2$ is a ring homomorphism, and if $I \subset R_2$ is an ideal, then $\varphi^{-1}(I)$ is an ideal of R_1,
00BU (40) if $\varphi : R_1 \to R_2$ is a ring homomorphism, and if $I \subset R_1$ is an ideal, then $\varphi(I) \cdot R_2$ (sometimes denoted $I \cdot R_2$, or IR_2) is the ideal of R_2 generated by $\varphi(I)$,
00BV (41) if $\varphi : R_1 \to R_2$ is a ring homomorphism, and if $\mathfrak p \subset R_2$ is a prime ideal, then $\varphi^{-1}(\mathfrak p)$ is a prime ideal of R_1,
00BW (42) M is an *R-module*,
055Y (43) for $m \in M$ the *annihilator* $I = \{f \in R \mid fm = 0\}$ of m in R,
00BX (44) $N \subset M$ is an *R-submodule*,
00BY (45) M is an *Noetherian R-module*,
00BZ (46) M is a *finite R-module*,
00C0 (47) M is a *finitely generated R-module*,
00C1 (48) M is a *finitely presented R-module*,
00C2 (49) M is a *free R-module*,
0516 (50) if $0 \to K \to L \to M \to 0$ is a short exact sequence of R-modules and K, M are free, then L is free,
00C3 (51) if $N \subset M \subset L$ are R-modules, then $L/M = (L/N)/(M/N)$,
00C4 (52) S is a *multiplicative subset of R*,
00C5 (53) the *localization* $R \to S^{-1}R$ of R,
00C6 (54) if R is a ring and S is a multiplicative subset of R then $S^{-1}R$ is the zero ring if and only if S contains 0,
00C7 (55) if R is a ring and if the multiplicative subset S consists completely of nonzerodivisors, then $R \to S^{-1}R$ is injective,
(56) if $\varphi : R_1 \to R_2$ is a ring homomorphism, and S is a multiplicative subsets of R_1, then $\varphi(S)$ is a multiplicative subset of R_2,
00C8 (57) if S, S' are multiplicative subsets of R, and if SS' denotes the set of products $SS' = \{r \in R \mid \exists s \in S, \exists s' \in S', r = ss'\}$ then SS' is a multiplicative subset of R,
00C9 (58) if S, S' are multiplicative subsets of R, and if $\overline{S}$ denotes the image of S in $(S')^{-1}R$, then $(SS')^{-1}R = \overline{S}^{-1}((S')^{-1}R)$,
00CA (59) the *localization* $S^{-1}M$ of the R-module M,
00CB (60) the functor $M \mapsto S^{-1}M$ preserves injective maps, surjective maps, and exactness,
00CC (61) if S, S' are multiplicative subsets of R, and if M is an R-module, then $(SS')^{-1}M = S^{-1}((S')^{-1}M)$,
00CD (62) if R is a ring, I and ideal of R and S a multiplicative subset of R, then $S^{-1}I$ is an ideal of $S^{-1}R$, and we have $S^{-1}R/S^{-1}I = \overline{S}^{-1}(R/I)$, where $\overline{S}$ is the image of S in R/I,

00CE (63) if R is a ring, and S a multiplicative subset of R, then any ideal I' of $S^{-1}R$ is of the form $S^{-1}I$, where one can take I to be the inverse image of I' in R,

00CF (64) if R is a ring, M an R-module, and S a multiplicative subset of R, then any submodule N' of $S^{-1}M$ is of the form $S^{-1}N$ for some submodule $N \subset M$, where one can take N to be the inverse image of N' in M,

00CG (65) if $S = \{1, f, f^2, \ldots\}$ then $R_f = S^{-1}R$ and $M_f = S^{-1}M$,

00CH (66) if $S = R \setminus \mathfrak{p} = \{x \in R \mid x \notin \mathfrak{p}\}$ for some prime ideal $\mathfrak{p}$, then it is customary to denote $R_\mathfrak{p} = S^{-1}R$ and $M_\mathfrak{p} = S^{-1}M$,

00CI (67) a *local ring* is a ring with exactly one maximal ideal,

03C0 (68) a *semi-local ring* is a ring with finitely many maximal ideals,

00CJ (69) if $\mathfrak{p}$ is a prime in R, then $R_\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{p}R_\mathfrak{p}$,

00CK (70) the *residue field*, denoted $\kappa(\mathfrak{p})$, of the prime $\mathfrak{p}$ in the ring R is the field of fractions of the domain $R/\mathfrak{p}$; it is equal to $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} = (R \setminus \mathfrak{p})^{-1}R/\mathfrak{p}$,

00CL (71) given R and M_1, M_2 the *tensor product* $M_1 \otimes_R M_2$,

0F0K (72) given matrices A and B in a ring R of sizes $m \times n$ and $n \times m$ we have $\det(AB) = \sum \det(A_S)\det({}_SB)$ in R where the sum is over subsets $S \subset \{1, \ldots, n\}$ of size m and A_S is the $m \times m$ submatrix of A with columns corresponding to S and ${}_SB$ is the $m \times m$ submatrix of B with rows corresponding to S,

(73) etc.

4. Snake lemma

07JV The snake lemma and its variants are discussed in the setting of abelian categories in Homology, Section 5.

07JW **Lemma 4.1.** *Suppose given a commutative diagram* [CE56, III, Lemma 3.3]

$$\begin{tikzcd} & X \arrow[r] \arrow[d, "\alpha"] & Y \arrow[r] \arrow[d, "\beta"] & Z \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & U \arrow[r] & V \arrow[r] & W & \end{tikzcd}$$

of abelian groups with exact rows, then there is a canonical exact sequence

$$\mathrm{Ker}(\alpha) \to \mathrm{Ker}(\beta) \to \mathrm{Ker}(\gamma) \to \mathrm{Coker}(\alpha) \to \mathrm{Coker}(\beta) \to \mathrm{Coker}(\gamma)$$

Moreover, if $X \to Y$ is injective, then the first map is injective, and if $V \to W$ is surjective, then the last map is surjective.

Proof. The map $\partial : \mathrm{Ker}(\gamma) \to \mathrm{Coker}(\alpha)$ is defined as follows. Take $z \in \mathrm{Ker}(\gamma)$. Choose $y \in Y$ mapping to z. Then $\beta(y) \in V$ maps to zero in W. Hence $\beta(y)$ is the image of some $u \in U$. Set $\partial z = \overline{u}$ the class of u in the cokernel of α. Proof of exactness is omitted. $\square$

5. Finite modules and finitely presented modules

0517 Just some basic notation and lemmas.

0518 **Definition 5.1.** Let R be a ring. Let M be an R-module.

(1) We say M is a *finite R-module*, or a *finitely generated R-module* if there exist $n \in \mathbf{N}$ and $x_1, \ldots, x_n \in M$ such that every element of M is a R-linear combination of the x_i. Equivalently, this means there exists a surjection $R^{\oplus n} \to M$ for some $n \in \mathbf{N}$.
(2) We say M is a *finitely presented R-module* or an *R-module of finite presentation* if there exist integers $n, m \in \mathbf{N}$ and an exact sequence

$$R^{\oplus m} \longrightarrow R^{\oplus n} \longrightarrow M \longrightarrow 0$$

Informally, M is a finitely presented R-module if and only if it is finitely generated and the module of relations among these generators is finitely generated as well. A choice of an exact sequence as in the definition is called a *presentation* of M.

07JX **Lemma 5.2.** *Let R be a ring. Let $\alpha : R^{\oplus n} \to M$ and $\beta : N \to M$ be module maps. If $\mathrm{Im}(\alpha) \subset \mathrm{Im}(\beta)$, then there exists an R-module map $\gamma : R^{\oplus n} \to N$ such that $\alpha = \beta \circ \gamma$.*

Proof. Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the ith basis vector of $R^{\oplus n}$. Let $x_i \in N$ be an element with $\alpha(e_i) = \beta(x_i)$ which exists by assumption. Set $\gamma(a_1, \ldots, a_n) = \sum a_i x_i$. By construction $\alpha = \beta \circ \gamma$. □

0519 **Lemma 5.3.** *Let R be a ring. Let*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be a short exact sequence of R-modules.

(1) *If M_1 and M_3 are finite R-modules, then M_2 is a finite R-module.*
(2) *If M_1 and M_3 are finitely presented R-modules, then M_2 is a finitely presented R-module.*
(3) *If M_2 is a finite R-module, then M_3 is a finite R-module.*
(4) *If M_2 is a finitely presented R-module and M_1 is a finite R-module, then M_3 is a finitely presented R-module.*
(5) *If M_3 is a finitely presented R-module and M_2 is a finite R-module, then M_1 is a finite R-module.*

Proof. Proof of (1). If $x_1, \ldots, x_n$ are generators of M_1 and $y_1, \ldots, y_m \in M_2$ are elements whose images in M_3 are generators of M_3, then $x_1, \ldots, x_n, y_1, \ldots, y_m$ generate M_2.

Part (3) is immediate from the definition.

Proof of (5). Assume M_3 is finitely presented and M_2 finite. Choose a presentation

$$R^{\oplus m} \to R^{\oplus n} \to M_3 \to 0$$

By Lemma 5.2 there exists a map $R^{\oplus n} \to M_2$ such that the solid diagram

$$\begin{tikzcd} & R^{\oplus m} \arrow[r] \arrow[d, dotted] & R^{\oplus n} \arrow[r] \arrow[d] & M_3 \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & M_1 \arrow[r] & M_2 \arrow[r] & M_3 \arrow[r] & 0 \end{tikzcd}$$

commutes. This produces the dotted arrow. By the snake lemma (Lemma 4.1) we see that we get an isomorphism

$$\mathrm{Coker}(R^{\oplus m} \to M_1) \cong \mathrm{Coker}(R^{\oplus n} \to M_2)$$

In particular we conclude that $\mathrm{Coker}(R^{\oplus m} \to M_1)$ is a finite R-module. Since $\mathrm{Im}(R^{\oplus m} \to M_1)$ is finite by (3), we see that M_1 is finite by part (1).

Proof of (4). Assume M_2 is finitely presented and M_1 is finite. Choose a presentation $R^{\oplus m} \to R^{\oplus n} \to M_2 \to 0$. Choose a surjection $R^{\oplus k} \to M_1$. By Lemma 5.2 there exists a factorization $R^{\oplus k} \to R^{\oplus n} \to M_2$ of the composition $R^{\oplus k} \to M_1 \to M_2$. Then $R^{\oplus k+m} \to R^{\oplus n} \to M_3 \to 0$ is a presentation.

Proof of (2). Assume that M_1 and M_3 are finitely presented. The argument in the proof of part (1) produces a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & R^{\oplus n} \arrow[r] \arrow[d] & R^{\oplus n+m} \arrow[r] \arrow[d] & R^{\oplus m} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M_1 \arrow[r] & M_2 \arrow[r] & M_3 \arrow[r] & 0 \end{tikzcd}$$

with surjective vertical arrows. By the snake lemma we obtain a short exact sequence

$$0 \to \mathrm{Ker}(R^{\oplus n} \to M_1) \to \mathrm{Ker}(R^{\oplus n+m} \to M_2) \to \mathrm{Ker}(R^{\oplus m} \to M_3) \to 0$$

By part (5) we see that the outer two modules are finite. Hence the middle one is finite too. By (4) we see that M_2 is of finite presentation. $\square$

00KZ **Lemma 5.4.** *Let R be a ring, and let M be a finite R-module. There exists a filtration by R-submodules*

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that each quotient M_i/M_{i-1} is isomorphic to R/I_i for some ideal I_i of R.

Proof. By induction on the number of generators of M. Let $x_1, \ldots, x_r \in M$ be a minimal number of generators. Let $M' = Rx_1 \subset M$. Then M/M' has $r-1$ generators and the induction hypothesis applies. And clearly $M' \cong R/I_1$ with $I_1 = \{f \in R \mid fx_1 = 0\}$. $\square$

0560 **Lemma 5.5.** *Let $R \to S$ be a ring map. Let M be an S-module. If M is finite as an R-module, then M is finite as an S-module.*

Proof. In fact, any R-generating set of M is also an S-generating set of M, since the R-module structure is induced by the image of R in S. $\square$

6. Ring maps of finite type and of finite presentation

00F2

00F3 **Definition 6.1.** Let $R \to S$ be a ring map.

(1) We say $R \to S$ is of *finite type*, or that S *is a finite type R-algebra* if there exist an $n \in \mathbf{N}$ and an surjection of R-algebras $R[x_1, \ldots, x_n] \to S$.
(2) We say $R \to S$ is of *finite presentation* if there exist integers $n, m \in \mathbf{N}$ and polynomials $f_1, \ldots, f_m \in R[x_1, \ldots, x_n]$ and an isomorphism of R-algebras $R[x_1, \ldots, x_n]/(f_1, \ldots, f_m) \cong S$.

Informally, $R \to S$ is of finite presentation if and only if S is finitely generated as an R-algebra and the ideal of relations among the generators is finitely generated. A choice of a surjection $R[x_1, \ldots, x_n] \to S$ as in the definition is sometimes called a *presentation* of S.

00F4 **Lemma 6.2.** *The notions finite type and finite presentation have the following permanence properties.*

(1) *A composition of ring maps of finite type is of finite type.*
(2) *A composition of ring maps of finite presentation is of finite presentation.*
(3) *Given $R \to S' \to S$ with $R \to S$ of finite type, then $S' \to S$ is of finite type.*
(4) *Given $R \to S' \to S$, with $R \to S$ of finite presentation, and $R \to S'$ of finite type, then $S' \to S$ is of finite presentation.*

Proof. We only prove the last assertion. Write $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and $S' = R[y_1, \ldots, y_a]/I$. Say that the class $\bar{y}_i$ of y_i maps to h_i mod $(f_1, \ldots, f_m)$ in S. Then it is clear that $S = S'[x_1, \ldots, x_n]/(f_1, \ldots, f_m, h_1 - \bar{y}_1, \ldots, h_a - \bar{y}_a)$. □

00R2 **Lemma 6.3.** *Let $R \to S$ be a ring map of finite presentation. For any surjection $\alpha : R[x_1, \ldots, x_n] \to S$ the kernel of α is a finitely generated ideal in $R[x_1, \ldots, x_n]$.*

Proof. Write $S = R[y_1, \ldots, y_m]/(f_1, \ldots, f_k)$. Choose $g_i \in R[y_1, \ldots, y_m]$ which are lifts of $\alpha(x_i)$. Then we see that $S = R[x_i, y_j]/(f_l, x_i - g_i)$. Choose $h_j \in R[x_1, \ldots, x_n]$ such that $\alpha(h_j)$ corresponds to y_j mod $(f_1, \ldots, f_k)$. Consider the map $\psi : R[x_i, y_j] \to R[x_i]$, $x_i \mapsto x_i$, $y_j \mapsto h_j$. Then the kernel of α is the image of $(f_l, x_i - g_i)$ under ψ and we win. □

0561 **Lemma 6.4.** *Let $R \to S$ be a ring map. Let M be an S-module. Assume $R \to S$ is of finite type and M is finitely presented as an R-module. Then M is finitely presented as an S-module.*

Proof. This is similar to the proof of part (4) of Lemma 6.2. We may assume $S = R[x_1, \ldots, x_n]/J$. Choose $y_1, \ldots, y_m \in M$ which generate M as an R-module and choose relations $\sum a_{ij} y_j = 0$, $i = 1, \ldots, t$ which generate the kernel of $R^{\oplus m} \to M$. For any $i = 1, \ldots, n$ and $j = 1, \ldots, m$ write

$$x_i y_j = \sum a_{ijk} y_k$$

for some $a_{ijk} \in R$. Consider the S-module N generated by $y_1, \ldots, y_m$ subject to the relations $\sum a_{ij} y_j = 0$, $i = 1, \ldots, t$ and $x_i y_j = \sum a_{ijk} y_k$, $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Then N has a presentation

$$S^{\oplus nm+t} \longrightarrow S^{\oplus m} \longrightarrow N \longrightarrow 0$$

By construction there is a surjective map $\varphi : N \to M$. To finish the proof we show φ is injective. Suppose $z = \sum b_j y_j \in N$ for some $b_j \in S$. We may think of b_j as a polynomial in $x_1, \ldots, x_n$ with coefficients in R. By applying the relations of the form $x_i y_j = \sum a_{ijk} y_k$ we can inductively lower the degree of the polynomials. Hence we see that $z = \sum c_j y_j$ for some $c_j \in R$. Hence if $\varphi(z) = 0$ then the vector $(c_1, \ldots, c_m)$ is an R-linear combination of the vectors $(a_{i1}, \ldots, a_{im})$ and we conclude that $z = 0$ as desired. □

7. Finite ring maps

0562 Here is the definition.

0563 **Definition 7.1.** Let $\varphi : R \to S$ be a ring map. We say $\varphi : R \to S$ is *finite* if S is finite as an R-module.

00GJ **Lemma 7.2.** *Let $R \to S$ be a finite ring map. Let M be an S-module. Then M is finite as an R-module if and only if M is finite as an S-module.*

Proof. One of the implications follows from Lemma 5.5. To see the other assume that M is finite as an S-module. Pick $x_1, \ldots, x_n \in S$ which generate S as an R-module. Pick $y_1, \ldots, y_m \in M$ which generate M as an S-module. Then $x_i y_j$ generate M as an R-module. □

00GL **Lemma 7.3.** *Suppose that $R \to S$ and $S \to T$ are finite ring maps. Then $R \to T$ is finite.*

Proof. If t_i generate T as an S-module and s_j generate S as an R-module, then $t_i s_j$ generate T as an R-module. (Also follows from Lemma 7.2.) □

0D46 **Lemma 7.4.** *Let $\varphi : R \to S$ be a ring map.*

1. *If φ is finite, then φ is of finite type.*
2. *If S is of finite presentation as an R-module, then φ is of finite presentation.*

Proof. For (1) if $x_1, \ldots, x_n \in S$ generate S as an R-module, then $x_1, \ldots, x_n$ generate S as an R-algebra. For (2), suppose that $\sum r^i_j x_i = 0$, $j = 1, \ldots, m$ is a set of generators of the relations among the x_i when viewed as R-module generators of S. Furthermore, write $1 = \sum r_i x_i$ for some $r_i \in R$ and $x_i x_j = \sum r^k_{ij} x_k$ for some $r^k_{ij} \in R$. Then

$$S = R[t_1, \ldots, t_n]/(\sum r^i_j t_i,\ 1 - \sum r_i t_i,\ t_i t_j - \sum r^k_{ij} t_k)$$

as an R-algebra which proves (2). □

For more information on finite ring maps, please see Section 36.

8. Colimits

07N7 Some of the material in this section overlaps with the general discussion on colimits in Categories, Sections 14 – 21. The notion of a preordered set is defined in Categories, Definition 21.1. It is a slightly weaker notion than a partially ordered set.

00D4 **Definition 8.1.** Let $(I, \leq)$ be a preordered set. A *system (M_i, μ_{ij}) of R-modules over I* consists of a family of R-modules $\{M_i\}_{i \in I}$ indexed by I and a family of R-module maps $\{\mu_{ij} : M_i \to M_j\}_{i \leq j}$ such that for all $i \leq j \leq k$

$$\mu_{ii} = \mathrm{id}_{M_i} \quad \mu_{ik} = \mu_{jk} \circ \mu_{ij}$$

We say (M_i, μ_{ij}) is a *directed system* if I is a directed set.

This is the same as the notion defined in Categories, Definition 21.2 and Section 21. We refer to Categories, Definition 14.2 for the definition of a colimit of a diagram/system in any category.

00D5 **Lemma 8.2.** *Let (M_i, μ_{ij}) be a system of R-modules over the preordered set I. The colimit of the system (M_i, μ_{ij}) is the quotient R-module $(\bigoplus_{i \in I} M_i)/Q$ where Q is the R-submodule generated by all elements*

$$\iota_i(x_i) - \iota_j(\mu_{ij}(x_i))$$

where $\iota_i : M_i \to \bigoplus_{i \in I} M_i$ is the natural inclusion. We denote the colimit $M = \mathrm{colim}_i M_i$. We denote $\pi : \bigoplus_{i \in I} M_i \to M$ the projection map and $\phi_i = \pi \circ \iota_i : M_i \to M$.

Proof. This lemma is a special case of Categories, Lemma 14.12 but we will also prove it directly in this case. Namely, note that $\phi_i = \phi_j \circ \mu_{ij}$ in the above construction. To show the pair (M, ϕ_i) is the colimit we have to show it satisfies the universal property: for any other such pair (Y, ψ_i) with $\psi_i : M_i \to Y$, $\psi_i = \psi_j \circ \mu_{ij}$, there is a unique R-module homomorphism $g : M \to Y$ such that the following diagram commutes:

$$\begin{tikzcd}
M_i \arrow[rr, "\mu_{ij}"] \arrow[dr, "\phi_i"] \arrow[ddr, "\psi_i"'] & & M_j \arrow[dl, "\phi_j"'] \arrow[ddl, "\psi_j"] \\
& M \arrow[d, "g"] & \\
& Y &
\end{tikzcd}$$

And this is clear because we can define g by taking the map ψ_i on the summand M_i in the direct sum $\bigoplus M_i$. $\square$

00D6 **Lemma 8.3.** *Let (M_i, μ_{ij}) be a system of R-modules over the preordered set I. Assume that I is directed. The colimit of the system (M_i, μ_{ij}) is canonically isomorphic to the module M defined as follows:*

(1) *as a set let*
$$M = \left(\coprod\nolimits_{i \in I} M_i\right) / \sim$$
where for $m \in M_i$ and $m' \in M_{i'}$ we have
$$m \sim m' \Leftrightarrow \mu_{ij}(m) = \mu_{i'j}(m') \text{ for some } j \geq i, i'$$
(2) *as an abelian group for $m \in M_i$ and $m' \in M_{i'}$ we define the sum of the classes of m and m' in M to be the class of $\mu_{ij}(m) + \mu_{i'j}(m')$ where $j \in I$ is any index with $i \leq j$ and $i' \leq j$, and*
(3) *as an R-module define for $m \in M_i$ and $x \in R$ the product of x and the class of m in M to be the class of xm in M.*

The canonical maps $\phi_i : M_i \to M$ are induced by the canonical maps $M_i \to \coprod_{i \in I} M_i$.

Proof. Omitted. Compare with Categories, Section 19. $\square$

00D7 **Lemma 8.4.** *Let (M_i, μ_{ij}) be a directed system. Let $M = \operatorname{colim} M_i$ with $\mu_i : M_i \to M$. Then, $\mu_i(x_i) = 0$ for $x_i \in M_i$ if and only if there exists $j \geq i$ such that $\mu_{ij}(x_i) = 0$.*

Proof. This is clear from the description of the directed colimit in Lemma 8.3. $\square$

00D8 **Example 8.5.** Consider the partially ordered set $I = \{a, b, c\}$ with $a < b$ and $a < c$ and no other strict inequalities. A system $(M_a, M_b, M_c, \mu_{ab}, \mu_{ac})$ over I consists of three R-modules M_a, M_b, M_c and two R-module homomorphisms $\mu_{ab} : M_a \to M_b$ and $\mu_{ac} : M_a \to M_c$. The colimit of the system is just

$$M := \operatorname{colim}_{i \in I} M_i = \operatorname{Coker}(M_a \to M_b \oplus M_c)$$

where the map is $\mu_{ab} \oplus -\mu_{ac}$. Thus the kernel of the canonical map $M_a \to M$ is $\operatorname{Ker}(\mu_{ab}) + \operatorname{Ker}(\mu_{ac})$. And the kernel of the canonical map $M_b \to M$ is the image of $\operatorname{Ker}(\mu_{ac})$ under the map μ_{ab}. Hence clearly the result of Lemma 8.4 is false for general systems.

00D9 **Definition 8.6.** Let (M_i, μ_{ij}), (N_i, ν_{ij}) be systems of R-modules over the same preordered set I. A *homomorphism of systems* Φ from (M_i, μ_{ij}) to (N_i, ν_{ij}) is by definition a family of R-module homomorphisms $\phi_i : M_i \to N_i$ such that $\phi_j \circ \mu_{ij} = \nu_{ij} \circ \phi_i$ for all $i \leq j$.

This is the same notion as a transformation of functors between the associated diagrams $M : I \to \text{Mod}_R$ and $N : I \to \text{Mod}_R$, in the language of categories. The following lemma is a special case of Categories, Lemma 14.8.

00DA **Lemma 8.7.** *Let (M_i, μ_{ij}), (N_i, ν_{ij}) be systems of R-modules over the same preordered set. A morphism of systems $\Phi = (\phi_i)$ from (M_i, μ_{ij}) to (N_i, ν_{ij}) induces a unique homomorphism*

$$\operatorname{colim} \phi_i : \operatorname{colim} M_i \longrightarrow \operatorname{colim} N_i$$

such that

$$\begin{tikzcd} M_i \arrow[r] \arrow[d, "\phi_i"'] & \operatorname{colim} M_i \arrow[d, "\operatorname{colim} \phi_i"] \\ N_i \arrow[r] & \operatorname{colim} N_i \end{tikzcd}$$

commutes for all $i \in I$.

Proof. Write $M = \operatorname{colim} M_i$ and $N = \operatorname{colim} N_i$ and $\phi = \operatorname{colim} \phi_i$ (as yet to be constructed). We will use the explicit description of M and N in Lemma 8.2 without further mention. The condition of the lemma is equivalent to the condition that

$$\begin{tikzcd} \bigoplus_{i \in I} M_i \arrow[r] \arrow[d, "\bigoplus \phi_i"'] & M \arrow[d, "\phi"] \\ \bigoplus_{i \in I} N_i \arrow[r] & N \end{tikzcd}$$

commutes. Hence it is clear that if ϕ exists, then it is unique. To see that ϕ exists, it suffices to show that the kernel of the upper horizontal arrow is mapped by $\bigoplus \phi_i$ to the kernel of the lower horizontal arrow. To see this, let $j \leq k$ and $x_j \in M_j$. Then

$$(\bigoplus \phi_i)(x_j - \mu_{jk}(x_j)) = \phi_j(x_j) - \phi_k(\mu_{jk}(x_j)) = \phi_j(x_j) - \nu_{jk}(\phi_j(x_j))$$

which is in the kernel of the lower horizontal arrow as required. □

00DB **Lemma 8.8.** *Let I be a directed set. Let (L_i, λ_{ij}), (M_i, μ_{ij}), and (N_i, ν_{ij}) be systems of R-modules over I. Let $\varphi_i : L_i \to M_i$ and $\psi_i : M_i \to N_i$ be morphisms of systems over I. Assume that for all $i \in I$ the sequence of R-modules*

$$L_i \xrightarrow{\varphi_i} M_i \xrightarrow{\psi_i} N_i$$

is a complex with homology H_i. Then the R-modules H_i form a system over I, the sequence of R-modules

$$\operatorname{colim}_i L_i \xrightarrow{\varphi} \operatorname{colim}_i M_i \xrightarrow{\psi} \operatorname{colim}_i N_i$$

is a complex as well, and denoting H its homology we have

$$H = \operatorname{colim}_i H_i.$$

Proof. It is clear that $\operatorname{colim}_i L_i \xrightarrow{\varphi} \operatorname{colim}_i M_i \xrightarrow{\psi} \operatorname{colim}_i N_i$ is a complex. For each $i \in I$, there is a canonical R-module morphism $H_i \to H$ (sending each $[m] \in H_i = \operatorname{Ker}(\psi_i)/\operatorname{Im}(\varphi_i)$ to the residue class in $H = \operatorname{Ker}(\psi)/\operatorname{Im}(\varphi)$ of the image of m in $\operatorname{colim}_i M_i$). These give rise to a morphism $\operatorname{colim}_i H_i \to H$. It remains to show that this morphism is surjective and injective.

We are going to repeatedly use the description of colimits over I as in Lemma 8.3 without further mention. Let $h \in H$. Since $H = \operatorname{Ker}(\psi)/\operatorname{Im}(\varphi)$ we see that h is the class mod $\operatorname{Im}(\varphi)$ of an element $[m]$ in $\operatorname{Ker}(\psi) \subset \operatorname{colim}_i M_i$. Choose an i such that $[m]$ comes from an element $m \in M_i$. Choose a $j \geq i$ such that $\nu_{ij}(\psi_i(m)) = 0$ which is possible since $[m] \in \operatorname{Ker}(\psi)$. After replacing i by j and m by $\mu_{ij}(m)$ we see that we may assume $m \in \operatorname{Ker}(\psi_i)$. This shows that the map $\operatorname{colim}_i H_i \to H$ is surjective.

Suppose that $h_i \in H_i$ has image zero on H. Since $H_i = \operatorname{Ker}(\psi_i)/\operatorname{Im}(\varphi_i)$ we may represent h_i by an element $m \in \operatorname{Ker}(\psi_i) \subset M_i$. The assumption on the vanishing of h_i in H means that the class of m in $\operatorname{colim}_i M_i$ lies in the image of φ. Hence there exists a $j \geq i$ and an $l \in L_j$ such that $\varphi_j(l) = \mu_{ij}(m)$. Clearly this shows that the image of h_i in H_j is zero. This proves the injectivity of $\operatorname{colim}_i H_i \to H$. $\square$

00DC **Example 8.9.** Taking colimits is not exact in general. Consider the partially ordered set $I = \{a, b, c\}$ with $a < b$ and $a < c$ and no other strict inequalities, as in Example 8.5. Consider the map of systems $(0, \mathbf{Z}, \mathbf{Z}, 0, 0) \to (\mathbf{Z}, \mathbf{Z}, \mathbf{Z}, 1, 1)$. From the description of the colimit in Example 8.5 we see that the associated map of colimits is not injective, even though the map of systems is injective on each object. Hence the result of Lemma 8.8 is false for general systems.

04B0 **Lemma 8.10.** *Let $\mathcal{I}$ be an index category satisfying the assumptions of Categories, Lemma 19.8. Then taking colimits of diagrams of abelian groups over $\mathcal{I}$ is exact (i.e., the analogue of Lemma 8.8 holds in this situation).*

Proof. By Categories, Lemma 19.8 we may write $\mathcal{I} = \coprod_{j \in J} \mathcal{I}_j$ with each $\mathcal{I}_j$ a filtered category, and J possibly empty. By Categories, Lemma 21.5 taking colimits over the index categories $\mathcal{I}_j$ is the same as taking the colimit over some directed set. Hence Lemma 8.8 applies to these colimits. This reduces the problem to showing that coproducts in the category of R-modules over the set J are exact. In other words, exact sequences $L_j \to M_j \to N_j$ of R modules we have to show that

$$\bigoplus\nolimits_{j \in J} L_j \longrightarrow \bigoplus\nolimits_{j \in J} M_j \longrightarrow \bigoplus\nolimits_{j \in J} N_j$$

is exact. This can be verified by hand, and holds even if J is empty. $\square$

9. Localization

00CM

00CN **Definition 9.1.** Let R be a ring, S a subset of R. We say S is a *multiplicative subset of* R if $1 \in S$ and S is closed under multiplication, i.e., $s, s' \in S \Rightarrow ss' \in S$.

Given a ring A and a multiplicative subset S, we define a relation on $A \times S$ as follows:

$$(x, s) \sim (y, t) \Leftrightarrow \exists u \in S \text{ such that } (xt - ys)u = 0$$

It is easily checked that this is an equivalence relation. Let x/s (or $\frac{x}{s}$) be the equivalence class of (x, s) and $S^{-1}A$ be the set of all equivalence classes. Define addition and multiplication in $S^{-1}A$ as follows:

$$x/s + y/t = (xt + ys)/st, \quad x/s \cdot y/t = xy/st$$

One can check that $S^{-1}A$ becomes a ring under these operations.

00CO **Definition 9.2.** This ring is called the *localization of A with respect to S*.

We have a natural ring map from A to its localization $S^{-1}A$,

$$A \longrightarrow S^{-1}A, \quad x \longmapsto x/1$$

which is sometimes called the *localization map*. In general the localization map is not injective, unless S contains no zerodivisors. For, if $x/1 = 0$, then there is a $u \in S$ such that $xu = 0$ in A and hence $x = 0$ since there are no zerodivisors in S. The localization of a ring has the following universal property.

00CP **Proposition 9.3.** *Let $f : A \to B$ be a ring map that sends every element in S to a unit of B. Then there is a unique homomorphism $g : S^{-1}A \to B$ such that the following diagram commutes.*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr] & & B \\ & S^{-1}A \arrow[ur, "g"'] & \end{tikzcd}$$

Proof. Existence. We define a map g as follows. For $x/s \in S^{-1}A$, let $g(x/s) = f(x)f(s)^{-1} \in B$. It is easily checked from the definition that this is a well-defined ring map. And it is also clear that this makes the diagram commutative.

Uniqueness. We now show that if $g' : S^{-1}A \to B$ satisfies $g'(x/1) = f(x)$, then $g = g'$. Hence $f(s) = g'(s/1)$ for $s \in S$ by the commutativity of the diagram. But then $g'(1/s)f(s) = 1$ in B, which implies that $g'(1/s) = f(s)^{-1}$ and hence $g'(x/s) = g'(x/1)g'(1/s) = f(x)f(s)^{-1} = g(x/s)$. $\square$

00CQ **Lemma 9.4.** *The localization $S^{-1}A$ is the zero ring if and only if $0 \in S$.*

Proof. If $0 \in S$, any pair $(a, s) \sim (0, 1)$ by definition. If $0 \notin S$, then clearly $1/1 \neq 0/1$ in $S^{-1}A$. $\square$

07JY **Lemma 9.5.** *Let R be a ring. Let $S \subset R$ be a multiplicative subset. The category of $S^{-1}R$-modules is equivalent to the category of R-modules N with the property that every $s \in S$ acts as an automorphism on N.*

Proof. The functor which defines the equivalence associates to an $S^{-1}R$-module M the same module but now viewed as an R-module via the localization map $R \to S^{-1}R$. Conversely, if N is an R-module, such that every $s \in S$ acts via an automorphism s_N, then we can think of N as an $S^{-1}R$-module by letting x/s act via $x_N \circ s_N^{-1}$. We omit the verification that these two functors are quasi-inverse to each other. $\square$

The notion of localization of a ring can be generalized to the localization of a module. Let A be a ring, S a multiplicative subset of A and M an A-module. We define a relation on $M \times S$ as follows

$$(m, s) \sim (n, t) \Leftrightarrow \exists u \in S \text{ such that } (mt - ns)u = 0$$

This is clearly an equivalence relation. Denote by m/s (or $\frac{m}{s}$) be the equivalence class of (m, s) and $S^{-1}M$ be the set of all equivalence classes. Define the addition and scalar multiplication as follows

$$m/s + n/t = (mt + ns)/st, \quad m/s \cdot n/t = mn/st$$

It is clear that this makes $S^{-1}M$ an $S^{-1}A$ module.

07JZ **Definition 9.6.** The $S^{-1}A$-module $S^{-1}M$ is called the *localization* of M at S.

Note that there is an A-module map $M \to S^{-1}M$, $m \mapsto m/1$ which is sometimes called the *localization map*. It satisfies the following universal property.

07K0 **Lemma 9.7.** *Let R be a ring. Let $S \subset R$ a multiplicative subset. Let M, N be R-modules. Assume all the elements of S act as automorphisms on N. Then the canonical map*

$$\mathrm{Hom}_R(S^{-1}M, N) \longrightarrow \mathrm{Hom}_R(M, N)$$

induced by the localization map, is an isomorphism.

Proof. It is clear that the map is well-defined and R-linear. Injectivity: Let $\alpha \in \mathrm{Hom}_R(S^{-1}M, N)$ and take an arbitrary element $m/s \in S^{-1}M$. Then, since $s \cdot \alpha(m/s) = \alpha(m/1)$, we have $\alpha(m/s) = s^{-1}(\alpha(m/1))$, so α is completely determined by what it does on the image of M in $S^{-1}M$. Surjectivity: Let $\beta : M \to N$ be a given R-linear map. We need to show that it can be "extended" to $S^{-1}M$. Define a map of sets

$$M \times S \to N, \quad (m, s) \mapsto s^{-1}\beta(m)$$

Clearly, this map respects the equivalence relation from above, so it descends to a well-defined map $\alpha : S^{-1}M \to N$. It remains to show that this map is R-linear, so take $r, r' \in R$ as well as $s, s' \in S$ and $m, m' \in M$. Then

$$\begin{aligned} \alpha(r \cdot m/s + r' \cdot m'/s') &= \alpha((r \cdot s' \cdot m + r' \cdot s \cdot m')/(ss')) \\ &= (ss')^{-1}\beta(r \cdot s' \cdot m + r' \cdot s \cdot m') \\ &= (ss')^{-1}(r \cdot s'\beta(m) + r' \cdot s\beta(m')) \\ &= r\alpha(m/s) + r'\alpha(m'/s') \end{aligned}$$

and we win. □

02C5 **Example 9.8.** Let A be a ring and let M be an A-module. Here are some important examples of localizations.

(1) Given $\mathfrak{p}$ a prime ideal of A consider $S = A \setminus \mathfrak{p}$. It is immediately checked that S is a multiplicative set. In this case we denote $A_\mathfrak{p}$ and $M_\mathfrak{p}$ the localization of A and M with respect to S respectively. These are called the *localization of A, resp. M at $\mathfrak{p}$*.
(2) Let $f \in A$. Consider $S = \{1, f, f^2, \ldots\}$. This is clearly a multiplicative subset of A. In this case we denote A_f (resp. M_f) the localization $S^{-1}A$ (resp. $S^{-1}M$). This is called the *localization of A, resp. M with respect to f*. Note that $A_f = 0$ if and only if f is nilpotent in A.

(3) Let $S = \{f \in A \mid f \text{ is not a zerodivisor in } A\}$. This is a multiplicative subset of A. In this case the ring $Q(A) = S^{-1}A$ is called either the *total quotient ring*, or the *total ring of fractions* of A.
(4) If A is a domain, then the total quotient ring $Q(A)$ is the field of fractions of A. Please see Fields, Example 3.4.

00CR **Lemma 9.9.** *Let R be a ring. Let $S \subset R$ be a multiplicative subset. Let M be an R-module. Then*

$$S^{-1}M = \operatorname{colim}_{f \in S} M_f$$

where the preorder on S is given by $f \geq f' \Leftrightarrow f = f'f''$ for some $f'' \in R$ in which case the map $M_{f'} \to M_f$ is given by $m/(f')^e \mapsto m(f'')^e/f^e$.

Proof. Omitted. Hint: Use the universal property of Lemma 9.7. □

In the following paragraph, let A denote a ring, and M, N denote modules over A.

If S and S' are multiplicative sets of A, then it is clear that

$$SS' = \{ss' : s \in S,\ s' \in S'\}$$

is also a multiplicative set of A. Then the following holds.

02C6 **Proposition 9.10.** *Let $\overline{S}$ be the image of S in $S'^{-1}A$, then $(SS')^{-1}A$ is isomorphic to $\overline{S}^{-1}(S'^{-1}A)$.*

Proof. The map sending $x \in A$ to $x/1 \in (SS')^{-1}A$ induces a map sending $x/s \in S'^{-1}A$ to $x/s \in (SS')^{-1}A$, by universal property. The image of the elements in $\overline{S}$ are invertible in $(SS')^{-1}A$. By the universal property we get a map $f : \overline{S}^{-1}(S'^{-1}A) \to (SS')^{-1}A$ which maps $(x/t')/(s/s')$ to $(x/t') \cdot (s/s')^{-1}$.

On the other hand, the map from A to $\overline{S}^{-1}(S'^{-1}A)$ sending $x \in A$ to $(x/1)/(1/1)$ also induces a map $g : (SS')^{-1}A \to \overline{S}^{-1}(S'^{-1}A)$ which sends x/ss' to $(x/s')/(s/1)$, by the universal property again. It is immediately checked that f and g are inverse to each other, hence they are both isomorphisms. □

For the module M we have

02C7 **Proposition 9.11.** *View $S'^{-1}M$ as an A-module, then $S^{-1}(S'^{-1}M)$ is isomorphic to $(SS')^{-1}M$.*

Proof. Note that given a A-module M, we have not proved any universal property for $S^{-1}M$. Hence we cannot reason as in the preceding proof; we have to construct the isomorphism explicitly.

We define the maps as follows

$$f : S^{-1}(S'^{-1}M) \longrightarrow (SS')^{-1}M, \quad \frac{x/s'}{s} \mapsto x/ss'$$

$$g : (SS')^{-1}M \longrightarrow S^{-1}(S'^{-1}M), \quad x/t \mapsto \frac{x/s'}{s} \text{ for some } s \in S, s' \in S', \text{ and } t = ss'$$

We have to check that these homomorphisms are well-defined, that is, independent the choice of the fraction. This is easily checked and it is also straightforward to show that they are inverse to each other. □

If $u : M \to N$ is an A homomorphism, then the localization indeed induces a well-defined $S^{-1}A$ homomorphism $S^{-1}u : S^{-1}M \to S^{-1}N$ which sends x/s to $u(x)/s$. It is immediately checked that this construction is functorial, so that S^{-1} is actually a functor from the category of A-modules to the category of $S^{-1}A$-modules. Moreover this functor is exact, as we show in the following proposition.

00CS **Proposition 9.12.** *Let $L \xrightarrow{u} M \xrightarrow{v} N$ be an exact sequence of R-modules. Then $S^{-1}L \to S^{-1}M \to S^{-1}N$ is also exact.*

Proof. First it is clear that $S^{-1}L \to S^{-1}M \to S^{-1}N$ is a complex since localization is a functor. Next suppose that x/s maps to zero in $S^{-1}N$ for some $x/s \in S^{-1}M$. Then by definition there is a $t \in S$ such that $v(xt) = v(x)t = 0$ in M, which means $xt \in \mathrm{Ker}(v)$. By the exactness of $L \to M \to N$ we have $xt = u(y)$ for some y in L. Then x/s is the image of y/st. This proves the exactness. $\square$

02C8 **Lemma 9.13.** *Localization respects quotients, i.e. if N is a submodule of M, then $S^{-1}(M/N) \simeq (S^{-1}M)/(S^{-1}N)$.*

Proof. From the exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

we have

$$0 \longrightarrow S^{-1}N \longrightarrow S^{-1}M \longrightarrow S^{-1}(M/N) \longrightarrow 0$$

The corollary then follows. $\square$

If, in the preceding Corollary, we take $N = I$ and $M = A$ for an ideal I of A, we see that $S^{-1}A/S^{-1}I \simeq S^{-1}(A/I)$ as A-modules. The next proposition shows that they are isomorphic as rings.

00CT **Proposition 9.14.** *Let I be an ideal of A, S a multiplicative set of A. Then $S^{-1}I$ is an ideal of $S^{-1}A$ and $\overline{S}^{-1}(A/I)$ is isomorphic to $S^{-1}A/S^{-1}I$, where $\overline{S}$ is the image of S in A/I.*

Proof. The fact that $S^{-1}I$ is an ideal is clear since I itself is an ideal. Define

$$f : S^{-1}A \longrightarrow \overline{S}^{-1}(A/I), \quad x/s \mapsto \overline{x}/\overline{s}$$

where $\overline{x}$ and $\overline{s}$ are the images of x and s in A/I. We shall keep similar notations in this proof. This map is well-defined by the universal property of $S^{-1}A$, and $S^{-1}I$ is contained in the kernel of it, therefore it induces a map

$$\overline{f} : S^{-1}A/S^{-1}I \longrightarrow \overline{S}^{-1}(A/I), \quad \overline{x/s} \mapsto \overline{x}/\overline{s}$$

On the other hand, the map $A \to S^{-1}A/S^{-1}I$ sending x to $\overline{x/1}$ induces a map $A/I \to S^{-1}A/S^{-1}I$ sending $\overline{x}$ to $\overline{x/1}$. The image of $\overline{S}$ is invertible in $S^{-1}A/S^{-1}I$, thus induces a map

$$g : \overline{S}^{-1}(A/I) \longrightarrow S^{-1}A/S^{-1}I, \quad \frac{\overline{x}}{\overline{s}} \mapsto \overline{x/s}$$

by the universal property. It is then clear that $\overline{f}$ and g are inverse to each other, hence are both isomorphisms. $\square$

We now consider how submodules behave in localization.

00CU **Lemma 9.15.** *Any submodule N' of $S^{-1}M$ is of the form $S^{-1}N$ for some $N \subset M$. Indeed one can take N to be the inverse image of N' in M.*

Proof. Let N be the inverse image of N' in M. Then one can see that $S^{-1}N \supset N'$. To show they are equal, take x/s in $S^{-1}N$, where $s \in S$ and $x \in N$. This yields that $x/1 \in N'$. Since N' is an $S^{-1}R$-submodule we have $x/s = x/1 \cdot 1/s \in N'$. This finishes the proof. □

Taking $M = A$ and $N = I$ an ideal of A, we have the following corollary, which can be viewed as a converse of the first part of Proposition 9.14.

02C9 **Lemma 9.16.** *Each ideal I' of $S^{-1}A$ takes the form $S^{-1}I$, where one can take I to be the inverse image of I' in A.*

Proof. Immediate from Lemma 9.15. □

10. Internal Hom

0581 If R is a ring, and M, N are R-modules, then

$$\mathrm{Hom}_R(M, N) = \{\varphi : M \to N\}$$

is the set of R-linear maps from M to N. This set comes with the structure of an abelian group by setting $(\varphi+\psi)(m) = \varphi(m)+\psi(m)$, as usual. In fact, $\mathrm{Hom}_R(M, N)$ is also an R-module via the rule $(x\varphi)(m) = x\varphi(m) = \varphi(xm)$.

Given maps $a : M \to M'$ and $b : N \to N'$ of R-modules, we can pre-compose and post-compose homomorphisms by a and b. This leads to the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_R(M', N) \arrow[r, "b\circ-"'] \arrow[d, "-\circ a"'] & \mathrm{Hom}_R(M', N') \arrow[d, "-\circ a"] \\
\mathrm{Hom}_R(M, N) \arrow[r, "b\circ-"] & \mathrm{Hom}_R(M, N')
\end{tikzcd}$$

In fact, the maps in this diagram are R-module maps. Thus Hom_R defines an additive functor

$$\mathrm{Mod}_R^{opp} \times \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_R, \quad (M, N) \longmapsto \mathrm{Hom}_R(M, N)$$

0582 **Lemma 10.1.** *Exactness and Hom_R. Let R be a ring.*

1. *Let $M_1 \to M_2 \to M_3 \to 0$ be a complex of R-modules. Then $M_1 \to M_2 \to M_3 \to 0$ is exact if and only if $0 \to \mathrm{Hom}_R(M_3, N) \to \mathrm{Hom}_R(M_2, N) \to \mathrm{Hom}_R(M_1, N)$ is exact for all R-modules N.*
2. *Let $0 \to M_1 \to M_2 \to M_3$ be a complex of R-modules. Then $0 \to M_1 \to M_2 \to M_3$ is exact if and only if $0 \to \mathrm{Hom}_R(N, M_1) \to \mathrm{Hom}_R(N, M_2) \to \mathrm{Hom}_R(N, M_3)$ is exact for all R-modules N.*

Proof. Omitted. □

0583 **Lemma 10.2.** *Let R be a ring. Let M be a finitely presented R-module. Let N be an R-module.*

1. *For $f \in R$ we have $\mathrm{Hom}_R(M, N)_f = \mathrm{Hom}_{R_f}(M_f, N_f) = \mathrm{Hom}_R(M_f, N_f)$,*
2. *for a multiplicative subset S of R we have*

$$S^{-1}\mathrm{Hom}_R(M, N) = \mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N) = \mathrm{Hom}_R(S^{-1}M, S^{-1}N).$$

Proof. Part (1) is a special case of part (2). The second equality in (2) follows from Lemma 9.7. Choose a presentation

$$\bigoplus_{j=1,\ldots,m} R \longrightarrow \bigoplus_{i=1,\ldots,n} R \to M \to 0.$$

By Lemma 10.1 this gives an exact sequence

$$0 \to \mathrm{Hom}_R(M, N) \to \bigoplus_{i=1,\ldots,n} N \longrightarrow \bigoplus_{j=1,\ldots,m} N.$$

Inverting S and using Proposition 9.12 we get an exact sequence

$$0 \to S^{-1}\mathrm{Hom}_R(M, N) \to \bigoplus_{i=1,\ldots,n} S^{-1}N \longrightarrow \bigoplus_{j=1,\ldots,m} S^{-1}N$$

and the result follows since $S^{-1}M$ sits in an exact sequence

$$\bigoplus_{j=1,\ldots,m} S^{-1}R \longrightarrow \bigoplus_{i=1,\ldots,n} S^{-1}R \to S^{-1}M \to 0$$

which induces (by Lemma 10.1) the exact sequence

$$0 \to \mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N) \to \bigoplus_{i=1,\ldots,n} S^{-1}N \longrightarrow \bigoplus_{j=1,\ldots,m} S^{-1}N$$

which is the same as the one above. □

11. Characterizing finite and finitely presented modules

0G8M Given a module N over a ring R, you can characterize whether or not N is a finite module or a finitely presented module in terms of the functor $\mathrm{Hom}_R(N, -)$.

0G8N **Lemma 11.1.** *Let R be a ring. Let N be an R-module. The following are equivalent*

1. *N is a finite R-module,*
2. *for any filtered colimit $M = \mathrm{colim}\, M_i$ of R-modules the map $\mathrm{colim}\,\mathrm{Hom}_R(N, M_i) \to \mathrm{Hom}_R(N, M)$ is injective.*

Proof. Assume (1) and choose generators $x_1, \ldots, x_m$ for N. If $N \to M_i$ is a module map and the composition $N \to M_i \to M$ is zero, then because $M = \mathrm{colim}_{i' \geq i} M_{i'}$ for each $j \in \{1, \ldots, m\}$ we can find a $i' \geq i$ such that x_j maps to zero in $M_{i'}$. Since there are finitely many x_j we can find a single i' which works for all of them. Then the composition $N \to M_i \to M_{i'}$ is zero and we conclude the map is injective, i.e., part (2) holds.

Assume (2). For a finite subset $E \subset N$ denote $N_E \subset N$ the R-submodule generated by the elements of E. Then $0 = \mathrm{colim}\, N/N_E$ is a filtered colimit. Hence we see that $\mathrm{id} : N \to N$ maps into N_E for some E, i.e., N is finitely generated. □

For purposes of reference, we define what it means to have a relation between elements of a module.

07N8 **Definition 11.2.** Let R be a ring. Let M be an R-module. Let $n \geq 0$ and $x_i \in M$ for $i = 1, \ldots, n$. A *relation* between $x_1, \ldots, x_n$ in M is a sequence of elements $f_1, \ldots, f_n \in R$ such that $\sum_{i=1,\ldots,n} f_i x_i = 0$.

00HA **Lemma 11.3.** *Let R be a ring and let M be an R-module. Then M is the colimit of a directed system (M_i, μ_{ij}) of R-modules with all M_i finitely presented R-modules.*

Proof. Consider any finite subset $S \subset M$ and any finite collection of relations E among the elements of S. So each $s \in S$ corresponds to $x_s \in M$ and each $e \in E$ consists of a vector of elements $f_{e,s} \in R$ such that $\sum f_{e,s} x_s = 0$. Let $M_{S,E}$ be the cokernel of the map

$$R^{\#E} \longrightarrow R^{\#S}, \quad (g_e)_{e \in E} \longmapsto (\sum g_e f_{e,s})_{s \in S}.$$

There are canonical maps $M_{S,E} \to M$. If $S \subset S'$ and if the elements of E correspond, via this map, to relations in E', then there is an obvious map $M_{S,E} \to M_{S',E'}$ commuting with the maps to M. Let I be the set of pairs (S, E) with ordering by inclusion as above. It is clear that the colimit of this directed system is M. $\square$

0G8P **Lemma 11.4.** *Let R be a ring. Let N be an R-module. The following are equivalent*

1. *N is a finitely presented R-module,*
2. *for any filtered colimit $M = \mathrm{colim}\, M_i$ of R-modules the map $\mathrm{colim}\, \mathrm{Hom}_R(N, M_i) \to \mathrm{Hom}_R(N, M)$ is bijective.*

Proof. Assume (1) and choose an exact sequence $F_{-1} \to F_0 \to N \to 0$ with F_i finite free. Then we have an exact sequence

$$0 \to \mathrm{Hom}_R(N, M) \to \mathrm{Hom}_R(F_0, M) \to \mathrm{Hom}_R(F_{-1}, M)$$

functorial in the R-module M. The functors $\mathrm{Hom}_R(F_i, M)$ commute with filtered colimits as $\mathrm{Hom}_R(R^{\oplus n}, M) = M^{\oplus n}$. Since filtered colimits are exact (Lemma 8.8) we see that (2) holds.

Assume (2). By Lemma 11.3 we can write $M = \mathrm{colim}\, M_i$ as a filtered colimit such that M_i is of finite presentation for all i. Thus id_M factors through M_i for some i. This means that M is a direct summand of a finitely presented R-module (namely M_i) and hence finitely presented. $\square$

12. Tensor products

00CV

00CW **Definition 12.1.** Let R be a ring, M, N, P be three R-modules. A mapping $f : M \times N \to P$ (where $M \times N$ is viewed only as Cartesian product of two R-modules) is said to be *R-bilinear* if for each $x \in M$ the mapping $y \mapsto f(x, y)$ of N into P is R-linear, and for each $y \in N$ the mapping $x \mapsto f(x, y)$ is also R-linear.

00CX **Lemma 12.2.** *Let M, N be R-modules. Then there exists a pair (T, g) where T is an R-module, and $g : M \times N \to T$ an R-bilinear mapping, with the following universal property: For any R-module P and any R-bilinear mapping $f : M \times N \to P$, there exists a unique R-linear mapping $\tilde{f} : T \to P$ such that $f = \tilde{f} \circ g$. In other words, the following diagram commutes:*

$$\begin{tikzcd}
M \times N \arrow[rr, "f"] \arrow[dr, "g"'] & & P \\
& T \arrow[ur, "\tilde{f}"'] &
\end{tikzcd}$$

Moreover, if (T, g) and (T', g') are two pairs with this property, then there exists a unique isomorphism $j : T \to T'$ such that $j \circ g = g'$.

The R-module T which satisfies the above universal property is called the *tensor product* of R-modules M and N, denoted as $M \otimes_R N$.

Proof. We first prove the existence of such R-module T. Let M, N be R-modules. Let T be the quotient module P/Q, where P is the free R-module $R^{(M \times N)}$ and Q is the R-module generated by all elements of the following types: $(x \in M, y \in N)$

$$\begin{aligned}(x + x', y) - (x, y) - (x', y),\\ (x, y + y') - (x, y) - (x, y'),\\ (ax, y) - a(x, y),\\ (x, ay) - a(x, y)\end{aligned}$$

Let $\pi : M \times N \to T$ denote the natural map. This map is R-bilinear, as implied by the above relations when we check the bilinearity conditions. Denote the image $\pi(x, y) = x \otimes y$, then these elements generate T. Now let $f : M \times N \to P$ be an R-bilinear map, then we can define $f' : T \to P$ by extending the mapping $f'(x \otimes y) = f(x, y)$. Clearly $f = f' \circ \pi$. Moreover, f' is uniquely determined by the value on the generating sets $\{x \otimes y : x \in M, y \in N\}$. Suppose there is another pair (T', g') satisfying the same properties. Then there is a unique $j : T \to T'$ and also $j' : T' \to T$ such that $g' = j \circ g$, $g = j' \circ g'$. But then both the maps $(j \circ j') \circ g$ and g satisfies the universal properties, so by uniqueness they are equal, and hence $j' \circ j$ is identity on T. Similarly $(j' \circ j) \circ g' = g'$ and $j \circ j'$ is identity on T'. So j is an isomorphism. ☐

00CY **Lemma 12.3.** *Let M, N, P be R-modules, then the bilinear maps*

$$\begin{aligned}(x, y) &\mapsto y \otimes x\\ (x + y, z) &\mapsto x \otimes z + y \otimes z\\ (r, x) &\mapsto rx\end{aligned}$$

induce unique isomorphisms

$$\begin{aligned}M \otimes_R N &\to N \otimes_R M,\\ (M \oplus N) \otimes_R P &\to (M \otimes_R P) \oplus (N \otimes_R P),\\ R \otimes_R M &\to M\end{aligned}$$

Proof. Omitted. ☐

We may generalize the tensor product of two R-modules to finitely many R-modules, and set up a correspondence between the multi-tensor product with multilinear mappings. Using almost the same construction one can prove that:

00CZ **Lemma 12.4.** *Let $M_1, \ldots, M_r$ be R-modules. Then there exists a pair (T, g) consisting of an R-module T and an R-multilinear mapping $g : M_1 \times \ldots \times M_r \to T$ with the universal property: For any R-multilinear mapping $f : M_1 \times \ldots \times M_r \to P$ there exists a unique R-module homomorphism $f' : T \to P$ such that $f' \circ g = f$. Such a module T is unique up to unique isomorphism. We denote it $M_1 \otimes_R \ldots \otimes_R M_r$ and we denote the universal multilinear map $(m_1, \ldots, m_r) \mapsto m_1 \otimes \ldots \otimes m_r$.*

Proof. Omitted. ☐

00D0 **Lemma 12.5.** *The homomorphisms*

$$(M \otimes_R N) \otimes_R P \to M \otimes_R N \otimes_R P \to M \otimes_R (N \otimes_R P)$$

such that $f((x \otimes y) \otimes z) = x \otimes y \otimes z$ *and* $g(x \otimes y \otimes z) = x \otimes (y \otimes z)$, $x \in M, y \in N, z \in P$ *are well-defined and are isomorphisms.*

Proof. We shall prove f is well-defined and is an isomorphism, and this proof carries analogously to g. Fix any $z \in P$, then the mapping $(x, y) \mapsto x \otimes y \otimes z$, $x \in M, y \in N$, is R-bilinear in x and y, and hence induces homomorphism $f_z : M \otimes N \to M \otimes N \otimes P$ which sends $f_z(x \otimes y) = x \otimes y \otimes z$. Then consider $(M \otimes N) \times P \to M \otimes N \otimes P$ given by $(w, z) \mapsto f_z(w)$. The map is R-bilinear and thus induces $f : (M \otimes_R N) \otimes_R P \to M \otimes_R N \otimes_R P$ and $f((x \otimes y) \otimes z) = x \otimes y \otimes z$. To construct the inverse, we note that the map $\pi : M \times N \times P \to (M \otimes N) \otimes P$ is R-trilinear. Therefore, it induces an R-linear map $h : M \otimes N \otimes P \to (M \otimes N) \otimes P$ which agrees with the universal property. Here we see that $h(x \otimes y \otimes z) = (x \otimes y) \otimes z$. From the explicit expression of f and h, $f \circ h$ and $h \circ f$ are identity maps of $M \otimes N \otimes P$ and $(M \otimes N) \otimes P$ respectively, hence f is our desired isomorphism. $\square$

Doing induction we see that this extends to multi-tensor products. Combined with Lemma 12.3 we see that the tensor product operation on the category of R-modules is associative, commutative and distributive.

00D1 **Definition 12.6.** An abelian group N is called an (A, B)*-bimodule* if it is both an A-module and a B-module, and the actions $A \to End(M)$ and $B \to End(M)$ are compatible in the sense that $(ax)b = a(xb)$ for all $a \in A, b \in B, x \in N$. Usually we denote it as ${}_AN_B$.

00D2 **Lemma 12.7.** *For A-module M, B-module P and (A, B)-bimodule N, the modules $(M \otimes_A N) \otimes_B P$ and $M \otimes_A (N \otimes_B P)$ can both be given (A, B)-bimodule structure, and moreover*

$$(M \otimes_A N) \otimes_B P \cong M \otimes_A (N \otimes_B P).$$

Proof. A priori $M \otimes_A N$ is an A-module, but we can give it a B-module structure by letting

$$(x \otimes y)b = x \otimes yb, \quad x \in M, y \in N, b \in B$$

Thus $M \otimes_A N$ becomes an (A, B)-bimodule. Similarly for $N \otimes_B P$, and thus for $(M \otimes_A N) \otimes_B P$ and $M \otimes_A (N \otimes_B P)$. By Lemma 12.5, these two modules are isomorphic as both as A-module and B-module via the same mapping. $\square$

00DE **Lemma 12.8.** *For any three R-modules M, N, P,*

$$\mathrm{Hom}_R(M \otimes_R N, P) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$$

Proof. An R-linear map $\hat{f} \in \mathrm{Hom}_R(M \otimes_R N, P)$ corresponds to an R-bilinear map $f : M \times N \to P$. For each $x \in M$ the mapping $y \mapsto f(x, y)$ is R-linear by the universal property. Thus f corresponds to a map $\phi_f : M \to \mathrm{Hom}_R(N, P)$. This map is R-linear since

$$\phi_f(ax + y)(z) = f(ax + y, z) = af(x, z) + f(y, z) = (a\phi_f(x) + \phi_f(y))(z),$$

for all $a \in R$, $x \in M$, $y \in M$ and $z \in N$. Conversely, any $f \in \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$ defines an R-bilinear map $M \times N \to P$, namely $(x, y) \mapsto f(x)(y)$. So this is a natural one-to-one correspondence between the two modules $\mathrm{Hom}_R(M \otimes_R N, P)$ and $\mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$. $\square$

00DD **Lemma 12.9** (Tensor products commute with colimits). *Let (M_i, μ_{ij}) be a system over the preordered set I. Let N be an R-module. Then*

$$\mathrm{colim}(M_i \otimes N) \cong (\mathrm{colim}\, M_i) \otimes N.$$

Moreover, the isomorphism is induced by the homomorphisms $\mu_i \otimes 1 : M_i \otimes N \to M \otimes N$ where $M = \mathrm{colim}_i M_i$ with natural maps $\mu_i : M_i \to M$.

Proof. First proof. The functor $M' \mapsto M' \otimes_R N$ is left adjoint to the functor $N' \mapsto \mathrm{Hom}_R(N, N')$ by Lemma 12.8. Thus $M' \mapsto M' \otimes_R N$ commutes with all colimits, see Categories, Lemma 24.5.

Second direct proof. Let $P = \mathrm{colim}(M_i \otimes N)$ with coprojections $\lambda_i : M_i \otimes N \to P$. Let $M = \mathrm{colim}\, M_i$ with coprojections $\mu_i : M_i \to M$. Then for all $i \leq j$, the following diagram commutes:

$$\begin{tikzcd} M_i \otimes N \arrow[r, "\mu_i \otimes 1"'] \arrow[d, "\mu_{ij} \otimes 1"'] & M \otimes N \arrow[d, "\mathrm{id}"] \\ M_j \otimes N \arrow[r, "\mu_j \otimes 1"] & M \otimes N \end{tikzcd}$$

By Lemma 8.7 these maps induce a unique homomorphism $\psi : P \to M \otimes N$ such that $\lambda_i = \psi \circ (\mu_i \otimes 1)$.

To construct the inverse map, for each $i \in I$, there is the canonical R-bilinear mapping $g_i : M_i \times N \to M_i \otimes N$. This induces a unique mapping $\widehat{\phi} : M \times N \to P$ such that $\widehat{\phi} \circ (\mu_i \times 1) = \lambda_i \circ g_i$. It is R-bilinear. Thus it induces an R-linear mapping $\phi : M \otimes N \to P$. From the commutative diagram below:

$$\begin{tikzcd} M_i \times N \arrow[r, "g_i"] \arrow[d, "\mu_i \times \mathrm{id}"] & M_i \otimes N \arrow[r, "\mathrm{id}"'] \arrow[d, "\lambda_i"'] & M_i \otimes N \arrow[d, "\mu_i \otimes \mathrm{id}"'] \arrow[dr, "\lambda_i"] & \\ M \times N \arrow[r, "\widehat{\phi}"] & P \arrow[r, "\psi"] & M \otimes N \arrow[r, "\phi"] & P \end{tikzcd}$$

we see that $\psi \circ \widehat{\phi} = g$, the canonical R-bilinear mapping $g : M \times N \to M \otimes N$. So $\psi \circ \phi$ is identity on $M \otimes N$. From the right-hand square and triangle, $\phi \circ \psi$ is also identity on P. $\square$

00DF **Lemma 12.10.** *Let*

$$M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$$

be an exact sequence of R-modules and homomorphisms, and let N be any R-module. Then the sequence

00DG (12.10.1) $$M_1 \otimes N \xrightarrow{f \otimes 1} M_2 \otimes N \xrightarrow{g \otimes 1} M_3 \otimes N \to 0$$

is exact. In other words, the functor $- \otimes_R N$ is right exact, in the sense that tensoring each term in the original right exact sequence preserves the exactness.

Proof. We apply the functor $\mathrm{Hom}(-, \mathrm{Hom}(N, P))$ to the first exact sequence. We obtain

$$0 \to \mathrm{Hom}(M_3, \mathrm{Hom}(N, P)) \to \mathrm{Hom}(M_2, \mathrm{Hom}(N, P)) \to \mathrm{Hom}(M_1, \mathrm{Hom}(N, P))$$

By Lemma 12.8, we have

$$0 \to \mathrm{Hom}(M_3 \otimes N, P) \to \mathrm{Hom}(M_2 \otimes N, P) \to \mathrm{Hom}(M_1 \otimes N, P)$$

Using the pullback property again, we arrive at the desired exact sequence. □

00DH **Remark 12.11.** However, tensor product does NOT preserve exact sequences in general. In other words, if $M_1 \to M_2 \to M_3$ is exact, then it is not necessarily true that $M_1 \otimes N \to M_2 \otimes N \to M_3 \otimes N$ is exact for arbitrary R-module N.

00DI **Example 12.12.** Consider the injective map $2 : \mathbf{Z} \to \mathbf{Z}$ viewed as a map of $\mathbf{Z}$-modules. Let $N = \mathbf{Z}/2$. Then the induced map $\mathbf{Z} \otimes \mathbf{Z}/2 \to \mathbf{Z} \otimes \mathbf{Z}/2$ is NOT injective. This is because for $x \otimes y \in \mathbf{Z} \otimes \mathbf{Z}/2$,

$$(2 \otimes 1)(x \otimes y) = 2x \otimes y = x \otimes 2y = x \otimes 0 = 0$$

Therefore the induced map is the zero map while $\mathbf{Z} \otimes N \neq 0$.

00DJ **Remark 12.13.** For R-modules N, if the functor $- \otimes_R N$ is exact, i.e. tensoring with N preserves all exact sequences, then N is said to be *flat* R-module. We will discuss this later in Section 39.

05BS **Lemma 12.14.** *Let R be a ring. Let M and N be R-modules.*

(1) *If N and M are finite, then so is $M \otimes_R N$.*
(2) *If N and M are finitely presented, then so is $M \otimes_R N$.*

Proof. Suppose M is finite. Then choose a presentation $0 \to K \to R^{\oplus n} \to M \to 0$. This gives an exact sequence $K \otimes_R N \to N^{\oplus n} \to M \otimes_R N \to 0$ by Lemma 12.10. We conclude that if N is finite too then $M \otimes_R N$ is a quotient of a finite module, hence finite, see Lemma 5.3. Similarly, if both N and M are finitely presented, then we see that K is finite and that $M \otimes_R N$ is a quotient of the finitely presented module $N^{\oplus n}$ by a finite module, namely $K \otimes_R N$, and hence finitely presented, see Lemma 5.3. □

00DK **Lemma 12.15.** *Let M be an R-module. Then the $S^{-1}R$-modules $S^{-1}M$ and $S^{-1}R \otimes_R M$ are canonically isomorphic, and the canonical isomorphism $f : S^{-1}R \otimes_R M \to S^{-1}M$ is given by*

$$f((a/s) \otimes m) = am/s, \forall a \in R, m \in M, s \in S$$

Proof. Obviously, the map $f' : S^{-1}R \times M \to S^{-1}M$ given by $f((a/s, m)) = am/s$ is bilinear, and thus by the universal property, this map induces a unique $S^{-1}R$-module homomorphism $f : S^{-1}R \otimes_R M \to S^{-1}M$ as in the statement of the lemma. Actually every element in $S^{-1}M$ is of the form m/s, $m \in M, s \in S$ and every element in $S^{-1}R \otimes_R M$ is of the form $1/s \otimes m$. To see the latter fact, write an element in $S^{-1}R \otimes_R M$ as

$$\sum_k \frac{a_k}{s_k} \otimes m_k = \sum_k \frac{a_k t_k}{s} \otimes m_k = \frac{1}{s} \otimes \sum_k a_k t_k m_k = \frac{1}{s} \otimes m$$

Where $m = \sum_k a_k t_k m_k$. Then it is obvious that f is surjective, and if $f(\frac{1}{s} \otimes m) = m/s = 0$ then there exists $t' \in S$ with $tm = 0$ in M. Then we have

$$\frac{1}{s} \otimes m = \frac{1}{st} \otimes tm = \frac{1}{st} \otimes 0 = 0$$

Therefore f is injective. □

00DL **Lemma 12.16.** *Let M, N be R-modules, then there is a canonical $S^{-1}R$-module isomorphism $f : S^{-1}M \otimes_{S^{-1}R} S^{-1}N \to S^{-1}(M \otimes_R N)$, given by*

$$f((m/s) \otimes (n/t)) = (m \otimes n)/st$$

Proof. We may use Lemma 12.7 and Lemma 12.15 repeatedly to see that these two $S^{-1}R$-modules are isomorphic, noting that $S^{-1}R$ is an $(R, S^{-1}R)$-bimodule:

$$\begin{aligned} S^{-1}(M \otimes_R N) &\cong S^{-1}R \otimes_R (M \otimes_R N) \\ &\cong S^{-1}M \otimes_R N \\ &\cong (S^{-1}M \otimes_{S^{-1}R} S^{-1}R) \otimes_R N \\ &\cong S^{-1}M \otimes_{S^{-1}R} (S^{-1}R \otimes_R N) \\ &\cong S^{-1}M \otimes_{S^{-1}R} S^{-1}N \end{aligned}$$

This isomorphism is easily seen to be the one stated in the lemma. □

13. Tensor algebra

00DM Let R be a ring. Let M be an R-module. We define the *tensor algebra of M over R* to be the noncommutative R-algebra

$$\mathrm{T}(M) = \mathrm{T}_R(M) = \bigoplus\nolimits_{n \geq 0} \mathrm{T}^n(M)$$

with $\mathrm{T}^0(M) = R$, $\mathrm{T}^1(M) = M$, $\mathrm{T}^2(M) = M \otimes_R M$, $\mathrm{T}^3(M) = M \otimes_R M \otimes_R M$, and so on. Multiplication is defined by the rule that on pure tensors we have

$$(x_1 \otimes x_2 \otimes \ldots \otimes x_n) \cdot (y_1 \otimes y_2 \otimes \ldots \otimes y_m) = x_1 \otimes x_2 \otimes \ldots \otimes x_n \otimes y_1 \otimes y_2 \otimes \ldots \otimes y_m$$

and we extend this by linearity.

We define the *exterior algebra $\wedge(M)$ of M over R* to be the quotient of $\mathrm{T}(M)$ by the two sided ideal generated by the elements $x \otimes x \in \mathrm{T}^2(M)$. The image of a pure tensor $x_1 \otimes \ldots \otimes x_n$ in $\wedge^n(M)$ is denoted $x_1 \wedge \ldots \wedge x_n$. These elements generate $\wedge^n(M)$, they are R-linear in each x_i and they are zero when two of the x_i are equal (i.e., they are alternating as functions of $x_1, x_2, \ldots, x_n$). The multiplication on $\wedge(M)$ is graded commutative, i.e., every $x \in M$ and $y \in M$ satisfy $x \wedge y = -y \wedge x$.

An example of this is when $M = Rx_1 \oplus \ldots \oplus Rx_n$ is a finite free module. In this case $\wedge(M)$ is free over R with basis the elements

$$x_{i_1} \wedge \ldots \wedge x_{i_r}$$

with $0 \leq r \leq n$ and $1 \leq i_1 < i_2 < \ldots < i_r \leq n$.

We define the *symmetric algebra $\mathrm{Sym}(M)$ of M over R* to be the quotient of $\mathrm{T}(M)$ by the two sided ideal generated by the elements $x \otimes y - y \otimes x \in \mathrm{T}^2(M)$. The image of a pure tensor $x_1 \otimes \ldots \otimes x_n$ in $\mathrm{Sym}^n(M)$ is denoted just $x_1 \ldots x_n$. These elements generate $\mathrm{Sym}^n(M)$, these are R-linear in each x_i and $x_1 \ldots x_n = x'_1 \ldots x'_n$ if the sequence of elements $x_1, \ldots, x_n$ is a permutation of the sequence $x'_1, \ldots, x'_n$. Thus we see that $\mathrm{Sym}(M)$ is commutative.

An example of this is when $M = Rx_1 \oplus \ldots \oplus Rx_n$ is a finite free module. In this case $\mathrm{Sym}(M) = R[x_1, \ldots, x_n]$ is a polynomial algebra.

00DN **Lemma 13.1.** *Let R be a ring. Let M be an R-module. If M is a free R-module, so is each symmetric and exterior power.*

Proof. Omitted, but see above for the finite free case. □

00DO **Lemma 13.2.** *Let R be a ring. Let $M_2 \to M_1 \to M \to 0$ be an exact sequence of R-modules. There are exact sequences*

$$M_2 \otimes_R \mathrm{Sym}^{n-1}(M_1) \to \mathrm{Sym}^n(M_1) \to \mathrm{Sym}^n(M) \to 0$$

and similarly

$$M_2 \otimes_R \wedge^{n-1}(M_1) \to \wedge^n(M_1) \to \wedge^n(M) \to 0$$

Proof. Omitted. □

00DP **Lemma 13.3.** *Let R be a ring. Let M be an R-module. Let x_i, $i \in I$ be a given system of generators of M as an R-module. Let $n \geq 2$. There exists a canonical exact sequence*

$$\bigoplus_{1 \leq j_1 < j_2 \leq n} \bigoplus_{i_1, i_2 \in I} T^{n-2}(M) \oplus \bigoplus_{1 \leq j_1 < j_2 \leq n} \bigoplus_{i \in I} T^{n-2}(M) \to T^n(M) \to \wedge^n(M) \to 0$$

where the pure tensor $m_1 \otimes \ldots \otimes m_{n-2}$ in the first summand maps to

$$\begin{aligned} & \underbrace{m_1 \otimes \ldots \otimes x_{i_1} \otimes \ldots \otimes x_{i_2} \otimes \ldots \otimes m_{n-2}}_{\text{with } x_{i_1} \text{ and } x_{i_2} \text{ occupying slots } j_1 \text{ and } j_2 \text{ in the tensor}} \\ + \quad & \underbrace{m_1 \otimes \ldots \otimes x_{i_2} \otimes \ldots \otimes x_{i_1} \otimes \ldots \otimes m_{n-2}}_{\text{with } x_{i_2} \text{ and } x_{i_1} \text{ occupying slots } j_1 \text{ and } j_2 \text{ in the tensor}} \end{aligned}$$

and $m_1 \otimes \ldots \otimes m_{n-2}$ in the second summand maps to

$$\underbrace{m_1 \otimes \ldots \otimes x_i \otimes \ldots \otimes x_i \otimes \ldots \otimes m_{n-2}}_{\text{with } x_i \text{ and } x_i \text{ occupying slots } j_1 \text{ and } j_2 \text{ in the tensor}}$$

There is also a canonical exact sequence

$$\bigoplus_{1 \leq j_1 < j_2 \leq n} \bigoplus_{i_1, i_2 \in I} T^{n-2}(M) \to T^n(M) \to \mathrm{Sym}^n(M) \to 0$$

where the pure tensor $m_1 \otimes \ldots \otimes m_{n-2}$ maps to

$$\begin{aligned} & \underbrace{m_1 \otimes \ldots \otimes x_{i_1} \otimes \ldots \otimes x_{i_2} \otimes \ldots \otimes m_{n-2}}_{\text{with } x_{i_1} \text{ and } x_{i_2} \text{ occupying slots } j_1 \text{ and } j_2 \text{ in the tensor}} \\ - \quad & \underbrace{m_1 \otimes \ldots \otimes x_{i_2} \otimes \ldots \otimes x_{i_1} \otimes \ldots \otimes m_{n-2}}_{\text{with } x_{i_2} \text{ and } x_{i_1} \text{ occupying slots } j_1 \text{ and } j_2 \text{ in the tensor}} \end{aligned}$$

Proof. Omitted. □

00DQ **Lemma 13.4.** *Let R be a ring. Let M_i be a directed system of R-modules. Then $\mathrm{colim}_i\, T(M_i) = T(\mathrm{colim}_i\, M_i)$ and similarly for the symmetric and exterior algebras.*

Proof. Omitted. Hint: Apply Lemma 12.9. □

0C6F **Lemma 13.5.** *Let R be a ring and let $S \subset R$ be a multiplicative subset. Then $S^{-1}T_R(M) = T_{S^{-1}R}(S^{-1}M)$ for any R-module M. Similar for symmetric and exterior algebras.*

Proof. Omitted. Hint: Apply Lemma 12.16. □

14. Base change

05G3 We formally introduce base change in algebra as follows.

05G4 **Definition 14.1.** Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Let $R \to R'$ be any ring map. The *base change* of φ by $R \to R'$ is the ring map $R' \to S \otimes_R R'$. In this situation we often write $S' = S \otimes_R R'$. The *base change* of the S-module M is the S'-module $M \otimes_R R'$.

If $S = R[x_i]/(f_j)$ for some collection of variables x_i, $i \in I$ and some collection of polynomials $f_j \in R[x_i]$, $j \in J$, then $S \otimes_R R' = R'[x_i]/(f'_j)$, where $f'_j \in R'[x_i]$ is the image of f_j under the map $R[x_i] \to R'[x_i]$ induced by $R \to R'$. This simple remark is the key to understanding base change.

05G5 **Lemma 14.2.** *Let $R \to S$ be a ring map. Let M be an S-module. Let $R \to R'$ be a ring map and let $S' = S \otimes_R R'$ and $M' = M \otimes_R R'$ be the base changes.*

(1) *If M is a finite S-module, then the base change M' is a finite S'-module.*
(2) *If M is an S-module of finite presentation, then the base change M' is an S'-module of finite presentation.*
(3) *If $R \to S$ is of finite type, then the base change $R' \to S'$ is of finite type.*
(4) *If $R \to S$ is of finite presentation, then the base change $R' \to S'$ is of finite presentation.*

Proof. Proof of (1). Take a surjective, S-linear map $S^{\oplus n} \to M \to 0$. By Lemma 12.3 and 12.10 the result after tensoring with R' is a surjection ${S'}^{\oplus n} \to M' \to 0$, so M' is a finitely generated S'-module. Proof of (2). Take a presentation $S^{\oplus m} \to S^{\oplus n} \to M \to 0$. By Lemma 12.3 and 12.10 the result after tensoring with R' gives a finite presentation ${S'}^{\oplus m} \to {S'}^{\oplus n} \to M' \to 0$, of the S'-module M'. Proof of (3). This follows by the remark preceding the lemma as we can take I to be finite by assumption. Proof of (4). This follows by the remark preceding the lemma as we can take I and J to be finite by assumption. $\square$

Let $\varphi : R \to S$ be a ring map. Given an S-module N we obtain an R-module N_R by the rule $r \cdot n = \varphi(r)n$. This is sometimes called the *restriction* of N to R.

05DQ **Lemma 14.3.** *Let $R \to S$ be a ring map. The functors $Mod_S \to Mod_R$, $N \mapsto N_R$ (restriction) and $Mod_R \to Mod_S$, $M \mapsto M \otimes_R S$ (base change) are adjoint functors. In a formula*

$$\mathrm{Hom}_R(M, N_R) = \mathrm{Hom}_S(M \otimes_R S, N)$$

Proof. If $\alpha : M \to N_R$ is an R-module map, then we define $\alpha' : M \otimes_R S \to N$ by the rule $\alpha'(m \otimes s) = s\alpha(m)$. If $\beta : M \otimes_R S \to N$ is an S-module map, we define $\beta' : M \to N_R$ by the rule $\beta'(m) = \beta(m \otimes 1)$. We omit the verification that these constructions are mutually inverse. $\square$

The lemma above tells us that restriction has a left adjoint, namely base change. It also has a right adjoint.

08YP **Lemma 14.4.** *Let $R \to S$ be a ring map. The functors $Mod_S \to Mod_R$, $N \mapsto N_R$ (restriction) and $Mod_R \to Mod_S$, $M \mapsto \mathrm{Hom}_R(S, M)$ are adjoint functors. In a formula*

$$\mathrm{Hom}_R(N_R, M) = \mathrm{Hom}_S(N, \mathrm{Hom}_R(S, M))$$

Proof. If $\alpha : N_R \to M$ is an R-module map, then we define $\alpha' : N \to \mathrm{Hom}_R(S, M)$ by the rule $\alpha'(n) = (s \mapsto \alpha(sn))$. If $\beta : N \to \mathrm{Hom}_R(S, M)$ is an S-module map, we define $\beta' : N_R \to M$ by the rule $\beta'(n) = \beta(n)(1)$. We omit the verification that these constructions are mutually inverse. $\square$

08YQ **Lemma 14.5.** *Let $R \to S$ be a ring map. Given S-modules M, N and an R-module P we have*
$$\mathrm{Hom}_R(M \otimes_S N, P) = \mathrm{Hom}_S(M, \mathrm{Hom}_R(N, P))$$

Proof. This can be proved directly, but it is also a consequence of Lemmas 14.4 and 12.8. Namely, we have
$$\begin{aligned} \mathrm{Hom}_R(M \otimes_S N, P) &= \mathrm{Hom}_S(M \otimes_S N, \mathrm{Hom}_R(S, P)) \\ &= \mathrm{Hom}_S(M, \mathrm{Hom}_S(N, \mathrm{Hom}_R(S, P))) \\ &= \mathrm{Hom}_S(M, \mathrm{Hom}_R(N, P)) \end{aligned}$$
as desired. $\square$

15. Miscellany

00DR The proofs in this section should not refer to any results except those from the section on basic notions, Section 3.

07K1 **Lemma 15.1.** *Let R be a ring, I and J two ideals and $\mathfrak{p}$ a prime ideal containing the product IJ. Then $\mathfrak{p}$ contains I or J.*

Proof. Assume the contrary and take $x \in I \setminus \mathfrak{p}$ and $y \in J \setminus \mathfrak{p}$. Their product is an element of $IJ \subset \mathfrak{p}$, which contradicts the assumption that $\mathfrak{p}$ was prime. $\square$

00DS **Lemma 15.2** (Prime avoidance)**.** *Let R be a ring. Let $I_i \subset R$, $i = 1, \ldots, r$, and $J \subset R$ be ideals. Assume*

(1) *$J \not\subset I_i$ for $i = 1, \ldots, r$, and*
(2) *all but two of I_i are prime ideals.*

Then there exists an $x \in J$, $x \notin I_i$ for all i.

Proof. The result is true for $r = 1$. If $r = 2$, then let $x, y \in J$ with $x \notin I_1$ and $y \notin I_2$. We are done unless $x \in I_2$ and $y \in I_1$. Then the element $x + y$ cannot be in I_1 (since that would mean $x + y - y \in I_1$) and it also cannot be in I_2.

For $r \geq 3$, assume the result holds for $r - 1$. After renumbering we may assume that I_r is prime. We may also assume there are no inclusions among the I_i. Pick $x \in J$, $x \notin I_i$ for all $i = 1, \ldots, r - 1$. If $x \notin I_r$ we are done. So assume $x \in I_r$. If $JI_1 \ldots I_{r-1} \subset I_r$ then $J \subset I_r$ (by Lemma 15.1) a contradiction. Pick $y \in JI_1 \ldots I_{r-1}$, $y \notin I_r$. Then $x + y$ works. $\square$

0EHL **Lemma 15.3.** *Let R be a ring. Let $x \in R$, $I \subset R$ an ideal, and $\mathfrak{p}_i$, $i = 1, \ldots, r$ be prime ideals. Suppose that $x + I \not\subset \mathfrak{p}_i$ for $i = 1, \ldots, r$. Then there exists an $y \in I$ such that $x + y \notin \mathfrak{p}_i$ for all i.*

Proof. We may assume there are no inclusions among the $\mathfrak{p}_i$. After reordering we may assume $x \notin \mathfrak{p}_i$ for $i < s$ and $x \in \mathfrak{p}_i$ for $i \geq s$. If $s = r + 1$ then we are done. If not, then we can find $y \in I$ with $y \notin \mathfrak{p}_s$. Choose $f \in \bigcap_{i<s} \mathfrak{p}_i$ with $f \notin \mathfrak{p}_s$. Then $x + fy$ is not contained in $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$. Thus we win by induction on s. $\square$

00DT **Lemma 15.4** (Chinese remainder)**.** *Let R be a ring.*

(1) *If $I_1, \ldots, I_r$ are ideals such that $I_a + I_b = R$ when $a \neq b$, then $I_1 \cap \ldots \cap I_r = I_1 I_2 \ldots I_r$ and $R/(I_1 I_2 \ldots I_r) \cong R/I_1 \times \ldots \times R/I_r$.*
(2) *If $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ are pairwise distinct maximal ideals then $\mathfrak{m}_a + \mathfrak{m}_b = R$ for $a \neq b$ and the above applies.*

Proof. Let us first prove $I_1 \cap \ldots \cap I_r = I_1 \ldots I_r$ as this will also imply the injectivity of the induced ring homomorphism $R/(I_1 \ldots I_r) \to R/I_1 \times \ldots \times R/I_r$. The inclusion $I_1 \cap \ldots \cap I_r \supset I_1 \ldots I_r$ is always fulfilled since ideals are closed under multiplication with arbitrary ring elements. To prove the other inclusion, we claim that the ideals

$$I_1 \ldots \hat{I}_i \ldots I_r, \quad i = 1, \ldots, r$$

generate the ring R. We prove this by induction on r. It holds when $r = 2$. If $r > 2$, then we see that R is the sum of the ideals $I_1 \ldots \hat{I}_i \ldots I_{r-1}$, $i = 1, \ldots, r-1$. Hence I_r is the sum of the ideals $I_1 \ldots \hat{I}_i \ldots I_r$, $i = 1, \ldots, r-1$. Applying the same argument with the reverse ordering on the ideals we see that I_1 is the sum of the ideals $I_1 \ldots \hat{I}_i \ldots I_r$, $i = 2, \ldots, r$. Since $R = I_1 + I_r$ by assumption we see that R is the sum of the ideals displayed above. Therefore we can find elements $a_i \in I_1 \ldots \hat{I}_i \ldots I_r$ such that their sum is one. Multiplying this equation by an element of $I_1 \cap \ldots \cap I_r$ gives the other inclusion. It remains to show that the canonical map $R/(I_1 \ldots I_r) \to R/I_1 \times \ldots \times R/I_r$ is surjective. For this, consider its action on the equation $1 = \sum_{i=1}^r a_i$ we derived above. On the one hand, a ring morphism sends 1 to 1 and on the other hand, the image of any a_i is zero in R/I_j for $j \neq i$. Therefore, the image of a_i in R/I_i is the identity. So given any element $(\bar{b_1}, \ldots, \bar{b_r}) \in R/I_1 \times \ldots \times R/I_r$, the element $\sum_{i=1}^r a_i \cdot b_i$ is an inverse image in R.

To see (2), by the very definition of being distinct maximal ideals, we have $\mathfrak{m}_a + \mathfrak{m}_b = R$ for $a \neq b$ and so the above applies. □

07DQ **Lemma 15.5.** *Let R be a ring. Let $n \geq m$. Let A be an $n \times m$ matrix with coefficients in R. Let $J \subset R$ be the ideal generated by the $m \times m$ minors of A.*

(1) *For any $f \in J$ there exists a $m \times n$ matrix B such that $BA = f 1_{m \times m}$.*
(2) *If $f \in R$ and $BA = f 1_{m \times m}$ for some $m \times n$ matrix B, then $f^m \in J$.*

Proof. For $I \subset \{1, \ldots, n\}$ with $|I| = m$, we denote by E_I the $m \times n$ matrix of the projection

$$R^{\oplus n} = \bigoplus\nolimits_{i \in \{1, \ldots, n\}} R \longrightarrow \bigoplus\nolimits_{i \in I} R$$

and set $A_I = E_I A$, i.e., A_I is the $m \times m$ matrix whose rows are the rows of A with indices in I. Let B_I be the adjugate (transpose of cofactor) matrix to A_I, i.e., such that $A_I B_I = B_I A_I = \det(A_I) 1_{m \times m}$. The $m \times m$ minors of A are the determinants $\det A_I$ for all the $I \subset \{1, \ldots, n\}$ with $|I| = m$. If $f \in J$ then we can write $f = \sum c_I \det(A_I)$ for some $c_I \in R$. Set $B = \sum c_I B_I E_I$ to see that (1) holds.

If $f 1_{m \times m} = BA$ then by the Cauchy-Binet formula (72) we have $f^m = \sum b_I \det(A_I)$ where b_I is the determinant of the $m \times m$ matrix whose columns are the columns of B with indices in I. □

080R **Lemma 15.6.** *Let R be a ring. Let $n \geq m$. Let $A = (a_{ij})$ be an $n \times m$ matrix with coefficients in R, written in block form as*

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

where A_1 has size $m \times m$. Let B be the adjugate (transpose of cofactor) matrix to A_1. Then

$$AB = \begin{pmatrix} f1_{m\times m} \\ C \end{pmatrix}$$

where $f = \det(A_1)$ and c_{ij} is (up to sign) the determinant of the $m \times m$ minor of A corresponding to the rows $1, \ldots, \hat{j}, \ldots, m, i$.

Proof. Since the adjugate has the property $A_1B = BA_1 = f$ the first block of the expression for AB is correct. Note that

$$c_{ij} = \sum\nolimits_k a_{ik}b_{kj} = \sum (-1)^{j+k} a_{ik} \det(A_1^{jk})$$

where A_1^{ij} means A_1 with the jth row and kth column removed. This last expression is the row expansion of the determinant of the matrix in the statement of the lemma. □

05WI **Lemma 15.7.** *Let R be a nonzero ring. Let $n \geq 1$. Let M be an R-module generated by $< n$ elements. Then any R-module map $f : R^{\oplus n} \to M$ has a nonzero kernel.*

Proof. Choose a surjection $R^{\oplus n-1} \to M$. We may lift the map f to a map $f' : R^{\oplus n} \to R^{\oplus n-1}$ (Lemma 5.2). It suffices to prove f' has a nonzero kernel. The map $f' : R^{\oplus n} \to R^{\oplus n-1}$ is given by a matrix $A = (a_{ij})$. If one of the a_{ij} is not nilpotent, say $a = a_{ij}$ is not, then we can replace R by the localization R_a and we may assume a_{ij} is a unit. Since if we find a nonzero kernel after localization then there was a nonzero kernel to start with as localization is exact, see Proposition 9.12. In this case we can do a base change on both $R^{\oplus n}$ and $R^{\oplus n-1}$ and reduce to the case where

$$A = \begin{pmatrix} 1 & 0 & 0 & \ldots \\ 0 & a_{22} & a_{23} & \ldots \\ 0 & a_{32} & \ldots & \\ \ldots & \ldots & & \end{pmatrix}$$

Hence in this case we win by induction on n. If not then each a_{ij} is nilpotent. Set $I = (a_{ij}) \subset R$. Note that $I^{m+1} = 0$ for some $m \geq 0$. Let m be the largest integer such that $I^m \neq 0$. Then we see that $(I^m)^{\oplus n}$ is contained in the kernel of the map and we win. □

0FJ7 **Lemma 15.8.** *Let R be a nonzero ring. Let $n, m \geq 0$ be integers. If $R^{\oplus n}$ is isomorphic to $R^{\oplus m}$ as R-modules, then $n = m$.*

Proof. Immediate from Lemma 15.7. □

16. Cayley-Hamilton

05G6
00DX **Lemma 16.1.** *Let R be a ring. Let $A = (a_{ij})$ be an $n \times n$ matrix with coefficients in R. Let $P(x) \in R[x]$ be the characteristic polynomial of A (defined as $\det(xid_{n\times n} - A)$). Then $P(A) = 0$ in $Mat(n \times n, R)$.*

Proof. We reduce the question to the well-known Cayley-Hamilton theorem from linear algebra in several steps:

(1) If $\phi : S \to R$ is a ring morphism and b_{ij} are inverse images of the a_{ij} under this map, then it suffices to show the statement for S and (b_{ij}) since ϕ is a ring morphism.
(2) If $\psi : R \hookrightarrow S$ is an injective ring morphism, it clearly suffices to show the result for S and the a_{ij} considered as elements of S.
(3) Thus we may first reduce to the case $R = \mathbf{Z}[X_{ij}]$, $a_{ij} = X_{ij}$ of a polynomial ring and then further to the case $R = \mathbf{Q}(X_{ij})$ where we may finally apply Cayley-Hamilton.

□

05BT **Lemma 16.2.** *Let R be a ring. Let M be a finite R-module. Let $\varphi : M \to M$ be an endomorphism. Then there exists a monic polynomial $P \in R[T]$ such that $P(\varphi) = 0$ as an endomorphism of M.*

Proof. Choose a surjective R-module map $R^{\oplus n} \to M$, given by $(a_1, \ldots, a_n) \mapsto \sum a_i x_i$ for some generators $x_i \in M$. Choose $(a_{i1}, \ldots, a_{in}) \in R^{\oplus n}$ such that $\varphi(x_i) = \sum a_{ij} x_j$. In other words the diagram

$$\begin{tikzcd} R^{\oplus n} \arrow[r] \arrow[d, "A"'] & M \arrow[d, "\varphi"] \\ R^{\oplus n} \arrow[r] & M \end{tikzcd}$$

is commutative where $A = (a_{ij})$. By Lemma 16.1 there exists a monic polynomial P such that $P(A) = 0$. Then it follows that $P(\varphi) = 0$. □

05G7 **Lemma 16.3.** *Let R be a ring. Let $I \subset R$ be an ideal. Let M be a finite R-module. Let $\varphi : M \to M$ be an endomorphism such that $\varphi(M) \subset IM$. Then there exists a monic polynomial $P = t^n + a_1 t^{n-1} + \ldots + a_n \in R[T]$ such that $a_j \in I^j$ and $P(\varphi) = 0$ as an endomorphism of M.*

Proof. Choose a surjective R-module map $R^{\oplus n} \to M$, given by $(a_1, \ldots, a_n) \mapsto \sum a_i x_i$ for some generators $x_i \in M$. Choose $(a_{i1}, \ldots, a_{in}) \in I^{\oplus n}$ such that $\varphi(x_i) = \sum a_{ij} x_j$. In other words the diagram

$$\begin{tikzcd} R^{\oplus n} \arrow[r] \arrow[d, "A"'] & M \arrow[d, "\varphi"] \\ I^{\oplus n} \arrow[r] & M \end{tikzcd}$$

is commutative where $A = (a_{ij})$. By Lemma 16.1 the polynomial $P(t) = \det(t\text{id}_{n \times n} - A)$ has all the desired properties. □

As a fun example application we prove the following surprising lemma.

05G8 **Lemma 16.4.** *Let R be a ring. Let M be a finite R-module. Let $\varphi : M \to M$ be a surjective R-module map. Then φ is an isomorphism.*

First proof. Write $R' = R[x]$ and think of M as a finite R'-module with x acting via φ. Set $I = (x) \subset R'$. By our assumption that φ is surjective we have $IM = M$. Hence we may apply Lemma 16.3 to M as an R'-module, the ideal I and the endomorphism id_M. We conclude that $(1 + a_1 + \ldots + a_n)\text{id}_M = 0$ with $a_j \in I$. Write $a_j = b_j(x)x$ for some $b_j(x) \in R[x]$. Translating back into φ we see that $\text{id}_M = -(\sum_{j=1,\ldots,n} b_j(\varphi))\varphi$, and hence φ is invertible. □

Second proof. We perform induction on the number of generators of M over R. If M is generated by one element, then $M \cong R/I$ for some ideal $I \subset R$. In this case we may replace R by R/I so that $M = R$. In this case $\varphi : R \to R$ is given by multiplication on M by an element $r \in R$. The surjectivity of φ forces r invertible, since φ must hit 1, which implies that φ is invertible.

Now assume that we have proven the lemma in the case of modules generated by $n-1$ elements, and are examining a module M generated by n elements. Let A mean the ring $R[t]$, and regard the module M as an A-module by letting t act via φ; since M is finite over R, it is finite over $R[t]$ as well, and since we're trying to prove φ injective, a set-theoretic property, we might as well prove the endomorphism $t : M \to M$ over A injective. We have reduced our problem to the case our endomorphism is multiplication by an element of the ground ring. Let $M' \subset M$ denote the sub-A-module generated by the first $n-1$ of the generators of M, and consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d, "\varphi|_{M'}"] & M \arrow[r] \arrow[d, "\varphi"] & M/M' \arrow[r] \arrow[d, "\varphi \bmod M'"] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M/M' \arrow[r] & 0,
\end{tikzcd}$$

where the restriction of φ to M' and the map induced by φ on the quotient M/M' are well-defined since φ is multiplication by an element in the base, and M' and M/M' are A-modules in their own right. By the case $n = 1$ the map $M/M' \to M/M'$ is an isomorphism. A diagram chase implies that $\varphi|_{M'}$ is surjective hence by induction $\varphi|_{M'}$ is an isomorphism. This forces the middle column to be an isomorphism by the snake lemma. □

17. The spectrum of a ring

00DY We arbitrarily decide that the spectrum of a ring as a topological space is part of the algebra chapter, whereas an affine scheme is part of the chapter on schemes.

00DZ **Definition 17.1.** Let R be a ring.

(1) The *spectrum* of R is the set of prime ideals of R. It is usually denoted $\text{Spec}(R)$.
(2) Given a subset $T \subset R$ we let $V(T) \subset \text{Spec}(R)$ be the set of primes containing T, i.e., $V(T) = \{\mathfrak{p} \in \text{Spec}(R) \mid \forall f \in T, f \in \mathfrak{p}\}$.
(3) Given an element $f \in R$ we let $D(f) \subset \text{Spec}(R)$ be the set of primes not containing f.

00E0 **Lemma 17.2.** *Let R be a ring.*

(1) *The spectrum of a ring R is empty if and only if R is the zero ring.*
(2) *Every nonzero ring has a maximal ideal.*
(3) *Every nonzero ring has a minimal prime ideal.*
(4) *Given an ideal $I \subset R$ and a prime ideal $I \subset \mathfrak{p}$ there exists a prime $I \subset \mathfrak{q} \subset \mathfrak{p}$ such that $\mathfrak{q}$ is minimal over I.*
(5) *If $T \subset R$, and if (T) is the ideal generated by T in R, then $V((T)) = V(T)$.*
(6) *If I is an ideal and $\sqrt{I}$ is its radical, see basic notion (27), then $V(I) = V(\sqrt{I})$.*
(7) *Given an ideal I of R we have $\sqrt{I} = \bigcap_{I \subset \mathfrak{p}} \mathfrak{p}$.*

(8) *If I is an ideal then $V(I) = \emptyset$ if and only if I is the unit ideal.*
(9) *If I, J are ideals of R then $V(I) \cup V(J) = V(I \cap J)$.*
(10) *If $(I_a)_{a\in A}$ is a set of ideals of R then $\bigcap_{a\in A} V(I_a) = V(\bigcup_{a\in A} I_a)$.*
(11) *If $f \in R$, then $D(f) \amalg V(f) = \operatorname{Spec}(R)$.*
(12) *If $f \in R$ then $D(f) = \emptyset$ if and only if f is nilpotent.*
(13) *If $f = uf'$ for some unit $u \in R$, then $D(f) = D(f')$.*
(14) *If $I \subset R$ is an ideal, and $\mathfrak{p}$ is a prime of R with $\mathfrak{p} \notin V(I)$, then there exists an $f \in R$ such that $\mathfrak{p} \in D(f)$, and $D(f) \cap V(I) = \emptyset$.*
(15) *If $f, g \in R$, then $D(fg) = D(f) \cap D(g)$.*
(16) *If $f_i \in R$ for $i \in I$, then $\bigcup_{i\in I} D(f_i)$ is the complement of $V(\{f_i\}_{i\in I})$ in $\operatorname{Spec}(R)$.*
(17) *If $f \in R$ and $D(f) = \operatorname{Spec}(R)$, then f is a unit.*

Proof. We address each part in the corresponding item below.

(1) This is a direct consequence of (2) or (3).
(2) Let $\mathfrak{A}$ be the set of all proper ideals of R. This set is ordered by inclusion and is non-empty, since $(0) \in \mathfrak{A}$ is a proper ideal. Let A be a totally ordered subset of $\mathfrak{A}$. Then $\bigcup_{I\in A} I$ is in fact an ideal. Since $1 \notin I$ for all $I \in A$, the union does not contain 1 and thus is proper. Hence $\bigcup_{I\in A} I$ is in $\mathfrak{A}$ and is an upper bound for the set A. Thus by Zorn's lemma $\mathfrak{A}$ has a maximal element, which is the sought-after maximal ideal.
(3) Since R is nonzero, it contains a maximal ideal which is a prime ideal. Thus the set $\mathfrak{A}$ of all prime ideals of R is nonempty. $\mathfrak{A}$ is ordered by reverse-inclusion. Let A be a totally ordered subset of $\mathfrak{A}$. It's pretty clear that $J = \bigcap_{I\in A} I$ is in fact an ideal. Not so clear, however, is that it is prime. Let $xy \in J$. Then $xy \in I$ for all $I \in A$. Now let $B = \{I \in A | y \in I\}$. Let $K = \bigcap_{I\in B} I$. Since A is totally ordered, either $K = J$ (and we're done, since then $y \in J$) or $K \supset J$ and for all $I \in A$ such that I is properly contained in K, we have $y \notin I$. But that means that for all those $I, x \in I$, since they are prime. Hence $x \in J$. In either case, J is prime as desired. Hence by Zorn's lemma we get a maximal element which in this case is a minimal prime ideal.
(4) This is the same exact argument as (3) except you only consider prime ideals contained in $\mathfrak{p}$ and containing I.
(5) (T) is the smallest ideal containing T. Hence if $T \subset I$, some ideal, then $(T) \subset I$ as well. Hence if $I \in V(T)$, then $I \in V((T))$ as well. The other inclusion is obvious.
(6) Since $I \subset \sqrt{I}, V(\sqrt{I}) \subset V(I)$. Now let $\mathfrak{p} \in V(I)$. Let $x \in \sqrt{I}$. Then $x^n \in I$ for some n. Hence $x^n \in \mathfrak{p}$. But since $\mathfrak{p}$ is prime, a boring induction argument gets you that $x \in \mathfrak{p}$. Hence $\sqrt{I} \subset \mathfrak{p}$ and $\mathfrak{p} \in V(\sqrt{I})$.
(7) Let $f \in R \setminus \sqrt{I}$. Then $f^n \notin I$ for all n. Hence $S = \{1, f, f^2, \ldots\}$ is a multiplicative subset, not containing 0. Take a prime ideal $\bar{\mathfrak{p}} \subset S^{-1}R$ containing $S^{-1}I$. Then the pull-back $\mathfrak{p}$ in R of $\bar{\mathfrak{p}}$ is a prime ideal containing I that does not intersect S. This shows that $\bigcap_{I\subset\mathfrak{p}} \mathfrak{p} \subset \sqrt{I}$. Now if $a \in \sqrt{I}$, then $a^n \in I$ for some n. Hence if $I \subset \mathfrak{p}$, then $a^n \in \mathfrak{p}$. But since $\mathfrak{p}$ is prime, we have $a \in \mathfrak{p}$. Thus the equality is shown.
(8) I is not the unit ideal if and only if I is contained in some maximal ideal (to see this, apply (2) to the ring R/I) which is therefore prime.

(9) If $\mathfrak{p} \in V(I) \cup V(J)$, then $I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$ which means that $I \cap J \subset \mathfrak{p}$. Now if $I \cap J \subset \mathfrak{p}$, then $IJ \subset \mathfrak{p}$ and hence either I of J is in $\mathfrak{p}$, since $\mathfrak{p}$ is prime.
(10) $\mathfrak{p} \in \bigcap_{a \in A} V(I_a) \Leftrightarrow I_a \subset \mathfrak{p}, \forall a \in A \Leftrightarrow \mathfrak{p} \in V(\bigcup_{a \in A} I_a)$
(11) If $\mathfrak{p}$ is a prime ideal and $f \in R$, then either $f \in \mathfrak{p}$ or $f \notin \mathfrak{p}$ (strictly) which is what the disjoint union says.
(12) If $a \in R$ is nilpotent, then $a^n = 0$ for some n. Hence $a^n \in \mathfrak{p}$ for any prime ideal. Thus $a \in \mathfrak{p}$ as can be shown by induction and $D(f) = \emptyset$. Now, as shown in (7), if $a \in R$ is not nilpotent, then there is a prime ideal that does not contain it.
(13) $f \in \mathfrak{p} \Leftrightarrow uf \in \mathfrak{p}$, since u is invertible.
(14) If $\mathfrak{p} \notin V(I)$, then $\exists f \in I \setminus \mathfrak{p}$. Then $f \notin \mathfrak{p}$ so $\mathfrak{p} \in D(f)$. Also if $\mathfrak{q} \in D(f)$, then $f \notin \mathfrak{q}$ and thus I is not contained in $\mathfrak{q}$. Thus $D(f) \cap V(I) = \emptyset$.
(15) If $fg \in \mathfrak{p}$, then $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$. Hence if $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$, then $fg \notin \mathfrak{p}$. Since $\mathfrak{p}$ is an ideal, if $fg \notin \mathfrak{p}$, then $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$.
(16) $\mathfrak{p} \in \bigcup_{i \in I} D(f_i) \Leftrightarrow \exists i \in I, f_i \notin \mathfrak{p} \Leftrightarrow \mathfrak{p} \in \text{Spec}(R) \setminus V(\{f_i\}_{i \in I})$
(17) If $D(f) = \text{Spec}(R)$, then $V(f) = \emptyset$ and hence $fR = R$, so f is a unit.

□

The lemma implies that the subsets $V(T)$ from Definition 17.1 form the closed subsets of a topology on $\text{Spec}(R)$. And it also shows that the sets $D(f)$ are open and form a basis for this topology.

00E1 **Definition 17.3.** Let R be a ring. The topology on $\text{Spec}(R)$ whose closed sets are the sets $V(T)$ is called the *Zariski* topology. The open subsets $D(f)$ are called the *standard opens* of $\text{Spec}(R)$.

It should be clear from context whether we consider $\text{Spec}(R)$ just as a set or as a topological space.

00E2 **Lemma 17.4.** *Suppose that $\varphi : R \to R'$ is a ring homomorphism. The induced map*

$$\text{Spec}(\varphi) : \text{Spec}(R') \longrightarrow \text{Spec}(R), \quad \mathfrak{p}' \longmapsto \varphi^{-1}(\mathfrak{p}')$$

is continuous for the Zariski topologies. In fact, for any element $f \in R$ we have $\text{Spec}(\varphi)^{-1}(D(f)) = D(\varphi(f))$.

Proof. It is basic notion (41) that $\mathfrak{p} := \varphi^{-1}(\mathfrak{p}')$ is indeed a prime ideal of R. The last assertion of the lemma follows directly from the definitions, and implies the first. □

If $\varphi' : R' \to R''$ is a second ring homomorphism then the composition

$$\text{Spec}(R'') \longrightarrow \text{Spec}(R') \longrightarrow \text{Spec}(R)$$

equals $\text{Spec}(\varphi' \circ \varphi)$. In other words, Spec is a contravariant functor from the category of rings to the category of topological spaces.

00E3 **Lemma 17.5.** *Let R be a ring. Let $S \subset R$ be a multiplicative subset. The map $R \to S^{-1}R$ induces via the functoriality of* Spec *a homeomorphism*

$$\text{Spec}(S^{-1}R) \longrightarrow \{\mathfrak{p} \in \text{Spec}(R) \mid S \cap \mathfrak{p} = \emptyset\}$$

where the topology on the right hand side is that induced from the Zariski topology on $\text{Spec}(R)$. The inverse map is given by $\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$.

Proof. Denote the right hand side of the arrow of the lemma by D. Choose a prime $\mathfrak{p}' \subset S^{-1}R$ and let $\mathfrak{p}$ the inverse image of $\mathfrak{p}'$ in R. Since $\mathfrak{p}'$ does not contain 1 we see that $\mathfrak{p}$ does not contain any element of S. Hence $\mathfrak{p} \in D$ and we see that the image is contained in D. Let $\mathfrak{p} \in D$. By assumption the image $\overline{S}$ does not contain 0. By basic notion (54) $\overline{S}^{-1}(R/\mathfrak{p})$ is not the zero ring. By basic notion (62) we see $S^{-1}R/S^{-1}\mathfrak{p} = \overline{S}^{-1}(R/\mathfrak{p})$ is a domain, and hence $S^{-1}\mathfrak{p}$ is a prime. The equality of rings also shows that the inverse image of $S^{-1}\mathfrak{p}$ in R is equal to $\mathfrak{p}$, because $R/\mathfrak{p} \to \overline{S}^{-1}(R/\mathfrak{p})$ is injective by basic notion (55). This proves that the map $\mathrm{Spec}(S^{-1}R) \to \mathrm{Spec}(R)$ is bijective onto D with inverse as given. It is continuous by Lemma 17.4. Finally, let $D(g) \subset \mathrm{Spec}(S^{-1}R)$ be a standard open. Write $g = h/s$ for some $h \in R$ and $s \in S$. Since g and $h/1$ differ by a unit we have $D(g) = D(h/1)$ in $\mathrm{Spec}(S^{-1}R)$. Hence by Lemma 17.4 and the bijectivity above the image of $D(g) = D(h/1)$ is $D \cap D(h)$. This proves the map is open as well. □

00E4 **Lemma 17.6.** *Let R be a ring. Let $f \in R$. The map $R \to R_f$ induces via the functoriality of* Spec *a homeomorphism*

$$\mathrm{Spec}(R_f) \longrightarrow D(f) \subset \mathrm{Spec}(R).$$

The inverse is given by $\mathfrak{p} \mapsto \mathfrak{p} \cdot R_f$.

Proof. This is a special case of Lemma 17.5. □

It is not the case that every "affine open" of a spectrum is a standard open. See Example 27.4.

00E5 **Lemma 17.7.** *Let R be a ring. Let $I \subset R$ be an ideal. The map $R \to R/I$ induces via the functoriality of* Spec *a homeomorphism*

$$\mathrm{Spec}(R/I) \longrightarrow V(I) \subset \mathrm{Spec}(R).$$

The inverse is given by $\mathfrak{p} \mapsto \mathfrak{p}/I$.

Proof. It is immediate that the image is contained in $V(I)$. On the other hand, if $\mathfrak{p} \in V(I)$ then $\mathfrak{p} \supset I$ and we may consider the ideal $\mathfrak{p}/I \subset R/I$. Using basic notion (51) we see that $(R/I)/(\mathfrak{p}/I) = R/\mathfrak{p}$ is a domain and hence $\mathfrak{p}/I$ is a prime ideal. From this it is immediately clear that the image of $D(f + I)$ is $D(f) \cap V(I)$, and hence the map is a homeomorphism. □

00E6 **Remark 17.8.** A fundamental commutative diagram associated to a ring map $\varphi : R \to S$, a prime $\mathfrak{q} \subset S$ and the corresponding prime $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$ of R is the following

$$\begin{tikzcd}
\kappa(\mathfrak{q}) = S_\mathfrak{q}/\mathfrak{q}S_\mathfrak{q} & S_\mathfrak{q} \arrow[l] & S \arrow[l] \arrow[r] & S/\mathfrak{q} \arrow[r] & \kappa(\mathfrak{q}) \\
\kappa(\mathfrak{p}) \otimes_R S = S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p} \arrow[u] & S_\mathfrak{p} \arrow[l] \arrow[u] & S \arrow[l] \arrow[r] \arrow[u] & S/\mathfrak{p}S \arrow[r] \arrow[u] & (R \setminus \mathfrak{p})^{-1}S/\mathfrak{p}S \arrow[u] \\
\kappa(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \arrow[u] & R_\mathfrak{p} \arrow[l] \arrow[u] & R \arrow[l] \arrow[r] \arrow[u] & R/\mathfrak{p} \arrow[r] \arrow[u] & \kappa(\mathfrak{p}) \arrow[u]
\end{tikzcd}$$

In this diagram the arrows in the outer left and outer right columns are identical. The horizontal maps induce on the associated spectra always a homeomorphism onto the image. The lower two rows of the diagram make sense without assuming $\mathfrak{q}$

exists. The lower squares induce fibre squares of topological spaces. This diagram shows that $\mathfrak{p}$ is in the image of the map on Spec if and only if $S \otimes_R \kappa(\mathfrak{p})$ is not the zero ring.

00E7 **Lemma 17.9.** *Let $\varphi : R \to S$ be a ring map. Let $\mathfrak{p}$ be a prime of R. The following are equivalent*

(1) $\mathfrak{p}$ *is in the image of* $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$,
(2) $S \otimes_R \kappa(\mathfrak{p}) \neq 0$,
(3) $S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p} \neq 0$,
(4) $(S/\mathfrak{p}S)_\mathfrak{p} \neq 0$, *and*
(5) $\mathfrak{p} = \varphi^{-1}(\mathfrak{p}S)$.

Proof. We have already seen the equivalence of the first two in Remark 17.8. The others are just reformulations of this. $\square$

00E8 **Lemma 17.10.** *Let R be a ring. The space* $\mathrm{Spec}(R)$ *is quasi-compact.*

Proof. It suffices to prove that any covering of $\mathrm{Spec}(R)$ by standard opens can be refined by a finite covering. Thus suppose that $\mathrm{Spec}(R) = \cup D(f_i)$ for a set of elements $\{f_i\}_{i \in I}$ of R. This means that $\cap V(f_i) = \emptyset$. According to Lemma 17.2 this means that $V(\{f_i\}) = \emptyset$. According to the same lemma this means that the ideal generated by the f_i is the unit ideal of R. This means that we can write 1 as a *finite* sum: $1 = \sum_{i \in J} r_i f_i$ with $J \subset I$ finite. And then it follows that $\mathrm{Spec}(R) = \cup_{i \in J} D(f_i)$. $\square$

04PM **Lemma 17.11.** *Let R be a ring. The topology on $X = \mathrm{Spec}(R)$ has the following properties:*

(1) *X is quasi-compact,*
(2) *X has a basis for the topology consisting of quasi-compact opens, and*
(3) *the intersection of any two quasi-compact opens is quasi-compact.*

Proof. The spectrum of a ring is quasi-compact, see Lemma 17.10. It has a basis for the topology consisting of the standard opens $D(f) = \mathrm{Spec}(R_f)$ (Lemma 17.6) which are quasi-compact by the first remark. The intersection of two standard opens is quasi-compact as $D(f) \cap D(g) = D(fg)$. Given any two quasi-compact opens $U, V \subset X$ we may write $U = D(f_1) \cup \ldots \cup D(f_n)$ and $V = D(g_1) \cup \ldots \cup D(g_m)$. Then $U \cap V = \bigcup D(f_i g_j)$ which is quasi-compact. $\square$

18. Local rings

07BH Local rings are the bread and butter of algebraic geometry.

07BI **Definition 18.1.** A *local ring* is a ring with exactly one maximal ideal. The maximal ideal is often denoted $\mathfrak{m}_R$ in this case. We often say "let $(R, \mathfrak{m}, \kappa)$ be a local ring" to indicate that R is local, $\mathfrak{m}$ is its unique maximal ideal and $\kappa = R/\mathfrak{m}$ is its residue field. A *local homomorphism of local rings* is a ring map $\varphi : R \to S$ such that R and S are local rings and such that $\varphi(\mathfrak{m}_R) \subset \mathfrak{m}_S$. If it is given that R and S are local rings, then the phrase "*local ring map $\varphi : R \to S$*" means that φ is a local homomorphism of local rings.

A field is a local ring. Any ring map between fields is a local homomorphism of local rings.

00E9 **Lemma 18.2.** *Let R be a ring. The following are equivalent:*

(1) *R is a local ring,*
(2) $\mathrm{Spec}(R)$ *has exactly one closed point,*
(3) *R has a maximal ideal $\mathfrak{m}$ and every element of $R \setminus \mathfrak{m}$ is a unit, and*
(4) *R is not the zero ring and for every $x \in R$ either x or $1 - x$ is invertible or both.*

Proof. Let R be a ring, and $\mathfrak{m}$ a maximal ideal. If $x \in R \setminus \mathfrak{m}$, and x is not a unit then there is a maximal ideal $\mathfrak{m}'$ containing x. Hence R has at least two maximal ideals. Conversely, if $\mathfrak{m}'$ is another maximal ideal, then choose $x \in \mathfrak{m}'$, $x \notin \mathfrak{m}$. Clearly x is not a unit. This proves the equivalence of (1) and (3). The equivalence (1) and (2) is tautological. If R is local then (4) holds since x is either in $\mathfrak{m}$ or not. If (4) holds, and $\mathfrak{m}$, $\mathfrak{m}'$ are distinct maximal ideals then we may choose $x \in R$ such that $x \bmod \mathfrak{m}' = 0$ and $x \bmod \mathfrak{m} = 1$ by the Chinese remainder theorem (Lemma 15.4). This element x is not invertible and neither is $1 - x$ which is a contradiction. Thus (4) and (1) are equivalent. □

The localization $R_\mathfrak{p}$ of a ring R at a prime $\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{p}R_\mathfrak{p}$. Namely, the quotient $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is the fraction field of the domain $R/\mathfrak{p}$ and every element of $R_\mathfrak{p}$ which is not contained in $\mathfrak{p}R_\mathfrak{p}$ is invertible.

07BJ **Lemma 18.3.** *Let $\varphi : R \to S$ be a ring map. Assume R and S are local rings. The following are equivalent:*

(1) *φ is a local ring map,*
(2) *$\varphi(\mathfrak{m}_R) \subset \mathfrak{m}_S$, and*
(3) *$\varphi^{-1}(\mathfrak{m}_S) = \mathfrak{m}_R$.*
(4) *For any $x \in R$, if $\varphi(x)$ is invertible in S, then x is invertible in R.*

Proof. Conditions (1) and (2) are equivalent by definition. If (3) holds then (2) holds. Conversely, if (2) holds, then $\varphi^{-1}(\mathfrak{m}_S)$ is a prime ideal containing the maximal ideal $\mathfrak{m}_R$, hence $\varphi^{-1}(\mathfrak{m}_S) = \mathfrak{m}_R$. Finally, (4) is the contrapositive of (2) by Lemma 18.2. □

Let $\varphi : R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime and set $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$. Then the induced ring map $R_\mathfrak{p} \to S_\mathfrak{q}$ is a local ring map.

19. The Jacobson radical of a ring

0AMD We recall that the *Jacobson radical* $\mathrm{rad}(R)$ of a ring R is the intersection of all maximal ideals of R. If R is local then $\mathrm{rad}(R)$ is the maximal ideal of R.

0AME **Lemma 19.1.** *Let R be a ring with Jacobson radical $rad(R)$. Let $I \subset R$ be an ideal. The following are equivalent*

(1) *$I \subset rad(R)$, and*
(2) *every element of $1 + I$ is a unit in R.*

In this case every element of R which maps to a unit of R/I is a unit.

Proof. If $f \in \mathrm{rad}(R)$, then $f \in \mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of R. Hence $1 + f \notin \mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of R. Thus the closed subset $V(1+f)$ of $\mathrm{Spec}(R)$ is empty. This implies that $1 + f$ is a unit, see Lemma 17.2.

Conversely, assume that $1 + f$ is a unit for all $f \in I$. If $\mathfrak{m}$ is a maximal ideal and $I \not\subset \mathfrak{m}$, then $I + \mathfrak{m} = R$. Hence $1 = f + g$ for some $g \in \mathfrak{m}$ and $f \in I$. Then $g = 1 + (-f)$ is not a unit, contradiction.

For the final statement let $f \in R$ map to a unit in R/I. Then we can find $g \in R$ mapping to the multiplicative inverse of $f \bmod I$. Then $fg = 1 \bmod I$. Hence fg is a unit of R by (2) which implies that f is a unit. $\square$

0B7C **Lemma 19.2.** *Let $\varphi : R \to S$ be a ring map such that the induced map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective. Then an element $x \in R$ is a unit if and only if $\varphi(x) \in S$ is a unit.*

Proof. If x is a unit, then so is $\varphi(x)$. Conversely, if $\varphi(x)$ is a unit, then $\varphi(x) \notin \mathfrak{q}$ for all $\mathfrak{q} \in \mathrm{Spec}(S)$. Hence $x \notin \varphi^{-1}(\mathfrak{q}) = \mathrm{Spec}(\varphi)(\mathfrak{q})$ for all $\mathfrak{q} \in \mathrm{Spec}(S)$. Since $\mathrm{Spec}(\varphi)$ is surjective we conclude that x is a unit by part (17) of Lemma 17.2. $\square$

20. Nakayama's lemma

07RC We quote from [Mat70]: "This simple but important lemma is due to T. Nakayama, G. Azumaya and W. Krull. Priority is obscure, and although it is usually called the Lemma of Nakayama, late Prof. Nakayama did not like the name."

00DV **Lemma 20.1** (Nakayama's lemma)**.** *Let R be a ring with Jacobson radical $rad(R)$. Let M be an R-module. Let $I \subset R$ be an ideal.* [Mat70, 1.M Lemma (NAK) page 11]

00DW (1) *If $IM = M$ and M is finite, then there exists an $f \in 1 + I$ such that $fM = 0$.*
(2) *If $IM = M$, M is finite, and $I \subset rad(R)$, then $M = 0$.*
(3) *If $N, N' \subset M$, $M = N + IN'$, and N' is finite, then there exists an $f \in 1+I$ such that $fM \subset N$ and $M_f = N_f$.*
(4) *If $N, N' \subset M$, $M = N + IN'$, N' is finite, and $I \subset rad(R)$, then $M = N$.*
(5) *If $N \to M$ is a module map, $N/IN \to M/IM$ is surjective, and M is finite, then there exists an $f \in 1 + I$ such that $N_f \to M_f$ is surjective.*
(6) *If $N \to M$ is a module map, $N/IN \to M/IM$ is surjective, M is finite, and $I \subset rad(R)$, then $N \to M$ is surjective.*
(7) *If $x_1, \ldots, x_n \in M$ generate M/IM and M is finite, then there exists an $f \in 1 + I$ such that $x_1, \ldots, x_n$ generate M_f over R_f.*
(8) *If $x_1, \ldots, x_n \in M$ generate M/IM, M is finite, and $I \subset rad(R)$, then M is generated by $x_1, \ldots, x_n$.*
(9) *If $IM = M$, I is nilpotent, then $M = 0$.*
(10) *If $N, N' \subset M$, $M = N + IN'$, and I is nilpotent then $M = N$.*
(11) *If $N \to M$ is a module map, I is nilpotent, and $N/IN \to M/IM$ is surjective, then $N \to M$ is surjective.*
(12) *If $\{x_\alpha\}_{\alpha \in A}$ is a set of elements of M which generate M/IM and I is nilpotent, then M is generated by the x_α.*

Proof. Proof of (1). Choose generators $y_1, \ldots, y_m$ of M over R. For each i we can write $y_i = \sum z_{ij} y_j$ with $z_{ij} \in I$ (since $M = IM$). In other words $\sum_j (\delta_{ij} - z_{ij}) y_j = 0$. Let f be the determinant of the $m \times m$ matrix $A = (\delta_{ij} - z_{ij})$. Note that $f \in 1 + I$ (since the matrix A is entrywise congruent to the $m \times m$ identity matrix modulo I). By Lemma 15.5 (1), there exists an $m \times m$ matrix B such that $BA = f 1_{m \times m}$. Writing out we see that $\sum_i b_{hi} a_{ij} = f \delta_{hj}$ for all h and j; hence, $\sum_{i,j} b_{hi} a_{ij} y_j = \sum_j f \delta_{hj} y_j = f y_h$ for every h. In other words, $0 = f y_h$ for every h (since each i satisfies $\sum_j a_{ij} y_j = 0$). This implies that f annihilates M.

By Lemma 19.1 an element of $1 + \mathrm{rad}(R)$ is invertible element of R. Hence we see that (1) implies (2). We obtain (3) by applying (1) to M/N which is finite as N'

is finite. We obtain (4) by applying (2) to M/N which is finite as N' is finite. We obtain (5) by applying (3) to M and the submodules $\text{Im}(N \to M)$ and M. We obtain (6) by applying (4) to M and the submodules $\text{Im}(N \to M)$ and M. We obtain (7) by applying (5) to the map $R^{\oplus n} \to M$, $(a_1, \ldots, a_n) \mapsto a_1x_1 + \ldots + a_nx_n$. We obtain (8) by applying (6) to the map $R^{\oplus n} \to M$, $(a_1, \ldots, a_n) \mapsto a_1x_1 + \ldots + a_nx_n$.

Part (9) holds because if $M = IM$ then $M = I^nM$ for all $n \geq 0$ and I being nilpotent means $I^n = 0$ for some $n \gg 0$. Parts (10), (11), and (12) follow from (9) by the arguments used above. □

0GLX **Lemma 20.2.** *Let R be a ring, let $S \subset R$ be a multiplicative subset, let $I \subset R$ be an ideal, and let M be a finite R-module. If $x_1, \ldots, x_r \in M$ generate $S^{-1}(M/IM)$ as an $S^{-1}(R/I)$-module, then there exists an $f \in S+I$ such that $x_1, \ldots, x_r$ generate M_f as an R_f-module.*[1]

Proof. Special case $I = 0$. Let $y_1, \ldots, y_s$ be generators for M over R. Since $S^{-1}M$ is generated by $x_1, \ldots, x_r$, for each i we can write $y_i = \sum(a_{ij}/s_{ij})x_j$ for some $a_{ij} \in R$ and $s_{ij} \in S$. Let $s \in S$ be the product of all of the s_{ij}. Then we see that y_i is contained in the R_s-submodule of M_s generated by $x_1, \ldots, x_r$. Hence $x_1, \ldots, x_r$ generates M_s.

General case. By the special case, we can find an $s \in S$ such that $x_1, \ldots, x_r$ generate $(M/IM)_s$ over $(R/I)_s$. By Lemma 20.1 we can find a $g \in 1 + I_s \subset R_s$ such that $x_1, \ldots, x_r$ generate $(M_s)_g$ over $(R_s)_g$. Write $g = 1 + i/s'$. Then $f = ss' + is$ works; details omitted. □

0E8M **Lemma 20.3.** *Let $A \to B$ be a local homomorphism of local rings. Assume*

1. *B is finite as an A-module,*
2. *$\mathfrak{m}_B$ is a finitely generated ideal,*
3. *$A \to B$ induces an isomorphism on residue fields, and*
4. *$\mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective.*

Then $A \to B$ is surjective.

Proof. To show that $A \to B$ is surjective, we view it as a map of A-modules and apply Lemma 20.1 (6). We conclude it suffices to show that $A/\mathfrak{m}_A \to B/\mathfrak{m}_AB$ is surjective. As $A/\mathfrak{m}_A = B/\mathfrak{m}_B$ it suffices to show that $\mathfrak{m}_AB \to \mathfrak{m}_B$ is surjective. View $\mathfrak{m}_AB \to \mathfrak{m}_B$ as a map of B-modules and apply Lemma 20.1 (6). We conclude it suffices to see that $\mathfrak{m}_AB/\mathfrak{m}_A\mathfrak{m}_B \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective. This follows from assumption (4). □

21. Open and closed subsets of spectra

04PN It turns out that open and closed subsets of a spectrum correspond to idempotents of the ring.

00EC **Lemma 21.1.** *Let R be a ring. Let $e \in R$ be an idempotent. In this case*

$$\text{Spec}(R) = D(e) \amalg D(1-e).$$

[1]Special cases: (I) $I = 0$. The lemma says if $x_1, \ldots, x_r$ generate $S^{-1}M$, then $x_1, \ldots, x_r$ generate M_f for some $f \in S$. (II) $I = \mathfrak{p}$ is a prime ideal and $S = R \setminus \mathfrak{p}$. The lemma says if $x_1, \ldots, x_r$ generate $M \otimes_R \kappa(\mathfrak{p})$ then $x_1, \ldots, x_r$ generate M_f for some $f \in R$, $f \notin \mathfrak{p}$.

Proof. Note that an idempotent e of a domain is either 1 or 0. Hence we see that

$$\begin{aligned} D(e) &= \{\mathfrak{p} \in \mathrm{Spec}(R) \mid e \not\in \mathfrak{p}\} \\ &= \{\mathfrak{p} \in \mathrm{Spec}(R) \mid e \neq 0 \text{ in } \kappa(\mathfrak{p})\} \\ &= \{\mathfrak{p} \in \mathrm{Spec}(R) \mid e = 1 \text{ in } \kappa(\mathfrak{p})\} \end{aligned}$$

Similarly we have

$$\begin{aligned} D(1-e) &= \{\mathfrak{p} \in \mathrm{Spec}(R) \mid 1-e \not\in \mathfrak{p}\} \\ &= \{\mathfrak{p} \in \mathrm{Spec}(R) \mid e \neq 1 \text{ in } \kappa(\mathfrak{p})\} \\ &= \{\mathfrak{p} \in \mathrm{Spec}(R) \mid e = 0 \text{ in } \kappa(\mathfrak{p})\} \end{aligned}$$

Since the image of e in any residue field is either 1 or 0 we deduce that $D(e)$ and $D(1-e)$ cover all of $\mathrm{Spec}(R)$. ☐

00ED **Lemma 21.2.** *Let R_1 and R_2 be rings. Let $R = R_1 \times R_2$. The maps $R \to R_1$, $(x, y) \mapsto x$ and $R \to R_2$, $(x, y) \mapsto y$ induce continuous maps $\mathrm{Spec}(R_1) \to \mathrm{Spec}(R)$ and $\mathrm{Spec}(R_2) \to \mathrm{Spec}(R)$. The induced map*

$$\mathrm{Spec}(R_1) \amalg \mathrm{Spec}(R_2) \longrightarrow \mathrm{Spec}(R)$$

is a homeomorphism. In other words, the spectrum of $R = R_1 \times R_2$ is the disjoint union of the spectrum of R_1 and the spectrum of R_2.

Proof. Write $1 = e_1 + e_2$ with $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Note that e_1 and $e_2 = 1 - e_1$ are idempotents. We leave it to the reader to show that $R_1 = R_{e_1}$ is the localization of R at e_1. Similarly for e_2. Thus the statement of the lemma follows from Lemma 21.1 combined with Lemma 17.6. ☐

We reprove the following lemma later after introducing a glueing lemma for functions. See Section 24.

00EE **Lemma 21.3.** *Let R be a ring. For each $U \subset \mathrm{Spec}(R)$ which is open and closed there exists a unique idempotent $e \in R$ such that $U = D(e)$. This induces a 1-1 correspondence between open and closed subsets $U \subset \mathrm{Spec}(R)$ and idempotents $e \in R$.*

Proof. Let $U \subset \mathrm{Spec}(R)$ be open and closed. Since U is closed it is quasi-compact by Lemma 17.10, and similarly for its complement. Write $U = \bigcup_{i=1}^n D(f_i)$ as a finite union of standard opens. Similarly, write $\mathrm{Spec}(R) \setminus U = \bigcup_{j=1}^m D(g_j)$ as a finite union of standard opens. Since $\emptyset = D(f_i) \cap D(g_j) = D(f_i g_j)$ we see that $f_i g_j$ is nilpotent by Lemma 17.2. Let $I = (f_1, \ldots, f_n) \subset R$ and let $J = (g_1, \ldots, g_m) \subset R$. Note that $V(J)$ equals U, that $V(I)$ equals the complement of U, so $\mathrm{Spec}(R) = V(I) \amalg V(J)$. By the remark on nilpotency above, we see that $(IJ)^N = (0)$ for some sufficiently large integer N. Since $\bigcup D(f_i) \cup \bigcup D(g_j) = \mathrm{Spec}(R)$ we see that $I + J = R$, see Lemma 17.2. By raising this equation to the $2N$th power we conclude that $I^N + J^N = R$. Write $1 = x + y$ with $x \in I^N$ and $y \in J^N$. Then $0 = xy = x(1-x)$ as $I^N J^N = (0)$. Thus $x = x^2$ is idempotent and contained in $I^N \subset I$. The idempotent $y = 1 - x$ is contained in $J^N \subset J$. This shows that the idempotent x maps to 1 in every residue field $\kappa(\mathfrak{p})$ for $\mathfrak{p} \in V(J)$ and that x maps to 0 in $\kappa(\mathfrak{p})$ for every $\mathfrak{p} \in V(I)$.

To see uniqueness suppose that e_1, e_2 are distinct idempotents in R. We have to show there exists a prime $\mathfrak{p}$ such that $e_1 \in \mathfrak{p}$ and $e_2 \not\in \mathfrak{p}$, or conversely. Write

$e_i' = 1 - e_i$. If $e_1 \neq e_2$, then $0 \neq e_1 - e_2 = e_1(e_2 + e_2') - (e_1 + e_1')e_2 = e_1e_2' - e_1'e_2$. Hence either the idempotent $e_1e_2' \neq 0$ or $e_1'e_2 \neq 0$. An idempotent is not nilpotent, and hence we find a prime $\mathfrak{p}$ such that either $e_1e_2' \notin \mathfrak{p}$ or $e_1'e_2 \notin \mathfrak{p}$, by Lemma 17.2. It is easy to see this gives the desired prime. □

00EF **Lemma 21.4.** *Let R be a nonzero ring. Then* $\text{Spec}(R)$ *is connected if and only if R has no nontrivial idempotents.*

Proof. Obvious from Lemma 21.3 and the definition of a connected topological space. □

00EH **Lemma 21.5.** *Let $I \subset R$ be a finitely generated ideal of a ring R such that $I = I^2$. Then*

1. *there exists an idempotent $e \in R$ such that $I = (e)$,*
2. *$R/I \cong R_{e'}$ for the idempotent $e' = 1 - e \in R$, and*
3. *$V(I)$ is open and closed in* $\text{Spec}(R)$.

Proof. By Nakayama's Lemma 20.1 there exists an element $f = 1 + i$, $i \in I$ such that $fI = 0$. Then $f^2 = f + fi = f$ is an idempotent. Consider the idempotent $e = 1 - f = -i \in I$. For $j \in I$ we have $ej = j - fj = j$ hence $I = (e)$. This proves (1).

Parts (2) and (3) follow from (1). Namely, we have $V(I) = V(e) = \text{Spec}(R) \setminus D(e)$ which is open and closed by either Lemma 21.1 or Lemma 21.3. This proves (3). For (2) observe that the map $R \to R_{e'}$ is surjective since $x/(e')^n = x/e' = xe'/(e')^2 = xe'/e' = x/1$ in $R_{e'}$. The kernel of the map $R \to R_{e'}$ is the set of elements of R annihilated by a positive power of e'. Since e' is idempotent this is the ideal of elements annihilated by e' which is the ideal $I = (e)$ as $e + e' = 1$ is a pair of orthognal idempotents. This proves (2). □

22. Connected components of spectra

00EB Connected components of spectra are not as easy to understand as one may think at first. This is because we are used to the topology of locally connected spaces, but the spectrum of a ring is in general not locally connected.

04PP **Lemma 22.1.** *Let R be a ring. Let $T \subset \text{Spec}(R)$ be a subset of the spectrum. The following are equivalent*

1. *T is closed and is a union of connected components of* $\text{Spec}(R)$,
2. *T is an intersection of open and closed subsets of* $\text{Spec}(R)$, *and*
3. *$T = V(I)$ where $I \subset R$ is an ideal generated by idempotents.*

Moreover, the ideal in (3) if it exists is unique.

Proof. By Lemma 17.11 and Topology, Lemma 12.12 we see that (1) and (2) are equivalent. Assume (2) and write $T = \bigcap U_\alpha$ with $U_\alpha \subset \text{Spec}(R)$ open and closed. Then $U_\alpha = D(e_\alpha)$ for some idempotent $e_\alpha \in R$ by Lemma 21.3. Then setting $I = (1 - e_\alpha)$ we see that $T = V(I)$, i.e., (3) holds. Finally, assume (3). Write $T = V(I)$ and $I = (e_\alpha)$ for some collection of idempotents e_α. Then it is clear that $T = \bigcap V(e_\alpha) = \bigcap D(1 - e_\alpha)$.

Suppose that I is an ideal generated by idempotents. Let $e \in R$ be an idempotent such that $V(I) \subset V(e)$. Then by Lemma 17.2 we see that $e^n \in I$ for some $n \geq 1$. As e is an idempotent this means that $e \in I$. Hence we see that I is generated by exactly those idempotents e such that $T \subset V(e)$. In other words, the ideal I is completely determined by the closed subset T which proves uniqueness. $\square$

00EG **Lemma 22.2.** *Let R be a ring. A connected component of* $\text{Spec}(R)$ *is of the form $V(I)$, where I is an ideal generated by idempotents such that every idempotent of R either maps to 0 or 1 in R/I.*

Proof. Let $\mathfrak{p}$ be a prime of R. By Lemma 17.11 we have see that the hypotheses of Topology, Lemma 12.10 are satisfied for the topological space $\text{Spec}(R)$. Hence the connected component of $\mathfrak{p}$ in $\text{Spec}(R)$ is the intersection of open and closed subsets of $\text{Spec}(R)$ containing $\mathfrak{p}$. Hence it equals $V(I)$ where I is generated by the idempotents $e \in R$ such that e maps to 0 in $\kappa(\mathfrak{p})$, see Lemma 21.3. Any idempotent e which is not in this collection clearly maps to 1 in R/I. $\square$

23. Glueing properties

00EN In this section we put a number of standard results of the form: if something is true for all members of a standard open covering then it is true. In fact, it often suffices to check things on the level of local rings as in the following lemma.

00HN **Lemma 23.1.** *Let R be a ring.*

1. *For an element x of an R-module M the following are equivalent*
   (a) $x = 0$,
   (b) *x maps to zero in $M_{\mathfrak{p}}$ for all* $\mathfrak{p} \in \text{Spec}(R)$,
   (c) *x maps to zero in $M_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of R.*
   In other words, the map $M \to \prod_{\mathfrak{m}} M_{\mathfrak{m}}$ is injective.
2. *Given an R-module M the following are equivalent*
   (a) *M is zero,*
   (b) *$M_{\mathfrak{p}}$ is zero for all* $\mathfrak{p} \in \text{Spec}(R)$,
   (c) *$M_{\mathfrak{m}}$ is zero for all maximal ideals $\mathfrak{m}$ of R.*
3. *Given a complex $M_1 \to M_2 \to M_3$ of R-modules the following are equivalent*
   (a) *$M_1 \to M_2 \to M_3$ is exact,*
   (b) *for every prime $\mathfrak{p}$ of R the localization $M_{1,\mathfrak{p}} \to M_{2,\mathfrak{p}} \to M_{3,\mathfrak{p}}$ is exact,*
   (c) *for every maximal ideal $\mathfrak{m}$ of R the localization $M_{1,\mathfrak{m}} \to M_{2,\mathfrak{m}} \to M_{3,\mathfrak{m}}$ is exact.*
4. *Given a map $f : M \to M'$ of R-modules the following are equivalent*
   (a) *f is injective,*
   (b) *$f_{\mathfrak{p}} : M_{\mathfrak{p}} \to M'_{\mathfrak{p}}$ is injective for all primes $\mathfrak{p}$ of R,*
   (c) *$f_{\mathfrak{m}} : M_{\mathfrak{m}} \to M'_{\mathfrak{m}}$ is injective for all maximal ideals $\mathfrak{m}$ of R.*
5. *Given a map $f : M \to M'$ of R-modules the following are equivalent*
   (a) *f is surjective,*
   (b) *$f_{\mathfrak{p}} : M_{\mathfrak{p}} \to M'_{\mathfrak{p}}$ is surjective for all primes $\mathfrak{p}$ of R,*
   (c) *$f_{\mathfrak{m}} : M_{\mathfrak{m}} \to M'_{\mathfrak{m}}$ is surjective for all maximal ideals $\mathfrak{m}$ of R.*
6. *Given a map $f : M \to M'$ of R-modules the following are equivalent*
   (a) *f is bijective,*
   (b) *$f_{\mathfrak{p}} : M_{\mathfrak{p}} \to M'_{\mathfrak{p}}$ is bijective for all primes $\mathfrak{p}$ of R,*
   (c) *$f_{\mathfrak{m}} : M_{\mathfrak{m}} \to M'_{\mathfrak{m}}$ is bijective for all maximal ideals $\mathfrak{m}$ of R.*

Proof. Let $x \in M$ as in (1). Let $I = \{f \in R \mid fx = 0\}$. It is easy to see that I is an ideal (it is the annihilator of x). Condition (1)(c) means that for all maximal ideals $\mathfrak{m}$ there exists an $f \in R \setminus \mathfrak{m}$ such that $fx = 0$. In other words, $V(I)$ does not contain a closed point. By Lemma 17.2 we see I is the unit ideal. Hence x is zero, i.e., (1)(a) holds. This proves (1).

Part (2) follows by applying (1) to all elements of M simultaneously.

Proof of (3). Let H be the homology of the sequence, i.e., $H = \mathrm{Ker}(M_2 \to M_3)/\mathrm{Im}(M_1 \to M_2)$. By Proposition 9.12 we have that $H_\mathfrak{p}$ is the homology of the sequence $M_{1,\mathfrak{p}} \to M_{2,\mathfrak{p}} \to M_{3,\mathfrak{p}}$. Hence (3) is a consequence of (2).

Parts (4) and (5) are special cases of (3). Part (6) follows formally on combining (4) and (5). ☐

00EO **Lemma 23.2.** *Let R be a ring. Let M be an R-module. Let S be an R-algebra. Suppose that $f_1, \ldots, f_n$ is a finite list of elements of R such that $\bigcup D(f_i) = \mathrm{Spec}(R)$, in other words $(f_1, \ldots, f_n) = R$.*

1. *If each $M_{f_i} = 0$ then $M = 0$.*
2. *If each M_{f_i} is a finite R_{f_i}-module, then M is a finite R-module.*
3. *If each M_{f_i} is a finitely presented R_{f_i}-module, then M is a finitely presented R-module.*
4. *Let $M \to N$ be a map of R-modules. If $M_{f_i} \to N_{f_i}$ is an isomorphism for each i then $M \to N$ is an isomorphism.*
5. *Let $0 \to M'' \to M \to M' \to 0$ be a complex of R-modules. If $0 \to M''_{f_i} \to M_{f_i} \to M'_{f_i} \to 0$ is exact for each i, then $0 \to M'' \to M \to M' \to 0$ is exact.*
6. *If each R_{f_i} is Noetherian, then R is Noetherian.*
7. *If each S_{f_i} is a finite type R-algebra, so is S.*
8. *If each S_{f_i} is of finite presentation over R, so is S.*

Proof. We prove each of the parts in turn.

1. By Proposition 9.10 this implies $M_\mathfrak{p} = 0$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$, so we conclude by Lemma 23.1.
2. For each i take a finite generating set X_i of M_{f_i}. Without loss of generality, we may assume that the elements of X_i are in the image of the localization map $M \to M_{f_i}$, so we take a finite set Y_i of preimages of the elements of X_i in M. Let Y be the union of these sets. This is still a finite set. Consider the obvious R-linear map $R^Y \to M$ sending the basis element e_y to y. By assumption this map is surjective after localizing at an arbitrary prime ideal $\mathfrak{p}$ of R, so it is surjective by Lemma 23.1 and M is finitely generated.
3. By (2) we have a short exact sequence
$$0 \to K \to R^n \to M \to 0$$
Since localization is an exact functor and M_{f_i} is finitely presented we see that K_{f_i} is finitely generated for all $1 \leq i \leq n$ by Lemma 5.3. By (2) this implies that K is a finite R-module and therefore M is finitely presented.
4. By Proposition 9.10 the assumption implies that the induced morphism on localizations at all prime ideals is an isomorphism, so we conclude by Lemma 23.1.
5. By Proposition 9.10 the assumption implies that the induced sequence of localizations at all prime ideals is short exact, so we conclude by Lemma 23.1.

(6) We will show that every ideal of R has a finite generating set: For this, let $I \subset R$ be an arbitrary ideal. By Proposition 9.12 each $I_{f_i} \subset R_{f_i}$ is an ideal. These are all finitely generated by assumption, so we conclude by (2).

(7) For each i take a finite generating set X_i of S_{f_i}. Without loss of generality, we may assume that the elements of X_i are in the image of the localization map $S \to S_{f_i}$, so we take a finite set Y_i of preimages of the elements of X_i in S. Let Y be the union of these sets. This is still a finite set. Consider the algebra homomorphism $R[X_y]_{y\in Y} \to S$ induced by Y. Since it is an algebra homomorphism, the image T is an R-submodule of the R-module S, so we can consider the quotient module S/T. By assumption, this is zero if we localize at the f_i, so it is zero by (1) and therefore S is an R-algebra of finite type.

(8) By the previous item, there exists a surjective R-algebra homomorphism $R[X_1, \ldots, X_n] \to S$. Let K be the kernel of this map. This is an ideal in $R[X_1, \ldots, X_n]$, finitely generated in each localization at f_i. Since the f_i generate the unit ideal in R, they also generate the unit ideal in $R[X_1, \ldots, X_n]$, so an application of (2) finishes the proof.

□

00EP **Lemma 23.3.** *Let $R \to S$ be a ring map. Suppose that $g_1, \ldots, g_n$ is a finite list of elements of S such that $\bigcup D(g_i) = \operatorname{Spec}(S)$ in other words $(g_1, \ldots, g_n) = S$.*

(1) *If each S_{g_i} is of finite type over R, then S is of finite type over R.*
(2) *If each S_{g_i} is of finite presentation over R, then S is of finite presentation over R.*

Proof. Choose $h_1, \ldots, h_n \in S$ such that $\sum h_i g_i = 1$.

Proof of (1). For each i choose a finite list of elements $x_{i,j} \in S_{g_i}$, $j = 1, \ldots, m_i$ which generate S_{g_i} as an R-algebra. Write $x_{i,j} = y_{i,j}/g_i^{n_{i,j}}$ for some $y_{i,j} \in S$ and some $n_{i,j} \geq 0$. Consider the R-subalgebra $S' \subset S$ generated by $g_1, \ldots, g_n$, $h_1, \ldots, h_n$ and $y_{i,j}$, $i = 1, \ldots, n$, $j = 1, \ldots, m_i$. Since localization is exact (Proposition 9.12), we see that $S'_{g_i} \to S_{g_i}$ is injective. On the other hand, it is surjective by our choice of $y_{i,j}$. The elements $g_1, \ldots, g_n$ generate the unit ideal in S' as $h_1, \ldots, h_n \in S'$. Thus $S' \to S$ viewed as an S'-module map is an isomorphism by Lemma 23.2.

Proof of (2). We already know that S is of finite type. Write $S = R[x_1, \ldots, x_m]/J$ for some ideal J. For each i choose a lift $g'_i \in R[x_1, \ldots, x_m]$ of g_i and we choose a lift $h'_i \in R[x_1, \ldots, x_m]$ of h_i. Then we see that

$$S_{g_i} = R[x_1, \ldots, x_m, y_i]/(J_i + (1 - y_i g'_i))$$

where J_i is the ideal of $R[x_1, \ldots, x_m, y_i]$ generated by J. Small detail omitted. By Lemma 6.3 we may choose a finite list of elements $f_{i,j} \in J$, $j = 1, \ldots, m_i$ such that the images of $f_{i,j}$ in J_i and $1 - y_i g'_i$ generate the ideal $J_i + (1 - y_i g'_i)$. Set

$$S' = R[x_1, \ldots, x_m]/\left(\sum h'_i g'_i - 1, f_{i,j}; i = 1, \ldots, n, j = 1, \ldots, m_i\right)$$

There is a surjective R-algebra map $S' \to S$. The classes of the elements $g'_1, \ldots, g'_n$ in S' generate the unit ideal and by construction the maps $S'_{g'_i} \to S_{g_i}$ are injective. Thus we conclude as in part (1). □

24. Glueing functions

00EI In this section we show that given an open covering

$$\operatorname{Spec}(R) = \bigcup\nolimits_{i=1}^{n} D(f_i)$$

by standard opens, and given an element $h_i \in R_{f_i}$ for each i such that $h_i = h_j$ as elements of $R_{f_i f_j}$ then there exists a unique $h \in R$ such that the image of h in R_{f_i} is h_i. This result can be interpreted in two ways:

(1) The rule $D(f) \mapsto R_f$ is a sheaf of rings on the standard opens, see Sheaves, Section 30.
(2) If we think of elements of R_f as the "algebraic" or "regular" functions on $D(f)$, then these glue as would continuous, resp. differentiable functions on a topological, resp. differentiable manifold.

00EK **Lemma 24.1.** *Let R be a ring. Let $f_1, \ldots, f_n$ be elements of R generating the unit ideal. Let M be an R-module. The sequence*

$$0 \to M \xrightarrow{\alpha} \bigoplus\nolimits_{i=1}^{n} M_{f_i} \xrightarrow{\beta} \bigoplus\nolimits_{i,j=1}^{n} M_{f_i f_j}$$

is exact, where $\alpha(m) = (m/1, \ldots, m/1)$ and $\beta(m_1/f_1^{e_1}, \ldots, m_n/f_n^{e_n}) = (m_i/f_i^{e_i} - m_j/f_j^{e_j})_{(i,j)}$.

Proof. It suffices to show that the localization of the sequence at any maximal ideal $\mathfrak{m}$ is exact, see Lemma 23.1. Since $f_1, \ldots, f_n$ generate the unit ideal, there is an i such that $f_i \notin \mathfrak{m}$. After renumbering we may assume $i = 1$. Note that $(M_{f_i})_{\mathfrak{m}} = (M_{\mathfrak{m}})_{f_i}$ and $(M_{f_i f_j})_{\mathfrak{m}} = (M_{\mathfrak{m}})_{f_i f_j}$, see Proposition 9.11. In particular $(M_{f_1})_{\mathfrak{m}} = M_{\mathfrak{m}}$ and $(M_{f_1 f_i})_{\mathfrak{m}} = (M_{\mathfrak{m}})_{f_i}$, because f_1 is a unit. Note that the maps in the sequence are the canonical ones coming from Lemma 9.7 and the identity map on M. Having said all of this, after replacing R by $R_{\mathfrak{m}}$, M by $M_{\mathfrak{m}}$, and f_i by their image in $R_{\mathfrak{m}}$, and f_1 by $1 \in R_{\mathfrak{m}}$, we reduce to the case where $f_1 = 1$.

Assume $f_1 = 1$. Injectivity of α is now trivial. Let $m = (m_i) \in \bigoplus_{i=1}^{n} M_{f_i}$ be in the kernel of β. Then $m_1 \in M_{f_1} = M$. Moreover, $\beta(m) = 0$ implies that m_1 and m_i map to the same element of $M_{f_1 f_i} = M_{f_i}$. Thus $\alpha(m_1) = m$ and the proof is complete. □

00EJ **Lemma 24.2.** *Let R be a ring, and let $f_1, f_2, \ldots f_n \in R$ generate the unit ideal in R. Then the following sequence is exact:*

$$0 \longrightarrow R \longrightarrow \bigoplus\nolimits_{i} R_{f_i} \longrightarrow \bigoplus\nolimits_{i,j} R_{f_i f_j}$$

where the maps $\alpha : R \longrightarrow \bigoplus_i R_{f_i}$ and $\beta : \bigoplus_i R_{f_i} \longrightarrow \bigoplus_{i,j} R_{f_i f_j}$ are defined as

$$\alpha(x) = \left(\frac{x}{1}, \ldots, \frac{x}{1}\right) \text{ and } \beta\left(\frac{x_1}{f_1^{r_1}}, \ldots, \frac{x_n}{f_n^{r_n}}\right) = \left(\frac{x_i}{f_i^{r_i}} - \frac{x_j}{f_j^{r_j}} \text{ in } R_{f_i f_j}\right).$$

Proof. Special case of Lemma 24.1. □

The following we have already seen above, but we state it explicitly here for convenience.

00EM **Lemma 24.3.** *Let R be a ring. If $\operatorname{Spec}(R) = U \amalg V$ with both U and V open then $R \cong R_1 \times R_2$ with $U \cong \operatorname{Spec}(R_1)$ and $V \cong \operatorname{Spec}(R_2)$ via the maps in Lemma 21.2. Moreover, both R_1 and R_2 are localizations as well as quotients of the ring R.*

Proof. By Lemma 21.3 we have $U = D(e)$ and $V = D(1-e)$ for some idempotent e. By Lemma 24.2 we see that $R \cong R_e \times R_{1-e}$ (since clearly $R_{e(1-e)} = 0$ so the glueing condition is trivial; of course it is trivial to prove the product decomposition directly in this case). The lemma follows. $\square$

0565 **Lemma 24.4.** *Let R be a ring. Let $f_1, \ldots, f_n \in R$. Let M be an R-module. Then $M \to \bigoplus M_{f_i}$ is injective if and only if*

$$M \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} M, \quad m \longmapsto (f_1 m, \ldots, f_n m)$$

is injective.

Proof. The map $M \to \bigoplus M_{f_i}$ is injective if and only if for all $m \in M$ and $e_1, \ldots, e_n \geq 1$ such that $f_i^{e_i} m = 0$, $i = 1, \ldots, n$ we have $m = 0$. This clearly implies the displayed map is injective. Conversely, suppose the displayed map is injective and $m \in M$ and $e_1, \ldots, e_n \geq 1$ are such that $f_i^{e_i} m = 0$, $i = 1, \ldots, n$. If $e_i = 1$ for all i, then we immediately conclude that $m = 0$ from the injectivity of the displayed map. Next, we prove this holds for any such data by induction on $e = \sum e_i$. The base case is $e = n$, and we have just dealt with this. If some $e_i > 1$, then set $m' = f_i m$. By induction we see that $m' = 0$. Hence we see that $f_i m = 0$, i.e., we may take $e_i = 1$ which decreases e and we win. $\square$

The following lemma is better stated and proved in the more general context of flat descent. However, it makes sense to state it here since it fits well with the above.

00EQ **Lemma 24.5.** *Let R be a ring. Let $f_1, \ldots, f_n \in R$. Suppose we are given the following data:*

1. *For each i an R_{f_i}-module M_i.*
2. *For each pair i, j an $R_{f_i f_j}$-module isomorphism $\psi_{ij} : (M_i)_{f_j} \to (M_j)_{f_i}$.*

which satisfy the "cocycle condition" that all the diagrams

$$\begin{tikzcd} (M_i)_{f_j f_k} \arrow[rr, "\psi_{ik}"] \arrow[dr, "\psi_{ij}"'] & & (M_k)_{f_i f_j} \\ & (M_j)_{f_i f_k} \arrow[ur, "\psi_{jk}"'] & \end{tikzcd}$$

commute (for all triples i, j, k). Given this data define

$$M = \operatorname{Ker}\left(\bigoplus\nolimits_{1 \leq i \leq n} M_i \longrightarrow \bigoplus\nolimits_{1 \leq i,j \leq n} (M_i)_{f_j}\right)$$

where $(m_1, \ldots, m_n)$ maps to the element whose (i,j)th entry is $m_i/1 - \psi_{ji}(m_j/1)$. Then the natural map $M \to M_i$ induces an isomorphism $M_{f_i} \to M_i$. Moreover $\psi_{ij}(m/1) = m/1$ for all $m \in M$ (with obvious notation).

Proof. To show that $M_{f_1} \to M_1$ is an isomorphism, it suffices to show that its localization at every prime $\mathfrak{p}'$ of R_{f_1} is an isomorphism, see Lemma 23.1. Write $\mathfrak{p}' = \mathfrak{p} R_{f_1}$ for some prime $\mathfrak{p} \subset R$, $f_1 \notin \mathfrak{p}$, see Lemma 17.6. Since localization is exact (Proposition 9.12), we see that

$$\begin{aligned}(M_{f_1})_{\mathfrak p'} &= M_{\mathfrak p} \\ &= \text{Ker}\left(\bigoplus\nolimits_{1\leq i\leq n} M_{i,\mathfrak p} \longrightarrow \bigoplus\nolimits_{1\leq i,j\leq n}((M_i)_{f_j})_{\mathfrak p}\right) \\ &= \text{Ker}\left(\bigoplus\nolimits_{1\leq i\leq n} M_{i,\mathfrak p} \longrightarrow \bigoplus\nolimits_{1\leq i,j\leq n}(M_{i,\mathfrak p})_{f_j}\right)\end{aligned}$$

Here we also used Proposition 9.11. Since f_1 is a unit in $R_{\mathfrak p}$, this reduces us to the case where $f_1 = 1$ by replacing R by $R_{\mathfrak p}$, f_i by the image of f_i in $R_{\mathfrak p}$, M by $M_{\mathfrak p}$, and f_1 by 1.

Assume $f_1 = 1$. Then $\psi_{1j} : (M_1)_{f_j} \to M_j$ is an isomorphism for $j = 2, \ldots, n$. If we use these isomorphisms to identify $M_j = (M_1)_{f_j}$, then we see that $\psi_{ij} : (M_1)_{f_if_j} \to (M_1)_{f_if_j}$ is the canonical identification. Thus the complex

$$0 \to M_1 \to \bigoplus\nolimits_{1\leq i\leq n}(M_1)_{f_i} \longrightarrow \bigoplus\nolimits_{1\leq i,j\leq n}(M_1)_{f_if_j}$$

is exact by Lemma 24.1. Thus the first map identifies M_1 with M in this case and everything is clear. □

25. Zerodivisors and total rings of fractions

02LV The local ring at a minimal prime has the following properties.

00EU **Lemma 25.1.** *Let $\mathfrak p$ be a minimal prime of a ring R. Every element of the maximal ideal of $R_{\mathfrak p}$ is nilpotent. If R is reduced then $R_{\mathfrak p}$ is a field.*

Proof. If some element x of $\mathfrak pR_{\mathfrak p}$ is not nilpotent, then $D(x) \neq \emptyset$, see Lemma 17.2. This contradicts the minimality of $\mathfrak p$. If R is reduced, then $\mathfrak pR_{\mathfrak p} = 0$ and hence it is a field. □

00EW **Lemma 25.2.** *Let R be a reduced ring. Then*

(1) *R is a subring of a product of fields,*
(2) *$R \to \prod_{\mathfrak p \text{ minimal}} R_{\mathfrak p}$ is an embedding into a product of fields,*
(3) *$\bigcup_{\mathfrak p \text{ minimal}} \mathfrak p$ is the set of zerodivisors of R.*

Proof. By Lemma 25.1 each of the rings $R_{\mathfrak p}$ is a field. In particular, the kernel of the ring map $R \to R_{\mathfrak p}$ is $\mathfrak p$. By Lemma 17.2 we have $\bigcap_{\mathfrak p} \mathfrak p = (0)$. Hence (2) and (1) are true. If $xy = 0$ and $y \neq 0$, then $y \notin \mathfrak p$ for some minimal prime $\mathfrak p$. Hence $x \in \mathfrak p$. Thus every zerodivisor of R is contained in $\bigcup_{\mathfrak p \text{ minimal}} \mathfrak p$. Conversely, suppose that $x \in \mathfrak p$ for some minimal prime $\mathfrak p$. Then x maps to zero in $R_{\mathfrak p}$, hence there exists $y \in R$, $y \notin \mathfrak p$ such that $xy = 0$. In other words, x is a zerodivisor. This finishes the proof of (3) and the lemma. □

The total ring of fractions $Q(R)$ of a ring R was introduced in Example 9.8.

02LW **Lemma 25.3.** *Let R be a ring. Let $S \subset R$ be a multiplicative subset consisting of nonzerodivisors. Then $Q(R) \cong Q(S^{-1}R)$. In particular $Q(R) \cong Q(Q(R))$.*

Proof. If $x \in S^{-1}R$ is a nonzerodivisor, and $x = r/f$ for some $r \in R$, $f \in S$, then r is a nonzerodivisor in R. Whence the lemma. □

We can apply glueing results to prove something about total rings of fractions $Q(R)$ which we introduced in Example 9.8.

02LX **Lemma 25.4.** *Let R be a ring. Assume that R has finitely many minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$, and that $\mathfrak{q}_1 \cup \ldots \cup \mathfrak{q}_t$ is the set of zerodivisors of R. Then the total ring of fractions $Q(R)$ is equal to $R_{\mathfrak{q}_1} \times \ldots \times R_{\mathfrak{q}_t}$.*

Proof. There are natural maps $Q(R) \to R_{\mathfrak{q}_i}$ since any nonzerodivisor is contained in $R \setminus \mathfrak{q}_i$. Hence a natural map $Q(R) \to R_{\mathfrak{q}_1} \times \ldots \times R_{\mathfrak{q}_t}$. For any nonminimal prime $\mathfrak{p} \subset R$ we see that $\mathfrak{p} \not\subset \mathfrak{q}_1 \cup \ldots \cup \mathfrak{q}_t$ by Lemma 15.2. Hence $\text{Spec}(Q(R)) = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$ (as subsets of $\text{Spec}(R)$, see Lemma 17.5). Therefore $\text{Spec}(Q(R))$ is a finite discrete set and it follows that $Q(R) = A_1 \times \ldots \times A_t$ with $\text{Spec}(A_i) = \{q_i\}$, see Lemma 24.3. Moreover A_i is a local ring, which is a localization of R. Hence $A_i \cong R_{\mathfrak{q}_i}$. $\square$

26. Irreducible components of spectra

00ER We show that irreducible components of the spectrum of a ring correspond to the minimal primes in the ring.

00ES **Lemma 26.1.** *Let R be a ring.*

1. *For a prime $\mathfrak{p} \subset R$ the closure of $\{\mathfrak{p}\}$ in the Zariski topology is $V(\mathfrak{p})$. In a formula $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$.*
2. *The irreducible closed subsets of $\text{Spec}(R)$ are exactly the subsets $V(\mathfrak{p})$, with $\mathfrak{p} \subset R$ a prime.*
3. *The irreducible components (see Topology, Definition 8.1) of $\text{Spec}(R)$ are exactly the subsets $V(\mathfrak{p})$, with $\mathfrak{p} \subset R$ a minimal prime.*

Proof. Note that if $\mathfrak{p} \in V(I)$, then $I \subset \mathfrak{p}$. Hence, clearly $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$. In particular $V(\mathfrak{p})$ is the closure of a singleton and hence irreducible. The second assertion implies the third. To show the second, let $V(I) \subset \text{Spec}(R)$ with I a radical ideal. If I is not prime, then choose $a, b \in R$, $a, b \notin I$ with $ab \in I$. In this case $V(I, a) \cup V(I, b) = V(I)$, but neither $V(I, b) = V(I)$ nor $V(I, a) = V(I)$, by Lemma 17.2. Hence $V(I)$ is not irreducible. $\square$

In other words, this lemma shows that every irreducible closed subset of $\text{Spec}(R)$ is of the form $V(\mathfrak{p})$ for some prime $\mathfrak{p}$. Since $V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$ we see that each irreducible closed subset has a unique generic point, see Topology, Definition 8.6. In particular, $\text{Spec}(R)$ is a sober topological space. We record this fact in the following lemma.

090M **Lemma 26.2.** *The spectrum of a ring is a spectral space, see Topology, Definition 23.1.*

Proof. Formally this follows from Lemma 26.1 and Lemma 17.11. See also discussion above. $\square$

00ET **Lemma 26.3.** *Let R be a ring. Let $\mathfrak{p} \subset R$ be a prime.*

1. *the set of irreducible closed subsets of $\text{Spec}(R)$ passing through $\mathfrak{p}$ is in one-to-one correspondence with primes $\mathfrak{q} \subset R_{\mathfrak{p}}$.*
2. *The set of irreducible components of $\text{Spec}(R)$ passing through $\mathfrak{p}$ is in one-to-one correspondence with minimal primes $\mathfrak{q} \subset R_{\mathfrak{p}}$.*

Proof. Follows from Lemma 26.1 and the description of $\text{Spec}(R_{\mathfrak{p}})$ in Lemma 17.5 which shows that $\text{Spec}(R_{\mathfrak{p}})$ corresponds to primes $\mathfrak{q}$ in R with $\mathfrak{q} \subset \mathfrak{p}$. $\square$

00EV **Lemma 26.4.** *Let R be a ring. Let $\mathfrak{p}$ be a minimal prime of R. Let $W \subset \text{Spec}(R)$ be a quasi-compact open not containing the point $\mathfrak{p}$. Then there exists an $f \in R$, $f \notin \mathfrak{p}$ such that $D(f) \cap W = \emptyset$.*

Proof. Since W is quasi-compact we may write it as a finite union of standard affine opens $D(g_i)$, $i = 1, \ldots, n$. Since $\mathfrak p \not\in W$ we have $g_i \in \mathfrak p$ for all i. By Lemma 25.1 each g_i is nilpotent in $R_{\mathfrak p}$. Hence we can find an $f \in R$, $f \not\in \mathfrak p$ such that for all i we have $f g_i^{n_i} = 0$ for some $n_i > 0$. Then $D(f)$ works. $\square$

04MG **Lemma 26.5.** *Let R be a ring. Let $X = \mathrm{Spec}(R)$ as a topological space. The following are equivalent*

1. *X is profinite,*
2. *X is Hausdorff,*
3. *X is totally disconnected.*
4. *every quasi-compact open of X is closed,*
5. *there are no nontrivial inclusions between its prime ideals,*
6. *every prime ideal is a maximal ideal,*
7. *every prime ideal is minimal,*
8. *every standard open $D(f) \subset X$ is closed, and*
9. *add more here.*

Proof. First proof. It is clear that (5), (6), and (7) are equivalent. It is clear that (4) and (8) are equivalent as every quasi-compact open is a finite union of standard opens. The implication (7) $\Rightarrow$ (4) follows from Lemma 26.4. Assume (4) holds. Let $\mathfrak p, \mathfrak p'$ be distinct primes of R. Choose an $f \in \mathfrak p'$, $f \not\in \mathfrak p$ (if needed switch $\mathfrak p$ with $\mathfrak p'$). Then $\mathfrak p' \not\in D(f)$ and $\mathfrak p \in D(f)$. By (4) the open $D(f)$ is also closed. Hence $\mathfrak p$ and $\mathfrak p'$ are in disjoint open neighbourhoods whose union is X. Thus X is Hausdorff and totally disconnected. Thus (4) $\Rightarrow$ (2) and (3). If (3) holds then there cannot be any specializations between points of $\mathrm{Spec}(R)$ and we see that (5) holds. If X is Hausdorff then every point is closed, so (2) implies (6). Thus (2), (3), (4), (5), (6), (7) and (8) are equivalent. Any profinite space is Hausdorff, so (1) implies (2). If X satisfies (2) and (3), then X (being quasi-compact by Lemma 17.10) is profinite by Topology, Lemma 22.2.

Second proof. Besides the equivalence of (4) and (8) this follows from Lemma 26.2 and purely topological facts, see Topology, Lemma 23.8. $\square$

27. Examples of spectra of rings

00EX In this section we put some examples of spectra.

00EY **Example 27.1.** In this example we describe $X = \mathrm{Spec}(\mathbf{Z}[x]/(x^2 - 4))$. Let $\mathfrak p$ be an arbitrary prime in X. Let $\phi : \mathbf{Z} \to \mathbf{Z}[x]/(x^2 - 4)$ be the natural ring map. Then, $\phi^{-1}(\mathfrak p)$ is a prime in $\mathbf{Z}$. If $\phi^{-1}(\mathfrak p) = (2)$, then since $\mathfrak p$ contains 2, it corresponds to a prime ideal in $\mathbf{Z}[x]/(x^2 - 4, 2) \cong (\mathbf{Z}/2\mathbf{Z})[x]/(x^2)$ via the map $\mathbf{Z}[x]/(x^2 - 4) \to \mathbf{Z}[x]/(x^2 - 4, 2)$. Any prime in $(\mathbf{Z}/2\mathbf{Z})[x]/(x^2)$ corresponds to a prime in $(\mathbf{Z}/2\mathbf{Z})[x]$ containing (x^2). Such primes will then contain x. Since $(\mathbf{Z}/2\mathbf{Z}) \cong (\mathbf{Z}/2\mathbf{Z})[x]/(x)$ is a field, (x) is a maximal ideal. Since any prime contains (x) and (x) is maximal, the ring contains only one prime (x). Thus, in this case, $\mathfrak p = (2, x)$. Now, if $\phi^{-1}(\mathfrak p) = (q)$ for $q > 2$, then since $\mathfrak p$ contains q, it corresponds to a prime ideal in $\mathbf{Z}[x]/(x^2 - 4, q) \cong (\mathbf{Z}/q\mathbf{Z})[x]/(x^2 - 4)$ via the map $\mathbf{Z}[x]/(x^2 - 4) \to \mathbf{Z}[x]/(x^2 - 4, q)$. Any prime in $(\mathbf{Z}/q\mathbf{Z})[x]/(x^2 - 4)$ corresponds to a prime in $(\mathbf{Z}/q\mathbf{Z})[x]$ containing $(x^2 - 4) = (x - 2)(x + 2)$. Hence, these primes must contain either $x - 2$ or $x + 2$. Since $(\mathbf{Z}/q\mathbf{Z})[x]$ is a PID, all nonzero primes are maximal, and so there are precisely 2 primes in $(\mathbf{Z}/q\mathbf{Z})[x]$ containing $(x - 2)(x + 2)$, namely $(x - 2)$ and $(x + 2)$. In conclusion, there exist two primes $(q, x-2)$ and $(q, x+2)$ since $2 \neq -2 \in \mathbf{Z}/(q)$. Finally, we treat the case where $\phi^{-1}(\mathfrak{p}) = (0)$. Notice that $\mathfrak{p}$ corresponds to a prime ideal in $\mathbf{Z}[x]$ that contains $(x^2-4) = (x-2)(x+2)$. Hence, $\mathfrak{p}$ contains either $(x-2)$ or $(x+2)$. Hence, $\mathfrak{p}$ corresponds to a prime in $\mathbf{Z}[x]/(x-2)$ or one in $\mathbf{Z}[x]/(x+2)$ that intersects $\mathbf{Z}$ only at 0, by assumption. Since $\mathbf{Z}[x]/(x-2) \cong \mathbf{Z}$ and $\mathbf{Z}[x]/(x+2) \cong \mathbf{Z}$, this means that $\mathfrak{p}$ must correspond to 0 in one of these rings. Thus, $\mathfrak{p} = (x-2)$ or $\mathfrak{p} = (x+2)$ in the original ring.

00EZ **Example 27.2.** In this example we describe $X = \mathrm{Spec}(\mathbf{Z}[x])$. Fix $\mathfrak{p} \in X$. Let $\phi : \mathbf{Z} \to \mathbf{Z}[x]$ and notice that $\phi^{-1}(\mathfrak{p}) \in \mathrm{Spec}(\mathbf{Z})$. If $\phi^{-1}(\mathfrak{p}) = (q)$ for q a prime number $q > 0$, then $\mathfrak{p}$ corresponds to a prime in $(\mathbf{Z}/(q))[x]$, which must be generated by a polynomial that is irreducible in $(\mathbf{Z}/(q))[x]$. If we choose a representative of this polynomial with minimal degree, then it will also be irreducible in $\mathbf{Z}[x]$. Hence, in this case $\mathfrak{p} = (q, f_q)$ where f_q is an irreducible polynomial in $\mathbf{Z}[x]$ that is irreducible when viewed in $(\mathbf{Z}/(q)[x])$. Now, assume that $\phi^{-1}(\mathfrak{p}) = (0)$. In this case, $\mathfrak{p}$ must be generated by nonconstant polynomials which, since $\mathfrak{p}$ is prime, may be assumed to be irreducible in $\mathbf{Z}[x]$. By Gauss' lemma, these polynomials are also irreducible in $\mathbf{Q}[x]$. Since $\mathbf{Q}[x]$ is a Euclidean domain, if there are at least two distinct irreducibles f, g generating $\mathfrak{p}$, then $1 = af + bg$ for $a, b \in \mathbf{Q}[x]$. Multiplying through by a common denominator, we see that $m = \bar{a}f + \bar{b}g$ for $\bar{a}, \bar{b} \in \mathbf{Z}[x]$ and nonzero $m \in \mathbf{Z}$. This is a contradiction. Hence, $\mathfrak{p}$ is generated by one irreducible polynomial in $\mathbf{Z}[x]$.

00F0 **Example 27.3.** In this example we describe $X = \mathrm{Spec}(k[x, y])$ when k is an arbitrary field. Clearly (0) is prime, and any principal ideal generated by an irreducible polynomial will also be a prime since $k[x, y]$ is a unique factorization domain. Now assume $\mathfrak{p}$ is an element of X that is not principal. Since $k[x, y]$ is a Noetherian UFD, the prime ideal $\mathfrak{p}$ can be generated by a finite number of irreducible polynomials $(f_1, \ldots, f_n)$. Now, I claim that if f, g are irreducible polynomials in $k[x, y]$ that are not associates, then $(f, g) \cap k[x] \neq 0$. To do this, it is enough to show that f and g are relatively prime when viewed in $k(x)[y]$. In this case, $k(x)[y]$ is a Euclidean domain, so by applying the Euclidean algorithm and clearing denominators, we obtain $p = af + bg$ for $p, a, b \in k[x]$. Thus, assume this is not the case, that is, that some nonunit $h \in k(x)[y]$ divides both f and g. Then, by Gauss's lemma, for some $a, b \in k(x)$ we have $ah | f$ and $bh | g$ for $ah, bh \in k[x]$. By irreducibility, $ah = f$ and $bh = g$ (since $h \notin k(x)$). So, back in $k(x)[y]$, f, g are associates, as $\frac{a}{b} g = f$. Since $k(x)$ is the fraction field of $k[x]$, we can write $g = \frac{r}{s} f$ for elements $r, s \in k[x]$ sharing no common factors. This implies that $sg = rf$ in $k[x, y]$ and so s must divide f since $k[x, y]$ is a UFD. Hence, $s = 1$ or $s = f$. If $s = f$, then $r = g$, implying $f, g \in k[x]$ and thus must be units in $k(x)$ and relatively prime in $k(x)[y]$, contradicting our hypothesis. If $s = 1$, then $g = rf$, another contradiction. Thus, we must have f, g relatively prime in $k(x)[y]$, a Euclidean domain. Thus, we have reduced to the case $\mathfrak{p}$ contains some irreducible polynomial $p \in k[x] \subset k[x, y]$. By the above, $\mathfrak{p}$ corresponds to a prime in the ring $k[x, y]/(p) = k(\alpha)[y]$, where α is an element algebraic over k with minimum polynomial p. This is a PID, and so any prime ideal corresponds to (0) or an irreducible polynomial in $k(\alpha)[y]$. Thus, $\mathfrak{p}$ is of the form (p) or (p, f) where f is a polynomial in $k[x, y]$ that is irreducible in the quotient $k[x, y]/(p)$.

00F1 **Example 27.4.** Consider the ring

$$R = \{f \in \mathbf{Q}[z] \text{ with } f(0) = f(1)\}.$$

Consider the map

$$\varphi : \mathbf{Q}[A, B] \to R$$

defined by $\varphi(A) = z^2 - z$ and $\varphi(B) = z^3 - z^2$. It is easily checked that $(A^3 - B^2 + AB) \subset \mathrm{Ker}(\varphi)$ and that $A^3 - B^2 + AB$ is irreducible. Assume that φ is surjective; then since R is an integral domain (it is a subring of an integral domain), $\mathrm{Ker}(\varphi)$ must be a prime ideal of $\mathbf{Q}[A, B]$. The prime ideals which contain $(A^3 - B^2 + AB)$ are $(A^3 - B^2 + AB)$ itself and any maximal ideal (f, g) with $f, g \in \mathbf{Q}[A, B]$ such that f is irreducible mod g. But R is not a field, so the kernel must be $(A^3 - B^2 + AB)$; hence φ gives an isomorphism $R \to \mathbf{Q}[A, B]/(A^3 - B^2 + AB)$.

To see that φ is surjective, we must express any $f \in R$ as a $\mathbf{Q}$-coefficient polynomial in $A(z) = z^2 - z$ and $B(z) = z^3 - z^2$. Note the relation $zA(z) = B(z)$. Let $a = f(0) = f(1)$. Then $z(z - 1)$ must divide $f(z) - a$, so we can write $f(z) = z(z - 1)g(z) + a = A(z)g(z) + a$. If $\deg(g) < 2$, then $h(z) = c_1z + c_0$ and $f(z) = A(z)(c_1z + c_0) + a = c_1B(z) + c_0A(z) + a$, so we are done. If $\deg(g) \geq 2$, then by the polynomial division algorithm, we can write $g(z) = A(z)h(z) + b_1z + b_0$ ($\deg(h) \leq \deg(g) - 2$), so $f(z) = A(z)^2h(z) + b_1B(z) + b_0A(z)$. Applying division to $h(z)$ and iterating, we obtain an expression for $f(z)$ as a polynomial in $A(z)$ and $B(z)$; hence φ is surjective.

Now let $a \in \mathbf{Q}$, $a \neq 0, \frac{1}{2}, 1$ and consider

$$R_a = \{f \in \mathbf{Q}[z, \frac{1}{z-a}] \text{ with } f(0) = f(1)\}.$$

This is a finitely generated $\mathbf{Q}$-algebra as well: it is easy to check that the functions $z^2 - z$, $z^3 - z$, and $\frac{a^2-a}{z-a} + z$ generate R_a as an $\mathbf{Q}$-algebra. We have the following inclusions:

$$R \subset R_a \subset \mathbf{Q}[z, \frac{1}{z-a}], \quad R \subset \mathbf{Q}[z] \subset \mathbf{Q}[z, \frac{1}{z-a}].$$

Recall (Lemma 17.5) that for a ring T and a multiplicative subset $S \subset T$, the ring map $T \to S^{-1}T$ induces a map on spectra $\mathrm{Spec}(S^{-1}T) \to \mathrm{Spec}(T)$ which is a homeomorphism onto the subset

$$\{\mathfrak{p} \in \mathrm{Spec}(T) \mid S \cap \mathfrak{p} = \emptyset\} \subset \mathrm{Spec}(T).$$

When $S = \{1, f, f^2, \ldots\}$ for some $f \in T$, this is the open set $D(f) \subset T$. We now verify a corresponding property for the ring map $R \to R_a$: we will show that the map $\theta : \mathrm{Spec}(R_a) \to \mathrm{Spec}(R)$ induced by inclusion $R \subset R_a$ is a homeomorphism onto an open subset of $\mathrm{Spec}(R)$ by verifying that θ is an injective local homeomorphism. We do so with respect to an open cover of $\mathrm{Spec}(R_a)$ by two distinguished opens, as we now describe. For any $r \in \mathbf{Q}$, let $\mathrm{ev}_r : R \to \mathbf{Q}$ be the homomorphism given by evaluation at r. Note that for $r = 0$ and $r = 1 - a$, this can be extended to a homomorphism $\mathrm{ev}'_r : R_a \to \mathbf{Q}$ (the latter because $\frac{1}{z-a}$ is well-defined at $z = 1 - a$, since $a \neq \frac{1}{2}$). However, ev_a does not extend to R_a. Write $\mathfrak{m}_r = \mathrm{Ker}(\mathrm{ev}_r)$. We have

$$\mathfrak{m}_0 = (z^2 - z, z^3 - z),$$

$$\mathfrak{m}_a = ((z - 1 + a)(z - a), (z^2 - 1 + a)(z - a)), \text{ and}$$

$$\mathfrak{m}_{1-a} = ((z - 1 + a)(z - a), (z - 1 + a)(z^2 - a)).$$

To verify this, note that the right-hand sides are clearly contained in the left-hand sides. Then check that the right-hand sides are maximal ideals by writing the generators in terms of A and B, and viewing R as $\mathbf{Q}[A,B]/(A^3 - B^2 + AB)$. Note that $\mathfrak{m}_a$ is not in the image of θ: we have

$$(z^2 - z)^2(z-a)\left(\frac{a^2-a}{z-a} + z\right) = (z^2-z)^2(a^2-a) + (z^2-z)^2(z-a)z$$

The left hand side is in $\mathfrak{m}_a R_a$ because $(z^2 - z)(z-a)$ is in $\mathfrak{m}_a$ and because $(z^2 - z)(\frac{a^2-a}{z-a} + z)$ is in R_a. Similarly the element $(z^2-z)^2(z-a)z$ is in $\mathfrak{m}_a R_a$ because $(z^2 - z)$ is in R_a and $(z^2-z)(z-a)$ is in $\mathfrak{m}_a$. As $a \not\in \{0,1\}$ we conclude that $(z^2-z)^2 \in \mathfrak{m}_a R_a$. Hence no ideal I of R_a can satisfy $I \cap R = \mathfrak{m}_a$, as such an I would have to contain $(z^2-z)^2$, which is in R but not in $\mathfrak{m}_a$. The distinguished open set $D((z-1+a)(z-a)) \subset \text{Spec}(R)$ is equal to the complement of the closed set $\{\mathfrak{m}_a, \mathfrak{m}_{1-a}\}$. Then check that $R_{(z-1+a)(z-a)} = (R_a)_{(z-1+a)(z-a)}$; calling this localized ring R', then, it follows that the map $R \to R'$ factors as $R \to R_a \to R'$. By Lemma 17.5, then, these maps express $\text{Spec}(R') \subset \text{Spec}(R_a)$ and $\text{Spec}(R') \subset \text{Spec}(R)$ as open subsets; hence $\theta : \text{Spec}(R_a) \to \text{Spec}(R)$, when restricted to $D((z-1+a)(z-a))$, is a homeomorphism onto an open subset. Similarly, θ restricted to $D((z^2+z+2a-2)(z-a)) \subset \text{Spec}(R_a)$ is a homeomorphism onto the open subset $D((z^2+z+2a-2)(z-a)) \subset \text{Spec}(R)$. Depending on whether $z^2+z+2a-2$ is irreducible or not over $\mathbf{Q}$, this former distinguished open set has complement equal to one or two closed points along with the closed point $\mathfrak{m}_a$. Furthermore, the ideal in R_a generated by the elements $(z^2+z+2a-a)(z-a)$ and $(z-1+a)(z-a)$ is all of R_a, so these two distinguished open sets cover $\text{Spec}(R_a)$. Hence in order to show that θ is a homeomorphism onto $\text{Spec}(R) - \{\mathfrak{m}_a\}$, it suffices to show that these one or two points can never equal $\mathfrak{m}_{1-a}$. And this is indeed the case, since $1-a$ is a root of $z^2+z+2a-2$ if and only if $a = 0$ or $a = 1$, both of which do not occur.

Despite this homeomorphism which mimics the behavior of a localization at an element of R, while $\mathbf{Q}[z, \frac{1}{z-a}]$ is the localization of $\mathbf{Q}[z]$ at the maximal ideal $(z-a)$, the ring R_a is *not* a localization of R: Any localization $S^{-1}R$ results in more units than the original ring R. The units of R are $\mathbf{Q}^\times$, the units of $\mathbf{Q}$. In fact, it is easy to see that the units of R_a are $\mathbf{Q}^*$. Namely, the units of $\mathbf{Q}[z, \frac{1}{z-a}]$ are $c(z-a)^n$ for $c \in \mathbf{Q}^*$ and $n \in \mathbf{Z}$ and it is clear that these are in R_a only if $n = 0$. Hence R_a has no more units than R does, and thus cannot be a localization of R.

We used the fact that $a \neq 0, 1$ to ensure that $\frac{1}{z-a}$ makes sense at $z = 0, 1$. We used the fact that $a \neq 1/2$ in a few places: (1) In order to be able to talk about the kernel of ev_{1-a} on R_a, which ensures that $\mathfrak{m}_{1-a}$ is a point of R_a (i.e., that R_a is missing just one point of R). (2) At the end in order to conclude that $(z-a)^{k+\ell}$ can only be in R for $k = \ell = 0$; indeed, if $a = 1/2$, then this is in R as long as $k + \ell$ is even. Hence there would indeed be more units in R_a than in R, and R_a could possibly be a localization of R.

28. A meta-observation about prime ideals

05K7 This section is taken from the CRing project. Let R be a ring and let $S \subset R$ be a multiplicative subset. A consequence of Lemma 17.5 is that an ideal $I \subset R$ maximal with respect to the property of not intersecting S is prime. The reason is that $I = R \cap \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of the ring $S^{-1}R$. It turns out that for many properties of ideals, the maximal ones are prime. A general method of seeing this was developed in [LR08]. In this section, we digress to explain this phenomenon.

Let R be a ring. If I is an ideal of R and $a \in R$, we define

$$(I : a) = \{x \in R \mid xa \in I\}.$$

More generally, if $J \subset R$ is an ideal, we define

$$(I : J) = \{x \in R \mid xJ \subset I\}.$$

05K8 **Lemma 28.1.** *Let R be a ring. For a principal ideal $J \subset R$, and for any ideal $I \subset J$ we have $I = J(I : J)$.*

Proof. Say $J = (a)$. Then $(I : J) = (I : a)$. Since $I \subset J$ we see that any $y \in I$ is of the form $y = xa$ for some $x \in (I : a)$. Hence $I \subset J(I : J)$. Conversely, if $x \in (I : a)$, then $xJ = (xa) \subset I$, which proves the other inclusion. $\square$

Let $\mathcal{F}$ be a collection of ideals of R. We are interested in conditions that will guarantee that the maximal elements in the complement of $\mathcal{F}$ are prime.

05K9 **Definition 28.2.** Let R be a ring. Let $\mathcal{F}$ be a set of ideals of R. We say $\mathcal{F}$ is an *Oka family* if $R \in \mathcal{F}$ and whenever $I \subset R$ is an ideal and $(I : a), (I, a) \in \mathcal{F}$ for some $a \in R$, then $I \in \mathcal{F}$.

Let us give some examples of Oka families. The first example is the basic example discussed in the introduction to this section.

05KA **Example 28.3.** Let R be a ring and let S be a multiplicative subset of R. We claim that $\mathcal{F} = \{I \subset R \mid I \cap S \neq \emptyset\}$ is an Oka family. Namely, suppose that $(I : a), (I, a) \in \mathcal{F}$ for some $a \in R$. Then pick $s \in (I, a) \cap S$ and $s' \in (I : a) \cap S$. Then $ss' \in I \cap S$ and hence $I \in \mathcal{F}$. Thus $\mathcal{F}$ is an Oka family.

05KB **Example 28.4.** Let R be a ring, $I \subset R$ an ideal, and $a \in R$. If $(I : a)$ is generated by $a_1, \ldots, a_n$ and (I, a) is generated by $a, b_1, \ldots, b_m$ with $b_1, \ldots, b_m \in I$, then I is generated by $aa_1, \ldots, aa_n, b_1, \ldots, b_m$. To see this, note that if $x \in I$, then $x \in (I, a)$ is a linear combination of $a, b_1, \ldots, b_m$, but the coefficient of a must lie in $(I : a)$. As a result, we deduce that the family of finitely generated ideals is an Oka family.

05KC **Example 28.5.** Let us show that the family of principal ideals of a ring R is an Oka family. Indeed, suppose $I \subset R$ is an ideal, $a \in R$, and (I, a) and $(I : a)$ are principal. Note that $(I : a) = (I : (I, a))$. Setting $J = (I, a)$, we find that J is principal and $(I : J)$ is too. By Lemma 28.1 we have $I = J(I : J)$. Thus we find in our situation that since $J = (I, a)$ and $(I : J)$ are principal, I is principal.

05KD **Example 28.6.** Let R be a ring. Let κ be an infinite cardinal. The family of ideals which can be generated by at most κ elements is an Oka family. The argument is analogous to the argument in Example 28.4 and is omitted.

0G1N **Example 28.7.** Let A be a ring, $I \subset A$ an ideal, and $a \in A$ an element. There is a short exact sequence $0 \to A/(I : a) \to A/I \to A/(I, a) \to 0$ where the first arrow is given by multiplication by a. Thus if P is a property of A-modules that is stable under extensions and holds for 0, then the family of ideals I such that A/I has P is an Oka family.

05KE **Proposition 28.8.** *If $\mathcal{F}$ is an Oka family of ideals, then any maximal element of the complement of $\mathcal{F}$ is prime.*

Proof. Suppose $I \notin \mathcal{F}$ is maximal with respect to not being in $\mathcal{F}$ but I is not prime. Note that $I \neq R$ because $R \in \mathcal{F}$. Since I is not prime we can find $a, b \in R - I$ with $ab \in I$. It follows that $(I, a) \neq I$ and $(I : a)$ contains $b \notin I$ so also $(I : a) \neq I$. Thus $(I : a), (I, a)$ both strictly contain I, so they must belong to $\mathcal{F}$. By the Oka condition, we have $I \in \mathcal{F}$, a contradiction. $\square$

At this point we are able to turn most of the examples above into a lemma about prime ideals in a ring.

05KF **Lemma 28.9.** *Let R be a ring. Let S be a multiplicative subset of R. An ideal $I \subset R$ which is maximal with respect to the property that $I \cap S = \emptyset$ is prime.*

Proof. This is the example discussed in the introduction to this section. For an alternative proof, combine Example 28.3 with Proposition 28.8. $\square$

05KG **Lemma 28.10.** *Let R be a ring.*

1. *An ideal $I \subset R$ maximal with respect to not being finitely generated is prime.*
2. *If every prime ideal of R is finitely generated, then every ideal of R is finitely generated*[2].

Proof. The first assertion is an immediate consequence of Example 28.4 and Proposition 28.8. For the second, suppose that there exists an ideal $I \subset R$ which is not finitely generated. The union of a totally ordered chain $\{I_\alpha\}$ of ideals that are not finitely generated is not finitely generated; indeed, if $I = \bigcup I_\alpha$ were generated by $a_1, \ldots, a_n$, then all the generators would belong to some I_α and would consequently generate it. By Zorn's lemma, there is an ideal maximal with respect to being not finitely generated. By the first part this ideal is prime. $\square$

05KH **Lemma 28.11.** *Let R be a ring.*

1. *An ideal $I \subset R$ maximal with respect to not being principal is prime.*
2. *If every prime ideal of R is principal, then every ideal of R is principal.*

Proof. The first part follows from Example 28.5 and Proposition 28.8. For the second, suppose that there exists an ideal $I \subset R$ which is not principal. The union of a totally ordered chain $\{I_\alpha\}$ of ideals that not principal is not principal; indeed, if $I = \bigcup I_\alpha$ were generated by a, then a would belong to some I_α and a would generate it. By Zorn's lemma, there is an ideal maximal with respect to not being principal. This ideal is necessarily prime by the first part. $\square$

05KI **Lemma 28.12.** *Let R be a ring.*

1. *An ideal maximal among the ideals which do not contain a nonzerodivisor is prime.*
2. *If R is nonzero and every nonzero prime ideal in R contains a nonzerodivisor, then R is a domain.*

Proof. Consider the set S of nonzerodivisors. It is a multiplicative subset of R. Hence any ideal maximal with respect to not intersecting S is prime, see Lemma 28.9. Thus, if every nonzero prime ideal contains a nonzerodivisor, then (0) is prime, i.e., R is a domain. $\square$

[2]Later we will say that R is Noetherian.

05KJ **Remark 28.13.** Let R be a ring. Let κ be an infinite cardinal. By applying Example 28.6 and Proposition 28.8 we see that any ideal maximal with respect to the property of not being generated by κ elements is prime. This result is not so useful because there exists a ring for which every prime ideal of R can be generated by $\aleph_0$ elements, but some ideal cannot. Namely, let k be a field, let T be a set whose cardinality is greater than $\aleph_0$ and let

$$R = k[\{x_n\}_{n \geq 1}, \{z_{t,n}\}_{t \in T, n \geq 0}]/(x_n^2, z_{t,n}^2, x_n z_{t,n} - z_{t,n-1})$$

This is a local ring with unique prime ideal $\mathfrak{m} = (x_n)$. But the ideal $(z_{t,n})$ cannot be generated by countably many elements.

0G2Z **Example 28.14.** Let R be a ring and $X = \operatorname{Spec}(R)$. Since closed subsets of X correspond to radical ideas of R (Lemma 17.2) we see that X is a Noetherian topological space if and only if we have ACC for radical ideals. This holds if and only if every radical ideal is the radical of a finitely generated ideal (details omitted). Let Comment by Lukas Heger of November 12, 2020.

$$\mathcal{F} = \{I \subset R \mid \sqrt{I} = \sqrt{(f_1, \ldots, f_n)} \text{ for some } n \text{ and } f_1, \ldots, f_n \in R\}.$$

The reader can show that $\mathcal{F}$ is an Oka family by using the identity

$$\sqrt{I} = \sqrt{(I, a)(I : a)}$$

which holds for any ideal $I \subset R$ and any element $a \in R$. On the other hand, if we have a totally ordered chain of ideals $\{I_\alpha\}$ none of which are in $\mathcal{F}$, then the union $I = \bigcup I_\alpha$ cannot be in $\mathcal{F}$ either. Otherwise $\sqrt{I} = \sqrt{(f_1, \ldots, f_n)}$, then $f_i^e \in I$ for some e, then $f_i^e \in I_\alpha$ for some α independent of i, then $\sqrt{I_\alpha} = \sqrt{(f_1, \ldots, f_n)}$, contradiction. Thus if the set of ideals not in $\mathcal{F}$ is nonempty, then it has maximal elements and exactly as in Lemma 28.10 we conclude that X is a Noetherian topological space if and only if every prime ideal of R is equal to $\sqrt{(f_1, \ldots, f_n)}$ for some $f_1, \ldots, f_n \in R$. If we ever need this result we will carefully state and prove this result here.

29. Images of ring maps of finite presentation

00F5 In this section we prove some results on the topology of maps $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ induced by ring maps $R \to S$, mainly Chevalley's Theorem. In order to do this we will use the notions of constructible sets, quasi-compact sets, retrocompact sets, and so on which are defined in Topology, Section 15.

00F6 **Lemma 29.1.** *Let $U \subset \operatorname{Spec}(R)$ be open. The following are equivalent:*

(1) *U is retrocompact in $\operatorname{Spec}(R)$,*
(2) *U is quasi-compact,*
(3) *U is a finite union of standard opens, and*
(4) *there exists a finitely generated ideal $I \subset R$ such that $X \setminus V(I) = U$.*

Proof. We have (1) $\Rightarrow$ (2) because $\operatorname{Spec}(R)$ is quasi-compact, see Lemma 17.10. We have (2) $\Rightarrow$ (3) because standard opens form a basis for the topology. Proof of (3) $\Rightarrow$ (1). Let $U = \bigcup_{i=1\ldots n} D(f_i)$. To show that U is retrocompact in $\operatorname{Spec}(R)$ it suffices to show that $U \cap V$ is quasi-compact for any quasi-compact open V of $\operatorname{Spec}(R)$. Write $V = \bigcup_{j=1\ldots m} D(g_j)$ which is possible by (2) $\Rightarrow$ (3). Each standard open is homeomorphic to the spectrum of a ring and hence quasi-compact, see Lemmas 17.6 and 17.10. Thus $U \cap V = (\bigcup_{i=1\ldots n} D(f_i)) \cap (\bigcup_{j=1\ldots m} D(g_j)) = \bigcup_{i,j} D(f_i g_j)$ is a finite union of quasi-compact opens hence quasi-compact. To finish the proof note that (4) is equivalent to (3) by Lemma 17.2. $\square$

00F7 **Lemma 29.2.** *Let $\varphi : R \to S$ be a ring map. The induced continuous map $f : \text{Spec}(S) \to \text{Spec}(R)$ is quasi-compact. For any constructible set $E \subset \text{Spec}(R)$ the inverse image $f^{-1}(E)$ is constructible in $\text{Spec}(S)$.*

Proof. We first show that the inverse image of any quasi-compact open $U \subset \text{Spec}(R)$ is quasi-compact. By Lemma 29.1 we may write U as a finite open of standard opens. Thus by Lemma 17.4 we see that $f^{-1}(U)$ is a finite union of standard opens. Hence $f^{-1}(U)$ is quasi-compact by Lemma 29.1 again. The second assertion now follows from Topology, Lemma 15.3. □

0G1P **Lemma 29.3.** *Let R be a ring. A subset of $\text{Spec}(R)$ is constructible if and only if it can be written as a finite union of subsets of the form $D(f) \cap V(g_1, \ldots, g_m)$ for $f, g_1, \ldots, g_m \in R$.*

Proof. By Lemma 29.1 the subset $D(f)$ and the complement of $V(g_1, \ldots, g_m)$ are retro-compact open. Hence $D(f) \cap V(g_1, \ldots, g_m)$ is a constructible subset and so is any finite union of such. Conversely, let $T \subset \text{Spec}(R)$ be constructible. By Topology, Definition 15.1, we may assume that $T = U \cap V^c$, where $U, V \subset \text{Spec}(R)$ are retrocompact open. By Lemma 29.1 we may write $U = \bigcup_{i=1,\ldots,n} D(f_i)$ and $V = \bigcup_{j=1,\ldots,m} D(g_j)$. Then $T = \bigcup_{i=1,\ldots,n} \big(D(f_i) \cap V(g_1, \ldots, g_m)\big)$. □

00F8 **Lemma 29.4.** *Let R be a ring and let $T \subset \text{Spec}(R)$ be constructible. Then there exists a ring map $R \to S$ of finite presentation such that T is the image of $\text{Spec}(S)$ in $\text{Spec}(R)$.*

Proof. The spectrum of a finite product of rings is the disjoint union of the spectra, see Lemma 21.2. Hence if $T = T_1 \cup T_2$ and the result holds for T_1 and T_2, then the result holds for T. By Lemma 29.3 we may assume that $T = D(f) \cap V(g_1, \ldots, g_m)$. In this case T is the image of the map $\text{Spec}((R/(g_1, \ldots, g_m))_f) \to \text{Spec}(R)$, see Lemmas 17.6 and 17.7. □

00F9 **Lemma 29.5.** *Let R be a ring. Let f be an element of R. Let $S = R_f$. Then the image of a constructible subset of $\text{Spec}(S)$ is constructible in $\text{Spec}(R)$.*

Proof. We repeatedly use Lemma 29.1 without mention. Let U, V be quasi-compact open in $\text{Spec}(S)$. We will show that the image of $U \cap V^c$ is constructible. Under the identification $\text{Spec}(S) = D(f)$ of Lemma 17.6 the sets U, V correspond to quasi-compact opens U', V' of $\text{Spec}(R)$. Hence it suffices to show that $U' \cap (V')^c$ is constructible in $\text{Spec}(R)$ which is clear. □

00FA **Lemma 29.6.** *Let R be a ring. Let I be a finitely generated ideal of R. Let $S = R/I$. Then the image of a constructible subset of $\text{Spec}(S)$ is constructible in $\text{Spec}(R)$.*

Proof. If $I = (f_1, \ldots, f_m)$, then we see that $V(I)$ is the complement of $\bigcup D(f_i)$, see Lemma 17.2. Hence it is constructible, by Lemma 29.1. Denote the map $R \to S$ by $f \mapsto \overline{f}$. We have to show that if $\overline{U}, \overline{V}$ are retrocompact opens of $\text{Spec}(S)$, then the image of $\overline{U} \cap \overline{V}^c$ in $\text{Spec}(R)$ is constructible. By Lemma 29.1 we may write $\overline{U} = \bigcup D(\overline{g_i})$. Setting $U = \bigcup D(g_i)$ we see $\overline{U}$ has image $U \cap V(I)$ which is constructible in $\text{Spec}(R)$. Similarly the image of $\overline{V}$ equals $V \cap V(I)$ for some retrocompact open V of $\text{Spec}(R)$. Hence the image of $\overline{U} \cap \overline{V}^c$ equals $U \cap V(I) \cap V^c$ as desired. □

00FB **Lemma 29.7.** *Let R be a ring. The map $\mathrm{Spec}(R[x]) \to \mathrm{Spec}(R)$ is open, and the image of any standard open is a quasi-compact open.*

Proof. It suffices to show that the image of a standard open $D(f)$, $f \in R[x]$ is quasi-compact open. The image of $D(f)$ is the image of $\mathrm{Spec}(R[x]_f) \to \mathrm{Spec}(R)$. Let $\mathfrak p \subset R$ be a prime ideal. Let $\overline{f}$ be the image of f in $\kappa(\mathfrak p)[x]$. Recall, see Lemma 17.9, that $\mathfrak p$ is in the image if and only if $R[x]_f \otimes_R \kappa(\mathfrak p) = \kappa(\mathfrak p)[x]_{\overline{f}}$ is not the zero ring. This is exactly the condition that f does not map to zero in $\kappa(\mathfrak p)[x]$, in other words, that some coefficient of f is not in $\mathfrak p$. Hence we see: if $f = a_d x^d + \ldots + a_0$, then the image of $D(f)$ is $D(a_d) \cup \ldots \cup D(a_0)$. $\square$

We prove a property of characteristic polynomials which will be used below.

00FC **Lemma 29.8.** *Let $R \to A$ be a ring homomorphism. Assume $A \cong R^{\oplus n}$ as an R-module. Let $f \in A$. The multiplication map $m_f : A \to A$ is R-linear and hence has a characteristic polynomial $P(T) = T^n + r_{n-1}T^{n-1} + \ldots + r_0 \in R[T]$. For any prime $\mathfrak p \in \mathrm{Spec}(R)$, f acts nilpotently on $A \otimes_R \kappa(\mathfrak p)$ if and only if $\mathfrak p \in V(r_0, \ldots, r_{n-1})$.*

Proof. This follows quite easily once we prove that the characteristic polynomial $\bar P(T) \in \kappa(\mathfrak p)[T]$ of the multiplication map $m_{\bar f} : A \otimes_R \kappa(\mathfrak p) \to A \otimes_R \kappa(\mathfrak p)$ which multiplies elements of $A \otimes_R \kappa(\mathfrak p)$ by $\bar f$, the image of f viewed in $\kappa(\mathfrak p)$, is just the image of $P(T)$ in $\kappa(\mathfrak p)[T]$. Let (a_{ij}) be the matrix of the map m_f with entries in R, using a basis $e_1, \ldots, e_n$ of A as an R-module. Then, $A \otimes_R \kappa(\mathfrak p) \cong (R \otimes_R \kappa(\mathfrak p))^{\oplus n} = \kappa(\mathfrak p)^n$, which is an n-dimensional vector space over $\kappa(\mathfrak p)$ with basis $e_1 \otimes 1, \ldots, e_n \otimes 1$. The image $\bar f = f \otimes 1$, and so the multiplication map $m_{\bar f}$ has matrix $(a_{ij} \otimes 1)$. Thus, the characteristic polynomial is precisely the image of $P(T)$.

From linear algebra, we know that a linear transformation acts nilpotently on an n-dimensional vector space if and only if the characteristic polynomial is T^n (since the characteristic polynomial divides some power of the minimal polynomial). Hence, f acts nilpotently on $A \otimes_R \kappa(\mathfrak p)$ if and only if $\bar P(T) = T^n$. This occurs if and only if $r_i \in \mathfrak p$ for all $0 \le i \le n-1$, that is when $\mathfrak p \in V(r_0, \ldots, r_{n-1})$. $\square$

00FD **Lemma 29.9.** *Let R be a ring. Let $f, g \in R[x]$ be polynomials. Assume the leading coefficient of g is a unit of R. There exists elements $r_i \in R$, $i = 1 \ldots, n$ such that the image of $D(f) \cap V(g)$ in $\mathrm{Spec}(R)$ is $\bigcup_{i=1,\ldots,n} D(r_i)$.*

Proof. Write $g = ux^d + a_{d-1}x^{d-1} + \ldots + a_0$, where d is the degree of g, and hence $u \in R^*$. Consider the ring $A = R[x]/(g)$. It is, as an R-module, finite free with basis the images of $1, x, \ldots, x^{d-1}$. Consider multiplication by (the image of) f on A. This is an R-module map. Hence we can let $P(T) \in R[T]$ be the characteristic polynomial of this map. Write $P(T) = T^d + r_{d-1}T^{d-1} + \ldots + r_0$. We claim that $r_0, \ldots, r_{d-1}$ have the desired property. We will use below the property of characteristic polynomials that

$$\mathfrak p \in V(r_0, \ldots, r_{d-1}) \Leftrightarrow \text{multiplication by } f \text{ is nilpotent on } A \otimes_R \kappa(\mathfrak p).$$

This was proved in Lemma 29.8.

Suppose $\mathfrak q \in D(f) \cap V(g)$, and let $\mathfrak p = \mathfrak q \cap R$. Then there is a nonzero map $A \otimes_R \kappa(\mathfrak p) \to \kappa(\mathfrak q)$ which is compatible with multiplication by f. And f acts as a unit on $\kappa(\mathfrak q)$. Thus we conclude $\mathfrak p \notin V(r_0, \ldots, r_{d-1})$.

On the other hand, suppose that $r_i \not\in \mathfrak{p}$ for some prime $\mathfrak{p}$ of R and some $0 \leq i \leq d-1$. Then multiplication by f is not nilpotent on the algebra $A \otimes_R \kappa(\mathfrak{p})$. Hence there exists a prime ideal $\overline{\mathfrak{q}} \subset A \otimes_R \kappa(\mathfrak{p})$ not containing the image of f. The inverse image of $\overline{\mathfrak{q}}$ in $R[x]$ is an element of $D(f) \cap V(g)$ mapping to $\mathfrak{p}$. $\square$

00FE **Theorem 29.10** (Chevalley's Theorem)**.** *Suppose that $R \to S$ is of finite presentation. The image of a constructible subset of* $\text{Spec}(S)$ *in* $\text{Spec}(R)$ *is constructible.*

Proof. Write $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. We may factor $R \to S$ as $R \to R[x_1] \to R[x_1, x_2] \to \ldots \to R[x_1, \ldots, x_{n-1}] \to S$. Hence we may assume that $S = R[x]/(f_1, \ldots, f_m)$. In this case we factor the map as $R \to R[x] \to S$, and by Lemma 29.6 we reduce to the case $S = R[x]$. By Lemma 29.1 suffices to show that if $T = (\bigcup_{i=1\ldots n} D(f_i)) \cap V(g_1, \ldots, g_m)$ for $f_i, g_j \in R[x]$ then the image in $\text{Spec}(R)$ is constructible. Since finite unions of constructible sets are constructible, it suffices to deal with the case $n = 1$, i.e., when $T = D(f) \cap V(g_1, \ldots, g_m)$.

Note that if $c \in R$, then we have

$$\text{Spec}(R) = V(c) \amalg D(c) = \text{Spec}(R/(c)) \amalg \text{Spec}(R_c),$$

and correspondingly $\text{Spec}(R[x]) = V(c) \amalg D(c) = \text{Spec}(R/(c)[x]) \amalg \text{Spec}(R_c[x])$. The intersection of $T = D(f) \cap V(g_1, \ldots, g_m)$ with each part still has the same shape, with f, g_i replaced by their images in $R/(c)[x]$, respectively $R_c[x]$. Note that the image of T in $\text{Spec}(R)$ is the union of the image of $T \cap V(c)$ and $T \cap D(c)$. Using Lemmas 29.5 and 29.6 it suffices to prove the images of both parts are constructible in $\text{Spec}(R/(c))$, respectively $\text{Spec}(R_c)$.

Let us assume we have $T = D(f) \cap V(g_1, \ldots, g_m)$ as above, with $\deg(g_1) \leq \deg(g_2) \leq \ldots \leq \deg(g_m)$. We are going to use induction on m, and on the degrees of the g_i. Let $d = \deg(g_1)$, i.e., $g_1 = cx^{d_1} + l.o.t$ with $c \in R$ not zero. Cutting R up into the pieces $R/(c)$ and R_c we either lower the degree of g_1 (and this is covered by induction) or we reduce to the case where c is invertible. If c is invertible, and $m > 1$, then write $g_2 = c'x^{d_2} + l.o.t.$ In this case consider $g_2' = g_2 - (c'/c)x^{d_2 - d_1} g_1$. Since the ideals $(g_1, g_2, \ldots, g_m)$ and $(g_1, g_2', g_3, \ldots, g_m)$ are equal we see that $T = D(f) \cap V(g_1, g_2', g_3 \ldots, g_m)$. But here the degree of g_2' is strictly less than the degree of g_2 and hence this case is covered by induction.

The bases case for the induction above are the cases (a) $T = D(f) \cap V(g)$ where the leading coefficient of g is invertible, and (b) $T = D(f)$. These two cases are dealt with in Lemmas 29.9 and 29.7. $\square$

30. More on images

00FF In this section we collect a few additional lemmas concerning the image on Spec for ring maps. See also Section 41 for example.

00FG **Lemma 30.1.** *Let $R \subset S$ be an inclusion of domains. Assume that $R \to S$ is of finite type. There exists a nonzero $f \in R$, and a nonzero $g \in S$ such that $R_f \to S_{fg}$ is of finite presentation.*

Proof. By induction on the number of generators of S over R. During the proof we may replace R by R_f and S by S_f for some nonzero $f \in R$.

Suppose that S is generated by a single element over R. Then $S = R[x]/\mathfrak{q}$ for some prime ideal $\mathfrak{q} \subset R[x]$. If $\mathfrak{q} = (0)$ there is nothing to prove. If $\mathfrak{q} \neq (0)$, then let $h \in \mathfrak{q}$

be a nonzero element with minimal degree in x. Write $h = fx^d + a_{d-1}x^{d-1} + \ldots + a_0$ with $a_i \in R$ and $f \neq 0$. After inverting f in R and S we may assume that h is monic. We obtain a surjective R-algebra map $R[x]/(h) \to S$. We have $R[x]/(h) = R \oplus Rx \oplus \ldots \oplus Rx^{d-1}$ as an R-module and by minimality of d we see that $R[x]/(h)$ maps injectively into S. Thus $R[x]/(h) \cong S$ is finitely presented over R.

Suppose that S is generated by $n > 1$ elements over R. Say $x_1, \ldots, x_n \in S$ generate S. Denote $S' \subset S$ the subring generated by $x_1, \ldots, x_{n-1}$. By induction hypothesis we see that there exist $f \in R$ and $g \in S'$ nonzero such that $R_f \to S'_{fg}$ is of finite presentation. Next we apply the induction hypothesis to $S'_{fg} \to S_{fg}$ to see that there exist $f' \in S'_{fg}$ and $g' \in S_{fg}$ such that $S'_{fgf'} \to S_{fgf'g'}$ is of finite presentation. We leave it to the reader to conclude. □

00FH **Lemma 30.2.** *Let $R \to S$ be a finite type ring map. Denote $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S)$. Write $f : Y \to X$ the induced map of spectra. Let $E \subset Y = \mathrm{Spec}(S)$ be a constructible set. If a point $\xi \in X$ is in $f(E)$, then $\overline{\{\xi\}} \cap f(E)$ contains an open dense subset of $\overline{\{\xi\}}$.*

Proof. Let $\xi \in X$ be a point of $f(E)$. Choose a point $\eta \in E$ mapping to ξ. Let $\mathfrak{p} \subset R$ be the prime corresponding to ξ and let $\mathfrak{q} \subset S$ be the prime corresponding to η. Consider the diagram

$$\begin{tikzcd} \eta \arrow[r] \arrow[d, mapsto] & E \cap Y' \arrow[r] \arrow[d] & Y' = \mathrm{Spec}(S/\mathfrak{q}) \arrow[r] \arrow[d] & Y \arrow[d] \\ \xi \arrow[r] & f(E) \cap X' \arrow[r] & X' = \mathrm{Spec}(R/\mathfrak{p}) \arrow[r] & X \end{tikzcd}$$

By Lemma 29.2 the set $E \cap Y'$ is constructible in Y'. It follows that we may replace X by X' and Y by Y'. Hence we may assume that $R \subset S$ is an inclusion of domains, ξ is the generic point of X, and η is the generic point of Y. By Lemma 30.1 combined with Chevalley's theorem (Theorem 29.10) we see that there exist dense opens $U \subset X$, $V \subset Y$ such that $f(V) \subset U$ and such that $f : V \to U$ maps constructible sets to constructible sets. Note that $E \cap V$ is constructible in V, see Topology, Lemma 15.4. Hence $f(E \cap V)$ is constructible in U and contains ξ. By Topology, Lemma 15.15 we see that $f(E \cap V)$ contains a dense open $U' \subset U$. □

At the end of this section we present a few more results on images of maps on Spectra that have nothing to do with constructible sets.

00FI **Lemma 30.3.** *Let $\varphi : R \to S$ be a ring map. The following are equivalent:*

1. *The map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective.*
2. *For any ideal $I \subset R$ the inverse image of $\sqrt{IS}$ in R is equal to $\sqrt{I}$.*
3. *For any radical ideal $I \subset R$ the inverse image of IS in R is equal to I.*
4. *For every prime $\mathfrak{p}$ of R the inverse image of $\mathfrak{p}S$ in R is $\mathfrak{p}$.*

In this case the same is true after any base change: Given a ring map $R \to R'$ the ring map $R' \to R' \otimes_R S$ has the equivalent properties (1), (2), (3) as well.

Proof. If $J \subset S$ is an ideal, then $\sqrt{\varphi^{-1}(J)} = \varphi^{-1}(\sqrt{J})$. This shows that (2) and (3) are equivalent. The implication (3) ⇒ (4) is immediate. If $I \subset R$ is a radical ideal, then Lemma 17.2 guarantees that $I = \bigcap_{I \subset \mathfrak{p}} \mathfrak{p}$. Hence (4) ⇒ (2). By Lemma 17.9 we have $\mathfrak{p} = \varphi^{-1}(\mathfrak{p}S)$ if and only if $\mathfrak{p}$ is in the image. Hence (1) ⇔ (4). Thus (1), (2), (3), and (4) are equivalent.

Assume (1) holds. Let $R \to R'$ be a ring map. Let $\mathfrak{p}' \subset R'$ be a prime ideal lying over the prime $\mathfrak{p}$ of R. To see that $\mathfrak{p}'$ is in the image of $\mathrm{Spec}(R' \otimes_R S) \to \mathrm{Spec}(R')$ we have to show that $(R' \otimes_R S) \otimes_{R'} \kappa(\mathfrak{p}')$ is not zero, see Lemma 17.9. But we have

$$(R' \otimes_R S) \otimes_{R'} \kappa(\mathfrak{p}') = S \otimes_R \kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}')$$

which is not zero as $S \otimes_R \kappa(\mathfrak{p})$ is not zero by assumption and $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{p}')$ is an extension of fields. □

00FJ **Lemma 30.4.** *Let R be a domain. Let $\varphi : R \to S$ be a ring map. The following are equivalent:*

1. *The ring map $R \to S$ is injective.*
2. *The image $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ contains a dense set of points.*
3. *There exists a prime ideal $\mathfrak{q} \subset S$ whose inverse image in R is (0).*

Proof. Let K be the field of fractions of the domain R. Assume that $R \to S$ is injective. Since localization is exact we see that $K \to S \otimes_R K$ is injective. Hence there is a prime mapping to (0) by Lemma 17.9.

Note that (0) is dense in $\mathrm{Spec}(R)$, so that the last condition implies the second.

Suppose the second condition holds. Let $f \in R$, $f \neq 0$. As R is a domain we see that $V(f)$ is a proper closed subset of R. By assumption there exists a prime $\mathfrak{q}$ of S such that $\varphi(f) \notin \mathfrak{q}$. Hence $\varphi(f) \neq 0$. Hence $R \to S$ is injective. □

00FK **Lemma 30.5.** *Let $R \subset S$ be an injective ring map. Then $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ hits all the minimal primes.*

Proof. Let $\mathfrak{p} \subset R$ be a minimal prime. In this case $R_\mathfrak{p}$ has a unique prime ideal. Hence it suffices to show that $S_\mathfrak{p}$ is not zero. And this follows from the fact that localization is exact, see Proposition 9.12. □

00FL **Lemma 30.6.** *Let $R \to S$ be a ring map. The following are equivalent:*

1. *The kernel of $R \to S$ consists of nilpotent elements.*
2. *The minimal primes of R are in the image of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$.*
3. *The image of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is dense in $\mathrm{Spec}(R)$.*

Proof. Let $I = \mathrm{Ker}(R \to S)$. Note that $\sqrt{(0)} = \bigcap_{\mathfrak{q} \subset S} \mathfrak{q}$, see Lemma 17.2. Hence $\sqrt{I} = \bigcap_{\mathfrak{q} \subset S} R \cap \mathfrak{q}$. Thus $V(I) = V(\sqrt{I})$ is the closure of the image of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$. This shows that (1) is equivalent to (3). It is clear that (2) implies (3). Finally, assume (1). We may replace R by R/I and S by S/IS without affecting the topology of the spectra and the map. Hence the implication (1) $\Rightarrow$ (2) follows from Lemma 30.5. □

0CAN **Lemma 30.7.** *Let $R \to S$ be a ring map. If a minimal prime $\mathfrak{p} \subset R$ is in the image of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$, then it is the image of a minimal prime.*

Proof. Say $\mathfrak{p} = \mathfrak{q} \cap R$. Then choose a minimal prime $\mathfrak{r} \subset S$ with $\mathfrak{r} \subset \mathfrak{q}$, see Lemma 17.2. By minimality of $\mathfrak{p}$ we see that $\mathfrak{p} = \mathfrak{r} \cap R$. □

31. Noetherian rings

00FM A ring R is *Noetherian* if any ideal of R is finitely generated. This is clearly equivalent to the ascending chain condition for ideals of R. By Lemma 28.10 it suffices to check that every prime ideal of R is finitely generated.

00FN **Lemma 31.1.** *Any finitely generated ring over a Noetherian ring is Noetherian. Any localization of a Noetherian ring is Noetherian.*

Proof. The statement on localizations follows from the fact that any ideal $J \subset S^{-1}R$ is of the form $I \cdot S^{-1}R$. Any quotient R/I of a Noetherian ring R is Noetherian because any ideal $\overline{J} \subset R/I$ is of the form J/I for some ideal $I \subset J \subset R$. Thus it suffices to show that if R is Noetherian so is $R[X]$. Suppose $J_1 \subset J_2 \subset \ldots$ is an ascending chain of ideals in $R[X]$. Consider the ideals $I_{i,d}$ defined as the ideal of elements of R which occur as leading coefficients of degree d polynomials in J_i. Clearly $I_{i,d} \subset I_{i',d'}$ whenever $i \leq i'$ and $d \leq d'$. By the ascending chain condition in R there are at most finitely many distinct ideals among all of the $I_{i,d}$. (Hint: Any infinite set of elements of $\mathbf{N} \times \mathbf{N}$ contains an increasing infinite sequence.) Take i_0 so large that $I_{i,d} = I_{i_0,d}$ for all $i \geq i_0$ and all d. Suppose $f \in J_i$ for some $i \geq i_0$. By induction on the degree $d = \deg(f)$ we show that $f \in J_{i_0}$. Namely, there exists a $g \in J_{i_0}$ whose degree is d and which has the same leading coefficient as f. By induction $f - g \in J_{i_0}$ and we win. $\square$

0306 **Lemma 31.2.** *If R is a Noetherian ring, then so is the formal power series ring $R[[x_1, \ldots, x_n]]$.*

Proof. Since $R[[x_1, \ldots, x_{n+1}]] \cong R[[x_1, \ldots, x_n]][[x_{n+1}]]$ it suffices to prove the statement that $R[[x]]$ is Noetherian if R is Noetherian. Let $I \subset R[[x]]$ be a ideal. We have to show that I is a finitely generated ideal. For each integer d denote $I_d = \{a \in R \mid ax^d + \text{h.o.t.} \in I\}$. Then we see that $I_0 \subset I_1 \subset \ldots$ stabilizes as R is Noetherian. Choose d_0 such that $I_{d_0} = I_{d_0+1} = \ldots$. For each $d \leq d_0$ choose elements $f_{d,j} \in I \cap (x^d)$, $j = 1, \ldots, n_d$ such that if we write $f_{d,j} = a_{d,j}x^d + \text{h.o.t}$ then $I_d = (a_{d,j})$. Denote $I' = (\{f_{d,j}\}_{d=0,\ldots,d_0,j=1,\ldots,n_d})$. Then it is clear that $I' \subset I$. Pick $f \in I$. First we may choose $c_{d,i} \in R$ such that

$$f - \sum c_{d,i} f_{d,i} \in (x^{d_0+1}) \cap I.$$

Next, we can choose $c_{i,1} \in R$, $i = 1, \ldots, n_{d_0}$ such that

$$f - \sum c_{d,i} f_{d,i} - \sum c_{i,1} x f_{d_0,i} \in (x^{d_0+2}) \cap I.$$

Next, we can choose $c_{i,2} \in R$, $i = 1, \ldots, n_{d_0}$ such that

$$f - \sum c_{d,i} f_{d,i} - \sum c_{i,1} x f_{d_0,i} - \sum c_{i,2} x^2 f_{d_0,i} \in (x^{d_0+3}) \cap I.$$

And so on. In the end we see that

$$f = \sum c_{d,i} f_{d,i} + \sum\nolimits_i (\sum\nolimits_e c_{i,e} x^e) f_{d_0,i}$$

is contained in I' as desired. $\square$

The following lemma, although easy, is useful because finite type $\mathbf{Z}$-algebras come up quite often in a technique called "absolute Noetherian reduction".

00FO **Lemma 31.3.** *Any finite type algebra over a field is Noetherian. Any finite type algebra over $\mathbf{Z}$ is Noetherian.*

Proof. This is immediate from Lemma 31.1 and the fact that fields are Noetherian rings and that $\mathbf{Z}$ is Noetherian ring (because it is a principal ideal domain). □

00FP **Lemma 31.4.** *Let R be a Noetherian ring.*

1. *Any finite R-module is of finite presentation.*
2. *Any submodule of a finite R-module is finite.*
3. *Any finite type R-algebra is of finite presentation over R.*

Proof. Let M be a finite R-module. By Lemma 5.4 we can find a finite filtration of M whose successive quotients are of the form R/I. Since any ideal is finitely generated, each of the quotients R/I is finitely presented. Hence M is finitely presented by Lemma 5.3. This proves (1).

Let $N \subset M$ be a submodule. As M is finite, the quotient M/N is finite. Thus M/N is of finite presentation by part (1). Thus we see that N is finite by Lemma 5.3 part (5). This proves part (2).

To see (3) note that any ideal of $R[x_1, \ldots, x_n]$ is finitely generated by Lemma 31.1. □

00FQ **Lemma 31.5.** *If R is a Noetherian ring then* $\text{Spec}(R)$ *is a Noetherian topological space, see Topology, Definition 9.1.*

Proof. This is because any closed subset of $\text{Spec}(R)$ is uniquely of the form $V(I)$ with I a radical ideal, see Lemma 17.2. And this correspondence is inclusion reversing. Thus the result follows from the definitions. □

00FR **Lemma 31.6.** *If R is a Noetherian ring then* $\text{Spec}(R)$ *has finitely many irreducible components. In other words R has finitely many minimal primes.*

Proof. By Lemma 31.5 and Topology, Lemma 9.2 we see there are finitely many irreducible components. By Lemma 26.1 these correspond to minimal primes of R. □

0CY6 **Lemma 31.7.** *Let $R \to S$ be a ring map. Let $R \to R'$ be of finite type. If S is Noetherian, then the base change $S' = R' \otimes_R S$ is Noetherian.*

Proof. By Lemma 14.2 finite type is stable under base change. Thus $S \to S'$ is of finite type. Since S is Noetherian we can apply Lemma 31.1. □

045I **Lemma 31.8.** *Let k be a field and let R be a Noetherian k-algebra. If K/k is a finitely generated field extension then $K \otimes_k R$ is Noetherian.*

Proof. Since K/k is a finitely generated field extension, there exists a finitely generated k-algebra $B \subset K$ such that K is the fraction field of B. In other words, $K = S^{-1}B$ with $S = B \setminus \{0\}$. Then $K \otimes_k R = S^{-1}(B \otimes_k R)$. Then $B \otimes_k R$ is Noetherian by Lemma 31.7. Finally, $K \otimes_k R = S^{-1}(B \otimes_k R)$ is Noetherian by Lemma 31.1. □

Here are some fun lemmas that are sometimes useful.

0BX1 **Lemma 31.9.** *Let R be a ring and $\mathfrak{p} \subset R$ be a prime. There exists an $f \in R$, $f \notin \mathfrak{p}$ such that $R_f \to R_\mathfrak{p}$ is injective in each of the following cases*

1. *R is a domain,*
2. *R is Noetherian, or*

(3) *R is reduced and has finitely many minimal primes.*

Proof. If R is a domain, then $R \subset R_{\mathfrak p}$, hence $f = 1$ works. If R is Noetherian, then the kernel I of $R \to R_{\mathfrak p}$ is a finitely generated ideal and we can find $f \in R$, $f \not\in \mathfrak p$ such that $IR_f = 0$. For this f the map $R_f \to R_{\mathfrak p}$ is injective and f works. If R is reduced with finitely many minimal primes $\mathfrak p_1, \ldots, \mathfrak p_n$, then we can choose $f \in \bigcap_{\mathfrak p_i \not\subset \mathfrak p} \mathfrak p_i$, $f \not\in \mathfrak p$. Indeed, if $\mathfrak p_i \not\subset \mathfrak p$ then there exist $f_i \in \mathfrak p_i$, $f_i \not\in \mathfrak p$ and $f = \prod f_i$ works. For this f we have $R_f \subset R_{\mathfrak p}$ because the minimal primes of R_f correspond to minimal primes of $R_{\mathfrak p}$ and we can apply Lemma 25.2 (some details omitted). $\square$

06RN **Lemma 31.10.** *Any surjective endomorphism of a Noetherian ring is an isomorphism.*

Proof. If $f : R \to R$ were such an endomorphism but not injective, then

$$\mathrm{Ker}(f) \subset \mathrm{Ker}(f \circ f) \subset \mathrm{Ker}(f \circ f \circ f) \subset \ldots$$

would be a strictly increasing chain of ideals. $\square$

32. Locally nilpotent ideals

0AMF Here is the definition.

00IL **Definition 32.1.** Let R be a ring. Let $I \subset R$ be an ideal. We say I is *locally nilpotent* if for every $x \in I$ there exists an $n \in \mathbf{N}$ such that $x^n = 0$. We say I is *nilpotent* if there exists an $n \in \mathbf{N}$ such that $I^n = 0$.

0EGG **Example 32.2.** Let $R = k[x_n | n \in \mathbf{N}]$ be the polynomial ring in infinitely many variables over a field k. Let I be the ideal generated by the elements x_n^n for $n \in \mathbf{N}$ and $S = R/I$. Then the ideal $J \subset S$ generated by the images of x_n, $n \in \mathbf{N}$ is locally nilpotent, but not nilpotent. Indeed, since S-linear combinations of nilpotents are nilpotent, to prove that J is locally nilpotent it is enough to observe that all its generators are nilpotent (which they obviously are). On the other hand, for each $n \in \mathbf{N}$ it holds that $x_{n+1}^n \not\in I$, so that $J^n \neq 0$. It follows that J is not nilpotent.

0544 **Lemma 32.3.** *Let $R \to R'$ be a ring map and let $I \subset R$ be a locally nilpotent ideal. Then IR' is a locally nilpotent ideal of R'.*

Proof. This follows from the fact that if $x, y \in R'$ are nilpotent, then $x + y$ is nilpotent too. Namely, if $x^n = 0$ and $y^m = 0$, then $(x+y)^{n+m-1} = 0$. $\square$

0AMG **Lemma 32.4.** *Let R be a ring and let $I \subset R$ be a locally nilpotent ideal. An element x of R is a unit if and only if the image of x in R/I is a unit.*

Proof. If x is a unit in R, then its image is clearly a unit in R/I. It remains to prove the converse. Assume the image of $y \in R$ in R/I is the inverse of the image of x. Then $xy = 1 - z$ for some $z \in I$. This means that $1 \equiv z$ modulo xR. Since z lies in the locally nilpotent ideal I, we have $z^N = 0$ for some sufficiently large N. It follows that $1 = 1^N \equiv z^N = 0$ modulo xR. In other words, x divides 1 and is hence a unit. $\square$

00IM **Lemma 32.5.** *Let R be a Noetherian ring. Let I, J be ideals of R. Suppose $J \subset \sqrt{I}$. Then $J^n \subset I$ for some n. In particular, in a Noetherian ring the notions of "locally nilpotent ideal" and "nilpotent ideal" coincide.*

Proof. Say $J = (f_1, \ldots, f_s)$. By assumption $f_i^{d_i} \in I$. Take $n = d_1 + d_2 + \ldots + d_s + 1$. □

00J9 **Lemma 32.6.** *Let R be a ring. Let $I \subset R$ be a locally nilpotent ideal. Then $R \to R/I$ induces a bijection on idempotents.*

First proof of Lemma 32.6. As I is locally nilpotent it is contained in every prime ideal. Hence $\text{Spec}(R/I) = V(I) = \text{Spec}(R)$. Hence the lemma follows from Lemma 21.3. □

Second proof of Lemma 32.6. Suppose $\overline{e} \in R/I$ is an idempotent. We have to lift $\overline{e}$ to an idempotent of R.

First, choose any lift $f \in R$ of $\overline{e}$, and set $x = f^2 - f$. Then, $x \in I$, so x is nilpotent (since I is locally nilpotent). Let now J be the ideal of R generated by x. Then, J is nilpotent (not just locally nilpotent), since it is generated by the nilpotent x.

Now, assume that we have found a lift $e \in R$ of $\overline{e}$ such that $e^2 - e \in J^k$ for some $k \geq 1$. Let $e' = e - (2e-1)(e^2 - e) = 3e^2 - 2e^3$, which is another lift of $\overline{e}$ (since the idempotency of $\overline{e}$ yields $e^2 - e \in I$). Then

$$(e')^2 - e' = (4e^2 - 4e - 3)(e^2 - e)^2 \in J^{2k}$$

by a simple computation.

We thus have started with a lift e of $\overline{e}$ such that $e^2 - e \in J^k$, and obtained a lift e' of $\overline{e}$ such that $(e')^2 - e' \in J^{2k}$. This way we can successively improve the approximation (starting with $e = f$, which fits the bill for $k = 1$). Eventually, we reach a stage where $J^k = 0$, and at that stage we have a lift e of $\overline{e}$ such that $e^2 - e \in J^k = 0$, that is, this e is idempotent.

We thus have seen that if $\overline{e} \in R/I$ is any idempotent, then there exists a lift of $\overline{e}$ which is an idempotent of R. It remains to prove that this lift is unique. Indeed, let e_1 and e_2 be two such lifts. We need to show that $e_1 = e_2$.

By definition of e_1 and e_2, we have $e_1 \equiv e_2 \mod I$, and both e_1 and e_2 are idempotent. From $e_1 \equiv e_2 \mod I$, we see that $e_1 - e_2 \in I$, so that $e_1 - e_2$ is nilpotent (since I is locally nilpotent). A straightforward computation (using the idempotency of e_1 and e_2) reveals that $(e_1 - e_2)^3 = e_1 - e_2$. Using this and induction, we obtain $(e_1 - e_2)^k = e_1 - e_2$ for any positive odd integer k. Since all high enough k satisfy $(e_1 - e_2)^k = 0$ (since $e_1 - e_2$ is nilpotent), this shows $e_1 - e_2 = 0$, so that $e_1 = e_2$, which completes our proof. □

05BU **Lemma 32.7.** *Let A be a possibly noncommutative algebra. Let $e \in A$ be an element such that $x = e^2 - e$ is nilpotent. Then there exists an idempotent of the form $e' = e + x(\sum a_{i,j} e^i x^j) \in A$ with $a_{i,j} \in \mathbf{Z}$.*

Proof. Consider the ring $R_n = \mathbf{Z}[e]/((e^2 - e)^n)$. It is clear that if we can prove the result for each R_n then the lemma follows. In R_n consider the ideal $I = (e^2 - e)$ and apply Lemma 32.6. □

0CAP **Lemma 32.8.** *Let R be a ring. Let $I \subset R$ be a locally nilpotent ideal. Let $n \geq 1$ be an integer which is invertible in R/I. Then*

1. *the nth power map $1 + I \to 1 + I$, $1 + x \mapsto (1 + x)^n$ is a bijection,*
2. *a unit of R is a nth power if and only if its image in R/I is an nth power.*

Proof. Let $a \in R$ be a unit whose image in R/I is the same as the image of b^n with $b \in R$. Then b is a unit (Lemma 32.4) and $ab^{-n} = 1 + x$ for some $x \in I$. Hence $ab^{-n} = c^n$ by part (1). Thus (2) follows from (1).

Proof of (1). This is true because there is an inverse to the map $1 + x \mapsto (1 + x)^n$. Namely, we can consider the map which sends $1 + x$ to

$$\begin{aligned}(1+x)^{1/n} &= 1 + \binom{1/n}{1}x + \binom{1/n}{2}x^2 + \binom{1/n}{3}x^3 + \ldots \\ &= 1 + \frac{1}{n}x + \frac{1-n}{2n^2}x^2 + \frac{(1-n)(1-2n)}{6n^3}x^3 + \ldots\end{aligned}$$

as in elementary calculus. This makes sense because the series is finite as $x^k = 0$ for all $k \gg 0$ and each coefficient $\binom{1/n}{k} \in \mathbf{Z}[1/n]$ (details omitted; observe that n is invertible in R by Lemma 32.4). □

33. Curiosity

02JG Lemma 24.3 explains what happens if $V(I)$ is open for some ideal $I \subset R$. But what if $\text{Spec}(S^{-1}R)$ is closed in $\text{Spec}(R)$? The next two lemmas give a partial answer. For more information see Section 108.

02JH **Lemma 33.1.** *Let R be a ring. Let $S \subset R$ be a multiplicative subset. Assume the image of the map $\text{Spec}(S^{-1}R) \to \text{Spec}(R)$ is closed. Then $S^{-1}R \cong R/I$ for some ideal $I \subset R$.*

Proof. Let $I = \text{Ker}(R \to S^{-1}R)$ so that $V(I)$ contains the image. Say the image is the closed subset $V(I') \subset \text{Spec}(R)$ for some ideal $I' \subset R$. So $V(I') \subset V(I)$. For $f \in I'$ we see that $f/1 \in S^{-1}R$ is contained in every prime ideal. Hence f^n maps to zero in $S^{-1}R$ for some $n \geq 1$ (Lemma 17.2). Hence $V(I') = V(I)$. Then this implies every $g \in S$ is invertible mod I. Hence we get ring maps $R/I \to S^{-1}R$ and $S^{-1}R \to R/I$. The first map is injective by choice of I. The second is the map $S^{-1}R \to S^{-1}(R/I) = R/I$ which has kernel $S^{-1}I$ because localization is exact. Since $S^{-1}I = 0$ we see also the second map is injective. Hence $S^{-1}R \cong R/I$. □

02JI **Lemma 33.2.** *Let R be a ring. Let $S \subset R$ be a multiplicative subset. Assume the image of the map $\text{Spec}(S^{-1}R) \to \text{Spec}(R)$ is closed. If R is Noetherian, or $\text{Spec}(R)$ is a Noetherian topological space, or S is finitely generated as a monoid, then $R \cong S^{-1}R \times R'$ for some ring R'.*

Proof. By Lemma 33.1 we have $S^{-1}R \cong R/I$ for some ideal $I \subset R$. By Lemma 24.3 it suffices to show that $V(I)$ is open. If R is Noetherian then $\text{Spec}(R)$ is a Noetherian topological space, see Lemma 31.5. If $\text{Spec}(R)$ is a Noetherian topological space, then the complement $\text{Spec}(R) \setminus V(I)$ is quasi-compact, see Topology, Lemma 12.13. Hence there exist finitely many $f_1, \ldots, f_n \in I$ such that $V(I) = V(f_1, \ldots, f_n)$. Since each f_i maps to zero in $S^{-1}R$ there exists a $g \in S$ such that $gf_i = 0$ for $i = 1, \ldots, n$. Hence $D(g) = V(I)$ as desired. In case S is finitely generated as a monoid, say S is generated by $g_1, \ldots, g_m$, then $S^{-1}R \cong R_{g_1 \ldots g_m}$ and we conclude that $V(I) = D(g_1 \ldots g_m)$. □

34. Hilbert Nullstellensatz

00FS

00FV **Theorem 34.1** (Hilbert Nullstellensatz)**.** *Let k be a field.*

00FW (1) *For any maximal ideal $\mathfrak{m} \subset k[x_1, \ldots, x_n]$ the field extension $\kappa(\mathfrak{m})/k$ is finite.*
00FX (2) *Any radical ideal $I \subset k[x_1, \ldots, x_n]$ is the intersection of maximal ideals containing it.*

The same is true in any finite type k-algebra.

Proof. It is enough to prove part (1) of the theorem for the case of a polynomial algebra $k[x_1, \ldots, x_n]$, because any finitely generated k-algebra is a quotient of such a polynomial algebra. We prove this by induction on n. The case $n = 0$ is clear. Suppose that $\mathfrak{m}$ is a maximal ideal in $k[x_1, \ldots, x_n]$. Let $\mathfrak{p} \subset k[x_n]$ be the intersection of $\mathfrak{m}$ with $k[x_n]$.

If $\mathfrak{p} \neq (0)$, then $\mathfrak{p}$ is maximal and generated by an irreducible monic polynomial P (because of the Euclidean algorithm in $k[x_n]$). Then $k' = k[x_n]/\mathfrak{p}$ is a finite field extension of k and contained in $\kappa(\mathfrak{m})$. In this case we get a surjection

$$k'[x_1, \ldots, x_{n-1}] \to k'[x_1, \ldots, x_n] = k' \otimes_k k[x_1, \ldots, x_n] \longrightarrow \kappa(\mathfrak{m})$$

and hence we see that $\kappa(\mathfrak{m})$ is a finite extension of k' by induction hypothesis. Thus $\kappa(\mathfrak{m})$ is finite over k as well.

If $\mathfrak{p} = (0)$ we consider the ring extension $k[x_n] \subset k[x_1, \ldots, x_n]/\mathfrak{m}$. This is a finitely generated ring extension, hence of finite presentation by Lemmas 31.3 and 31.4. Thus the image of $\mathrm{Spec}(k[x_1, \ldots, x_n]/\mathfrak{m})$ in $\mathrm{Spec}(k[x_n])$ is constructible by Theorem 29.10. Since the image contains (0) we conclude that it contains a standard open $D(f)$ for some $f \in k[x_n]$ nonzero. Since clearly $D(f)$ is infinite we get a contradiction with the assumption that $k[x_1, \ldots, x_n]/\mathfrak{m}$ is a field (and hence has a spectrum consisting of one point).

Proof of (2). Let $I \subset R$ be a radical ideal, with R of finite type over k. Let $f \in R$, $f \notin I$. We have to find a maximal ideal $\mathfrak{m} \subset R$ with $I \subset \mathfrak{m}$ and $f \notin \mathfrak{m}$. The ring $(R/I)_f$ is nonzero, since $1 = 0$ in this ring would mean $f^n \in I$ and since I is radical this would mean $f \in I$ contrary to our assumption on f. Thus we may choose a maximal ideal $\mathfrak{m}'$ in $(R/I)_f$, see Lemma 17.2. Let $\mathfrak{m} \subset R$ be the inverse image of $\mathfrak{m}'$ in R. We see that $I \subset \mathfrak{m}$ and $f \notin \mathfrak{m}$. If we show that $\mathfrak{m}$ is a maximal ideal of R, then we are done. We clearly have

$$k \subset R/\mathfrak{m} \subset \kappa(\mathfrak{m}').$$

By part (1) the field extension $\kappa(\mathfrak{m}')/k$ is finite. Hence $R/\mathfrak{m}$ is a field by Fields, Lemma 8.10. Thus $\mathfrak{m}$ is maximal and the proof is complete. □

00FY **Lemma 34.2.** *Let R be a ring. Let K be a field. If $R \subset K$ and K is of finite type over R, then there exists an $f \in R$ such that R_f is a field, and K/R_f is a finite field extension.*

Proof. By Lemma 30.2 there exist a nonempty open $U \subset \mathrm{Spec}(R)$ contained in the image $\{(0)\}$ of $\mathrm{Spec}(K) \to \mathrm{Spec}(R)$. Choose $f \in R$, $f \neq 0$ such that $D(f) \subset U$, i.e., $D(f) = \{(0)\}$. Then R_f is a domain whose spectrum has exactly one point and R_f is a field. Then K is a finitely generated algebra over the field R_f and hence a finite field extension of R_f by the Hilbert Nullstellensatz (Theorem 34.1). □

35. Jacobson rings

00FZ Let R be a ring. The closed points of $\mathrm{Spec}(R)$ are the maximal ideals of R. Often rings which occur naturally in algebraic geometry have lots of maximal ideals. For example finite type algebras over a field or over $\mathbf{Z}$. We will show that these are examples of Jacobson rings.

00G0 **Definition 35.1.** Let R be a ring. We say that R is a *Jacobson ring* if every radical ideal I is the intersection of the maximal ideals containing it.

00G1 **Lemma 35.2.** *Any algebra of finite type over a field is Jacobson.*

Proof. This follows from Theorem 34.1 and Definition 35.1. $\square$

00G2 **Lemma 35.3.** *Let R be a ring. If every prime ideal of R is the intersection of the maximal ideals containing it, then R is Jacobson.*

Proof. This is immediately clear from the fact that every radical ideal $I \subset R$ is the intersection of the primes containing it. See Lemma 17.2. $\square$

00G3 **Lemma 35.4.** *A ring R is Jacobson if and only if $\mathrm{Spec}(R)$ is Jacobson, see Topology, Definition 18.1.*

Proof. Suppose R is Jacobson. Let $Z \subset \mathrm{Spec}(R)$ be a closed subset. We have to show that the set of closed points in Z is dense in Z. Let $U \subset \mathrm{Spec}(R)$ be an open such that $U \cap Z$ is nonempty. We have to show $Z \cap U$ contains a closed point of $\mathrm{Spec}(R)$. We may assume $U = D(f)$ as standard opens form a basis for the topology on $\mathrm{Spec}(R)$. According to Lemma 17.2 we may assume that $Z = V(I)$, where I is a radical ideal. We see also that $f \notin I$. By assumption, there exists a maximal ideal $\mathfrak{m} \subset R$ such that $I \subset \mathfrak{m}$ but $f \notin \mathfrak{m}$. Hence $\mathfrak{m} \in D(f) \cap V(I) = U \cap Z$ as desired.

Conversely, suppose that $\mathrm{Spec}(R)$ is Jacobson. Let $I \subset R$ be a radical ideal. Let $J = \cap_{I \subset \mathfrak{m}} \mathfrak{m}$ be the intersection of the maximal ideals containing I. Clearly J is a radical ideal, $V(J) \subset V(I)$, and $V(J)$ is the smallest closed subset of $V(I)$ containing all the closed points of $V(I)$. By assumption we see that $V(J) = V(I)$. But Lemma 17.2 shows there is a bijection between Zariski closed sets and radical ideals, hence $I = J$ as desired. $\square$

034J **Lemma 35.5.** *Let R be a ring. If R is not Jacobson there exist a prime $\mathfrak{p} \subset R$, an element $f \in R$ such that the following hold*

1. *$\mathfrak{p}$ is not a maximal ideal,*
2. *$f \notin \mathfrak{p}$,*
3. *$V(\mathfrak{p}) \cap D(f) = \{\mathfrak{p}\}$, and*
4. *$(R/\mathfrak{p})_f$ is a field.*

On the other hand, if R is Jacobson, then for any pair $(\mathfrak{p}, f)$ such that (1) and (2) hold the set $V(\mathfrak{p}) \cap D(f)$ is infinite.

Proof. Assume R is not Jacobson. By Lemma 35.4 this means there exists an closed subset $T \subset \mathrm{Spec}(R)$ whose set $T_0 \subset T$ of closed points is not dense in T. Choose an $f \in R$ such that $T_0 \subset V(f)$ but $T \not\subset V(f)$. Note that $T \cap D(f)$ is homeomorphic to $\mathrm{Spec}((R/I)_f)$ if $T = V(I)$, see Lemmas 17.7 and 17.6. As any ring has a maximal ideal (Lemma 17.2) we can choose a closed point t of space $T \cap D(f)$. Then t corresponds to a prime ideal $\mathfrak{p} \subset R$ which is not maximal (as

$t \notin T_0$). Thus (1) holds. By construction $f \notin \mathfrak{p}$, hence (2). As t is a closed point of $T \cap D(f)$ we see that $V(\mathfrak{p}) \cap D(f) = \{\mathfrak{p}\}$, i.e., (3) holds. Hence we conclude that $(R/\mathfrak{p})_f$ is a domain whose spectrum has one point, hence (4) holds (for example combine Lemmas 18.2 and 25.1).

Conversely, suppose that R is Jacobson and $(\mathfrak{p}, f)$ satisfy (1) and (2). If $V(\mathfrak{p}) \cap D(f) = \{\mathfrak{p}, \mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$ then $\mathfrak{p} \neq \mathfrak{q}_i$ implies there exists an element $g \in R$ such that $g \notin \mathfrak{p}$ but $g \in \mathfrak{q}_i$ for all i. Hence $V(\mathfrak{p}) \cap D(fg) = \{\mathfrak{p}\}$ which is impossible since each locally closed subset of $\mathrm{Spec}(R)$ contains at least one closed point as $\mathrm{Spec}(R)$ is a Jacobson topological space. □

00G4 **Lemma 35.6.** *The ring $\mathbf{Z}$ is a Jacobson ring. More generally, let R be a ring such that*

1. *R is a domain,*
2. *R is Noetherian,*
3. *any nonzero prime ideal is a maximal ideal, and*
4. *R has infinitely many maximal ideals.*

Then R is a Jacobson ring.

Proof. Let R satisfy (1), (2), (3) and (4). The statement means that $(0) = \bigcap_{\mathfrak{m} \subset R} \mathfrak{m}$. Since R has infinitely many maximal ideals it suffices to show that any nonzero $x \in R$ is contained in at most finitely many maximal ideals, in other words that $V(x)$ is finite. By Lemma 17.7 we see that $V(x)$ is homeomorphic to $\mathrm{Spec}(R/xR)$. By assumption (3) every prime of R/xR is minimal and hence corresponds to an irreducible component of $\mathrm{Spec}(R)$ (Lemma 26.1). As R/xR is Noetherian, the topological space $\mathrm{Spec}(R/xR)$ is Noetherian (Lemma 31.5) and has finitely many irreducible components (Topology, Lemma 9.2). Thus $V(x)$ is finite as desired. □

02CC **Example 35.7.** Let A be an infinite set. For each $\alpha \in A$, let k_α be a field. We claim that $R = \prod_{\alpha \in A} k_\alpha$ is Jacobson. First, note that any element $f \in R$ has the form $f = ue$, with $u \in R$ a unit and $e \in R$ an idempotent (left to the reader). Hence $D(f) = D(e)$, and $R_f = R_e = R/(1-e)$ is a quotient of R. Actually, any ring with this property is Jacobson. Namely, say $\mathfrak{p} \subset R$ is a prime ideal and $f \in R$, $f \notin \mathfrak{p}$. We have to find a maximal ideal $\mathfrak{m}$ of R such that $\mathfrak{p} \subset \mathfrak{m}$ and $f \notin \mathfrak{m}$. Because R_f is a quotient of R we see that any maximal ideal of R_f corresponds to a maximal ideal of R not containing f. Hence the result follows by choosing a maximal ideal of R_f containing $\mathfrak{p}R_f$.

00G5 **Example 35.8.** A domain R with finitely many maximal ideals $\mathfrak{m}_i$, $i = 1, \ldots, n$ is not a Jacobson ring, except when it is a field. Namely, in this case (0) is not the intersection of the maximal ideals $(0) \neq \mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \ldots \cap \mathfrak{m}_n \supset \mathfrak{m}_1 \cdot \mathfrak{m}_2 \cdot \ldots \cdot \mathfrak{m}_n \neq 0$. In particular a discrete valuation ring, or any local ring with at least two prime ideals is not a Jacobson ring.

00GA **Lemma 35.9.** *Let $R \to S$ be a ring map. Let $\mathfrak{m} \subset R$ be a maximal ideal. Let $\mathfrak{q} \subset S$ be a prime ideal lying over $\mathfrak{m}$ such that $\kappa(\mathfrak{q})/\kappa(\mathfrak{m})$ is an algebraic field extension. Then $\mathfrak{q}$ is a maximal ideal of S.*

Proof. Consider the diagram

$$\begin{tikzcd} S \arrow[r] & S/\mathfrak q \arrow[r] & \kappa(\mathfrak q) \\ R \arrow[u] \arrow[r] & R/\mathfrak m \arrow[u] & \end{tikzcd}$$

We see that $\kappa(\mathfrak m) \subset S/\mathfrak q \subset \kappa(\mathfrak q)$. Because the field extension $\kappa(\mathfrak m) \subset \kappa(\mathfrak q)$ is algebraic, any ring between $\kappa(\mathfrak m)$ and $\kappa(\mathfrak q)$ is a field (Fields, Lemma 8.10). Thus $S/\mathfrak q$ is a field, and a posteriori equal to $\kappa(\mathfrak q)$. $\square$

00FT **Lemma 35.10.** *Suppose that k is a field and suppose that V is a nonzero vector space over k. Assume the dimension of V (which is a cardinal number) is smaller than the cardinality of k. Then for any linear operator $T : V \to V$ there exists some monic polynomial $P(t) \in k[t]$ such that $P(T)$ is not invertible.*

Proof. If not then V inherits the structure of a vector space over the field $k(t)$. But the dimension of $k(t)$ over k is at least the cardinality of k for example due to the fact that the elements $\frac{1}{t-\lambda}$ are k-linearly independent. $\square$

Here is another version of Hilbert's Nullstellensatz.

00FU **Theorem 35.11.** *Let k be a field. Let S be a k-algebra generated over k by the elements $\{x_i\}_{i\in I}$. Assume the cardinality of I is smaller than the cardinality of k. Then*

1. *for all maximal ideals $\mathfrak m \subset S$ the field extension $\kappa(\mathfrak m)/k$ is algebraic, and*
2. *S is a Jacobson ring.*

Proof. If I is finite then the result follows from the Hilbert Nullstellensatz, Theorem 34.1. In the rest of the proof we assume I is infinite. It suffices to prove the result for $\mathfrak m \subset k[\{x_i\}_{i\in I}]$ maximal in the polynomial ring on variables x_i, since S is a quotient of this. As I is infinite the set of monomials $x_{i_1}^{e_1}\ldots x_{i_r}^{e_r}$, $i_1, \ldots, i_r \in I$ and $e_1, \ldots, e_r \geq 0$ has cardinality at most equal to the cardinality of I. Because the cardinality of $I \times \ldots \times I$ is the cardinality of I, and also the cardinality of $\bigcup_{n\geq 0} I^n$ has the same cardinality. (If I is finite, then this is not true and in that case this proof only works if k is uncountable.)

To arrive at a contradiction pick $T \in \kappa(\mathfrak m)$ transcendental over k. Note that the k-linear map $T : \kappa(\mathfrak m) \to \kappa(\mathfrak m)$ given by multiplication by T has the property that $P(T)$ is invertible for all monic polynomials $P(t) \in k[t]$. Also, $\kappa(\mathfrak m)$ has dimension at most the cardinality of I over k since it is a quotient of the vector space $k[\{x_i\}_{i\in I}]$ over k (whose dimension is $\#I$ as we saw above). This is impossible by Lemma 35.10.

To show that S is Jacobson we argue as follows. If not then there exists a prime $\mathfrak q \subset S$ and an element $f \in S$, $f \notin \mathfrak q$ such that $\mathfrak q$ is not maximal and $(S/\mathfrak q)_f$ is a field, see Lemma 35.5. But note that $(S/\mathfrak q)_f$ is generated by at most $\#I + 1$ elements. Hence the field extension $(S/\mathfrak q)_f/k$ is algebraic (by the first part of the proof). This implies that $\kappa(\mathfrak q)$ is an algebraic extension of k hence $\mathfrak q$ is maximal by Lemma 35.9. This contradiction finishes the proof. $\square$

046V **Lemma 35.12.** *Let k be a field. Let S be a k-algebra. For any field extension K/k whose cardinality is larger than the cardinality of S we have*

(1) *for every maximal ideal $\mathfrak{m}$ of S_K the field $\kappa(\mathfrak{m})$ is algebraic over K, and*
(2) *S_K is a Jacobson ring.*

Proof. Choose $k \subset K$ such that the cardinality of K is greater than the cardinality of S. Since the elements of S generate the K-algebra S_K we see that Theorem 35.11 applies. $\square$

02CB **Example 35.13.** The trick in the proof of Theorem 35.11 really does not work if k is a countable field and I is countable too. Let k be a countable field. Let x be a variable, and let $k(x)$ be the field of rational functions in x. Consider the polynomial algebra $R = k[x, \{x_f\}_{f \in k[x]-\{0\}}]$. Let $I = (\{fx_f - 1\}_{f \in k[x]-\{0\}})$. Note that I is a proper ideal in R. Choose a maximal ideal $I \subset \mathfrak{m}$. Then $k \subset R/\mathfrak{m}$ is isomorphic to $k(x)$, and is not algebraic over k.

00G6 **Lemma 35.14.** *Let R be a Jacobson ring. Let $f \in R$. The ring R_f is Jacobson and maximal ideals of R_f correspond to maximal ideals of R not containing f.*

Proof. By Topology, Lemma 18.5 we see that $D(f) = \text{Spec}(R_f)$ is Jacobson and that closed points of $D(f)$ correspond to closed points in $\text{Spec}(R)$ which happen to lie in $D(f)$. Thus R_f is Jacobson by Lemma 35.4. $\square$

00G7 **Example 35.15.** Here is a simple example that shows Lemma 35.14 to be false if R is not Jacobson. Consider the ring $R = \mathbf{Z}_{(2)}$, i.e., the localization of $\mathbf{Z}$ at the prime (2). The localization of R at the element 2 is isomorphic to $\mathbf{Q}$, in a formula: $R_2 \cong \mathbf{Q}$. Clearly the map $R \to R_2$ maps the closed point of $\text{Spec}(\mathbf{Q})$ to the generic point of $\text{Spec}(R)$.

00G8 **Example 35.16.** Here is a simple example that shows Lemma 35.14 is false if R is Jacobson but we localize at infinitely many elements. Namely, let $R = \mathbf{Z}$ and consider the localization $(R \setminus \{0\})^{-1}R \cong \mathbf{Q}$ of R at the set of all nonzero elements. Clearly the map $\mathbf{Z} \to \mathbf{Q}$ maps the closed point of $\text{Spec}(\mathbf{Q})$ to the generic point of $\text{Spec}(\mathbf{Z})$.

00G9 **Lemma 35.17.** *Let R be a Jacobson ring. Let $I \subset R$ be an ideal. The ring R/I is Jacobson and maximal ideals of R/I correspond to maximal ideals of R containing I.*

Proof. The proof is the same as the proof of Lemma 35.14. $\square$

0CY7 **Lemma 35.18.** *Let R be a Jacobson ring. Let K be a field. Let $R \subset K$ and K is of finite type over R. Then R is a field and K/R is a finite field extension.*

Proof. First note that R is a domain. By Lemma 34.2 we see that R_f is a field and K/R_f is a finite field extension for some nonzero $f \in R$. Hence (0) is a maximal ideal of R_f and by Lemma 35.14 we conclude (0) is a maximal ideal of R. $\square$

00GB **Proposition 35.19.** *Let R be a Jacobson ring. Let $R \to S$ be a ring map of finite type. Then*

(1) *The ring S is Jacobson.*
(2) *The map $\text{Spec}(S) \to \text{Spec}(R)$ transforms closed points to closed points.*
(3) *For $\mathfrak{m}' \subset S$ maximal lying over $\mathfrak{m} \subset R$ the field extension $\kappa(\mathfrak{m}')/\kappa(\mathfrak{m})$ is finite.*

Proof. Let $\mathfrak{m}' \subset S$ be a maximal ideal and $R \cap \mathfrak{m}' = \mathfrak{m}$. Then $R/\mathfrak{m} \to S/\mathfrak{m}'$ satisfies the conditions of Lemma 35.18 by Lemma 35.17. Hence $R/\mathfrak{m}$ is a field and $\mathfrak{m}$ a maximal ideal and the induced residue field extension is finite. This proves (2) and (3).

If S is not Jacobson, then by Lemma 35.5 there exists a non-maximal prime ideal $\mathfrak{q}$ of S and an $g \in S$, $g \notin \mathfrak{q}$ such that $(S/\mathfrak{q})_g$ is a field. To arrive at a contradiction we show that $\mathfrak{q}$ is a maximal ideal. Let $\mathfrak{p} = \mathfrak{q} \cap R$. Then $R/\mathfrak{p} \to (S/\mathfrak{q})_g$ satisfies the conditions of Lemma 35.18 by Lemma 35.17. Hence $R/\mathfrak{p}$ is a field and the field extension $\kappa(\mathfrak{p}) \to (S/\mathfrak{q})_g = \kappa(\mathfrak{q})$ is finite, thus algebraic. Then $\mathfrak{q}$ is a maximal ideal of S by Lemma 35.9. Contradiction. $\square$

00GC **Lemma 35.20.** *Any finite type algebra over $\mathbf{Z}$ is Jacobson.*

Proof. Combine Lemma 35.6 and Proposition 35.19. $\square$

00GD **Lemma 35.21.** *Let $R \to S$ be a finite type ring map of Jacobson rings. Denote $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S)$. Write $f : Y \to X$ the induced map of spectra. Let $E \subset Y = \mathrm{Spec}(S)$ be a constructible set. Denote with a subscript $_0$ the set of closed points of a topological space.*

1. *We have $f(E)_0 = f(E_0) = X_0 \cap f(E)$.*
2. *A point $\xi \in X$ is in $f(E)$ if and only if $\overline{\{\xi\}} \cap f(E_0)$ is dense in $\overline{\{\xi\}}$.*

Proof. We have a commutative diagram of continuous maps

$$\begin{tikzcd} E \arrow[r] \arrow[d] & Y \arrow[d] \\ f(E) \arrow[r] & X \end{tikzcd}$$

Suppose $x \in f(E)$ is closed in $f(E)$. Then $f^{-1}(\{x\}) \cap E$ is nonempty and closed in E. Applying Topology, Lemma 18.5 to both inclusions

$$f^{-1}(\{x\}) \cap E \subset E \subset Y$$

we find there exists a point $y \in f^{-1}(\{x\}) \cap E$ which is closed in Y. In other words, there exists $y \in Y_0$ and $y \in E_0$ mapping to x. Hence $x \in f(E_0)$. This proves that $f(E)_0 \subset f(E_0)$. Proposition 35.19 implies that $f(E_0) \subset X_0 \cap f(E)$. The inclusion $X_0 \cap f(E) \subset f(E)_0$ is trivial. This proves the first assertion.

Suppose that $\xi \in f(E)$. According to Lemma 30.2 the set $f(E) \cap \overline{\{\xi\}}$ contains a dense open subset of $\overline{\{\xi\}}$. Since X is Jacobson we conclude that $f(E) \cap \overline{\{\xi\}}$ contains a dense set of closed points, see Topology, Lemma 18.5. We conclude by part (1) of the lemma.

On the other hand, suppose that $\overline{\{\xi\}} \cap f(E_0)$ is dense in $\overline{\{\xi\}}$. By Lemma 29.4 there exists a ring map $S \to S'$ of finite presentation such that E is the image of $Y' := \mathrm{Spec}(S') \to Y$. Then E_0 is the image of Y'_0 by the first part of the lemma applied to the ring map $S \to S'$. Thus we may assume that $E = Y$ by replacing S

by S'. Suppose ξ corresponds to $\mathfrak p \subset R$. Consider the diagram

$$\begin{tikzcd} S \arrow[r] & S/\mathfrak pS \\ R \arrow[u] \arrow[r] & R/\mathfrak p \arrow[u] \end{tikzcd}$$

This diagram and the density of $f(Y_0) \cap V(\mathfrak p)$ in $V(\mathfrak p)$ shows that the morphism $R/\mathfrak p \to S/\mathfrak pS$ satisfies condition (2) of Lemma 30.4. Hence we conclude there exists a prime $\overline{\mathfrak q} \subset S/\mathfrak pS$ mapping to (0). In other words the inverse image $\mathfrak q$ of $\overline{\mathfrak q}$ in S maps to $\mathfrak p$ as desired. □

The conclusion of the lemma above is that we can read off the image of f from the set of closed points of the image. This is a little nicer in case the map is of finite presentation because then we know that images of a constructible is constructible. Before we state it we introduce some notation. Denote $\text{Constr}(X)$ the set of constructible sets. Let $R \to S$ be a ring map. Denote $X = \text{Spec}(R)$ and $Y = \text{Spec}(S)$. Write $f : Y \to X$ the induced map of spectra. Denote with a subscript $_0$ the set of closed points of a topological space.

00GE **Lemma 35.22.** *With notation as above. Assume that R is a Noetherian Jacobson ring. Further assume $R \to S$ is of finite type. There is a commutative diagram*

$$\begin{tikzcd} Constr(Y) \arrow[r, "E\mapsto E_0"] \arrow[d, "E\mapsto f(E)"] & Constr(Y_0) \arrow[d, "E\mapsto f(E)"] \\ Constr(X) \arrow[r, "E\mapsto E_0"] & Constr(X_0) \end{tikzcd}$$

where the horizontal arrows are the bijections from Topology, Lemma 18.8.

Proof. Since $R \to S$ is of finite type, it is of finite presentation, see Lemma 31.4. Thus the image of a constructible set in X is constructible in Y by Chevalley's theorem (Theorem 29.10). Combined with Lemma 35.21 the lemma follows. □

To illustrate the use of Jacobson rings, we give the following two examples.

00GF **Example 35.23.** Let k be a field. The space $\text{Spec}(k[x, y]/(xy))$ has two irreducible components: namely the x-axis and the y-axis. As a generalization, let

$$R = k[x_{11}, x_{12}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, y_{22}]/\mathfrak a,$$

where $\mathfrak a$ is the ideal in $k[x_{11}, x_{12}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, y_{22}]$ generated by the entries of the 2×2 product matrix

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}.$$

In this example we will describe $\text{Spec}(R)$.

To prove the statement about $\text{Spec}(k[x, y]/(xy))$ we argue as follows. If $\mathfrak p \subset k[x, y]$ is any ideal containing xy, then either x or y would be contained in $\mathfrak p$. Hence the minimal such prime ideals are just (x) and (y). In case k is algebraically closed, the max-Spec of these components can then be visualized as the point sets of y- and x-axis.

For the generalization, note that we may identify the closed points of the spectrum of $k[x_{11}, x_{12}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, y_{22}])$ with the space of matrices

$$\left\{(X,Y) \in \mathrm{Mat}(2,k) \times \mathrm{Mat}(2,k) \mid X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}\right\}$$

at least if k is algebraically closed. Now define a group action of $\mathrm{GL}(2,k) \times \mathrm{GL}(2,k) \times \mathrm{GL}(2,k)$ on the space of matrices $\{(X,Y)\}$ by

$$(g_1, g_2, g_3) \times (X,Y) \mapsto ((g_1 X g_2^{-1}, g_2 Y g_3^{-1})).$$

Here, also observe that the algebraic set

$$\mathrm{GL}(2,k) \times \mathrm{GL}(2,k) \times \mathrm{GL}(2,k) \subset \mathrm{Mat}(2,k) \times \mathrm{Mat}(2,k) \times \mathrm{Mat}(2,k)$$

is irreducible since it is the max spectrum of the domain

$$k[x_{11}, x_{12}, \dots, z_{21}, z_{22}, (x_{11}x_{22} - x_{12}x_{21})^{-1}, (y_{11}y_{22} - y_{12}y_{21})^{-1}, (z_{11}z_{22} - z_{12}z_{21})^{-1}].$$

Since the image of irreducible an algebraic set is still irreducible, it suffices to classify the orbits of the set $\{(X,Y) \in \mathrm{Mat}(2,k) \times \mathrm{Mat}(2,k) | XY = 0\}$ and take their closures. From standard linear algebra, we are reduced to the following three cases:

(1) $\exists (g_1, g_2)$ such that $g_1 X g_2^{-1} = I_{2\times 2}$. Then Y is necessarily 0, which as an algebraic set is invariant under the group action. It follows that this orbit is contained in the irreducible algebraic set defined by the prime ideal $(y_{11}, y_{12}, y_{21}, y_{22})$. Taking the closure, we see that $(y_{11}, y_{12}, y_{21}, y_{22})$ is actually a component.

(2) $\exists (g_1, g_2)$ such that

$$g_1 X g_2^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

This case occurs if and only if X is a rank 1 matrix, and furthermore, Y is killed by such an X if and only if

$$x_{11}y_{11} + x_{12}y_{21} = 0; \quad x_{11}y_{12} + x_{12}y_{22} = 0;$$
$$x_{21}y_{11} + x_{22}y_{21} = 0; \quad x_{21}y_{12} + x_{22}y_{22} = 0.$$

Fix a rank 1 X, such non zero Y's satisfying the above equations form an irreducible algebraic set for the following reason($Y = 0$ is contained the previous case): $0 = g_1 X g_2^{-1} g_2 Y$ implies that

$$g_2 Y = \begin{pmatrix} 0 & 0 \\ y'_{21} & y'_{22} \end{pmatrix}.$$

With a further $\mathrm{GL}(2,k)$-action on the right by g_3, $g_2 Y$ can be brought into

$$g_2 Y g_3^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and thus such Y's form an irreducible algebraic set isomorphic to the image of $\mathrm{GL}(2,k)$ under this action. Finally, notice that the "rank 1" condition for X's forms an open dense subset of the irreducible algebraic set $\det X = x_{11}x_{22} - x_{12}x_{21} = 0$. It now follows that all the five equations define an irreducible component $(x_{11}y_{11} + x_{12}y_{21}, x_{11}y_{12} + x_{12}y_{22}, x_{21}y_{11} + x_{22}y_{21}, x_{21}y_{12} + x_{22}y_{22}, x_{11}x_{22} - x_{12}x_{21})$ in the open subset of the space of pairs of nonzero matrices. It can be shown that the pair of equations

$\det X = 0$, $\det Y = 0$ cuts $\mathrm{Spec}(R)$ in an irreducible component with the above locus an open dense subset.

(3) $\exists(g_1, g_2)$ such that $g_1 X g_2^{-1} = 0$, or equivalently, $X = 0$. Then Y can be arbitrary and this component is thus defined by $(x_{11}, x_{12}, x_{21}, x_{22})$.

00GG **Example 35.24.** For another example, consider $R = k[\{t_{ij}\}_{i,j=1}^n]/\mathfrak{a}$, where $\mathfrak{a}$ is the ideal generated by the entries of the product matrix $T^2 - T$, $T = (t_{ij})$. From linear algebra, we know that under the $GL(n,k)$-action defined by $g, T \mapsto gTg^{-1}$, T is classified by the its rank and each T is conjugate to some $\mathrm{diag}(1, \ldots, 1, 0, \ldots, 0)$, which has r 1's and $n-r$ 0's. Thus each orbit of such a $\mathrm{diag}(1, \ldots, 1, 0, \ldots, 0)$ under the group action forms an irreducible component and every idempotent matrix is contained in one such orbit. Next we will show that any two different orbits are necessarily disjoint. For this purpose we only need to cook up polynomial functions that take different values on different orbits. In characteristic 0 cases, such a function can be taken to be $f(t_{ij}) = trace(T) = \sum_{i=1}^n t_{ii}$. In positive characteristic cases, things are slightly more tricky since we might have $trace(T) = 0$ even if $T \neq 0$. For instance, $char = 3$

$$trace \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = 3 = 0$$

Anyway, these components can be separated using other functions. For instance, in the characteristic 3 case, $tr(\wedge^3 T)$ takes value 1 on the components corresponding to $diag(1,1,1)$ and 0 on other components.

36. Finite and integral ring extensions

00GH Trivial lemmas concerning finite and integral ring maps. We recall the definition.

00GI **Definition 36.1.** Let $\varphi : R \to S$ be a ring map.

(1) An element $s \in S$ is *integral over R* if there exists a monic polynomial $P(x) \in R[x]$ such that $P^\varphi(s) = 0$, where $P^\varphi(x) \in S[x]$ is the image of P under $\varphi : R[x] \to S[x]$.
(2) The ring map φ is *integral* if every $s \in S$ is integral over R.

052I **Lemma 36.2.** *Let $\varphi : R \to S$ be a ring map. Let $y \in S$. If there exists a finite R-submodule M of S such that $1 \in M$ and $yM \subset M$, then y is integral over R.*

Proof. Consider the map $\varphi : M \to M$, $x \mapsto y \cdot x$. By Lemma 16.2 there exist a monic polynomial $P \in R[T]$ with $P(\varphi) = 0$. In the ring S we get $P(y) = P(y) \cdot 1 = P(\varphi)(1) = 0$. □

00GK **Lemma 36.3.** *A finite ring extension is integral.*

Proof. Let $R \to S$ be finite. Let $y \in S$. Apply Lemma 36.2 to $M = S$ to see that y is integral over R. □

00GM **Lemma 36.4.** *Let $\varphi : R \to S$ be a ring map. Let $s_1, \ldots, s_n$ be a finite set of elements of S. In this case s_i is integral over R for all $i = 1, \ldots, n$ if and only if there exists an R-subalgebra $S' \subset S$ finite over R containing all of the s_i.*

Proof. If each s_i is integral, then the subalgebra generated by $\varphi(R)$ and the s_i is finite over R. Namely, if s_i satisfies a monic equation of degree d_i over R, then this subalgebra is generated as an R-module by the elements $s_1^{e_1} \ldots s_n^{e_n}$ with

$0 \leq e_i \leq d_i - 1$. Conversely, suppose given a finite R-subalgebra S' containing all the s_i. Then all of the s_i are integral by Lemma 36.3. □

02JJ **Lemma 36.5.** *Let $R \to S$ be a ring map. The following are equivalent*

1. *$R \to S$ is finite,*
2. *$R \to S$ is integral and of finite type, and*
3. *there exist $x_1, \ldots, x_n \in S$ which generate S as an algebra over R such that each x_i is integral over R.*

Proof. Clear from Lemma 36.4. □

00GN **Lemma 36.6.** *Suppose that $R \to S$ and $S \to T$ are integral ring maps. Then $R \to T$ is integral.*

Proof. Let $t \in T$. Let $P(x) \in S[x]$ be a monic polynomial such that $P(t) = 0$. Apply Lemma 36.4 to the finite set of coefficients of P. Hence t is integral over some subalgebra $S' \subset S$ finite over R. Apply Lemma 36.4 again to find a subalgebra $T' \subset T$ finite over S' and containing t. Lemma 7.3 applied to $R \to S' \to T'$ shows that T' is finite over R. The integrality of t over R now follows from Lemma 36.3. □

00GO **Lemma 36.7.** *Let $R \to S$ be a ring homomorphism. The set*

$$S' = \{s \in S \mid s \text{ is integral over } R\}$$

is an R-subalgebra of S.

Proof. This is clear from Lemmas 36.4 and 36.3. □

0CY8 **Lemma 36.8.** *Let $R_i \to S_i$ be ring maps $i = 1, \ldots, n$. Let R and S denote the product of the R_i and S_i respectively. Then an element $s = (s_1, \ldots, s_n) \in S$ is integral over R if and only if each s_i is integral over R_i.*

Proof. Omitted. □

00GP **Definition 36.9.** Let $R \to S$ be a ring map. The ring $S' \subset S$ of elements integral over R, see Lemma 36.7, is called the *integral closure* of R in S. If $R \subset S$ we say that R is *integrally closed* in S if $R = S'$.

In particular, we see that $R \to S$ is integral if and only if the integral closure of R in S is all of S.

0CY9 **Lemma 36.10.** *Let $R_i \to S_i$ be ring maps $i = 1, \ldots, n$. Denote the integral closure of R_i in S_i by S'_i. Further let R and S denote the product of the R_i and S_i respectively. Then the integral closure of R in S is the product of the S'_i. In particular $R \to S$ is integrally closed if and only if each $R_i \to S_i$ is integrally closed.*

Proof. This follows immediately from Lemma 36.8. □

0307 **Lemma 36.11.** *Integral closure commutes with localization: If $A \to B$ is a ring map, and $S \subset A$ is a multiplicative subset, then the integral closure of $S^{-1}A$ in $S^{-1}B$ is $S^{-1}B'$, where $B' \subset B$ is the integral closure of A in B.*

Proof. Since localization is exact we see that $S^{-1}B' \subset S^{-1}B$. Suppose $x \in B'$ and $f \in S$. Then $x^d + \sum_{i=1,\ldots,d} a_i x^{d-i} = 0$ in B for some $a_i \in A$. Hence also

$$(x/f)^d + \sum\nolimits_{i=1,\ldots,d} a_i/f^i (x/f)^{d-i} = 0$$

in $S^{-1}B$. In this way we see that $S^{-1}B'$ is contained in the integral closure of $S^{-1}A$ in $S^{-1}B$. Conversely, suppose that $x/f \in S^{-1}B$ is integral over $S^{-1}A$. Then we have

$$(x/f)^d + \sum\nolimits_{i=1,\ldots,d} (a_i/f_i)(x/f)^{d-i} = 0$$

in $S^{-1}B$ for some $a_i \in A$ and $f_i \in S$. This means that

$$(f'f_1 \ldots f_d x)^d + \sum\nolimits_{i=1,\ldots,d} f^i (f')^i f_1^i \ldots f_i^{i-1} \ldots f_d^i a_i (f'f_1 \ldots f_d x)^{d-i} = 0$$

for a suitable $f' \in S$. Hence $f'f_1 \ldots f_d x \in B'$ and thus $x/f \in S^{-1}B'$ as desired. □

034K **Lemma 36.12.** *Let $\varphi : R \to S$ be a ring map. Let $x \in S$. The following are equivalent:*

1. *x is integral over R, and*
2. *for every prime ideal $\mathfrak p \subset R$ the element $x \in S_{\mathfrak p}$ is integral over $R_{\mathfrak p}$.*

Proof. It is clear that (1) implies (2). Assume (2). Consider the R-algebra $S' \subset S$ generated by $\varphi(R)$ and x. Let $\mathfrak p$ be a prime ideal of R. Then we know that $x^d + \sum_{i=1,\ldots,d} \varphi(a_i)x^{d-i} = 0$ in $S_{\mathfrak p}$ for some $a_i \in R_{\mathfrak p}$. Hence we see, by looking at which denominators occur, that for some $f \in R$, $f \notin \mathfrak p$ we have $a_i \in R_f$ and $x^d + \sum_{i=1,\ldots,d} \varphi(a_i)x^{d-i} = 0$ in S_f. This implies that S'_f is finite over R_f. Since $\mathfrak p$ was arbitrary and $\text{Spec}(R)$ is quasi-compact (Lemma 17.10) we can find finitely many elements $f_1, \ldots, f_n \in R$ which generate the unit ideal of R such that S'_{f_i} is finite over R_{f_i}. Hence we conclude from Lemma 23.2 that S' is finite over R. Hence x is integral over R by Lemma 36.4. □

02JK **Lemma 36.13.** *Let $R \to S$ and $R \to R'$ be ring maps. Set $S' = R' \otimes_R S$.*

1. *If $R \to S$ is integral so is $R' \to S'$.*
2. *If $R \to S$ is finite so is $R' \to S'$.*

Proof. We prove (1). Let $s_i \in S$ be generators for S over R. Each of these satisfies a monic polynomial equation P_i over R. Hence the elements $1 \otimes s_i \in S'$ generate S' over R' and satisfy the corresponding polynomial P_i' over R'. Since these elements generate S' over R' we see that S' is integral over R'. Proof of (2) omitted. □

02JL **Lemma 36.14.** *Let $R \to S$ be a ring map. Let $f_1, \ldots, f_n \in R$ generate the unit ideal.*

1. *If each $R_{f_i} \to S_{f_i}$ is integral, so is $R \to S$.*
2. *If each $R_{f_i} \to S_{f_i}$ is finite, so is $R \to S$.*

Proof. Proof of (1). Let $s \in S$. Consider the ideal $I \subset R[x]$ of polynomials P such that $P(s) = 0$. Let $J \subset R$ denote the ideal (!) of leading coefficients of elements of I. By assumption and clearing denominators we see that $f_i^{n_i} \in J$ for all i and certain $n_i \geq 0$. Hence J contains 1 and we see s is integral over R. Proof of (2) omitted. □

02JM **Lemma 36.15.** *Let $A \to B \to C$ be ring maps.*

1. *If $A \to C$ is integral so is $B \to C$.*
2. *If $A \to C$ is finite so is $B \to C$.*

Proof. Omitted. □

0308 **Lemma 36.16.** *Let $A \to B \to C$ be ring maps. Let B' be the integral closure of A in B, let C' be the integral closure of B' in C. Then C' is the integral closure of A in C.*

Proof. Omitted. □

00GQ **Lemma 36.17.** *Suppose that $R \to S$ is an integral ring extension with $R \subset S$. Then $\varphi : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective.*

Proof. Let $\mathfrak{p} \subset R$ be a prime ideal. We have to show $\mathfrak{p}S_{\mathfrak{p}} \neq S_{\mathfrak{p}}$, see Lemma 17.9. The localization $R_{\mathfrak{p}} \to S_{\mathfrak{p}}$ is injective (as localization is exact) and integral by Lemma 36.11 or 36.13. Hence we may replace R, S by $R_{\mathfrak{p}}$, $S_{\mathfrak{p}}$ and we may assume R is local with maximal ideal $\mathfrak{m}$ and it suffices to show that $\mathfrak{m}S \neq S$. Suppose $1 = \sum f_i s_i$ with $f_i \in \mathfrak{m}$ and $s_i \in S$ in order to get a contradiction. Let $R \subset S' \subset S$ be such that $R \to S'$ is finite and $s_i \in S'$, see Lemma 36.4. The equation $1 = \sum f_i s_i$ implies that the finite R-module S' satisfies $S' = \mathfrak{m}S'$. Hence by Nakayama's Lemma 20.1 we see $S' = 0$. Contradiction. □

00GR **Lemma 36.18.** *Let R be a ring. Let K be a field. If $R \subset K$ and K is integral over R, then R is a field and K is an algebraic extension. If $R \subset K$ and K is finite over R, then R is a field and K is a finite algebraic extension.*

Proof. Assume that $R \subset K$ is integral. By Lemma 36.17 we see that $\mathrm{Spec}(R)$ has 1 point. Since clearly R is a domain we see that $R = R_{(0)}$ is a field (Lemma 25.1). The other assertions are immediate from this. □

00GS **Lemma 36.19.** *Let k be a field. Let S be a k-algebra over k.*

(1) *If S is a domain and finite dimensional over k, then S is a field.*
(2) *If S is integral over k and a domain, then S is a field.*
(3) *If S is integral over k then every prime of S is a maximal ideal (see Lemma 26.5 for more consequences).*

Proof. The statement on primes follows from the statement "integral + domain ⇒ field". Let S integral over k and assume S is a domain, Take $s \in S$. By Lemma 36.4 we may find a finite dimensional k-subalgebra $k \subset S' \subset S$ containing s. Hence S is a field if we can prove the first statement. Assume S finite dimensional over k and a domain. Pick $s \in S$. Since S is a domain the multiplication map $s : S \to S$ is surjective by dimension reasons. Hence there exists an element $s_1 \in S$ such that $ss_1 = 1$. So S is a field. □

00GT **Lemma 36.20.** *Suppose $R \to S$ is integral. Let $\mathfrak{q}, \mathfrak{q}' \in \mathrm{Spec}(S)$ be distinct primes having the same image in $\mathrm{Spec}(R)$. Then neither $\mathfrak{q} \subset \mathfrak{q}'$ nor $\mathfrak{q}' \subset \mathfrak{q}$.*

Proof. Let $\mathfrak{p} \subset R$ be the image. By Remark 17.8 the primes $\mathfrak{q}, \mathfrak{q}'$ correspond to ideals in $S \otimes_R \kappa(\mathfrak{p})$. Thus the lemma follows from Lemma 36.19. □

05DR **Lemma 36.21.** *Suppose $R \to S$ is finite. Then the fibres of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ are finite.*

Proof. By the discussion in Remark 17.8 the fibres are the spectra of the rings $S \otimes_R \kappa(\mathfrak{p})$. As $R \to S$ is finite, these fibre rings are finite over $\kappa(\mathfrak{p})$ hence Noetherian by Lemma 31.1. By Lemma 36.20 every prime of $S \otimes_R \kappa(\mathfrak{p})$ is a minimal prime. Hence by Lemma 31.6 there are at most finitely many. □

00GU **Lemma 36.22.** *Let $R \to S$ be a ring map such that S is integral over R. Let $\mathfrak p \subset \mathfrak p' \subset R$ be primes. Let $\mathfrak q$ be a prime of S mapping to $\mathfrak p$. Then there exists a prime $\mathfrak q'$ with $\mathfrak q \subset \mathfrak q'$ mapping to $\mathfrak p'$.*

Proof. We may replace R by $R/\mathfrak p$ and S by $S/\mathfrak q$. This reduces us to the situation of having an integral extension of domains $R \subset S$ and a prime $\mathfrak p' \subset R$. By Lemma 36.17 we win. □

The property expressed in the lemma above is called the "going up property" for the ring map $R \to S$, see Definition 41.1.

0564 **Lemma 36.23.** *Let $R \to S$ be a finite and finitely presented ring map. Let M be an S-module. Then M is finitely presented as an R-module if and only if M is finitely presented as an S-module.*

Proof. One of the implications follows from Lemma 6.4. To see the other assume that M is finitely presented as an S-module. Pick a presentation

$$S^{\oplus m} \longrightarrow S^{\oplus n} \longrightarrow M \longrightarrow 0$$

As S is finite as an R-module, the kernel of $S^{\oplus n} \to M$ is a finite R-module. Thus from Lemma 5.3 we see that it suffices to prove that S is finitely presented as an R-module.

Pick $y_1, \ldots, y_n \in S$ such that $y_1, \ldots, y_n$ generate S as an R-module. By Lemma 36.2 each y_i is integral over R. Choose monic polynomials $P_i(x) \in R[x]$ with $P_i(y_i) = 0$. Consider the ring

$$S' = R[x_1, \ldots, x_n]/(P_1(x_1), \ldots, P_n(x_n))$$

Then we see that S is of finite presentation as an S'-algebra by Lemma 6.2. Since $S' \to S$ is surjective, the kernel $J = \text{Ker}(S' \to S)$ is finitely generated as an ideal by Lemma 6.3. Hence J is a finite S'-module (immediate from the definitions). Thus $S = \text{Coker}(J \to S')$ is of finite presentation as an S'-module by Lemma 5.3. Hence, arguing as in the first paragraph, it suffices to show that S' is of finite presentation as an R-module. Actually, S' is free as an R-module with basis the monomials $x_1^{e_1} \ldots x_n^{e_n}$ for $0 \leq e_i < \deg(P_i)$. Namely, write $R \to S'$ as the composition

$$R \to R[x_1]/(P_1(x_1)) \to R[x_1, x_2]/(P_1(x_1), P_2(x_2)) \to \ldots \to S'$$

This shows that the ith ring in this sequence is free as a module over the $(i-1)$st one with basis $1, x_i, \ldots, x_i^{\deg(P_i)-1}$. The result follows easily from this by induction. Some details omitted. □

052J **Lemma 36.24.** *Let R be a ring. Let $x, y \in R$ be nonzerodivisors. Let $R[x/y] \subset R_{xy}$ be the R-subalgebra generated by x/y, and similarly for the subalgebras $R[y/x]$ and $R[x/y, y/x]$. If R is integrally closed in R_x or R_y, then the sequence*

$$0 \to R \xrightarrow{(-1,1)} R[x/y] \oplus R[y/x] \xrightarrow{(1,1)} R[x/y, y/x] \to 0$$

is a short exact sequence of R-modules.

Proof. Since $x/y \cdot y/x = 1$ it is clear that the map $R[x/y] \oplus R[y/x] \to R[x/y, y/x]$ is surjective. Let $\alpha \in R[x/y] \cap R[y/x]$. To show exactness in the middle we have to prove that $\alpha \in R$. By assumption we may write

$$\alpha = a_0 + a_1 x/y + \ldots + a_n (x/y)^n = b_0 + b_1 y/x + \ldots + b_m (y/x)^m$$

for some $n, m \geq 0$ and $a_i, b_j \in R$. Pick some $N > \max(n, m)$. Consider the finite R-submodule M of R_{xy} generated by the elements

$$(x/y)^N, (x/y)^{N-1}, \ldots, x/y, 1, y/x, \ldots, (y/x)^{N-1}, (y/x)^N$$

We claim that $\alpha M \subset M$. Namely, it is clear that $(x/y)^i(b_0 + b_1 y/x + \ldots + b_m(y/x)^m) \in M$ for $0 \leq i \leq N$ and that $(y/x)^i(a_0 + a_1 x/y + \ldots + a_n(x/y)^n) \in M$ for $0 \leq i \leq N$. Hence α is integral over R by Lemma 36.2. Note that $\alpha \in R_x$, so if R is integrally closed in R_x then $\alpha \in R$ as desired. $\square$

37. Normal rings

037B We first introduce the notion of a normal domain, and then we introduce the (very general) notion of a normal ring.

0309 **Definition 37.1.** A domain R is called *normal* if it is integrally closed in its field of fractions.

034L **Lemma 37.2.** *Let $R \to S$ be a ring map. If S is a normal domain, then the integral closure of R in S is a normal domain.*

Proof. Omitted. $\square$

The following notion is occasionally useful when studying normality.

00GW **Definition 37.3.** Let R be a domain.

(1) An element g of the fraction field of R is called *almost integral over R* if there exists an element $r \in R$, $r \neq 0$ such that $rg^n \in R$ for all $n \geq 0$.
(2) The domain R is called *completely normal* if every almost integral element of the fraction field of R is contained in R.

The following lemma shows that a Noetherian domain is normal if and only if it is completely normal.

00GX **Lemma 37.4.** *Let R be a domain with fraction field K. If $u, v \in K$ are almost integral over R, then so are $u+v$ and uv. Any element $g \in K$ which is integral over R is almost integral over R. If R is Noetherian then the converse holds as well.*

Proof. If $ru^n \in R$ for all $n \geq 0$ and $v^n r' \in R$ for all $n \geq 0$, then $(uv)^n rr'$ and $(u+v)^n rr'$ are in R for all $n \geq 0$. Hence the first assertion. Suppose $g \in K$ is integral over R. In this case there exists an $d > 0$ such that the ring $R[g]$ is generated by $1, g, \ldots, g^d$ as an R-module. Let $r \in R$ be a common denominator of the elements $1, g, \ldots, g^d \in K$. It is follows that $rR[g] \subset R$, and hence g is almost integral over R.

Suppose R is Noetherian and $g \in K$ is almost integral over R. Let $r \in R$, $r \neq 0$ be as in the definition. Then $R[g] \subset \frac{1}{r}R$ as an R-module. Since R is Noetherian this implies that $R[g]$ is finite over R. Hence g is integral over R, see Lemma 36.3. $\square$

00GY **Lemma 37.5.** *Any localization of a normal domain is normal.*

Proof. Let R be a normal domain, and let $S \subset R$ be a multiplicative subset. Suppose g is an element of the fraction field of R which is integral over $S^{-1}R$. Let $P = x^d + \sum_{j<d} a_j x^j$ be a polynomial with $a_i \in S^{-1}R$ such that $P(g) = 0$. Choose $s \in S$ such that $sa_i \in R$ for all i. Then sg satisfies the monic polynomial $x^d + \sum_{j<d} s^{d-j} a_j x^j$ which has coefficients $s^{d-j} a_j$ in R. Hence $sg \in R$ because R is normal. Hence $g \in S^{-1}R$. $\square$

00GZ **Lemma 37.6.** *A principal ideal domain is normal.*

Proof. Let R be a principal ideal domain. Let $g = a/b$ be an element of the fraction field of R integral over R. Because R is a principal ideal domain we may divide out a common factor of a and b and assume $(a, b) = R$. In this case, any equation $(a/b)^n + r_{n-1}(a/b)^{n-1} + \ldots + r_0 = 0$ with $r_i \in R$ would imply $a^n \in (b)$. This contradicts $(a, b) = R$ unless b is a unit in R. □

00H0 **Lemma 37.7.** *Let R be a domain with fraction field K. Suppose $f = \sum \alpha_i x^i$ is an element of $K[x]$.*

(1) *If f is integral over $R[x]$ then all α_i are integral over R, and*
(2) *If f is almost integral over $R[x]$ then all α_i are almost integral over R.*

Proof. We first prove the second statement. Write $f = \alpha_0 + \alpha_1 x + \ldots + \alpha_r x^r$ with $\alpha_r \neq 0$. By assumption there exists $h = b_0 + b_1 x + \ldots + b_s x^s \in R[x]$, $b_s \neq 0$ such that $f^n h \in R[x]$ for all $n \geq 0$. This implies that $b_s \alpha_r^n \in R$ for all $n \geq 0$. Hence α_r is almost integral over R. Since the set of almost integral elements form a subring (Lemma 37.4) we deduce that $f - \alpha_r x^r = \alpha_0 + \alpha_1 x + \ldots + \alpha_{r-1} x^{r-1}$ is almost integral over $R[x]$. By induction on r we win.

In order to prove the first statement we will use absolute Noetherian reduction. Namely, write $\alpha_i = a_i/b_i$ and let $P(t) = t^d + \sum_{j<d} f_j t^j$ be a polynomial with coefficients $f_j \in R[x]$ such that $P(f) = 0$. Let $f_j = \sum f_{ji} x^i$. Consider the subring $R_0 \subset R$ generated by the finite list of elements a_i, b_i, f_{ji} of R. It is a domain; let K_0 be its field of fractions. Since R_0 is a finite type $\mathbf{Z}$-algebra it is Noetherian, see Lemma 31.3. It is still the case that $f \in K_0[x]$ is integral over $R_0[x]$, because all the identities in R among the elements a_i, b_i, f_{ji} also hold in R_0. By Lemma 37.4 the element f is almost integral over $R_0[x]$. By the second statement of the lemma, the elements α_i are almost integral over R_0. And since R_0 is Noetherian, they are integral over R_0, see Lemma 37.4. Of course, then they are integral over R. □

030A **Lemma 37.8.** *Let R be a normal domain. Then $R[x]$ is a normal domain.*

Proof. The result is true if R is a field K because $K[x]$ is a euclidean domain and hence a principal ideal domain and hence normal by Lemma 37.6. Let g be an element of the fraction field of $R[x]$ which is integral over $R[x]$. Because g is integral over $K[x]$ where K is the fraction field of R we may write $g = \alpha_d x^d + \alpha_{d-1} x^{d-1} + \ldots + \alpha_0$ with $\alpha_i \in K$. By Lemma 37.7 the elements α_i are integral over R and hence are in R. □

0BI0 **Lemma 37.9.** *Let R be a Noetherian normal domain. Then $R[[x]]$ is a Noetherian normal domain.*

Proof. The power series ring is Noetherian by Lemma 31.2. Let $f, g \in R[[x]]$ be nonzero elements such that $w = f/g$ is integral over $R[[x]]$. Let K be the fraction field of R. Since the ring of Laurent series $K((x)) = K[[x]][1/x]$ is a field, we can write $w = a_n x^n + a_{n+1} x^{n+1} + \ldots)$ for some $n \in \mathbf{Z}$, $a_i \in K$, and $a_n \neq 0$. By Lemma 37.4 we see there exists a nonzero element $h = b_m x^m + b_{m+1} x^{m+1} + \ldots$ in $R[[x]]$ with $b_m \neq 0$ such that $w^e h \in R[[x]]$ for all $e \geq 1$. We conclude that $n \geq 0$ and that $b_m a_n^e \in R$ for all $e \geq 1$. Since R is Noetherian this implies that $a_n \in R$ by the same lemma. Now, if $a_n, a_{n+1}, \ldots, a_{N-1} \in R$, then we can apply the same argument to $w - a_n x^n - \ldots - a_{N-1} x^{N-1} = a_N x^N + \ldots$. In this way we see that all $a_i \in R$ and the lemma is proved. □

030B **Lemma 37.10.** *Let R be a domain. The following are equivalent:*

(1) *The domain R is a normal domain,*
(2) *for every prime $\mathfrak{p} \subset R$ the local ring $R_\mathfrak{p}$ is a normal domain, and*
(3) *for every maximal ideal $\mathfrak{m}$ the ring $R_\mathfrak{m}$ is a normal domain.*

Proof. This follows easily from the fact that for any domain R we have

$$R = \bigcap\nolimits_{\mathfrak{m}} R_{\mathfrak{m}}$$

inside the fraction field of R. Namely, if g is an element of the right hand side then the ideal $I = \{x \in R \mid xg \in R\}$ is not contained in any maximal ideal $\mathfrak{m}$, whence $I = R$. □

Lemma 37.10 shows that the following definition is compatible with Definition 37.1. (It is the definition from EGA – see [DG67, IV, 5.13.5 and 0, 4.1.4].)

00GV **Definition 37.11.** A ring R is called *normal* if for every prime $\mathfrak{p} \subset R$ the localization $R_\mathfrak{p}$ is a normal domain (see Definition 37.1).

Note that a normal ring is a reduced ring, as R is a subring of the product of its localizations at all primes (see for example Lemma 23.1).

034M **Lemma 37.12.** *A normal ring is integrally closed in its total ring of fractions.*

Proof. Let R be a normal ring. Let $x \in Q(R)$ be an element of the total ring of fractions of R integral over R. Set $I = \{f \in R, fx \in R\}$. Let $\mathfrak{p} \subset R$ be a prime. As $R \subset R_\mathfrak{p}$ is flat we see that $R_\mathfrak{p} \subset Q(R) \otimes_R R_\mathfrak{p}$. As $R_\mathfrak{p}$ is a normal domain we see that $x \otimes 1$ is an element of $R_\mathfrak{p}$. Hence we can find $a, f \in R$, $f \notin \mathfrak{p}$ such that $x \otimes 1 = a \otimes 1/f$. This means that $fx - a$ maps to zero in $Q(R) \otimes_R R_\mathfrak{p} = Q(R)_\mathfrak{p}$, which in turn means that there exists an $f' \in R$, $f' \notin \mathfrak{p}$ such that $f'fx = f'a$ in R. In other words, $ff' \in I$. Thus I is an ideal which isn't contained in any of the prime ideals of R, i.e., $I = R$ and $x \in R$. □

037C **Lemma 37.13.** *A localization of a normal ring is a normal ring.*

Proof. Omitted. □

00H1 **Lemma 37.14.** *Let R be a normal ring. Then $R[x]$ is a normal ring.*

Proof. Let $\mathfrak{q}$ be a prime of $R[x]$. Set $\mathfrak{p} = R \cap \mathfrak{q}$. Then we see that $R_\mathfrak{p}[x]$ is a normal domain by Lemma 37.8. Hence $(R[x])_\mathfrak{q}$ is a normal domain by Lemma 37.5. □

0CYA **Lemma 37.15.** *A finite product of normal rings is normal.*

Proof. It suffices to show that the product of two normal rings, say R and S, is normal. By Lemma 21.3 the prime ideals of $R \times S$ are of the form $\mathfrak{p} \times S$ and $R \times \mathfrak{q}$, where $\mathfrak{p}$ and $\mathfrak{q}$ are primes of R and S respectively. Localization yields $(R \times S)_{\mathfrak{p} \times S} = R_\mathfrak{p}$ which is a normal domain by assumption. Similarly for S. □

030C **Lemma 37.16.** *Let R be a ring. Assume R is reduced and has finitely many minimal primes. Then the following are equivalent:*

(1) *R is a normal ring,*
(2) *R is integrally closed in its total ring of fractions, and*
(3) *R is a finite product of normal domains.*

Proof. The implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) hold in general, see Lemmas 37.12 and 37.15.

Let $\mathfrak p_1, \ldots, \mathfrak p_n$ be the minimal primes of R. By Lemmas 25.2 and 25.4 we have $Q(R) = R_{\mathfrak p_1} \times \ldots \times R_{\mathfrak p_n}$, and by Lemma 25.1 each factor is a field. Denote $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ the ith idempotent of $Q(R)$.

If R is integrally closed in $Q(R)$, then it contains in particular the idempotents e_i, and we see that R is a product of n domains (see Sections 22 and 24). Each factor is of the form $R/\mathfrak p_i$ with field of fractions $R_{\mathfrak p_i}$. By Lemma 36.10 each map $R/\mathfrak p_i \to R_{\mathfrak p_i}$ is integrally closed. Hence R is a finite product of normal domains. ☐

037D **Lemma 37.17.** *Let $(R_i, \varphi_{ii'})$ be a directed system (Categories, Definition 8.1) of rings. If each R_i is a normal ring so is $R = \mathrm{colim}_i\, R_i$.*

Proof. Let $\mathfrak p \subset R$ be a prime ideal. Set $\mathfrak p_i = R_i \cap \mathfrak p$ (usual abuse of notation). Then we see that $R_{\mathfrak p} = \mathrm{colim}_i (R_i)_{\mathfrak p_i}$. Since each $(R_i)_{\mathfrak p_i}$ is a normal domain we reduce to proving the statement of the lemma for normal domains. If $a, b \in R$ and a/b satisfies a monic polynomial $P(T) \in R[T]$, then we can find a (sufficiently large) $i \in I$ such that a, b come from objects a_i, b_i over R_i, P comes from a monic polynomial $P_i \in R_i[T]$ and $P_i(a_i/b_i) = 0$. Since R_i is normal we see $a_i/b_i \in R_i$ and hence also $a/b \in R$. ☐

38. Going down for integral over normal

037E We first play around a little bit with the notion of elements integral over an ideal, and then we prove the theorem referred to in the section title.

00H2 **Definition 38.1.** Let $\varphi : R \to S$ be a ring map. Let $I \subset R$ be an ideal. We say an element $g \in S$ is *integral over I* if there exists a monic polynomial $P = x^d + \sum_{j<d} a_j x^j$ with coefficients $a_j \in I^{d-j}$ such that $P^\varphi(g) = 0$ in S.

This is mostly used when $\varphi = \mathrm{id}_R : R \to R$. In this case the set I' of elements integral over I is called the *integral closure of I*. We will see that I' is an ideal of R (and of course $I \subset I'$).

00H3 **Lemma 38.2.** *Let $\varphi : R \to S$ be a ring map. Let $I \subset R$ be an ideal. Let $A = \sum I^n t^n \subset R[t]$ be the subring of the polynomial ring generated by $R \oplus It \subset R[t]$. An element $s \in S$ is integral over I if and only if the element $st \in S[t]$ is integral over A.*

Proof. Suppose st is integral over A. Let $P = x^d + \sum_{j<d} a_j x^j$ be a monic polynomial with coefficients in A such that $P^\varphi(st) = 0$. Let $a'_j \in A$ be the degree $d-j$ part of a_i, in other words $a'_j = a''_j t^{d-j}$ with $a''_j \in I^{d-j}$. For degree reasons we still have $(st)^d + \sum_{j<d} \varphi(a''_j) t^{d-j} (st)^j = 0$. Hence we see that s is integral over I.

Suppose that s is integral over I. Say $P = x^d + \sum_{j<d} a_j x^j$ with $a_j \in I^{d-j}$. Then we immediately find a polynomial $Q = x^d + \sum_{j<d} (a_j t^{d-j}) x^j$ with coefficients in A which proves that st is integral over A. ☐

00H4 **Lemma 38.3.** *Let $\varphi : R \to S$ be a ring map. Let $I \subset R$ be an ideal. The set of elements of S which are integral over I form a R-submodule of S. Furthermore, if $s \in S$ is integral over R, and s' is integral over I, then ss' is integral over I.*

Proof. Closure under addition is clear from the characterization of Lemma 38.2. Any element $s \in S$ which is integral over R corresponds to the degree 0 element s of $S[x]$ which is integral over A (because $R \subset A$). Hence we see that multiplication by s on $S[x]$ preserves the property of being integral over A, by Lemma 36.7. $\square$

00H5 **Lemma 38.4.** *Suppose $\varphi : R \to S$ is integral. Suppose $I \subset R$ is an ideal. Then every element of IS is integral over I.*

Proof. Immediate from Lemma 38.3. $\square$

00H6 **Lemma 38.5.** *Let K be a field. Let $n, m \in \mathbf{N}$ and $a_0, \ldots, a_{n-1}, b_0, \ldots, b_{m-1} \in K$. If the polynomial $x^n + a_{n-1}x^{n-1} + \ldots + a_0$ divides the polynomial $x^m + b_{m-1}x^{m-1} + \ldots + b_0$ in $K[x]$ then*

(1) *$a_0, \ldots, a_{n-1}$ are integral over any subring R_0 of K containing the elements $b_0, \ldots, b_{m-1}$, and*
(2) *each a_i lies in $\sqrt{(b_0, \ldots, b_{m-1})R}$ for any subring $R \subset K$ containing the elements $a_0, \ldots, a_{n-1}, b_0, \ldots, b_{m-1}$.*

Proof. Let L/K be a field extension such that we can write $x^m + b_{m-1}x^{m-1} + \ldots + b_0 = \prod_{i=1}^m (x - \beta_i)$ with $\beta_i \in L$. See Fields, Section 16. Each β_i is integral over R_0. Since each a_i is a homogeneous polynomial in $\beta_1, \ldots, \beta_m$ we deduce the same for the a_i (use Lemma 36.7).

Choose $c_0, \ldots, c_{m-n-1} \in K$ such that

$$\begin{gathered} x^m + b_{m-1}x^{m-1} + \ldots + b_0 = \\ (x^n + a_{n-1}x^{n-1} + \ldots + a_0)(x^{m-n} + c_{m-n-1}x^{m-n-1} + \ldots + c_0). \end{gathered}$$

By part (1) the elements c_i are integral over R. Consider the integral extension

$$R \subset R' = R[c_0, \ldots, c_{m-n-1}] \subset K$$

By Lemmas 36.17 and 30.3 we see that $R \cap \sqrt{(b_0, \ldots, b_{m-1})R'} = \sqrt{(b_0, \ldots, b_{m-1})R}$. Thus we may replace R by R' and assume $c_i \in R$. Dividing out the radical $\sqrt{(b_0, \ldots, b_{m-1})}$ we get a reduced ring $\overline{R}$. We have to show that the images $\overline{a}_i \in \overline{R}$ are zero. And in $\overline{R}[x]$ we have the relation

$$\begin{gathered} x^m = x^m + \overline{b}_{m-1}x^{m-1} + \ldots + \overline{b}_0 = \\ (x^n + \overline{a}_{n-1}x^{n-1} + \ldots + \overline{a}_0)(x^{m-n} + \overline{c}_{m-n-1}x^{m-n-1} + \ldots + \overline{c}_0). \end{gathered}$$

It is easy to see that this implies $\overline{a}_i = 0$ for all i. Indeed by Lemma 25.1 the localization of $\overline{R}$ at a minimal prime $\mathfrak{p}$ is a field and $\overline{R}_{\mathfrak{p}}[x]$ a UFD. Thus $f = x^n + \sum \overline{a}_i x^i$ is associated to x^n and since f is monic $f = x^n$ in $\overline{R}_{\mathfrak{p}}[x]$. Then there exists an $s \in \overline{R}$, $s \notin \mathfrak{p}$ such that $s(f - x^n) = 0$. Therefore all $\overline{a}_i$ lie in $\mathfrak{p}$ and we conclude by Lemma 25.2. $\square$

00H7 **Lemma 38.6.** *Let $R \subset S$ be an inclusion of domains. Assume R is normal. Let $g \in S$ be integral over R. Then the minimal polynomial of g has coefficients in R.*

Proof. Let $P = x^m + b_{m-1}x^{m-1} + \ldots + b_0$ be a polynomial with coefficients in R such that $P(g) = 0$. Let $Q = x^n + a_{n-1}x^{n-1} + \ldots + a_0$ be the minimal polynomial for g over the fraction field K of R. Then Q divides P in $K[x]$. By Lemma 38.5 we see the a_i are integral over R. Since R is normal this means they are in R. $\square$

00H8 **Proposition 38.7.** *Let $R \subset S$ be an inclusion of domains. Assume R is normal and S integral over R. Let $\mathfrak p \subset \mathfrak p' \subset R$ be primes. Let $\mathfrak q'$ be a prime of S with $\mathfrak p' = R \cap \mathfrak q'$. Then there exists a prime $\mathfrak q$ with $\mathfrak q \subset \mathfrak q'$ such that $\mathfrak p = R \cap \mathfrak q$. In other words: the going down property holds for $R \to S$, see Definition 41.1.*

Proof. Let $\mathfrak p$, $\mathfrak p'$ and $\mathfrak q'$ be as in the statement. We have to show there is a prime $\mathfrak q$, with $\mathfrak q \subset \mathfrak q'$ and $R \cap \mathfrak q = \mathfrak p$. This is the same as finding a prime of $S_{\mathfrak q'}$ mapping to $\mathfrak p$. According to Lemma 17.9 we have to show that $\mathfrak p S_{\mathfrak q'} \cap R = \mathfrak p$. Pick $z \in \mathfrak p S_{\mathfrak q'} \cap R$. We may write $z = y/g$ with $y \in \mathfrak p S$ and $g \in S$, $g \notin \mathfrak q'$. Written differently we have $zg = y$.

By Lemma 38.4 there exists a monic polynomial $P = x^m + b_{m-1}x^{m-1} + \ldots + b_0$ with $b_i \in \mathfrak p$ such that $P(y) = 0$.

By Lemma 38.6 the minimal polynomial of g over K has coefficients in R. Write it as $Q = x^n + a_{n-1}x^{n-1} + \ldots + a_0$. Note that not all a_i, $i = n-1, \ldots, 0$ are in $\mathfrak p$ since that would imply $g^n = \sum_{j<n} a_j g^j \in \mathfrak p S \subset \mathfrak p' S \subset \mathfrak q'$ which is a contradiction.

Since $y = zg$ we see immediately from the above that $Q' = x^n + za_{n-1}x^{n-1} + \ldots + z^n a_0$ is the minimal polynomial for y. Hence Q' divides P and by Lemma 38.5 we see that $z^j a_{n-j} \in \sqrt{(b_0, \ldots, b_{m-1})} \subset \mathfrak p$, $j = 1, \ldots, n$. Because not all a_i, $i = n-1, \ldots, 0$ are in $\mathfrak p$ we conclude $z \in \mathfrak p$ as desired. $\square$

39. Flat modules and flat ring maps

00H9 One often used result is that if $M = \mathrm{colim}_{i \in \mathcal I} M_i$ is a colimit of R-modules and if N is an R-module then

$$M \otimes N = \mathrm{colim}_{i \in \mathcal I} M_i \otimes_R N,$$

see Lemma 12.9. This property is usually expressed by saying that $\otimes$ *commutes with colimits*. Another often used result is that if $0 \to N_1 \to N_2 \to N_3 \to 0$ is an exact sequence and if M is any R-module, then

$$M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3 \to 0$$

is still exact, see Lemma 12.10. Both of these properties tell us that the functor $N \mapsto M \otimes_R N$ *is right exact*. See Categories, Section 23 and Homology, Section 7. An R-module M is flat if $N \mapsto N \otimes_R M$ is also left exact, i.e., if it is exact. Here is the precise definition.

00HB **Definition 39.1.** Let R be a ring.

1. An R-module M is called *flat* if whenever $N_1 \to N_2 \to N_3$ is an exact sequence of R-modules the sequence $M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3$ is exact as well.
2. An R-module M is called *faithfully flat* if the complex of R-modules $N_1 \to N_2 \to N_3$ is exact if and only if the sequence $M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3$ is exact.
3. A ring map $R \to S$ is called *flat* if S is flat as an R-module.
4. A ring map $R \to S$ is called *faithfully flat* if S is faithfully flat as an R-module.

Here is an example of how you can use the flatness condition.

0BBY **Lemma 39.2.** *Let R be a ring. Let $I, J \subset R$ be ideals. Let M be a flat R-module. Then $IM \cap JM = (I \cap J)M$.*

Proof. Consider the exact sequence $0 \to I \cap J \to R \to R/I \oplus R/J$. Tensoring with the flat module M we obtain an exact sequence

$$0 \to (I \cap J) \otimes_R M \to M \to M/IM \oplus M/JM$$

Since the kernel of $M \to M/IM \oplus M/JM$ is equal to $IM \cap JM$ we conclude. □

05UT **Lemma 39.3.** *Let R be a ring. Let $\{M_i, \varphi_{ii'}\}$ be a directed system of flat R-modules. Then $\mathrm{colim}_i M_i$ is a flat R-module.*

Proof. This follows as $\otimes$ commutes with colimits and because directed colimits are exact, see Lemma 8.8. □

00HC **Lemma 39.4.** *A composition of (faithfully) flat ring maps is (faithfully) flat. If $R \to R'$ is (faithfully) flat, and M' is a (faithfully) flat R'-module, then M' is a (faithfully) flat R-module.*

Proof. The first statement of the lemma is a particular case of the second, so it is clearly enough to prove the latter. Let $R \to R'$ be a flat ring map, and M' a flat R'-module. We need to prove that M' is a flat R-module. Let $N_1 \to N_2 \to N_3$ be an exact complex of R-modules. Then, the complex $R' \otimes_R N_1 \to R' \otimes_R N_2 \to R' \otimes_R N_3$ is exact (since R' is flat as an R-module), and so the complex $M' \otimes_{R'} (R' \otimes_R N_1) \to M' \otimes_{R'} (R' \otimes_R N_2) \to M' \otimes_{R'} (R' \otimes_R N_3)$ is exact (since M' is a flat R'-module). Since $M' \otimes_{R'} (R' \otimes_R N) \cong (M' \otimes_{R'} R') \otimes_R N \cong M' \otimes_R N$ for any R-module N functorially (by Lemmas 12.7 and 12.3), this complex is isomorphic to the complex $M' \otimes_R N_1 \to M' \otimes_R N_2 \to M' \otimes_R N_3$, which is therefore also exact. This shows that M' is a flat R-module. Tracing this argument backwards, we can show that if $R \to R'$ is faithfully flat, and if M' is faithfully flat as an R'-module, then M' is faithfully flat as an R-module. □

00HD **Lemma 39.5.** *Let M be an R-module. The following are equivalent:*

00HE (1) *M is flat over R.*
00HF (2) *for every injection of R-modules $N \subset N'$ the map $N \otimes_R M \to N' \otimes_R M$ is injective.*
00HG (3) *for every ideal $I \subset R$ the map $I \otimes_R M \to R \otimes_R M = M$ is injective.*
00HH (4) *for every finitely generated ideal $I \subset R$ the map $I \otimes_R M \to R \otimes_R M = M$ is injective.*

Proof. The implications (1) implies (2) implies (3) implies (4) are all trivial. Thus we prove (4) implies (1). Suppose that $N_1 \to N_2 \to N_3$ is exact. Let $K = \mathrm{Ker}(N_2 \to N_3)$ and $Q = \mathrm{Im}(N_2 \to N_3)$. Then we get maps

$$N_1 \otimes_R M \to K \otimes_R M \to N_2 \otimes_R M \to Q \otimes_R M \to N_3 \otimes_R M$$

Observe that the first and third arrows are surjective. Thus if we show that the second and fourth arrows are injective, then we are done[3]. Hence it suffices to show that $- \otimes_R M$ transforms injective R-module maps into injective R-module maps.

[3] Here is the argument in more detail: Assume that we know that the second and fourth arrows are injective. Lemma 12.10 (applied to the exact sequence $K \to N_2 \to Q \to 0$) yields that the sequence $K \otimes_R M \to N_2 \otimes_R M \to Q \otimes_R M \to 0$ is exact. Hence, $\mathrm{Ker}\,(N_2 \otimes_R M \to Q \otimes_R M) = \mathrm{Im}\,(K \otimes_R M \to N_2 \otimes_R M)$. Since $\mathrm{Im}\,(K \otimes_R M \to N_2 \otimes_R M) = \mathrm{Im}\,(N_1 \otimes_R M \to N_2 \otimes_R M)$ (due to the surjectivity of $N_1 \otimes_R M \to K \otimes_R M$) and $\mathrm{Ker}\,(N_2 \otimes_R M \to Q \otimes_R M) = \mathrm{Ker}\,(N_2 \otimes_R M \to N_3 \otimes_R M)$ (due to the injectivity of $Q \otimes_R M \to N_3 \otimes_R M$), this becomes $\mathrm{Ker}\,(N_2 \otimes_R M \to N_3 \otimes_R M) = \mathrm{Im}\,(N_1 \otimes_R M \to N_2 \otimes_R M)$, which shows that the functor $- \otimes_R M$ is exact, whence M is flat.

Assume $K \to N$ is an injective R-module map and let $x \in \text{Ker}(K \otimes_R M \to N \otimes_R M)$. We have to show that x is zero. The R-module K is the union of its finite R-submodules; hence, $K \otimes_R M$ is the colimit of R-modules of the form $K_i \otimes_R M$ where K_i runs over all finite R-submodules of K (because tensor product commutes with colimits). Thus, for some i our x comes from an element $x_i \in K_i \otimes_R M$. Thus we may assume that K is a finite R-module. Assume this. We regard the injection $K \to N$ as an inclusion, so that $K \subset N$.

The R-module N is the union of its finite R-submodules that contain K. Hence, $N \otimes_R M$ is the colimit of R-modules of the form $N_i \otimes_R M$ where N_i runs over all finite R-submodules of N that contain K (again since tensor product commutes with colimits). Notice that this is a colimit over a directed system (since the sum of two finite submodules of N is again finite). Hence, (by Lemma 8.4) the element $x \in K \otimes_R M$ maps to zero in at least one of these R-modules $N_i \otimes_R M$ (since x maps to zero in $N \otimes_R M$). Thus we may assume N is a finite R-module.

Assume N is a finite R-module. Write $N = R^{\oplus n}/L$ and $K = L'/L$ for some $L \subset L' \subset R^{\oplus n}$. For any R-submodule $G \subset R^{\oplus n}$, we have a canonical map $G \otimes_R M \to M^{\oplus n}$ obtained by composing $G \otimes_R M \to R^n \otimes_R M = M^{\oplus n}$. It suffices to prove that $L \otimes_R M \to M^{\oplus n}$ and $L' \otimes_R M \to M^{\oplus n}$ are injective. Namely, if so, then we see that $K \otimes_R M = L' \otimes_R M/L \otimes_R M \to M^{\oplus n}/L \otimes_R M$ is injective too[4].

Thus it suffices to show that $L \otimes_R M \to M^{\oplus n}$ is injective when $L \subset R^{\oplus n}$ is an R-submodule. We do this by induction on n. The base case $n = 1$ we handle below. For the induction step assume $n > 1$ and set $L' = L \cap R \oplus 0^{\oplus n-1}$. Then $L'' = L/L'$ is a submodule of $R^{\oplus n-1}$. We obtain a diagram

$$\begin{tikzcd}
 & L' \otimes_R M \arrow[r] \arrow[d] & L \otimes_R M \arrow[r] \arrow[d] & L'' \otimes_R M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & M^{\oplus n} \arrow[r] & M^{\oplus n-1} \arrow[r] & 0
\end{tikzcd}$$

By induction hypothesis and the base case the left and right vertical arrows are injective. The rows are exact. It follows that the middle vertical arrow is injective too.

The base case of the induction above is when $L \subset R$ is an ideal. In other words, we have to show that $I \otimes_R M \to M$ is injective for any ideal I of R. We know this is true when I is finitely generated. However, $I = \bigcup I_\alpha$ is the union of the finitely generated ideals I_α contained in it. In other words, $I = \operatorname{colim} I_\alpha$. Since $\otimes$ commutes with colimits we see that $I \otimes_R M = \operatorname{colim} I_\alpha \otimes_R M$ and since all the morphisms $I_\alpha \otimes_R M \to M$ are injective by assumption, the same is true for $I \otimes_R M \to M$. □

05UU **Lemma 39.6.** *Let $\{R_i, \varphi_{ii'}\}$ be a system of rings over the directed set I. Let $R = \operatorname{colim}_i R_i$.*

(1) *If M is an R-module such that M is flat as an R_i-module for all i, then M is flat as an R-module.*

[4]This becomes obvious if we identify $L' \otimes_R M$ and $L \otimes_R M$ with submodules of $M^{\oplus n}$ (which is legitimate since the maps $L \otimes_R M \to M^{\oplus n}$ and $L' \otimes_R M \to M^{\oplus n}$ are injective and commute with the obvious map $L' \otimes_R M \to L \otimes_R M$).

(2) *For $i \in I$ let M_i be a flat R_i-module and for $i' \geq i$ let $f_{ii'} : M_i \to M_{i'}$ be a $\varphi_{ii'}$-linear map such that $f_{i'i''} \circ f_{ii'} = f_{ii''}$. Then $M = \text{colim}_{i \in I} M_i$ is a flat R-module.*

Proof. Part (1) is a special case of part (2) with $M_i = M$ for all i and $f_{ii'} = \text{id}_M$. Proof of (2). Let $\mathfrak{a} \subset R$ be a finitely generated ideal. By Lemma 39.5 it suffices to show that $\mathfrak{a} \otimes_R M \to M$ is injective. We can find an $i \in I$ and a finitely generated ideal $\mathfrak{a}' \subset R_i$ such that $\mathfrak{a} = \mathfrak{a}'R$. Then $\mathfrak{a} = \text{colim}_{i' \geq i} \mathfrak{a}'R_{i'}$. Since $\otimes$ commutes with colimits the map $\mathfrak{a} \otimes_R M \to M$ is the colimit of the maps

$$\mathfrak{a}'R_{i'} \otimes_{R_{i'}} M_{i'} \longrightarrow M_{i'}$$

These maps are all injective by assumption. Since colimits over I are exact by Lemma 8.8 we win. $\square$

00HI **Lemma 39.7.** *Suppose that M is (faithfully) flat over R, and that $R \to R'$ is a ring map. Then $M \otimes_R R'$ is (faithfully) flat over R'.*

Proof. For any R'-module N we have a canonical isomorphism $N \otimes_{R'} (R' \otimes_R M) = N \otimes_R M$. Hence the desired exactness properties of the functor $- \otimes_{R'} (R' \otimes_R M)$ follow from the corresponding exactness properties of the functor $- \otimes_R M$. $\square$

00HJ **Lemma 39.8.** *Let $R \to R'$ be a faithfully flat ring map. Let M be a module over R, and set $M' = R' \otimes_R M$. Then M is flat over R if and only if M' is flat over R'.*

Proof. By Lemma 39.7 we see that if M is flat then M' is flat. For the converse, suppose that M' is flat. Let $N_1 \to N_2 \to N_3$ be an exact sequence of R-modules. We want to show that $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact. We know that $N_1 \otimes_R R' \to N_2 \otimes_R R' \to N_3 \otimes_R R'$ is exact, because $R \to R'$ is flat. Flatness of M' implies that $N_1 \otimes_R R' \otimes_{R'} M' \to N_2 \otimes_R R' \otimes_{R'} M' \to N_3 \otimes_R R' \otimes_{R'} M'$ is exact. We may write this as $N_1 \otimes_R M \otimes_R R' \to N_2 \otimes_R M \otimes_R R' \to N_3 \otimes_R M \otimes_R R'$. Finally, faithful flatness implies that $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact. $\square$

0584 **Lemma 39.9.** *Let R be a ring. Let $S \to S'$ be a flat map of R-algebras. Let M be a module over S, and set $M' = S' \otimes_S M$.*

(1) *If M is flat over R, then M' is flat over R.*
(2) *If $S \to S'$ is faithfully flat, then M is flat over R if and only if M' is flat over R.*

Proof. Let $N \to N'$ be an injection of R-modules. By the flatness of $S \to S'$ we have

$$\text{Ker}(N \otimes_R M \to N' \otimes_R M) \otimes_S S' = \text{Ker}(N \otimes_R M' \to N' \otimes_R M')$$

If M is flat over R, then the left hand side is zero and we find that M' is flat over R by the second characterization of flatness in Lemma 39.5. If M' is flat over R then we have the vanishing of the right hand side and if in addition $S \to S'$ is faithfully flat, this implies that $\text{Ker}(N \otimes_R M \to N' \otimes_R M)$ is zero which in turn shows that M is flat over R. $\square$

039V **Lemma 39.10.** *Let $R \to S$ be a ring map. Let M be an S-module. If M is flat as an R-module and faithfully flat as an S-module, then $R \to S$ is flat.*

Proof. Let $N_1 \to N_2 \to N_3$ be an exact sequence of R-modules. By assumption $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact. We may write this as

$$N_1 \otimes_R S \otimes_S M \to N_2 \otimes_R S \otimes_S M \to N_3 \otimes_R S \otimes_S M.$$

By faithful flatness of M over S we conclude that $N_1 \otimes_R S \to N_2 \otimes_R S \to N_3 \otimes_R S$ is exact. Hence $R \to S$ is flat. $\square$

Let R be a ring. Let M be an R-module. Let $\sum f_i x_i = 0$ be a relation in M. We say the relation $\sum f_i x_i$ is *trivial* if there exist an integer $m \geq 0$, elements $y_j \in M$, $j = 1, \ldots, m$, and elements $a_{ij} \in R$, $i = 1, \ldots, n$, $j = 1, \ldots, m$ such that

$$x_i = \sum\nolimits_j a_{ij} y_j, \forall i, \quad \text{and} \quad 0 = \sum\nolimits_i f_i a_{ij}, \forall j.$$

00HK **Lemma 39.11** (Equational criterion of flatness)**.** *A module M over R is flat if and only if every relation in M is trivial.*

Proof. Assume M is flat and let $\sum f_i x_i = 0$ be a relation in M. Let $I = (f_1, \ldots, f_n)$, and let $K = \mathrm{Ker}(R^n \to I, (a_1, \ldots, a_n) \mapsto \sum_i a_i f_i)$. So we have the short exact sequence $0 \to K \to R^n \to I \to 0$. Then $\sum f_i \otimes x_i$ is an element of $I \otimes_R M$ which maps to zero in $R \otimes_R M = M$. By flatness $\sum f_i \otimes x_i$ is zero in $I \otimes_R M$. Thus there exists an element of $K \otimes_R M$ mapping to $\sum e_i \otimes x_i \in R^n \otimes_R M$ where e_i is the ith basis element of R^n. Write this element as $\sum k_j \otimes y_j$ and then write the image of k_j in R^n as $\sum a_{ij} e_i$ to get the result.

Assume every relation is trivial, let I be a finitely generated ideal, and let $x = \sum f_i \otimes x_i$ be an element of $I \otimes_R M$ mapping to zero in $R \otimes_R M = M$. This just means exactly that $\sum f_i x_i$ is a relation in M. And the fact that it is trivial implies easily that x is zero, because

$$x = \sum f_i \otimes x_i = \sum f_i \otimes \left(\sum a_{ij} y_j\right) = \sum \left(\sum f_i a_{ij}\right) \otimes y_j = 0$$

$\square$

00HL **Lemma 39.12.** *Suppose that R is a ring, $0 \to M'' \to M' \to M \to 0$ a short exact sequence, and N an R-module. If M is flat then $N \otimes_R M'' \to N \otimes_R M'$ is injective, i.e., the sequence*

$$0 \to N \otimes_R M'' \to N \otimes_R M' \to N \otimes_R M \to 0$$

is a short exact sequence.

Proof. Let $R^{(I)} \to N$ be a surjection from a free module onto N with kernel K. The result follows from the snake lemma applied to the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & M'' \otimes_R N & \to & M' \otimes_R N & \to & M \otimes_R N & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & (M'')^{(I)} & \to & (M')^{(I)} & \to & M^{(I)} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & M'' \otimes_R K & \to & M' \otimes_R K & \to & M \otimes_R K & \to & 0 \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}$$

with exact rows and columns. The middle row is exact because tensoring with the free module $R^{(I)}$ is exact. $\square$

00HM **Lemma 39.13.** *Suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of R-modules. If M' and M'' are flat so is M. If M and M'' are flat so is M'.*

Proof. We will use the criterion that a module N is flat if for every ideal $I \subset R$ the map $N \otimes_R I \to N$ is injective, see Lemma 39.5. Consider an ideal $I \subset R$. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \to & M' & \to & M & \to & M'' & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & M' \otimes_R I & \to & M \otimes_R I & \to & M'' \otimes_R I & \to & 0 \end{array}$$

with exact rows. This immediately proves the first assertion. The second follows because if M'' is flat then the lower left horizontal arrow is injective by Lemma 39.12. $\square$

00HO **Lemma 39.14.** *Let R be a ring. Let M be an R-module. The following are equivalent*

1. *M is faithfully flat, and*
2. *M is flat and for all R-module homomorphisms $\alpha : N \to N'$ we have $\alpha = 0$ if and only if $\alpha \otimes id_M = 0$.*

Proof. If M is faithfully flat, then $0 \to \mathrm{Ker}(\alpha) \to N \to N'$ is exact if and only if the same holds after tensoring with M. This proves (1) implies (2). For the other, assume (2). Let $N_1 \to N_2 \to N_3$ be a complex, and assume the complex $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact. Take $x \in \mathrm{Ker}(N_2 \to N_3)$, and consider the map $\alpha : R \to N_2/\mathrm{Im}(N_1)$, $r \mapsto rx + \mathrm{Im}(N_1)$. By the exactness of the complex $- \otimes_R M$ we see that $\alpha \otimes \mathrm{id}_M$ is zero. By assumption we get that α is zero. Hence x is in the image of $N_1 \to N_2$. $\square$

00HP **Lemma 39.15.** *Let M be a flat R-module. The following are equivalent:*

1. *M is faithfully flat,*
2. *for every nonzero R-module N, then tensor product $M \otimes_R N$ is nonzero,*
3. *for all $\mathfrak{p} \in \mathrm{Spec}(R)$ the tensor product $M \otimes_R \kappa(\mathfrak{p})$ is nonzero, and*
4. *for all maximal ideals $\mathfrak{m}$ of R the tensor product $M \otimes_R \kappa(\mathfrak{m}) = M/\mathfrak{m}M$ is nonzero.*

Proof. Assume M faithfully flat and $N \neq 0$. By Lemma 39.14 the nonzero map $1 : N \to N$ induces a nonzero map $M \otimes_R N \to M \otimes_R N$, so $M \otimes_R N \neq 0$. Thus (1) implies (2). The imlpications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are immediate.

Assume (4). Suppose that $N_1 \to N_2 \to N_3$ is a complex and suppose that $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact. Let H be the cohomology of the complex, so $H = \mathrm{Ker}(N_2 \to N_3)/\mathrm{Im}(N_1 \to N_2)$. To finish the proof we will show $H = 0$. By flatness we see that $H \otimes_R M = 0$. Take $x \in H$ and let $I = \{f \in R \mid fx = 0\}$ be its annihilator. Since $R/I \subset H$ we get $M/IM \subset H \otimes_R M = 0$ by flatness of M. If $I \neq R$ we may choose a maximal ideal $I \subset \mathfrak{m} \subset R$. This immediately gives a contradiction. $\square$

00HQ **Lemma 39.16.** *Let $R \to S$ be a flat ring map. The following are equivalent:*

1. *$R \to S$ is faithfully flat,*
2. *the induced map on Spec is surjective, and*
3. *any closed point $x \in \mathrm{Spec}(R)$ is in the image of the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$.*

Proof. This follows quickly from Lemma 39.15, because we saw in Remark 17.8 that $\mathfrak{p}$ is in the image if and only if the ring $S \otimes_R \kappa(\mathfrak{p})$ is nonzero. □

00HR **Lemma 39.17.** *A flat local ring homomorphism of local rings is faithfully flat.*

Proof. Immediate from Lemma 39.16. □

Flatness meshes well with localization.

00HT **Lemma 39.18.** *Let R be a ring. Let $S \subset R$ be a multiplicative subset.*

1. *The localization $S^{-1}R$ is a flat R-algebra.*
2. *If M is an $S^{-1}R$-module, then M is a flat R-module if and only if M is a flat $S^{-1}R$-module.*
3. *Suppose M is an R-module. Then M is a flat R-module if and only if $M_{\mathfrak{p}}$ is a flat $R_{\mathfrak{p}}$-module for all primes $\mathfrak{p}$ of R.*
4. *Suppose M is an R-module. Then M is a flat R-module if and only if $M_{\mathfrak{m}}$ is a flat $R_{\mathfrak{m}}$-module for all maximal ideals $\mathfrak{m}$ of R.*
5. *Suppose $R \to A$ is a ring map, M is an A-module, and $g_1, \ldots, g_m \in A$ are elements generating the unit ideal of A. Then M is flat over R if and only if each localization M_{g_i} is flat over R.*
6. *Suppose $R \to A$ is a ring map, and M is an A-module. Then M is a flat R-module if and only if the localization $M_{\mathfrak{q}}$ is a flat $R_{\mathfrak{p}}$-module (with $\mathfrak{p}$ the prime of R lying under $\mathfrak{q}$) for all primes $\mathfrak{q}$ of A.*
7. *Suppose $R \to A$ is a ring map, and M is an A-module. Then M is a flat R-module if and only if the localization $M_{\mathfrak{m}}$ is a flat $R_{\mathfrak{p}}$-module (with $\mathfrak{p} = R \cap \mathfrak{m}$) for all maximal ideals $\mathfrak{m}$ of A.*

Proof. Let us prove the last statement of the lemma. In the proof we will use repeatedly that localization is exact and commutes with tensor product, see Sections 9 and 12.

Suppose $R \to A$ is a ring map, and M is an A-module. Assume that $M_{\mathfrak{m}}$ is a flat $R_{\mathfrak{p}}$-module for all maximal ideals $\mathfrak{m}$ of A (with $\mathfrak{p} = R \cap \mathfrak{m}$). Let $I \subset R$ be an ideal. We have to show the map $I \otimes_R M \to M$ is injective. We can think of this as a map of A-modules. By assumption the localization $(I \otimes_R M)_{\mathfrak{m}} \to M_{\mathfrak{m}}$ is injective because $(I \otimes_R M)_{\mathfrak{m}} = I_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{m}}$. Hence the kernel of $I \otimes_R M \to M$ is zero by Lemma 23.1. Hence M is flat over R.

Conversely, assume M is flat over R. Pick a prime $\mathfrak{q}$ of A lying over the prime $\mathfrak{p}$ of R. Suppose that $I \subset R_{\mathfrak{p}}$ is an ideal. We have to show that $I \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{q}} \to M_{\mathfrak{q}}$ is injective. We can write $I = J_{\mathfrak{p}}$ for some ideal $J \subset R$. Then the map $I \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{q}} \to M_{\mathfrak{q}}$ is just the localization (at $\mathfrak{q}$) of the map $J \otimes_R M \to M$ which is injective. Since localization is exact we see that $M_{\mathfrak{q}}$ is a flat $R_{\mathfrak{p}}$-module.

This proves (7) and (6). The other statements follow in a straightforward way from the last statement (proofs omitted). □

00HS **Lemma 39.19.** *Let $R \to S$ be flat. Let $\mathfrak{p} \subset \mathfrak{p}'$ be primes of R. Let $\mathfrak{q}' \subset S$ be a prime of S mapping to $\mathfrak{p}'$. Then there exists a prime $\mathfrak{q} \subset \mathfrak{q}'$ mapping to $\mathfrak{p}$.*

Proof. By Lemma 39.18 the local ring map $R_{\mathfrak{p}'} \to S_{\mathfrak{q}'}$ is flat. By Lemma 39.17 this local ring map is faithfully flat. By Lemma 39.16 there is a prime mapping to $\mathfrak{p}R_{\mathfrak{p}'}$. The inverse image of this prime in S does the job. □

The property of $R \to S$ described in the lemma is called the "going down property". See Definition 41.1.

090N **Lemma 39.20.** *Let R be a ring. Let $\{S_i, \varphi_{ii'}\}$ be a directed system of faithfully flat R-algebras. Then $S = \mathrm{colim}_i\, S_i$ is a faithfully flat R-algebra.*

Proof. By Lemma 39.3 we see that S is flat. Let $\mathfrak{m} \subset R$ be a maximal ideal. By Lemma 39.16 none of the rings $S_i/\mathfrak{m}S_i$ is zero. Hence $S/\mathfrak{m}S = \mathrm{colim}\, S_i/\mathfrak{m}S_i$ is nonzero as well because 1 is not equal to zero. Thus the image of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ contains $\mathfrak{m}$ and we see that $R \to S$ is faithfully flat by Lemma 39.16. $\square$

40. Supports and annihilators

080S Some very basic definitions and lemmas.

00L1 **Definition 40.1.** Let R be a ring and let M be an R-module. The *support of M* is the set

$$\mathrm{Supp}(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid M_\mathfrak{p} \neq 0\}$$

0585 **Lemma 40.2.** *Let R be a ring. Let M be an R-module. Then*

$$M = (0) \Leftrightarrow Supp(M) = \emptyset.$$

Proof. Actually, Lemma 23.1 even shows that $\mathrm{Supp}(M)$ always contains a maximal ideal if M is not zero. $\square$

07T7 **Definition 40.3.** Let R be a ring. Let M be an R-module.

(1) Given an element $m \in M$ the *annihilator of m* is the ideal
$$\mathrm{Ann}_R(m) = \mathrm{Ann}(m) = \{f \in R \mid fm = 0\}.$$
(2) The *annihilator of M* is the ideal
$$\mathrm{Ann}_R(M) = \mathrm{Ann}(M) = \{f \in R \mid fm = 0\ \forall m \in M\}.$$

07T8 **Lemma 40.4.** *Let $R \to S$ be a flat ring map. Let M be an R-module and $m \in M$. Then $Ann_R(m)S = Ann_S(m \otimes 1)$. If M is a finite R-module, then $Ann_R(M)S = Ann_S(M \otimes_R S)$.*

Proof. Set $I = \mathrm{Ann}_R(m)$. By definition there is an exact sequence $0 \to I \to R \to M$ where the map $R \to M$ sends f to fm. Using flatness we obtain an exact sequence $0 \to I \otimes_R S \to S \to M \otimes_R S$ which proves the first assertion. If $m_1, \ldots, m_n$ is a set of generators of M then $\mathrm{Ann}_R(M) = \bigcap \mathrm{Ann}_R(m_i)$. Similarly $\mathrm{Ann}_S(M \otimes_R S) = \bigcap \mathrm{Ann}_S(m_i \otimes 1)$. Set $I_i = \mathrm{Ann}_R(m_i)$. Then it suffices to show that $\bigcap_{i=1,\ldots,n}(I_i S) = (\bigcap_{i=1,\ldots,n} I_i)S$. This is Lemma 39.2. $\square$

00L2 **Lemma 40.5.** *Let R be a ring and let M be an R-module. If M is finite, then $Supp(M)$ is closed. More precisely, if $I = Ann(M)$ is the annihilator of M, then $V(I) = Supp(M)$.*

Proof. We will show that $V(I) = \mathrm{Supp}(M)$.

Suppose $\mathfrak{p} \in \mathrm{Supp}(M)$. Then $M_\mathfrak{p} \neq 0$. Choose an element $m \in M$ whose image in $M_\mathfrak{p}$ is nonzero. Then the annihilator of m is contained in $\mathfrak{p}$ by construction of the localization $M_\mathfrak{p}$. Hence a fortiori $I = \mathrm{Ann}(M)$ must be contained in $\mathfrak{p}$.

Conversely, suppose that $\mathfrak{p} \notin \mathrm{Supp}(M)$. Then $M_\mathfrak{p} = 0$. Let $x_1, \ldots, x_r \in M$ be generators. By Lemma 9.9 there exists an $f \in R$, $f \notin \mathfrak{p}$ such that $x_i/1 = 0$ in M_f. Hence $f^{n_i} x_i = 0$ for some $n_i \geq 1$. Hence $f^n M = 0$ for $n = \max\{n_i\}$ as desired. $\square$

0BUR **Lemma 40.6.** *Let $R \to R'$ be a ring map and let M be a finite R-module. Then Supp($M \otimes_R R'$) is the inverse image of Supp(M).*

Proof. Let $\mathfrak{p} \in \text{Supp}(M)$. By Nakayama's lemma (Lemma 20.1) we see that

$$M \otimes_R \kappa(\mathfrak{p}) = M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$$

is a nonzero $\kappa(\mathfrak{p})$ vector space. Hence for every prime $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p}$ we see that

$$(M \otimes_R R')_{\mathfrak{p}'}/\mathfrak{p}'(M \otimes_R R')_{\mathfrak{p}'} = (M \otimes_R R') \otimes_{R'} \kappa(\mathfrak{p}') = M \otimes_R \kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}')$$

is nonzero. This implies $\mathfrak{p}' \in \text{Supp}(M \otimes_R R')$. For the converse, if $\mathfrak{p}' \subset R'$ is a prime lying over an arbitrary prime $\mathfrak{p} \subset R$, then

$$(M \otimes_R R')_{\mathfrak{p}'} = M_\mathfrak{p} \otimes_{R_\mathfrak{p}} R'_{\mathfrak{p}'}.$$

Hence if $\mathfrak{p}' \in \text{Supp}(M \otimes_R R')$ lies over the prime $\mathfrak{p} \subset R$, then $\mathfrak{p} \in \text{Supp}(M)$. $\square$

07Z5 **Lemma 40.7.** *Let R be a ring, let M be an R-module, and let $m \in M$. Then $\mathfrak{p} \in V(Ann(m))$ if and only if m does not map to zero in $M_\mathfrak{p}$.*

Proof. We may replace M by $Rm \subset M$. Then (1) $\text{Ann}(m) = \text{Ann}(M)$ and (2) x does not map to zero in $M_\mathfrak{p}$ if and only if $\mathfrak{p} \in \text{Supp}(M)$. The result now follows from Lemma 40.5. $\square$

051B **Lemma 40.8.** *Let R be a ring and let M be an R-module. If M is a finitely presented R-module, then Supp(M) is a closed subset of* $\text{Spec}(R)$ *whose complement is quasi-compact.*

Proof. Choose a presentation

$$R^{\oplus m} \longrightarrow R^{\oplus n} \longrightarrow M \to 0$$

Let $A \in \text{Mat}(n \times m, R)$ be the matrix of the first map. By Nakayama's Lemma 20.1 we see that

$$M_\mathfrak{p} \neq 0 \Leftrightarrow M \otimes \kappa(\mathfrak{p}) \neq 0 \Leftrightarrow \text{rank}(A \bmod \mathfrak{p}) < n.$$

Hence, if I is the ideal of R generated by the $n \times n$ minors of A, then $\text{Supp}(M) = V(I)$. Since I is finitely generated, say $I = (f_1, \ldots, f_t)$, we see that $\text{Spec}(R) \setminus V(I)$ is a finite union of the standard opens $D(f_i)$, hence quasi-compact. $\square$

00L3 **Lemma 40.9.** *Let R be a ring and let M be an R-module.*

1. *If M is finite then the support of M/IM is Supp(M) $\cap V(I)$.*
2. *If $N \subset M$, then Supp(N) $\subset$ Supp(M).*
3. *If Q is a quotient module of M then Supp(Q) $\subset$ Supp(M).*
4. *If $0 \to N \to M \to Q \to 0$ is a short exact sequence then Supp(M) $=$ Supp(Q) $\cup$ Supp(N).*

Proof. The functors $M \mapsto M_\mathfrak{p}$ are exact. This immediately implies all but the first assertion. For the first assertion we need to show that $M_\mathfrak{p} \neq 0$ and $I \subset \mathfrak{p}$ implies $(M/IM)_\mathfrak{p} = M_\mathfrak{p}/IM_\mathfrak{p} \neq 0$. This follows from Nakayama's Lemma 20.1. $\square$

41. Going up and going down

00HU Suppose $\mathfrak p$, $\mathfrak p'$ are primes of the ring R. Let $X = \text{Spec}(R)$ with the Zariski topology. Denote $x \in X$ the point corresponding to $\mathfrak p$ and $x' \in X$ the point corresponding to $\mathfrak p'$. Then we have:

$$x' \rightsquigarrow x \Leftrightarrow \mathfrak p' \subset \mathfrak p.$$

In words: x is a specialization of x' if and only if $\mathfrak p' \subset \mathfrak p$. See Topology, Section 19 for terminology and notation.

00HV **Definition 41.1.** Let $\varphi : R \to S$ be a ring map.

1. We say a $\varphi : R \to S$ satisfies *going up* if given primes $\mathfrak p \subset \mathfrak p'$ in R and a prime $\mathfrak q$ in S lying over $\mathfrak p$ there exists a prime $\mathfrak q'$ of S such that (a) $\mathfrak q \subset \mathfrak q'$, and (b) $\mathfrak q'$ lies over $\mathfrak p'$.
2. We say a $\varphi : R \to S$ satisfies *going down* if given primes $\mathfrak p \subset \mathfrak p'$ in R and a prime $\mathfrak q'$ in S lying over $\mathfrak p'$ there exists a prime $\mathfrak q$ of S such that (a) $\mathfrak q \subset \mathfrak q'$, and (b) $\mathfrak q$ lies over $\mathfrak p$.

So far we have see the following cases of this:

1. An integral ring map satisfies going up, see Lemma 36.22.
2. As a special case finite ring maps satisfy going up.
3. As a special case quotient maps $R \to R/I$ satisfy going up.
4. A flat ring map satisfies going down, see Lemma 39.19
5. As a special case any localization satisfies going down.
6. An extension $R \subset S$ of domains, with R normal and S integral over R satisfies going down, see Proposition 38.7.

Here is another case where going down holds.

0407 **Lemma 41.2.** *Let $R \to S$ be a ring map. If the induced map $\varphi : \text{Spec}(S) \to \text{Spec}(R)$ is open, then $R \to S$ satisfies going down.*

Proof. Suppose that $\mathfrak p \subset \mathfrak p' \subset R$ and $\mathfrak q' \subset S$ lies over $\mathfrak p'$. As φ is open, for every $g \in S$, $g \notin \mathfrak q'$ we see that $\mathfrak p$ is in the image of $D(g) \subset \text{Spec}(S)$. In other words $S_g \otimes_R \kappa(\mathfrak p)$ is not zero. Since $S_{\mathfrak q'}$ is the directed colimit of these S_g this implies that $S_{\mathfrak q'} \otimes_R \kappa(\mathfrak p)$ is not zero, see Lemmas 9.9 and 12.9. Hence $\mathfrak p$ is in the image of $\text{Spec}(S_{\mathfrak q'}) \to \text{Spec}(R)$ as desired. □

00HW **Lemma 41.3.** *Let $R \to S$ be a ring map.*

1. *$R \to S$ satisfies going down if and only if generalizations lift along the map $\text{Spec}(S) \to \text{Spec}(R)$, see Topology, Definition 19.4.*
2. *$R \to S$ satisfies going up if and only if specializations lift along the map $\text{Spec}(S) \to \text{Spec}(R)$, see Topology, Definition 19.4.*

Proof. Omitted. □

00HX **Lemma 41.4.** *Suppose $R \to S$ and $S \to T$ are ring maps satisfying going down. Then so does $R \to T$. Similarly for going up.*

Proof. According to Lemma 41.3 this follows from Topology, Lemma 19.5 □

00HY **Lemma 41.5.** *Let $R \to S$ be a ring map. Let $T \subset \text{Spec}(R)$ be the image of $\text{Spec}(S)$. If T is stable under specialization, then T is closed.*

Proof. We give two proofs.

First proof. Let $\mathfrak p \subset R$ be a prime ideal such that the corresponding point of $\text{Spec}(R)$ is in the closure of T. This means that for every $f \in R$, $f \not\in \mathfrak p$ we have $D(f) \cap T \neq \emptyset$. Note that $D(f) \cap T$ is the image of $\text{Spec}(S_f)$ in $\text{Spec}(R)$. Hence we conclude that $S_f \neq 0$. In other words, $1 \neq 0$ in the ring S_f. Since $S_{\mathfrak p}$ is the directed colimit of the rings S_f we conclude that $1 \neq 0$ in $S_{\mathfrak p}$. In other words, $S_{\mathfrak p} \neq 0$ and considering the image of $\text{Spec}(S_{\mathfrak p}) \to \text{Spec}(S) \to \text{Spec}(R)$ we see there exists a $\mathfrak p' \in T$ with $\mathfrak p' \subset \mathfrak p$. As we assumed T closed under specialization we conclude $\mathfrak p$ is a point of T as desired.

Second proof. Let $I = \text{Ker}(R \to S)$. We may replace R by R/I. In this case the ring map $R \to S$ is injective. By Lemma 30.5 all the minimal primes of R are contained in the image T. Hence if T is stable under specialization then it contains all primes. □

00HZ **Lemma 41.6.** *Let $R \to S$ be a ring map. The following are equivalent:*

(1) *Going up holds for $R \to S$, and*
(2) *the map $\text{Spec}(S) \to \text{Spec}(R)$ is closed.*

Proof. It is a general fact that specializations lift along a closed map of topological spaces, see Topology, Lemma 19.7. Hence the second condition implies the first.

Assume that going up holds for $R \to S$. Let $V(I) \subset \text{Spec}(S)$ be a closed set. We want to show that the image of $V(I)$ in $\text{Spec}(R)$ is closed. The ring map $S \to S/I$ obviously satisfies going up. Hence $R \to S \to S/I$ satisfies going up, by Lemma 41.4. Replacing S by S/I it suffices to show the image T of $\text{Spec}(S)$ in $\text{Spec}(R)$ is closed. By Topology, Lemmas 19.2 and 19.6 this image is stable under specialization. Thus the result follows from Lemma 41.5. □

00I0 **Lemma 41.7.** *Let R be a ring. Let $E \subset \text{Spec}(R)$ be a constructible subset.*

(1) *If E is stable under specialization, then E is closed.*
(2) *If E is stable under generalization, then E is open.*

Proof. First proof. The first assertion follows from Lemma 41.5 combined with Lemma 29.4. The second follows because the complement of a constructible set is constructible (see Topology, Lemma 15.2), the first part of the lemma and Topology, Lemma 19.2.

Second proof. Since $\text{Spec}(R)$ is a spectral space by Lemma 26.2 this is a special case of Topology, Lemma 23.6. □

00I1 **Proposition 41.8.** *Let $R \to S$ be flat and of finite presentation. Then $\text{Spec}(S) \to \text{Spec}(R)$ is open. More generally this holds for any ring map $R \to S$ of finite presentation which satisfies going down.*

Proof. Assume that $R \to S$ has finite presentation and satisfies going down. It suffices to prove that the image of a standard open $D(f)$ is open. Since $S \to S_f$ satisfies going down as well, we see that $R \to S_f$ satisfies going down. Thus after replacing S by S_f we see it suffices to prove the image is open. By Chevalley's theorem (Theorem 29.10) the image is a constructible set E. And E is stable under generalization because $R \to S$ satisfies going down, see Topology, Lemmas 19.2 and 19.6. Hence E is open by Lemma 41.7. □

037F **Lemma 41.9.** *Let k be a field, and let R, S be k-algebras. Let $S' \subset S$ be a sub k-algebra, and let $f \in S' \otimes_k R$. In the commutative diagram*

$$\begin{tikzcd} \mathrm{Spec}((S \otimes_k R)_f) \arrow[rr] \arrow[dr] & & \mathrm{Spec}((S' \otimes_k R)_f) \arrow[dl] \\ & \mathrm{Spec}(R) & \end{tikzcd}$$

the images of the diagonal arrows are the same.

Proof. Let $\mathfrak{p} \subset R$ be in the image of the south-west arrow. This means (Lemma 17.9) that

$$(S' \otimes_k R)_f \otimes_R \kappa(\mathfrak{p}) = (S' \otimes_k \kappa(\mathfrak{p}))_f$$

is not the zero ring, i.e., $S' \otimes_k \kappa(\mathfrak{p})$ is not the zero ring and the image of f in it is not nilpotent. The ring map $S' \otimes_k \kappa(\mathfrak{p}) \to S \otimes_k \kappa(\mathfrak{p})$ is injective. Hence also $S \otimes_k \kappa(\mathfrak{p})$ is not the zero ring and the image of f in it is not nilpotent. Hence $(S \otimes_k R)_f \otimes_R \kappa(\mathfrak{p})$ is not the zero ring. Thus (Lemma 17.9) we see that $\mathfrak{p}$ is in the image of the south-east arrow as desired. □

037G **Lemma 41.10.** *Let k be a field. Let R and S be k-algebras. The map* $\mathrm{Spec}(S \otimes_k R) \to \mathrm{Spec}(R)$ *is open.*

Proof. Let $f \in S \otimes_k R$. It suffices to prove that the image of the standard open $D(f)$ is open. Let $S' \subset S$ be a finite type k-subalgebra such that $f \in S' \otimes_k R$. The map $R \to S' \otimes_k R$ is flat and of finite presentation, hence the image U of $\mathrm{Spec}((S' \otimes_k R)_f) \to \mathrm{Spec}(R)$ is open by Proposition 41.8. By Lemma 41.9 this is also the image of $D(f)$ and we win. □

Here is a tricky lemma that is sometimes useful.

00EA **Lemma 41.11.** *Let $R \to S$ be a ring map. Let $\mathfrak{p} \subset R$ be a prime. Assume that*

1. *there exists a unique prime $\mathfrak{q} \subset S$ lying over $\mathfrak{p}$, and*
2. *either*
    - (a) *going up holds for $R \to S$, or*
    - (b) *going down holds for $R \to S$ and there is at most one prime of S above every prime of R.*

Then $S_{\mathfrak{p}} = S_{\mathfrak{q}}$.

Proof. Consider any prime $\mathfrak{q}' \subset S$ which corresponds to a point of $\mathrm{Spec}(S_{\mathfrak{p}})$. This means that $\mathfrak{p}' = R \cap \mathfrak{q}'$ is contained in $\mathfrak{p}$. Here is a picture

$$\begin{tikzcd}[arrows=dash] \mathfrak{q}' \arrow[r] \arrow[d] & ? \arrow[r] \arrow[d] & S \arrow[d] \\ \mathfrak{p}' \arrow[r] & \mathfrak{p} \arrow[r] & R \end{tikzcd}$$

Assume (1) and (2)(a). By going up there exists a prime $\mathfrak{q}'' \subset S$ with $\mathfrak{q}' \subset \mathfrak{q}''$ and $\mathfrak{q}''$ lying over $\mathfrak{p}$. By the uniqueness of $\mathfrak{q}$ we conclude that $\mathfrak{q}'' = \mathfrak{q}$. In other words $\mathfrak{q}'$ defines a point of $\mathrm{Spec}(S_{\mathfrak{q}})$.

Assume (1) and (2)(b). By going down there exists a prime $\mathfrak{q}'' \subset \mathfrak{q}$ lying over $\mathfrak{p}'$. By the uniqueness of primes lying over $\mathfrak{p}'$ we see that $\mathfrak{q}' = \mathfrak{q}''$. In other words $\mathfrak{q}'$ defines a point of $\mathrm{Spec}(S_{\mathfrak{q}})$.

In both cases we conclude that the map $\text{Spec}(S_{\mathfrak q}) \to \text{Spec}(S_{\mathfrak p})$ is bijective. Clearly this means all the elements of $S - \mathfrak q$ are all invertible in $S_{\mathfrak p}$, in other words $S_{\mathfrak p} = S_{\mathfrak q}$. □

The following lemma is a generalization of going down for flat ring maps.

080T **Lemma 41.12.** *Let $R \to S$ be a ring map. Let N be a finite S-module flat over R. Endow Supp$(N) \subset \text{Spec}(S)$ with the induced topology. Then generalizations lift along Supp$(N) \to \text{Spec}(R)$.*

Proof. The meaning of the statement is as follows. Let $\mathfrak p \subset \mathfrak p' \subset R$ be primes. Let $\mathfrak q' \subset S$ be a prime $\mathfrak q' \in \text{Supp}(N)$ Then there exists a prime $\mathfrak q \subset \mathfrak q'$, $\mathfrak q \in \text{Supp}(N)$ lying over $\mathfrak p$. As N is flat over R we see that $N_{\mathfrak q'}$ is flat over $R_{\mathfrak p'}$, see Lemma 39.18. As $N_{\mathfrak q'}$ is finite over $S_{\mathfrak q'}$ and not zero since $\mathfrak q' \in \text{Supp}(N)$ we see that $N_{\mathfrak q'} \otimes_{S_{\mathfrak q'}} \kappa(\mathfrak q')$ is nonzero by Nakayama's Lemma 20.1. Thus $N_{\mathfrak q'} \otimes_{R_{\mathfrak p'}} \kappa(\mathfrak p')$ is also not zero. We conclude from Lemma 39.15 that $N_{\mathfrak q'} \otimes_{R_{\mathfrak p'}} \kappa(\mathfrak p)$ is nonzero. Let $J \subset S_{\mathfrak q'} \otimes_{R_{\mathfrak p'}} \kappa(\mathfrak p)$ be the annihilator of the finite nonzero module $N_{\mathfrak q'} \otimes_{R_{\mathfrak p'}} \kappa(\mathfrak p)$. Since J is a proper ideal we can choose a prime $\mathfrak q \subset S$ which corresponds to a prime of $S_{\mathfrak q'} \otimes_{R_{\mathfrak p'}} \kappa(\mathfrak p)/J$. This prime is in the support of N, lies over $\mathfrak p$, and is contained in $\mathfrak q'$ as desired. □

42. Separable extensions

030I In this section we talk about separability for nonalgebraic field extensions. This is closely related to the concept of geometrically reduced algebras, see Definition 43.1.

030O **Definition 42.1.** Let K/k be a field extension.

(1) We say K is *separably generated over k* if there exists a transcendence basis $\{x_i; i \in I\}$ of K/k such that the extension $K/k(x_i; i \in I)$ is a separable algebraic extension.
(2) We say K is *separable over k* if for every subextension $k \subset K' \subset K$ with K' finitely generated over k, the extension K'/k is separably generated.

With this awkward definition it is not clear that a separably generated field extension is itself separable. It will turn out that this is the case, see Lemma 44.2.

030P **Lemma 42.2.** *Let K/k be a separable field extension. For any subextension $K/K'/k$ the field extension K'/k is separable.*

Proof. This is direct from the definition. □

030Q **Lemma 42.3.** *Let K/k be a separably generated, and finitely generated field extension. Set $r = trdeg_k(K)$. Then there exist elements $x_1, \ldots, x_{r+1}$ of K such that*

(1) *$x_1, \ldots, x_r$ is a transcendence basis of K over k,*
(2) *$K = k(x_1, \ldots, x_{r+1})$, and*
(3) *x_{r+1} is separable over $k(x_1, \ldots, x_r)$.*

Proof. Combine the definition with Fields, Lemma 19.1. □

04KM **Lemma 42.4.** *Let K/k be a finitely generated field extension. There exists a diagram*

$$\begin{tikzcd} K \arrow[r] & K' \\ k \arrow[u] \arrow[r] & k' \arrow[u] \end{tikzcd}$$

where k'/k, K'/K are finite purely inseparable field extensions such that K'/k' is a separably generated field extension.

Proof. This lemma is only interesting when the characteristic of k is $p > 0$. Choose $x_1, \ldots, x_r$ a transcendence basis of K over k. As K is finitely generated over k the extension $k(x_1, \ldots, x_r) \subset K$ is finite. Let $K/K_{sep}/k(x_1, \ldots, x_r)$ be the subextension found in Fields, Lemma 14.6. If $K = K_{sep}$ then we are done. We will use induction on $d = [K : K_{sep}]$.

Assume that $d > 1$. Choose a $\beta \in K$ with $\alpha = \beta^p \in K_{sep}$ and $\beta \notin K_{sep}$. Let $P = T^n + a_1T^{n-1} + \ldots + a_n$ be the minimal polynomial of α over $k(x_1, \ldots, x_r)$. Let k'/k be a finite purely inseparable extension obtained by adjoining pth roots such that each a_i is a pth power in $k'(x_1^{1/p}, \ldots, x_r^{1/p})$. Such an extension exists; details omitted. Let L be a field fitting into the diagram

$$\begin{tikzcd} K \arrow[r] & L \\ k(x_1, \ldots, x_r) \arrow[u] \arrow[r] & k'(x_1^{1/p}, \ldots, x_r^{1/p}) \arrow[u] \end{tikzcd}$$

We may and do assume L is the compositum of K and $k'(x_1^{1/p}, \ldots, x_r^{1/p})$. Let $L/L_{sep}/k'(x_1^{1/p}, \ldots, x_r^{1/p})$ be the subextension found in Fields, Lemma 14.6. Then L_{sep} is the compositum of K_{sep} and $k'(x_1^{1/p}, \ldots, x_r^{1/p})$. The element $\alpha \in L_{sep}$ is a zero of the polynomial P all of whose coefficients are pth powers in $k'(x_1^{1/p}, \ldots, x_r^{1/p})$ and whose roots are pairwise distinct. By Fields, Lemma 28.2 we see that $\alpha = (\alpha')^p$ for some $\alpha' \in L_{sep}$. Clearly, this means that β maps to $\alpha' \in L_{sep}$. In other words, we get the tower of fields

$$\begin{tikzcd} K \arrow[r] & L \\ K_{sep}(\beta) \arrow[u] \arrow[r] & L_{sep} \arrow[u] \\ K_{sep} \arrow[u] \arrow[r] & L_{sep} \arrow[u, equal] \\ k(x_1, \ldots, x_r) \arrow[u] \arrow[r] & k'(x_1^{1/p}, \ldots, x_r^{1/p}) \arrow[u] \\ k \arrow[u] \arrow[r] & k' \arrow[u] \end{tikzcd}$$

Thus this construction leads to a new situation with $[L : L_{sep}] < [K : K_{sep}]$. By induction we can find $k' \subset k''$ and $L \subset L'$ as in the lemma for the extension L/k'. Then the extensions k''/k and L'/K work for the extension K/k. This proves the lemma. □

43. Geometrically reduced algebras

05DS The main result on geometrically reduced algebras is Lemma 44.3. We suggest the reader skip to the lemma after reading the definition.

030S **Definition 43.1.** Let k be a field. Let S be a k-algebra. We say S is *geometrically reduced over k* if for every field extension K/k the K-algebra $K \otimes_k S$ is reduced.

Let k be a field and let S be a reduced k-algebra. To check that S is geometrically reduced it will suffice to check that $\overline{k} \otimes_k S$ is reduced (where $\overline{k}$ denotes the algebraic closure of k). In fact it is enough to check this for finite purely inseparable field extensions k'/k. See Lemma 44.3.

030T **Lemma 43.2.** *Elementary properties of geometrically reduced algebras. Let k be a field. Let S be a k-algebra.*

1. *If S is geometrically reduced over k so is every k-subalgebra.*
2. *If all finitely generated k-subalgebras of S are geometrically reduced, then S is geometrically reduced.*
3. *A directed colimit of geometrically reduced k-algebras is geometrically reduced.*
4. *If S is geometrically reduced over k, then any localization of S is geometrically reduced over k.*

Proof. Omitted. The second and third property follow from the fact that tensor product commutes with colimits. □

04KN **Lemma 43.3.** *Let k be a field. If R is geometrically reduced over k, and $S \subset R$ is a multiplicative subset, then the localization $S^{-1}R$ is geometrically reduced over k. If R is geometrically reduced over k, then $R[x]$ is geometrically reduced over k.*

Proof. Omitted. Hints: A localization of a reduced ring is reduced, and localization commutes with tensor products. □

In the proofs of the following lemmas we will repeatedly use the following observation: Suppose that $R' \subset R$ and $S' \subset S$ are inclusions of k-algebras. Then the map $R' \otimes_k S' \to R \otimes_k S$ is injective.

00I3 **Lemma 43.4.** *Let k be a field. Let R, S be k-algebras.*

1. *If $R \otimes_k S$ is nonreduced, then there exist finitely generated subalgebras $R' \subset R$, $S' \subset S$ such that $R' \otimes_k S'$ is not reduced.*
2. *If $R \otimes_k S$ contains a nonzero zerodivisor, then there exist finitely generated subalgebras $R' \subset R$, $S' \subset S$ such that $R' \otimes_k S'$ contains a nonzero zerodivisor.*
3. *If $R \otimes_k S$ contains a nontrivial idempotent, then there exist finitely generated subalgebras $R' \subset R$, $S' \subset S$ such that $R' \otimes_k S'$ contains a nontrivial idempotent.*

Proof. Suppose $z \in R \otimes_k S$ is nilpotent. We may write $z = \sum_{i=1,\ldots,n} x_i \otimes y_i$. Thus we may take R' the k-subalgebra generated by the x_i and S' the k-subalgebra generated by the y_i. The second and third statements are proved in the same way. $\square$

034N **Lemma 43.5.** *Let k be a field. Let S be a geometrically reduced k-algebra. Let R be any reduced k-algebra. Then $R \otimes_k S$ is reduced.*

Proof. By Lemma 43.4 we may assume that R is of finite type over k. Then R, as a reduced Noetherian ring, embeds into a finite product of fields (see Lemmas 25.4, 31.6, and 25.1). Hence we may assume R is a finite product of fields. In this case it follows from Definition 43.1 that $R \otimes_k S$ is reduced. $\square$

030U **Lemma 43.6.** *Let k be a field. Let S be a reduced k-algebra. Let K/k be either a separable field extension, or a separably generated field extension. Then $K \otimes_k S$ is reduced.*

Proof. Assume $k \subset K$ is separable. By Lemma 43.4 we may assume that S is of finite type over k and K is finitely generated over k. Then S embeds into a finite product of fields, namely its total ring of fractions (see Lemmas 25.1 and 25.4). Hence we may actually assume that S is a domain. We choose $x_1, \ldots, x_{r+1} \in K$ as in Lemma 42.3. Let $P \in k(x_1, \ldots, x_r)[T]$ be the minimal polynomial of x_{r+1}. It is a separable polynomial. It is easy to see that $k[x_1, \ldots, x_r] \otimes_k S = S[x_1, \ldots, x_r]$ is a domain. This implies $k(x_1, \ldots, x_r) \otimes_k S$ is a domain as it is a localization of $S[x_1, \ldots, x_r]$. The ring extension $k(x_1, \ldots, x_r) \otimes_k S \subset K \otimes_k S$ is generated by a single element x_{r+1} with a single equation, namely P. Hence $K \otimes_k S$ embeds into $F[T]/(P)$ where F is the fraction field of $k(x_1, \ldots, x_r) \otimes_k S$. Since P is separable this is a finite product of fields and we win.

At this point we do not yet know that a separably generated field extension is separable, so we have to prove the lemma in this case also. To do this suppose that $\{x_i\}_{i \in I}$ is a separating transcendence basis for K over k. For any finite set of elements $\lambda_j \in K$ there exists a finite subset $T \subset I$ such that $k(\{x_i\}_{i \in T}) \subset k(\{x_i\}_{i \in T} \cup \{\lambda_j\})$ is finite separable. Hence we see that K is a directed colimit of finitely generated and separably generated extensions of k. Thus the argument of the preceding paragraph applies to this case as well. $\square$

07K2 **Lemma 43.7.** *Let k be a field and let S be a k-algebra. Assume that S is reduced and that $S_\mathfrak{p}$ is geometrically reduced for every minimal prime $\mathfrak{p}$ of S. Then S is geometrically reduced.*

Proof. Since S is reduced the map $S \to \prod_{\mathfrak{p} \text{ minimal}} S_\mathfrak{p}$ is injective, see Lemma 25.2. If K/k is a field extension, then the maps

$$S \otimes_k K \to (\prod S_\mathfrak{p}) \otimes_k K \to \prod S_\mathfrak{p} \otimes_k K$$

are injective: the first as $k \to K$ is flat and the second by inspection because K is a free k-module. As $S_\mathfrak{p}$ is geometrically reduced the ring on the right is reduced. Thus we see that $S \otimes_k K$ is reduced as a subring of a reduced ring. $\square$

0C2X **Lemma 43.8.** *Let k'/k be a separable algebraic extension. Then there exists a multiplicative subset $S \subset k' \otimes_k k'$ such that the multiplication map $k' \otimes_k k' \to k'$ is identified with $k' \otimes_k k' \to S^{-1}(k' \otimes_k k')$.*

Proof. First assume k'/k is finite separable. Then $k' = k(\alpha)$, see Fields, Lemma 19.1. Let $P \in k[x]$ be the minimal polynomial of α over k. Then P is an irreducible, separable, monic polynomial, see Fields, Section 12. Then $k'[x]/(P) \to k' \otimes_k k'$, $\sum \alpha_i x^i \mapsto \alpha_i \otimes \alpha^i$ is an isomorphism. We can factor $P = (x - \alpha)Q$ in $k'[x]$ and since P is separable we see that $Q(\alpha) \neq 0$. Then it is clear that the multiplicative set S' generated by Q in $k'[x]/(P)$ works, i.e., that $k' = (S')^{-1}(k'[x]/(P))$. By transport of structure the image S of S' in $k' \otimes_k k'$ works.

In the general case we write $k' = \bigcup k_i$ as the union of its finite subfield extensions over k. For each i there is a multiplicative subset $S_i \subset k_i \otimes_k k_i$ such that $k_i = S_i^{-1}(k_i \otimes_k k_i)$. Then $S = \bigcup S_i \subset k' \otimes_k k'$ works. $\square$

0C2Y **Lemma 43.9.** *Let k'/k be a separable algebraic field extension. Let A be an algebra over k'. Then A is geometrically reduced over k if and only if it is geometrically reduced over k'.*

Proof. Assume A is geometrically reduced over k'. Let K/k be a field extension. Then $K \otimes_k k'$ is a reduced ring by Lemma 43.6. Hence by Lemma 43.5 we find that $K \otimes_k A = (K \otimes_k k') \otimes_{k'} A$ is reduced.

Assume A is geometrically reduced over k. Let K/k' be a field extension. Then

$$K \otimes_{k'} A = (K \otimes_k A) \otimes_{(k' \otimes_k k')} k'$$

Since $k' \otimes_k k' \to k'$ is a localization by Lemma 43.8, we see that $K \otimes_{k'} A$ is a localization of a reduced algebra, hence reduced. $\square$

44. Separable extensions, continued

05DT In this section we continue the discussion started in Section 42. Let p be a prime number and let k be a field of characteristic p. In this case we write $k^{1/p}$ for the extension of k gotten by adjoining pth roots of all the elements of k to k. (In other words it is the subfield of an algebraic closure of k generated by the pth roots of elements of k.)

030W **Lemma 44.1.** *Let k be a field of characteristic $p > 0$. Let K/k be a field extension. The following are equivalent:*

1. *K is separable over k,*
2. *the ring $K \otimes_k k^{1/p}$ is reduced, and*
3. *K is geometrically reduced over k.*

Proof. The implication (1) $\Rightarrow$ (3) follows from Lemma 43.6. The implication (3) $\Rightarrow$ (2) is immediate.

Assume (2). Let $K/L/k$ be a subextension such that L is a finitely generated field extension of k. We have to show that we can find a separating transcendence basis of L. The assumption implies that $L \otimes_k k^{1/p}$ is reduced. Let $x_1, \ldots, x_r$ be a transcendence basis of L over k such that the degree of inseparability of the finite extension $k(x_1, \ldots, x_r) \subset L$ is minimal. If L is separable over $k(x_1, \ldots, x_r)$ then we win. Assume this is not the case to get a contradiction. Then there exists an element $\alpha \in L$ which is not separable over $k(x_1, \ldots, x_r)$. Let $P(T) \in k(x_1, \ldots, x_r)[T]$ be the minimal polynomial of α over $k(x_1, \ldots, x_r)$. After replacing α by $f\alpha$ for some nonzero $f \in k[x_1, \ldots, x_r]$ we may and do assume that P lies in $k[x_1, \ldots, x_r, T]$. Because α is not separable P is a polynomial in T^p, see Fields, Lemma 12.1. Let

dp be the degree of P as a polynomial in T. Since P is the minimal polynomial of α the monomials

$$x_1^{e_1} \dots x_r^{e_r} \alpha^e$$

for $e < dp$ are linearly independent over k in L. We claim that the element $\partial P/\partial x_i \in k[x_1, \dots, x_r, T]$ is not zero for at least one i. Namely, if this was not the case, then P is actually a polynomial in $x_1^p, \dots, x_r^p, T^p$. In that case we can consider $P^{1/p} \in k^{1/p}[x_1, \dots, x_r, T]$. This would map to $P^{1/p}(x_1, \dots, x_r, \alpha)$ which is a nilpotent element of $k^{1/p} \otimes_k L$ and hence zero. On the other hand, $P^{1/p}(x_1, \dots, x_r, \alpha)$ is a $k^{1/p}$-linear combination the monomials listed above, hence nonzero in $k^{1/p} \otimes_k L$. This is a contradiction which proves our claim.

Thus, after renumbering, we may assume that $\partial P/\partial x_1$ is not zero. As P is an irreducible polynomial in T over $k(x_1, \dots, x_r)$ it is irreducible as a polynomial in $x_1, \dots, x_r, T$, hence by Gauss's lemma it is irreducible as a polynomial in x_1 over $k(x_2, \dots, x_r, T)$. Since the transcendence degree of L is r we see that $x_2, \dots, x_r, \alpha$ are algebraically independent. Hence $P(X, x_2, \dots, x_r, \alpha) \in k(x_2, \dots, x_r, \alpha)[X]$ is irreducible. It follows that x_1 is separably algebraic over $k(x_2, \dots, x_r, \alpha)$. This means that the degree of inseparability of the finite extension $k(x_2, \dots, x_r, \alpha) \subset L$ is less than the degree of inseparability of the finite extension $k(x_1, \dots, x_r) \subset L$, which is a contradiction. □

030X **Lemma 44.2.** *A separably generated field extension is separable.*

Proof. Combine Lemma 43.6 with Lemma 44.1. □

In the following lemma we will use the notion of the perfect closure which is defined in Definition 45.5.

030V **Lemma 44.3.** *Let k be a field. Let S be a k-algebra. The following are equivalent:*

1. *$k' \otimes_k S$ is reduced for every finite purely inseparable extension k' of k,*
2. *$k^{1/p} \otimes_k S$ is reduced,*
3. *$k^{perf} \otimes_k S$ is reduced, where k^{perf} is the perfect closure of k,*
4. *$\overline{k} \otimes_k S$ is reduced, where $\overline{k}$ is the algebraic closure of k, and*
5. *S is geometrically reduced over k.*

Proof. Note that any finite purely inseparable extension k'/k embeds in k^{perf}. Moreover, $k^{1/p}$ embeds into k^{perf} which embeds into $\overline{k}$. Thus it is clear that (5) ⇒ (4) ⇒ (3) ⇒ (2) and that (3) ⇒ (1).

We prove that (1) ⇒ (5). Assume $k' \otimes_k S$ is reduced for every finite purely inseparable extension k' of k. Let K/k be an extension of fields. We have to show that $K \otimes_k S$ is reduced. By Lemma 43.4 we reduce to the case where K/k is a finitely generated field extension. Choose a diagram

$$\begin{tikzcd} K \arrow[r] & K' \\ k \arrow[u] \arrow[r] & k' \arrow[u] \end{tikzcd}$$

as in Lemma 42.4. By assumption $k' \otimes_k S$ is reduced. By Lemma 43.6 it follows that $K' \otimes_k S$ is reduced. Hence we conclude that $K \otimes_k S$ is reduced as desired.

Finally we prove that (2) ⇒ (5). Assume $k^{1/p} \otimes_k S$ is reduced. Then S is reduced. Moreover, for each localization $S_{\mathfrak{p}}$ at a minimal prime $\mathfrak{p}$, the ring $k^{1/p} \otimes_k S_{\mathfrak{p}}$ is a localization of $k^{1/p} \otimes_k S$ hence is reduced. But $S_\mathfrak{p}$ is a field by Lemma 25.1, hence $S_\mathfrak{p}$ is geometrically reduced by Lemma 44.1. It follows from Lemma 43.7 that S is geometrically reduced. □

45. Perfect fields

05DU Here is the definition.

030Y **Definition 45.1.** Let k be a field. We say k is *perfect* if every field extension of k is separable over k.

030Z **Lemma 45.2.** *A field k is perfect if and only if it is a field of characteristic 0 or a field of characteristic $p > 0$ such that every element has a pth root.*

Proof. The characteristic zero case is clear. Assume the characteristic of k is $p > 0$. If k is perfect, then all the field extensions where we adjoin a pth root of an element of k have to be trivial, hence every element of k has a pth root. Conversely if every element has a pth root, then $k = k^{1/p}$ and every field extension of k is separable by Lemma 44.1. □

030R **Lemma 45.3.** *Let K/k be a finitely generated field extension. There exists a diagram*

$$\begin{tikzcd} K \arrow[r] & K' \\ k \arrow[u] \arrow[r] & k' \arrow[u] \end{tikzcd}$$

where k'/k, K'/K are finite purely inseparable field extensions such that K'/k' is a separable field extension. In this situation we can assume that $K' = k'K$ is the compositum, and also that $K' = (k' \otimes_k K)_{red}$.

Proof. By Lemma 42.4 we can find such a diagram with K'/k' separably generated. By Lemma 44.2 this implies that K' is separable over k'. The compositum $k'K$ is a subextension of K'/k' and hence $k' \subset k'K$ is separable by Lemma 42.2. The ring $(k' \otimes_k K)_{red}$ is a domain as for some $n \gg 0$ the map $x \mapsto x^{p^n}$ maps it into K. Hence it is a field by Lemma 36.19. Thus $(k' \otimes_k K)_{red} \to K'$ maps it isomorphically onto $k'K$. □

046W **Lemma 45.4.** *For every field k there exists a purely inseparable extension k'/k such that k' is perfect. The field extension k'/k is unique up to unique isomorphism.*

Proof. If the characteristic of k is zero, then $k' = k$ is the unique choice. Assume the characteristic of k is $p > 0$. For every $n > 0$ there exists a unique algebraic extension $k \subset k^{1/p^n}$ such that (a) every element $\lambda \in k$ has a p^nth root in k^{1/p^n} and (b) for every element $\mu \in k^{1/p^n}$ we have $\mu^{p^n} \in k$. Namely, consider the ring map $k \to k^{1/p^n} = k$, $x \mapsto x^{p^n}$. This is injective and satisfies (a) and (b). It is clear that $k^{1/p^n} \subset k^{1/p^{n+1}}$ as extensions of k via the map $y \mapsto y^p$. Then we can take $k' = \bigcup k^{1/p^n}$. Some details omitted. □

046X **Definition 45.5.** Let k be a field. The field extension k'/k of Lemma 45.4 is called the *perfect closure* of k. Notation k^{perf}/k.

Note that if k'/k is any algebraic purely inseparable extension, then k' is a subextension of k^{perf}, i.e., $k^{perf}/k'/k$. Namely, $(k')^{perf}$ is isomorphic to k^{perf} by the uniqueness of Lemma 45.4.

00I4 **Lemma 45.6.** *Let k be a perfect field. Any reduced k algebra is geometrically reduced over k. Let R, S be k-algebras. Assume both R and S are reduced. Then the k-algebra $R \otimes_k S$ is reduced.*

Proof. The first statement follows from Lemma 44.3. For the second statement use the first statement and Lemma 43.5. □

46. Universal homeomorphisms

0BR5 Let k'/k be an algebraic purely inseparable field extension. Then for any k-algebra R the ring map $R \to k' \otimes_k R$ induces a homeomorphism of spectra. The reason for this is the slightly more general Lemma 46.7 below.

0BR6 **Lemma 46.1.** *Let $\varphi : R \to S$ be a surjective map with locally nilpotent kernel. Then φ induces a homeomorphism of spectra and isomorphisms on residue fields. For any ring map $R \to R'$ the ring map $R' \to R' \otimes_R S$ is surjective with locally nilpotent kernel.*

Proof. By Lemma 17.7 the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is a homeomorphism onto the closed subset $V(\mathrm{Ker}(\varphi))$. Of course $V(\mathrm{Ker}(\varphi)) = \mathrm{Spec}(R)$ because every prime ideal of R contains every nilpotent element of R. This also implies the statement on residue fields. By right exactness of tensor product we see that $\mathrm{Ker}(\varphi)R'$ is the kernel of the surjective map $R' \to R' \otimes_R S$. Hence the final statement by Lemma 32.3. □

0BR7 **Lemma 46.2.** *Let k'/k be a field extension. The following are equivalent* [Alp14, Lemma 3.1.6]

1. *for each $x \in k'$ there exists an $n > 0$ such that $x^n \in k$, and*
2. *$k' = k$ or k and k' have characteristic $p > 0$ and either k'/k is a purely inseparable extension or k and k' are algebraic extensions of $\mathbf{F}_p$.*

Proof. Observe that each of the possibilities listed in (2) satisfies (1). Thus we assume k'/k satisfies (1) and we prove that we are in one of the cases of (2). Discarding the case $k = k'$ we may assume $k' \neq k$. It is clear that k'/k is algebraic. Hence we may assume that k'/k is a nontrivial finite extension. Let $k'/k'_{sep}/k$ be the separable subextension found in Fields, Lemma 14.6. We have to show that $k = k'_{sep}$ or that k is an algebraic over $\mathbf{F}_p$. Thus we may assume that k'/k is a nontrivial finite separable extension and we have to show k is algebraic over $\mathbf{F}_p$.

Pick $x \in k'$, $x \notin k$. Pick $n, m > 0$ such that $x^n \in k$ and $(x+1)^m \in k$. Let $\overline{k}$ be an algebraic closure of k. We can choose embeddings $\sigma, \tau : k' \to \overline{k}$ with $\sigma(x) \neq \tau(x)$. This follows from the discussion in Fields, Section 12 (more precisely, after replacing k' by the k-extension generated by x it follows from Fields, Lemma 12.8). Then we see that $\sigma(x) = \zeta\tau(x)$ for some nth root of unity ζ in $\overline{k}$. Similarly, we see that $\sigma(x+1) = \zeta'\tau(x+1)$ for some mth root of unity $\zeta' \in \overline{k}$. Since $\sigma(x+1) \neq \tau(x+1)$ we see $\zeta' \neq 1$. Then

$$\zeta'(\tau(x) + 1) = \zeta'\tau(x+1) = \sigma(x+1) = \sigma(x) + 1 = \zeta\tau(x) + 1$$

implies that

$$\tau(x)(\zeta' - \zeta) = 1 - \zeta'$$

hence $\zeta' \neq \zeta$ and

$$\tau(x) = (1 - \zeta')/(\zeta' - \zeta)$$

Hence every element of k' which is not in k is algebraic over the prime subfield. Since k' is generated over the prime subfield by the elements of k' which are not in k, we conclude that k' (and hence k) is algebraic over the prime subfield.

Finally, if the characteristic of k is 0, the above leads to a contradiction as follows (we encourage the reader to find their own proof). For every rational number y we similarly get a root of unity ζ_y such that $\sigma(x+y) = \zeta_y \tau(x+y)$. Then we find

$$\zeta\tau(x) + y = \zeta_y(\tau(x) + y)$$

and by our formula for $\tau(x)$ above we conclude $\zeta_y \in \mathbf{Q}(\zeta, \zeta')$. Since the number field $\mathbf{Q}(\zeta, \zeta')$ contains only a finite number of roots of unity we find two distinct rational numbers y, y' with $\zeta_y = \zeta_{y'}$. Then we conclude that

$$y - y' = \sigma(x+y) - \sigma(x+y') = \zeta_y(\tau(x+y)) - \zeta_{y'}\tau(x+y') = \zeta_y(y-y')$$

which implies $\zeta_y = 1$ a contradiction. □

0BR8 **Lemma 46.3.** *Let $\varphi : R \to S$ be a ring map. If*

(1) *for any $x \in S$ there exists $n > 0$ such that x^n is in the image of φ, and*
(2) $\text{Ker}(\varphi)$ *is locally nilpotent,*

then φ induces a homeomorphism on spectra and induces residue field extensions satisfying the equivalent conditions of Lemma 46.2.

Proof. Assume (1) and (2). Let $\mathfrak{q}, \mathfrak{q}'$ be primes of S lying over the same prime ideal $\mathfrak{p}$ of R. Suppose $x \in S$ with $x \in \mathfrak{q}$, $x \notin \mathfrak{q}'$. Then $x^n \in \mathfrak{q}$ and $x^n \notin \mathfrak{q}'$ for all $n > 0$. If $x^n = \varphi(y)$ with $y \in R$ for some $n > 0$ then

$$x^n \in \mathfrak{q} \Rightarrow y \in \mathfrak{p} \Rightarrow x^n \in \mathfrak{q}'$$

which is a contradiction. Hence there does not exist an x as above and we conclude that $\mathfrak{q} = \mathfrak{q}'$, i.e., the map on spectra is injective. By assumption (2) the kernel $I = \text{Ker}(\varphi)$ is contained in every prime, hence $\text{Spec}(R) = \text{Spec}(R/I)$ as topological spaces. As the induced map $R/I \to S$ is integral by assumption (1) Lemma 36.17 shows that $\text{Spec}(S) \to \text{Spec}(R/I)$ is surjective. Combining the above we see that $\text{Spec}(S) \to \text{Spec}(R)$ is bijective. If $x \in S$ is arbitrary, and we pick $y \in R$ such that $\varphi(y) = x^n$ for some $n > 0$, then we see that the open $D(x) \subset \text{Spec}(S)$ corresponds to the open $D(y) \subset \text{Spec}(R)$ via the bijection above. Hence we see that the map $\text{Spec}(S) \to \text{Spec}(R)$ is a homeomorphism.

To see the statement on residue fields, let $\mathfrak{q} \subset S$ be a prime lying over a prime ideal $\mathfrak{p} \subset R$. Let $x \in \kappa(\mathfrak{q})$. If we think of $\kappa(\mathfrak{q})$ as the residue field of the local ring $S_\mathfrak{q}$, then we see that x is the image of some $y/z \in S_\mathfrak{q}$ with $y \in S$, $z \in S$, $z \notin \mathfrak{q}$. Choose $n, m > 0$ such that y^n, z^m are in the image of φ. Then x^{nm} is the residue of $(y/z)^{nm} = (y^n)^m/(z^m)^n$ which is in the image of $R_\mathfrak{p} \to S_\mathfrak{q}$. Hence x^{nm} is in the image of $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{q})$. □

0EUH **Lemma 46.4.** *Let $\varphi : R \to S$ be a ring map. Assume*

(a) *S is generated as an R-algebra by elements x such that $x^2, x^3 \in \varphi(R)$, and*
(b) $\text{Ker}(\varphi)$ *is locally nilpotent,*

Then φ induces isomorphisms on residue fields and a homeomorphism of spectra. For any ring map $R \to R'$ the ring map $R' \to R' \otimes_R S$ also satisfies (a) and (b).

Proof. Assume (a) and (b). The map on spectra is closed as S is integral over R, see Lemmas 41.6 and 36.22. The image is dense by Lemma 30.6. Thus $\text{Spec}(S) \to \text{Spec}(R)$ is surjective. If $\mathfrak{q} \subset S$ is a prime lying over $\mathfrak{p} \subset R$ then the field extension $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ is generated by elements $\alpha \in \kappa(\mathfrak{q})$ whose square and cube are in $\kappa(\mathfrak{p})$. Thus clearly $\alpha \in \kappa(\mathfrak{p})$ and we find that $\kappa(\mathfrak{q}) = \kappa(\mathfrak{p})$. If $\mathfrak{q}, \mathfrak{q}'$ were two distinct primes lying over $\mathfrak{p}$, then at least one of the generators x of S as in (a) would have distinct images in $\kappa(\mathfrak{q}) = \kappa(\mathfrak{p})$ and $\kappa(\mathfrak{q}') = \kappa(\mathfrak{p})$. This would contradict the fact that both x^2 and x^3 do have the same image. This proves that $\text{Spec}(S) \to \text{Spec}(R)$ is injective hence a homeomorphism (by what was already shown).

Since φ induces a homeomorphism on spectra, it is in particular surjective on spectra which is a property preserved under any base change, see Lemma 30.3. Therefore for any $R \to R'$ the kernel of the ring map $R' \to R' \otimes_R S$ consists of nilpotent elements, see Lemma 30.6, in other words (b) holds for $R' \to R' \otimes_R S$. It is clear that (a) is preserved under base change. □

0545 **Lemma 46.5.** *Let p be a prime number. Let $n, m > 0$ be two integers. There exists an integer a such that $(x+y)^{p^a}, p^a(x+y) \in \mathbf{Z}[x^{p^n}, p^n x, y^{p^m}, p^m y]$.*

Proof. This is clear for $p^a(x+y)$ as soon as $a \geq n, m$. In fact, pick $a \gg n, m$. Write

$$(x+y)^{p^a} = \sum\nolimits_{i,j \geq 0, i+j=p^a} \binom{p^a}{i, j} x^i y^j$$

For every $i, j \geq 0$ with $i + j = p^a$ write $i = qp^n + r$ with $r \in \{0, \ldots, p^n - 1\}$ and $j = q'p^m + r'$ with $r' \in \{0, \ldots, p^m - 1\}$. The condition $(x+y)^{p^a} \in \mathbf{Z}[x^{p^n}, p^n x, y^{p^m}, p^m y]$ holds if

$$p^{nr+mr'} \text{ divides } \binom{p^a}{i, j}$$

If $r = r' = 0$ then the divisibility holds. If $r \neq 0$, then we write

$$\binom{p^a}{i, j} = \frac{p^a}{i} \binom{p^a - 1}{i - 1, j}$$

Since $r \neq 0$ the rational number p^a/i has p-adic valuation at least $a-(n-1)$ (because i is not divisible by p^n). Thus $\binom{p^a}{i,j}$ is divisible by p^{a-n+1} in this case. Similarly, we see that if $r' \neq 0$, then $\binom{p^a}{i,j}$ is divisible by p^{a-m+1}. Picking $a = np^n + mp^m + n + m$ will work. □

0BR9 **Lemma 46.6.** *Let k'/k be a field extension. Let p be a prime number. The following are equivalent*

1. *k' is generated as a field extension of k by elements x such that there exists an $n > 0$ with $x^{p^n} \in k$ and $p^n x \in k$, and*
2. *$k = k'$ or the characteristic of k and k' is p and k'/k is purely inseparable.*

Proof. Let $x \in k'$. If there exists an $n > 0$ with $x^{p^n} \in k$ and $p^n x \in k$ and if the characteristic is not p, then $x \in k$. If the characteristic is p, then we find $x^{p^n} \in k$ and hence x is purely inseparable over k. □

0BRA **Lemma 46.7.** *Let $\varphi : R \to S$ be a ring map. Let p be a prime number. Assume*

(a) *S is generated as an R-algebra by elements x such that there exists an $n > 0$ with $x^{p^n} \in \varphi(R)$ and $p^n x \in \varphi(R)$, and*
(b) $\text{Ker}(\varphi)$ *is locally nilpotent,*

Then φ induces a homeomorphism of spectra and induces residue field extensions satisfying the equivalent conditions of Lemma 46.6. For any ring map $R \to R'$ the ring map $R' \to R' \otimes_R S$ also satisfies (a) and (b).

Proof. Assume (a) and (b). Note that (b) is equivalent to condition (2) of Lemma 46.3. Let $T \subset S$ be the set of elements $x \in S$ such that there exists an integer $n > 0$ such that $x^{p^n}, p^n x \in \varphi(R)$. We claim that $T = S$. This will prove that condition (1) of Lemma 46.3 holds and hence φ induces a homeomorphism on spectra. By assumption (a) it suffices to show that $T \subset S$ is an R-sub algebra. If $x \in T$ and $y \in R$, then it is clear that $yx \in T$. Suppose $x, y \in T$ and $n, m > 0$ such that $x^{p^n}, y^{p^m}, p^n x, p^m y \in \varphi(R)$. Then $(xy)^{p^{n+m}}, p^{n+m}xy \in \varphi(R)$ hence $xy \in T$. We have $x + y \in T$ by Lemma 46.5 and the claim is proved.

Since φ induces a homeomorphism on spectra, it is in particular surjective on spectra which is a property preserved under any base change, see Lemma 30.3. Therefore for any $R \to R'$ the kernel of the ring map $R' \to R' \otimes_R S$ consists of nilpotent elements, see Lemma 30.6, in other words (b) holds for $R' \to R' \otimes_R S$. It is clear that (a) is preserved under base change. Finally, the condition on residue fields follows from (a) as generators for S as an R-algebra map to generators for the residue field extensions. $\square$

0BRB **Lemma 46.8.** *Let $\varphi : R \to S$ be a ring map. Assume*

1. *φ induces an injective map of spectra,*
2. *φ induces purely inseparable residue field extensions.*

Then for any ring map $R \to R'$ properties (1) and (2) are true for $R' \to R' \otimes_R S$.

Proof. Set $S' = R' \otimes_R S$ so that we have a commutative diagram of continuous maps of spectra of rings

$$\begin{tikzcd} \mathrm{Spec}(S') \arrow[r] \arrow[d] & \mathrm{Spec}(S) \arrow[d] \\ \mathrm{Spec}(R') \arrow[r] & \mathrm{Spec}(R) \end{tikzcd}$$

Let $\mathfrak{p}' \subset R'$ be a prime ideal lying over $\mathfrak{p} \subset R$. If there is no prime ideal of S lying over $\mathfrak{p}$, then there is no prime ideal of S' lying over $\mathfrak{p}'$. Otherwise, by Remark 17.8 there is a unique prime ideal $\mathfrak{r}$ of $F = S \otimes_R \kappa(\mathfrak{p})$ whose residue field is purely inseparable over $\kappa(\mathfrak{p})$. Consider the ring maps

$$\kappa(\mathfrak{p}) \to F \to \kappa(\mathfrak{r})$$

By Lemma 25.1 the ideal $\mathfrak{r} \subset F$ is locally nilpotent, hence we may apply Lemma 46.1 to the ring map $F \to \kappa(\mathfrak{r})$. We may apply Lemma 46.7 to the ring map $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{r})$. Hence the composition and the second arrow in the maps

$$\kappa(\mathfrak{p}') \to \kappa(\mathfrak{p}') \otimes_{\kappa(\mathfrak{p})} F \to \kappa(\mathfrak{p}') \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{r})$$

induces bijections on spectra and purely inseparable residue field extensions. This implies the same thing for the first map. Since

$$\kappa(\mathfrak{p}') \otimes_{\kappa(\mathfrak{p})} F = \kappa(\mathfrak{p}') \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}) \otimes_R S = \kappa(\mathfrak{p}') \otimes_R S = \kappa(\mathfrak{p}') \otimes_{R'} R' \otimes_R S$$

we conclude by the discussion in Remark 17.8. $\square$

0BRC **Lemma 46.9.** *Let $\varphi : R \to S$ be a ring map. Assume*

(1) *φ is integral,*
(2) *φ induces an injective map of spectra,*
(3) *φ induces purely inseparable residue field extensions.*

Then φ induces a homeomorphism from $\mathrm{Spec}(S)$ *onto a closed subset of* $\mathrm{Spec}(R)$ *and for any ring map $R \to R'$ properties (1), (2), (3) are true for $R' \to R' \otimes_R S$.*

Proof. The map on spectra is closed by Lemmas 41.6 and 36.22. The properties are preserved under base change by Lemmas 46.8 and 36.13. □

0BRD **Lemma 46.10.** *Let $\varphi : R \to S$ be a ring map. Assume*

(1) *φ is integral,*
(2) *φ induces an bijective map of spectra,*
(3) *φ induces purely inseparable residue field extensions.*

Then φ induces a homeomorphism on spectra and for any ring map $R \to R'$ properties (1), (2), (3) are true for $R' \to R' \otimes_R S$.

Proof. Follows from Lemmas 46.9 and 30.3. □

09EF **Lemma 46.11.** *Let $\varphi : R \to S$ be a ring map such that*

(1) *the kernel of φ is locally nilpotent, and*
(2) *S is generated as an R-algebra by elements x such that there exist $n > 0$ and a polynomial $P(T) \in R[T]$ whose image in $S[T]$ is $(T - x)^n$.*

Then $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ *is a homeomorphism and $R \to S$ induces purely inseparable extensions of residue fields. Moreover, conditions (1) and (2) remain true on arbitrary base change.*

Proof. We may replace R by $R/\mathrm{Ker}(\varphi)$, see Lemma 46.1. Assumption (2) implies S is generated over R by elements which are integral over R. Hence $R \subset S$ is integral (Lemma 36.7). In particular $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective and closed (Lemmas 36.17, 41.6, and 36.22).

Let $x \in S$ be one of the generators in (2), i.e., there exists an $n > 0$ be such that $(T - x)^n \in R[T]$. Let $\mathfrak{p} \subset R$ be a prime. The $\kappa(\mathfrak{p}) \otimes_R S$ ring is nonzero by the above and Lemma 17.9. If the characteristic of $\kappa(\mathfrak{p})$ is zero then we see that $nx \in R$ implies $1 \otimes x$ is in the image of $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{p}) \otimes_R S$. Hence $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{p}) \otimes_R S$ is an isomorphism. If the characteristic of $\kappa(\mathfrak{p})$ is $p > 0$, then write $n = p^k m$ with m prime to p. In $\kappa(\mathfrak{p}) \otimes_R S[T]$ we have

$$(T - 1 \otimes x)^n = ((T - 1 \otimes x)^{p^k})^m = (T^{p^k} - 1 \otimes x^{p^k})^m$$

and we see that $mx^{p^k} \in R$. This implies that $1 \otimes x^{p^k}$ is in the image of $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{p}) \otimes_R S$. Hence Lemma 46.7 applies to $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{p}) \otimes_R S$. In both cases we conclude that $\kappa(\mathfrak{p}) \otimes_R S$ has a unique prime ideal with residue field purely inseparable over $\kappa(\mathfrak{p})$. By Remark 17.8 we conclude that φ is bijective on spectra.

The statement on base change is immediate. □

47. Geometrically irreducible algebras

00I2 An algebra S over a field k is geometrically irreducible if the algebra $S \otimes_k k'$ has a unique minimal prime for every field extension k'/k. In this section we develop a bit of theory relevant to this notion.

00I6 **Lemma 47.1.** *Let $R \to S$ be a ring map. Assume*

(a) $\operatorname{Spec}(R)$ *is irreducible,*
(b) $R \to S$ *is flat,*
(c) $R \to S$ *is of finite presentation,*
(d) *the fibre rings* $S \otimes_R \kappa(\mathfrak{p})$ *have irreducible spectra for a dense collection of primes* $\mathfrak{p}$ *of* R.

Then $\operatorname{Spec}(S)$ *is irreducible. This is true more generally with (b) + (c) replaced by "the map* $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ *is open".*

Proof. The assumptions (b) and (c) imply that the map on spectra is open, see Proposition 41.8. Hence the lemma follows from Topology, Lemma 8.14. □

00I7 **Lemma 47.2.** *Let k be a separably closed field. Let R, S be k-algebras. If R, S have a unique minimal prime, so does $R \otimes_k S$.*

Proof. Let $k \subset \overline{k}$ be a perfect closure, see Definition 45.5. By assumption $\overline{k}$ is algebraically closed. The ring maps $R \to R \otimes_k \overline{k}$ and $S \to S \otimes_k \overline{k}$ and $R \otimes_k S \to (R \otimes_k S) \otimes_k \overline{k} = (R \otimes_k \overline{k}) \otimes_{\overline{k}} (S \otimes_k \overline{k})$ satisfy the assumptions of Lemma 46.7. Hence we may assume k is algebraically closed.

We may replace R and S by their reductions. Hence we may assume that R and S are domains. By Lemma 45.6 we see that $R \otimes_k S$ is reduced. Hence its spectrum is reducible if and only if it contains a nonzero zerodivisor. By Lemma 43.4 we reduce to the case where R and S are domains of finite type over k algebraically closed.

Note that the ring map $R \to R \otimes_k S$ is of finite presentation and flat. Moreover, for every maximal ideal $\mathfrak{m}$ of R we have $(R \otimes_k S) \otimes_R R/\mathfrak{m} \cong S$ because $k \cong R/\mathfrak{m}$ by the Hilbert Nullstellensatz Theorem 34.1. Moreover, the set of maximal ideals is dense in the spectrum of R since $\operatorname{Spec}(R)$ is Jacobson, see Lemma 35.2. Hence we see that Lemma 47.1 applies to the ring map $R \to R \otimes_k S$ and we conclude that the spectrum of $R \otimes_k S$ is irreducible as desired. □

037K **Lemma 47.3.** *Let k be a field. Let R be a k-algebra. The following are equivalent*

(1) *for every field extension k'/k the spectrum of $R \otimes_k k'$ is irreducible,*
(2) *for every finite separable field extension k'/k the spectrum of $R \otimes_k k'$ is irreducible,*
(3) *the spectrum of $R \otimes_k \overline{k}$ is irreducible where $\overline{k}$ is the separable algebraic closure of k, and*
(4) *the spectrum of $R \otimes_k \overline{k}$ is irreducible where $\overline{k}$ is the algebraic closure of k.*

Proof. It is clear that (1) implies (2).

Assume (2) and let $\overline{k}$ is the separable algebraic closure of k. Suppose $\mathfrak{q}_i \subset R \otimes_k \overline{k}$, $i = 1, 2$ are two minimal prime ideals. For every finite subextension $\overline{k}/k'/k$ the extension k'/k is separable and the ring map $R \otimes_k k' \to R \otimes_k \overline{k}$ is flat. Hence $\mathfrak{p}_i = (R \otimes_k k') \cap \mathfrak{q}_i$ are minimal prime ideals (as we have going down for flat ring maps by Lemma 39.19). Thus we see that $\mathfrak{p}_1 = \mathfrak{p}_2$ by assumption (2). Since $\overline{k} = \bigcup k'$ we conclude $\mathfrak{q}_1 = \mathfrak{q}_2$. Hence $\operatorname{Spec}(R \otimes_k \overline{k})$ is irreducible.

Assume (3) and let $\overline{k}$ be the algebraic closure of k. Let $\overline{k}/\overline{k}'/k$ be the corresponding separable algebraic closure of k. Then $\overline{k}/\overline{k}'$ is purely inseparable (in positive characteristic) or trivial. Hence $R \otimes_k \overline{k}' \to R \otimes_k \overline{k}$ induces a homeomorphism on spectra, for example by Lemma 46.7. Thus we have (4).

Assume (4). Let k'/k be an arbitrary field extension and let $\overline{k}$ be the algebraic closure of k. We may choose a field F such that both k' and $\overline{k}$ are isomorphic to subfields of F. Then

$$R \otimes_k F = (R \otimes_k \overline{k}) \otimes_{\overline{k}} F$$

and hence we see from Lemma 47.2 that $R \otimes_k F$ has a unique minimal prime. Finally, the ring map $R \otimes_k k' \to R \otimes_k F$ is flat and injective and hence any minimal prime of $R \otimes_k k'$ is the image of a minimal prime of $R \otimes_k F$ (by Lemma 30.5 and going down). We conclude that there is only one such minimal prime and the proof is complete. □

037L **Definition 47.4.** Let k be a field. Let S be a k-algebra. We say S is *geometrically irreducible over k* if for every field extension k'/k the spectrum of $S \otimes_k k'$ is irreducible[5].

By Lemma 47.3 it suffices to check this for finite separable field extensions k'/k or for k' equal to the separable algebraic closure of k.

037M **Lemma 47.5.** *Let k be a field. Let R be a k-algebra. If k is separably algebraically closed then R is geometrically irreducible over k if and only if the spectrum of R is irreducible.*

Proof. Immediate from the remark following Definition 47.4. □

037N **Lemma 47.6.** *Let k be a field. Let S be a k-algebra.*

1. *If S is geometrically irreducible over k so is every k-subalgebra.*
2. *If all finitely generated k-subalgebras of S are geometrically irreducible, then S is geometrically irreducible.*
3. *A directed colimit of geometrically irreducible k-algebras is geometrically irreducible.*

Proof. Let $S' \subset S$ be a subalgebra. Then for any extension k'/k the ring map $S' \otimes_k k' \to S \otimes_k k'$ is injective also. Hence (1) follows from Lemma 30.5 (and the fact that the image of an irreducible space under a continuous map is irreducible). The second and third property follow from the fact that tensor product commutes with colimits. □

037O **Lemma 47.7.** *Let k be a field. Let S be a geometrically irreducible k-algebra. Let R be any k-algebra. The map*

$$\mathrm{Spec}(R \otimes_k S) \longrightarrow \mathrm{Spec}(R)$$

induces a bijection on irreducible components.

Proof. Recall that irreducible components correspond to minimal primes (Lemma 26.1). As $R \to R \otimes_k S$ is flat we see by going down (Lemma 39.19) that any minimal prime of $R \otimes_k S$ lies over a minimal prime of R. Conversely, if $\mathfrak{p} \subset R$ is a (minimal) prime then

$$R \otimes_k S/\mathfrak{p}(R \otimes_k S) = (R/\mathfrak{p}) \otimes_k S \subset \kappa(\mathfrak{p}) \otimes_k S$$

by flatness of $R \to R \otimes_k S$. The ring $\kappa(\mathfrak{p}) \otimes_k S$ has irreducible spectrum by assumption. It follows that $R \otimes_k S/\mathfrak{p}(R \otimes_k S)$ has a single minimal prime (Lemma 30.5). In other words, the inverse image of the irreducible set $V(\mathfrak{p})$ is irreducible. Hence the lemma follows. □

[5]An irreducible space is nonempty.

Let us make some remarks on the notion of geometrically irreducible field extensions.

037P **Lemma 47.8.** *Let K/k be a field extension. If k is algebraically closed in K, then K is geometrically irreducible over k.*

Proof. Assume k is algebraically closed in K. By Definition 47.4 and Lemma 47.3 it suffices to show that the spectrum of $K \otimes_k k'$ is irreducible for every finite separable extension k'/k. Say k' is generated by $\alpha \in k'$ over k, see Fields, Lemma 19.1. Let $P = T^d + a_1 T^{d-1} + \ldots + a_d \in k[T]$ be the minimal polynomial of α. Then $K \otimes_k k' \cong K[T]/(P)$. The only way the spectrum of $K[T]/(P)$ can be reducible is if P is reducible in $K[T]$. Assume $P = P_1 P_2$ is a nontrivial factorization in $K[T]$ to get a contradiction. By Lemma 38.5 we see that the coefficients of P_1 and P_2 are algebraic over k. Our assumption implies the coefficients of P_1 and P_2 are in k which contradicts the fact that P is irreducible over k. $\square$

0G30 **Lemma 47.9.** *Let K/k be a geometrically irreducible field extension. Let S be a geometrically irreducible K-algebra. Then S is geometrically irreducible over k.*

Proof. By Definition 47.4 and Lemma 47.3 it suffices to show that the spectrum of $S \otimes_k k'$ is irreducible for every finite separable extension k'/k. Since K is geometrically irreducible over k we see that $K' = K \otimes_k k'$ is a finite, separable field extension of K. Hence the spectrum of $S \otimes_k k' = S \otimes_K K'$ is irreducible as S is assumed geometrically irreducible over K. $\square$

0G31 **Lemma 47.10.** *Let K/k be a field extension. The following are equivalent*

1. *K is geometrically irreducible over k, and*
2. *the induced extension $K(t)/k(t)$ of purely transcendental extensions is geometrically irreducible.*

Proof. Assume (1). Denote Ω an algebraic closure of $k(t)$. By Definition 47.4 we find that the spectrum of

$$K \otimes_k \Omega = K \otimes_k k(t) \otimes_{k(t)} \Omega$$

is irreducible. Since $K(t)$ is a localization of $K \otimes_k k(T)$ we conclude that the spectrum of $K(t) \otimes_{k(t)} \Omega$ is irreducible. Thus by Lemma 47.3 we find that $K(t)/k(t)$ is geometrically irreducible.

Assume (2). Let k'/k be a field extension. We have to show that $K \otimes_k k'$ has a unique minimal prime. We know that the spectrum of

$$K(t) \otimes_{k(t)} k'(t)$$

is irreducible, i.e., has a unique minimal prime. Since there is an injective map $K \otimes_k k' \to K(t) \otimes_{k(t)} k'(t)$ (details omitted) we conclude by Lemmas 30.5 and 30.7. $\square$

0G32 **Lemma 47.11.** *Let $K/L/M$ be a tower of fields with L/M geometrically irreducible. Let $x \in K$ be transcendental over L. Then $L(x)/M(x)$ is geometrically irreducible.*

Proof. This follows from Lemma 47.10 because the fields $L(x)$ and $M(x)$ are purely transcendental extensions of L and M. $\square$

0G33 **Lemma 47.12.** *Let K/k be a field extension. The following are equivalent*

(1) *K/k is geometrically irreducible, and*
(2) *every element $\alpha \in K$ separably algebraic over k is in k.*

Proof. Assume (1) and let $\alpha \in K$ be separably algebraic over k. Then $k' = k(\alpha)$ is a finite separable extension of k contained in K. By Lemma 47.6 the extension k'/k is geometrically irreducible. In particular, we see that the spectrum of $k' \otimes_k \overline{k}$ is irreducible (and hence if it is a product of fields, then there is exactly one factor). By Fields, Lemma 13.4 it follows that $\mathrm{Hom}_k(k', \overline{k})$ has one element which in turn implies that $k' = k$ by Fields, Lemma 12.11. Thus (2) holds.

Assume (2). Let $k' \subset K$ be the subfield consisting of elements algebraic over k. By Lemma 47.8 the extension K/k' is geometrically irreducible. By assumption k'/k is a purely inseparable extension. By Lemma 46.7 the extension k'/k is geometrically irreducible. Hence by Lemma 47.9 we see that K/k is geometrically irreducible. $\square$

037Q **Lemma 47.13.** *Let K/k be a field extension. Consider the subextension $K/k'/k$ consisting of elements separably algebraic over k. Then K is geometrically irreducible over k'. If K/k is a finitely generated field extension, then $[k' : k] < \infty$.*

Proof. The first statement is immediate from Lemma 47.12 and the fact that elements separably algebraic over k' are in k' by the transitivity of separable algebraic extensions, see Fields, Lemma 12.12. If K/k is finitely generated, then k' is finite over k by Fields, Lemma 26.11. $\square$

04KP **Lemma 47.14.** *Let K/k be an extension of fields. Let $\overline{k}/k$ be a separable algebraic closure. Then $Gal(\overline{k}/k)$ acts transitively on the primes of $\overline{k} \otimes_k K$.*

Proof. Let $K/k'/k$ be the subextension found in Lemma 47.13. Note that as $k \subset \overline{k}$ is integral all the prime ideals of $\overline{k} \otimes_k K$ and $\overline{k} \otimes_k k'$ are maximal, see Lemma 36.20. By Lemma 47.7 the map

$$\mathrm{Spec}(\overline{k} \otimes_k K) \to \mathrm{Spec}(\overline{k} \otimes_k k')$$

is bijective because (1) all primes are minimal primes, (2) $\overline{k} \otimes_k K = (\overline{k} \otimes_k k') \otimes_{k'} K$, and (3) K is geometrically irreducible over k'. Hence it suffices to prove the lemma for the action of $\mathrm{Gal}(\overline{k}/k)$ on the primes of $\overline{k} \otimes_k k'$.

As every prime of $\overline{k} \otimes_k k'$ is maximal, the residue fields are isomorphic to $\overline{k}$. Hence the prime ideals of $\overline{k} \otimes_k k'$ correspond one to one to elements of $\mathrm{Hom}_k(k', \overline{k})$ with $\sigma \in \mathrm{Hom}_k(k', \overline{k})$ corresponding to the kernel $\mathfrak{p}_\sigma$ of $1 \otimes \sigma : \overline{k} \otimes_k k' \to \overline{k}$. In particular $\mathrm{Gal}(\overline{k}/k)$ acts transitively on this set as desired. $\square$

48. Geometrically connected algebras

05DV

037R **Lemma 48.1.** *Let k be a separably algebraically closed field. Let R, S be k-algebras. If $\mathrm{Spec}(R)$, and $\mathrm{Spec}(S)$ are connected, then so is $\mathrm{Spec}(R \otimes_k S)$.*

Proof. Recall that $\mathrm{Spec}(R)$ is connected if and only if R has no nontrivial idempotents, see Lemma 21.4. Hence, by Lemma 43.4 we may assume R and S are of finite type over k. In this case R and S are Noetherian, and have finitely many minimal primes, see Lemma 31.6. Thus we may argue by induction on $n+m$ where n, resp. m is the number of irreducible components of $\mathrm{Spec}(R)$, resp. $\mathrm{Spec}(S)$. Of course the case where either n or m is zero is trivial. If $n = m = 1$, i.e., $\mathrm{Spec}(R)$ and $\mathrm{Spec}(S)$

both have one irreducible component, then the result holds by Lemma 47.2. Suppose that $n > 1$. Let $\mathfrak{p} \subset R$ be a minimal prime corresponding to the irreducible closed subset $T \subset \text{Spec}(R)$. Let $T' \subset \text{Spec}(R)$ be the union of the other $n-1$ irreducible components. Choose an ideal $I \subset R$ such that $T' = V(I) = \text{Spec}(R/I)$ (Lemma 17.7). By choosing our minimal prime carefully we may in addition arrange it so that T' is connected, see Topology, Lemma 8.17. Then $T \cup T' = \text{Spec}(R)$ and $T \cap T' = V(\mathfrak{p} + I) = \text{Spec}(R/(\mathfrak{p} + I))$ is not empty as $\text{Spec}(R)$ is assumed connected. The inverse image of T in $\text{Spec}(R \otimes_k S)$ is $\text{Spec}(R/\mathfrak{p} \otimes_k S)$, and the inverse of T' in $\text{Spec}(R \otimes_k S)$ is $\text{Spec}(R/I \otimes_k S)$. By induction these are both connected. The inverse image of $T \cap T'$ is $\text{Spec}(R/(\mathfrak{p} + I) \otimes_k S)$ which is nonempty. Hence $\text{Spec}(R \otimes_k S)$ is connected. □

037S **Lemma 48.2.** *Let k be a field. Let R be a k-algebra. The following are equivalent*

(1) *for every field extension k'/k the spectrum of $R \otimes_k k'$ is connected, and*
(2) *for every finite separable field extension k'/k the spectrum of $R \otimes_k k'$ is connected.*

Proof. For any extension of fields k'/k the connectivity of the spectrum of $R \otimes_k k'$ is equivalent to $R \otimes_k k'$ having no nontrivial idempotents, see Lemma 21.4. Assume (2). Let $k \subset \overline{k}$ be a separable algebraic closure of k. Using Lemma 43.4 we see that (2) is equivalent to $R \otimes_k \overline{k}$ having no nontrivial idempotents. For any field extension k'/k, there exists a field extension $\overline{k}'/\overline{k}$ with $k' \subset \overline{k}'$. By Lemma 48.1 we see that $R \otimes_k \overline{k}'$ has no nontrivial idempotents. If $R \otimes_k k'$ has a nontrivial idempotent, then also $R \otimes_k \overline{k}'$, contradiction. □

037T **Definition 48.3.** Let k be a field. Let S be a k-algebra. We say S is *geometrically connected over k* if for every field extension k'/k the spectrum of $S \otimes_k k'$ is connected.

By Lemma 48.2 it suffices to check this for finite separable field extensions k'/k.

037U **Lemma 48.4.** *Let k be a field. Let R be a k-algebra. If k is separably algebraically closed then R is geometrically connected over k if and only if the spectrum of R is connected.*

Proof. Immediate from the remark following Definition 48.3. □

037V **Lemma 48.5.** *Let k be a field. Let S be a k-algebra.*

(1) *If S is geometrically connected over k so is every k-subalgebra.*
(2) *If all finitely generated k-subalgebras of S are geometrically connected, then S is geometrically connected.*
(3) *A directed colimit of geometrically connected k-algebras is geometrically connected.*

Proof. This follows from the characterization of connectedness in terms of the nonexistence of nontrivial idempotents. The second and third property follow from the fact that tensor product commutes with colimits. □

The following lemma will be superseded by the more general Varieties, Lemma 7.4.

037W **Lemma 48.6.** *Let k be a field. Let S be a geometrically connected k-algebra. Let R be any k-algebra. The map*

$$R \longrightarrow R \otimes_k S$$

induces a bijection on idempotents, and the map

$$\mathrm{Spec}(R \otimes_k S) \longrightarrow \mathrm{Spec}(R)$$

induces a bijection on connected components.

Proof. The second assertion follows from the first combined with Lemma 22.2. By Lemmas 48.5 and 43.4 we may assume that R and S are of finite type over k. Then we see that also $R \otimes_k S$ is of finite type over k. Note that in this case all the rings are Noetherian and hence their spectra have finitely many connected components (since they have finitely many irreducible components, see Lemma 31.6). In particular, all connected components in question are open! Hence via Lemma 24.3 we see that the first statement of the lemma in this case is equivalent to the second. Let's prove this. As the algebra S is geometrically connected and nonzero we see that all fibres of $X = \mathrm{Spec}(R \otimes_k S) \to \mathrm{Spec}(R) = Y$ are connected and nonempty. Also, as $R \to R \otimes_k S$ is flat of finite presentation the map $X \to Y$ is open (Proposition 41.8). Topology, Lemma 7.6 shows that $X \to Y$ induces bijection on connected components. $\square$

49. Geometrically integral algebras

05DW Here is the definition.

05DX **Definition 49.1.** Let k be a field. Let S be a k-algebra. We say S is *geometrically integral over k* if for every field extension k'/k the ring of $S \otimes_k k'$ is a domain.

Any question about geometrically integral algebras can be translated in a question about geometrically reduced and irreducible algebras.

05DY **Lemma 49.2.** *Let k be a field. Let S be a k-algebra. In this case S is geometrically integral over k if and only if S is geometrically irreducible as well as geometrically reduced over k.*

Proof. Omitted. $\square$

0FWF **Lemma 49.3.** *Let k be a field. Let S be a k-algebra. The following are equivalent*

1. *S is geometrically integral over k,*
2. *for every finite extension k'/k of fields the ring $S \otimes_k k'$ is a domain,*
3. *$S \otimes_k \overline{k}$ is a domain where $\overline{k}$ is the algebraic closure of k.*

Proof. Follows from Lemmas 49.2, 44.3, and 47.3. $\square$

09P9 **Lemma 49.4.** *Let k be a field. Let S be a geometrically integral k-algebra. Let R be a k-algebra and an integral domain. Then $R \otimes_k S$ is an integral domain.*

Proof. By Lemma 43.5 the ring $R \otimes_k S$ is reduced and by Lemma 47.7 the ring $R \otimes_k S$ is irreducible (the spectrum has just one irreducible component), so $R \otimes_k S$ is an integral domain. $\square$

50. Valuation rings

00I8 Here are some definitions.

00I9 **Definition 50.1.** Valuation rings.

1. Let K be a field. Let A, B be local rings contained in K. We say that B *dominates* A if $A \subset B$ and $\mathfrak{m}_A = A \cap \mathfrak{m}_B$.

(2) Let A be a ring. We say A is a *valuation ring* if A is a local domain and if A is maximal for the relation of domination among local rings contained in the fraction field of A.
(3) Let A be a valuation ring with fraction field K. If $R \subset K$ is a subring of K, then we say A is *centered* on R if $R \subset A$.

With this definition a field is a valuation ring.

00IA **Lemma 50.2.** *Let K be a field. Let $A \subset K$ be a local subring. Then there exists a valuation ring with fraction field K dominating A.*

Proof. We consider the collection of local subrings of K as a partially ordered set using the relation of domination. Suppose that $\{A_i\}_{i \in I}$ is a totally ordered collection of local subrings of K. Then $B = \bigcup A_i$ is a local subring which dominates all of the A_i. Hence by Zorn's Lemma, it suffices to show that if $A \subset K$ is a local ring whose fraction field is not K, then there exists a local ring $B \subset K$, $B \neq A$ dominating A.

Pick $t \in K$ which is not in the fraction field of A. If t is transcendental over A, then $A[t] \subset K$ and hence $A[t]_{(t,\mathfrak{m})} \subset K$ is a local ring distinct from A dominating A. Suppose t is algebraic over A. Then for some $a \in A$ the element at is integral over A. In this case the subring $A' \subset K$ generated by A and ta is finite over A. By Lemma 36.17 there exists a prime ideal $\mathfrak{m}' \subset A'$ lying over $\mathfrak{m}$. Then $A'_{\mathfrak{m}'}$ dominates A. If $A = A'_{\mathfrak{m}'}$, then t is in the fraction field of A which we assumed not to be the case. Thus $A \neq A'_{\mathfrak{m}'}$ as desired. $\square$

00IB **Lemma 50.3.** *Let A be a valuation ring with maximal ideal $\mathfrak{m}$ and fraction field K. Let $x \in K$. Then either $x \in A$ or $x^{-1} \in A$ or both.*

Proof. Assume that x is not in A. Let A' denote the subring of K generated by A and x. Since A is a valuation ring we see that there is no prime of A' lying over $\mathfrak{m}$. Since $\mathfrak{m}$ is maximal we see that $V(\mathfrak{m}A') = \emptyset$. Then $\mathfrak{m}A' = A'$ by Lemma 17.2. Hence we can write $1 = \sum_{i=0}^{d} t_i x^i$ with $t_i \in \mathfrak{m}$. This implies that $(1-t_0)(x^{-1})^d - \sum t_i (x^{-1})^{d-i} = 0$. In particular we see that x^{-1} is integral over A. Thus the subring A'' of K generated by A and x^{-1} is finite over A and we see there exists a prime ideal $\mathfrak{m}'' \subset A''$ lying over $\mathfrak{m}$ by Lemma 36.17. Since A is a valuation ring we conclude that $A = (A'')_{\mathfrak{m}''}$ and hence $x^{-1} \in A$. $\square$

052K **Lemma 50.4.** *Let $A \subset K$ be a subring of a field K such that for all $x \in K$ either $x \in A$ or $x^{-1} \in A$ or both. Then A is a valuation ring with fraction field K.*

Proof. If A is not K, then A is not a field and there is a nonzero maximal ideal $\mathfrak{m}$. If $\mathfrak{m}'$ is a second maximal ideal, then choose $x, y \in A$ with $x \in \mathfrak{m}$, $y \notin \mathfrak{m}$, $x \notin \mathfrak{m}'$, and $y \in \mathfrak{m}'$ (see Lemma 15.2). Then neither $x/y \in A$ nor $y/x \in A$ contradicting the assumption of the lemma. Thus we see that A is a local ring. Suppose that A' is a local ring contained in K which dominates A. Let $x \in A'$. We have to show that $x \in A$. If not, then $x^{-1} \in A$, and of course $x^{-1} \in \mathfrak{m}_A$. But then $x^{-1} \in \mathfrak{m}_{A'}$ which contradicts $x \in A'$. $\square$

0AS4 **Lemma 50.5.** *Let I be a directed set. Let (A_i, φ_{ij}) be a system of valuation rings over I. Then $A = \operatorname{colim} A_i$ is a valuation ring.*

Proof. It is clear that A is a domain. Let $a, b \in A$. Lemma 50.4 tells us we have to show that either $a|b$ or $b|a$ in A. Choose i so large that there exist $a_i, b_i \in A_i$ mapping to a, b. Then Lemma 50.3 applied to a_i, b_i in A_i implies the result for a, b in A. $\square$

052L **Lemma 50.6.** *Let L/K be an extension of fields. If $B \subset L$ is a valuation ring, then $A = K \cap B$ is a valuation ring.*

Proof. We can replace L by the fraction field F of B and K by $K \cap F$. Then the lemma follows from a combination of Lemmas 50.3 and 50.4. $\square$

0AAV **Lemma 50.7.** *Let L/K be an algebraic extension of fields. If $B \subset L$ is a valuation ring with fraction field L and not a field, then $A = K \cap B$ is a valuation ring and not a field.*

Proof. By Lemma 50.6 the ring A is a valuation ring. If A is a field, then $A = K$. Then $A = K \subset B$ is an integral extension, hence there are no proper inclusions among the primes of B (Lemma 36.20). This contradicts the assumption that B is a local domain and not a field. $\square$

088Y **Lemma 50.8.** *Let A be a valuation ring. For any prime ideal $\mathfrak{p} \subset A$ the quotient $A/\mathfrak{p}$ is a valuation ring. The same is true for the localization $A_\mathfrak{p}$ and in fact any localization of A.*

Proof. Use the characterization of valuation rings given in Lemma 50.4. $\square$

088Z **Lemma 50.9.** *Let A' be a valuation ring with residue field K. Let A be a valuation ring with fraction field K. Then $C = \{\lambda \in A' \mid \lambda \bmod \mathfrak{m}_{A'} \in A\}$ is a valuation ring.*

Proof. Note that $\mathfrak{m}_{A'} \subset C$ and $C/\mathfrak{m}_{A'} = A$. In particular, the fraction field of C is equal to the fraction field of A'. We will use the criterion of Lemma 50.4 to prove the lemma. Let x be an element of the fraction field of C. By the lemma we may assume $x \in A'$. If $x \in \mathfrak{m}_{A'}$, then we see $x \in C$. If not, then x is a unit of A' and we also have $x^{-1} \in A'$. Hence either x or x^{-1} maps to an element of A by the lemma again. $\square$

00IC **Lemma 50.10.** *Let A be a valuation ring. Then A is a normal domain.*

Proof. Suppose x is in the field of fractions of A and integral over A, say $x^{d+1} + \sum_{i \leq d} a_i x^i = 0$. By Lemma 50.4 either $x \in A$ (and we're done) or $x^{-1} \in A$. In the second case we see that $x = -\sum a_i x^{i-d} \in A$ as well. $\square$

090P **Lemma 50.11.** *Let A be a normal domain with fraction field K.*

1. *For every $x \in K$, $x \notin A$ there exists a valuation ring $A \subset V \subset K$ with fraction field K such that $x \notin V$.*
2. *If A is local, we can moreover choose V which dominates A.*

In other words, A is the intersection of all valuation rings in K containing A and if A is local, then A is the intersection of all valuation rings in K dominating A.

Proof. Suppose $x \in K$, $x \notin A$. Consider $B = A[x^{-1}]$. Then $x \notin B$. Namely, if $x = a_0 + a_1 x^{-1} + \ldots + a_d x^{-d}$ then $x^{d+1} - a_0 x^d - \ldots - a_d = 0$ and x is integral over A in contradiction with the fact that A is normal. Thus x^{-1} is not a unit in B. Thus $V(x^{-1}) \subset \mathrm{Spec}(B)$ is not empty (Lemma 17.2), and we can choose a prime

$\mathfrak{p} \subset B$ with $x^{-1} \in \mathfrak{p}$. Choose a valuation ring $V \subset K$ dominating $B_{\mathfrak{p}}$ (Lemma 50.2). Then $x \notin V$ as $x^{-1} \in \mathfrak{m}_V$.

If A is local, then we claim that $x^{-1}B + \mathfrak{m}_A B \neq B$. Namely, if $1 = (a_0 + a_1 x^{-1} + \ldots + a_d x^{-d})x^{-1} + a'_0 + \ldots + a'_d x^{-d}$ with $a_i \in A$ and $a'_i \in \mathfrak{m}_A$, then we'd get

$$(1 - a'_0)x^{d+1} - (a_0 + a'_1)x^d - \ldots - a_d = 0$$

Since $a'_0 \in \mathfrak{m}_A$ we see that $1 - a'_0$ is a unit in A and we conclude that x would be integral over A, a contradiction as before. Then choose the prime $\mathfrak{p} \supset x^{-1}B + \mathfrak{m}_A B$ we find V dominating A. □

An *totally ordered abelian group* is a pair $(\Gamma, \geq)$ consisting of an abelian group Γ endowed with a total ordering $\geq$ such that $\gamma \geq \gamma' \Rightarrow \gamma + \gamma'' \geq \gamma' + \gamma''$ for all $\gamma, \gamma', \gamma'' \in \Gamma$.

00ID **Lemma 50.12.** *Let A be a valuation ring with field of fractions K. Set $\Gamma = K^*/A^*$ (with group law written additively). For $\gamma, \gamma' \in \Gamma$ define $\gamma \geq \gamma'$ if and only if $\gamma - \gamma'$ is in the image of $A - \{0\} \to \Gamma$. Then $(\Gamma, \geq)$ is a totally ordered abelian group.*

Proof. Omitted, but follows easily from Lemma 50.3. Note that in case $A = K$ we obtain the zero group $\Gamma = \{0\}$ endowed with its unique total ordering. □

00IE **Definition 50.13.** Let A be a valuation ring.

1. The totally ordered abelian group $(\Gamma, \geq)$ of Lemma 50.12 is called the *value group* of the valuation ring A.
2. The map $v : A - \{0\} \to \Gamma$ and also $v : K^* \to \Gamma$ is called the *valuation* associated to A.
3. The valuation ring A is called a *discrete valuation ring* if $\Gamma \cong \mathbf{Z}$.

Note that if $\Gamma \cong \mathbf{Z}$ then there is a unique such isomorphism such that $1 \geq 0$. If the isomorphism is chosen in this way, then the ordering becomes the usual ordering of the integers.

00IF **Lemma 50.14.** *Let A be a valuation ring. The valuation $v : A - \{0\} \to \Gamma_{\geq 0}$ has the following properties:*

1. $v(a) = 0 \Leftrightarrow a \in A^*$,
2. $v(ab) = v(a) + v(b)$,
3. $v(a + b) \geq \min(v(a), v(b))$.

Proof. Omitted. □

090Q **Lemma 50.15.** *Let A be a ring. The following are equivalent*

1. *A is a valuation ring,*
2. *A is a local domain and every finitely generated ideal of A is principal.*

Proof. Assume A is a valuation ring and let $f_1, \ldots, f_n \in A$. Choose i such that $v(f_i)$ is minimal among $v(f_j)$. Then $(f_i) = (f_1, \ldots, f_n)$. Conversely, assume A is a local domain and every finitely generated ideal of A is principal. Pick $f, g \in A$ and write $(f, g) = (h)$. Then $f = ah$ and $g = bh$ and $h = cf + dg$ for some $a, b, c, d \in A$. Thus $ac + bd = 1$ and we see that either a or b is a unit, i.e., either g/f or f/g is an element of A. This shows A is a valuation ring by Lemma 50.4. □

00IG **Lemma 50.16.** *Let $(\Gamma, \geq)$ be a totally ordered abelian group. Let K be a field. Let $v : K^* \to \Gamma$ be a homomorphism of abelian groups such that $v(a+b) \geq \min(v(a), v(b))$ for $a, b \in K$ with $a, b, a+b$ not zero. Then*

$$A = \{x \in K \mid x = 0 \text{ or } v(x) \geq 0\}$$

is a valuation ring with value group $\mathrm{Im}(v) \subset \Gamma$, with maximal ideal

$$\mathfrak{m} = \{x \in K \mid x = 0 \text{ or } v(x) > 0\}$$

and with group of units

$$A^* = \{x \in K^* \mid v(x) = 0\}.$$

Proof. Omitted. □

Let $(\Gamma, \geq)$ be a totally ordered abelian group. An *ideal of* Γ is a subset $I \subset \Gamma$ such that all elements of I are ≥ 0 and $\gamma \in I$, $\gamma' \geq \gamma$ implies $\gamma' \in I$. We say that such an ideal is *prime* if $\gamma + \gamma' \in I, \gamma, \gamma' \geq 0 \Rightarrow \gamma \in I$ or $\gamma' \in I$.

00IH **Lemma 50.17.** *Let A be a valuation ring. Ideals in A correspond $1-1$ with ideals of Γ. This bijection is inclusion preserving, and maps prime ideals to prime ideals.*

Proof. Omitted. □

00II **Lemma 50.18.** *A valuation ring is Noetherian if and only if it is a discrete valuation ring or a field.*

Proof. Suppose A is a discrete valuation ring with valuation $v : A \setminus \{0\} \to \mathbf{Z}$ normalized so that $\mathrm{Im}(v) = \mathbf{Z}_{\geq 0}$. By Lemma 50.17 the ideals of A are the subsets $I_n = \{0\} \cup v^{-1}(\mathbf{Z}_{\geq n})$. It is clear that any element $x \in A$ with $v(x) = n$ generates I_n. Hence A is a PID so certainly Noetherian.

Suppose A is a Noetherian valuation ring with value group Γ. By Lemma 50.17 we see the ascending chain condition holds for ideals in Γ. We may assume A is not a field, i.e., there is a $\gamma \in \Gamma$ with $\gamma > 0$. Applying the ascending chain condition to the subsets $\gamma + \Gamma_{\geq 0}$ with $\gamma > 0$ we see there exists a smallest element γ_0 which is bigger than 0. Let $\gamma \in \Gamma$ be an element $\gamma > 0$. Consider the sequence of elements γ, $\gamma - \gamma_0$, $\gamma - 2\gamma_0$, etc. By the ascending chain condition these cannot all be > 0. Let $\gamma - n\gamma_0$ be the last one ≥ 0. By minimality of γ_0 we see that $0 = \gamma - n\gamma_0$. Hence Γ is a cyclic group as desired. □

51. More Noetherian rings

00IJ

00IK **Lemma 51.1.** *Let R be a Noetherian ring. Any finite R-module is of finite presentation. Any submodule of a finite R-module is finite. The ascending chain condition holds for R-submodules of a finite R-module.*

Proof. We first show that any submodule N of a finite R-module M is finite. We do this by induction on the number of generators of M. If this number is 1, then $N = J/I \subset M = R/I$ for some ideals $I \subset J \subset R$. Thus the definition of Noetherian implies the result. If the number of generators of M is greater than 1, then we can find a short exact sequence $0 \to M' \to M \to M'' \to 0$ where M' and M'' have fewer generators. Note that setting $N' = M' \cap N$ and $N'' = \mathrm{Im}(N \to M'')$ gives a similar short exact sequence for N. Hence the result follows from the induction hypothesis since the number of generators of N is at most the number of generators of N' plus the number of generators of N''.

To show that M is finitely presented just apply the previous result to the kernel of a presentation $R^n \to M$.

It is well known and easy to prove that the ascending chain condition for R-submodules of M is equivalent to the condition that every submodule of M is a finite R-module. We omit the proof. $\square$

00IN **Lemma 51.2** (Artin-Rees)**.** *Suppose that R is Noetherian, $I \subset R$ an ideal. Let $N \subset M$ be finite R-modules. There exists a constant $c > 0$ such that $I^n M \cap N = I^{n-c}(I^c M \cap N)$ for all $n \geq c$.*

Proof. Consider the ring $S = R \oplus I \oplus I^2 \oplus \ldots = \bigoplus_{n \geq 0} I^n$. Convention: $I^0 = R$. Multiplication maps $I^n \times I^m$ into I^{n+m} by multiplication in R. Note that if $I = (f_1, \ldots, f_t)$ then S is a quotient of the Noetherian ring $R[X_1, \ldots, X_t]$. The map just sends the monomial $X_1^{e_1} \ldots X_t^{e_t}$ to $f_1^{e_1} \ldots f_t^{e_t}$. Thus S is Noetherian. Similarly, consider the module $M \oplus IM \oplus I^2M \oplus \ldots = \bigoplus_{n \geq 0} I^n M$. This is a finitely generated S-module. Namely, if $x_1, \ldots, x_r$ generate M over R, then they also generate $\bigoplus_{n \geq 0} I^n M$ over S. Next, consider the submodule $\bigoplus_{n \geq 0} I^n M \cap N$. This is an S-submodule, as is easily verified. By Lemma 51.1 it is finitely generated as an S-module, say by $\xi_j \in \bigoplus_{n \geq 0} I^n M \cap N$, $j = 1, \ldots, s$. We may assume by decomposing each ξ_j into its homogeneous pieces that each $\xi_j \in I^{d_j} M \cap N$ for some d_j. Set $c = \max\{d_j\}$. Then for all $n \geq c$ every element in $I^n M \cap N$ is of the form $\sum h_j \xi_j$ with $h_j \in I^{n-d_j}$. The lemma now follows from this and the trivial observation that $I^{n-d_j}(I^{d_j} M \cap N) \subset I^{n-c}(I^c M \cap N)$. $\square$

00IO **Lemma 51.3.** *Suppose that $0 \to K \to M \xrightarrow{f} N$ is an exact sequence of finitely generated modules over a Noetherian ring R. Let $I \subset R$ be an ideal. Then there exists a c such that*

$$f^{-1}(I^n N) = K + I^{n-c} f^{-1}(I^c N) \quad \textit{and} \quad f(M) \cap I^n N \subset f(I^{n-c} M)$$

for all $n \geq c$.

Proof. Apply Lemma 51.2 to $\text{Im}(f) \subset N$ and note that $f : I^{n-c} M \to I^{n-c} f(M)$ is surjective. $\square$

00IP **Lemma 51.4** (Krull's intersection theorem)**.** *Let R be a Noetherian local ring. Let $I \subset R$ be a proper ideal. Let M be a finite R-module. Then $\bigcap_{n \geq 0} I^n M = 0$.*

Proof. Let $N = \bigcap_{n \geq 0} I^n M$. Then $N = I^n M \cap N$ for all $n \geq 0$. By the Artin-Rees Lemma 51.2 we see that $N = I^n M \cap N \subset IN$ for some suitably large n. By Nakayama's Lemma 20.1 we see that $N = 0$. $\square$

00IQ **Lemma 51.5.** *Let R be a Noetherian ring. Let $I \subset R$ be an ideal. Let M be a finite R-module. Let $N = \bigcap_n I^n M$.*

1. *For every prime $\mathfrak{p}$, $I \subset \mathfrak{p}$ there exists a $f \in R$, $f \notin \mathfrak{p}$ such that $N_f = 0$.*
2. *If I is contained in the Jacobson radical of R, then $N = 0$.*

Proof. Proof of (1). Let $x_1, \ldots, x_n$ be generators for the module N, see Lemma 51.1. For every prime $\mathfrak{p}$, $I \subset \mathfrak{p}$ we see that the image of N in the localization $M_{\mathfrak{p}}$ is zero, by Lemma 51.4. Hence we can find $g_i \in R$, $g_i \not\in \mathfrak p$ such that x_i maps to zero in N_{g_i}. Thus $N_{g_1 g_2 \ldots g_n} = 0$.

Part (2) follows from (1) and Lemma 23.1. $\square$

00IR **Remark 51.6.** Lemma 51.4 in particular implies that $\bigcap_n I^n = (0)$ when $I \subset R$ is a non-unit ideal in a Noetherian local ring R. More generally, let R be a Noetherian ring and $I \subset R$ an ideal. Suppose that $f \in \bigcap_{n \in \mathbf{N}} I^n$. Then Lemma 51.5 says that for every prime ideal $I \subset \mathfrak p$ there exists a $g \in R$, $g \not\in \mathfrak p$ such that f maps to zero in R_g. In algebraic geometry we express this by saying that "f is zero in an open neighbourhood of the closed set $V(I)$ of $\text{Spec}(R)$".

00IS **Lemma 51.7** (Artin-Tate)**.** *Let R be a Noetherian ring. Let S be a finitely generated R-algebra. If $T \subset S$ is an R-subalgebra such that S is finitely generated as a T-module, then T is of finite type over R.*

Proof. Choose elements $x_1, \ldots, x_n \in S$ which generate S as an R-algebra. Choose $y_1, \ldots, y_m$ in S which generate S as a T-module. Thus there exist $a_{ij} \in T$ such that $x_i = \sum a_{ij} y_j$. There also exist $b_{ijk} \in T$ such that $y_i y_j = \sum b_{ijk} y_k$. Let $T' \subset T$ be the sub R-algebra generated by a_{ij} and b_{ijk}. This is a finitely generated R-algebra, hence Noetherian. Consider the algebra

$$S' = T'[Y_1, \ldots, Y_m]/(Y_i Y_j - \sum b_{ijk} Y_k).$$

Note that S' is finite over T', namely as a T'-module it is generated by the classes of $1, Y_1, \ldots, Y_m$. Consider the T'-algebra homomorphism $S' \to S$ which maps Y_i to y_i. Because $a_{ij} \in T'$ we see that x_j is in the image of this map. Thus $S' \to S$ is surjective. Therefore S is finite over T' as well. Since T' is Noetherian and we conclude that $T \subset S$ is finite over T' and we win. $\square$

52. Length

00IU

02LY **Definition 52.1.** Let R be a ring. For any R-module M we define the *length* of M over R by the formula

$$\text{length}_R(M) = \sup\{n \mid \exists\ 0 = M_0 \subset M_1 \subset \ldots \subset M_n = M,\ M_i \neq M_{i+1}\}.$$

In other words it is the supremum of the lengths of chains of submodules. There is an obvious notion of when a chain of submodules is a refinement of another. This gives a partial ordering on the collection of all chains of submodules, with the smallest chain having the shape $0 = M_0 \subset M_1 = M$ if M is not zero. We note the obvious fact that if the length of M is finite, then every chain can be refined to a maximal chain. But it is not as obvious that all maximal chains have the same length (as we will see later).

02LZ **Lemma 52.2.** *Let R be a ring. Let M be an R-module. If $\text{length}_R(M) < \infty$ then M is a finite R-module.*

Proof. Omitted. $\square$

00IV **Lemma 52.3.** *If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of modules over R then the length of M is the sum of the lengths of M' and M''.*

Proof. Given filtrations of M' and M'' of lengths n', n'' it is easy to make a corresponding filtration of M of length $n' + n''$. Thus we see that $\text{length}_R M \geq \text{length}_R M' + \text{length}_R M''$. Conversely, given a filtration $M_0 \subset M_1 \subset \ldots \subset M_n$ of M consider the induced filtrations $M_i' = M_i \cap M'$ and $M_i'' = \text{Im}(M_i \to M'')$. Let n' (resp. n'') be the number of steps in the filtration $\{M_i'\}$ (resp. $\{M_i''\}$). If $M_i' = M_{i+1}'$ and $M_i'' = M_{i+1}''$ then $M_i = M_{i+1}$. Hence we conclude that $n' + n'' \geq n$. Combined with the earlier result we win. $\square$

00IW **Lemma 52.4.** *Let R be a local ring with maximal ideal $\mathfrak m$. Let M be an R-module.*

(1) *If M is a finite module and $\mathfrak m^n M \neq 0$ for all $n \geq 0$, then $\text{length}_R(M) = \infty$.*
(2) *If M has finite length then $\mathfrak m^n M = 0$ for some n.*

Proof. Assume $\mathfrak m^n M \neq 0$ for all $n \geq 0$. Choose $x \in M$ and $f_1, \ldots, f_n \in \mathfrak m$ such that $f_1 f_2 \ldots f_n x \neq 0$. By Nakayama's Lemma 20.1 the first n steps in the filtration

$$0 \subset R f_1 \ldots f_n x \subset R f_1 \ldots f_{n-1} x \subset \ldots \subset R x \subset M$$

are distinct. This can also be seen directly. For example, if $R f_1 x = R f_1 f_2 x$, then $f_1 x = g f_1 f_2 x$ for some g, hence $(1 - g f_2) f_1 x = 0$ hence $f_1 x = 0$ as $1 - g f_2$ is a unit which is a contradiction with the choice of x and $f_1, \ldots, f_n$. Hence the length is infinite, i.e., (1) holds. Combine (1) and Lemma 52.2 to see (2). $\square$

00IX **Lemma 52.5.** *Let $R \to S$ be a ring map. Let M be an S-module. We always have $\text{length}_R(M) \geq \text{length}_S(M)$. If $R \to S$ is surjective then equality holds.*

Proof. A filtration of M by S-submodules gives rise a filtration of M by R-submodules. This proves the inequality. And if $R \to S$ is surjective, then any R-submodule of M is automatically an S-submodule. Hence equality in this case. $\square$

00IY **Lemma 52.6.** *Let R be a ring with maximal ideal $\mathfrak m$. Suppose that M is an R-module with $\mathfrak m M = 0$. Then the length of M as an R-module agrees with the dimension of M as a $R/\mathfrak m$ vector space. The length is finite if and only if M is a finite R-module.*

Proof. The first part is a special case of Lemma 52.5. Thus the length is finite if and only if M has a finite basis as a $R/\mathfrak m$-vector space if and only if M has a finite set of generators as an R-module. $\square$

00IZ **Lemma 52.7.** *Let R be a ring. Let M be an R-module. Let $S \subset R$ be a multiplicative subset. Then $\text{length}_R(M) \geq \text{length}_{S^{-1}R}(S^{-1}M)$.*

Proof. Any submodule $N' \subset S^{-1}M$ is of the form $S^{-1}N$ for some R-submodule $N \subset M$, by Lemma 9.15. The lemma follows. $\square$

00J0 **Lemma 52.8.** *Let R be a ring with finitely generated maximal ideal $\mathfrak m$. (For example R Noetherian.) Suppose that M is a finite R-module with $\mathfrak m^n M = 0$ for some n. Then $\text{length}_R(M) < \infty$.*

Proof. Consider the filtration $0 = \mathfrak m^n M \subset \mathfrak m^{n-1} M \subset \ldots \subset \mathfrak m M \subset M$. All of the subquotients are finitely generated R-modules to which Lemma 52.6 applies. We conclude by additivity, see Lemma 52.3. $\square$

00J1 **Definition 52.9.** Let R be a ring. Let M be an R-module. We say M is *simple* if $M \neq 0$ and every submodule of M is either equal to M or to 0.

00J2 **Lemma 52.10.** *Let R be a ring. Let M be an R-module. The following are equivalent:*

(1) *M is simple,*
(2) *$\text{length}_R(M) = 1$, and*
(3) *$M \cong R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m} \subset R$.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of R. By Lemma 52.6 the module $R/\mathfrak{m}$ has length 1. The equivalence of the first two assertions is tautological. Suppose that M is simple. Choose $x \in M$, $x \neq 0$. As M is simple we have $M = R \cdot x$. Let $I \subset R$ be the annihilator of x, i.e., $I = \{f \in R \mid fx = 0\}$. The map $R/I \to M$, $f \bmod I \mapsto fx$ is an isomorphism, hence R/I is a simple R-module. Since $R/I \neq 0$ we see $I \neq R$. Let $I \subset \mathfrak{m}$ be a maximal ideal containing I. If $I \neq \mathfrak{m}$, then $\mathfrak{m}/I \subset R/I$ is a nontrivial submodule contradicting the simplicity of R/I. Hence we see $I = \mathfrak{m}$ as desired. □

00J3 **Lemma 52.11.** *Let R be a ring. Let M be a finite length R-module. Choose any maximal chain of submodules*

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_n = M$$

with $M_i \neq M_{i-1}$, $i = 1, \ldots, n$. Then

(1) *$n = \text{length}_R(M)$,*
(2) *each M_i/M_{i-1} is simple,*
(3) *each M_i/M_{i-1} is of the form $R/\mathfrak{m}_i$ for some maximal ideal $\mathfrak{m}_i$,*
(4) *given a maximal ideal $\mathfrak{m} \subset R$ we have*

$$\#\{i \mid \mathfrak{m}_i = \mathfrak{m}\} = \text{length}_{R_\mathfrak{m}}(M_\mathfrak{m}).$$

Proof. If M_i/M_{i-1} is not simple then we can refine the filtration and the filtration is not maximal. Thus we see that M_i/M_{i-1} is simple. By Lemma 52.10 the modules M_i/M_{i-1} have length 1 and are of the form $R/\mathfrak{m}_i$ for some maximal ideals $\mathfrak{m}_i$. By additivity of length, Lemma 52.3, we see $n = \text{length}_R(M)$. Since localization is exact, we see that

$$0 = (M_0)_\mathfrak{m} \subset (M_1)_\mathfrak{m} \subset (M_2)_\mathfrak{m} \subset \ldots \subset (M_n)_\mathfrak{m} = M_\mathfrak{m}$$

is a filtration of $M_\mathfrak{m}$ with successive quotients $(M_i/M_{i-1})_\mathfrak{m}$. Thus the last statement follows directly from the fact that given maximal ideals $\mathfrak{m}$, $\mathfrak{m}'$ of R we have

$$(R/\mathfrak{m}')_\mathfrak{m} \cong \begin{cases} 0 & \text{if } \mathfrak{m} \neq \mathfrak{m}', \\ R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m} & \text{if } \mathfrak{m} = \mathfrak{m}' \end{cases}$$

This we leave to the reader. □

02M0 **Lemma 52.12.** *Let A be a local ring with maximal ideal $\mathfrak{m}$. Let B be a semi-local ring with maximal ideals $\mathfrak{m}_i$, $i = 1, \ldots, n$. Suppose that $A \to B$ is a homomorphism such that each $\mathfrak{m}_i$ lies over $\mathfrak{m}$ and such that*

$$[\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})] < \infty.$$

Let M be a B-module of finite length. Then

$$\text{length}_A(M) = \sum\nolimits_{i=1,\ldots,n} [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})] \text{length}_{B_{\mathfrak{m}_i}}(M_{\mathfrak{m}_i}),$$

in particular $\text{length}_A(M) < \infty$.

Proof. Choose a maximal chain

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_m = M$$

by B-submodules as in Lemma 52.11. Then each quotient M_j/M_{j-1} is isomorphic to $\kappa(\mathfrak{m}_{i(j)})$ for some $i(j) \in \{1, \ldots, n\}$. Moreover $\text{length}_A(\kappa(\mathfrak{m}_i)) = [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})]$ by Lemma 52.6. The lemma follows by additivity of lengths (Lemma 52.3). $\square$

02M1 **Lemma 52.13.** *Let $A \to B$ be a flat local homomorphism of local rings. Then for any A-module M we have*

$$\text{length}_A(M)\text{length}_B(B/\mathfrak{m}_A B) = \text{length}_B(M \otimes_A B).$$

In particular, if $\text{length}_B(B/\mathfrak{m}_A B) < \infty$ then M has finite length if and only if $M \otimes_A B$ has finite length.

Proof. The ring map $A \to B$ is faithfully flat by Lemma 39.17. Hence if $0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$ is a chain of length n in M, then the corresponding chain $0 = M_0 \otimes_A B \subset M_1 \otimes_A B \subset \ldots \subset M_n \otimes_A B = M \otimes_A B$ has length n also. This proves $\text{length}_A(M) = \infty \Rightarrow \text{length}_B(M \otimes_A B) = \infty$. Next, assume $\text{length}_A(M) < \infty$. In this case we see that M has a filtration of length $\ell = \text{length}_A(M)$ whose quotients are $A/\mathfrak{m}_A$. Arguing as above we see that $M \otimes_A B$ has a filtration of length ℓ whose quotients are isomorphic to $B \otimes_A A/\mathfrak{m}_A = B/\mathfrak{m}_A B$. Thus the lemma follows. $\square$

02M2 **Lemma 52.14.** *Let $A \to B \to C$ be flat local homomorphisms of local rings. Then*

$$\text{length}_B(B/\mathfrak{m}_A B)\text{length}_C(C/\mathfrak{m}_B C) = \text{length}_C(C/\mathfrak{m}_A C)$$

Proof. Follows from Lemma 52.13 applied to the ring map $B \to C$ and the B-module $M = B/\mathfrak{m}_A B$ $\square$

53. Artinian rings

00J4 Artinian rings, and especially local Artinian rings, play an important role in algebraic geometry, for example in deformation theory.

00J5 **Definition 53.1.** A ring R is *Artinian* if it satisfies the descending chain condition for ideals.

00J6 **Lemma 53.2.** *Suppose R is a finite dimensional algebra over a field. Then R is Artinian.*

Proof. The descending chain condition for ideals obviously holds. $\square$

00J7 **Lemma 53.3.** *If R is Artinian then R has only finitely many maximal ideals.*

Proof. Suppose that $\mathfrak{m}_i$, $i = 1, 2, 3, \ldots$ are pairwise distinct maximal ideals. Then $\mathfrak{m}_1 \supset \mathfrak{m}_1 \cap \mathfrak{m}_2 \supset \mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \mathfrak{m}_3 \supset \ldots$ is an infinite descending sequence (because by the Chinese remainder theorem all the maps $R \to \oplus_{i=1}^n R/\mathfrak{m}_i$ are surjective). $\square$

00J8 **Lemma 53.4.** *Let R be Artinian. The Jacobson radical of R is a nilpotent ideal.*

Proof. Let $I \subset R$ be the Jacobson radical. Note that $I \supset I^2 \supset I^3 \supset \ldots$ is a descending sequence. Thus $I^n = I^{n+1}$ for some n. Set $J = \{x \in R \mid xI^n = 0\}$. We have to show $J = R$. If not, choose an ideal $J' \neq J$, $J \subset J'$ minimal (possible by the Artinian property). Then $J' = J + Rx$ for some $x \in R$. By NAK, Lemma 20.1, we have $IJ' \subset J$. Hence $xI^{n+1} \subset xI \cdot I^n \subset J \cdot I^n = 0$. Since $I^{n+1} = I^n$ we conclude $x \in J$. Contradiction. $\square$

00JA **Lemma 53.5.** *Any ring with finitely many maximal ideals and locally nilpotent Jacobson radical is the product of its localizations at its maximal ideals. Also, all primes are maximal.*

Proof. Let R be a ring with finitely many maximal ideals $\mathfrak m_1, \ldots, \mathfrak m_n$. Let $I = \bigcap_{i=1}^n \mathfrak m_i$ be the Jacobson radical of R. Assume I is locally nilpotent. Let $\mathfrak p$ be a prime ideal of R. Since every prime contains every nilpotent element of R we see $\mathfrak p \supset \mathfrak m_1 \cap \ldots \cap \mathfrak m_n$. Since $\mathfrak m_1 \cap \ldots \cap \mathfrak m_n \supset \mathfrak m_1 \ldots \mathfrak m_n$ we conclude $\mathfrak p \supset \mathfrak m_1 \ldots \mathfrak m_n$. Hence $\mathfrak p \supset \mathfrak m_i$ for some i, and so $\mathfrak p = \mathfrak m_i$. By the Chinese remainder theorem (Lemma 15.4) we have $R/I \cong \bigoplus R/\mathfrak m_i$ which is a product of fields. Hence by Lemma 32.6 there are idempotents e_i, $i = 1, \ldots, n$ with $e_i \bmod \mathfrak m_j = \delta_{ij}$. Hence $R = \prod Re_i$, and each Re_i is a ring with exactly one maximal ideal. ☐

00JB **Lemma 53.6.** *A ring R is Artinian if and only if it has finite length as a module over itself. Any such ring R is both Artinian and Noetherian, any prime ideal of R is a maximal ideal, and R is equal to the (finite) product of its localizations at its maximal ideals.*

Proof. If R has finite length over itself then it satisfies both the ascending chain condition and the descending chain condition for ideals. Hence it is both Noetherian and Artinian. Any Artinian ring is equal to product of its localizations at maximal ideals by Lemmas 53.3, 53.4, and 53.5.

Suppose that R is Artinian. We will show R has finite length over itself. It suffices to exhibit a chain of submodules whose successive quotients have finite length. By what we said above we may assume that R is local, with maximal ideal $\mathfrak m$. By Lemma 53.4 we have $\mathfrak m^n = 0$ for some n. Consider the sequence $0 = \mathfrak m^n \subset \mathfrak m^{n-1} \subset \ldots \subset \mathfrak m \subset R$. By Lemma 52.6 the length of each subquotient $\mathfrak m^j/\mathfrak m^{j+1}$ is the dimension of this as a vector space over $\kappa(\mathfrak m)$. This has to be finite since otherwise we would have an infinite descending chain of sub vector spaces which would correspond to an infinite descending chain of ideals in R. ☐

54. Homomorphisms essentially of finite type

07DR Some simple remarks on localizations of finite type ring maps.

00QM **Definition 54.1.** Let $R \to S$ be a ring map.

1. We say that $R \to S$ is *essentially of finite type* if S is the localization of an R-algebra of finite type.
2. We say that $R \to S$ is *essentially of finite presentation* if S is the localization of an R-algebra of finite presentation.

07DS **Lemma 54.2.** *The class of ring maps which are essentially of finite type is preserved under composition. Similarly for essentially of finite presentation.*

Proof. Omitted. ☐

0AUF **Lemma 54.3.** *The class of ring maps which are essentially of finite type is preserved by base change. Similarly for essentially of finite presentation.*

Proof. Omitted. ☐

07DT **Lemma 54.4.** *Let $R \to S$ be a ring map. Assume S is an Artinian local ring with maximal ideal $\mathfrak m$. Then*

(1) *$R \to S$ is finite if and only if $R \to S/\mathfrak{m}$ is finite,*
(2) *$R \to S$ is of finite type if and only if $R \to S/\mathfrak{m}$ is of finite type.*
(3) *$R \to S$ is essentially of finite type if and only if the composition $R \to S/\mathfrak{m}$ is essentially of finite type.*

Proof. If $R \to S$ is finite, then $R \to S/\mathfrak{m}$ is finite by Lemma 7.3. Conversely, assume $R \to S/\mathfrak{m}$ is finite. As S has finite length over itself (Lemma 53.6) we can choose a filtration

$$0 \subset I_1 \subset \ldots \subset I_n = S$$

by ideals such that $I_i/I_{i-1} \cong S/\mathfrak{m}$ as S-modules. Thus S has a filtration by R-submodules I_i such that each successive quotient is a finite R-module. Thus S is a finite R-module by Lemma 5.3.

If $R \to S$ is of finite type, then $R \to S/\mathfrak{m}$ is of finite type by Lemma 6.2. Conversely, assume that $R \to S/\mathfrak{m}$ is of finite type. Choose $f_1, \ldots, f_n \in S$ which map to generators of $S/\mathfrak{m}$. Then $A = R[x_1, \ldots, x_n] \to S$, $x_i \mapsto f_i$ is a ring map such that $A \to S/\mathfrak{m}$ is surjective (in particular finite). Hence $A \to S$ is finite by part (1) and we see that $R \to S$ is of finite type by Lemma 6.2.

If $R \to S$ is essentially of finite type, then $R \to S/\mathfrak{m}$ is essentially of finite type by Lemma 54.2. Conversely, assume that $R \to S/\mathfrak{m}$ is essentially of finite type. Suppose $S/\mathfrak{m}$ is the localization of $R[x_1, \ldots, x_n]/I$. Choose $f_1, \ldots, f_n \in S$ whose congruence classes modulo $\mathfrak{m}$ correspond to the congruence classes of $x_1, \ldots, x_n$ modulo I. Consider the map $R[x_1, \ldots, x_n] \to S$, $x_i \mapsto f_i$ with kernel J. Set $A = R[x_1, \ldots, x_n]/J \subset S$ and $\mathfrak{p} = A \cap \mathfrak{m}$. Note that $A/\mathfrak{p} \subset S/\mathfrak{m}$ is equal to the image of $R[x_1, \ldots, x_n]/I$ in $S/\mathfrak{m}$. Hence $\kappa(\mathfrak{p}) = S/\mathfrak{m}$. Thus $A_\mathfrak{p} \to S$ is finite by part (1). We conclude that S is essentially of finite type by Lemma 54.2. □

The following lemma can be proven using properness of projective space instead of the algebraic argument we give here.

0AUG **Lemma 54.5.** *Let $\varphi : R \to S$ be essentially of finite type with R and S local (but not necessarily φ local). Then there exists an n and a maximal ideal $\mathfrak{m} \subset R[x_1, \ldots, x_n]$ lying over $\mathfrak{m}_R$ such that S is a localization of a quotient of $R[x_1, \ldots, x_n]_\mathfrak{m}$.*

Proof. We can write S as a localization of a quotient of $R[x_1, \ldots, x_n]$. Hence it suffices to prove the lemma in case $S = R[x_1, \ldots, x_n]_\mathfrak{q}$ for some prime $\mathfrak{q} \subset R[x_1, \ldots, x_n]$. If $\mathfrak{q} + \mathfrak{m}_R R[x_1, \ldots, x_n] \neq R[x_1, \ldots, x_n]$ then we can find a maximal ideal $\mathfrak{m}$ as in the statement of the lemma with $\mathfrak{q} \subset \mathfrak{m}$ and the result is clear.

Choose a valuation ring $A \subset \kappa(\mathfrak{q})$ which dominates the image of $R \to \kappa(\mathfrak{q})$ (Lemma 50.2). If the image $\lambda_i \in \kappa(\mathfrak{q})$ of x_i is contained in A, then $\mathfrak{q}$ is contained in the inverse image of $\mathfrak{m}_A$ via $R[x_1, \ldots, x_n] \to A$ which means we are back in the preceding case. Hence there exists an i such that $\lambda_i^{-1} \in A$ and such that $\lambda_j/\lambda_i \in A$ for all $j = 1, \ldots, n$ (because the value group of A is totally ordered, see Lemma 50.12). Then we consider the map

$$R[y_0, y_1, \ldots, \hat{y}_i, \ldots, y_n] \to R[x_1, \ldots, x_n]_\mathfrak{q}, \quad y_0 \mapsto 1/x_i, \quad y_j \mapsto x_j/x_i$$

Let $\mathfrak{q}' \subset R[y_0, \ldots, \hat{y}_i, \ldots, y_n]$ be the inverse image of $\mathfrak{q}$. Since $y_0 \notin \mathfrak{q}'$ it is easy to see that the displayed arrow defines an isomorphism on localizations. On the other hand, the result of the first paragraph applies to $R[y_0, \ldots, \hat{y}_i, \ldots, y_n]$ because y_j maps to an element of A. This finishes the proof. □

55. K-groups

00JC Let R be a ring. We will introduce two abelian groups associated to R. The first of the two is denoted $K'_0(R)$ and has the following properties[6]:

(1) For every finite R-module M there is given an element $[M]$ in $K'_0(R)$,
(2) for every short exact sequence $0 \to M' \to M \to M'' \to 0$ of finite R-modules we have the relation $[M] = [M'] + [M'']$,
(3) the group $K'_0(R)$ is generated by the elements $[M]$, and
(4) all relations in $K'_0(R)$ among the generators $[M]$ are $\mathbf{Z}$-linear combinations of the relations coming from exact sequences as above.

The actual construction is a bit more annoying since one has to take care that the collection of all finitely generated R-modules is a proper class. However, this problem can be overcome by taking as set of generators of the group $K'_0(R)$ the elements $[R^n/K]$ where n ranges over all integers and K ranges over all submodules $K \subset R^n$. The generators for the subgroup of relations imposed on these elements will be the relations coming from short exact sequences whose terms are of the form R^n/K. The element $[M]$ is defined by choosing n and K such that $M \cong R^n/K$ and putting $[M] = [R^n/K]$. Details left to the reader.

00JD **Lemma 55.1.** *If R is an Artinian local ring then the length function defines a natural abelian group homomorphism $\text{length}_R : K'_0(R) \to \mathbf{Z}$.*

Proof. The length of any finite R-module is finite, because it is the quotient of R^n which has finite length by Lemma 53.6. And the length function is additive, see Lemma 52.3. $\square$

The second of the two is denoted $K_0(R)$ and has the following properties:

(1) For every finite projective R-module M there is given an element $[M]$ in $K_0(R)$,
(2) for every short exact sequence $0 \to M' \to M \to M'' \to 0$ of finite projective R-modules we have the relation $[M] = [M'] + [M'']$,
(3) the group $K_0(R)$ is generated by the elements $[M]$, and
(4) all relations in $K_0(R)$ are $\mathbf{Z}$-linear combinations of the relations coming from exact sequences as above.

The construction of this group is done as above.

We note that there is an obvious map $K_0(R) \to K'_0(R)$ which is not an isomorphism in general.

00JE **Example 55.2.** Note that if $R = k$ is a field then we clearly have $K_0(k) = K'_0(k) \cong \mathbf{Z}$ with the isomorphism given by the dimension function (which is also the length function).

0FJ8 **Example 55.3.** Let R be a PID. We claim $K_0(R) = K'_0(R) = \mathbf{Z}$. Namely, any finite projective R-module is finite free. A finite free module has a well defined rank by Lemma 15.8. Given a short exact sequence of finite free modules

$$0 \to M' \to M \to M'' \to 0$$

we have $\text{rank}(M) = \text{rank}(M') + \text{rank}(M'')$ because we have $M \cong M' \oplus M'$ in this case (for example we have a splitting by Lemma 5.2). We conclude $K_0(R) = \mathbf{Z}$.

[6]The definition makes sense for any ring but is rarely used unless R is Noetherian.

The structure theorem for modules of a PID says that any finitely generated R-module is of the form $M = R^{\oplus r} \oplus R/(d_1) \oplus \ldots \oplus R/(d_k)$. Consider the short exact sequence

$$0 \to (d_i) \to R \to R/(d_i) \to 0$$

Since the ideal (d_i) is isomorphic to R as a module (it is free with generator d_i), in $K'_0(R)$ we have $[(d_i)] = [R]$. Then $[R/(d_i)] = [(d_i)] - [R] = 0$. From this it follows that a torsion module has zero class in $K'_0(R)$. Using the rank of the free part gives an identification $K'_0(R) = \mathbf{Z}$ and the canonical homomorphism from $K_0(R) \to K'_0(R)$ is an isomorphism.

00JF **Example 55.4.** Let k be a field. Then $K_0(k[x]) = K'_0(k[x]) = \mathbf{Z}$. This follows from Example 55.3 as $R = k[x]$ is a PID.

00JG **Example 55.5.** Let k be a field. Let $R = \{f \in k[x] \mid f(0) = f(1)\}$, compare Example 27.4. In this case $K_0(R) \cong k^* \oplus \mathbf{Z}$, but $K'_0(R) = \mathbf{Z}$.

00JH **Lemma 55.6.** *Let $R = R_1 \times R_2$. Then $K_0(R) = K_0(R_1) \times K_0(R_2)$ and $K'_0(R) = K'_0(R_1) \times K'_0(R_2)$*

Proof. Omitted. □

00JI **Lemma 55.7.** *Let R be an Artinian local ring. The map $length_R : K'_0(R) \to \mathbf{Z}$ of Lemma 55.1 is an isomorphism.*

Proof. Omitted. □

00JJ **Lemma 55.8.** *Let $(R, \mathfrak{m})$ be a local ring. Every finite projective R-module is finite free. The map $rank_R : K_0(R) \to \mathbf{Z}$ defined by $[M] \to rank_R(M)$ is well defined and an isomorphism.*

Proof. Let P be a finite projective R-module. Choose elements $x_1, \ldots, x_n \in P$ which map to a basis of $P/\mathfrak{m}P$. By Nakayama's Lemma 20.1 these elements generate P. The corresponding surjection $u : R^{\oplus n} \to P$ has a splitting as P is projective. Hence $R^{\oplus n} = P \oplus Q$ with $Q = \text{Ker}(u)$. It follows that $Q/\mathfrak{m}Q = 0$, hence Q is zero by Nakayama's lemma. In this way we see that every finite projective R-module is finite free. A finite free module has a well defined rank by Lemma 15.8. Given a short exact sequence of finite free R-modules

$$0 \to M' \to M \to M'' \to 0$$

we have $\text{rank}(M) = \text{rank}(M') + \text{rank}(M'')$ because we have $M \cong M' \oplus M'$ in this case (for example we have a splitting by Lemma 5.2). We conclude $K_0(R) = \mathbf{Z}$. □

00JK **Lemma 55.9.** *Let R be a local Artinian ring. There is a commutative diagram*

$$\begin{tikzcd}
K_0(R) \arrow[r] \arrow[d, "rank_R"'] & K'_0(R) \arrow[d, "length_R"] \\
\mathbf{Z} \arrow[r, "length_R(R)"] & \mathbf{Z}
\end{tikzcd}$$

where the vertical maps are isomorphisms by Lemmas 55.7 and 55.8.

Proof. Let P be a finite projective R-module. We have to show that $\text{length}_R(P) = \text{rank}_R(P)\text{length}_R(R)$. By Lemma 55.8 the module P is finite free. So $P \cong R^{\oplus n}$ for some $n \geq 0$. Then $\text{rank}_R(P) = n$ and $\text{length}_R(R^{\oplus n}) = n\text{length}_R(R)$ by additivity of lenghts (Lemma 52.3). Thus the result holds. □

56. Graded rings

00JL A *graded ring* will be for us a ring S endowed with a direct sum decomposition $S = \bigoplus_{d \geq 0} S_d$ of the underlying abelian group such that $S_d \cdot S_e \subset S_{d+e}$. Note that we do not allow nonzero elements in negative degrees. The *irrelevant ideal* is the ideal $S_+ = \bigoplus_{d > 0} S_d$. A *graded module* will be an S-module M endowed with a direct sum decomposition $M = \bigoplus_{n \in \mathbf{Z}} M_n$ of the underlying abelian group such that $S_d \cdot M_e \subset M_{d+e}$. Note that for modules we do allow nonzero elements in negative degrees. We think of S as a graded S-module by setting $S_{-k} = (0)$ for $k > 0$. An element x (resp. f) of M (resp. S) is called *homogeneous* if $x \in M_d$ (resp. $f \in S_d$) for some d. A *map of graded S-modules* is a map of S-modules $\varphi : M \to M'$ such that $\varphi(M_d) \subset M'_d$. We do not allow maps to shift degrees. Let us denote $\text{GrHom}_0(M, N)$ the S_0-module of homomorphisms of graded modules from M to N.

At this point there are the notions of graded ideal, graded quotient ring, graded submodule, graded quotient module, graded tensor product, etc. We leave it to the reader to find the relevant definitions, and lemmas. For example: A short exact sequence of graded modules is short exact in every degree.

Given a graded ring S, a graded S-module M and $n \in \mathbf{Z}$ we denote $M(n)$ the graded S-module with $M(n)_d = M_{n+d}$. This is called the *twist of M by n*. In particular we get modules $S(n)$, $n \in \mathbf{Z}$ which will play an important role in the study of projective schemes. There are some obvious functorial isomorphisms such as $(M \oplus N)(n) = M(n) \oplus N(n)$, $(M \otimes_S N)(n) = M \otimes_S N(n) = M(n) \otimes_S N$. In addition we can define a graded S-module structure on the S_0-module

$$\text{GrHom}(M, N) = \bigoplus\nolimits_{n \in \mathbf{Z}} \text{GrHom}_n(M, N), \quad \text{GrHom}_n(M, N) = \text{GrHom}_0(M, N(n)).$$

We omit the definition of the multiplication.

0EKB **Lemma 56.1.** *Let S be a graded ring. Let M be a graded S-module.*

1. *If $S_+M = M$ and M is finite, then $M = 0$.*
2. *If $N, N' \subset M$ are graded submodules, $M = N + S_+N'$, and N' is finite, then $M = N$.*
3. *If $N \to M$ is a map of graded modules, $N/S_+N \to M/S_+M$ is surjective, and M is finite, then $N \to M$ is surjective.*
4. *If $x_1, \ldots, x_n \in M$ are homogeneous and generate M/S_+M and M is finite, then $x_1, \ldots, x_n$ generate M.*

Proof. Proof of (1). Choose generators $y_1, \ldots, y_r$ of M over S. We may assume that y_i is homogeneous of degree d_i. After renumbering we may assume $d_r = \min(d_i)$. Then the condition that $S_+M = M$ implies $y_r = 0$. Hence $M = 0$ by induction on r. Part (2) follows by applying (1) to M/N. Part (3) follows by applying (2) to the submodules $\text{Im}(N \to M)$ and M. Part (4) follows by applying (3) to the module map $\bigoplus S(-d_i) \to M$, $(s_1, \ldots, s_n) \mapsto \sum s_i x_i$. $\square$

Let S be a graded ring. Let $d \geq 1$ be an integer. We set $S^{(d)} = \bigoplus_{n \geq 0} S_{nd}$. We think of $S^{(d)}$ as a graded ring with degree n summand $(S^{(d)})_n = S_{nd}$. Given a graded S-module M we can similarly consider $M^{(d)} = \bigoplus_{n \in \mathbf{Z}} M_{nd}$ which is a graded $S^{(d)}$-module.

0EGH **Lemma 56.2.** *Let S be a graded ring, which is finitely generated over S_0. Then for all sufficiently divisible d the algebra $S^{(d)}$ is generated in degree 1 over S_0.*

Proof. Say S is generated by $f_1, \ldots, f_r \in S$ over S_0. After replacing f_i by their homogeneous parts, we may assume f_i is homogeneous of degree $d_i > 0$. Then any element of S_n is a linear combination with coefficients in S_0 of monomials $f_1^{e_1} \ldots f_r^{e_r}$ with $\sum e_i d_i = n$. Let m be a multiple of $\text{lcm}(d_i)$. For any $N \geq r$ if

$$\sum e_i d_i = Nm$$

then for some i we have $e_i \geq m/d_i$ by an elementary argument. Hence every monomial of degree Nm is a product of a monomial of degree m, namely f_i^{m/d_i}, and a monomial of degree $(N-1)m$. It follows that any monomial of degree nrm with $n \geq 2$ is a product of monomials of degree rm. Thus $S^{(rm)}$ is generated in degree 1 over S_0. □

077G **Lemma 56.3.** *Let $R \to S$ be a homomorphism of graded rings. Let $S' \subset S$ be the integral closure of R in S. Then*

$$S' = \bigoplus\nolimits_{d \geq 0} S' \cap S_d,$$

i.e., S' is a graded R-subalgebra of S.

Proof. We have to show the following: If $s = s_n + s_{n+1} + \ldots + s_m \in S'$, then each homogeneous part $s_j \in S'$. We will prove this by induction on $m - n$ over all homomorphisms $R \to S$ of graded rings. First note that it is immediate that s_0 is integral over R_0 (hence over R) as there is a ring map $S \to S_0$ compatible with the ring map $R \to R_0$. Thus, after replacing s by $s - s_0$, we may assume $n > 0$. Consider the extension of graded rings $R[t, t^{-1}] \to S[t, t^{-1}]$ where t has degree 0. There is a commutative diagram

$$\begin{tikzcd}
S[t, t^{-1}] \arrow[r, "s \mapsto t^{\deg(s)} s"'] & S[t, t^{-1}] \\
R[t, t^{-1}] \arrow[u] \arrow[r, "r \mapsto t^{\deg(r)} r"] & R[t, t^{-1}] \arrow[u]
\end{tikzcd}$$

where the horizontal maps are ring automorphisms. Hence the integral closure C of $S[t, t^{-1}]$ over $R[t, t^{-1}]$ maps into itself. Thus we see that

$$t^m(s_n + s_{n+1} + \ldots + s_m) - (t^n s_n + t^{n+1} s_{n+1} + \ldots + t^m s_m) \in C$$

which implies by induction hypothesis that each $(t^m - t^i)s_i \in C$ for $i = n, \ldots, m-1$. Note that for any ring A and $m > i \geq n > 0$ we have $A[t, t^{-1}]/(t^m - t^i - 1) \cong A[t]/(t^m - t^i - 1) \supset A$ because $t(t^{m-1} - t^{i-1}) = 1$ in $A[t]/(t^m - t^i - 1)$. Since $t^m - t^i$ maps to 1 we see the image of s_i in the ring $S[t]/(t^m - t^i - 1)$ is integral over $R[t]/(t^m - t^i - 1)$ for $i = n, \ldots, m-1$. Since $R \to R[t]/(t^m - t^i - 1)$ is finite we see that s_i is integral over R by transitivity, see Lemma 36.6. Finally, we also conclude that $s_m = s - \sum_{i=n,\ldots,m-1} s_i$ is integral over R. □

57. Proj of a graded ring

00JM

Let S be a graded ring. A *homogeneous ideal* is simply an ideal $I \subset S$ which is also a graded submodule of S. Equivalently, it is an ideal generated by homogeneous elements. Equivalently, if $f \in I$ and

$$f = f_0 + f_1 + \ldots + f_n$$

is the decomposition of f into homogeneous parts in S then $f_i \in I$ for each i. To check that a homogeneous ideal $\mathfrak{p}$ is prime it suffices to check that if $ab \in \mathfrak{p}$ with a, b homogeneous then either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.

00JN **Definition 57.1.** Let S be a graded ring. We define $\text{Proj}(S)$ to be the set of homogeneous prime ideals $\mathfrak{p}$ of S such that $S_+ \not\subset \mathfrak{p}$. The set $\text{Proj}(S)$ is a subset of $\text{Spec}(S)$ and we endow it with the induced topology. The topological space $\text{Proj}(S)$ is called the *homogeneous spectrum* of the graded ring S.

Note that by construction there is a continuous map

$$\text{Proj}(S) \longrightarrow \text{Spec}(S_0).$$

Let $S = \oplus_{d \geq 0} S_d$ be a graded ring. Let $f \in S_d$ and assume that $d \geq 1$. We define $S_{(f)}$ to be the subring of S_f consisting of elements of the form r/f^n with r homogeneous and $\deg(r) = nd$. If M is a graded S-module, then we define the $S_{(f)}$-module $M_{(f)}$ as the sub module of M_f consisting of elements of the form x/f^n with x homogeneous of degree nd.

00JO **Lemma 57.2.** *Let S be a $\mathbf{Z}$-graded ring containing a homogeneous invertible element of positive degree. Then the set $G \subset \text{Spec}(S)$ of $\mathbf{Z}$-graded primes of S (with induced topology) maps homeomorphically to $\text{Spec}(S_0)$.*

Proof. First we show that the map is a bijection by constructing an inverse. Let $f \in S_d$, $d > 0$ be invertible in S. If $\mathfrak{p}_0$ is a prime of S_0, then $\mathfrak{p}_0 S$ is a $\mathbf{Z}$-graded ideal of S such that $\mathfrak{p}_0 S \cap S_0 = \mathfrak{p}_0$. And if $ab \in \mathfrak{p}_0 S$ with a, b homogeneous, then $a^d b^d / f^{\deg(a)+\deg(b)} \in \mathfrak{p}_0$. Thus either $a^d/f^{\deg(a)} \in \mathfrak{p}_0$ or $b^d/f^{\deg(b)} \in \mathfrak{p}_0$, in other words either $a^d \in \mathfrak{p}_0 S$ or $b^d \in \mathfrak{p}_0 S$. It follows that $\sqrt{\mathfrak{p}_0 S}$ is a $\mathbf{Z}$-graded prime ideal of S whose intersection with S_0 is $\mathfrak{p}_0$.

To show that the map is a homeomorphism we show that the image of $G \cap D(g)$ is open. If $g = \sum g_i$ with $g_i \in S_i$, then by the above $G \cap D(g)$ maps onto the set $\bigcup D(g_i^d / f^i)$ which is open. $\square$

For $f \in S$ homogeneous of degree > 0 we define

$$D_+(f) = \{\mathfrak{p} \in \text{Proj}(S) \mid f \notin \mathfrak{p}\}.$$

Finally, for a homogeneous ideal $I \subset S$ we define

$$V_+(I) = \{\mathfrak{p} \in \text{Proj}(S) \mid I \subset \mathfrak{p}\}.$$

We will use more generally the notation $V_+(E)$ for any set E of homogeneous elements $E \subset S$.

00JP **Lemma 57.3** (Topology on Proj). *Let $S = \oplus_{d \geq 0} S_d$ be a graded ring.*

(1) *The sets $D_+(f)$ are open in Proj(S).*
(2) *We have $D_+(ff') = D_+(f) \cap D_+(f')$.*
(3) *Let $g = g_0 + \ldots + g_m$ be an element of S with $g_i \in S_i$. Then*

$$D(g) \cap \mathit{Proj}(S) = (D(g_0) \cap \mathit{Proj}(S)) \cup \bigcup\nolimits_{i \geq 1} D_+(g_i).$$

(4) *Let $g_0 \in S_0$ be a homogeneous element of degree* 0. *Then*
$$D(g_0) \cap Proj(S) = \bigcup\nolimits_{f \in S_d,\ d \geq 1} D_+(g_0 f).$$
(5) *The open sets $D_+(f)$ form a basis for the topology of Proj(S).*
(6) *Let $f \in S$ be homogeneous of positive degree. The ring S_f has a natural* **Z***-grading. The ring maps $S \to S_f \leftarrow S_{(f)}$ induce homeomorphisms*
$$D_+(f) \leftarrow \{\mathbf{Z}\text{-graded primes of } S_f\} \to \mathrm{Spec}(S_{(f)}).$$
(7) *There exists an S such that Proj(S) is not quasi-compact.*
(8) *The sets $V_+(I)$ are closed.*
(9) *Any closed subset $T \subset Proj(S)$ is of the form $V_+(I)$ for some homogeneous ideal $I \subset S$.*
(10) *For any graded ideal $I \subset S$ we have $V_+(I) = \emptyset$ if and only if $S_+ \subset \sqrt{I}$.*

Proof. Since $D_+(f) = \mathrm{Proj}(S) \cap D(f)$, these sets are open. This proves (1). Also (2) follows as $D(ff') = D(f) \cap D(f')$. Similarly the sets $V_+(I) = \mathrm{Proj}(S) \cap V(I)$ are closed. This proves (8).

Suppose that $T \subset \mathrm{Proj}(S)$ is closed. Then we can write $T = \mathrm{Proj}(S) \cap V(J)$ for some ideal $J \subset S$. By definition of a homogeneous ideal if $g \in J$, $g = g_0 + \ldots + g_m$ with $g_d \in S_d$ then $g_d \in \mathfrak{p}$ for all $\mathfrak{p} \in T$. Thus, letting $I \subset S$ be the ideal generated by the homogeneous parts of the elements of J we have $T = V_+(I)$. This proves (9).

The formula for $\mathrm{Proj}(S) \cap D(g)$, with $g \in S$ is direct from the definitions. This proves (3). Consider the formula for $\mathrm{Proj}(S) \cap D(g_0)$. The inclusion of the right hand side in the left hand side is obvious. For the other inclusion, suppose $g_0 \notin \mathfrak{p}$ with $\mathfrak{p} \in \mathrm{Proj}(S)$. If all $g_0 f \in \mathfrak{p}$ for all homogeneous f of positive degree, then we see that $S_+ \subset \mathfrak{p}$ which is a contradiction. This gives the other inclusion. This proves (4).

The collection of opens $D(g) \cap \mathrm{Proj}(S)$ forms a basis for the topology since the standard opens $D(g) \subset \mathrm{Spec}(S)$ form a basis for the topology on $\mathrm{Spec}(S)$. By the formulas above we can express $D(g) \cap \mathrm{Proj}(S)$ as a union of opens $D_+(f)$. Hence the collection of opens $D_+(f)$ forms a basis for the topology also. This proves (5).

Proof of (6). First we note that $D_+(f)$ may be identified with a subset (with induced topology) of $D(f) = \mathrm{Spec}(S_f)$ via Lemma 17.6. Note that the ring S_f has a **Z**-grading. The homogeneous elements are of the form r/f^n with $r \in S$ homogeneous and have degree $\deg(r/f^n) = \deg(r) - n\deg(f)$. The subset $D_+(f)$ corresponds exactly to those prime ideals $\mathfrak{p} \subset S_f$ which are **Z**-graded ideals (i.e., generated by homogeneous elements). Hence we have to show that the set of **Z**-graded prime ideals of S_f maps homeomorphically to $\mathrm{Spec}(S_{(f)})$. This follows from Lemma 57.2.

Let $S = \mathbf{Z}[X_1, X_2, X_3, \ldots]$ with grading such that each X_i has degree 1. Then it is easy to see that
$$\mathrm{Proj}(S) = \bigcup\nolimits_{i=1}^{\infty} D_+(X_i)$$
does not have a finite refinement. This proves (7).

Let $I \subset S$ be a graded ideal. If $\sqrt{I} \supset S_+$ then $V_+(I) = \emptyset$ since every prime $\mathfrak{p} \in \mathrm{Proj}(S)$ does not contain S_+ by definition. Conversely, suppose that $S_+ \not\subset \sqrt{I}$.

Then we can find an element $f \in S_+$ such that f is not nilpotent modulo I. Clearly this means that one of the homogeneous parts of f is not nilpotent modulo I, in other words we may (and do) assume that f is homogeneous. This implies that $IS_f \neq S_f$, in other words that $(S/I)_f$ is not zero. Hence $(S/I)_{(f)} \neq 0$ since it is a ring which maps into $(S/I)_f$. Pick a prime $\mathfrak{q} \subset (S/I)_{(f)}$. This corresponds to a graded prime of S/I, not containing the irrelevant ideal $(S/I)_+$. And this in turn corresponds to a graded prime ideal $\mathfrak{p}$ of S, containing I but not containing S_+ as desired. This proves (10) and finishes the proof. □

00JQ **Example 57.4.** Let R be a ring. If $S = R[X]$ with $\deg(X) = 1$, then the natural map $\text{Proj}(S) \to \text{Spec}(R)$ is a bijection and in fact a homeomorphism. Namely, suppose $\mathfrak{p} \in \text{Proj}(S)$. Since $S_+ \not\subset \mathfrak{p}$ we see that $X \notin \mathfrak{p}$. Thus if $aX^n \in \mathfrak{p}$ with $a \in R$ and $n > 0$, then $a \in \mathfrak{p}$. It follows that $\mathfrak{p} = \mathfrak{p}_0 S$ with $\mathfrak{p}_0 = \mathfrak{p} \cap R$.

If $\mathfrak{p} \in \text{Proj}(S)$, then we define $S_{(\mathfrak{p})}$ to be the ring whose elements are fractions r/f where $r, f \in S$ are homogeneous elements of the same degree such that $f \notin \mathfrak{p}$. As usual we say $r/f = r'/f'$ if and only if there exists some $f'' \in S$ homogeneous, $f'' \notin \mathfrak{p}$ such that $f''(rf' - r'f) = 0$. Given a graded S-module M we let $M_{(\mathfrak{p})}$ be the $S_{(\mathfrak{p})}$-module whose elements are fractions x/f with $x \in M$ and $f \in S$ homogeneous of the same degree such that $f \notin \mathfrak{p}$. We say $x/f = x'/f'$ if and only if there exists some $f'' \in S$ homogeneous, $f'' \notin \mathfrak{p}$ such that $f''(xf' - x'f) = 0$.

00JR **Lemma 57.5.** *Let S be a graded ring. Let M be a graded S-module. Let $\mathfrak{p}$ be an element of Proj(S). Let $f \in S$ be a homogeneous element of positive degree such that $f \notin \mathfrak{p}$, i.e., $\mathfrak{p} \in D_+(f)$. Let $\mathfrak{p}' \subset S_{(f)}$ be the element of* $\text{Spec}(S_{(f)})$ *corresponding to $\mathfrak{p}$ as in Lemma 57.3. Then $S_{(\mathfrak{p})} = (S_{(f)})_{\mathfrak{p}'}$ and compatibly $M_{(\mathfrak{p})} = (M_{(f)})_{\mathfrak{p}'}$.*

Proof. We define a map $\psi : M_{(\mathfrak{p})} \to (M_{(f)})_{\mathfrak{p}'}$. Let $x/g \in M_{(\mathfrak{p})}$. We set

$$\psi(x/g) = (xg^{\deg(f)-1}/f^{\deg(x)})/(g^{\deg(f)}/f^{\deg(g)}).$$

This makes sense since $\deg(x) = \deg(g)$ and since $g^{\deg(f)}/f^{\deg(g)} \notin \mathfrak{p}'$. We omit the verification that ψ is well defined, a module map and an isomorphism. Hint: the inverse sends $(x/f^n)/(g/f^m)$ to $(xf^m)/(gf^n)$. □

Here is a graded variant of Lemma 15.2.

00JS **Lemma 57.6.** *Suppose S is a graded ring, $\mathfrak{p}_i$, $i = 1, \ldots, r$ homogeneous prime ideals and $I \subset S_+$ a graded ideal. Assume $I \not\subset \mathfrak{p}_i$ for all i. Then there exists a homogeneous element $x \in I$ of positive degree such that $x \notin \mathfrak{p}_i$ for all i.*

Proof. We may assume there are no inclusions among the $\mathfrak{p}_i$. The result is true for $r = 1$. Suppose the result holds for $r - 1$. Pick $x \in I$ homogeneous of positive degree such that $x \notin \mathfrak{p}_i$ for all $i = 1, \ldots, r - 1$. If $x \notin \mathfrak{p}_r$ we are done. So assume $x \in \mathfrak{p}_r$. If $I\mathfrak{p}_1 \ldots \mathfrak{p}_{r-1} \subset \mathfrak{p}_r$ then $I \subset \mathfrak{p}_r$ a contradiction. Pick $y \in I\mathfrak{p}_1 \ldots \mathfrak{p}_{r-1}$ homogeneous and $y \notin \mathfrak{p}_r$. Then $x^{\deg(y)} + y^{\deg(x)}$ works. □

00JT **Lemma 57.7.** *Let S be a graded ring. Let $\mathfrak{p} \subset S$ be a prime. Let $\mathfrak{q}$ be the homogeneous ideal of S generated by the homogeneous elements of $\mathfrak{p}$. Then $\mathfrak{q}$ is a prime ideal of S.*

Proof. Suppose $f, g \in S$ are such that $fg \in \mathfrak{q}$. Let f_d (resp. g_e) be the homogeneous part of f (resp. g) of degree d (resp. e). Assume d, e are maxima such that $f_d \neq 0$ and $g_e \neq 0$. By assumption we can write $fg = \sum a_i f_i$ with $f_i \in \mathfrak{p}$ homogeneous.

Say $\deg(f_i) = d_i$. Then $f_d g_e = \sum a'_i f_i$ with a'_i to homogeneous par of degree $d + e - d_i$ of a_i (or 0 if $d + e - d_i < 0$). Hence $f_d \in \mathfrak{p}$ or $g_e \in \mathfrak{p}$. Hence $f_d \in \mathfrak{q}$ or $g_e \in \mathfrak{q}$. In the first case replace f by $f - f_d$, in the second case replace g by $g - g_e$. Then still $fg \in \mathfrak{q}$ but the discrete invariant $d + e$ has been decreased. Thus we may continue in this fashion until either f or g is zero. This clearly shows that $fg \in \mathfrak{q}$ implies either $f \in \mathfrak{q}$ or $g \in \mathfrak{q}$ as desired. □

00JU **Lemma 57.8.** *Let S be a graded ring.*

1. *Any minimal prime of S is a homogeneous ideal of S.*
2. *Given a homogeneous ideal $I \subset S$ any minimal prime over I is homogeneous.*

Proof. The first assertion holds because the prime $\mathfrak{q}$ constructed in Lemma 57.7 satisfies $\mathfrak{q} \subset \mathfrak{p}$. The second because we may consider S/I and apply the first part. □

07Z2 **Lemma 57.9.** *Let R be a ring. Let S be a graded R-algebra. Let $f \in S_+$ be homogeneous. Assume that S is of finite type over R. Then*

1. *the ring $S_{(f)}$ is of finite type over R, and*
2. *for any finite graded S-module M the module $M_{(f)}$ is a finite $S_{(f)}$-module.*

Proof. Choose $f_1, \ldots, f_n \in S$ which generate S as an R-algebra. We may assume that each f_i is homogeneous (by decomposing each f_i into its homogeneous components). An element of $S_{(f)}$ is a sum of the form

$$\sum\nolimits_{e \deg(f) = \sum e_i \deg(f_i)} \lambda_{e_1 \ldots e_n} f_1^{e_1} \ldots f_n^{e_n} / f^e$$

with $\lambda_{e_1 \ldots e_n} \in R$. Thus $S_{(f)}$ is generated as an R-algebra by the $f_1^{e_1} \ldots f_n^{e_n}/f^e$ with the property that $e \deg(f) = \sum e_i \deg(f_i)$. If $e_i \geq \deg(f)$ then we can write this as

$$f_1^{e_1} \ldots f_n^{e_n}/f^e = f_i^{\deg(f)}/f^{\deg(f_i)} \cdot f_1^{e_1} \ldots f_i^{e_i - \deg(f)} \ldots f_n^{e_n}/f^{e - \deg(f_i)}$$

Thus we only need the elements $f_i^{\deg(f)}/f^{\deg(f_i)}$ as well as the elements $f_1^{e_1} \ldots f_n^{e_n}/f^e$ with $e \deg(f) = \sum e_i \deg(f_i)$ and $e_i < \deg(f)$. This is a finite list and we see that (1) is true.

To see (2) suppose that M is generated by homogeneous elements $x_1, \ldots, x_m$. Then arguing as above we find that $M_{(f)}$ is generated as an $S_{(f)}$-module by the finite list of elements of the form $f_1^{e_1} \ldots f_n^{e_n} x_j / f^e$ with $e \deg(f) = \sum e_i \deg(f_i) + \deg(x_j)$ and $e_i < \deg(f)$. □

052N **Lemma 57.10.** *Let R be a ring. Let R' be a finite type R-algebra, and let M be a finite R'-module. There exists a graded R-algebra S, a graded S-module N and an element $f \in S$ homogeneous of degree 1 such that*

1. *$R' \cong S_{(f)}$ and $M \cong N_{(f)}$ (as modules),*
2. *$S_0 = R$ and S is generated by finitely many elements of degree 1 over R, and*
3. *N is a finite S-module.*

Proof. We may write $R' = R[x_1, \ldots, x_n]/I$ for some ideal I. For an element $g \in R[x_1, \ldots, x_n]$ denote $\tilde{g} \in R[X_0, \ldots, X_n]$ the element homogeneous of minimal degree such that $g = \tilde{g}(1, x_1, \ldots, x_n)$. Let $\tilde{I} \subset R[X_0, \ldots, X_n]$ generated by all elements $\tilde{g}$, $g \in I$. Set $S = R[X_0, \ldots, X_n]/\tilde{I}$ and denote f the image of X_0 in S. By construction we have an isomorphism

$$S_{(f)} \longrightarrow R', \quad X_i/X_0 \longmapsto x_i.$$

To do the same thing with the module M we choose a presentation

$$M = (R')^{\oplus r} / \sum\nolimits_{j \in J} R' k_j$$

with $k_j = (k_{1j}, \ldots, k_{rj})$. Let $d_{ij} = \deg(\tilde{k}_{ij})$. Set $d_j = \max\{d_{ij}\}$. Set $K_{ij} = X_0^{d_j - d_{ij}} \tilde{k}_{ij}$ which is homogeneous of degree d_j. With this notation we set

$$N = \text{Coker}\left(\bigoplus\nolimits_{j \in J} S(-d_j) \xrightarrow{(K_{ij})} S^{\oplus r} \right)$$

which works. Some details omitted. $\square$

58. Noetherian graded rings

00JV A bit of theory on Noetherian graded rings including some material on Hilbert polynomials.

07Z4 **Lemma 58.1.** *Let S be a graded ring. A set of homogeneous elements $f_i \in S_+$ generates S as an algebra over S_0 if and only if they generate S_+ as an ideal of S.*

Proof. If the f_i generate S as an algebra over S_0 then every element in S_+ is a polynomial without constant term in the f_i and hence S_+ is generated by the f_i as an ideal. Conversely, suppose that $S_+ = \sum S f_i$. We will prove that any element f of S can be written as a polynomial in the f_i with coefficients in S_0. It suffices to do this for homogeneous elements. Say f has degree d. Then we may perform induction on d. The case $d = 0$ is immediate. If $d > 0$ then $f \in S_+$ hence we can write $f = \sum g_i f_i$ for some $g_i \in S$. As S is graded we can replace g_i by its homogeneous component of degree $d - \deg(f_i)$. By induction we see that each g_i is a polynomial in the f_i and we win. $\square$

00JW **Lemma 58.2.** *A graded ring S is Noetherian if and only if S_0 is Noetherian and S_+ is finitely generated as an ideal of S.*

Proof. It is clear that if S is Noetherian then $S_0 = S/S_+$ is Noetherian and S_+ is finitely generated. Conversely, assume S_0 is Noetherian and S_+ finitely generated as an ideal of S. Pick generators $S_+ = (f_1, \ldots, f_n)$. By decomposing the f_i into homogeneous pieces we may assume each f_i is homogeneous. By Lemma 58.1 we see that $S_0[X_1, \ldots X_n] \to S$ sending X_i to f_i is surjective. Thus S is Noetherian by Lemma 31.1. $\square$

00JX **Definition 58.3.** Let A be an abelian group. We say that a function $f : n \mapsto f(n) \in A$ defined for all sufficient large integers n is a *numerical polynomial* if there exists $r \geq 0$, elements $a_0, \ldots, a_r \in A$ such that

$$f(n) = \sum\nolimits_{i=0}^{r} \binom{n}{i} a_i$$

for all $n \gg 0$.

The reason for using the binomial coefficients is the elementary fact that any polynomial $P \in \mathbf{Q}[T]$ all of whose values at integer points are integers, is equal to a sum $P(T) = \sum a_i \binom{T}{i}$ with $a_i \in \mathbf{Z}$. Note that in particular the expressions $\binom{T+1}{i+1}$ are of this form.

00JY **Lemma 58.4.** *If $A \to A'$ is a homomorphism of abelian groups and if $f : n \mapsto f(n) \in A$ is a numerical polynomial, then so is the composition.*

Proof. This is immediate from the definitions. □

00JZ **Lemma 58.5.** *Suppose that $f : n \mapsto f(n) \in A$ is defined for all n sufficiently large and suppose that $n \mapsto f(n) - f(n-1)$ is a numerical polynomial. Then f is a numerical polynomial.*

Proof. Let $f(n) - f(n-1) = \sum_{i=0}^{r} \binom{n}{i} a_i$ for all $n \gg 0$. Set $g(n) = f(n) - \sum_{i=0}^{r} \binom{n+1}{i+1} a_i$. Then $g(n) - g(n-1) = 0$ for all $n \gg 0$. Hence g is eventually constant, say equal to a_{-1}. We leave it to the reader to show that $a_{-1} + \sum_{i=0}^{r} \binom{n+1}{i+1} a_i$ has the required shape (see remark above the lemma). □

00K0 **Lemma 58.6.** *If M is a finitely generated graded S-module, and if S is finitely generated over S_0, then each M_n is a finite S_0-module.*

Proof. Suppose the generators of M are m_i and the generators of S are f_i. By taking homogeneous components we may assume that the m_i and the f_i are homogeneous and we may assume $f_i \in S_+$. In this case it is clear that each M_n is generated over S_0 by the "monomials" $\prod f_i^{e_i} m_j$ whose degree is n. □

00K1 **Proposition 58.7.** *Suppose that S is a Noetherian graded ring and M a finite graded S-module. Consider the function*

$$\mathbf{Z} \longrightarrow K'_0(S_0), \quad n \longmapsto [M_n]$$

see Lemma 58.6. If S_+ is generated by elements of degree 1, then this function is a numerical polynomial.

Proof. We prove this by induction on the minimal number of generators of S_1. If this number is 0, then $M_n = 0$ for all $n \gg 0$ and the result holds. To prove the induction step, let $x \in S_1$ be one of a minimal set of generators, such that the induction hypothesis applies to the graded ring $S/(x)$.

First we show the result holds if x is nilpotent on M. This we do by induction on the minimal integer r such that $x^r M = 0$. If $r = 1$, then M is a module over S/xS and the result holds (by the other induction hypothesis). If $r > 1$, then we can find a short exact sequence $0 \to M' \to M \to M'' \to 0$ such that the integers r', r'' are strictly smaller than r. Thus we know the result for M'' and M'. Hence we get the result for M because of the relation $[M_d] = [M'_d] + [M''_d]$ in $K'_0(S_0)$.

If x is not nilpotent on M, let $M' \subset M$ be the largest submodule on which x is nilpotent. Consider the exact sequence $0 \to M' \to M \to M/M' \to 0$ we see again it suffices to prove the result for M/M'. In other words we may assume that multiplication by x is injective.

Let $\overline{M} = M/xM$. Note that the map $x : M \to M$ is *not* a map of graded S-modules, since it does not map M_d into M_d. Namely, for each d we have the following short exact sequence

$$0 \to M_d \xrightarrow{x} M_{d+1} \to \overline{M}_{d+1} \to 0$$

This proves that $[M_{d+1}] - [M_d] = [\overline{M}_{d+1}]$. Hence we win by Lemma 58.5. □

02CD **Remark 58.8.** If S is still Noetherian but S is not generated in degree 1, then the function associated to a graded S-module is a periodic polynomial (i.e., it is a numerical polynomial on the congruence classes of integers modulo n for some n).

00K2 **Example 58.9.** Suppose that $S = k[X_1, \ldots, X_d]$. By Example 55.2 we may identify $K_0(k) = K'_0(k) = \mathbf{Z}$. Hence any finitely generated graded $k[X_1, \ldots, X_d]$-module gives rise to a numerical polynomial $n \mapsto \dim_k(M_n)$.

00K3 **Lemma 58.10.** *Let k be a field. Suppose that $I \subset k[X_1, \ldots, X_d]$ is a nonzero graded ideal. Let $M = k[X_1, \ldots, X_d]/I$. Then the numerical polynomial $n \mapsto \dim_k(M_n)$ (see Example 58.9) has degree $< d - 1$ (or is zero if $d = 1$).*

Proof. The numerical polynomial associated to the graded module $k[X_1, \ldots, X_d]$ is $n \mapsto \binom{n-1+d}{d-1}$. For any nonzero homogeneous $f \in I$ of degree e and any degree $n >> e$ we have $I_n \supset f \cdot k[X_1, \ldots, X_d]_{n-e}$ and hence $\dim_k(I_n) \geq \binom{n-e-1+d}{d-1}$. Hence $\dim_k(M_n) \leq \binom{n-1+d}{d-1} - \binom{n-e-1+d}{d-1}$. We win because the last expression has degree $< d - 1$ (or is zero if $d = 1$). $\square$

59. Noetherian local rings

00K4 In all of this section $(R, \mathfrak{m}, \kappa)$ is a Noetherian local ring. We develop some theory on Hilbert functions of modules in this section. Let M be a finite R-module. We define the *Hilbert function* of M to be the function

$$\varphi_M : n \longmapsto \text{length}_R(\mathfrak{m}^n M/\mathfrak{m}^{n+1}M)$$

defined for all integers $n \geq 0$. Another important invariant is the function

$$\chi_M : n \longmapsto \text{length}_R(M/\mathfrak{m}^{n+1}M)$$

defined for all integers $n \geq 0$. Note that we have by Lemma 52.3 that

$$\chi_M(n) = \sum\nolimits_{i=0}^{n} \varphi_M(i).$$

There is a variant of this construction which uses an ideal of definition.

07DU **Definition 59.1.** Let $(R, \mathfrak{m})$ be a local Noetherian ring. An ideal $I \subset R$ such that $\sqrt{I} = \mathfrak{m}$ is called *an ideal of definition of R*.

Let $I \subset R$ be an ideal of definition. Because R is Noetherian this means that $\mathfrak{m}^r \subset I$ for some r, see Lemma 32.5. Hence any finite R-module annihilated by a power of I has a finite length, see Lemma 52.8. Thus it makes sense to define

$$\varphi_{I,M}(n) = \text{length}_R(I^n M/I^{n+1}M) \quad \text{and} \quad \chi_{I,M}(n) = \text{length}_R(M/I^{n+1}M)$$

for all $n \geq 0$. Again we have that

$$\chi_{I,M}(n) = \sum\nolimits_{i=0}^{n} \varphi_{I,M}(i).$$

00K5 **Lemma 59.2.** *Suppose that $M' \subset M$ are finite R-modules with finite length quotient. Then there exists a constants c_1, c_2 such that for all $n \geq c_2$ we have*

$$c_1 + \chi_{I,M'}(n - c_2) \leq \chi_{I,M}(n) \leq c_1 + \chi_{I,M'}(n)$$

Proof. Since M/M' has finite length there is a $c_2 \geq 0$ such that $I^{c_2}M \subset M'$. Let $c_1 = \text{length}_R(M/M')$. For $n \geq c_2$ we have

$$\begin{aligned} \chi_{I,M}(n) &= \text{length}_R(M/I^{n+1}M) \\ &= c_1 + \text{length}_R(M'/I^{n+1}M) \\ &\leq c_1 + \text{length}_R(M'/I^{n+1}M') \\ &= c_1 + \chi_{I,M'}(n) \end{aligned}$$

On the other hand, since $I^{c_2}M \subset M'$, we have $I^nM \subset I^{n-c_2}M'$ for $n \geq c_2$. Thus for $n \geq c_2$ we get

$$\begin{aligned} \chi_{I,M}(n) &= \text{length}_R(M/I^{n+1}M) \\ &= c_1 + \text{length}_R(M'/I^{n+1}M) \\ &\geq c_1 + \text{length}_R(M'/I^{n+1-c_2}M') \\ &= c_1 + \chi_{I,M'}(n - c_2) \end{aligned}$$

which finishes the proof. $\square$

00K6 **Lemma 59.3.** *Suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of finite R-modules. Then there exists a submodule $N \subset M'$ with finite colength l and $c \geq 0$ such that*

$$\chi_{I,M}(n) = \chi_{I,M''}(n) + \chi_{I,N}(n - c) + l$$

and

$$\varphi_{I,M}(n) = \varphi_{I,M''}(n) + \varphi_{I,N}(n - c)$$

for all $n \geq c$.

Proof. Note that $M/I^nM \to M''/I^nM''$ is surjective with kernel $M'/M' \cap I^nM$. By the Artin-Rees Lemma 51.2 there exists a constant c such that $M' \cap I^nM = I^{n-c}(M' \cap I^cM)$. Denote $N = M' \cap I^cM$. Note that $I^cM' \subset N \subset M'$. Hence $\text{length}_R(M'/M' \cap I^nM) = \text{length}_R(M'/N) + \text{length}_R(N/I^{n-c}N)$ for $n \geq c$. From the short exact sequence

$$0 \to M'/M' \cap I^nM \to M/I^nM \to M''/I^nM'' \to 0$$

and additivity of lengths (Lemma 52.3) we obtain the equality

$$\chi_{I,M}(n - 1) = \chi_{I,M''}(n - 1) + \chi_{I,N}(n - c - 1) + \text{length}_R(M'/N)$$

for $n \geq c$. We have $\varphi_{I,M}(n) = \chi_{I,M}(n) - \chi_{I,M}(n-1)$ and similarly for the modules M'' and N. Hence we get $\varphi_{I,M}(n) = \varphi_{I,M''}(n) + \varphi_{I,N}(n - c)$ for $n \geq c$. $\square$

00K7 **Lemma 59.4.** *Suppose that I, I' are two ideals of definition for the Noetherian local ring R. Let M be a finite R-module. There exists a constant a such that $\chi_{I,M}(n) \leq \chi_{I',M}(an)$ for $n \geq 1$.*

Proof. There exists an integer $c \geq 1$ such that $(I')^c \subset I$. Hence we get a surjection $M/(I')^{c(n+1)}M \to M/I^{n+1}M$. Whence the result with $a = 2c - 1$. $\square$

00K8 **Proposition 59.5.** *Let R be a Noetherian local ring. Let M be a finite R-module. Let $I \subset R$ be an ideal of definition. The Hilbert function $\varphi_{I,M}$ and the function $\chi_{I,M}$ are numerical polynomials.*

Proof. Consider the graded ring $S = R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \ldots = \bigoplus_{d \geq 0} I^d/I^{d+1}$. Consider the graded S-module $N = M/IM \oplus IM/I^2M \oplus \ldots = \bigoplus_{d \geq 0} I^dM/I^{d+1}M$. This pair (S, N) satisfies the hypotheses of Proposition 58.7. Hence the result for $\varphi_{I,M}$ follows from that proposition and Lemma 55.1. The result for $\chi_{I,M}$ follows from this and Lemma 58.5. ☐

09CA **Definition 59.6.** Let R be a Noetherian local ring. Let M be a finite R-module. The *Hilbert polynomial* of M over R is the element $P(t) \in \mathbf{Q}[t]$ such that $P(n) = \varphi_M(n)$ for $n \gg 0$.

By Proposition 59.5 we see that the Hilbert polynomial exists.

00K9 **Lemma 59.7.** *Let R be a Noetherian local ring. Let M be a finite R-module.*

1. *The degree of the numerical polynomial $\varphi_{I,M}$ is independent of the ideal of definition I.*
2. *The degree of the numerical polynomial $\chi_{I,M}$ is independent of the ideal of definition I.*

Proof. Part (2) follows immediately from Lemma 59.4. Part (1) follows from (2) because $\varphi_{I,M}(n) = \chi_{I,M}(n) - \chi_{I,M}(n-1)$ for $n \geq 1$. ☐

00KA **Definition 59.8.** Let R be a local Noetherian ring and M a finite R-module. We denote $d(M)$ the element of $\{-\infty, 0, 1, 2, \ldots\}$ defined as follows:

1. If $M = 0$ we set $d(M) = -\infty$,
2. if $M \neq 0$ then $d(M)$ is the degree of the numerical polynomial χ_M.

If $\mathfrak{m}^nM \neq 0$ for all n, then we see that $d(M)$ is the degree $+1$ of the Hilbert polynomial of M.

00KB **Lemma 59.9.** *Let R be a Noetherian local ring. Let $I \subset R$ be an ideal of definition. Let M be a finite R-module which does not have finite length. If $M' \subset M$ is a submodule with finite colength, then $\chi_{I,M} - \chi_{I,M'}$ is a polynomial of degree $<$ degree of either polynomial.*

Proof. Follows from Lemma 59.2 by elementary calculus. ☐

00KC **Lemma 59.10.** *Let R be a Noetherian local ring. Let $I \subset R$ be an ideal of definition. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of finite R-modules. Then*

1. *if M' does not have finite length, then $\chi_{I,M} - \chi_{I,M''} - \chi_{I,M'}$ is a numerical polynomial of degree $<$ the degree of $\chi_{I,M'}$,*
2. $\max\{\deg(\chi_{I,M'}), \deg(\chi_{I,M''})\} = \deg(\chi_{I,M})$, *and*
3. $\max\{d(M'), d(M'')\} = d(M)$,

Proof. We first prove (1). Let $N \subset M'$ be as in Lemma 59.3. By Lemma 59.9 the numerical polynomial $\chi_{I,M'} - \chi_{I,N}$ has degree $<$ the common degree of $\chi_{I,M'}$ and $\chi_{I,N}$. By Lemma 59.3 the difference

$$\chi_{I,M}(n) - \chi_{I,M''}(n) - \chi_{I,N}(n-c)$$

is constant for $n \gg 0$. By elementary calculus the difference $\chi_{I,N}(n) - \chi_{I,N}(n-c)$ has degree $<$ the degree of $\chi_{I,N}$ which is bigger than zero (see above). Putting everything together we obtain (1).

Note that the leading coefficients of $\chi_{I,M'}$ and $\chi_{I,M''}$ are nonnegative. Thus the degree of $\chi_{I,M'} + \chi_{I,M''}$ is equal to the maximum of the degrees. Thus if M' does not have finite length, then (2) follows from (1). If M' does have finite length, then $I^nM \to I^nM''$ is an isomorphism for all $n \gg 0$ by Artin-Rees (Lemma 51.2). Thus $M/I^nM \to M''/I^nM''$ is a surjection with kernel M' for $n \gg 0$ and we see that $\chi_{I,M}(n) - \chi_{I,M''}(n) = \text{length}(M')$ for all $n \gg 0$. Thus (2) holds in this case also.

Proof of (3). This follows from (2) except if one of M, M', or M'' is zero. We omit the proof in these special cases. □

60. Dimension

00KD Please compare with Topology, Section 10.

0GIE **Definition 60.1.** Let R be a ring. A *chain of prime ideals* is a sequence $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n$ of prime ideals of R such that $\mathfrak{p}_i \neq \mathfrak{p}_{i+1}$ for $i = 0, \ldots, n-1$. The *length* of this chain of prime ideals is n.

Recall that we have an inclusion reversing bijection between prime ideals of a ring R and irreducible closed subsets of $\text{Spec}(R)$, see Lemma 26.1.

00KE **Definition 60.2.** The *Krull dimension* of the ring R is the Krull dimension of the topological space $\text{Spec}(R)$, see Topology, Definition 10.1. In other words it is the supremum of the integers $n \geq 0$ such that R has a chain of prime ideals

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n, \quad \mathfrak{p}_i \neq \mathfrak{p}_{i+1}.$$

of length n.

00KF **Definition 60.3.** The *height* of a prime ideal $\mathfrak{p}$ of a ring R is the dimension of the local ring $R_{\mathfrak{p}}$.

00KG **Lemma 60.4.** *The Krull dimension of R is the supremum of the heights of its (maximal) primes.*

Proof. This is so because we can always add a maximal ideal at the end of a chain of prime ideals. □

00KH **Lemma 60.5.** *A Noetherian ring of dimension 0 is Artinian. Conversely, any Artinian ring is Noetherian of dimension zero.*

Proof. By Lemma 31.5 the space $\text{Spec}(R)$ is Noetherian. By Topology, Lemma 9.2 we see that $\text{Spec}(R)$ has finitely many irreducible components, say $\text{Spec}(R) = Z_1 \cup \ldots \cup Z_r$. According to Lemma 26.1, each $Z_i = V(\mathfrak{p}_i)$ with $\mathfrak{p}_i$ a minimal ideal. Since the dimension is 0 these $\mathfrak{p}_i$ are also maximal. Thus $\text{Spec}(R)$ is the discrete topological space with elements $\mathfrak{p}_i$. All elements f of the Jacobson radical $I = \cap \mathfrak{p}_i$ are nilpotent since otherwise R_f would not be the zero ring and we would have another prime. Since I is finitely generated we conclude that I is nilpotent, Lemma 32.5. By Lemma 53.5 R is the product of its local rings. By Lemma 52.8 each of these has finite length over R. Hence we conclude that R is Artinian by Lemma 53.6.

If R is Artinian then by Lemma 53.6 it is Noetherian. All of its primes are maximal by a combination of Lemmas 53.3, 53.4 and 53.5. □

In the following we will use the invariant $d(-)$ defined in Definition 59.8. Here is a warm up lemma.

00KI **Lemma 60.6.** *Let R be a Noetherian local ring. Then $\dim(R) = 0 \Leftrightarrow d(R) = 0$.*

Proof. This is because $d(R) = 0$ if and only if R has finite length as an R-module. See Lemma 53.6. $\square$

00KJ **Proposition 60.7.** *Let R be a ring. The following are equivalent:*

(1) *R is Artinian,*
(2) *R is Noetherian and $\dim(R) = 0$,*
(3) *R has finite length as a module over itself,*
(4) *R is a finite product of Artinian local rings,*
(5) *R is Noetherian and $\text{Spec}(R)$ is a finite discrete topological space,*
(6) *R is a finite product of Noetherian local rings of dimension 0,*
(7) *R is a finite product of Noetherian local rings R_i with $d(R_i) = 0$,*
(8) *R is a finite product of Noetherian local rings R_i whose maximal ideals are nilpotent,*
(9) *R is Noetherian, has finitely many maximal ideals and its Jacobson radical ideal is nilpotent, and*
(10) *R is Noetherian and there are no strict inclusions among its primes.*

Proof. This is a combination of Lemmas 53.5, 53.6, 60.5, and 60.6. $\square$

00KK **Lemma 60.8.** *Let R be a local Noetherian ring. The following are equivalent:*

00KL (1) $\dim(R) = 1$,
00KM (2) $d(R) = 1$,
00KN (3) *there exists an $x \in \mathfrak{m}$, x not nilpotent such that $V(x) = \{\mathfrak{m}\}$,*
00KO (4) *there exists an $x \in \mathfrak{m}$, x not nilpotent such that $\mathfrak{m} = \sqrt{(x)}$, and*
00KP (5) *there exists an ideal of definition generated by 1 element, and no ideal of definition is generated by 0 elements.*

Proof. First, assume that $\dim(R) = 1$. Let $\mathfrak{p}_i$ be the minimal primes of R. Because the dimension is 1 the only other prime of R is $\mathfrak{m}$. According to Lemma 31.6 there are finitely many. Hence we can find $x \in \mathfrak{m}$, $x \notin \mathfrak{p}_i$, see Lemma 15.2. Thus the only prime containing x is $\mathfrak{m}$ and hence (3).

If (3) then $\mathfrak{m} = \sqrt{(x)}$ by Lemma 17.2, and hence (4). The converse is clear as well. The equivalence of (4) and (5) follows from directly the definitions.

Assume (5). Let $I = (x)$ be an ideal of definition. Note that I^n/I^{n+1} is a quotient of R/I via multiplication by x^n and hence $\text{length}_R(I^n/I^{n+1})$ is bounded. Thus $d(R) = 0$ or $d(R) = 1$, but $d(R) = 0$ is excluded by the assumption that 0 is not an ideal of definition.

Assume (2). To get a contradiction, assume there exist primes $\mathfrak{p} \subset \mathfrak{q} \subset \mathfrak{m}$, with both inclusions strict. Pick some ideal of definition $I \subset R$. We will repeatedly use Lemma 59.10. First of all it implies, via the exact sequence $0 \to \mathfrak{p} \to R \to R/\mathfrak{p} \to 0$, that $d(R/\mathfrak{p}) \leq 1$. But it clearly cannot be zero. Pick $x \in \mathfrak{q}$, $x \notin \mathfrak{p}$. Consider the short exact sequence

$$0 \to R/\mathfrak{p} \to R/\mathfrak{p} \to R/(xR + \mathfrak{p}) \to 0.$$

This implies that $\chi_{I,R/\mathfrak{p}} - \chi_{I,R/\mathfrak{p}} - \chi_{I,R/(xR+\mathfrak{p})} = -\chi_{I,R/(xR+\mathfrak{p})}$ has degree < 1. In other words, $d(R/(xR + \mathfrak{p})) = 0$, and hence $\dim(R/(xR + \mathfrak{p})) = 0$, by Lemma 60.6. But $R/(xR + \mathfrak{p})$ has the distinct primes $\mathfrak{q}/(xR + \mathfrak{p})$ and $\mathfrak{m}/(xR + \mathfrak{p})$ which gives the desired contradiction. $\square$

00KQ **Proposition 60.9.** *Let R be a local Noetherian ring. Let $d \geq 0$ be an integer. The following are equivalent:*

00KR (1) $\dim(R) = d$,

00KS (2) $d(R) = d$,

00KT (3) *there exists an ideal of definition generated by d elements, and no ideal of definition is generated by fewer than d elements.*

Proof. This proof is really just the same as the proof of Lemma 60.8. We will prove the proposition by induction on d. By Lemmas 60.6 and 60.8 we may assume that $d > 1$. Denote the minimal number of generators for an ideal of definition of R by $d'(R)$. We will prove the inequalities $\dim(R) \geq d'(R) \geq d(R) \geq \dim(R)$, and hence they are all equal.

First, assume that $\dim(R) = d$. Let $\mathfrak{p}_i$ be the minimal primes of R. According to Lemma 31.6 there are finitely many. Hence we can find $x \in \mathfrak{m}$, $x \not\in \mathfrak{p}_i$, see Lemma 15.2. Note that every maximal chain of primes starts with some $\mathfrak{p}_i$, hence the dimension of R/xR is at most $d-1$. By induction there are $x_2, \ldots, x_d$ which generate an ideal of definition in R/xR. Hence R has an ideal of definition generated by (at most) d elements.

Assume $d'(R) = d$. Let $I = (x_1, \ldots, x_d)$ be an ideal of definition. Note that I^n/I^{n+1} is a quotient of a direct sum of $\binom{d+n-1}{d-1}$ copies R/I via multiplication by all degree n monomials in $x_1, \ldots, x_n$. Hence $\text{length}_R(I^n/I^{n+1})$ is bounded by a polynomial of degree $d-1$. Thus $d(R) \leq d$.

Assume $d(R) = d$. Consider a chain of primes $\mathfrak{p} \subset \mathfrak{q} \subset \mathfrak{q}_2 \subset \ldots \subset \mathfrak{q}_e = \mathfrak{m}$, with all inclusions strict, and $e \geq 2$. Pick some ideal of definition $I \subset R$. We will repeatedly use Lemma 59.10. First of all it implies, via the exact sequence $0 \to \mathfrak{p} \to R \to R/\mathfrak{p} \to 0$, that $d(R/\mathfrak{p}) \leq d$. But it clearly cannot be zero. Pick $x \in \mathfrak{q}$, $x \not\in \mathfrak{p}$. Consider the short exact sequence

$$0 \to R/\mathfrak{p} \to R/\mathfrak{p} \to R/(xR + \mathfrak{p}) \to 0.$$

This implies that $\chi_{I,R/\mathfrak{p}} - \chi_{I,R/\mathfrak{p}} - \chi_{I,R/(xR+\mathfrak{p})} = -\chi_{I,R/(xR+\mathfrak{p})}$ has degree $< d$. In other words, $d(R/(xR + \mathfrak{p})) \leq d - 1$, and hence $\dim(R/(xR + \mathfrak{p})) \leq d - 1$, by induction. Now $R/(xR+\mathfrak{p})$ has the chain of prime ideals $\mathfrak{q}/(xR+\mathfrak{p}) \subset \mathfrak{q}_2/(xR+\mathfrak{p}) \subset \ldots \subset \mathfrak{q}_e/(xR + \mathfrak{p})$ which gives $e - 1 \leq d - 1$. Since we started with an arbitrary chain of primes this proves that $\dim(R) \leq d(R)$.

Reading back the reader will see we proved the circular inequalities as desired. $\square$

Let $(R, \mathfrak{m})$ be a Noetherian local ring. From the above it is clear that $\mathfrak{m}$ cannot be generated by fewer than $\dim(R)$ variables. By Nakayama's Lemma 20.1 the minimal number of generators of $\mathfrak{m}$ equals $\dim_{\kappa(\mathfrak{m})} \mathfrak{m}/\mathfrak{m}^2$. Hence we have the following fundamental inequality

$$\dim(R) \leq \dim_{\kappa(\mathfrak{m})} \mathfrak{m}/\mathfrak{m}^2.$$

It turns out that the rings where equality holds have a lot of good properties. They are called regular local rings.

00KU **Definition 60.10.** Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d.

(1) A *system of parameters of* R is a sequence of elements $x_1, \ldots, x_d \in \mathfrak{m}$ which generates an ideal of definition of R,

(2) if there exist $x_1, \ldots, x_d \in \mathfrak{m}$ such that $\mathfrak{m} = (x_1, \ldots, x_d)$ then we call R a *regular local ring* and $x_1, \ldots, x_d$ a *regular system of parameters*.

The following lemmas are clear from the proofs of the lemmas and proposition above, but we spell them out so we have convenient references.

00KV **Lemma 60.11.** *Let R be a Noetherian ring. Let $x \in R$.*

(1) *If $\mathfrak{p}$ is minimal over (x) then the height of $\mathfrak{p}$ is 0 or 1.*
(2) *If $\mathfrak{p}, \mathfrak{q} \in \text{Spec}(R)$ and $\mathfrak{q}$ is minimal over $(\mathfrak{p}, x)$, then there is no prime strictly between $\mathfrak{p}$ and $\mathfrak{q}$.*

Proof. Proof of (1). If $\mathfrak{p}$ is minimal over x, then the only prime ideal of $R_\mathfrak{p}$ containing x is the maximal ideal $\mathfrak{p}R_\mathfrak{p}$. This is true because the primes of $R_\mathfrak{p}$ correspond 1-to-1 with the primes of R contained in $\mathfrak{p}$, see Lemma 17.5. Hence Lemma 60.8 shows $\dim(R_\mathfrak{p}) = 1$ if x is not nilpotent in $R_\mathfrak{p}$. Of course, if x is nilpotent in $R_\mathfrak{p}$ the argument gives that $\mathfrak{p}R_\mathfrak{p}$ is the only prime ideal and we see that the height is 0.

Proof of (2). By part (1) we see that $\mathfrak{q}/\mathfrak{p}$ is a prime of height 1 or 0 in $R/\mathfrak{p}$. This immediately implies there cannot be a prime strictly between $\mathfrak{p}$ and $\mathfrak{q}$. $\square$

0BBZ **Lemma 60.12.** *Let R be a Noetherian ring. Let $f_1, \ldots, f_r \in R$.*

(1) *If $\mathfrak{p}$ is minimal over $(f_1, \ldots, f_r)$ then the height of $\mathfrak{p}$ is $\leq r$.*
(2) *If $\mathfrak{p}, \mathfrak{q} \in \text{Spec}(R)$ and $\mathfrak{q}$ is minimal over $(\mathfrak{p}, f_1, \ldots, f_r)$, then every chain of primes between $\mathfrak{p}$ and $\mathfrak{q}$ has length at most r.*

Proof. Proof of (1). If $\mathfrak{p}$ is minimal over $f_1, \ldots, f_r$, then the only prime ideal of $R_\mathfrak{p}$ containing $f_1, \ldots, f_r$ is the maximal ideal $\mathfrak{p}R_\mathfrak{p}$. This is true because the primes of $R_\mathfrak{p}$ correspond 1-to-1 with the primes of R contained in $\mathfrak{p}$, see Lemma 17.5. Hence Proposition 60.9 shows $\dim(R_\mathfrak{p}) \leq r$.

Proof of (2). By part (1) we see that $\mathfrak{q}/\mathfrak{p}$ is a prime of height $\leq r$. This immediately implies the statement about chains of primes between $\mathfrak{p}$ and $\mathfrak{q}$. $\square$

00KW **Lemma 60.13.** *Suppose that R is a Noetherian local ring and $x \in \mathfrak{m}$ an element of its maximal ideal. Then $\dim R \leq \dim R/xR + 1$. If x is not contained in any of the minimal primes of R then equality holds. (For example if x is a nonzerodivisor.)*

Proof. If $x_1, \ldots, x_{\dim R/xR} \in R$ map to elements of R/xR which generate an ideal of definition for R/xR, then $x, x_1, \ldots, x_{\dim R/xR}$ generate an ideal of definition for R. Hence the inequality by Proposition 60.9. On the other hand, if x is not contained in any minimal prime of R, then the chains of primes in R/xR all give rise to chains in R which are at least one step away from being maximal. $\square$

02IE **Lemma 60.14.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Suppose $x_1, \ldots, x_d \in \mathfrak{m}$ generate an ideal of definition and $d = \dim(R)$. Then $\dim(R/(x_1, \ldots, x_i)) = d - i$ for all $i = 1, \ldots, d$.*

Proof. Follows either from the proof of Proposition 60.9, or by using induction on d and Lemma 60.13. $\square$

61. Applications of dimension theory

02IF We can use the results on dimension to prove certain rings have infinite spectra and to produce more Jacobson rings.

02IG **Lemma 61.1.** *Let R be a Noetherian local domain of dimension ≥ 2. A nonempty open subset $U \subset \text{Spec}(R)$ is infinite.*

Proof. To get a contradiction, assume that $U \subset \text{Spec}(R)$ is finite. In this case $(0) \in U$ and $\{(0)\}$ is an open subset of U (because the complement of $\{(0)\}$ is the union of the closures of the other points). Thus we may assume $U = \{(0)\}$. Let $\mathfrak{m} \subset R$ be the maximal ideal. We can find an $x \in \mathfrak{m}$, $x \neq 0$ such that $V(x) \cup U = \text{Spec}(R)$. In other words we see that $D(x) = \{(0)\}$. In particular we see that $\dim(R/xR) = \dim(R) - 1 \geq 1$, see Lemma 60.13. Let $\overline{y}_2, \ldots, \overline{y}_{\dim(R)} \in R/xR$ generate an ideal of definition of R/xR, see Proposition 60.9. Choose lifts $y_2, \ldots, y_{\dim(R)} \in R$, so that $x, y_2, \ldots, y_{\dim(R)}$ generate an ideal of definition in R. This implies that $\dim(R/(y_2)) = \dim(R) - 1$ and $\dim(R/(y_2, x)) = \dim(R) - 2$, see Lemma 60.14. Hence there exists a prime $\mathfrak{p}$ containing y_2 but not x. This contradicts the fact that $D(x) = \{(0)\}$. $\square$

The rings $k[[t]]$ where k is a field, or the ring of p-adic numbers are Noetherian rings of dimension 1 with finitely many primes. This is the maximum dimension for which this can happen.

0ALV **Lemma 61.2.** *A Noetherian ring with finitely many primes has dimension ≤ 1.*

Proof. Let R be a Noetherian ring with finitely many primes. If R is a local domain, then the lemma follows from Lemma 61.1. If R is a domain, then $R_\mathfrak{m}$ has dimension ≤ 1 for all maximal ideals $\mathfrak{m}$ by the local case. Hence $\dim(R) \leq 1$ by Lemma 60.4. If R is general, then $\dim(R/\mathfrak{q}) \leq 1$ for every minimal prime $\mathfrak{q}$ of R. Since every prime contains a minimal prime (Lemma 17.2), this implies $\dim(R) \leq 1$. $\square$

0ALW **Lemma 61.3.** *Let S be a nonzero finite type algebra over a field k. Then* $\dim(S) = 0$ *if and only if S has finitely many primes.*

Proof. Recall that $\text{Spec}(S)$ is sober, Noetherian, and Jacobson, see Lemmas 26.2, 31.5, 35.2, and 35.4. If it has dimension 0, then every point defines an irreducible component and there are only a finite number of irreducible components (Topology, Lemma 9.2). Conversely, if $\text{Spec}(S)$ is finite, then it is discrete by Topology, Lemma 18.6 and hence the dimension is 0. $\square$

00KX **Lemma 61.4.** *Noetherian Jacobson rings.*

1. *Any Noetherian domain R of dimension 1 with infinitely many primes is Jacobson.*
2. *Any Noetherian ring such that every prime $\mathfrak{p}$ is either maximal or contained in infinitely many prime ideals is Jacobson.*

Proof. Part (1) is a reformulation of Lemma 35.6.

Let R be a Noetherian ring such that every non-maximal prime $\mathfrak{p}$ is contained in infinitely many prime ideals. Assume $\text{Spec}(R)$ is not Jacobson to get a contradiction. By Lemmas 26.1 and 31.5 we see that $\text{Spec}(R)$ is a sober, Noetherian topological space. By Topology, Lemma 18.3 we see that there exists a non-maximal ideal

$\mathfrak{p} \subset R$ such that $\{\mathfrak{p}\}$ is a locally closed subset of $\text{Spec}(R)$. In other words, $\mathfrak{p}$ is not maximal and $\{\mathfrak{p}\}$ is an open subset of $V(\mathfrak{p})$. Consider a prime $\mathfrak{q} \subset R$ with $\mathfrak{p} \subset \mathfrak{q}$. Recall that the topology on the spectrum of $(R/\mathfrak{p})_{\mathfrak{q}} = R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}$ is induced from that of $\text{Spec}(R)$, see Lemmas 17.5 and 17.7. Hence we see that $\{(0)\}$ is a locally closed subset of $\text{Spec}((R/\mathfrak{p})_{\mathfrak{q}})$. By Lemma 61.1 we conclude that $\dim((R/\mathfrak{p})_{\mathfrak{q}}) = 1$. Since this holds for every $\mathfrak{q} \supset \mathfrak{p}$ we conclude that $\dim(R/\mathfrak{p}) = 1$. At this point we use the assumption that $\mathfrak{p}$ is contained in infinitely many primes to see that $\text{Spec}(R/\mathfrak{p})$ is infinite. Hence by part (1) of the lemma we see that $V(\mathfrak{p}) \cong \text{Spec}(R/\mathfrak{p})$ is the closure of its closed points. This is the desired contradiction since it means that $\{\mathfrak{p}\} \subset V(\mathfrak{p})$ cannot be open. □

62. Support and dimension of modules

00KY Some basic results on the support and dimension of modules.

00L0 **Lemma 62.1.** *Let R be a Noetherian ring, and let M be a finite R-module. There exists a filtration by R-submodules*

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that each quotient M_i/M_{i-1} is isomorphic to $R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ of R.

First proof. By Lemma 5.4 it suffices to do the case $M = R/I$ for some ideal I. Consider the set S of ideals J such that the lemma does not hold for the module R/J, and order it by inclusion. To arrive at a contradiction, assume that S is not empty. Because R is Noetherian, S has a maximal element J. By definition of S, the ideal J cannot be prime. Pick $a, b \in R$ such that $ab \in J$, but neither $a \in J$ nor $b \in J$. Consider the filtration $0 \subset aR/(J \cap aR) \subset R/J$. Note that both the submodule $aR/(J \cap aR)$ and the quotient module $(R/J)/(aR/(J \cap aR))$ are cyclic modules; write them as R/J' and R/J'' so we have a short exact sequence $0 \to R/J' \to R/J \to R/J'' \to 0$. The inclusion $J \subset J'$ is strict as $b \in J'$ and the inclusion $J \subset J''$ is strict as $a \in J''$. Hence by maximality of J, both R/J' and R/J'' have a filtration as above and hence so does R/J. Contradiction. □

Second proof. For an R-module M we say $P(M)$ holds if there exists a filtration as in the statement of the lemma. Observe that P is stable under extensions and holds for 0. By Lemma 5.4 it suffices to prove $P(R/I)$ holds for every ideal I. If not then because R is Noetherian, there is a maximal counter example J. By Example 28.7 and Proposition 28.8 the ideal J is prime which is a contradiction. □

00L4 **Lemma 62.2.** *Let R, M, M_i, $\mathfrak{p}_i$ as in Lemma 62.1. Then $\text{Supp}(M) = \bigcup V(\mathfrak{p}_i)$ and in particular $\mathfrak{p}_i \in \text{Supp}(M)$.*

Proof. This follows from Lemmas 40.5 and 40.9. □

00L5 **Lemma 62.3.** *Suppose that R is a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let M be a nonzero finite R-module. Then $\text{Supp}(M) = \{\mathfrak{m}\}$ if and only if M has finite length over R.*

Proof. Assume that $\text{Supp}(M) = \{\mathfrak{m}\}$. It suffices to show that all the primes $\mathfrak{p}_i$ in the filtration of Lemma 62.1 are the maximal ideal. This is clear by Lemma 62.2.

Suppose that M has finite length over R. Then $\mathfrak{m}^n M = 0$ by Lemma 52.4. Since some element of $\mathfrak{m}$ maps to a unit in $R_{\mathfrak{p}}$ for any prime $\mathfrak{p} \neq \mathfrak{m}$ in R we see $M_{\mathfrak{p}} = 0$. □

00L6 **Lemma 62.4.** *Let R be a Noetherian ring. Let $I \subset R$ be an ideal. Let M be a finite R-module. Then $I^n M = 0$ for some $n \geq 0$ if and only if $Supp(M) \subset V(I)$.*

Proof. Indeed, $I^n M = 0$ is equivalent to $I^n \subset \text{Ann}(M)$. Since R is Noetherian, this is equivalent to $I \subset \sqrt{\text{Ann}(M)}$, see Lemma 32.5. This in turn is equivalent to $V(I) \supset V(\text{Ann}(M))$, see Lemma 17.2. By Lemma 40.5 this is equivalent to $V(I) \supset \text{Supp}(M)$. □

00L7 **Lemma 62.5.** *Let R, M, M_i, $\mathfrak{p}_i$ as in Lemma 62.1. The minimal elements of the set $\{\mathfrak{p}_i\}$ are the minimal elements of $Supp(M)$. The number of times a minimal prime $\mathfrak{p}$ occurs is*

$$\#\{i \mid \mathfrak{p}_i = \mathfrak{p}\} = length_{R_\mathfrak{p}} M_\mathfrak{p}.$$

Proof. The first statement follows because $\text{Supp}(M) = \bigcup V(\mathfrak{p}_i)$, see Lemma 62.2. Let $\mathfrak{p} \in \text{Supp}(M)$ be minimal. The support of $M_\mathfrak{p}$ is the set consisting of the maximal ideal $\mathfrak{p}R_\mathfrak{p}$. Hence by Lemma 62.3 the length of $M_\mathfrak{p}$ is finite and > 0. Next we note that $M_\mathfrak{p}$ has a filtration with subquotients $(R/\mathfrak{p}_i)_\mathfrak{p} = R_\mathfrak{p}/\mathfrak{p}_i R_\mathfrak{p}$. These are zero if $\mathfrak{p}_i \not\subset \mathfrak{p}$ and equal to $\kappa(\mathfrak{p})$ if $\mathfrak{p}_i \subset \mathfrak{p}$ because by minimality of $\mathfrak{p}$ we have $\mathfrak{p}_i = \mathfrak{p}$ in this case. The result follows since $\kappa(\mathfrak{p})$ has length 1. □

00L8 **Lemma 62.6.** *Let R be a Noetherian local ring. Let M be a finite R-module. Then $d(M) = \dim(Supp(M))$ where $d(M)$ is as in Definition 59.8.*

Proof. Let $M_i, \mathfrak{p}_i$ be as in Lemma 62.1. By Lemma 59.10 we obtain the equality $d(M) = \max\{d(R/\mathfrak{p}_i)\}$. By Proposition 60.9 we have $d(R/\mathfrak{p}_i) = \dim(R/\mathfrak{p}_i)$. Trivially $\dim(R/\mathfrak{p}_i) = \dim V(\mathfrak{p}_i)$. Since all minimal primes of $\text{Supp}(M)$ occur among the $\mathfrak{p}_i$ (Lemma 62.5) we win. □

0B51 **Lemma 62.7.** *Let R be a Noetherian ring. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of finite R-modules. Then $\max\{\dim(Supp(M')), \dim(Supp(M''))\} = \dim(Supp(M))$.*

Proof. If R is local, this follows immediately from Lemmas 62.6 and 59.10. A more elementary argument, which works also if R is not local, is to use that $\text{Supp}(M')$, $\text{Supp}(M'')$, and $\text{Supp}(M)$ are closed (Lemma 40.5) and that $\text{Supp}(M) = \text{Supp}(M') \cup \text{Supp}(M'')$ (Lemma 40.9). □

63. Associated primes

00L9 Here is the standard definition. For non-Noetherian rings and non-finite modules it may be more appropriate to use the definition in Section 66.

00LA **Definition 63.1.** Let R be a ring. Let M be an R-module. A prime $\mathfrak{p}$ of R is *associated* to M if there exists an element $m \in M$ whose annihilator is $\mathfrak{p}$. The set of all such primes is denoted $\text{Ass}_R(M)$ or $\text{Ass}(M)$.

0586 **Lemma 63.2.** *Let R be a ring. Let M be an R-module. Then $Ass(M) \subset Supp(M)$.*

Proof. If $m \in M$ has annihilator $\mathfrak{p}$, then in particular no element of $R \setminus \mathfrak{p}$ annihilates m. Hence m is a nonzero element of $M_\mathfrak{p}$, i.e., $\mathfrak{p} \in \text{Supp}(M)$. □

02M3 **Lemma 63.3.** *Let R be a ring. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R-modules. Then $Ass(M') \subset Ass(M)$ and $Ass(M) \subset Ass(M') \cup Ass(M'')$. Also $Ass(M' \oplus M'') = Ass(M') \cup Ass(M'')$.*

Proof. If $m' \in M'$, then the annihilator of m' viewed as an element of M' is the same as the annihilator of m' viewed as an element of M. Hence the inclusion $\text{Ass}(M') \subset \text{Ass}(M)$. Let $m \in M$ be an element whose annihilator is a prime ideal $\mathfrak{p}$. If there exists a $g \in R$, $g \notin \mathfrak{p}$ such that $m' = gm \in M'$ then the annihilator of m' is $\mathfrak{p}$. If there does not exist a $g \in R$, $g \notin \mathfrak{p}$ such that $gm \in M'$, then the annilator of the image $m'' \in M''$ of m is $\mathfrak{p}$. This proves the inclusion $\text{Ass}(M) \subset \text{Ass}(M') \cup \text{Ass}(M'')$. We omit the proof of the final statement. $\square$

00LB **Lemma 63.4.** *Let R be a ring, and M an R-module. Suppose there exists a filtration by R-submodules*

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that each quotient M_i/M_{i-1} is isomorphic to $R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ of R. Then $Ass(M) \subset \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$.

Proof. By induction on the length n of the filtration $\{M_i\}$. Pick $m \in M$ whose annihilator is a prime $\mathfrak{p}$. If $m \in M_{n-1}$ we are done by induction. If not, then m maps to a nonzero element of $M/M_{n-1} \cong R/\mathfrak{p}_n$. Hence we have $\mathfrak{p} \subset \mathfrak{p}_n$. If equality does not hold, then we can find $f \in \mathfrak{p}_n$, $f \notin \mathfrak{p}$. In this case the annihilator of fm is still $\mathfrak{p}$ and $fm \in M_{n-1}$. Thus we win by induction. $\square$

00LC **Lemma 63.5.** *Let R be a Noetherian ring. Let M be a finite R-module. Then $Ass(M)$ is finite.*

Proof. Immediate from Lemma 63.4 and Lemma 62.1. $\square$

02CE **Proposition 63.6.** *Let R be a Noetherian ring. Let M be a finite R-module. The following sets of primes are the same:*

(1) *The minimal primes in the support of M.*
(2) *The minimal primes in $Ass(M)$.*
(3) *For any filtration $0 = M_0 \subset M_1 \subset \ldots \subset M_{n-1} \subset M_n = M$ with $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ the minimal primes of the set $\{\mathfrak{p}_i\}$.*

Proof. Choose a filtration as in (3). In Lemma 62.5 we have seen that the sets in (1) and (3) are equal.

Let $\mathfrak{p}$ be a minimal element of the set $\{\mathfrak{p}_i\}$. Let i be minimal such that $\mathfrak{p} = \mathfrak{p}_i$. Pick $m \in M_i$, $m \notin M_{i-1}$. The annihilator of m is contained in $\mathfrak{p}_i = \mathfrak{p}$ and contains $\mathfrak{p}_1\mathfrak{p}_2 \ldots \mathfrak{p}_i$. By our choice of i and $\mathfrak{p}$ we have $\mathfrak{p}_j \not\subset \mathfrak{p}$ for $j < i$ and hence we have $\mathfrak{p}_1\mathfrak{p}_2 \ldots \mathfrak{p}_{i-1} \not\subset \mathfrak{p}_i$. Pick $f \in \mathfrak{p}_1\mathfrak{p}_2 \ldots \mathfrak{p}_{i-1}$, $f \notin \mathfrak{p}$. Then fm has annihilator $\mathfrak{p}$. In this way we see that $\mathfrak{p}$ is an associated prime of M. By Lemma 63.2 we have $\text{Ass}(M) \subset \text{Supp}(M)$ and hence $\mathfrak{p}$ is minimal in $\text{Ass}(M)$. Thus the set of primes in (1) is contained in the set of primes of (2).

Let $\mathfrak{p}$ be a minimal element of $\text{Ass}(M)$. Since $\text{Ass}(M) \subset \text{Supp}(M)$ there is a minimal element $\mathfrak{q}$ of $\text{Supp}(M)$ with $\mathfrak{q} \subset \mathfrak{p}$. We have just shown that $\mathfrak{q} \in \text{Ass}(M)$. Hence $\mathfrak{q} = \mathfrak{p}$ by minimality of $\mathfrak{p}$. Thus the set of primes in (2) is contained in the set of primes of (1). $\square$

0587 **Lemma 63.7.** *Let R be a Noetherian ring. Let M be an R-module. Then*

$$M = (0) \Leftrightarrow Ass(M) = \emptyset.$$

Proof. If $M = (0)$, then $\text{Ass}(M) = \emptyset$ by definition. If $M \neq 0$, pick any nonzero finitely generated submodule $M' \subset M$, for example a submodule generated by a single nonzero element. By Lemma 40.2 we see that $\text{Supp}(M')$ is nonempty. By Proposition 63.6 this implies that $\text{Ass}(M')$ is nonempty. By Lemma 63.3 this implies $\text{Ass}(M) \neq \emptyset$. $\square$

05BV **Lemma 63.8.** *Let R be a Noetherian ring. Let M be an R-module. Any $\mathfrak p \in$ Supp(M) which is minimal among the elements of Supp(M) is an element of Ass(M).*

Proof. If M is a finite R-module, then this is a consequence of Proposition 63.6. In general write $M = \bigcup M_\lambda$ as the union of its finite submodules, and use that $\text{Supp}(M) = \bigcup \text{Supp}(M_\lambda)$ and $\text{Ass}(M) = \bigcup \text{Ass}(M_\lambda)$. $\square$

00LD **Lemma 63.9.** *Let R be a Noetherian ring. Let M be an R-module. The union $\bigcup_{\mathfrak q \in Ass(M)} \mathfrak q$ is the set of elements of R which are zerodivisors on M.*

Proof. Any element in any associated prime clearly is a zerodivisor on M. Conversely, suppose $x \in R$ is a zerodivisor on M. Consider the submodule $N = \{m \in M \mid xm = 0\}$. Since N is not zero it has an associated prime $\mathfrak q$ by Lemma 63.7. Then $x \in \mathfrak q$ and $\mathfrak q$ is an associated prime of M by Lemma 63.3. $\square$

0B52 **Lemma 63.10.** *Let R is a Noetherian local ring, M a finite R-module, and $f \in \mathfrak m$ an element of the maximal ideal of R. Then*

$$\dim(Supp(M/fM)) \leq \dim(Supp(M)) \leq \dim(Supp(M/fM)) + 1$$

If f is not in any of the minimal primes of the support of M (for example if f is a nonzerodivisor on M), then equality holds for the right inequality.

Proof. (The parenthetical statement follows from Lemma 63.9.) The first inequality follows from $\text{Supp}(M/fM) \subset \text{Supp}(M)$, see Lemma 40.9. For the second inequality, note that $\text{Supp}(M/fM) = \text{Supp}(M) \cap V(f)$, see Lemma 40.9. It follows, for example by Lemma 62.2 and elementary properties of dimension, that it suffices to show $\dim V(\mathfrak p) \leq \dim(V(\mathfrak p) \cap V(f)) + 1$ for primes $\mathfrak p$ of R. This is a consequence of Lemma 60.13. Finally, if f is not contained in any minimal prime of the support of M, then the chains of primes in $\text{Supp}(M/fM)$ all give rise to chains in $\text{Supp}(M)$ which are at least one step away from being maximal. $\square$

05BW **Lemma 63.11.** *Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Then* $\text{Spec}(\varphi)(Ass_S(M)) \subset Ass_R(M)$.

Proof. If $\mathfrak q \in \text{Ass}_S(M)$, then there exists an m in M such that the annihilator of m in S is $\mathfrak q$. Then the annihilator of m in R is $\mathfrak q \cap R$. $\square$

05BX **Remark 63.12.** Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Then it is not always the case that $\text{Spec}(\varphi)(\text{Ass}_S(M)) \supset \text{Ass}_R(M)$. For example, consider the ring map $R = k \to S = k[x_1, x_2, x_3, \ldots]/(x_i^2)$ and $M = S$. Then $\text{Ass}_R(M)$ is not empty, but $\text{Ass}_S(S)$ is empty.

05DZ **Lemma 63.13.** *Let $\varphi : R \to S$ be a ring map. Let M be an S-module. If S is Noetherian, then* $\text{Spec}(\varphi)(Ass_S(M)) = Ass_R(M)$.

Proof. We have already seen in Lemma 63.11 that $\text{Spec}(\varphi)(\text{Ass}_S(M)) \subset \text{Ass}_R(M)$. For the converse, choose a prime $\mathfrak{p} \in \text{Ass}_R(M)$. Let $m \in M$ be an element such that the annihilator of m in R is $\mathfrak{p}$. Let $I = \{g \in S \mid gm = 0\}$ be the annihilator of m in S. Then $R/\mathfrak{p} \subset S/I$ is injective. Combining Lemmas 30.5 and 30.7 we see that there is a prime $\mathfrak{q} \subset S$ minimal over I mapping to $\mathfrak{p}$. By Proposition 63.6 we see that $\mathfrak{q}$ is an associated prime of S/I, hence $\mathfrak{q}$ is an associated prime of M by Lemma 63.3 and we win. □

05BY **Lemma 63.14.** *Let R be a ring. Let I be an ideal. Let M be an R/I-module. Via the canonical injection $\text{Spec}(R/I) \to \text{Spec}(R)$ we have $Ass_{R/I}(M) = Ass_R(M)$.*

Proof. Omitted. □

0310 **Lemma 63.15.** *Let R be a ring. Let M be an R-module. Let $\mathfrak{p} \subset R$ be a prime.*

1. *If $\mathfrak{p} \in Ass(M)$ then $\mathfrak{p}R_\mathfrak{p} \in Ass(M_\mathfrak{p})$.*
2. *If $\mathfrak{p}$ is finitely generated then the converse holds as well.*

Proof. If $\mathfrak{p} \in \text{Ass}(M)$ there exists an element $m \in M$ whose annihilator is $\mathfrak{p}$. As localization is exact (Proposition 9.12) we see that the annihilator of $m/1$ in $M_\mathfrak{p}$ is $\mathfrak{p}R_\mathfrak{p}$ hence (1) holds. Assume $\mathfrak{p}R_\mathfrak{p} \in \text{Ass}(M_\mathfrak{p})$ and $\mathfrak{p} = (f_1, \ldots, f_n)$. Let m/g be an element of $M_\mathfrak{p}$ whose annihilator is $\mathfrak{p}R_\mathfrak{p}$. This implies that the annihilator of m is contained in $\mathfrak{p}$. As $f_i m/g = 0$ in $M_\mathfrak{p}$ we see there exists a $g_i \in R$, $g_i \notin \mathfrak{p}$ such that $g_i f_i m = 0$ in M. Combined we see the annihilator of $g_1 \ldots g_n m$ is $\mathfrak{p}$. Hence $\mathfrak{p} \in \text{Ass}(M)$. □

05BZ **Lemma 63.16.** *Let R be a ring. Let M be an R-module. Let $S \subset R$ be a multiplicative subset. Via the canonical injection $\text{Spec}(S^{-1}R) \to \text{Spec}(R)$ we have*

1. *$Ass_R(S^{-1}M) = Ass_{S^{-1}R}(S^{-1}M)$,*
2. *$Ass_R(M) \cap \text{Spec}(S^{-1}R) \subset Ass_R(S^{-1}M)$, and*
3. *if R is Noetherian this inclusion is an equality.*

Proof. The first equality follows, since if $m \in S^{-1}M$, then the annihilator of m in R is the intersection of the annihilator of m in $S^{-1}R$ with R. The displayed inclusion and equality in the Noetherian case follows from Lemma 63.15 since for $\mathfrak{p} \in R$, $S \cap \mathfrak{p} = \emptyset$ we have $M_\mathfrak{p} = (S^{-1}M)_{S^{-1}\mathfrak{p}}$. □

05C0 **Lemma 63.17.** *Let R be a ring. Let M be an R-module. Let $S \subset R$ be a multiplicative subset. Assume that every $s \in S$ is a nonzerodivisor on M. Then*

$$Ass_R(M) = Ass_R(S^{-1}M).$$

Proof. As $M \subset S^{-1}M$ by assumption we get the inclusion $\text{Ass}(M) \subset \text{Ass}(S^{-1}M)$ from Lemma 63.3. Conversely, suppose that $n/s \in S^{-1}M$ is an element whose annihilator is a prime ideal $\mathfrak{p}$. Then the annihilator of $n \in M$ is also $\mathfrak{p}$. □

00LL **Lemma 63.18.** *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let $I \subset \mathfrak{m}$ be an ideal. Let M be a finite R-module. The following are equivalent:*

1. *There exists an $x \in I$ which is not a zerodivisor on M.*
2. *We have $I \not\subset \mathfrak{q}$ for all $\mathfrak{q} \in Ass(M)$.*

Proof. If there exists a nonzerodivisor x in I, then x clearly cannot be in any associated prime of M. Conversely, suppose $I \not\subset \mathfrak{q}$ for all $\mathfrak{q} \in \text{Ass}(M)$. In this case we can choose $x \in I$, $x \notin \mathfrak{q}$ for all $\mathfrak{q} \in \text{Ass}(M)$ by Lemmas 63.5 and 15.2. By Lemma 63.9 the element x is not a zerodivisor on M. □

0311 **Lemma 63.19.** *Let R be a ring. Let M be an R-module. If R is Noetherian the map*
$$M \longrightarrow \prod\nolimits_{\mathfrak p \in Ass(M)} M_{\mathfrak p}$$
is injective.

Proof. Let $x \in M$ be an element of the kernel of the map. Then if $\mathfrak p$ is an associated prime of $Rx \subset M$ we see on the one hand that $\mathfrak p \in \text{Ass}(M)$ (Lemma 63.3) and on the other hand that $(Rx)_{\mathfrak p} \subset M_{\mathfrak p}$ is not zero. This contradiction shows that $\text{Ass}(Rx) = \emptyset$. Hence $Rx = 0$ by Lemma 63.7. □

64. Symbolic powers

05G9 Here is the definition.

0313 **Definition 64.1.** Let R be a ring. Let $\mathfrak p$ be a prime ideal. For $n \geq 0$ the nth *symbolic power* of $\mathfrak p$ is the ideal $\mathfrak p^{(n)} = \text{Ker}(R \to R_{\mathfrak p}/\mathfrak p^n R_{\mathfrak p})$.

Note that $\mathfrak p^n \subset \mathfrak p^{(n)}$ but equality does not always hold.

0314 **Lemma 64.2.** *Let R be a Noetherian ring. Let $\mathfrak p$ be a prime ideal. Let $n > 0$. Then $Ass(R/\mathfrak p^{(n)}) = \{\mathfrak p\}$.*

Proof. If $\mathfrak q$ is an associated prime of $R/\mathfrak p^{(n)}$ then clearly $\mathfrak p \subset \mathfrak q$. On the other hand, any element $x \in R$, $x \not\in \mathfrak p$ is a nonzerodivisor on $R/\mathfrak p^{(n)}$. Namely, if $y \in R$ and $xy \in \mathfrak p^{(n)} = R \cap \mathfrak p^n R_{\mathfrak p}$ then $y \in \mathfrak p^n R_{\mathfrak p}$, hence $y \in \mathfrak p^{(n)}$. Hence the lemma follows. □

0BC0 **Lemma 64.3.** *Let $R \to S$ be flat ring map. Let $\mathfrak p \subset R$ be a prime such that $\mathfrak q = \mathfrak p S$ is a prime of S. Then $\mathfrak p^{(n)} S = \mathfrak q^{(n)}$.*

Proof. Since $\mathfrak p^{(n)} = \text{Ker}(R \to R_{\mathfrak p}/\mathfrak p^n R_{\mathfrak p})$ we see using flatness that $\mathfrak p^{(n)} S$ is the kernel of the map $S \to S_{\mathfrak p}/\mathfrak p^n S_{\mathfrak p}$. On the other hand $\mathfrak q^{(n)}$ is the kernel of the map $S \to S_{\mathfrak q}/\mathfrak q^n S_{\mathfrak q} = S_{\mathfrak q}/\mathfrak p^n S_{\mathfrak q}$. Hence it suffices to show that
$$S_{\mathfrak p}/\mathfrak p^n S_{\mathfrak p} \longrightarrow S_{\mathfrak q}/\mathfrak p^n S_{\mathfrak q}$$
is injective. Observe that the right hand module is the localization of the left hand module by elements $f \in S$, $f \not\in \mathfrak q$. Thus it suffices to show these elements are nonzerodivisors on $S_{\mathfrak p}/\mathfrak p^n S_{\mathfrak p}$. By flatness, the module $S_{\mathfrak p}/\mathfrak p^n S_{\mathfrak p}$ has a finite filtration whose subquotients are
$$\mathfrak p^i S_{\mathfrak p}/\mathfrak p^{i+1} S_{\mathfrak p} \cong \mathfrak p^i R_{\mathfrak p}/\mathfrak p^{i+1} R_{\mathfrak p} \otimes_{R_{\mathfrak p}} S_{\mathfrak p} \cong V \otimes_{\kappa(\mathfrak p)} (S/\mathfrak q)_{\mathfrak p}$$
where V is a $\kappa(\mathfrak p)$ vector space. Thus f acts invertibly as desired. □

65. Relative assassin

05GA Discussion of relative assassins. Let $R \to S$ be a ring map. Let N be an S-module. In this situation we can introduce the following sets of primes $\mathfrak q$ of S:

1. A: with $\mathfrak p = R \cap \mathfrak q$ we have that $\mathfrak q \in \text{Ass}_S(N \otimes_R \kappa(\mathfrak p))$,
2. A': with $\mathfrak p = R \cap \mathfrak q$ we have that $\mathfrak q$ is in the image of $\text{Ass}_{S \otimes \kappa(\mathfrak p)}(N \otimes_R \kappa(\mathfrak p))$ under the canonical map $\text{Spec}(S \otimes_R \kappa(\mathfrak p)) \to \text{Spec}(S)$,
3. A_{fin}: with $\mathfrak p = R \cap \mathfrak q$ we have that $\mathfrak q \in \text{Ass}_S(N/\mathfrak p N)$,
4. A'_{fin}: for some prime $\mathfrak p' \subset R$ we have $\mathfrak q \in \text{Ass}_S(N/\mathfrak p' N)$,
5. B: for some R-module M we have $\mathfrak q \in \text{Ass}_S(N \otimes_R M)$, and
6. B_{fin}: for some finite R-module M we have $\mathfrak q \in \text{Ass}_S(N \otimes_R M)$.

Let us determine some of the relations between theses sets.

05GB **Lemma 65.1.** *Let $R \to S$ be a ring map. Let N be an S-module. Let A, A', A_{fin}, B, and B_{fin} be the subsets of* $\mathrm{Spec}(S)$ *introduced above.*

(1) *We always have $A = A'$.*
(2) *We always have $A_{fin} \subset A$, $B_{fin} \subset B$, $A_{fin} \subset A'_{fin} \subset B_{fin}$ and $A \subset B$.*
(3) *If S is Noetherian, then $A = A_{fin}$ and $B = B_{fin}$.*
(4) *If N is flat over R, then $A = A_{fin} = A'_{fin}$ and $B = B_{fin}$.*
(5) *If R is Noetherian and N is flat over R, then all of the sets are equal, i.e., $A = A' = A_{fin} = A'_{fin} = B = B_{fin}$.*

Proof. Some of the arguments in the proof will be repeated in the proofs of later lemmas which are more precise than this one (because they deal with a given module M or a given prime $\mathfrak{p}$ and not with the collection of all of them).

Proof of (1). Let $\mathfrak{p}$ be a prime of R. Then we have

$$\mathrm{Ass}_S(N \otimes_R \kappa(\mathfrak{p})) = \mathrm{Ass}_{S/\mathfrak{p}S}(N \otimes_R \kappa(\mathfrak{p})) = \mathrm{Ass}_{S \otimes_R \kappa(\mathfrak{p})}(N \otimes_R \kappa(\mathfrak{p}))$$

the first equality by Lemma 63.14 and the second by Lemma 63.16 part (1). This prove that $A = A'$. The inclusion $A_{fin} \subset A'_{fin}$ is clear.

Proof of (2). Each of the inclusions is immediate from the definitions except perhaps $A_{fin} \subset A$ which follows from Lemma 63.16 and the fact that we require $\mathfrak{p} = R \cap \mathfrak{q}$ in the formulation of A_{fin}.

Proof of (3). The equality $A = A_{fin}$ follows from Lemma 63.16 part (3) if S is Noetherian. Let $\mathfrak{q} = (g_1, \ldots, g_m)$ be a finitely generated prime ideal of S. Say $z \in N \otimes_R M$ is an element whose annihilator is $\mathfrak{q}$. We may pick a finite submodule $M' \subset M$ such that z is the image of $z' \in N \otimes_R M'$. Then $\mathrm{Ann}_S(z') \subset \mathfrak{q} = \mathrm{Ann}_S(z)$. Since $N \otimes_R -$ commutes with colimits and since M is the directed colimit of finite R-modules we can find $M' \subset M'' \subset M$ such that the image $z'' \in N \otimes_R M''$ is annihilated by $g_1, \ldots, g_m$. Hence $\mathrm{Ann}_S(z'') = \mathfrak{q}$. This proves that $B = B_{fin}$ if S is Noetherian.

Proof of (4). If N is flat, then the functor $N \otimes_R -$ is exact. In particular, if $M' \subset M$, then $N \otimes_R M' \subset N \otimes_R M$. Hence if $z \in N \otimes_R M$ is an element whose annihilator $\mathfrak{q} = \mathrm{Ann}_S(z)$ is a prime, then we can pick any finite R-submodule $M' \subset M$ such that $z \in N \otimes_R M'$ and we see that the annihilator of z as an element of $N \otimes_R M'$ is equal to $\mathfrak{q}$. Hence $B = B_{fin}$. Let $\mathfrak{p}'$ be a prime of R and let $\mathfrak{q}$ be a prime of S which is an associated prime of $N/\mathfrak{p}'N$. This implies that $\mathfrak{p}'S \subset \mathfrak{q}$. As N is flat over R we see that $N/\mathfrak{p}'N$ is flat over the integral domain $R/\mathfrak{p}'$. Hence every nonzero element of $R/\mathfrak{p}'$ is a nonzerodivisor on $N/\mathfrak{p}'$. Hence none of these elements can map to an element of $\mathfrak{q}$ and we conclude that $\mathfrak{p}' = R \cap \mathfrak{q}$. Hence $A_{fin} = A'_{fin}$. Finally, by Lemma 63.17 we see that $\mathrm{Ass}_S(N/\mathfrak{p}'N) = \mathrm{Ass}_S(N \otimes_R \kappa(\mathfrak{p}'))$, i.e., $A'_{fin} = A$.

Proof of (5). We only need to prove $A'_{fin} = B_{fin}$ as the other equalities have been proved in (4). To see this let M be a finite R-module. By Lemma 62.1 there exists a filtration by R-submodules

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that each quotient M_i/M_{i-1} is isomorphic to $R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ of R. Since N is flat we obtain a filtration by S-submodules

$$0 = N \otimes_R M_0 \subset N \otimes_R M_1 \subset \ldots \subset N \otimes_R M_n = N \otimes_R M$$

such that each subquotient is isomorphic to $N/\mathfrak{p}_iN$. By Lemma 63.3 we conclude that $\text{Ass}_S(N \otimes_R M) \subset \bigcup \text{Ass}_S(N/\mathfrak{p}_iN)$. Hence we see that $B_{fin} \subset A'_{fin}$. Since the other inclusion is part of (2) we win. $\square$

We define the relative assassin of N over S/R to be the set $A = A'$ above. As a motivation we point out that it depends only on the fibre modules $N \otimes_R \kappa(\mathfrak{p})$ over the fibre rings. As in the case of the assassin of a module we warn the reader that this notion makes most sense when the fibre rings $S \otimes_R \kappa(\mathfrak{p})$ are Noetherian, for example if $R \to S$ is of finite type.

05GC **Definition 65.2.** Let $R \to S$ be a ring map. Let N be an S-module. The *relative assassin of N over S/R* is the set

$$\text{Ass}_{S/R}(N) = \{\mathfrak{q} \subset S \mid \mathfrak{q} \in \text{Ass}_S(N \otimes_R \kappa(\mathfrak{p})) \text{ with } \mathfrak{p} = R \cap \mathfrak{q}\}.$$

This is the set named A in Lemma 65.1.

The spirit of the next few results is that they are about the relative assassin, even though this may not be apparent.

0312 **Lemma 65.3.** *Let $R \to S$ be a ring map. Let M be an R-module, and let N be an S-module. If N is flat as R-module, then*

$$Ass_S(M \otimes_R N) \supset \bigcup\nolimits_{\mathfrak{p} \in Ass_R(M)} Ass_S(N/\mathfrak{p}N)$$

and if R is Noetherian then we have equality.

Proof. If $\mathfrak{p} \in \text{Ass}_R(M)$ then there exists an injection $R/\mathfrak{p} \to M$. As N is flat over R we obtain an injection $R/\mathfrak{p} \otimes_R N \to M \otimes_R N$. Since $R/\mathfrak{p} \otimes_R N = N/\mathfrak{p}N$ we conclude that $\text{Ass}_S(N/\mathfrak{p}N) \subset \text{Ass}_S(M \otimes_R N)$, see Lemma 63.3. Hence the right hand side is contained in the left hand side.

Write $M = \bigcup M_\lambda$ as the union of its finitely generated R-submodules. Then also $N \otimes_R M = \bigcup N \otimes_R M_\lambda$ (as N is R-flat). By definition of associated primes we see that $\text{Ass}_S(N \otimes_R M) = \bigcup \text{Ass}_S(N \otimes_R M_\lambda)$ and $\text{Ass}_R(M) = \bigcup \text{Ass}(M_\lambda)$. Hence we may assume M is finitely generated.

Let $\mathfrak{q} \in \text{Ass}_S(M \otimes_R N)$, and assume R is Noetherian and M is a finite R-module. To finish the proof we have to show that $\mathfrak{q}$ is an element of the right hand side. First we observe that $\mathfrak{q}S_\mathfrak{q} \in \text{Ass}_{S_\mathfrak{q}}((M \otimes_R N)_\mathfrak{q})$, see Lemma 63.15. Let $\mathfrak{p}$ be the corresponding prime of R. Note that

$$(M \otimes_R N)_\mathfrak{q} = M \otimes_R N_\mathfrak{q} = M_\mathfrak{p} \otimes_{R_\mathfrak{p}} N_\mathfrak{q}$$

If $\mathfrak{p}R_\mathfrak{p} \not\in \text{Ass}_{R_\mathfrak{p}}(M_\mathfrak{p})$ then there exists an element $x \in \mathfrak{p}R_\mathfrak{p}$ which is a nonzerodivisor in $M_\mathfrak{p}$ (see Lemma 63.18). Since $N_\mathfrak{q}$ is flat over $R_\mathfrak{p}$ we see that the image of x in $\mathfrak{q}S_\mathfrak{q}$ is a nonzerodivisor on $(M \otimes_R N)_\mathfrak{q}$. This is a contradiction with the assumption that $\mathfrak{q}S_\mathfrak{q} \in \text{Ass}_S((M \otimes_R N)_\mathfrak{q})$. Hence we conclude that $\mathfrak{p}$ is one of the associated primes of M.

Continuing the argument we choose a filtration

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that each quotient M_i/M_{i-1} is isomorphic to $R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ of R, see Lemma 62.1. (By Lemma 63.4 we have $\mathfrak{p}_i = \mathfrak{p}$ for at least one i.) This gives a filtration

$$0 = M_0 \otimes_R N \subset M_1 \otimes_R N \subset \ldots \subset M_n \otimes_R N = M \otimes_R N$$

with subquotients isomorphic to $N/\mathfrak{p}_iN$. If $\mathfrak{p}_i \neq \mathfrak{p}$ then $\mathfrak{q}$ cannot be associated to the module $N/\mathfrak{p}_iN$ by the result of the preceding paragraph (as $\text{Ass}_R(R/\mathfrak{p}_i) = \{\mathfrak{p}_i\}$). Hence we conclude that $\mathfrak{q}$ is associated to $N/\mathfrak{p}N$ as desired. □

05C1 **Lemma 65.4.** *Let $R \to S$ be a ring map. Let N be an S-module. Assume N is flat as an R-module and R is a domain with fraction field K. Then*

$$Ass_S(N) = Ass_S(N \otimes_R K) = Ass_{S \otimes_R K}(N \otimes_R K)$$

via the canonical inclusion $\text{Spec}(S \otimes_R K) \subset \text{Spec}(S)$.

Proof. Note that $S \otimes_R K = (R \setminus \{0\})^{-1}S$ and $N \otimes_R K = (R \setminus \{0\})^{-1}N$. For any nonzero $x \in R$ multiplication by x on N is injective as N is flat over R. Hence the lemma follows from Lemma 63.17 combined with Lemma 63.16 part (1). □

05C2 **Lemma 65.5.** *Let $R \to S$ be a ring map. Let M be an R-module, and let N be an S-module. Assume N is flat as R-module. Then*

$$Ass_S(M \otimes_R N) \supset \bigcup\nolimits_{\mathfrak{p} \in Ass_R(M)} Ass_{S \otimes_R \kappa(\mathfrak{p})}(N \otimes_R \kappa(\mathfrak{p}))$$

where we use Remark 17.8 to think of the spectra of fibre rings as subsets of $\text{Spec}(S)$*. If R is Noetherian then this inclusion is an equality.*

Proof. This is equivalent to Lemma 65.3 by Lemmas 63.14, 39.7, and 65.4. □

05E0 **Remark 65.6.** Let $R \to S$ be a ring map. Let N be an S-module. Let $\mathfrak{p}$ be a prime of R. Then

$$\text{Ass}_S(N \otimes_R \kappa(\mathfrak{p})) = \text{Ass}_{S/\mathfrak{p}S}(N \otimes_R \kappa(\mathfrak{p})) = \text{Ass}_{S \otimes_R \kappa(\mathfrak{p})}(N \otimes_R \kappa(\mathfrak{p})).$$

The first equality by Lemma 63.14 and the second by Lemma 63.16 part (1).

66. Weakly associated primes

0546 This is a variant on the notion of an associated prime that is useful for non-Noetherian ring and non-finite modules.

0547 **Definition 66.1.** Let R be a ring. Let M be an R-module. A prime $\mathfrak{p}$ of R is *weakly associated* to M if there exists an element $m \in M$ such that $\mathfrak{p}$ is minimal among the prime ideals containing the annihilator $\text{Ann}(m) = \{f \in R \mid fm = 0\}$. The set of all such primes is denoted $\text{WeakAss}_R(M)$ or $\text{WeakAss}(M)$.

Thus an associated prime is a weakly associated prime. Here is a characterization in terms of the localization at the prime.

0566 **Lemma 66.2.** *Let R be a ring. Let M be an R-module. Let $\mathfrak{p}$ be a prime of R. The following are equivalent:*

1. *$\mathfrak{p}$ is weakly associated to M,*
2. *$\mathfrak{p}R_\mathfrak{p}$ is weakly associated to $M_\mathfrak{p}$, and*
3. *$M_\mathfrak{p}$ contains an element whose annihilator has radical equal to $\mathfrak{p}R_\mathfrak{p}$.*

Proof. Assume (1). Then there exists an element $m \in M$ such that $\mathfrak{p}$ is minimal among the primes containing the annihilator $I = \{x \in R \mid xm = 0\}$ of m. As localization is exact, the annihilator of m in $M_\mathfrak{p}$ is $I_\mathfrak{p}$. Hence $\mathfrak{p}R_\mathfrak{p}$ is a minimal prime of $R_\mathfrak{p}$ containing the annihilator $I_\mathfrak{p}$ of m in $M_\mathfrak{p}$. This implies (2) holds, and also (3) as it implies that $\sqrt{I_\mathfrak{p}} = \mathfrak{p}R_\mathfrak{p}$.

Applying the implication (1) $\Rightarrow$ (3) to $M_\mathfrak{p}$ over $R_\mathfrak{p}$ we see that (2) $\Rightarrow$ (3).

Finally, assume (3). This means there exists an element $m/f \in M_\mathfrak{p}$ whose annihilator has radical equal to $\mathfrak{p}R_\mathfrak{p}$. Then the annihilator $I = \{x \in R \mid xm = 0\}$ of m in M is such that $\sqrt{I_\mathfrak{p}} = \mathfrak{p}R_\mathfrak{p}$. Clearly this means that $\mathfrak{p}$ contains I and is minimal among the primes containing I, i.e., (1) holds. □

0EMA **Lemma 66.3.** *For a reduced ring the weakly associated primes of the ring are the minimal primes.*

Proof. Let $(R, \mathfrak{m})$ be a reduced local ring. Suppose $x \in R$ is an element whose annihilator has radical $\mathfrak{m}$. If $\mathfrak{m} \neq 0$, then x cannot be a unit, so $x \in \mathfrak{m}$. Then in particular $x^{1+n} = 0$ for some $n \geq 0$. Hence $x = 0$. Which contradicts the assumption that the annihilator of x is contained in $\mathfrak{m}$. Thus we see that $\mathfrak{m} = 0$, i.e., R is a field. By Lemma 66.2 this implies the statement of the lemma. □

0548 **Lemma 66.4.** *Let R be a ring. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R-modules. Then $WeakAss(M') \subset WeakAss(M)$ and $WeakAss(M) \subset WeakAss(M') \cup WeakAss(M'')$.*

Proof. We will use the characterization of weakly associated primes of Lemma 66.2. Let $\mathfrak{p}$ be a prime of R. As localization is exact we obtain the short exact sequence $0 \to M'_\mathfrak{p} \to M_\mathfrak{p} \to M''_\mathfrak{p} \to 0$. Suppose that $m \in M_\mathfrak{p}$ is an element whose annihilator has radical $\mathfrak{p}R_\mathfrak{p}$. Then either the image $\overline{m}$ of m in $M''_\mathfrak{p}$ is zero and $m \in M'_\mathfrak{p}$, or the radical of the annihilator of $\overline{m}$ is $\mathfrak{p}R_\mathfrak{p}$. This proves that $\text{WeakAss}(M) \subset \text{WeakAss}(M') \cup \text{WeakAss}(M'')$. The inclusion $\text{WeakAss}(M') \subset \text{WeakAss}(M)$ is immediate from the definitions. □

0588 **Lemma 66.5.** *Let R be a ring. Let M be an R-module. Then*

$$M = (0) \Leftrightarrow WeakAss(M) = \emptyset$$

Proof. If $M = (0)$ then $\text{WeakAss}(M) = \emptyset$ by definition. Conversely, suppose that $M \neq 0$. Pick a nonzero element $m \in M$. Write $I = \{x \in R \mid xm = 0\}$ the annihilator of m. Then $R/I \subset M$. Hence $\text{WeakAss}(R/I) \subset \text{WeakAss}(M)$ by Lemma 66.4. But as $I \neq R$ we have $V(I) = \text{Spec}(R/I)$ contains a minimal prime, see Lemmas 17.2 and 17.7, and we win. □

0589 **Lemma 66.6.** *Let R be a ring. Let M be an R-module. Then*

$$Ass(M) \subset WeakAss(M) \subset Supp(M).$$

Proof. The first inclusion is immediate from the definitions. If $\mathfrak{p} \in \text{WeakAss}(M)$, then by Lemma 66.2 we have $M_\mathfrak{p} \neq 0$, hence $\mathfrak{p} \in \text{Supp}(M)$. □

05C3 **Lemma 66.7.** *Let R be a ring. Let M be an R-module. The union $\bigcup_{\mathfrak{q} \in WeakAss(M)} \mathfrak{q}$ is the set elements of R which are zerodivisors on M.*

Proof. Suppose $f \in \mathfrak{q} \in \text{WeakAss}(M)$. Then there exists an element $m \in M$ such that $\mathfrak{q}$ is minimal over $I = \{x \in R \mid xm = 0\}$. Hence there exists a $g \in R$, $g \notin \mathfrak{q}$ and $n > 0$ such that $f^n gm = 0$. Note that $gm \neq 0$ as $g \notin I$. If we take n minimal as above, then $f(f^{n-1}gm) = 0$ and $f^{n-1}gm \neq 0$, so f is a zerodivisor on M. Conversely, suppose $f \in R$ is a zerodivisor on M. Consider the submodule $N = \{m \in M \mid fm = 0\}$. Since N is not zero it has a weakly associated prime $\mathfrak{q}$ by Lemma 66.5. Clearly $f \in \mathfrak{q}$ and by Lemma 66.4 $\mathfrak{q}$ is a weakly associated prime of M. □

05C4 **Lemma 66.8.** *Let R be a ring. Let M be an R-module. Any $\mathfrak{p} \in Supp(M)$ which is minimal among the elements of $Supp(M)$ is an element of $WeakAss(M)$.*

Proof. Note that $\text{Supp}(M_\mathfrak{p}) = \{\mathfrak{p}R_\mathfrak{p}\}$ in $\text{Spec}(R_\mathfrak{p})$. In particular $M_\mathfrak{p}$ is nonzero, and hence $\text{WeakAss}(M_\mathfrak{p}) \neq \emptyset$ by Lemma 66.5. Since $\text{WeakAss}(M_\mathfrak{p}) \subset \text{Supp}(M_\mathfrak{p})$ by Lemma 66.6 we conclude that $\text{WeakAss}(M_\mathfrak{p}) = \{\mathfrak{p}R_\mathfrak{p}\}$, whence $\mathfrak{p} \in \text{WeakAss}(M)$ by Lemma 66.2. □

058A **Lemma 66.9.** *Let R be a ring. Let M be an R-module. Let $\mathfrak{p}$ be a prime ideal of R which is finitely generated. Then*

$$\mathfrak{p} \in Ass(M) \Leftrightarrow \mathfrak{p} \in WeakAss(M).$$

In particular, if R is Noetherian, then $Ass(M) = WeakAss(M)$.

Proof. Write $\mathfrak{p} = (g_1, \ldots, g_n)$ for some $g_i \in R$. It is enough the prove the implication "$\Leftarrow$" as the other implication holds in general, see Lemma 66.6. Assume $\mathfrak{p} \in \text{WeakAss}(M)$. By Lemma 66.2 there exists an element $m \in M_\mathfrak{p}$ such that $I = \{x \in R_\mathfrak{p} \mid xm = 0\}$ has radical $\mathfrak{p}R_\mathfrak{p}$. Hence for each i there exists a smallest $e_i > 0$ such that $g_i^{e_i} m = 0$ in $M_\mathfrak{p}$. If $e_i > 1$ for some i, then we can replace m by $g_i^{e_i - 1} m \neq 0$ and decrease $\sum e_i$. Hence we may assume that the annihilator of $m \in M_\mathfrak{p}$ is $(g_1, \ldots, g_n)R_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}$. By Lemma 63.15 we see that $\mathfrak{p} \in \text{Ass}(M)$. □

05C5 **Remark 66.10.** Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Then it is not always the case that $\text{Spec}(\varphi)(\text{WeakAss}_S(M)) \subset \text{WeakAss}_R(M)$ contrary to the case of associated primes (see Lemma 63.11). An example is to consider the ring map

$$R = k[x_1, x_2, x_3, \ldots] \to S = k[x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots]/(x_1y_1, x_2y_2, x_3y_3, \ldots)$$

and $M = S$. In this case $\mathfrak{q} = \sum x_i S$ is a minimal prime of S, hence a weakly associated prime of $M = S$ (see Lemma 66.8). But on the other hand, for any nonzero element of S the annihilator in R is finitely generated, and hence does not have radical equal to $R \cap \mathfrak{q} = (x_1, x_2, x_3, \ldots)$ (details omitted).

05C6 **Lemma 66.11.** *Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Then we have* $\text{Spec}(\varphi)(\textit{WeakAss}_S(M)) \supset \textit{WeakAss}_R(M)$.

Proof. Let $\mathfrak{p}$ be an element of $\text{WeakAss}_R(M)$. Then there exists an $m \in M_\mathfrak{p}$ whose annihilator $I = \{x \in R_\mathfrak{p} \mid xm = 0\}$ has radical $\mathfrak{p}R_\mathfrak{p}$. Consider the annihilator $J = \{x \in S_\mathfrak{p} \mid xm = 0\}$ of m in $S_\mathfrak{p}$. As $IS_\mathfrak{p} \subset J$ we see that any minimal prime $\mathfrak{q} \subset S_\mathfrak{p}$ over J lies over $\mathfrak{p}$. Moreover such a $\mathfrak{q}$ corresponds to a weakly associated prime of M for example by Lemma 66.2. □

05C7 **Remark 66.12.** Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Denote $f : \text{Spec}(S) \to \text{Spec}(R)$ the associated map on spectra. Then we have

$$f(\text{Ass}_S(M)) \subset \text{Ass}_R(M) \subset \text{WeakAss}_R(M) \subset f(\text{WeakAss}_S(M))$$

see Lemmas 63.11, 66.11, and 66.6. In general all of the inclusions may be strict, see Remarks 63.12 and 66.10. If S is Noetherian, then all the inclusions are equalities as the outer two are equal by Lemma 66.9.

05E1 **Lemma 66.13.** *Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Denote $f : \text{Spec}(S) \to \text{Spec}(R)$ the associated map on spectra. If φ is a finite ring map, then*

$$\textit{WeakAss}_R(M) = f(\textit{WeakAss}_S(M)).$$

Proof. One of the inclusions has already been proved, see Remark 66.12. To prove the other assume $\mathfrak q \in \text{WeakAss}_S(M)$ and let $\mathfrak p$ be the corresponding prime of R. Let $m \in M$ be an element such that $\mathfrak q$ is a minimal prime over $J = \{g \in S \mid gm = 0\}$. Thus the radical of $JS_{\mathfrak q}$ is $\mathfrak q S_{\mathfrak q}$. As $R \to S$ is finite there are finitely many primes $\mathfrak q = \mathfrak q_1, \mathfrak q_2, \ldots, \mathfrak q_l$ over $\mathfrak p$, see Lemma 36.21. Pick $x \in \mathfrak q$ with $x \not\in \mathfrak q_i$ for $i > 1$, see Lemma 15.2. By the above there exists an element $y \in S$, $y \not\in \mathfrak q$ and an integer $t > 0$ such that $yx^t m = 0$. Thus the element $ym \in M$ is annihilated by x^t, hence ym maps to zero in $M_{\mathfrak q_i}$, $i = 2, \ldots, l$. To be sure, ym does not map to zero in $S_{\mathfrak q}$.

The ring $S_{\mathfrak p}$ is semi-local with maximal ideals $\mathfrak q_i S_{\mathfrak p}$ by going up for finite ring maps, see Lemma 36.22. If $f \in \mathfrak p R_{\mathfrak p}$ then some power of f ends up in $JS_{\mathfrak q}$ hence for some $n > 0$ we see that $f^t ym$ maps to zero in $M_{\mathfrak q}$. As ym vanishes at the other maximal ideals of $S_{\mathfrak p}$ we conclude that $f^t ym$ is zero in $M_{\mathfrak p}$, see Lemma 23.1. In this way we see that $\mathfrak p$ is a minimal prime over the annihilator of ym in R and we win. □

05C8 **Lemma 66.14.** *Let R be a ring. Let I be an ideal. Let M be an R/I-module. Via the canonical injection* $\text{Spec}(R/I) \to \text{Spec}(R)$ *we have $\text{WeakAss}_{R/I}(M) = \text{WeakAss}_R(M)$.*

Proof. Special case of Lemma 66.13. □

05C9 **Lemma 66.15.** *Let R be a ring. Let M be an R-module. Let $S \subset R$ be a multiplicative subset. Via the canonical injection* $\text{Spec}(S^{-1}R) \to \text{Spec}(R)$ *we have $\text{WeakAss}_R(S^{-1}M) = \text{WeakAss}_{S^{-1}R}(S^{-1}M)$ and*

$$\text{WeakAss}(M) \cap \text{Spec}(S^{-1}R) = \text{WeakAss}(S^{-1}M).$$

Proof. Suppose that $m \in S^{-1}M$. Let $I = \{x \in R \mid xm = 0\}$ and $I' = \{x' \in S^{-1}R \mid x'm = 0\}$. Then $I' = S^{-1}I$ and $I \cap S = \emptyset$ unless $I = R$ (verifications omitted). Thus primes in $S^{-1}R$ minimal over I' correspond bijectively to primes in R minimal over I and avoiding S. This proves the equality $\text{WeakAss}_R(S^{-1}M) = \text{WeakAss}_{S^{-1}R}(S^{-1}M)$. The second equality follows from Lemma 66.2 since for $\mathfrak p \in R$, $S \cap \mathfrak p = \emptyset$ we have $M_{\mathfrak p} = (S^{-1}M)_{S^{-1}\mathfrak p}$. □

05CA **Lemma 66.16.** *Let R be a ring. Let M be an R-module. Let $S \subset R$ be a multiplicative subset. Assume that every $s \in S$ is a nonzerodivisor on M. Then*

$$\text{WeakAss}(M) = \text{WeakAss}(S^{-1}M).$$

Proof. As $M \subset S^{-1}M$ by assumption we obtain $\text{WeakAss}(M) \subset \text{WeakAss}(S^{-1}M)$ from Lemma 66.4. Conversely, suppose that $n/s \in S^{-1}M$ is an element with annihilator I and $\mathfrak p$ a prime which is minimal over I. Then the annihilator of $n \in M$ is I and $\mathfrak p$ is a prime minimal over I. □

05CB **Lemma 66.17.** *Let R be a ring. Let M be an R-module. The map*

$$M \longrightarrow \prod\nolimits_{\mathfrak p \in \text{WeakAss}(M)} M_{\mathfrak p}$$

is injective.

Proof. Let $x \in M$ be an element of the kernel of the map. Set $N = Rx \subset M$. If $\mathfrak p$ is a weakly associated prime of N we see on the one hand that $\mathfrak p \in \text{WeakAss}(M)$ (Lemma 66.4) and on the other hand that $N_{\mathfrak p} \subset M_{\mathfrak p}$ is not zero. This contradiction shows that $\text{WeakAss}(N) = \emptyset$. Hence $N = 0$, i.e., $x = 0$ by Lemma 66.5. □

05CC **Lemma 66.18.** *Let $R \to S$ be a ring map. Let N be an S-module. Assume N is flat as an R-module and R is a domain with fraction field K. Then*

$$WeakAss_S(N) = WeakAss_{S \otimes_R K}(N \otimes_R K)$$

via the canonical inclusion $\operatorname{Spec}(S \otimes_R K) \subset \operatorname{Spec}(S)$.

Proof. Note that $S \otimes_R K = (R \setminus \{0\})^{-1}S$ and $N \otimes_R K = (R \setminus \{0\})^{-1}N$. For any nonzero $x \in R$ multiplication by x on N is injective as N is flat over R. Hence the lemma follows from Lemma 66.16. □

0CUB **Lemma 66.19.** *Let K/k be a field extension. Let R be a k-algebra. Let M be an R-module. Let $\mathfrak q \subset R \otimes_k K$ be a prime lying over $\mathfrak p \subset R$. If $\mathfrak q$ is weakly associated to $M \otimes_k K$, then $\mathfrak p$ is weakly associated to M.*

Proof. Let $z \in M \otimes_k K$ be an element such that $\mathfrak q$ is minimal over the annihilator $J \subset R \otimes_k K$ of z. Choose a finitely generated subextension $K/L/k$ such that $z \in M \otimes_k L$. Since $R \otimes_k L \to R \otimes_k K$ is flat we see that $J = I(R \otimes_k K)$ where $I \subset R \otimes_k L$ is the annihilator of z in the smaller ring (Lemma 40.4). Thus $\mathfrak q \cap (R \otimes_k L)$ is minimal over I by going down (Lemma 39.19). In this way we reduce to the case described in the next paragraph.

Assume K/k is a finitely generated field extension. Let $x_1, \ldots, x_r \in K$ be a transcendence basis of K over k, see Fields, Section 26. Set $L = k(x_1, \ldots, x_r)$. Say $[K : L] = n$. Then $R \otimes_k L \to R \otimes_k K$ is a finite ring map. Hence $\mathfrak q \cap (R \otimes_k L)$ is a weakly associated prime of $M \otimes_k K$ viewed as a $R \otimes_k L$-module by Lemma 66.13. Since $M \otimes_k K \cong (M \otimes_k L)^{\oplus n}$ as a $R \otimes_k L$-module, we see that $\mathfrak q \cap (R \otimes_k L)$ is a weakly associated prime of $M \otimes_k L$ (for example by using Lemma 66.4 and induction). In this way we reduce to the case discussed in the next paragraph.

Assume $K = k(x_1, \ldots, x_r)$ is a purely transcendental field extension. We may replace R by $R_{\mathfrak p}$, M by $M_{\mathfrak p}$ and $\mathfrak q$ by $\mathfrak q(R_{\mathfrak p} \otimes_k K)$. See Lemma 66.15. In this way we reduce to the case discussed in the next paragraph.

Assume $K = k(x_1, \ldots, x_r)$ is a purely transcendental field extension and R is local with maximal ideal $\mathfrak p$. We claim that any $f \in R \otimes_k K$, $f \notin \mathfrak p(R \otimes_k K)$ is a nonzerodivisor on $M \otimes_k K$. Namely, let $z \in M \otimes_k K$ be an element. There is a finite R-submodule $M' \subset M$ such that $z \in M' \otimes_k K$ and such that M' is minimal with this property: choose a basis $\{t_\alpha\}$ of K as a k-vector space, write $z = \sum m_\alpha \otimes t_\alpha$ and let M' be the R-submodule generated by the m_α. If $z \in \mathfrak p(M' \otimes_k K) = \mathfrak p M' \otimes_k K$, then $\mathfrak p M' = M'$ and $M' = 0$ by Lemma 20.1 a contradiction. Thus z has nonzero image $\overline{z}$ in $M'/\mathfrak p M' \otimes_k K$ But $R/\mathfrak p \otimes_k K$ is a domain as a localization of $\kappa(\mathfrak p)[x_1, \ldots, x_n]$ and $M'/\mathfrak p M' \otimes_k K$ is a free module, hence $f\overline{z} \neq 0$. This proves the claim.

Finally, pick $z \in M \otimes_k K$ such that $\mathfrak q$ is minimal over the annihilator $J \subset R \otimes_k K$ of z. For $f \in \mathfrak p$ there exists an $n \geq 1$ and a $g \in R \otimes_k K$, $g \notin \mathfrak q$ such that $gf^n z \in J$, i.e., $gf^n z = 0$. (This holds because $\mathfrak q$ lies over $\mathfrak p$ and $\mathfrak q$ is minimal over J.) Above we have seen that g is a nonzerodivisor hence $f^n z = 0$. This means that $\mathfrak p$ is a weakly associated prime of $M \otimes_k K$ viewed as an R-module. Since $M \otimes_k K$ is a direct sum of copies of M we conclude that $\mathfrak p$ is a weakly associated prime of M as before. □

67. Embedded primes

02M4 Here is the definition.

02M5 **Definition 67.1.** Let R be a ring. Let M be an R-module.

(1) The associated primes of M which are not minimal among the associated primes of M are called the *embedded associated primes* of M.
(2) The *embedded primes of* R are the embedded associated primes of R as an R-module.

Here is a way to get rid of these.

02M6 **Lemma 67.2.** *Let R be a Noetherian ring. Let M be a finite R-module. Consider the set of R-submodules*

$$\{K \subset M \mid Supp(K) \text{ nowhere dense in } Supp(M)\}.$$

This set has a maximal element K and the quotient $M' = M/K$ has the following properties

(1) *$Supp(M) = Supp(M')$,*
(2) *M' has no embedded associated primes,*
(3) *for any $f \in R$ which is contained in all embedded associated primes of M we have $M_f \cong M'_f$.*

Proof. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ denote the minimal primes in the support of M. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ denote the embedded associated primes of M. Then $\text{Ass}(M) = \{\mathfrak{q}_j, \mathfrak{p}_i\}$. There are finitely many of these, see Lemma 63.5. Set $I = \prod_{i=1,\ldots,s} \mathfrak{p}_i$. Then $I \not\subset \mathfrak{q}_j$ for any j. Hence by Lemma 15.2 we can find an $f \in I$ such that $f \not\in \mathfrak{q}_j$ for all $j = 1, \ldots, t$. Set $M' = \text{Im}(M \to M_f)$. This implies that $M_f \cong M'_f$. Since $M' \subset M_f$ we see that $\text{Ass}(M') \subset \text{Ass}(M_f) = \{\mathfrak{q}_j\}$. Thus M' has no embedded associated primes.

Moreover, the support of $K = \text{Ker}(M \to M')$ is contained in $V(\mathfrak{p}_1) \cup \ldots \cup V(\mathfrak{p}_s)$, because $\text{Ass}(K) \subset \text{Ass}(M)$ (see Lemma 63.3) and $\text{Ass}(K)$ contains none of the $\mathfrak{q}_i$ by construction. Clearly, K is in fact the largest submodule of M whose support is contained in $V(\mathfrak{p}_1) \cup \ldots \cup V(\mathfrak{p}_t)$. This implies that K is the maximal element of the set displayed in the lemma. $\square$

02M7 **Lemma 67.3.** *Let R be a Noetherian ring. Let M be a finite R-module. For any $f \in R$ we have $(M')_f = (M_f)'$ where $M \to M'$ and $M_f \to (M_f)'$ are the quotients constructed in Lemma 67.2.*

Proof. Omitted. $\square$

02M8 **Lemma 67.4.** *Let R be a Noetherian ring. Let M be a finite R-module without embedded associated primes. Let $I = \{x \in R \mid xM = 0\}$. Then the ring R/I has no embedded primes.*

Proof. We may replace R by R/I. Hence we may assume every nonzero element of R acts nontrivially on M. By Lemma 40.5 this implies that $\text{Spec}(R)$ equals the support of M. Suppose that $\mathfrak{p}$ is an embedded prime of R. Let $x \in R$ be an element whose annihilator is $\mathfrak{p}$. Consider the nonzero module $N = xM \subset M$. It is annihilated by $\mathfrak{p}$. Hence any associated prime $\mathfrak{q}$ of N contains $\mathfrak{p}$ and is also an associated prime of M. Then $\mathfrak{q}$ would be an embedded associated prime of M which contradicts the assumption of the lemma. $\square$

68. Regular sequences

0AUH In this section we develop some basic properties of regular sequences.

00LF **Definition 68.1.** Let R be a ring. Let M be an R-module. A sequence of elements $f_1, \ldots, f_r$ of R is called an *M-regular sequence* if the following conditions hold:

1. f_i is a nonzerodivisor on $M/(f_1, \ldots, f_{i-1})M$ for each $i = 1, \ldots, r$, and
2. the module $M/(f_1, \ldots, f_r)M$ is not zero.

If I is an ideal of R and $f_1, \ldots, f_r \in I$ then we call $f_1, \ldots, f_r$ an *M-regular sequence in I*. If $M = R$, we call $f_1, \ldots, f_r$ simply a *regular sequence* (in I).

Please pay attention to the fact that the definition depends on the order of the elements $f_1, \ldots, f_r$ (see examples below). Some papers/books drop the requirement that the module $M/(f_1, \ldots, f_r)M$ is nonzero. This has the advantage that being a regular sequence is preserved under localization. However, we will use this definition mainly to define the depth of a module in case R is local; in that case the f_i are required to be in the maximal ideal – a condition which is not preserved under going from R to a localization $R_\mathfrak{p}$.

00LG **Example 68.2.** Let k be a field. In the ring $k[x, y, z]$ the sequence $x, y(1-x), z(1-x)$ is regular but the sequence $y(1-x), z(1-x), x$ is not.

00LH **Example 68.3.** Let k be a field. Consider the ring $k[x, y, w_0, w_1, w_2, \ldots]/I$ where I is generated by yw_i, $i = 0, 1, 2, \ldots$ and $w_i - xw_{i+1}$, $i = 0, 1, 2, \ldots$. The sequence x, y is regular, but y is a zerodivisor. Moreover you can localize at the maximal ideal (x, y, w_i) and still get an example.

00LJ **Lemma 68.4.** *Let R be a local Noetherian ring. Let M be a finite R-module. Let $x_1, \ldots, x_c$ be an M-regular sequence. Then any permutation of the x_i is a regular sequence as well.*

Proof. First we do the case $c = 2$. Consider $K \subset M$ the kernel of $x_2 : M \to M$. For any $z \in K$ we know that $z = x_1 z'$ for some $z' \in M$ because x_2 is a nonzerodivisor on M/x_1M. Because x_1 is a nonzerodivisor on M we see that $x_2 z' = 0$ as well. Hence $x_1 : K \to K$ is surjective. Thus $K = 0$ by Nakayama's Lemma 20.1. Next, consider multiplication by x_1 on M/x_2M. If $z \in M$ maps to an element $\overline{z} \in M/x_2M$ in the kernel of this map, then $x_1 z = x_2 y$ for some $y \in M$. But then since x_1, x_2 is a regular sequence we see that $y = x_1 y'$ for some $y' \in M$. Hence $x_1(z - x_2 y') = 0$ and hence $z = x_2 y'$ and hence $\overline{z} = 0$ as desired.

For the general case, observe that any permutation is a composition of transpositions of adjacent indices. Hence it suffices to prove that

$$x_1, \ldots, x_{i-2}, x_i, x_{i-1}, x_{i+1}, \ldots, x_c$$

is an M-regular sequence. This follows from the case we just did applied to the module $M/(x_1, \ldots, x_{i-2})$ and the length 2 regular sequence x_{i-1}, x_i. $\square$

00LM **Lemma 68.5.** *Let R, S be local rings. Let $R \to S$ be a flat local ring homomorphism. Let $x_1, \ldots, x_r$ be a sequence in R. Let M be an R-module. The following are equivalent*

1. *$x_1, \ldots, x_r$ is an M-regular sequence in R, and*
2. *the images of $x_1, \ldots, x_r$ in S form a $M \otimes_R S$-regular sequence.*

Proof. This is so because $R \to S$ is faithfully flat by Lemma 39.17. $\square$

061L **Lemma 68.6.** *Let R be a Noetherian ring. Let M be a finite R-module. Let $\mathfrak{p}$ be a prime. Let $x_1, \ldots, x_r$ be a sequence in R whose image in $R_\mathfrak{p}$ forms an $M_\mathfrak{p}$-regular sequence. Then there exists a $g \in R$, $g \notin \mathfrak{p}$ such that the image of $x_1, \ldots, x_r$ in R_g forms an M_g-regular sequence.*

Proof. Set

$$K_i = \mathrm{Ker}\left(x_i : M/(x_1, \ldots, x_{i-1})M \to M/(x_1, \ldots, x_{i-1})M\right).$$

This is a finite R-module whose localization at $\mathfrak{p}$ is zero by assumption. Hence there exists a $g \in R$, $g \notin \mathfrak{p}$ such that $(K_i)_g = 0$ for all $i = 1, \ldots, r$. This g works. □

065K **Lemma 68.7.** *Let A be a ring. Let I be an ideal generated by a regular sequence $f_1, \ldots, f_n$ in A. Let $g_1, \ldots, g_m \in A$ be elements whose images $\overline{g}_1, \ldots, \overline{g}_m$ form a regular sequence in A/I. Then $f_1, \ldots, f_n, g_1, \ldots, g_m$ is a regular sequence in A.*

Proof. This follows immediately from the definitions. □

0F1T **Lemma 68.8.** *Let R be a ring. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of R-modules. Let $f_1, \ldots, f_r \in R$. If $f_1, \ldots, f_r$ is M_1-regular and M_3-regular, then $f_1, \ldots, f_r$ is M_2-regular.*

Proof. By Lemma 4.1, if $f_1 : M_1 \to M_1$ and $f_1 : M_3 \to M_3$ are injective, then so is $f_1 : M_2 \to M_2$ and we obtain a short exact sequence

$$0 \to M_1/f_1M_1 \to M_2/f_1M_2 \to M_3/f_1M_3 \to 0$$

The lemma follows from this and induction on r. Some details omitted. □

07DV **Lemma 68.9.** *Let R be a ring. Let M be an R-module. Let $f_1, \ldots, f_r \in R$ and $e_1, \ldots, e_r > 0$ integers. Then $f_1, \ldots, f_r$ is an M-regular sequence if and only if $f_1^{e_1}, \ldots, f_r^{e_r}$ is an M-regular sequence.*

Proof. We will prove this by induction on r. If $r = 1$ this follows from the following two easy facts: (a) a power of a nonzerodivisor on M is a nonzerodivisor on M and (b) a divisor of a nonzerodivisor on M is a nonzerodivisor on M. If $r > 1$, then by induction applied to M/f_1M we have that $f_1, f_2, \ldots, f_r$ is an M-regular sequence if and only if $f_1, f_2^{e_2}, \ldots, f_r^{e_r}$ is an M-regular sequence. Thus it suffices to show, given $e > 0$, that $f_1^e, f_2, \ldots, f_r$ is an M-regular sequence if and only if $f_1, \ldots, f_r$ is an M-regular sequence. We will prove this by induction on e. The case $e = 1$ is trivial. Since f_1 is a nonzerodivisor under both assumptions (by the case $r = 1$) we have a short exact sequence

$$0 \to M/f_1M \xrightarrow{f_1^{e-1}} M/f_1^eM \to M/f_1^{e-1}M \to 0$$

Suppose that $f_1, f_2, \ldots, f_r$ is an M-regular sequence. Then by induction the elements $f_2, \ldots, f_r$ are M/f_1M and $M/f_1^{e-1}M$-regular sequences. By Lemma 68.8 $f_2, \ldots, f_r$ is M/f_1^eM-regular. Hence $f_1^e, f_2, \ldots, f_r$ is M-regular. Conversely, suppose that $f_1^e, f_2, \ldots, f_r$ is an M-regular sequence. Then $f_2 : M/f_1^eM \to M/f_1^eM$ is injective, hence $f_2 : M/f_1M \to M/f_1M$ is injective, hence by induction(!) $f_2 : M/f_1^{e-1}M \to M/f_1^{e-1}M$ is injective, hence

$$0 \to M/(f_1, f_2)M \xrightarrow{f_1^{e-1}} M/(f_1^e, f_2)M \to M/(f_1^{e-1}, f_2)M \to 0$$

is a short exact sequence by Lemma 4.1. This proves the converse for $r = 2$. If $r > 2$, then we have $f_3 : M/(f_1^e, f_2)M \to M/(f_1^e, f_2)M$ is injective, hence $f_3 : M/(f_1, f_2)M \to M/(f_1, f_2)M$ is injective, and so on. Some details omitted. □

07DW **Lemma 68.10.** *Let R be a ring. Let $f_1, \ldots, f_r \in R$ which do not generate the unit ideal. The following are equivalent:*

(1) *any permutation of $f_1, \ldots, f_r$ is a regular sequence,*
(2) *any subsequence of $f_1, \ldots, f_r$ (in the given order) is a regular sequence, and*
(3) *$f_1x_1, \ldots, f_rx_r$ is a regular sequence in the polynomial ring $R[x_1, \ldots, x_r]$.*

Proof. It is clear that (1) implies (2). We prove (2) implies (1) by induction on r. The case $r = 1$ is trivial. The case $r = 2$ says that if $a, b \in R$ are a regular sequence and b is a nonzerodivisor, then b, a is a regular sequence. This is clear because the kernel of $a : R/(b) \to R/(b)$ is isomorphic to the kernel of $b : R/(a) \to R/(a)$ if both a and b are nonzerodivisors. The case $r > 2$. Assume (2) holds and say we want to prove $f_{\sigma(1)}, \ldots, f_{\sigma(r)}$ is a regular sequence for some permutation σ. We already know that $f_{\sigma(1)}, \ldots, f_{\sigma(r-1)}$ is a regular sequence by induction. Hence it suffices to show that f_s where $s = \sigma(r)$ is a nonzerodivisor modulo $f_1, \ldots, \hat{f}_s, \ldots, f_r$. If $s = r$ we are done. If $s < r$, then note that f_s and f_r are both nonzerodivisors in the ring $R/(f_1, \ldots, \hat{f}_s, \ldots, f_{r-1})$ (by induction hypothesis again). Since we know f_s, f_r is a regular sequence in that ring we conclude by the case of sequence of length 2 that f_r, f_s is too.

Note that $R[x_1, \ldots, x_r]/(f_1x_1, \ldots, f_ix_i)$ as an R-module is a direct sum of the modules

$$R/I_E \cdot x_1^{e_1} \ldots x_r^{e_r}$$

indexed by multi-indices $E = (e_1, \ldots, e_r)$ where I_E is the ideal generated by f_j for $1 \leq j \leq i$ with $e_j > 0$. Hence $f_{i+1}x_i$ is a nonzerodivisor on this if and only if f_{i+1} is a nonzerodivisor on R/I_E for all E. Taking E with all positive entries, we see that f_{i+1} is a nonzerodivisor on $R/(f_1, \ldots, f_i)$. Thus (3) implies (2). Conversely, if (2) holds, then any subsequence of $f_1, \ldots, f_i, f_{i+1}$ is a regular sequence in particular f_{i+1} is a nonzerodivisor on all R/I_E. In this way we see that (2) implies (3). □

69. Quasi-regular sequences

061M There is a notion of regular sequence which is slightly weaker than that of a regular sequence and easier to use. Let R be a ring and let $f_1, \ldots, f_c \in R$. Set $J = (f_1, \ldots, f_c)$. Let M be an R-module. Then there is a canonical map

061N (69.0.1)
$$M/JM \otimes_{R/J} R/J[X_1, \ldots, X_c] \longrightarrow \bigoplus\nolimits_{n \geq 0} J^nM/J^{n+1}M$$

of graded $R/J[X_1, \ldots, X_c]$-modules defined by the rule

$$\overline{m} \otimes X_1^{e_1} \ldots X_c^{e_c} \longmapsto f_1^{e_1} \ldots f_c^{e_c} m \bmod J^{e_1 + \ldots + e_c + 1}M.$$

Note that (69.0.1) is always surjective.

061P **Definition 69.1.** Let R be a ring. Let M be an R-module. A sequence of elements $f_1, \ldots, f_c$ of R is called *M-quasi-regular* if (69.0.1) is an isomorphism. If $M = R$, we call $f_1, \ldots, f_c$ simply a *quasi-regular sequence*.

So if $f_1, \ldots, f_c$ is a quasi-regular sequence, then

$$R/J[X_1, \ldots, X_c] = \bigoplus\nolimits_{n \geq 0} J^n/J^{n+1}$$

where $J = (f_1, \ldots, f_c)$. It is clear that being a quasi-regular sequence is independent of the order of $f_1, \ldots, f_c$.

00LN **Lemma 69.2.** *Let R be a ring.*

1. *A regular sequence $f_1, \ldots, f_c$ of R is a quasi-regular sequence.*
2. *Suppose that M is an R-module and that $f_1, \ldots, f_c$ is an M-regular sequence. Then $f_1, \ldots, f_c$ is an M-quasi-regular sequence.*

Proof. Set $J = (f_1, \ldots, f_c)$. We prove the first assertion by induction on c. We have to show that given any relation $\sum_{|I|=n} a_I f^I \in J^{n+1}$ with $a_I \in R$ we actually have $a_I \in J$ for all multi-indices I. Since any element of J^{n+1} is of the form $\sum_{|I|=n} b_I f^I$ with $b_I \in J$ we may assume, after replacing a_I by $a_I - b_I$, the relation reads $\sum_{|I|=n} a_I f^I = 0$. We can rewrite this as

$$\sum\nolimits_{e=0}^{n} \left(\sum\nolimits_{|I'|=n-e} a_{I',e} f^{I'} \right) f_c^e = 0$$

Here and below the "primed" multi-indices I' are required to be of the form $I' = (i_1, \ldots, i_{c-1}, 0)$. We will show by descending induction on $l \in \{0, \ldots, n\}$ that if we have a relation

$$\sum\nolimits_{e=0}^{l} \left(\sum\nolimits_{|I'|=n-e} a_{I',e} f^{I'} \right) f_c^e = 0$$

then $a_{I',e} \in J$ for all I', e. Namely, set $J' = (f_1, \ldots, f_{c-1})$. Observe that $\sum_{|I'|=n-l} a_{I',l} f^{I'}$ is mapped into $(J')^{n-l+1}$ by f_c^l. By induction hypothesis (for the induction on c) we see that $f_c^l a_{I',l} \in J'$. Because f_c is not a zerodivisor on R/J' (as $f_1, \ldots, f_c$ is a regular sequence) we conclude that $a_{I',l} \in J'$. This allows us to rewrite the term $(\sum_{|I'|=n-l} a_{I',l} f^{I'}) f_c^l$ in the form $(\sum_{|I'|=n-l+1} f_c b_{I',l-1} f^{I'}) f_c^{l-1}$. This gives a new relation of the form

$$\left(\sum\nolimits_{|I'|=n-l+1} (a_{I',l-1} + f_c b_{I',l-1}) f^{I'} \right) f_c^{l-1} + \sum\nolimits_{e=0}^{l-2} \left(\sum\nolimits_{|I'|=n-e} a_{I',e} f^{I'} \right) f_c^e = 0$$

Now by the induction hypothesis (on l this time) we see that all $a_{I',l-1} + f_c b_{I',l-1} \in J$ and all $a_{I',e} \in J$ for $e \leq l-2$. This, combined with $a_{I',l} \in J' \subset J$ seen above, finishes the proof of the induction step.

The second assertion means that given any formal expression $F = \sum_{|I|=n} m_I X^I$, $m_I \in M$ with $\sum m_I f^I \in J^{n+1} M$, then all the coefficients m_I are in J. This is proved in exactly the same way as we prove the corresponding result for the first assertion above. $\square$

065L **Lemma 69.3.** *Let $R \to R'$ be a flat ring map. Let M be an R-module. Suppose that $f_1, \ldots, f_r \in R$ form an M-quasi-regular sequence. Then the images of $f_1, \ldots, f_r$ in R' form a $M \otimes_R R'$-quasi-regular sequence.*

Proof. Set $J = (f_1, \ldots, f_r)$, $J' = JR'$ and $M' = M \otimes_R R'$. We have to show the canonical map $\mu : R'/J'[X_1, \ldots X_r] \otimes_{R'/J'} M'/J'M' \to \bigoplus (J')^n M'/(J')^{n+1} M'$ is an isomorphism. Because $R \to R'$ is flat the sequences $0 \to J^n M \to M$ and $0 \to J^{n+1} M \to J^n M \to J^n M / J^{n+1} M \to 0$ remain exact on tensoring with R'. This first implies that $J^n M \otimes_R R' = (J')^n M'$ and then that $(J')^n M'/(J')^{n+1} M' = J^n M / J^{n+1} M \otimes_R R'$. Thus μ is the tensor product of (69.0.1), which is an isomorphism by assumption, with $\mathrm{id}_{R'}$ and we conclude. $\square$

061Q **Lemma 69.4.** *Let R be a Noetherian ring. Let M be a finite R-module. Let $\mathfrak p$ be a prime. Let $x_1, \ldots, x_c$ be a sequence in R whose image in $R_{\mathfrak p}$ forms an $M_{\mathfrak p}$-quasi-regular sequence. Then there exists a $g \in R$, $g \notin \mathfrak p$ such that the image of $x_1, \ldots, x_c$ in R_g forms an M_g-quasi-regular sequence.*

Proof. Consider the kernel K of the map (69.0.1). As $M/JM \otimes_{R/J} R/J[X_1, \ldots, X_c]$ is a finite $R/J[X_1, \ldots, X_c]$-module and as $R/J[X_1, \ldots, X_c]$ is Noetherian, we see that K is also a finite $R/J[X_1, \ldots, X_c]$-module. Pick homogeneous generators $k_1, \ldots, k_t \in K$. By assumption for each $i = 1, \ldots, t$ there exists a $g_i \in R$, $g_i \notin \mathfrak p$ such that $g_i k_i = 0$. Hence $g = g_1 \ldots g_t$ works. $\square$

061R **Lemma 69.5.** *Let R be a ring. Let M be an R-module. Let $f_1, \ldots, f_c \in R$ be an M-quasi-regular sequence. For any i the sequence $\overline{f}_{i+1}, \ldots, \overline{f}_c$ of $\overline{R} = R/(f_1, \ldots, f_i)$ is an $\overline{M} = M/(f_1, \ldots, f_i)M$-quasi-regular sequence.*

Proof. It suffices to prove this for $i = 1$. Set $\overline{J} = (\overline{f}_2, \ldots, \overline{f}_c) \subset \overline{R}$. Then

$$\begin{aligned} \overline{J}^n\overline{M}/\overline{J}^{n+1}\overline{M} &= (J^nM + f_1M)/(J^{n+1}M + f_1M) \\ &= J^nM/(J^{n+1}M + J^nM \cap f_1M). \end{aligned}$$

Thus, in order to prove the lemma it suffices to show that $J^{n+1}M + J^nM \cap f_1M = J^{n+1}M + f_1J^{n-1}M$ because that will show that $\bigoplus_{n\geq 0} \overline{J}^n\overline{M}/\overline{J}^{n+1}\overline{M}$ is the quotient of $\bigoplus_{n\geq 0} J^nM/J^{n+1} \cong M/JM[X_1, \ldots, X_c]$ by X_1. Actually, we have $J^nM \cap f_1M = f_1J^{n-1}M$. Namely, if $m \notin J^{n-1}M$, then $f_1m \notin J^nM$ because $\bigoplus J^nM/J^{n+1}M$ is the polynomial algebra $M/J[X_1, \ldots, X_c]$ by assumption. $\square$

061S **Lemma 69.6.** *Let $(R, \mathfrak m)$ be a local Noetherian ring. Let M be a nonzero finite R-module. Let $f_1, \ldots, f_c \in \mathfrak m$ be an M-quasi-regular sequence. Then $f_1, \ldots, f_c$ is an M-regular sequence.*

Proof. Set $J = (f_1, \ldots, f_c)$. Let us show that f_1 is a nonzerodivisor on M. Suppose $x \in M$ is not zero. By Krull's intersection theorem there exists an integer r such that $x \in J^rM$ but $x \notin J^{r+1}M$, see Lemma 51.4. Then $f_1x \in J^{r+1}M$ is an element whose class in $J^{r+1}M/J^{r+2}M$ is nonzero by the assumed structure of $\bigoplus J^nM/J^{n+1}M$. Whence $f_1x \neq 0$.

Now we can finish the proof by induction on c using Lemma 69.5. $\square$

061T **Remark 69.7** (Koszul regular sequences)**.** In the paper [Kab71] the author introduces two more regularity conditions for sequences $x_1, \ldots, x_r$ of elements of a ring R. Namely, we say the sequence is *Koszul-regular* if $H_i(K_\bullet(R, x_\bullet)) = 0$ for $i \geq 1$ where $K_\bullet(R, x_\bullet)$ is the Koszul complex. The sequence is called *H_1-regular* if $H_1(K_\bullet(R, x_\bullet)) = 0$. If R is a local ring (possibly non-Noetherian) and the sequence consists of elements of the maximal ideal, then one has the implications regular $\Rightarrow$ Koszul-regular $\Rightarrow$ H_1-regular $\Rightarrow$ quasi-regular. By examples the author shows that these implications cannot be reversed in general. We introduce these notions in more detail in More on Algebra, Section 30.

065M **Remark 69.8.** Let k be a field. Consider the ring

$$A = k[x, y, w, z_0, z_1, z_2, \ldots]/(y^2z_0 - wx, z_0 - yz_1, z_1 - yz_2, \ldots)$$

In this ring x is a nonzerodivisor and the image of y in A/xA gives a quasi-regular sequence. But it is not true that x, y is a quasi-regular sequence in A because

$(x,y)/(x,y)^2$ isn't free of rank two over $A/(x,y)$ due to the fact that $wx = 0$ in $(x,y)/(x,y)^2$ but w isn't zero in $A/(x,y)$. Hence the analogue of Lemma 68.7 does not hold for quasi-regular sequences.

065N **Lemma 69.9.** *Let R be a ring. Let $J = (f_1, \ldots, f_r)$ be an ideal of R. Let M be an R-module. Set $\overline{R} = R/\bigcap_{n \geq 0} J^n$, $\overline{M} = M/\bigcap_{n \geq 0} J^n M$, and denote $\overline{f}_i$ the image of f_i in $\overline{R}$. Then $f_1, \ldots, f_r$ is M-quasi-regular if and only if $\overline{f}_1, \ldots, \overline{f}_r$ is $\overline{M}$-quasi-regular.*

Proof. This is true because $J^n M/J^{n+1}M \cong \overline{J}^n\overline{M}/\overline{J}^{n+1}\overline{M}$. $\square$

70. Blow up algebras

052P In this section we make some elementary observations about blowing up.

052Q **Definition 70.1.** Let R be a ring. Let $I \subset R$ be an ideal.

(1) The *blowup algebra*, or the *Rees algebra*, associated to the pair (R, I) is the graded R-algebra
$$\mathrm{Bl}_I(R) = \bigoplus\nolimits_{n \geq 0} I^n = R \oplus I \oplus I^2 \oplus \ldots$$
where the summand I^n is placed in degree n.
(2) Let $a \in I$ be an element. Denote $a^{(1)}$ the element a seen as an element of degree 1 in the Rees algebra. Then the *affine blowup algebra* $R[\frac{I}{a}]$ is the algebra $(\mathrm{Bl}_I(R))_{(a^{(1)})}$ constructed in Section 57.

In other words, an element of $R[\frac{I}{a}]$ is represented by an expression of the form x/a^n with $x \in I^n$. Two representatives x/a^n and y/a^m define the same element if and only if $a^k(a^m x - a^n y) = 0$ for some $k \geq 0$.

07Z3 **Lemma 70.2.** *Let R be a ring, $I \subset R$ an ideal, and $a \in I$. Let $R' = R[\frac{I}{a}]$ be the affine blowup algebra. Then*

(1) *the image of a in R' is a nonzerodivisor,*
(2) *$IR' = aR'$, and*
(3) *$(R')_a = R_a$.*

Proof. Immediate from the description of $R[\frac{I}{a}]$ above. $\square$

0BIP **Lemma 70.3.** *Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal and $a \in I$. Set $J = IS$ and let $b \in J$ be the image of a. Then $S[\frac{J}{b}]$ is the quotient of $S \otimes_R R[\frac{I}{a}]$ by the ideal of elements annihilated by some power of b.*

Proof. Let S' be the quotient of $S \otimes_R R[\frac{I}{a}]$ by its b-power torsion elements. The ring map
$$S \otimes_R R[\tfrac{I}{a}] \longrightarrow S[\tfrac{J}{b}]$$
is surjective and annihilates a-power torsion as b is a nonzerodivisor in $S[\frac{J}{b}]$. Hence we obtain a surjective map $S' \to S[\frac{J}{b}]$. To see that the kernel is trivial, we construct an inverse map. Namely, let $z = y/b^n$ be an element of $S[\frac{J}{b}]$, i.e., $y \in J^n$. Write $y = \sum x_i s_i$ with $x_i \in I^n$ and $s_i \in S$. We map z to the class of $\sum s_i \otimes x_i/a^n$ in S'. This is well defined because an element of the kernel of the map $S \otimes_R I^n \to J^n$ is annihilated by a^n, hence maps to zero in S'. $\square$

0G8Q **Example 70.4.** Let R be a ring. Let $P = R[t_1, \ldots, t_n]$ be the polynomial algebra. Let $I = (t_1, \ldots, t_n) \subset P$. With notation as in Definition 70.1 there is an isomorphism

$$P[T_1, \ldots, T_n]/(t_iT_j - t_jT_i) \longrightarrow \mathrm{Bl}_I(P)$$

sending T_i to $t_i^{(1)}$. We leave it to the reader to show that this map is well defined. Since I is generated by $t_1, \ldots, t_n$ we see that our map is surjective. To see that our map is injective one has to show: for each $e \geq 1$ the P-module I^e is generated by the monomials $t^E = t_1^{e_1} \ldots x_n^{e_n}$ for multiindices $E = (e_1, \ldots, e_n)$ of degree $|E| = e$ subject only to the relations $t_it^E = t_jt^{E'}$ when $|E| = |E'| = e$ and $e_a + \delta_{ai} = e'_a + \delta_{aj},\ a = 1, \ldots, n$ (Kronecker delta). We omit the details.

0G8R **Example 70.5.** Let R be a ring. Let $P = R[t_1, \ldots, t_n]$ be the polynomial algebra. Let $I = (t_1, \ldots, t_n) \subset P$. Let $a = t_1$. With notation as in Definition 70.1 there is an isomorphism

$$P[x_2, \ldots, x_n]/(t_1x_2 - t_2, \ldots, t_1x_n - t_n) \longrightarrow P[\tfrac{I}{a}] = P[\tfrac{I}{t_1}]$$

sending x_i to t_i/t_1. We leave it to the reader to show that this map is well defined. Since I is generated by $t_1, \ldots, t_n$ we see that our map is surjective. To see that our map is injective, the reader can argue that the source and target of our map are t_1-torsion free and that the map is an isomorphism after inverting t_1, see Lemma 70.2. Alternatively, the reader can use the description of the Rees algebra in Example 70.4. We omit the details.

0G8S **Lemma 70.6.** *Let R be a ring. Let $I = (a_1, \ldots, a_n)$ be an ideal of R. Let $a = a_1$. Then there is a surjection*

$$R[x_2, \ldots, x_n]/(ax_2 - a_2, \ldots, ax_n - a_n) \longrightarrow R[\tfrac{I}{a}]$$

whose kernel is the a-power torsion in the source.

Proof. Consider the ring map $P = \mathbf{Z}[t_1, \ldots, t_n] \to R$ sending t_i to a_i. Set $J = (t_1, \ldots, t_n)$. By Example 70.5 we have $P[\frac{J}{t_1}] = P[x_2, \ldots, x_n]/(t_1x_2 - t_2, \ldots, t_1x_n - t_n)$. Apply Lemma 70.3 to the map $P \to A$ to conclude. $\square$

080U **Lemma 70.7.** *Let R be a ring, $I \subset R$ an ideal, and $a \in I$. Set $R' = R[\frac{I}{a}]$. If $f \in R$ is such that $V(f) = V(I)$, then f maps to a nonzerodivisor in R' and $R'_f = R'_a = R_a$.*

Proof. We will use the results of Lemma 70.2 without further mention. The assumption $V(f) = V(I)$ implies $V(fR') = V(IR') = V(aR')$. Hence $a^n = fb$ and $f^m = ac$ for some $b, c \in R'$. The lemma follows. $\square$

0BBI **Lemma 70.8.** *Let R be a ring, $I \subset R$ an ideal, $a \in I$, and $f \in R$. Set $R' = R[\frac{I}{a}]$ and $R'' = R[\frac{fI}{fa}]$. Then there is a surjective R-algebra map $R' \to R''$ whose kernel is the set of f-power torsion elements of R'.*

Proof. The map is given by sending x/a^n for $x \in I^n$ to $f^nx/(fa)^n$. It is straightforward to check this map is well defined and surjective. Since af is a nonzero divisor in R'' (Lemma 70.2) we see that the set of f-power torsion elements are mapped to zero. Conversely, if $x \in R'$ and $f^nx \neq 0$ for all $n > 0$, then $(af)^nx \neq 0$ for all n as a is a nonzero divisor in R'. It follows that the image of x in R'' is not zero by the description of R'' following Definition 70.1. $\square$

052S **Lemma 70.9.** *If R is reduced then every (affine) blowup algebra of R is reduced.*

Proof. Let $I \subset R$ be an ideal and $a \in I$. Suppose x/a^n with $x \in I^n$ is a nilpotent element of $R[\frac{I}{a}]$. Then $(x/a^n)^m = 0$. Hence $a^N x^m = 0$ in R for some $N \geq 0$. After increasing N if necessary we may assume $N = me$ for some $e \geq 0$. Then $(a^e x)^m = 0$ and since R is reduced we find $a^e x = 0$. This means that $x/a^n = 0$ in $R[\frac{I}{a}]$. $\square$

052R **Lemma 70.10.** *Let R be a domain, $I \subset R$ an ideal, and $a \in I$ a nonzero element. Then the affine blowup algebra $R[\frac{I}{a}]$ is a domain.*

Proof. Suppose x/a^n, y/a^m with $x \in I^n$, $y \in I^m$ are elements of $R[\frac{I}{a}]$ whose product is zero. Then $a^N xy = 0$ in R. Since R is a domain we conclude that either $x = 0$ or $y = 0$. $\square$

052T **Lemma 70.11.** *Let R be a ring. Let $I \subset R$ be an ideal. Let $a \in I$. If a is not contained in any minimal prime of R, then $\mathrm{Spec}(R[\frac{I}{a}]) \to \mathrm{Spec}(R)$ has dense image.*

Proof. If $a^k x = 0$ for $x \in R$, then x is contained in all the minimal primes of R and hence nilpotent, see Lemma 17.2. Thus the kernel of $R \to R[\frac{I}{a}]$ consists of nilpotent elements. Hence the result follows from Lemma 30.6. $\square$

052M **Lemma 70.12.** *Let $(R, \mathfrak{m})$ be a local domain with fraction field K. Let $R \subset A \subset K$ be a valuation ring which dominates R. Then*

$$A = \mathrm{colim}\, R[\tfrac{I}{a}]$$

is a directed colimit of affine blowups $R \to R[\frac{I}{a}]$ with the following properties

1. *$a \in I \subset \mathfrak{m}$,*
2. *I is finitely generated, and*
3. *the fibre ring of $R \to R[\frac{I}{a}]$ at $\mathfrak{m}$ is not zero.*

Proof. Any blowup algebra $R[\frac{I}{a}]$ is a domain contained in K see Lemma 70.10. The lemma simply says that A is the directed union of the ones where $a \in I$ have properties (1), (2), (3). If $R[\frac{I}{a}] \subset A$ and $R[\frac{J}{b}] \subset A$, then we have

$$R[\tfrac{I}{a}] \cup R[\tfrac{J}{b}] \subset R[\tfrac{IJ}{ab}] \subset A$$

The first inclusion because $x/a^n = b^n x/(ab)^n$ and the second one because if $z \in (IJ)^n$, then $z = \sum x_i y_i$ with $x_i \in I^n$ and $y_i \in J^n$ and hence $z/(ab)^n = \sum (x_i/a^n)(y_i/b^n)$ is contained in A.

Consider a finite subset $E \subset A$. Say $E = \{e_1, \ldots, e_n\}$. Choose a nonzero $a \in R$ such that we can write $e_i = f_i/a$ for all $i = 1, \ldots, n$. Set $I = (f_1, \ldots, f_n, a)$. We claim that $R[\frac{I}{a}] \subset A$. This is clear as an element of $R[\frac{I}{a}]$ can be represented as a polynomial in the elements e_i. The lemma follows immediately from this observation. $\square$

71. Ext groups

00LO In this section we do a tiny bit of homological algebra, in order to establish some fundamental properties of depth over Noetherian local rings.

00LP **Lemma 71.1.** *Let R be a ring. Let M be an R-module.*

(1) *There exists an exact complex*

$$\ldots \to F_2 \to F_1 \to F_0 \to M \to 0.$$

with F_i free R-modules.

(2) *If R is Noetherian and M finite over R, then we can choose the complex such that F_i is finite free. In other words, we can find an exact complex*

$$\ldots \to R^{\oplus n_2} \to R^{\oplus n_1} \to R^{\oplus n_0} \to M \to 0.$$

Proof. Let us explain only the Noetherian case. As a first step choose a surjection $R^{n_0} \to M$. Then having constructed an exact complex of length e we simply choose a surjection $R^{n_{e+1}} \to \text{Ker}(R^{n_e} \to R^{n_{e-1}})$ which is possible because R is Noetherian. □

00LQ **Definition 71.2.** Let R be a ring. Let M be an R-module.

(1) A (left) *resolution* $F_\bullet \to M$ of M is an exact complex

$$\ldots \to F_2 \to F_1 \to F_0 \to M \to 0$$

of R-modules.

(2) A *resolution of M by free R-modules* is a resolution $F_\bullet \to M$ where each F_i is a free R-module.

(3) A *resolution of M by finite free R-modules* is a resolution $F_\bullet \to M$ where each F_i is a finite free R-module.

We often use the notation $F_\bullet$ to denote a complex of R-modules

$$\ldots \to F_i \to F_{i-1} \to \ldots$$

In this case we often use d_i or $d_{F,i}$ to denote the map $F_i \to F_{i-1}$. In this section we are always going to assume that F_0 is the last nonzero term in the complex. The *ith homology group of the complex $F_\bullet$* is the group $H_i = \text{Ker}(d_{F,i})/\text{Im}(d_{F,i+1})$. A *map of complexes* $\alpha : F_\bullet \to G_\bullet$ is given by maps $\alpha_i : F_i \to G_i$ such that $\alpha_{i-1} \circ d_{F,i} = d_{G,i-1} \circ \alpha_i$. Such a map induces a map on homology $H_i(\alpha) : H_i(F_\bullet) \to H_i(G_\bullet)$. If $\alpha, \beta : F_\bullet \to G_\bullet$ are maps of complexes, then a *homotopy* between α and β is given by a collection of maps $h_i : F_i \to G_{i+1}$ such that $\alpha_i - \beta_i = d_{G,i+1} \circ h_i + h_{i-1} \circ d_{F,i}$. Two maps $\alpha, \beta : F_\bullet \to G_\bullet$ are said to be *homotopic* if a homotopy between α and β exists.

We will use a very similar notation regarding complexes of the form $F^\bullet$ which look like

$$\ldots \to F^i \xrightarrow{d^i} F^{i+1} \to \ldots$$

There are maps of complexes, homotopies, etc. In this case we set $H^i(F^\bullet) = \text{Ker}(d^i)/\text{Im}(d^{i-1})$ and we call it the *ith cohomology group*.

00LR **Lemma 71.3.** *Any two homotopic maps of complexes induce the same maps on (co)homology groups.*

Proof. Omitted. □

00LS **Lemma 71.4.** *Let R be a ring. Let $M \to N$ be a map of R-modules. Let $N_\bullet \to N$ be an arbitrary resolution. Let*

$$\ldots \to F_2 \to F_1 \to F_0 \to M$$

be a complex of R-modules where each F_i is a free R-module. Then

(1) *there exists a map of complexes $F_\bullet \to N_\bullet$ such that*

$$\begin{tikzcd} F_0 \arrow[r] \arrow[d] & M \arrow[d] \\ N_0 \arrow[r] & N \end{tikzcd}$$

is commutative, and

(2) *any two maps $\alpha, \beta : F_\bullet \to N_\bullet$ as in (1) are homotopic.*

Proof. Proof of (1). Because F_0 is free we can find a map $F_0 \to N_0$ lifting the map $F_0 \to M \to N$. We obtain an induced map $F_1 \to F_0 \to N_0$ which ends up in the image of $N_1 \to N_0$. Since F_1 is free we may lift this to a map $F_1 \to N_1$. This in turn induces a map $F_2 \to F_1 \to N_1$ which maps to zero into N_0. Since $N_\bullet$ is exact we see that the image of this map is contained in the image of $N_2 \to N_1$. Hence we may lift to get a map $F_2 \to N_2$. Repeat.

Proof of (2). To show that α, β are homotopic it suffices to show the difference $\gamma = \alpha - \beta$ is homotopic to zero. Note that the image of $\gamma_0 : F_0 \to N_0$ is contained in the image of $N_1 \to N_0$. Hence we may lift γ_0 to a map $h_0 : F_0 \to N_1$. Consider the map $\gamma_1' = \gamma_1 - h_0 \circ d_{F,1}$. By our choice of h_0 we see that the image of γ_1' is contained in the kernel of $N_1 \to N_0$. Since $N_\bullet$ is exact we may lift γ_1' to a map $h_1 : F_1 \to N_2$. At this point we have $\gamma_1 = h_0 \circ d_{F,1} + d_{N,2} \circ h_1$. Repeat. $\square$

At this point we are ready to define the groups $\mathrm{Ext}^i_R(M, N)$. Namely, choose a resolution $F_\bullet$ of M by free R-modules, see Lemma 71.1. Consider the (cohomological) complex

$$\mathrm{Hom}_R(F_\bullet, N) : \mathrm{Hom}_R(F_0, N) \to \mathrm{Hom}_R(F_1, N) \to \mathrm{Hom}_R(F_2, N) \to \ldots$$

We define $\mathrm{Ext}^i_R(M, N)$ for $i \geq 0$ to be the ith cohomology group of this complex[7]. For $i < 0$ we set $\mathrm{Ext}^i_R(M, N) = 0$. Before we continue we point out that

$$\mathrm{Ext}^0_R(M, N) = \mathrm{Ker}(\mathrm{Hom}_R(F_0, N) \to \mathrm{Hom}_R(F_1, N)) = \mathrm{Hom}_R(M, N)$$

because we can apply part (1) of Lemma 10.1 to the exact sequence $F_1 \to F_0 \to M \to 0$. The following lemma explains in what sense this is well defined.

00LT **Lemma 71.5.** *Let R be a ring. Let M_1, M_2, N be R-modules. Suppose that $F_\bullet$ is a free resolution of the module M_1, and $G_\bullet$ is a free resolution of the module M_2. Let $\varphi : M_1 \to M_2$ be a module map. Let $\alpha : F_\bullet \to G_\bullet$ be a map of complexes inducing φ on $M_1 = \mathrm{Coker}(d_{F,1}) \to M_2 = \mathrm{Coker}(d_{G,1})$, see Lemma 71.4. Then the induced maps*

$$H^i(\alpha) : H^i(\mathrm{Hom}_R(F_\bullet, N)) \longrightarrow H^i(\mathrm{Hom}_R(G_\bullet, N))$$

are independent of the choice of α. If φ is an isomorphism, so are all the maps $H^i(\alpha)$. If $M_1 = M_2$, $F_\bullet = G_\bullet$, and φ is the identity, so are all the maps $H_i(\alpha)$.

Proof. Another map $\beta : F_\bullet \to G_\bullet$ inducing φ is homotopic to α by Lemma 71.4. Hence the maps $\mathrm{Hom}_R(F_\bullet, N) \to \mathrm{Hom}_R(G_\bullet, N)$ are homotopic. Hence the independence result follows from Lemma 71.3.

Suppose that φ is an isomorphism. Let $\psi : M_2 \to M_1$ be an inverse. Choose $\beta : G_\bullet \to F_\bullet$ be a map inducing $\psi : M_2 = \mathrm{Coker}(d_{G,1}) \to M_1 = \mathrm{Coker}(d_{F,1})$, see

[7] At this point it would perhaps be more appropriate to say “an” in stead of “the” Ext-group.

Lemma 71.4. OK, and now consider the map $H^i(\alpha) \circ H^i(\beta) = H^i(\alpha \circ \beta)$. By the above the map $H^i(\alpha \circ \beta)$ is the *same* as the map $H^i(\mathrm{id}_{G_\bullet}) = \mathrm{id}$. Similarly for the composition $H^i(\beta) \circ H^i(\alpha)$. Hence $H^i(\alpha)$ and $H^i(\beta)$ are inverses of each other. $\square$

00LU **Lemma 71.6.** *Let R be a ring. Let M be an R-module. Let $0 \to N' \to N \to N'' \to 0$ be a short exact sequence. Then we get a long exact sequence*

$$\begin{aligned} 0 \to \mathrm{Hom}_R(M, N') \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N'') \\ \to \mathrm{Ext}^1_R(M, N') \to \mathrm{Ext}^1_R(M, N) \to \mathrm{Ext}^1_R(M, N'') \to \ldots \end{aligned}$$

Proof. Pick a free resolution $F_\bullet \to M$. Since each of the F_i are free we see that we get a short exact sequence of complexes

$$0 \to \mathrm{Hom}_R(F_\bullet, N') \to \mathrm{Hom}_R(F_\bullet, N) \to \mathrm{Hom}_R(F_\bullet, N'') \to 0$$

Thus we get the long exact sequence from the snake lemma applied to this. $\square$

065P **Lemma 71.7.** *Let R be a ring. Let N be an R-module. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence. Then we get a long exact sequence*

$$\begin{aligned} 0 \to \mathrm{Hom}_R(M'', N) \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M', N) \\ \to \mathrm{Ext}^1_R(M'', N) \to \mathrm{Ext}^1_R(M, N) \to \mathrm{Ext}^1_R(M', N) \to \ldots \end{aligned}$$

Proof. Pick sets of generators $\{m'_{i'}\}_{i' \in I'}$ and $\{m''_{i''}\}_{i'' \in I''}$ of M' and M''. For each $i'' \in I''$ choose a lift $\tilde{m}''_{i''} \in M$ of the element $m''_{i''} \in M''$. Set $F' = \bigoplus_{i' \in I'} R$, $F'' = \bigoplus_{i'' \in I''} R$ and $F = F' \oplus F''$. Mapping the generators of these free modules to the corresponding chosen generators gives surjective R-module maps $F' \to M'$, $F'' \to M''$, and $F \to M$. We obtain a map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & M' & \to & M & \to & M'' & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & F' & \to & F & \to & F'' & \to & 0 \end{array}$$

By the snake lemma we see that the sequence of kernels $0 \to K' \to K \to K'' \to 0$ is short exact sequence of R-modules. Hence we can continue this process indefinitely. In other words we obtain a short exact sequence of resolutions fitting into the diagram

$$\begin{array}{ccccccccc} 0 & \to & M' & \to & M & \to & M'' & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & F'_\bullet & \to & F_\bullet & \to & F''_\bullet & \to & 0 \end{array}$$

Because each of the sequences $0 \to F'_n \to F_n \to F''_n \to 0$ is split exact (by construction) we obtain a short exact sequence of complexes

$$0 \to \mathrm{Hom}_R(F''_\bullet, N) \to \mathrm{Hom}_R(F_\bullet, N) \to \mathrm{Hom}_R(F'_\bullet, N) \to 0$$

by applying the $\mathrm{Hom}_R(-, N)$ functor. Thus we get the long exact sequence from the snake lemma applied to this. $\square$

00LV **Lemma 71.8.** *Let R be a ring. Let M, N be R-modules. Any $x \in R$ such that either $xN = 0$, or $xM = 0$ annihilates each of the modules $\mathrm{Ext}^i_R(M, N)$.*

Proof. Pick a free resolution $F_\bullet$ of M. Since $\mathrm{Ext}^i_R(M, N)$ is defined as the cohomology of the complex $\mathrm{Hom}_R(F_\bullet, N)$ the lemma is clear when $xN = 0$. If $xM = 0$, then we see that multiplication by x on $F_\bullet$ lifts the zero map on M. Hence by Lemma 71.5 we see that it induces the same map on Ext groups as the zero map. $\square$

08YR **Lemma 71.9.** *Let R be a Noetherian ring. Let M, N be finite R-modules. Then $\text{Ext}^i_R(M, N)$ is a finite R-module for all i.*

Proof. This holds because $\text{Ext}^i_R(M, N)$ is computed as the cohomology groups of a complex $\text{Hom}_R(F_\bullet, N)$ with each F_n a finite free R-module, see Lemma 71.1. □

72. Depth

00LE Here is our definition.

00LI **Definition 72.1.** Let R be a ring, and $I \subset R$ an ideal. Let M be a finite R-module. The *I-depth* of M, denoted $\text{depth}_I(M)$, is defined as follows:

(1) if $IM \neq M$, then $\text{depth}_I(M)$ is the supremum in $\{0, 1, 2, \ldots, \infty\}$ of the lengths of M-regular sequences in I,
(2) if $IM = M$ we set $\text{depth}_I(M) = \infty$.

If $(R, \mathfrak{m})$ is local we call $\text{depth}_{\mathfrak{m}}(M)$ simply the *depth* of M.

Explanation. By Definition 68.1 the empty sequence is not a regular sequence on the zero module, but for practical purposes it turns out to be convenient to set the depth of the 0 module equal to $+\infty$. Note that if $I = R$, then $\text{depth}_I(M) = \infty$ for all finite R-modules M. If I is contained in the Jacobson radical of R (e.g., if R is local and $I \subset \mathfrak{m}_R$), then $M \neq 0 \Rightarrow IM \neq M$ by Nakayama's lemma. A module M has I-depth 0 if and only if M is nonzero and I does not contain a nonzerodivisor on M.

Example 68.2 shows depth does not behave well even if the ring is Noetherian, and Example 68.3 shows that it does not behave well if the ring is local but non-Noetherian. We will see depth behaves well if the ring is local Noetherian.

0AUI **Lemma 72.2.** *Let R be a ring, $I \subset R$ an ideal, and M a finite R-module. Then $\text{depth}_I(M)$ is equal to the supremum of the lengths of sequences $f_1, \ldots, f_r \in I$ such that f_i is a nonzerodivisor on $M/(f_1, \ldots, f_{i-1})M$.*

Proof. Suppose that $IM = M$. Then Lemma 20.1 shows there exists an $f \in I$ such that $f : M \to M$ is id_M. Hence $f, 0, 0, 0, \ldots$ is an infinite sequence of successive nonzerodivisors and we see agreement holds in this case. If $IM \neq M$, then we see that a sequence as in the lemma is an M-regular sequence and we conclude that agreement holds as well. □

00LK **Lemma 72.3.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let M be a nonzero finite R-module. Then $\dim(\text{Supp}(M)) \geq \text{depth}(M)$.*

Proof. The proof is by induction on $\dim(\text{Supp}(M))$. If $\dim(\text{Supp}(M)) = 0$, then $\text{Supp}(M) = \{\mathfrak{m}\}$, whence $\text{Ass}(M) = \{\mathfrak{m}\}$ (by Lemmas 63.2 and 63.7), and hence the depth of M is zero for example by Lemma 63.18. For the induction step we assume $\dim(\text{Supp}(M)) > 0$. Let $f_1, \ldots, f_d$ be a sequence of elements of $\mathfrak{m}$ such that f_i is a nonzerodivisor on $M/(f_1, \ldots, f_{i-1})M$. According to Lemma 72.2 it suffices to prove $\dim(\text{Supp}(M)) \geq d$. We may assume $d > 0$ otherwise the lemma holds. By Lemma 63.10 we have $\dim(\text{Supp}(M/f_1M)) = \dim(\text{Supp}(M)) - 1$. By induction we conclude $\dim(\text{Supp}(M/f_1M)) \geq d - 1$ as desired. □

0AUJ **Lemma 72.4.** *Let R be a Noetherian ring, $I \subset R$ an ideal, and M a finite nonzero R-module such that $IM \neq M$. Then $\text{depth}_I(M) < \infty$.*

Proof. Since M/IM is nonzero we can choose $\mathfrak p \in \text{Supp}(M/IM)$ by Lemma 40.2. Then $(M/IM)_{\mathfrak p} \neq 0$ which implies $I \subset \mathfrak p$ and moreover implies $M_{\mathfrak p} \neq IM_{\mathfrak p}$ as localization is exact. Let $f_1, \ldots, f_r \in I$ be an M-regular sequence. Then $M_{\mathfrak p}/(f_1, \ldots, f_r)M_{\mathfrak p}$ is nonzero as $(f_1, \ldots, f_r) \subset I$. As localization is flat we see that the images of $f_1, \ldots, f_r$ form a $M_{\mathfrak p}$-regular sequence in $I_{\mathfrak p}$. Since this works for every M-regular sequence in I we conclude that $\text{depth}_I(M) \leq \text{depth}_{I_{\mathfrak p}}(M_{\mathfrak p})$. The latter is $\leq \text{depth}(M_{\mathfrak p})$ which is $< \infty$ by Lemma 72.3. □

00LW **Lemma 72.5.** *Let R be a Noetherian local ring with maximal ideal $\mathfrak m$. Let M be a nonzero finite R-module. Then depth(M) is equal to the smallest integer i such that $\text{Ext}^i_R(R/\mathfrak m, M)$ is nonzero.*

Proof. Let $\delta(M)$ denote the depth of M and let $i(M)$ denote the smallest integer i such that $\text{Ext}^i_R(R/\mathfrak m, M)$ is nonzero. We will see in a moment that $i(M) < \infty$. By Lemma 63.18 we have $\delta(M) = 0$ if and only if $i(M) = 0$, because $\mathfrak m \in \text{Ass}(M)$ exactly means that $i(M) = 0$. Hence if $\delta(M)$ or $i(M)$ is > 0, then we may choose $x \in \mathfrak m$ such that (a) x is a nonzerodivisor on M, and (b) $\text{depth}(M/xM) = \delta(M) - 1$. Consider the long exact sequence of Ext-groups associated to the short exact sequence $0 \to M \to M \to M/xM \to 0$ by Lemma 71.6:

$$\begin{aligned} 0 &\to \text{Hom}_R(\kappa, M) \to \text{Hom}_R(\kappa, M) \to \text{Hom}_R(\kappa, M/xM) \\ &\to \text{Ext}^1_R(\kappa, M) \to \text{Ext}^1_R(\kappa, M) \to \text{Ext}^1_R(\kappa, M/xM) \to \ldots \end{aligned}$$

Since $x \in \mathfrak m$ all the maps $\text{Ext}^i_R(\kappa, M) \to \text{Ext}^i_R(\kappa, M)$ are zero, see Lemma 71.8. Thus it is clear that $i(M/xM) = i(M) - 1$. Induction on $\delta(M)$ finishes the proof. □

00LX **Lemma 72.6.** *Let R be a local Noetherian ring. Let $0 \to N' \to N \to N'' \to 0$ be a short exact sequence of nonzero finite R-modules.*

(1) *depth(N) $\geq \min\{$depth(N'), depth(N'')$\}$*
(2) *depth(N'') $\geq \min\{$depth(N), depth(N') $- 1\}$*
(3) *depth(N') $\geq \min\{$depth(N), depth(N'') $+ 1\}$*

Proof. Use the characterization of depth using the Ext groups $\text{Ext}^i(\kappa, N)$, see Lemma 72.5, and use the long exact cohomology sequence

$$\begin{aligned} 0 &\to \text{Hom}_R(\kappa, N') \to \text{Hom}_R(\kappa, N) \to \text{Hom}_R(\kappa, N'') \\ &\to \text{Ext}^1_R(\kappa, N') \to \text{Ext}^1_R(\kappa, N) \to \text{Ext}^1_R(\kappa, N'') \to \ldots \end{aligned}$$

from Lemma 71.6. □

090R **Lemma 72.7.** *Let R be a local Noetherian ring and M a nonzero finite R-module.*

(1) *If $x \in \mathfrak m$ is a nonzerodivisor on M, then depth(M/xM) = depth(M) $- 1$.*
(2) *Any M-regular sequence $x_1, \ldots, x_r$ can be extended to an M-regular sequence of length depth(M).*

Proof. Part (2) is a formal consequence of part (1). Let $x \in R$ be as in (1). By the short exact sequence $0 \to M \to M \to M/xM \to 0$ and Lemma 72.6 we see that the depth drops by at most 1. On the other hand, if $x_1, \ldots, x_r \in \mathfrak m$ is a regular sequence for M/xM, then $x, x_1, \ldots, x_r$ is a regular sequence for M. Hence we see that the depth drops by at least 1. □

0CN5 **Lemma 72.8.** *Let $(R, \mathfrak m)$ be a local Noetherian ring and M a finite R-module. Let $x \in \mathfrak m$, $\mathfrak p \in$ Ass(M), and $\mathfrak q$ minimal over $\mathfrak p + (x)$. Then $\mathfrak q \in$ Ass(M/x^nM) for some $n \geq 1$.*

Proof. Pick a submodule $N \subset M$ with $N \cong R/\mathfrak p$. By the Artin-Rees lemma (Lemma 51.2) we can pick $n > 0$ such that $N \cap x^nM \subset xN$. Let $\overline{N} \subset M/x^nM$ be the image of $N \to M \to M/x^nM$. By Lemma 63.3 it suffices to show $\mathfrak q \in \text{Ass}(\overline{N})$. By our choice of n there is a surjection $\overline{N} \to N/xN = R/\mathfrak p + (x)$ and hence $\mathfrak q$ is in the support of $\overline{N}$. Since $\overline{N}$ is annihilated by x^n and $\mathfrak p$ we see that $\mathfrak q$ is minimal among the primes in the support of $\overline{N}$. Thus $\mathfrak q$ is an associated prime of $\overline{N}$ by Lemma 63.8. $\square$

0BK4 **Lemma 72.9.** *Let $(R, \mathfrak m)$ be a local Noetherian ring and M a finite R-module. For $\mathfrak p \in Ass(M)$ we have $\dim(R/\mathfrak p) \geq depth(M)$.*

Proof. If $\mathfrak m \in \text{Ass}(M)$ then there is a nonzero element $x \in M$ which is annihilated by all elements of $\mathfrak m$. Thus $\text{depth}(M) = 0$. In particular the lemma holds in this case.

If $\text{depth}(M) = 1$, then by the first paragraph we find that $\mathfrak m \notin \text{Ass}(M)$. Hence $\dim(R/\mathfrak p) \geq 1$ for all $\mathfrak p \in \text{Ass}(M)$ and the lemma is true in this case as well.

We will prove the lemma in general by induction on $\text{depth}(M)$ which we may and do assume to be > 1. Pick $x \in \mathfrak m$ which is a nonzerodivisor on M. Note $x \notin \mathfrak p$ (Lemma 63.9). By Lemma 60.13 we have $\dim(R/\mathfrak p + (x)) = \dim(R/\mathfrak p) - 1$. Thus there exists a prime $\mathfrak q$ minimal over $\mathfrak p + (x)$ with $\dim(R/\mathfrak q) = \dim(R/\mathfrak p) - 1$ (small argument omitted; hint: the dimension of a Noetherian local ring A is the maximum of the dimensions of $A/\mathfrak r$ taken over the minimal primes $\mathfrak r$ of A). Pick n as in Lemma 72.8 so that $\mathfrak q$ is an associated prime of M/x^nM. We may apply induction hypothesis to M/x^nM and $\mathfrak q$ because $\text{depth}(M/x^nM) = \text{depth}(M) - 1$ by Lemma 72.7. We find $\dim(R/\mathfrak q) \geq \text{depth}(M/x^nM)$ and we win. $\square$

0FCC **Lemma 72.10.** *Let R be a local Noetherian ring and M a finite R-module. For a prime ideal $\mathfrak p \subset R$ we have $depth(M_\mathfrak p) + \dim(R/\mathfrak p) \geq depth(M)$.*

Proof. If $M_\mathfrak p = 0$, then $\text{depth}(M_\mathfrak p) = \infty$ and the lemma holds. If $\text{depth}(M) \leq \dim(R/\mathfrak p)$, then the lemma is true. If $\text{depth}(M) > \dim(R/\mathfrak p)$, then $\mathfrak p$ is not contained in any associated prime $\mathfrak q$ of M by Lemma 72.9. Hence we can find an $x \in \mathfrak p$ not contained in any associated prime of M by Lemma 15.2 and Lemma 63.5. Then x is a nonzerodivisor on M, see Lemma 63.9. Hence $\text{depth}(M/xM) = \text{depth}(M) - 1$ and $\text{depth}(M_\mathfrak p/xM_\mathfrak p) = \text{depth}(M_\mathfrak p) - 1$ provided $M_\mathfrak p$ is nonzero, see Lemma 72.7. Thus we conclude by induction on $\text{depth}(M)$. $\square$

0AUK **Lemma 72.11.** *Let $(R, \mathfrak m)$ be a Noetherian local ring. Let $R \to S$ be a finite ring map. Let $\mathfrak m_1, \ldots, \mathfrak m_n$ be the maximal ideals of S. Let N be a finite S-module. Then*

$$\min\nolimits_{i=1,\ldots,n} depth(N_{\mathfrak m_i}) = depth_\mathfrak m(N)$$

Proof. By Lemmas 36.20, 36.22, and Lemma 36.21 the maximal ideals of S are exactly the primes of S lying over $\mathfrak m$ and there are finitely many of them. Hence the statement of the lemma makes sense. We will prove the lemma by induction on $k = \min_{i=1,\ldots,n} \text{depth}(N_{\mathfrak m_i})$. If $k = 0$, then $\text{depth}(N_{\mathfrak m_i}) = 0$ for some i. By Lemma 72.5 this means $\mathfrak m_i S_{\mathfrak m_i}$ is an associated prime of $N_{\mathfrak m_i}$ and hence $\mathfrak m_i$ is an associated prime of N (Lemma 63.16). By Lemma 63.13 we see that $\mathfrak m$ is an associated prime of N as an R-module. Whence $\text{depth}_\mathfrak m(N) = 0$. This proves the base case. If $k > 0$, then we see that $\mathfrak m_i \notin \text{Ass}_S(N)$. Hence $\mathfrak m \notin \text{Ass}_R(N)$, again by Lemma 63.13. Thus we can find $f \in \mathfrak m$ which is not a zerodivisor on N, see Lemma 63.18.

By Lemma 72.7 all the depths drop exactly by 1 when passing from N to N/fN and the induction hypothesis does the rest. $\square$

73. Functorialities for Ext

087M In this section we briefly discuss the functoriality of Ext with respect to change of ring, etc. Here is a list of items to work out.

(1) Given $R \to R'$, an R-module M and an R'-module N' the R-module $\text{Ext}^i_R(M, N')$ has a natural R'-module structure. Moreover, there is a canonical R'-linear map $\text{Ext}^i_{R'}(M \otimes_R R', N') \to \text{Ext}^i_R(M, N')$.
(2) Given $R \to R'$ and R-modules M, N there is a natural R-module map $\text{Ext}^i_R(M, N) \to \text{Ext}^i_R(M, N \otimes_R R')$.

087N **Lemma 73.1.** *Given a flat ring map $R \to R'$, an R-module M, and an R'-module N' the natural map*

$$\text{Ext}^i_{R'}(M \otimes_R R', N') \to \mathit{Ext}^i_R(M, N')$$

is an isomorphism for $i \geq 0$.

Proof. Choose a free resolution $F_\bullet$ of M. Since $R \to R'$ is flat we see that $F_\bullet \otimes_R R'$ is a free resolution of $M \otimes_R R'$ over R'. The statement is that the map

$$\text{Hom}_{R'}(F_\bullet \otimes_R R', N') \to \text{Hom}_R(F_\bullet, N')$$

induces an isomorphism on homology groups, which is true because it is an isomorphism of complexes by Lemma 14.3. $\square$

74. An application of Ext groups

02HN Here it is.

02HO **Lemma 74.1.** *Let R be a Noetherian ring. Let $I \subset R$ be an ideal contained in the Jacobson radical of R. Let $N \to M$ be a homomorphism of finite R-modules. Suppose that there exists arbitrarily large n such that $N/I^nN \to M/I^nM$ is a split injection. Then $N \to M$ is a split injection.*

Proof. Assume $\varphi : N \to M$ satisfies the assumptions of the lemma. Note that this implies that $\text{Ker}(\varphi) \subset I^nN$ for arbitrarily large n. Hence by Lemma 51.5 we see that φ is injection. Let $Q = M/N$ so that we have a short exact sequence

$$0 \to N \to M \to Q \to 0.$$

Let

$$F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to Q \to 0$$

be a finite free resolution of Q. We can choose a map $\alpha : F_0 \to M$ lifting the map $F_0 \to Q$. This induces a map $\beta : F_1 \to N$ such that $\beta \circ d_2 = 0$. The extension above is split if and only if there exists a map $\gamma : F_0 \to N$ such that $\beta = \gamma \circ d_1$. In other words, the class of β in $\text{Ext}^1_R(Q, N)$ is the obstruction to splitting the short exact sequence above.

Suppose n is a large integer such that $N/I^nN \to M/I^nM$ is a split injection. This implies

$$0 \to N/I^nN \to M/I^nM \to Q/I^nQ \to 0.$$

is still short exact. Also, the sequence

$$F_1/I^nF_1 \xrightarrow{d_1} F_0/I^nF_0 \to Q/I^nQ \to 0$$

is still exact. Arguing as above we see that the map $\overline{\beta} : F_1/I^nF_1 \to N/I^nN$ induced by β is equal to $\overline{\gamma_n} \circ d_1$ for some map $\overline{\gamma_n} : F_0/I^nF_0 \to N/I^nN$. Since F_0 is free we can lift $\overline{\gamma_n}$ to a map $\gamma_n : F_0 \to N$ and then we see that $\beta - \gamma_n \circ d_1$ is a map from F_1 into I^nN. In other words we conclude that

$$\beta \in \mathrm{Im}\Big(\mathrm{Hom}_R(F_0, N) \to \mathrm{Hom}_R(F_1, N)\Big) + I^n\,\mathrm{Hom}_R(F_1, N).$$

for this n.

Since we have this property for arbitrarily large n by assumption we conclude that the image of β in the cokernel of $\mathrm{Hom}_R(F_0, N) \to \mathrm{Hom}_R(F_1, N)$ is zero by Lemma 51.5. Hence β is in the image of the map $\mathrm{Hom}_R(F_0, N) \to \mathrm{Hom}_R(F_1, N)$ as desired. □

75. Tor groups and flatness

00LY In this section we use some of the homological algebra developed in the previous section to explain what Tor groups are. Namely, suppose that R is a ring and that M, N are two R-modules. Choose a resolution $F_\bullet$ of M by free R-modules. See Lemma 71.1. Consider the homological complex

$$F_\bullet \otimes_R N : \ldots \to F_2 \otimes_R N \to F_1 \otimes_R N \to F_0 \otimes_R N$$

We define $\mathrm{Tor}_i^R(M, N)$ to be the ith homology group of this complex. The following lemma explains in what sense this is well defined.

00LZ **Lemma 75.1.** *Let R be a ring. Let M_1, M_2, N be R-modules. Suppose that $F_\bullet$ is a free resolution of the module M_1 and that $G_\bullet$ is a free resolution of the module M_2. Let $\varphi : M_1 \to M_2$ be a module map. Let $\alpha : F_\bullet \to G_\bullet$ be a map of complexes inducing φ on $M_1 = \mathrm{Coker}(d_{F,1}) \to M_2 = \mathrm{Coker}(d_{G,1})$, see Lemma 71.4. Then the induced maps*

$$H_i(\alpha) : H_i(F_\bullet \otimes_R N) \longrightarrow H_i(G_\bullet \otimes_R N)$$

are independent of the choice of α. If φ is an isomorphism, so are all the maps $H_i(\alpha)$. If $M_1 = M_2$, $F_\bullet = G_\bullet$, and φ is the identity, so are all the maps $H_i(\alpha)$.

Proof. The proof of this lemma is identical to the proof of Lemma 71.5. □

Not only does this lemma imply that the Tor modules are well defined, but it also provides for the functoriality of the constructions $(M, N) \mapsto \mathrm{Tor}_i^R(M, N)$ in the first variable. Of course the functoriality in the second variable is evident. We leave it to the reader to see that each of the Tor_i^R is in fact a functor

$$\mathrm{Mod}_R \times \mathrm{Mod}_R \to \mathrm{Mod}_R.$$

Here Mod_R denotes the category of R-modules, and for the definition of the product category see Categories, Definition 2.20. Namely, given morphisms of R-modules $M_1 \to M_2$ and $N_1 \to N_2$ we get a commutative diagram

$$\begin{tikzcd}
\mathrm{Tor}_i^R(M_1, N_1) \arrow[r] \arrow[d] & \mathrm{Tor}_i^R(M_1, N_2) \arrow[d] \\
\mathrm{Tor}_i^R(M_2, N_1) \arrow[r] & \mathrm{Tor}_i^R(M_2, N_2)
\end{tikzcd}$$

00M0 **Lemma 75.2.** *Let R be a ring and let M be an R-module. Suppose that $0 \to N' \to N \to N'' \to 0$ is a short exact sequence of R-modules. There exists a long exact sequence*

$$Tor_1^R(M, N') \to Tor_1^R(M, N) \to Tor_1^R(M, N'') \to M \otimes_R N' \to M \otimes_R N \to M \otimes_R N'' \to 0$$

Proof. The proof of this is the same as the proof of Lemma 71.6. □

Consider a homological double complex of R-modules

$$\begin{tikzcd}
\ldots \arrow[r, "d"] & A_{2,0} \arrow[r, "d"] & A_{1,0} \arrow[r, "d"] & A_{0,0} \\
\ldots \arrow[r, "d"] & A_{2,1} \arrow[r, "d"] \arrow[u, "\delta"] & A_{1,1} \arrow[r, "d"] \arrow[u, "\delta"] & A_{0,1} \arrow[u, "\delta"] \\
\ldots \arrow[r, "d"] & A_{2,2} \arrow[r, "d"] \arrow[u, "\delta"] & A_{1,2} \arrow[r, "d"] \arrow[u, "\delta"] & A_{0,2} \arrow[u, "\delta"] \\
 & \ldots \arrow[u, "\delta"] & \ldots \arrow[u, "\delta"] & \ldots \arrow[u, "\delta"]
\end{tikzcd}$$

This means that $d_{i,j} : A_{i,j} \to A_{i-1,j}$ and $\delta_{i,j} : A_{i,j} \to A_{i,j-1}$ have the following properties

(1) Any composition of two $d_{i,j}$ is zero. In other words the rows of the double complex are complexes.
(2) Any composition of two $\delta_{i,j}$ is zero. In other words the columns of the double complex are complexes.
(3) For any pair (i, j) we have $\delta_{i-1,j} \circ d_{i,j} = d_{i,j-1} \circ \delta_{i,j}$. In other words, all the squares commute.

The correct thing to do is to associate a spectral sequence to any such double complex. However, for the moment we can get away with doing something slightly easier.

Namely, for the purposes of this section only, given a double complex $(A_{\bullet,\bullet}, d, \delta)$ set $R(A)_j = \text{Coker}(A_{1,j} \to A_{0,j})$ and $U(A)_i = \text{Coker}(A_{i,1} \to A_{i,0})$. (The letters R and U are meant to suggest Right and Up.) We endow $R(A)_\bullet$ with the structure of a complex using the maps δ. Similarly we endow $U(A)_\bullet$ with the structure of a complex using the maps d. In other words we obtain the following huge commutative diagram

$$\begin{tikzcd}
\ldots \arrow[r, "d"] & U(A)_2 \arrow[r, "d"] & U(A)_1 \arrow[r, "d"] & U(A)_0 & \\
\ldots \arrow[r, "d"] & A_{2,0} \arrow[r, "d"] \arrow[u] & A_{1,0} \arrow[r, "d"] \arrow[u] & A_{0,0} \arrow[r] \arrow[u] & R(A)_0 \\
\ldots \arrow[r, "d"] & A_{2,1} \arrow[r, "d"] \arrow[u, "\delta"] & A_{1,1} \arrow[r, "d"] \arrow[u, "\delta"] & A_{0,1} \arrow[r] \arrow[u, "\delta"] & R(A)_1 \arrow[u, "\delta"] \\
\ldots \arrow[r, "d"] & A_{2,2} \arrow[r, "d"] \arrow[u, "\delta"] & A_{1,2} \arrow[r, "d"] \arrow[u, "\delta"] & A_{0,2} \arrow[r] \arrow[u, "\delta"] & R(A)_2 \arrow[u, "\delta"] \\
 & \ldots \arrow[u, "\delta"] & \ldots \arrow[u, "\delta"] & \ldots \arrow[u, "\delta"] & \ldots \arrow[u, "\delta"]
\end{tikzcd}$$

(This is no longer a double complex of course.) It is clear what a morphism $\Phi : (A_{\bullet,\bullet}, d, \delta) \to (B_{\bullet,\bullet}, d, \delta)$ of double complexes is, and it is clear that this induces morphisms of complexes $R(\Phi) : R(A)_\bullet \to R(B)_\bullet$ and $U(\Phi) : U(A)_\bullet \to U(B)_\bullet$.

00M1 **Lemma 75.3.** *Let $(A_{\bullet,\bullet}, d, \delta)$ be a double complex such that*

(1) *Each row $A_{\bullet,j}$ is a resolution of $R(A)_j$.*
(2) *Each column $A_{i,\bullet}$ is a resolution of $U(A)_i$.*

Then there are canonical isomorphisms

$$H_i(R(A)_\bullet) \cong H_i(U(A)_\bullet).$$

The isomorphisms are functorial with respect to morphisms of double complexes with the properties above.

Proof. We will show that $H_i(R(A)_\bullet))$ and $H_i(U(A)_\bullet)$ are canonically isomorphic to a third group. Namely

$$\mathbf{H}_i(A) := \frac{\{(a_{i,0}, a_{i-1,1}, \ldots, a_{0,i}) \mid d(a_{i,0}) = \delta(a_{i-1,1}), \ldots, d(a_{1,i-1}) = \delta(a_{0,i})\}}{\{d(a_{i+1,0}) + \delta(a_{i,1}), d(a_{i,1}) + \delta(a_{i-1,2}), \ldots, d(a_{1,i}) + \delta(a_{0,i+1})\}}$$

Here we use the notational convention that $a_{i,j}$ denotes an element of $A_{i,j}$. In other words, an element of $\mathbf{H}_i$ is represented by a zig-zag, represented as follows for $i = 2$

$$\begin{tikzcd}
a_{2,0} \arrow[r, mapsto] & d(a_{2,0}) = \delta(a_{1,1}) & \\
 & a_{1,1} \arrow[u, mapsto] \arrow[r, mapsto] & d(a_{1,1}) = \delta(a_{0,2}) \\
 & & a_{0,2} \arrow[u, mapsto]
\end{tikzcd}$$

Naturally, we divide out by "trivial" zig-zags, namely the submodule generated by elements of the form $(0, \ldots, 0, -\delta(a_{t+1,t-i}), d(a_{t+1,t-i}), 0, \ldots, 0)$. Note that there are canonical homomorphisms

$$\mathbf{H}_i(A) \to H_i(R(A)_\bullet), \quad (a_{i,0}, a_{i-1,1}, \ldots, a_{0,i}) \mapsto \text{class of image of } a_{0,i}$$

and

$$\mathbf{H}_i(A) \to H_i(U(A)_\bullet), \quad (a_{i,0}, a_{i-1,1}, \ldots, a_{0,i}) \mapsto \text{class of image of } a_{i,0}$$

First we show that these maps are surjective. Suppose that $\overline{r} \in H_i(R(A)_\bullet)$. Let $r \in R(A)_i$ be a cocycle representing the class of $\overline{r}$. Let $a_{0,i} \in A_{0,i}$ be an element which maps to r. Because $\delta(r) = 0$, we see that $\delta(a_{0,i})$ is in the image of d. Hence there exists an element $a_{1,i-1} \in A_{1,i-1}$ such that $d(a_{1,i-1}) = \delta(a_{0,i})$. This in turn implies that $\delta(a_{1,i-1})$ is in the kernel of d (because $d(\delta(a_{1,i-1})) = \delta(d(a_{1,i-1})) = \delta(\delta(a_{0,i})) = 0$. By exactness of the rows we find an element $a_{2,i-2}$ such that $d(a_{2,i-2}) = \delta(a_{1,i-1})$. And so on until a full zig-zag is found. Of course surjectivity of $\mathbf{H}_i \to H_i(U(A))$ is shown similarly.

To prove injectivity we argue in exactly the same way. Namely, suppose we are given a zig-zag $(a_{i,0}, a_{i-1,1}, \ldots, a_{0,i})$ which maps to zero in $H_i(R(A)_\bullet)$. This means that $a_{0,i}$ maps to an element of $\text{Coker}(A_{i,1} \to A_{i,0})$ which is in the image of $\delta : \text{Coker}(A_{i+1,1} \to A_{i+1,0}) \to \text{Coker}(A_{i,1} \to A_{i,0})$. In other words, $a_{0,i}$ is in the image of $\delta \oplus d : A_{0,i+1} \oplus A_{1,i} \to A_{0,i}$. From the definition of trivial zig-zags we see that we may modify our zig-zag by a trivial one and assume that $a_{0,i} = 0$. This immediately implies that $d(a_{1,i-1}) = 0$. As the rows are exact this implies that $a_{1,i-1}$ is in the image of $d : A_{2,i-1} \to A_{1,i-1}$. Thus we may modify our zig-zag once again by a trivial zig-zag and assume that our zig-zag looks like $(a_{i,0}, a_{i-1,1}, \ldots, a_{2,i-2}, 0, 0)$. Continuing like this we obtain the desired injectivity.

If $\Phi : (A_{\bullet,\bullet}, d, \delta) \to (B_{\bullet,\bullet}, d, \delta)$ is a morphism of double complexes both of which satisfy the conditions of the lemma, then we clearly obtain a commutative diagram

$$\begin{tikzcd}
H_i(U(A)_\bullet) \arrow[d] & \mathbf{H}_i(A) \arrow[l] \arrow[r] \arrow[d] & H_i(R(A)_\bullet) \arrow[d] \\
H_i(U(B)_\bullet) & \mathbf{H}_i(B) \arrow[l] \arrow[r] & H_i(R(B)_\bullet)
\end{tikzcd}$$

This proves the functoriality. □

00M2 **Remark 75.4.** The isomorphism constructed above is the "correct" one only up to signs. A good part of homological algebra is concerned with choosing signs for various maps and showing commutativity of diagrams with intervention of suitable signs. For the moment we will simply use the isomorphism as given in the proof above, and worry about signs later.

00M3 **Lemma 75.5.** *Let R be a ring. For any $i \geq 0$ the functors $Mod_R \times Mod_R \to Mod_R$, $(M, N) \mapsto Tor_i^R(M, N)$ and $(M, N) \mapsto Tor_i^R(N, M)$ are canonically isomorphic.*

Proof. Let $F_\bullet$ be a free resolution of the module M and let $G_\bullet$ be a free resolution of the module N. Consider the double complex $(A_{i,j}, d, \delta)$ defined as follows:

(1) set $A_{i,j} = F_i \otimes_R G_j$,
(2) set $d_{i,j} : F_i \otimes_R G_j \to F_{i-1} \otimes G_j$ equal to $d_{F,i} \otimes \text{id}$, and
(3) set $\delta_{i,j} : F_i \otimes_R G_j \to F_i \otimes G_{j-1}$ equal to $\text{id} \otimes d_{G,j}$.

This double complex is usually simply denoted $F_\bullet \otimes_R G_\bullet$.

Since each G_j is free, and hence flat we see that each row of the double complex is exact except in homological degree 0. Since each F_i is free and hence flat we see that each column of the double complex is exact except in homological degree 0. Hence the double complex satisfies the conditions of Lemma 75.3.

To see what the lemma says we compute $R(A)_\bullet$ and $U(A)_\bullet$. Namely,

$$\begin{aligned} R(A)_i &= \mathrm{Coker}(A_{1,i} \to A_{0,i}) \\ &= \mathrm{Coker}(F_1 \otimes_R G_i \to F_0 \otimes_R G_i) \\ &= \mathrm{Coker}(F_1 \to F_0) \otimes_R G_i \\ &= M \otimes_R G_i \end{aligned}$$

In fact these isomorphisms are compatible with the differentials δ and we see that $R(A)_\bullet = M \otimes_R G_\bullet$ as homological complexes. In exactly the same way we see that $U(A)_\bullet = F_\bullet \otimes_R N$. We get

$$\begin{aligned} \mathrm{Tor}_i^R(M, N) &= H_i(F_\bullet \otimes_R N) \\ &= H_i(U(A)_\bullet) \\ &= H_i(R(A)_\bullet) \\ &= H_i(M \otimes_R G_\bullet) \\ &= H_i(G_\bullet \otimes_R M) \\ &= \mathrm{Tor}_i^R(N, M) \end{aligned}$$

Here the third equality is Lemma 75.3, and the fifth equality uses the isomorphism $V \otimes W = W \otimes V$ of the tensor product.

Functoriality. Suppose that we have R-modules M_ν, N_ν, $\nu = 1, 2$. Let $\varphi : M_1 \to M_2$ and $\psi : N_1 \to N_2$ be morphisms of R-modules. Suppose that we have free resolutions $F_{\nu,\bullet}$ for M_ν and free resolutions $G_{\nu,\bullet}$ for N_ν. By Lemma 71.4 we may choose maps of complexes $\alpha : F_{1,\bullet} \to F_{2,\bullet}$ and $\beta : G_{1,\bullet} \to G_{2,\bullet}$ compatible with φ and ψ. We claim that the pair (α, β) induces a morphism of double complexes

$$\alpha \otimes \beta : F_{1,\bullet} \otimes_R G_{1,\bullet} \longrightarrow F_{2,\bullet} \otimes_R G_{2,\bullet}$$

This is really a very straightforward check using the rule that $F_{1,i} \otimes_R G_{1,j} \to F_{2,i} \otimes_R G_{2,j}$ is given by $\alpha_i \otimes \beta_j$ where α_i, resp. β_j is the degree i, resp. j component of α, resp. β. The reader also readily verifies that the induced maps $R(F_{1,\bullet} \otimes_R G_{1,\bullet})_\bullet \to R(F_{2,\bullet} \otimes_R G_{2,\bullet})_\bullet$ agrees with the map $M_1 \otimes_R G_{1,\bullet} \to M_2 \otimes_R G_{2,\bullet}$ induced by $\varphi \otimes \beta$. Similarly for the map induced on the $U(-)_\bullet$ complexes. Thus the statement on functoriality follows from the statement on functoriality in Lemma 75.3. □

00M4 **Remark 75.6.** An interesting case occurs when $M = N$ in the above. In this case we get a canonical map $\mathrm{Tor}_i^R(M, M) \to \mathrm{Tor}_i^R(M, M)$. Note that this map is not the identity, because even when $i = 0$ this map is not the identity! For example, if V is a vector space of dimension n over a field, then the switch map $V \otimes_k V \to V \otimes_k V$ has $(n^2 + n)/2$ eigenvalues $+1$ and $(n^2 - n)/2$ eigenvalues -1. In characteristic 2 it is not even diagonalizable. Note that even changing the sign of the map will not get rid of this.

0AZ4 **Lemma 75.7.** *Let R be a Noetherian ring. Let M, N be finite R-modules. Then $Tor_p^R(M, N)$ is a finite R-module for all p.*

Proof. This holds because $\mathrm{Tor}_p^R(M, N)$ is computed as the cohomology groups of a complex $F_\bullet \otimes_R N$ with each F_n a finite free R-module, see Lemma 71.1. □

00M5 **Lemma 75.8.** *Let R be a ring. Let M be an R-module. The following are equivalent:*

1. *The module M is flat over R.*
2. *For all $i > 0$ the functor $Tor_i^R(M, -)$ is zero.*
3. *The functor $Tor_1^R(M, -)$ is zero.*
4. *For all ideals $I \subset R$ we have $Tor_1^R(M, R/I) = 0$.*
5. *For all finitely generated ideals $I \subset R$ we have $Tor_1^R(M, R/I) = 0$.*

Proof. Suppose M is flat. Let N be an R-module. Let $F_\bullet$ be a free resolution of N. Then $F_\bullet \otimes_R M$ is a resolution of $N \otimes_R M$, by flatness of M. Hence all higher Tor groups vanish.

It now suffices to show that the last condition implies that M is flat. Let $I \subset R$ be an ideal. Consider the short exact sequence $0 \to I \to R \to R/I \to 0$. Apply Lemma 75.2. We get an exact sequence

$$\mathrm{Tor}_1^R(M, R/I) \to M \otimes_R I \to M \otimes_R R \to M \otimes_R R/I \to 0$$

Since obviously $M \otimes_R R = M$ we conclude that the last hypothesis implies that $M \otimes_R I \to M$ is injective for every finitely generated ideal I. Thus M is flat by Lemma 39.5. ☐

00M6 **Remark 75.9.** The proof of Lemma 75.8 actually shows that

$$\mathrm{Tor}_1^R(M, R/I) = \mathrm{Ker}(I \otimes_R M \to M).$$

76. Functorialities for Tor

00M7 In this section we briefly discuss the functoriality of Tor with respect to change of ring, etc. Here is a list of items to work out.

1. Given a ring map $R \to R'$, an R-module M and an R'-module N' the R-modules $\mathrm{Tor}_i^R(M, N')$ have a natural R'-module structure.
2. Given a ring map $R \to R'$ and R-modules M, N there is a natural R-module map $\mathrm{Tor}_i^R(M, N) \to \mathrm{Tor}_i^{R'}(M \otimes_R R', N \otimes_R R')$.
3. Given a ring map $R \to R'$ an R-module M and an R'-module N' there exists a natural R'-module map $\mathrm{Tor}_i^R(M, N') \to \mathrm{Tor}_i^{R'}(M \otimes_R R', N')$.

00M8 **Lemma 76.1.** *Given a flat ring map $R \to R'$ and R-modules M, N the natural R-module map $Tor_i^R(M, N) \otimes_R R' \to Tor_i^{R'}(M \otimes_R R', N \otimes_R R')$ is an isomorphism for all i.*

Proof. Omitted. This is true because a free resolution $F_\bullet$ of M over R stays exact when tensoring with R' over R and hence $(F_\bullet \otimes_R N) \otimes_R R'$ computes the Tor groups over R'. ☐

The following lemma does not seem to fit anywhere else.

0BNF **Lemma 76.2.** *Let R be a ring. Let $M = \mathrm{colim}\, M_i$ be a filtered colimit of R-modules. Let N be an R-module. Then $Tor_n^R(M, N) = \mathrm{colim}\, Tor_n^R(M_i, N)$ for all n.*

Proof. Choose a free resolution $F_\bullet$ of N. Then $F_\bullet \otimes_R M = \mathrm{colim}\, F_\bullet \otimes_R M_i$ as complexes by Lemma 12.9. Thus the result by Lemma 8.8. ☐

77. Projective modules

05CD Some lemmas on projective modules.

05CE **Definition 77.1.** Let R be a ring. An R-module P is *projective* if and only if the functor $\mathrm{Hom}_R(P, -) : \mathrm{Mod}_R \to \mathrm{Mod}_R$ is an exact functor.

The functor $\mathrm{Hom}_R(M, -)$ is left exact for any R-module M, see Lemma 10.1. Hence the condition for P to be projective really signifies that given a surjection of R-modules $N \to N'$ the map $\mathrm{Hom}_R(P, N) \to \mathrm{Hom}_R(P, N')$ is surjective.

05CF **Lemma 77.2.** *Let R be a ring. Let P be an R-module. The following are equivalent*

1. *P is projective,*
2. *P is a direct summand of a free R-module, and*
3. *$\mathrm{Ext}^1_R(P, M) = 0$ for every R-module M.*

Proof. Assume P is projective. Choose a surjection $\pi : F \to P$ where F is a free R-module. As P is projective there exists a $i \in \mathrm{Hom}_R(P, F)$ such that $\pi \circ i = \mathrm{id}_P$. In other words $F \cong \mathrm{Ker}(\pi) \oplus i(P)$ and we see that P is a direct summand of F.

Conversely, assume that $P \oplus Q = F$ is a free R-module. Note that the free module $F = \bigoplus_{i \in I} R$ is projective as $\mathrm{Hom}_R(F, M) = \prod_{i \in I} M$ and the functor $M \mapsto \prod_{i \in I} M$ is exact. Then $\mathrm{Hom}_R(F, -) = \mathrm{Hom}_R(P, -) \times \mathrm{Hom}_R(Q, -)$ as functors, hence both P and Q are projective.

Assume $P \oplus Q = F$ is a free R-module. Then we have a free resolution $F_\bullet$ of the form

$$\ldots F \xrightarrow{a} F \xrightarrow{b} F \to P \to 0$$

where the maps a, b alternate and are equal to the projector onto P and Q. Hence the complex $\mathrm{Hom}_R(F_\bullet, M)$ is split exact in degrees ≥ 1, whence we see the vanishing in (3).

Assume $\mathrm{Ext}^1_R(P, M) = 0$ for every R-module M. Pick a free resolution $F_\bullet \to P$. Set $M = \mathrm{Im}(F_1 \to F_0) = \mathrm{Ker}(F_0 \to P)$. Consider the element $\xi \in \mathrm{Ext}^1_R(P, M)$ given by the class of the quotient map $\pi : F_1 \to M$. Since ξ is zero there exists a map $s : F_0 \to M$ such that $\pi = s \circ (F_1 \to F_0)$. Clearly, this means that

$$F_0 = \mathrm{Ker}(s) \oplus \mathrm{Ker}(F_0 \to P) = P \oplus \mathrm{Ker}(F_0 \to P)$$

and we win. □

0G8T **Lemma 77.3.** *Let R be a Noetherian ring. Let P be a finite R-module. If $\mathrm{Ext}^1_R(P, M) = 0$ for every finite R-module M, then P is projective.*

This lemma can be strengthened: There is a version for finitely presented R-modules if R is not assumed Noetherian. There is a version with M running through all finite length modules in the Noetherian case.

Proof. Choose a surjection $R^{\oplus n} \to P$ with kernel M. Since $\mathrm{Ext}^1_R(P, M) = 0$ this surjection is split and we conclude by Lemma 77.2. □

065Q **Lemma 77.4.** *A direct sum of projective modules is projective.*

Proof. This is true by the characterization of projectives as direct summands of free modules in Lemma 77.2. □

07LV **Lemma 77.5.** *Let R be a ring. Let $I \subset R$ be a nilpotent ideal. Let $\overline{P}$ be a projective R/I-module. Then there exists a projective R-module P such that $P/IP \cong \overline{P}$.*

Proof. By Lemma 77.2 we can choose a set A and a direct sum decomposition $\bigoplus_{\alpha \in A} R/I = \overline{P} \oplus \overline{K}$ for some R/I-module $\overline{K}$. Write $F = \bigoplus_{\alpha \in A} R$ for the free R-module on A. Choose a lift $p : F \to F$ of the projector $\overline{p}$ associated to the direct summand $\overline{P}$ of $\bigoplus_{\alpha \in A} R/I$. Note that $p^2 - p \in \text{End}_R(F)$ is a nilpotent endomorphism of F (as I is nilpotent and the matrix entries of $p^2 - p$ are in I; more precisely, if $I^n = 0$, then $(p^2 - p)^n = 0$). Hence by Lemma 32.7 we can modify our choice of p and assume that p is a projector. Set $P = \text{Im}(p)$. $\square$

0D47 **Lemma 77.6.** *Let R be a ring. Let $I \subset R$ be a locally nilpotent ideal. Let $\overline{P}$ be a finite projective R/I-module. Then there exists a finite projective R-module P such that $P/IP \cong \overline{P}$.*

Proof. Recall that $\overline{P}$ is a direct summand of a free R/I-module $\bigoplus_{\alpha \in A} R/I$ by Lemma 77.2. As $\overline{P}$ is finite, it follows that $\overline{P}$ is contained in $\bigoplus_{\alpha \in A'} R/I$ for some $A' \subset A$ finite. Hence we may assume we have a direct sum decomposition $(R/I)^{\oplus n} = \overline{P} \oplus \overline{K}$ for some n and some R/I-module $\overline{K}$. Choose a lift $p \in \text{Mat}(n \times n, R)$ of the projector $\overline{p}$ associated to the direct summand $\overline{P}$ of $(R/I)^{\oplus n}$. Note that $p^2 - p \in \text{Mat}(n \times n, R)$ is nilpotent: as I is locally nilpotent and the matrix entries c_{ij} of $p^2 - p$ are in I we have $c_{ij}^t = 0$ for some $t > 0$ and then $(p^2 - p)^{tn^2} = 0$ (by looking at the matrix coefficients). Hence by Lemma 32.7 we can modify our choice of p and assume that p is a projector. Set $P = \text{Im}(p)$. $\square$

05CG **Lemma 77.7.** *Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. Assume*

1. *I is nilpotent,*
2. *M/IM is a projective R/I-module,*
3. *M is a flat R-module.*

Then M is a projective R-module.

Proof. By Lemma 77.5 we can find a projective R-module P and an isomorphism $P/IP \to M/IM$. We are going to show that M is isomorphic to P which will finish the proof. Because P is projective we can lift the map $P \to P/IP \to M/IM$ to an R-module map $P \to M$ which is an isomorphism modulo I. Since $I^n = 0$ for some n, we can use the filtrations

$$0 = I^n M \subset I^{n-1} M \subset \ldots \subset IM \subset M$$
$$0 = I^n P \subset I^{n-1} P \subset \ldots \subset IP \subset P$$

to see that it suffices to show that the induced maps $I^a P / I^{a+1} P \to I^a M / I^{a+1} M$ are bijective. Since both P and M are flat R-modules we can identify this with the map

$$I^a / I^{a+1} \otimes_{R/I} P/IP \longrightarrow I^a / I^{a+1} \otimes_{R/I} M/IM$$

induced by $P \to M$. Since we chose $P \to M$ such that the induced map $P/IP \to M/IM$ is an isomorphism, we win. $\square$

78. Finite projective modules

00NV

00NW **Definition 78.1.** Let R be a ring and M an R-module.

(1) We say that M is *locally free* if we can cover $\text{Spec}(R)$ by standard opens $D(f_i)$, $i \in I$ such that M_{f_i} is a free R_{f_i}-module for all $i \in I$.
(2) We say that M is *finite locally free* if we can choose the covering such that each M_{f_i} is finite free.
(3) We say that M is *finite locally free of rank r* if we can choose the covering such that each M_{f_i} is isomorphic to $R_{f_i}^{\oplus r}$.

Note that a finite locally free R-module is automatically finitely presented by Lemma 23.2. Moreover, if M is a finite locally free module of rank r over a ring R and if R is nonzero, then r is uniquely determined by Lemma 15.8 (because at least one of the localizations R_{f_i} is a nonzero ring).

00NX **Lemma 78.2.** *Let R be a ring and let M be an R-module. The following are equivalent*

(1) *M is finitely presented and R-flat,*
(2) *M is finite projective,*
(3) *M is a direct summand of a finite free R-module,*
(4) *M is finitely presented and for all $\mathfrak{p} \in \text{Spec}(R)$ the localization $M_{\mathfrak{p}}$ is free,*
(5) *M is finitely presented and for all maximal ideals $\mathfrak{m} \subset R$ the localization $M_{\mathfrak{m}}$ is free,*
(6) *M is finite and locally free,*
(7) *M is finite locally free, and*
(8) *M is finite, for every prime $\mathfrak{p}$ the module $M_{\mathfrak{p}}$ is free, and the function*

$$\rho_M : \text{Spec}(R) \to \mathbf{Z}, \quad \mathfrak{p} \longmapsto \dim_{\kappa(\mathfrak{p})} M \otimes_R \kappa(\mathfrak{p})$$

is locally constant in the Zariski topology.

Proof. First suppose M is finite projective, i.e., (2) holds. Take a surjection $R^n \to M$ and let K be the kernel. Since M is projective, $0 \to K \to R^n \to M \to 0$ splits. Hence (2) $\Rightarrow$ (3). The implication (3) $\Rightarrow$ (2) follows from the fact that a direct summand of a projective is projective, see Lemma 77.2.

Assume (3), so we can write $K \oplus M \cong R^{\oplus n}$. So K is a direct summand of R^n and thus finitely generated. This shows $M = R^{\oplus n}/K$ is finitely presented. In other words, (3) $\Rightarrow$ (1).

Assume M is finitely presented and flat, i.e., (1) holds. We will prove that (7) holds. Pick any prime $\mathfrak{p}$ and $x_1, \ldots, x_r \in M$ which map to a basis of $M \otimes_R \kappa(\mathfrak{p})$. By Nakayama's lemma (in the form of Lemma 20.2) these elements generate M_g for some $g \in R$, $g \notin \mathfrak{p}$. The corresponding surjection $\varphi : R_g^{\oplus r} \to M_g$ has the following two properties: (a) $\text{Ker}(\varphi)$ is a finite R_g-module (see Lemma 5.3) and (b) $\text{Ker}(\varphi) \otimes \kappa(\mathfrak{p}) = 0$ by flatness of M_g over R_g (see Lemma 39.12). Hence by Nakayama's lemma again there exists a $g' \in R_g$ such that $\text{Ker}(\varphi)_{g'} = 0$. In other words, $M_{gg'}$ is free.

A finite locally free module is a finite module, see Lemma 23.2, hence (7) $\Rightarrow$ (6). It is clear that (6) $\Rightarrow$ (7) and that (7) $\Rightarrow$ (8).

A finite locally free module is a finitely presented module, see Lemma 23.2, hence (7) $\Rightarrow$ (4). Of course (4) implies (5). Since we may check flatness locally (see Lemma 39.18) we conclude that (5) implies (1). At this point we have

$$\begin{tikzcd}
(2) \arrow[r, Leftrightarrow] & (3) \arrow[r, Rightarrow] & (1) \arrow[r, Rightarrow] & (7) \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & (6) \\
 & & (5) \arrow[u, Rightarrow] & (4) \arrow[l, Rightarrow] & (8)
\end{tikzcd}$$

Suppose that M satisfies (1), (4), (5), (6), and (7). We will prove that (3) holds. It suffices to show that M is projective. We have to show that $\mathrm{Hom}_R(M, -)$ is exact. Let $0 \to N'' \to N \to N' \to 0$ be a short exact sequence of R-module. We have to show that $0 \to \mathrm{Hom}_R(M, N'') \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N') \to 0$ is exact. As M is finite locally free there exist a covering $\mathrm{Spec}(R) = \bigcup D(f_i)$ such that M_{f_i} is finite free. By Lemma 10.2 we see that

$$0 \to \mathrm{Hom}_R(M, N'')_{f_i} \to \mathrm{Hom}_R(M, N)_{f_i} \to \mathrm{Hom}_R(M, N')_{f_i} \to 0$$

is equal to $0 \to \mathrm{Hom}_{R_{f_i}}(M_{f_i}, N''_{f_i}) \to \mathrm{Hom}_{R_{f_i}}(M_{f_i}, N_{f_i}) \to \mathrm{Hom}_{R_{f_i}}(M_{f_i}, N'_{f_i}) \to 0$ which is exact as M_{f_i} is free and as the localization $0 \to N''_{f_i} \to N_{f_i} \to N'_{f_i} \to 0$ is exact (as localization is exact). Whence we see that $0 \to \mathrm{Hom}_R(M, N'') \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N') \to 0$ is exact by Lemma 23.2.

Finally, assume that (8) holds. Pick a maximal ideal $\mathfrak{m} \subset R$. Pick $x_1, \ldots, x_r \in M$ which map to a $\kappa(\mathfrak{m})$-basis of $M \otimes_R \kappa(\mathfrak{m}) = M/\mathfrak{m}M$. In particular $\rho_M(\mathfrak{m}) = r$. By Nakayama's Lemma 20.1 there exists an $f \in R$, $f \notin \mathfrak{m}$ such that $x_1, \ldots, x_r$ generate M_f over R_f. By the assumption that ρ_M is locally constant there exists a $g \in R$, $g \notin \mathfrak{m}$ such that ρ_M is constant equal to r on $D(g)$. We claim that

$$\Psi : R_{fg}^{\oplus r} \longrightarrow M_{fg}, \quad (a_1, \ldots, a_r) \longmapsto \sum a_i x_i$$

is an isomorphism. This claim will show that M is finite locally free, i.e., that (7) holds. To see the claim it suffices to show that the induced map on localizations $\Psi_\mathfrak{p} : R_\mathfrak{p}^{\oplus r} \to M_\mathfrak{p}$ is an isomorphism for all $\mathfrak{p} \in D(fg)$, see Lemma 23.1. By our choice of f the map $\Psi_\mathfrak{p}$ is surjective. By assumption (8) we have $M_\mathfrak{p} \cong R_\mathfrak{p}^{\oplus \rho_M(\mathfrak{p})}$ and by our choice of g we have $\rho_M(\mathfrak{p}) = r$. Hence $\Psi_\mathfrak{p}$ determines a surjection $R_\mathfrak{p}^{\oplus r} \to M_\mathfrak{p} \cong R_\mathfrak{p}^{\oplus r}$ whence is an isomorphism by Lemma 16.4. (Of course this last fact follows from a simple matrix argument also.) $\square$

0FWG **Lemma 78.3.** *Let R be a reduced ring and let M be an R-module. Then the equivalent conditions of Lemma 78.2 are also equivalent to*

(9) *M is finite and the function $\rho_M : \mathrm{Spec}(R) \to \mathbf{Z}$, $\mathfrak{p} \mapsto \dim_{\kappa(\mathfrak{p})} M \otimes_R \kappa(\mathfrak{p})$ is locally constant in the Zariski topology.*

Proof. Pick a maximal ideal $\mathfrak{m} \subset R$. Pick $x_1, \ldots, x_r \in M$ which map to a $\kappa(\mathfrak{m})$-basis of $M \otimes_R \kappa(\mathfrak{m}) = M/\mathfrak{m}M$. In particular $\rho_M(\mathfrak{m}) = r$. By Nakayama's Lemma 20.1 there exists an $f \in R$, $f \notin \mathfrak{m}$ such that $x_1, \ldots, x_r$ generate M_f over R_f. By the assumption that ρ_M is locally constant there exists a $g \in R$, $g \notin \mathfrak{m}$ such that ρ_M is constant equal to r on $D(g)$. We claim that

$$\Psi : R_{fg}^{\oplus r} \longrightarrow M_{fg}, \quad (a_1, \ldots, a_r) \longmapsto \sum a_i x_i$$

is an isomorphism. This claim will show that M is finite locally free, i.e., that (7) holds. Since Ψ is surjective, it suffices to show that Ψ is injective. Since R_{fg} is reduced, it suffices to show that Ψ is injective after localization at all minimal primes $\mathfrak{p}$ of R_{fg}, see Lemma 25.2. However, we know that $R_\mathfrak{p} = \kappa(\mathfrak{p})$ by Lemma 25.1 and $\rho_M(\mathfrak{p}) = r$ hence $\Psi_\mathfrak{p} : R_\mathfrak{p}^{\oplus r} \to M \otimes_R \kappa(\mathfrak{p})$ is an isomorphism as a surjective map of finite dimensional vector spaces of the same dimension. □

00NY **Remark 78.4.** It is not true that a finite R-module which is R-flat is automatically projective. A counter example is where $R = \mathcal{C}^\infty(\mathbf{R})$ is the ring of infinitely differentiable functions on $\mathbf{R}$, and $M = R_\mathfrak{m} = R/I$ where $\mathfrak{m} = \{f \in R \mid f(0) = 0\}$ and $I = \{f \in R \mid \exists \epsilon, \epsilon > 0 : f(x) = 0\ \forall x, |x| < \epsilon\}$.

00NZ **Lemma 78.5.** *(Warning: see Remark 78.4.) Suppose R is a local ring, and M is a finite flat R-module. Then M is finite free.*

Proof. Follows from the equational criterion of flatness, see Lemma 39.11. Namely, suppose that $x_1, \ldots, x_r \in M$ map to a basis of $M/\mathfrak{m}M$. By Nakayama's Lemma 20.1 these elements generate M. We want to show there is no relation among the x_i. Instead, we will show by induction on n that if $x_1, \ldots, x_n \in M$ are linearly independent in the vector space $M/\mathfrak{m}M$ then they are independent over R.

The base case of the induction is where we have $x \in M$, $x \notin \mathfrak{m}M$ and a relation $fx = 0$. By the equational criterion there exist $y_j \in M$ and $a_j \in R$ such that $x = \sum a_j y_j$ and $fa_j = 0$ for all j. Since $x \notin \mathfrak{m}M$ we see that at least one a_j is a unit and hence $f = 0$.

Suppose that $\sum f_i x_i$ is a relation among $x_1, \ldots, x_n$. By our choice of x_i we have $f_i \in \mathfrak{m}$. According to the equational criterion of flatness there exist $a_{ij} \in R$ and $y_j \in M$ such that $x_i = \sum a_{ij} y_j$ and $\sum f_i a_{ij} = 0$. Since $x_n \notin \mathfrak{m}M$ we see that $a_{nj} \notin \mathfrak{m}$ for at least one j. Since $\sum f_i a_{ij} = 0$ we get $f_n = \sum_{i=1}^{n-1} (-a_{ij}/a_{nj}) f_i$. The relation $\sum f_i x_i = 0$ now can be rewritten as $\sum_{i=1}^{n-1} f_i(x_i + (-a_{ij}/a_{nj})x_n) = 0$. Note that the elements $x_i + (-a_{ij}/a_{nj})x_n$ map to $n-1$ linearly independent elements of $M/\mathfrak{m}M$. By induction assumption we get that all the f_i, $i \leq n-1$ have to be zero, and also $f_n = \sum_{i=1}^{n-1} (-a_{ij}/a_{nj}) f_i$. This proves the induction step. □

00O1 **Lemma 78.6.** *Let $R \to S$ be a flat local homomorphism of local rings. Let M be a finite R-module. Then M is finite projective over R if and only if $M \otimes_R S$ is finite projective over S.*

Proof. By Lemma 78.2 being finite projective over a local ring is the same thing as being finite free. Suppose that $M \otimes_R S$ is a finite free S-module. Pick $x_1, \ldots, x_r \in M$ whose images in $M/\mathfrak{m}_R M$ form a basis over $\kappa(\mathfrak{m})$. Then we see that $x_1 \otimes 1, \ldots, x_r \otimes 1$ are a basis for $M \otimes_R S$. This implies that the map $R^{\oplus r} \to M, (a_i) \mapsto \sum a_i x_i$ becomes an isomorphism after tensoring with S. By faithful flatness of $R \to S$, see Lemma 39.17 we see that it is an isomorphism. □

02M9 **Lemma 78.7.** *Let R be a semi-local ring. Let M be a finite locally free module. If M has constant rank, then M is free. In particular, if R has connected spectrum, then M is free.*

Proof. Omitted. Hints: First show that $M/\mathfrak{m}_i M$ has the same dimension d for all maximal ideal $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ of R using the rank is constant. Next, show that there exist elements $x_1, \ldots, x_d \in M$ which form a basis for each $M/\mathfrak{m}_i M$ by the Chinese remainder theorem. Finally show that $x_1, \ldots, x_d$ is a basis for M. □

Here is a technical lemma that is used in the chapter on groupoids.

03C1 **Lemma 78.8.** *Let R be a local ring with maximal ideal $\mathfrak{m}$ and infinite residue field. Let $R \to S$ be a ring map. Let M be an S-module and let $N \subset M$ be an R-submodule. Assume*

1. *S is semi-local and $\mathfrak{m}S$ is contained in the Jacobson radical of S,*
2. *M is a finite free S-module, and*
3. *N generates M as an S-module.*

Then N contains an S-basis of M.

Proof. Assume M is free of rank n. Let $I \subset S$ be the Jacobson radical. By Nakayama's Lemma 20.1 a sequence of elements $m_1, \ldots, m_n$ is a basis for M if and only if $\overline{m}_i \in M/IM$ generate M/IM. Hence we may replace M by M/IM, N by $N/(N \cap IM)$, R by $R/\mathfrak{m}$, and S by S/IS. In this case we see that S is a finite product of fields $S = k_1 \times \ldots \times k_r$ and $M = k_1^{\oplus n} \times \ldots \times k_r^{\oplus n}$. The fact that $N \subset M$ generates M as an S-module means that there exist $x_j \in N$ such that a linear combination $\sum a_j x_j$ with $a_j \in S$ has a nonzero component in each factor $k_i^{\oplus n}$. Because $R = k$ is an infinite field, this means that also some linear combination $y = \sum c_j x_j$ with $c_j \in k$ has a nonzero component in each factor. Hence $y \in N$ generates a free direct summand $Sy \subset M$. By induction on n the result holds for M/Sy and the submodule $\overline{N} = N/(N \cap Sy)$. In other words there exist $\overline{y}_2, \ldots, \overline{y}_n$ in $\overline{N}$ which (freely) generate M/Sy. Then $y, y_2, \ldots, y_n$ (freely) generate M and we win. □

0DVB **Lemma 78.9.** *Let R be ring. Let L, M, N be R-modules. The canonical map*

$$\mathrm{Hom}_R(M, N) \otimes_R L \to \mathrm{Hom}_R(M, N \otimes_R L)$$

is an isomorphism if M is finite projective.

Proof. By Lemma 78.2 we see that M is finitely presented as well as finite locally free. By Lemmas 10.2 and 12.16 formation of the left and right hand side of the arrow commutes with localization. We may check that our map is an isomorphism after localization, see Lemma 23.2. Thus we may assume M is finite free. In this case the lemma is immediate. □

79. Open loci defined by module maps

05GD The set of primes where a given module map is surjective, or an isomorphism is sometimes open. In the case of finite projective modules we can look at the rank of the map.

05GE **Lemma 79.1.** *Let R be a ring. Let $\varphi : M \to N$ be a map of R-modules with N a finite R-module. Then we have the equality*

$$\begin{aligned} U &= \{\mathfrak{p} \subset R \mid \varphi_\mathfrak{p} : M_\mathfrak{p} \to N_\mathfrak{p} \text{ is surjective}\} \\ &= \{\mathfrak{p} \subset R \mid \varphi \otimes \kappa(\mathfrak{p}) : M \otimes \kappa(\mathfrak{p}) \to N \otimes \kappa(\mathfrak{p}) \text{ is surjective}\} \end{aligned}$$

and U is an open subset of $\mathrm{Spec}(R)$. Moreover, for any $f \in R$ such that $D(f) \subset U$ the map $M_f \to N_f$ is surjective.

Proof. The equality in the displayed formula follows from Nakayama's lemma. Nakayama's lemma also implies that U is open. See Lemma 20.1 especially part (3). If $D(f) \subset U$, then $M_f \to N_f$ is surjective on all localizations at primes of R_f, and hence it is surjective by Lemma 23.1. □

05GF **Lemma 79.2.** *Let R be a ring. Let $\varphi : M \to N$ be a map of R-modules with M finite and N finitely presented. Then*

$$U = \{\mathfrak p \subset R \mid \varphi_{\mathfrak p} : M_{\mathfrak p} \to N_{\mathfrak p} \text{ is an isomorphism}\}$$

is an open subset of $\mathrm{Spec}(R)$.

Proof. Let $\mathfrak p \in U$. Pick a presentation $N = R^{\oplus n}/\sum_{j=1,\ldots,m} Rk_j$. Denote e_i the image in N of the ith basis vector of $R^{\oplus n}$. For each $i \in \{1, \ldots, n\}$ choose an element $m_i \in M_{\mathfrak p}$ such that $\varphi(m_i) = f_i e_i$ for some $f_i \in R$, $f_i \notin \mathfrak p$. This is possible as $\varphi_{\mathfrak p}$ is an isomorphism. Set $f = f_1 \ldots f_n$ and let $\psi : R_f^{\oplus n} \to M_f$ be the map which maps the ith basis vector to m_i/f_i. Note that $\varphi_f \circ \psi$ is the localization at f of the given map $R^{\oplus n} \to N$. As $\varphi_{\mathfrak p}$ is an isomorphism we see that $\psi(k_j)$ is an element of M which maps to zero in $M_{\mathfrak p}$. Hence we see that there exist $g_j \in R$, $g_j \notin \mathfrak p$ such that $g_j \psi(k_j) = 0$. Setting $g = g_1 \ldots g_m$, we see that ψ_g factors through N_{fg} to give a map $\chi : N_{fg} \to M_{fg}$. By construction χ is a right inverse to φ_{fg}. It follows that $\chi_{\mathfrak p}$ is an isomorphism. By Lemma 79.1 there is an $h \in R$, $h \notin \mathfrak p$ such that $\chi_h : N_{fgh} \to M_{fgh}$ is surjective. Hence φ_{fgh} and χ_h are mutually inverse maps, which implies that $D(fgh) \subset U$ as desired. □

00O0 **Lemma 79.3.** *Let R be a ring. Let $\varphi : P_1 \to P_2$ be a map of finite projective modules. Then*

1. *The set U of primes $\mathfrak p \in \mathrm{Spec}(R)$ such that $\varphi \otimes \kappa(\mathfrak p)$ is injective is open and for any $f \in R$ such that $D(f) \subset U$ we have*
   (a) *$P_{1,f} \to P_{2,f}$ is injective, and*
   (b) *the module $\mathrm{Coker}(\varphi)_f$ is finite projective over R_f.*
2. *The set W of primes $\mathfrak p \in \mathrm{Spec}(R)$ such that $\varphi \otimes \kappa(\mathfrak p)$ is surjective is open and for any $f \in R$ such that $D(f) \subset W$ we have*
   (a) *$P_{1,f} \to P_{2,f}$ is surjective, and*
   (b) *the module $\mathrm{Ker}(\varphi)_f$ is finite projective over R_f.*
3. *The set V of primes $\mathfrak p \in \mathrm{Spec}(R)$ such that $\varphi \otimes \kappa(\mathfrak p)$ is an isomorphism is open and for any $f \in R$ such that $D(f) \subset V$ the map $\varphi : P_{1,f} \to P_{2,f}$ is an isomorphism of modules over R_f.*

Proof. To prove the set U is open we may work locally on $\mathrm{Spec}(R)$. Thus we may replace R by a suitable localization and assume that $P_1 = R^{n_1}$ and $P_2 = R^{n_2}$, see Lemma 78.2. In this case injectivity of $\varphi \otimes \kappa(\mathfrak p)$ is equivalent to $n_1 \leq n_2$ and some $n_1 \times n_1$ minor f of the matrix of φ being invertible in $\kappa(\mathfrak p)$. Thus $D(f) \subset U$. This argument also shows that $P_{1,\mathfrak p} \to P_{2,\mathfrak p}$ is injective for $\mathfrak p \in U$.

Now suppose $D(f) \subset U$. By the remark in the previous paragraph and Lemma 23.1 we see that $P_{1,f} \to P_{2,f}$ is injective, i.e., (1)(a) holds. By Lemma 78.2 to prove (1)(b) it suffices to prove that $\mathrm{Coker}(\varphi)$ is finite projective locally on $D(f)$. Thus, as we saw above, we may assume that $P_1 = R^{n_1}$ and $P_2 = R^{n_2}$ and that some minor of the matrix of φ is invertible in R. If the minor in question corresponds to the first n_1 basis vectors of R^{n_2}, then using the last $n_2 - n_1$ basis vectors we get a map $R^{n_2 - n_1} \to R^{n_2} \to \mathrm{Coker}(\varphi)$ which is easily seen to be an isomorphism.

Openness of W and (2)(a) for $D(f) \subset W$ follow from Lemma 79.1. Since $P_{2,f}$ is projective over R_f we see that $\varphi_f : P_{1,f} \to P_{2,f}$ has a section and it follows that $\mathrm{Ker}(\varphi)_f$ is a direct summand of $P_{2,f}$. Therefore $\mathrm{Ker}(\varphi)_f$ is finite projective. Thus (2)(b) holds as well.

It is clear that $V = U \cap W$ is open and the other statement in (3) follows from (1)(a) and (2)(a). □

80. Faithfully flat descent for projectivity of modules

058B In the next few sections we prove, following Raynaud and Gruson [GR71], that the projectivity of modules descends along faithfully flat ring maps. The idea of the proof is to use dévissage à la Kaplansky [Kap58] to reduce to the case of countably generated modules. Given a well-behaved filtration of a module M, dévissage allows us to express M as a direct sum of successive quotients of the filtering submodules (see Section 84). Using this technique, we prove that a projective module is a direct sum of countably generated modules (Theorem 84.5). To prove descent of projectivity for countably generated modules, we introduce a "Mittag-Leffler" condition on modules, prove that a countably generated module is projective if and only if it is flat and Mittag-Leffler (Theorem 93.3), and then show that the property of being a Mittag-Leffler module descends (Lemma 95.1). Finally, given an arbitrary module M whose base change by a faithfully flat ring map is projective, we filter M by submodules whose successive quotients are countably generated projective modules, and then by dévissage conclude M is a direct sum of projectives, hence projective itself (Theorem 95.6).

We note that there is an error in the proof of faithfully flat descent of projectivity in [GR71]. There, descent of projectivity along faithfully flat ring maps is deduced from descent of projectivity along a more general type of ring map ([GR71, Example 3.1.4(1) of Part II]). However, the proof of descent along this more general type of map is incorrect. In [Gru73], Gruson explains what went wrong, although he does not provide a fix for the case of interest. Patching this hole in the proof of faithfully flat descent of projectivity comes down to proving that the property of being a Mittag-Leffler module descends along faithfully flat ring maps. We do this in Lemma 95.1.

81. Characterizing flatness

058C In this section we discuss criteria for flatness. The main result in this section is Lazard's theorem (Theorem 81.4 below), which says that a flat module is the colimit of a directed system of free finite modules. We remind the reader of the "equational criterion for flatness", see Lemma 39.11. It turns out that this can be massaged into a seemingly much stronger property.

058D **Lemma 81.1.** *Let M be an R-module. The following are equivalent:*

(1) *M is flat.*
(2) *If $f : R^n \to M$ is a module map and $x \in \mathrm{Ker}(f)$, then there are module maps $h : R^n \to R^m$ and $g : R^m \to M$ such that $f = g \circ h$ and $x \in \mathrm{Ker}(h)$.*
(3) *Suppose $f : R^n \to M$ is a module map, $N \subset \mathrm{Ker}(f)$ any submodule, and $h : R^n \to R^m$ a map such that $N \subset \mathrm{Ker}(h)$ and f factors through h.*

Then given any $x \in \mathrm{Ker}(f)$ we can find a map $h' : R^n \to R^{m'}$ such that $N + Rx \subset \mathrm{Ker}(h')$ and f factors through h'.

(4) *If $f : R^n \to M$ is a module map and $N \subset \mathrm{Ker}(f)$ is a finitely generated submodule, then there are module maps $h : R^n \to R^m$ and $g : R^m \to M$ such that $f = g \circ h$ and $N \subset \mathrm{Ker}(h)$.*

Proof. That (1) is equivalent to (2) is just a reformulation of the equational criterion for flatness[8]. To show (2) implies (3), let $g : R^m \to M$ be the map such that f factors as $f = g \circ h$. By (2) find $h'' : R^m \to R^{m'}$ such that h'' kills $h(x)$ and $g : R^m \to M$ factors through h''. Then taking $h' = h'' \circ h$ works. (3) implies (4) by induction on the number of generators of $N \subset \mathrm{Ker}(f)$ in (4). Clearly (4) implies (2). $\square$

058E **Lemma 81.2.** *Let M be an R-module. Then M is flat if and only if the following condition holds: if P is a finitely presented R-module and $f : P \to M$ a module map, then there is a free finite R-module F and module maps $h : P \to F$ and $g : F \to M$ such that $f = g \circ h$.*

Proof. This is just a reformulation of condition (4) from Lemma 81.1. $\square$

058F **Lemma 81.3.** *Let M be an R-module. Then M is flat if and only if the following condition holds: for every finitely presented R-module P, if $N \to M$ is a surjective R-module map, then the induced map $\mathrm{Hom}_R(P, N) \to \mathrm{Hom}_R(P, M)$ is surjective.*

Proof. First suppose M is flat. We must show that if P is finitely presented, then given a map $f : P \to M$, it factors through the map $N \to M$. By Lemma 81.2 the map f factors through a map $F \to M$ where F is free and finite. Since F is free, this map factors through $N \to M$. Thus f factors through $N \to M$.

Conversely, suppose the condition of the lemma holds. Let $f : P \to M$ be a map from a finitely presented module P. Choose a free module N with a surjection $N \to M$ onto M. Then f factors through $N \to M$, and since P is finitely generated, f factors through a free finite submodule of N. Thus M satisfies the condition of Lemma 81.2, hence is flat. $\square$

058G **Theorem 81.4** (Lazard's theorem)**.** *Let M be an R-module. Then M is flat if and only if it is the colimit of a directed system of free finite R-modules.*

Proof. A colimit of a directed system of flat modules is flat, as taking directed colimits is exact and commutes with tensor product. Hence if M is the colimit of a directed system of free finite modules then M is flat.

For the converse, first recall that any module M can be written as the colimit of a directed system of finitely presented modules, in the following way. Choose a surjection $f : R^I \to M$ for some set I, and let K be the kernel. Let E be the set of ordered pairs (J, N) where J is a finite subset of I and N is a finitely generated submodule of $R^J \cap K$. Then E is made into a directed partially ordered set by defining $(J, N) \leq (J', N')$ if and only if $J \subset J'$ and $N \subset N'$. Define $M_e = R^J/N$

[8]In fact, a module map $f : R^n \to M$ corresponds to a choice of elements $x_1, x_2, \ldots, x_n$ of M (namely, the images of the standard basis elements $e_1, e_2, \ldots, e_n$); furthermore, an element $x \in \mathrm{Ker}(f)$ corresponds to a relation between these $x_1, x_2, \ldots, x_n$ (namely, the relation $\sum_i f_i x_i = 0$, where the f_i are the coordinates of x). The module map h (represented as an $m \times n$-matrix) corresponds to the matrix (a_{ij}) from Lemma 39.11, and the y_j of Lemma 39.11 are the images of the standard basis vectors of R^m under g.

for $e = (J, N)$, and define $f_{ee'} : M_e \to M_{e'}$ to be the natural map for $e \leq e'$. Then $(M_e, f_{ee'})$ is a directed system and the natural maps $f_e : M_e \to M$ induce an isomorphism $\mathrm{colim}_{e \in E} M_e \xrightarrow{\cong} M$.

Now suppose M is flat. Let $I = M \times \mathbf{Z}$, write (x_i) for the canonical basis of R^I, and take in the above discussion $f : R^I \to M$ to be the map sending x_i to the projection of i onto M. To prove the theorem it suffices to show that the $e \in E$ such that M_e is free form a cofinal subset of E. So let $e = (J, N) \in E$ be arbitrary. By Lemma 81.2 there is a free finite module F and maps $h : R^J/N \to F$ and $g : F \to M$ such that the natural map $f_e : R^J/N \to M$ factors as $R^J/N \xrightarrow{h} F \xrightarrow{g} M$. We are going to realize F as $M_{e'}$ for some $e' \geq e$.

Let $\{b_1, \ldots, b_n\}$ be a finite basis of F. Choose n distinct elements $i_1, \ldots, i_n \in I$ such that $i_\ell \notin J$ for all ℓ, and such that the image of x_{i_ℓ} under $f : R^I \to M$ equals the image of b_ℓ under $g : F \to M$. This is possible since every element of M can be written as $f(x_i)$ for infinitely many distinct $i \in I$ (by our choice of I). Now let $J' = J \cup \{i_1, \ldots, i_n\}$, and define $R^{J'} \to F$ by $x_i \mapsto h(x_i)$ for $i \in J$ and $x_{i_\ell} \mapsto b_\ell$ for $\ell = 1, \ldots, n$. Let $N' = \mathrm{Ker}(R^{J'} \to F)$. Observe:

(1) The square

$$\begin{tikzcd} R^{J'} \arrow[r] \arrow[d, hook] & F \arrow[d, "g"] \\ R^I \arrow[r, "f"'] & M \end{tikzcd}$$

is commutative, hence $N' \subset K = \mathrm{Ker}(f)$;

(2) $R^{J'} \to F$ is a surjection onto a free finite module, hence it splits and so N' is finitely generated;

(3) $J \subset J'$ and $N \subset N'$.

By (1) and (2) $e' = (J', N')$ is in E, by (3) $e' \geq e$, and by construction $M_{e'} = R^{J'}/N' \cong F$ is free. $\square$

82. Universally injective module maps

058H Next we discuss universally injective module maps, which are in a sense complementary to flat modules (see Lemma 82.5). We follow Lazard's thesis [Laz69]; also see [Lam99].

058I **Definition 82.1.** Let $f : M \to N$ be a map of R-modules. Then f is called *universally injective* if for every R-module Q, the map $f \otimes_R \mathrm{id}_Q : M \otimes_R Q \to N \otimes_R Q$ is injective. A sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of R-modules is called *universally exact* if it is exact and $M_1 \to M_2$ is universally injective.

058J **Example 82.2.** Examples of universally exact sequences.

(1) A split short exact sequence is universally exact since tensoring commutes with taking direct sums.

(2) The colimit of a directed system of universally exact sequences is universally exact. This follows from the fact that taking directed colimits is exact and that tensoring commutes with taking colimits. In particular the colimit of a directed system of split exact sequences is universally exact. We will see below that, conversely, any universally exact sequence arises in this way.

Next we give a list of criteria for a short exact sequence to be universally exact. They are analogues of criteria for flatness given above. Parts (3)-(6) below correspond, respectively, to the criteria for flatness given in Lemmas 39.11, 81.1, 81.3, and Theorem 81.4.

058K **Theorem 82.3.** *Let*

$$0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0$$

be an exact sequence of R-modules. The following are equivalent:

(1) *The sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ is universally exact.*
(2) *For every finitely presented R-module Q, the sequence*

$$0 \to M_1 \otimes_R Q \to M_2 \otimes_R Q \to M_3 \otimes_R Q \to 0$$

is exact.
(3) *Given elements $x_i \in M_1$ $(i = 1, \ldots, n)$, $y_j \in M_2$ $(j = 1, \ldots, m)$, and $a_{ij} \in R$ $(i = 1, \ldots, n, j = 1, \ldots, m)$ such that for all i*

$$f_1(x_i) = \sum\nolimits_j a_{ij} y_j,$$

there exists $z_j \in M_1$ $(j = 1, \ldots, m)$ such that for all i,

$$x_i = \sum\nolimits_j a_{ij} z_j.$$

(4) *Given a commutative diagram of R-module maps*

$$\begin{tikzcd} R^n \arrow[r] \arrow[d] & R^m \arrow[d] \\ M_1 \arrow[r, "f_1"] & M_2 \end{tikzcd}$$

where m and n are integers, there exists a map $R^m \to M_1$ making the top triangle commute.
(5) *For every finitely presented R-module P, the R-module map $\mathrm{Hom}_R(P, M_2) \to \mathrm{Hom}_R(P, M_3)$ is surjective.*
(6) *The sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ is the colimit of a directed system of split exact sequences of the form*

$$0 \to M_1 \to M_{2,i} \to M_{3,i} \to 0$$

where the $M_{3,i}$ are finitely presented.

Proof. Obviously (1) implies (2).

Next we show (2) implies (3). Let $f_1(x_i) = \sum_j a_{ij} y_j$ be relations as in (3). Let (d_j) be a basis for R^m, (e_i) a basis for R^n, and $R^m \to R^n$ the map given by $d_j \mapsto \sum_i a_{ij} e_i$. Let Q be the cokernel of $R^m \to R^n$. Then tensoring $R^m \to R^n \to Q \to 0$ by the map $f_1 : M_1 \to M_2$, we get a commutative diagram

$$\begin{tikzcd} M_1^{\oplus m} \arrow[r] \arrow[d] & M_1^{\oplus n} \arrow[r] \arrow[d] & M_1 \otimes_R Q \arrow[r] \arrow[d] & 0 \\ M_2^{\oplus m} \arrow[r] & M_2^{\oplus n} \arrow[r] & M_2 \otimes_R Q \arrow[r] & 0 \end{tikzcd}$$

where $M_1^{\oplus m} \to M_1^{\oplus n}$ is given by

$$(z_1, \ldots, z_m) \mapsto (\sum\nolimits_j a_{1j} z_j, \ldots, \sum\nolimits_j a_{nj} z_j),$$

and $M_2^{\oplus m} \to M_2^{\oplus n}$ is given similarly. We want to show $x = (x_1, \ldots, x_n) \in M_1^{\oplus n}$ is in the image of $M_1^{\oplus m} \to M_1^{\oplus n}$. By (2) the map $M_1 \otimes Q \to M_2 \otimes Q$ is injective, hence by exactness of the top row it is enough to show x maps to 0 in $M_2 \otimes Q$, and so by exactness of the bottom row it is enough to show the image of x in $M_2^{\oplus n}$ is in the image of $M_2^{\oplus m} \to M_2^{\oplus n}$. This is true by assumption.

Condition (4) is just a translation of (3) into diagram form.

Next we show (4) implies (5). Let $\varphi : P \to M_3$ be a map from a finitely presented R-module P. We must show that φ lifts to a map $P \to M_2$. Choose a presentation of P,

$$R^n \xrightarrow{g_1} R^m \xrightarrow{g_2} P \to 0.$$

Using freeness of R^n and R^m, we can construct $h_2 : R^m \to M_2$ and then $h_1 : R^n \to M_1$ such that the following diagram commutes

$$\begin{tikzcd}
 & R^n \arrow[r, "g_1"] \arrow[d, "h_1"] & R^m \arrow[r, "g_2"] \arrow[d, "h_2"] & P \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & M_1 \arrow[r, "f_1"] & M_2 \arrow[r, "f_2"] & M_3 \arrow[r] & 0.
\end{tikzcd}$$

By (4) there is a map $k_1 : R^m \to M_1$ such that $k_1 \circ g_1 = h_1$. Now define $h_2' : R^m \to M_2$ by $h_2' = h_2 - f_1 \circ k_1$. Then

$$h_2' \circ g_1 = h_2 \circ g_1 - f_1 \circ k_1 \circ g_1 = h_2 \circ g_1 - f_1 \circ h_1 = 0.$$

Hence by passing to the quotient h_2' defines a map $\varphi' : P \to M_2$ such that $\varphi' \circ g_2 = h_2'$. In a diagram, we have

$$\begin{tikzcd}
R^m \arrow[r, "g_2"] \arrow[d, "h_2'"'] & P \arrow[d, "\varphi"] \arrow[dl, "\varphi'"'] \\
M_2 \arrow[r, "f_2"] & M_3.
\end{tikzcd}$$

where the top triangle commutes. We claim that φ' is the desired lift, i.e. that $f_2 \circ \varphi' = \varphi$. From the definitions we have

$$f_2 \circ \varphi' \circ g_2 = f_2 \circ h_2' = f_2 \circ h_2 - f_2 \circ f_1 \circ k_1 = f_2 \circ h_2 = \varphi \circ g_2.$$

Since g_2 is surjective, this finishes the proof.

Now we show (5) implies (6). Write M_3 as the colimit of a directed system of finitely presented modules $M_{3,i}$, see Lemma 11.3. Let $M_{2,i}$ be the fiber product of $M_{3,i}$ and M_2 over M_3—by definition this is the submodule of $M_2 \times M_{3,i}$ consisting of elements whose two projections onto M_3 are equal. Let $M_{1,i}$ be the kernel of the projection $M_{2,i} \to M_{3,i}$. Then we have a directed system of exact sequences

$$0 \to M_{1,i} \to M_{2,i} \to M_{3,i} \to 0,$$

and for each i a map of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & M_{1,i} \arrow[r] \arrow[d] & M_{2,i} \arrow[r] \arrow[d] & M_{3,i} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M_1 \arrow[r] & M_2 \arrow[r] & M_3 \arrow[r] & 0 \end{tikzcd}$$

compatible with the directed system. From the definition of the fiber product $M_{2,i}$, it follows that the map $M_{1,i} \to M_1$ is an isomorphism. By (5) there is a map $M_{3,i} \to M_2$ lifting $M_{3,i} \to M_3$, and by the universal property of the fiber product this gives rise to a section of $M_{2,i} \to M_{3,i}$. Hence the sequences

$$0 \to M_{1,i} \to M_{2,i} \to M_{3,i} \to 0$$

split. Passing to the colimit, we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{colim} M_{1,i} \arrow[r] \arrow[d, "\cong"] & \operatorname{colim} M_{2,i} \arrow[r] \arrow[d] & \operatorname{colim} M_{3,i} \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & M_1 \arrow[r] & M_2 \arrow[r] & M_3 \arrow[r] & 0 \end{tikzcd}$$

with exact rows and outer vertical maps isomorphisms. Hence $\operatorname{colim} M_{2,i} \to M_2$ is also an isomorphism and (6) holds.

Condition (6) implies (1) by Example 82.2 (2). □

The previous theorem shows that a universally exact sequence is always a colimit of split short exact sequences. If the cokernel of a universally injective map is finitely presented, then in fact the map itself splits:

058L **Lemma 82.4.** *Let*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence of R-modules. Suppose M_3 is of finite presentation. Then

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is universally exact if and only if it is split.

Proof. A split short exact sequence is always universally exact, see Example 82.2. Conversely, if the sequence is universally exact, then by Theorem 82.3 (5) applied to $P = M_3$, the map $M_2 \to M_3$ admits a section. □

The following lemma shows how universally injective maps are complementary to flat modules.

058M **Lemma 82.5.** *Let M be an R-module. Then M is flat if and only if any exact sequence of R-modules*

$$0 \to M_1 \to M_2 \to M \to 0$$

is universally exact.

Proof. This follows from Lemma 81.3 and Theorem 82.3 (5). □

058N **Example 82.6.** Non-split and non-flat universally exact sequences.

(1) In spite of Lemma 82.4, it is possible to have a short exact sequence of R-modules
$$0 \to M_1 \to M_2 \to M_3 \to 0$$
that is universally exact but non-split. For instance, take $R = \mathbf{Z}$, let $M_1 = \bigoplus_{n=1}^{\infty} \mathbf{Z}$, let $M_2 = \prod_{n=1}^{\infty} \mathbf{Z}$, and let M_3 be the cokernel of the inclusion $M_1 \to M_2$. Then M_1, M_2, M_3 are all flat since they are torsion-free (More on Algebra, Lemma 22.11), so by Lemma 82.5,
$$0 \to M_1 \to M_2 \to M_3 \to 0$$
is universally exact. However there can be no section $s : M_3 \to M_2$. In fact, if x is the image of $(2, 2^2, 2^3, \ldots) \in M_2$ in M_3, then any module map $s : M_3 \to M_2$ must kill x. This is because $x \in 2^n M_3$ for any $n \geq 1$, hence $s(x)$ is divisible by 2^n for all $n \geq 1$ and so must be 0.

(2) In spite of Lemma 82.5, it is possible to have a short exact sequence of R-modules
$$0 \to M_1 \to M_2 \to M_3 \to 0$$
that is universally exact but with M_1, M_2, M_3 all non-flat. In fact if M is any non-flat module, just take the split exact sequence
$$0 \to M \to M \oplus M \to M \to 0.$$
For instance over $R = \mathbf{Z}$, take M to be any torsion module.

(3) Taking the direct sum of an exact sequence as in (1) with one as in (2), we get a short exact sequence of R-modules
$$0 \to M_1 \to M_2 \to M_3 \to 0$$
that is universally exact, non-split, and such that M_1, M_2, M_3 are all non-flat.

058P **Lemma 82.7.** *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a universally exact sequence of R-modules, and suppose M_2 is flat. Then M_1 and M_3 are flat.*

Proof. Let $0 \to N \to N' \to N'' \to 0$ be a short exact sequence of R-modules. Consider the commutative diagram
$$\begin{tikzcd}
M_1 \otimes_R N \arrow[r] \arrow[d] & M_2 \otimes_R N \arrow[r] \arrow[d] & M_3 \otimes_R N \arrow[d] \\
M_1 \otimes_R N' \arrow[r] \arrow[d] & M_2 \otimes_R N' \arrow[r] \arrow[d] & M_3 \otimes_R N' \arrow[d] \\
M_1 \otimes_R N'' \arrow[r] & M_2 \otimes_R N'' \arrow[r] & M_3 \otimes_R N''
\end{tikzcd}$$
(we have dropped the 0's on the boundary). By assumption the rows give short exact sequences and the arrow $M_2 \otimes N \to M_2 \otimes N'$ is injective. Clearly this implies that $M_1 \otimes N \to M_1 \otimes N'$ is injective and we see that M_1 is flat. In particular the left and middle columns give rise to short exact sequences. It follows from a diagram chase that the arrow $M_3 \otimes N \to M_3 \otimes N'$ is injective. Hence M_3 is flat. $\square$

05CH **Lemma 82.8.** *Let R be a ring. Let $M \to M'$ be a universally injective R-module map. Then for any R-module N the map $M \otimes_R N \to M' \otimes_R N$ is universally injective.*

Proof. Omitted. ☐

05CI **Lemma 82.9.** *Let R be a ring. A composition of universally injective R-module maps is universally injective.*

Proof. Omitted. ☐

05CJ **Lemma 82.10.** *Let R be a ring. Let $M \to M'$ and $M' \to M''$ be R-module maps. If their composition $M \to M''$ is universally injective, then $M \to M'$ is universally injective.*

Proof. Omitted. ☐

05CK **Lemma 82.11.** *Let $R \to S$ be a faithfully flat ring map. Then $R \to S$ is universally injective as a map of R-modules. In particular $R \cap IS = I$ for any ideal $I \subset R$.*

Proof. Let N be an R-module. We have to show that $N \to N \otimes_R S$ is injective. As S is faithfully flat as an R-module, it suffices to prove this after tensoring with S. Hence it suffices to show that $N \otimes_R S \to N \otimes_R S \otimes_R S$, $n \otimes s \mapsto n \otimes 1 \otimes s$ is injective. This is true because there is a retraction, namely, $n \otimes s \otimes s' \mapsto n \otimes ss'$. ☐

05CL **Lemma 82.12.** *Let $R \to S$ be a ring map. Let $M \to M'$ be a map of S-modules. The following are equivalent*

(1) *$M \to M'$ is universally injective as a map of R-modules,*
(2) *for each prime $\mathfrak{q}$ of S the map $M_{\mathfrak{q}} \to M'_{\mathfrak{q}}$ is universally injective as a map of R-modules,*
(3) *for each maximal ideal $\mathfrak{m}$ of S the map $M_{\mathfrak{m}} \to M'_{\mathfrak{m}}$ is universally injective as a map of R-modules,*
(4) *for each prime $\mathfrak{q}$ of S the map $M_{\mathfrak{q}} \to M'_{\mathfrak{q}}$ is universally injective as a map of $R_{\mathfrak{p}}$-modules, where $\mathfrak{p}$ is the inverse image of $\mathfrak{q}$ in R, and*
(5) *for each maximal ideal $\mathfrak{m}$ of S the map $M_{\mathfrak{m}} \to M'_{\mathfrak{m}}$ is universally injective as a map of $R_{\mathfrak{p}}$-modules, where $\mathfrak{p}$ is the inverse image of $\mathfrak{m}$ in R.*

Proof. Let N be an R-module. Let $\mathfrak{q}$ be a prime of S lying over the prime $\mathfrak{p}$ of R. Then we have

$$(M \otimes_R N)_{\mathfrak{q}} = M_{\mathfrak{q}} \otimes_R N = M_{\mathfrak{q}} \otimes_{R_{\mathfrak{p}}} N_{\mathfrak{p}}.$$

Moreover, the same thing holds for M' and localization is exact. Also, if N is an $R_{\mathfrak{p}}$-module, then $N_{\mathfrak{p}} = N$. Using this the equivalences can be proved in a straightforward manner.

For example, suppose that (5) holds. Let $K = \mathrm{Ker}(M \otimes_R N \to M' \otimes_R N)$. By the remarks above we see that $K_{\mathfrak{m}} = 0$ for each maximal ideal $\mathfrak{m}$ of S. Hence $K = 0$ by Lemma 23.1. Thus (1) holds. Conversely, suppose that (1) holds. Take any $\mathfrak{q} \subset S$ lying over $\mathfrak{p} \subset R$. Take any module N over $R_{\mathfrak{p}}$. Then by assumption $\mathrm{Ker}(M \otimes_R N \to M' \otimes_R N) = 0$. Hence by the formulae above and the fact that $N = N_{\mathfrak{p}}$ we see that $\mathrm{Ker}(M_{\mathfrak{q}} \otimes_{R_{\mathfrak{p}}} N \to M'_{\mathfrak{q}} \otimes_{R_{\mathfrak{p}}} N) = 0$. In other words (4) holds. Of course (4) $\Rightarrow$ (5) is immediate. Hence (1), (4) and (5) are all equivalent. We omit the proof of the other equivalences. ☐

05CM **Lemma 82.13.** *Let $\varphi : A \to B$ be a ring map. Let $S \subset A$ and $S' \subset B$ be multiplicative subsets such that $\varphi(S) \subset S'$. Let $M \to M'$ be a map of B-modules.*

(1) *If $M \to M'$ is universally injective as a map of A-modules, then $(S')^{-1}M \to (S')^{-1}M'$ is universally injective as a map of A-modules and as a map of $S^{-1}A$-modules.*
(2) *If M and M' are $(S')^{-1}B$-modules, then $M \to M'$ is universally injective as a map of A-modules if and only if it is universally injective as a map of $S^{-1}A$-modules.*

Proof. You can prove this using Lemma 82.12 but you can also prove it directly as follows. Assume $M \to M'$ is A-universally injective. Let Q be an A-module. Then $Q \otimes_A M \to Q \otimes_A M'$ is injective. Since localization is exact we see that $(S')^{-1}(Q \otimes_A M) \to (S')^{-1}(Q \otimes_A M')$ is injective. As $(S')^{-1}(Q \otimes_A M) = Q \otimes_A (S')^{-1}M$ and similarly for M' we see that $Q \otimes_A (S')^{-1}M \to Q \otimes_A (S')^{-1}M'$ is injective, hence $(S')^{-1}M \to (S')^{-1}M'$ is universally injective as a map of A-modules. This proves the first part of (1). To see (2) we can use the following two facts: (a) if Q is an $S^{-1}A$-module, then $Q \otimes_A S^{-1}A = Q$, i.e., tensoring with Q over A is the same thing as tensoring with Q over $S^{-1}A$, (b) if M is any A-module on which the elements of S are invertible, then $M \otimes_A Q = M \otimes_{S^{-1}A} S^{-1}Q$. Part (2) follows from this immediately. □

0AS5 **Lemma 82.14.** *Let R be a ring and let $M \to M'$ be a map of R-modules. If M' is flat, then $M \to M'$ is universally injective if and only if $M/IM \to M'/IM'$ is injective for every finitely generated ideal I of R.*

Proof. It suffices to show that $M \otimes_R Q \to M' \otimes_R Q$ is injective for every finite R-module Q, see Theorem 82.3. Then Q has a finite filtration $0 = Q_0 \subset Q_1 \subset \ldots \subset Q_n = Q$ by submodules whose subquotients are isomorphic to cyclic modules R/I_i, see Lemma 5.4. Since M' is flat, we obtain a filtration

$$\begin{tikzcd} M \otimes Q_1 \arrow[r] \arrow[d] & M \otimes Q_2 \arrow[r] \arrow[d] & \ldots \arrow[r] & M \otimes Q \arrow[d] \\ M' \otimes Q_1 \arrow[r, hook] & M' \otimes Q_2 \arrow[r, hook] & \ldots \arrow[r, hook] & M' \otimes Q \end{tikzcd}$$

of $M' \otimes_R Q$ by submodules $M' \otimes_R Q_i$ whose successive quotients are $M' \otimes_R R/I_i = M'/I_iM'$. A simple induction argument shows that it suffices to check $M/I_iM \to M'/I_iM'$ is injective. Note that the collection of finitely generated ideals $I'_i \subset I_i$ is a directed set. Thus $M/I_iM = \operatorname{colim} M/I'_iM$ is a filtered colimit, similarly for M', the maps $M/I'_iM \to M'/I'_iM'$ are injective by assumption, and since filtered colimits are exact (Lemma 8.8) we conclude. □

83. Descent for finite projective modules

058Q In this section we give an elementary proof of the fact that the property of being a *finite* projective module descends along faithfully flat ring maps. The proof does not apply when we drop the finiteness condition. However, the method is indicative of the one we shall use to prove descent for the property of being a *countably generated* projective module—see the comments at the end of this section.

058R **Lemma 83.1.** *Let M be an R-module. Then M is finite projective if and only if M is finitely presented and flat.*

Proof. This is part of Lemma 78.2. However, at this point we can give a more elegant proof of the implication (1) ⇒ (2) of that lemma as follows. If M is finitely presented and flat, then take a surjection $R^n \to M$. By Lemma 81.3 applied to $P = M$, the map $R^n \to M$ admits a section. So M is a direct summand of a free module and hence projective. $\square$

Here are some properties of modules that descend.

03C4 **Lemma 83.2.** *Let $R \to S$ be a faithfully flat ring map. Let M be an R-module. Then*

1. *if the S-module $M \otimes_R S$ is of finite type, then M is of finite type,*
2. *if the S-module $M \otimes_R S$ is of finite presentation, then M is of finite presentation,*
3. *if the S-module $M \otimes_R S$ is flat, then M is flat, and*
4. *add more here as needed.*

Proof. Assume $M \otimes_R S$ is of finite type. Let $y_1, \ldots, y_m$ be generators of $M \otimes_R S$ over S. Write $y_j = \sum x_i \otimes f_i$ for some $x_1, \ldots, x_n \in M$. Then we see that the map $\varphi : R^{\oplus n} \to M$ has the property that $\varphi \otimes \text{id}_S : S^{\oplus n} \to M \otimes_R S$ is surjective. Since $R \to S$ is faithfully flat we see that φ is surjective, and M is finitely generated.

Assume $M \otimes_R S$ is of finite presentation. By (1) we see that M is of finite type. Choose a surjection $R^{\oplus n} \to M$ and denote K the kernel. As $R \to S$ is flat we see that $K \otimes_R S$ is the kernel of the base change $S^{\oplus n} \to M \otimes_R S$. As $M \otimes_R S$ is of finite presentation we conclude that $K \otimes_R S$ is of finite type. Hence by (1) we see that K is of finite type and hence M is of finite presentation.

Part (3) is Lemma 39.8. $\square$

058S **Proposition 83.3.** *Let $R \to S$ be a faithfully flat ring map. Let M be an R-module. If the S-module $M \otimes_R S$ is finite projective, then M is finite projective.*

Proof. Follows from Lemmas 83.1 and 83.2. $\square$

The next few sections are about removing the finiteness assumption by using dévissage to reduce to the countably generated case. In the countably generated case, the strategy is to find a characterization of countably generated projective modules analogous to Lemma 83.1, and then to prove directly that this characterization descends. We do this by introducing the notion of a Mittag-Leffler module and proving that if a module M is countably generated, then it is projective if and only if it is flat and Mittag-Leffler (Theorem 93.3). When M is finitely generated, this statement reduces to Lemma 83.1 (since, according to Example 91.1 (1), a finitely generated module is Mittag-Leffler if and only if it is finitely presented).

84. Transfinite dévissage of modules

058T In this section we introduce a dévissage technique for decomposing a module into a direct sum. The main result is that a projective module is a direct sum of countably generated modules (Theorem 84.5 below). We follow [Kap58].

058U **Definition 84.1.** Let M be an R-module. A *direct sum dévissage* of M is a family of submodules $(M_\alpha)_{\alpha \in S}$, indexed by an ordinal S and increasing (with respect to inclusion), such that:

(0) $M_0 = 0$;
(1) $M = \bigcup_\alpha M_\alpha$;
(2) if $\alpha \in S$ is a limit ordinal, then $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$;

(3) if $\alpha + 1 \in S$, then M_α is a direct summand of $M_{\alpha+1}$.

If moreover

(4) $M_{\alpha+1}/M_\alpha$ is countably generated for $\alpha + 1 \in S$,

then $(M_\alpha)_{\alpha \in S}$ is called a *Kaplansky dévissage* of M.

The terminology is justified by the following lemma.

058V **Lemma 84.2.** *Let M be an R-module. If $(M_\alpha)_{\alpha \in S}$ is a direct sum dévissage of M, then $M \cong \bigoplus_{\alpha+1 \in S} M_{\alpha+1}/M_\alpha$.*

Proof. By property (3) of a direct sum dévissage, there is an inclusion $M_{\alpha+1}/M_\alpha \to M$ for each $\alpha \in S$. Consider the map

$$f : \bigoplus\nolimits_{\alpha+1 \in S} M_{\alpha+1}/M_\alpha \to M$$

given by the sum of these inclusions. Further consider the restrictions

$$f_\beta : \bigoplus\nolimits_{\alpha+1 \leq \beta} M_{\alpha+1}/M_\alpha \longrightarrow M$$

for $\beta \in S$. Transfinite induction on S shows that the image of f_β is M_β. For $\beta = 0$ this is true by (0). If $\beta + 1$ is a successor ordinal and it is true for β, then it is true for $\beta + 1$ by (3). And if β is a limit ordinal and it is true for $\alpha < \beta$, then it is true for β by (2). Hence f is surjective by (1).

Transfinite induction on S also shows that the restrictions f_β are injective. For $\beta = 0$ it is true. If $\beta+1$ is a successor ordinal and f_β is injective, then let x be in the kernel and write $x = (x_{\alpha+1})_{\alpha+1 \leq \beta+1}$ in terms of its components $x_{\alpha+1} \in M_{\alpha+1}/M_\alpha$. By property (3) and the fact that the image of f_β is M_β both $(x_{\alpha+1})_{\alpha+1 \leq \beta}$ and $x_{\beta+1}$ map to 0. Hence $x_{\beta+1} = 0$ and, by the assumption that the restriction f_β is injective also $x_{\alpha+1} = 0$ for every $\alpha + 1 \leq \beta$. So $x = 0$ and $f_{\beta+1}$ is injective. If β is a limit ordinal consider an element x of the kernel. Then x is already contained in the domain of f_α for some $\alpha < \beta$. Thus $x = 0$ which finishes the induction. We conclude that f is injective since f_β is for each $\beta \in S$. □

058W **Lemma 84.3.** *Let M be an R-module. Then M is a direct sum of countably generated R-modules if and only if it admits a Kaplansky dévissage.*

Proof. The lemma takes care of the "if" direction. Conversely, suppose $M = \bigoplus_{i \in I} N_i$ where each N_i is a countably generated R-module. Well-order I so that we can think of it as an ordinal. Then setting $M_i = \bigoplus_{j<i} N_j$ gives a Kaplansky dévissage $(M_i)_{i \in I}$ of M. □

058X **Theorem 84.4.** *Suppose M is a direct sum of countably generated R-modules. If P is a direct summand of M, then P is also a direct sum of countably generated R-modules.*

Proof. Write $M = P \oplus Q$. We are going to construct a Kaplansky dévissage $(M_\alpha)_{\alpha \in S}$ of M which, in addition to the defining properties (0)-(4), satisfies:

(5) Each M_α is a direct summand of M;

(6) $M_\alpha = P_\alpha \oplus Q_\alpha$, where $P_\alpha = P \cap M_\alpha$ and $Q = Q \cap M_\alpha$.

(Note: if properties (0)-(2) hold, then in fact property (3) is equivalent to property (5).)

To see how this implies the theorem, it is enough to show that $(P_\alpha)_{\alpha \in S}$ forms a Kaplansky dévissage of P. Properties (0), (1), and (2) are clear. By (5) and (6) for (M_α), each P_α is a direct summand of M. Since $P_\alpha \subset P_{\alpha+1}$, this implies P_α is a direct summand of $P_{\alpha+1}$; hence (3) holds for (P_α). For (4), note that

$$M_{\alpha+1}/M_\alpha \cong P_{\alpha+1}/P_\alpha \oplus Q_{\alpha+1}/Q_\alpha,$$

so $P_{\alpha+1}/P_\alpha$ is countably generated because this is true of $M_{\alpha+1}/M_\alpha$.

It remains to construct the M_α. Write $M = \bigoplus_{i \in I} N_i$ where each N_i is a countably generated R-module. Choose a well-ordering of I. By transfinite recursion we are going to define an increasing family of submodules M_α of M, one for each ordinal α, such that M_α is a direct sum of some subset of the N_i.

For $\alpha = 0$ let $M_0 = 0$. If α is a limit ordinal and M_β has been defined for all $\beta < \alpha$, then define $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$. Since each M_β for $\beta < \alpha$ is a direct sum of a subset of the N_i, the same will be true of M_α. If $\alpha + 1$ is a successor ordinal and M_α has been defined, then define $M_{\alpha+1}$ as follows. If $M_\alpha = M$, then let $M_{\alpha+1} = M$. If not, choose the smallest $j \in I$ such that N_j is not contained in M_α. We will construct an infinite matrix $(x_{mn}), m, n = 1, 2, 3, \ldots$ such that:

(1) N_j is contained in the submodule of M generated by the entries x_{mn};
(2) if we write any entry $x_{k\ell}$ in terms of its P- and Q-components, $x_{k\ell} = y_{k\ell} + z_{k\ell}$, then the matrix (x_{mn}) contains a set of generators for each N_i for which $y_{k\ell}$ or $z_{k\ell}$ has nonzero component.

Then we define $M_{\alpha+1}$ to be the submodule of M generated by M_α and all x_{mn}; by property (2) of the matrix (x_{mn}), $M_{\alpha+1}$ will be a direct sum of some subset of the N_i. To construct the matrix (x_{mn}), let $x_{11}, x_{12}, x_{13}, \ldots$ be a countable set of generators for N_j. Then if $x_{11} = y_{11} + z_{11}$ is the decomposition into P- and Q-components, let $x_{21}, x_{22}, x_{23}, \ldots$ be a countable set of generators for the sum of the N_i for which y_{11} or z_{11} have nonzero component. Repeat this process on x_{12} to get elements $x_{31}, x_{32}, \ldots$, the third row of our matrix. Repeat on x_{21} to get the fourth row, on x_{13} to get the fifth, and so on, going down along successive anti-diagonals as indicated below:

$$\left(\begin{array}{ccccccccc}
x_{11} & & x_{12} & & x_{13} & & x_{14} & & \ldots \\
 & \swarrow & & \swarrow & & \swarrow & & & \\
x_{21} & & x_{22} & & x_{23} & & \ldots & & \\
 & \swarrow & & \swarrow & & & & & \\
x_{31} & & x_{32} & & \ldots & & & & \\
 & \swarrow & & & & & & & \\
x_{41} & & \ldots & & & & & & \\
\ldots & & & & & & & &
\end{array}\right).$$

Transfinite induction on I (using the fact that we constructed $M_{\alpha+1}$ to contain N_j for the smallest j such that N_j is not contained in M_α) shows that for each $i \in I$, N_i is contained in some M_α. Thus, there is some large enough ordinal S satisfying: for each $i \in I$ there is $\alpha \in S$ such that N_i is contained in M_α. This means $(M_\alpha)_{\alpha \in S}$ satisfies property (1) of a Kaplansky dévissage of M. The family $(M_\alpha)_{\alpha \in S}$ moreover satisfies the other defining properties, and also (5) and (6) above: properties (0), (2), (4), and (6) are clear by construction; property (5) is true because each M_α is by construction a direct sum of some N_i; and (3) is implied by (5) and the fact that $M_\alpha \subset M_{\alpha+1}$. $\square$

As a corollary we get the result for projective modules stated at the beginning of the section.

058Y **Theorem 84.5.** *If P is a projective R-module, then P is a direct sum of countably generated projective R-modules.*

Proof. A module is projective if and only if it is a direct summand of a free module, so this follows from Theorem 84.4. $\square$

85. Projective modules over a local ring

058Z In this section we prove a very cute result: a projective module M over a local ring is free (Theorem 85.4 below). Note that with the additional assumption that M is finite, this result is Lemma 78.5. In general we have:

0590 **Lemma 85.1.** *Let R be a ring. Then every projective R-module is free if and only if every countably generated projective R-module is free.*

Proof. Follows immediately from Theorem 84.5. $\square$

Here is a criterion for a countably generated module to be free.

0591 **Lemma 85.2.** *Let M be a countably generated R-module with the following property: if $M = N \oplus N'$ with N' a finite free R-module, then any element of N is contained in a free direct summand of N. Then M is free.*

Proof. Let $x_1, x_2, \ldots$ be a countable set of generators for M. We inductively construct finite free direct summands $F_1, F_2, \ldots$ of M such that for all n we have that $F_1 \oplus \ldots \oplus F_n$ is a direct summand of M which contains $x_1, \ldots, x_n$. Namely, given $F_1, \ldots, F_n$ with the desired properties, write

$$M = F_1 \oplus \ldots \oplus F_n \oplus N$$

and let $x \in N$ be the image of x_{n+1}. Then we can find a free direct summand $F_{n+1} \subset N$ containing x by the assumption in the statement of the lemma. Of course we can replace F_{n+1} by a finite free direct summand of F_{n+1} and the induction step is complete. Then $M = \bigoplus_{i=1}^{\infty} F_i$ is free. $\square$

0592 **Lemma 85.3.** *Let P be a projective module over a local ring R. Then any element of P is contained in a free direct summand of P.*

Proof. Since P is projective it is a direct summand of some free R-module F, say $F = P \oplus Q$. Let $x \in P$ be the element that we wish to show is contained in a free direct summand of P. Let B be a basis of F such that the number of basis elements needed in the expression of x is minimal, say $x = \sum_{i=1}^{n} a_i e_i$ for some $e_i \in B$ and $a_i \in R$. Then no a_j can be expressed as a linear combination of the other a_i; for if $a_j = \sum_{i \neq j} a_i b_i$ for some $b_i \in R$, then replacing e_i by $e_i + b_i e_j$ for $i \neq j$ and leaving unchanged the other elements of B, we get a new basis for F in terms of which x has a shorter expression.

Let $e_i = y_i + z_i, y_i \in P, z_i \in Q$ be the decomposition of e_i into its P- and Q-components. Write $y_i = \sum_{j=1}^{n} b_{ij} e_j + t_i$, where t_i is a linear combination of elements in B other than $e_1, \ldots, e_n$. To finish the proof it suffices to show that the matrix (b_{ij}) is invertible. For then the map $F \to F$ sending $e_i \mapsto y_i$ for $i = 1, \ldots, n$ and fixing $B \setminus \{e_1, \ldots, e_n\}$ is an isomorphism, so that $y_1, \ldots, y_n$ together with $B \setminus \{e_1, \ldots, e_n\}$ form a basis for F. Then the submodule N spanned by $y_1, \ldots, y_n$

is a free submodule of P; N is a direct summand of P since $N \subset P$ and both N and P are direct summands of F; and $x \in N$ since $x \in P$ implies $x = \sum_{i=1}^n a_i e_i = \sum_{i=1}^n a_i y_i$.

Now we prove that (b_{ij}) is invertible. Plugging $y_i = \sum_{j=1}^n b_{ij} e_j + t_i$ into $\sum_{i=1}^n a_i e_i = \sum_{i=1}^n a_i y_i$ and equating the coefficients of e_j gives $a_j = \sum_{i=1}^n a_i b_{ij}$. But as noted above, our choice of B guarantees that no a_j can be written as a linear combination of the other a_i. Thus b_{ij} is a non-unit for $i \neq j$, and $1 - b_{ii}$ is a non-unit—so in particular b_{ii} is a unit—for all i. But a matrix over a local ring having units along the diagonal and non-units elsewhere is invertible, as its determinant is a unit. □

0593 **Theorem 85.4.** *If P is a projective module over a local ring R, then P is free.*

Proof. Follows from Lemmas 85.1, 85.2, and 85.3. □

86. Mittag-Leffler systems

0594 The purpose of this section is to define Mittag-Leffler systems and why this is a useful notion.

In the following, I will be a directed set, see Categories, Definition 21.1. Let $(A_i, \varphi_{ji} : A_j \to A_i)$ be an inverse system of sets or of modules indexed by I, see Categories, Definition 21.4. This is a directed inverse system as we assumed I directed (Categories, Definition 21.4). For each $i \in I$, the images $\varphi_{ji}(A_j) \subset A_i$ for $j \geq i$ form a decreasing directed family of subsets (or submodules) of A_i. Let $A'_i = \bigcap_{j \geq i} \varphi_{ji}(A_j)$. Then $\varphi_{ji}(A'_j) \subset A'_i$ for $j \geq i$, hence by restricting we get a directed inverse system $(A'_i, \varphi_{ji}|_{A'_j})$. From the construction of the limit of an inverse system in the category of sets or modules, we have $\lim A_i = \lim A'_i$. The Mittag-Leffler condition on (A_i, φ_{ji}) is that A'_i equals $\varphi_{ji}(A_j)$ for some $j \geq i$ (and hence equals $\varphi_{ki}(A_k)$ for all $k \geq j$):

0595 **Definition 86.1.** Let (A_i, φ_{ji}) be a directed inverse system of sets over I. Then we say (A_i, φ_{ji}) is *Mittag-Leffler* if for each $i \in I$, the family $\varphi_{ji}(A_j) \subset A_i$ for $j \geq i$ stabilizes. Explicitly, this means that for each $i \in I$, there exists $j \geq i$ such that for $k \geq j$ we have $\varphi_{ki}(A_k) = \varphi_{ji}(A_j)$. If (A_i, φ_{ji}) is a directed inverse system of modules over a ring R, we say that it is Mittag-Leffler if the underlying inverse system of sets is Mittag-Leffler.

0596 **Example 86.2.** If (A_i, φ_{ji}) is a directed inverse system of sets or of modules and the maps φ_{ji} are surjective, then clearly the system is Mittag-Leffler. Conversely, suppose (A_i, φ_{ji}) is Mittag-Leffler. Let $A'_i \subset A_i$ be the stable image of $\varphi_{ji}(A_j)$ for $j \geq i$. Then $\varphi_{ji}|_{A'_j} : A'_j \to A'_i$ is surjective for $j \geq i$ and $\lim A_i = \lim A'_i$. Hence the limit of the Mittag-Leffler system (A_i, φ_{ji}) can also be written as the limit of a directed inverse system over I with surjective maps.

0597 **Lemma 86.3.** *Let (A_i, φ_{ji}) be a directed inverse system over I. Suppose I is countable. If (A_i, φ_{ji}) is Mittag-Leffler and the A_i are nonempty, then* $\lim A_i$ *is nonempty.*

Proof. Let $i_1, i_2, i_3, \ldots$ be an enumeration of the elements of I. Define inductively a sequence of elements $j_n \in I$ for $n = 1, 2, 3, \ldots$ by the conditions: $j_1 = i_1$, and $j_n \geq i_n$ and $j_n \geq j_m$ for $m < n$. Then the sequence j_n is increasing and forms a cofinal subset of I. Hence we may assume $I = \{1, 2, 3, \ldots\}$. So by Example 86.2 we are reduced to showing that the limit of an inverse system of nonempty sets with surjective maps indexed by the positive integers is nonempty. This is obvious. □

The Mittag-Leffler condition will be important for us because of the following exactness property.

0598 **Lemma 86.4.** *Let*

$$0 \to A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \to 0$$

be an exact sequence of directed inverse systems of abelian groups over I. Suppose I is countable. If (A_i) is Mittag-Leffler, then

$$0 \to \lim A_i \to \lim B_i \to \lim C_i \to 0$$

is exact.

Proof. Taking limits of directed inverse systems is left exact, hence we only need to prove surjectivity of $\lim B_i \to \lim C_i$. So let $(c_i) \in \lim C_i$. For each $i \in I$, let $E_i = g_i^{-1}(c_i)$, which is nonempty since $g_i : B_i \to C_i$ is surjective. The system of maps $\varphi_{ji} : B_j \to B_i$ for (B_i) restrict to maps $E_j \to E_i$ which make (E_i) into an inverse system of nonempty sets. It is enough to show that (E_i) is Mittag-Leffler. For then Lemma 86.3 would show $\lim E_i$ is nonempty, and taking any element of $\lim E_i$ would give an element of $\lim B_i$ mapping to (c_i).

By the injection $f_i : A_i \to B_i$ we will regard A_i as a subset of B_i. Since (A_i) is Mittag-Leffler, if $i \in I$ then there exists $j \geq i$ such that $\varphi_{ki}(A_k) = \varphi_{ji}(A_j)$ for $k \geq j$. We claim that also $\varphi_{ki}(E_k) = \varphi_{ji}(E_j)$ for $k \geq j$. Always $\varphi_{ki}(E_k) \subset \varphi_{ji}(E_j)$ for $k \geq j$. For the reverse inclusion let $e_j \in E_j$, and we need to find $x_k \in E_k$ such that $\varphi_{ki}(x_k) = \varphi_{ji}(e_j)$. Let $e'_k \in E_k$ be any element, and set $e'_j = \varphi_{kj}(e'_k)$. Then $g_j(e_j - e'_j) = c_j - c_j = 0$, hence $e_j - e'_j = a_j \in A_j$. Since $\varphi_{ki}(A_k) = \varphi_{ji}(A_j)$, there exists $a_k \in A_k$ such that $\varphi_{ki}(a_k) = \varphi_{ji}(a_j)$. Hence

$$\varphi_{ki}(e'_k + a_k) = \varphi_{ji}(e'_j) + \varphi_{ji}(a_j) = \varphi_{ji}(e_j),$$

so we can take $x_k = e'_k + a_k$. □

87. Inverse systems

03C9 In many papers (and in this section) the term *inverse system* is used to indicate an inverse system over the partially ordered set $(\mathbf{N}, \geq)$. We briefly discuss such systems in this section. This material will be discussed more broadly in Homology, Section 31. Suppose we are given a ring R and a sequence of R-modules

$$M_1 \xleftarrow{\varphi_2} M_2 \xleftarrow{\varphi_3} M_3 \leftarrow \ldots$$

with maps as indicated. By composing successive maps we obtain maps $\varphi_{ii'} : M_i \to M_{i'}$ whenever $i \geq i'$ such that moreover $\varphi_{ii''} = \varphi_{i'i''} \circ \varphi_{ii'}$ whenever $i \geq i' \geq i''$. Conversely, given the system of maps $\varphi_{ii'}$ we can set $\varphi_i = \varphi_{i(i-1)}$ and recover the maps displayed above. In this case

$$\lim M_i = \{(x_i) \in \prod M_i \mid \varphi_i(x_i) = x_{i-1},\ i = 2, 3, \ldots\}$$

compare with Categories, Section 15. As explained in Homology, Section 31 this is actually a limit in the category of R-modules, as defined in Categories, Section 14.

03CA **Lemma 87.1.** *Let R be a ring. Let $0 \to K_i \to L_i \to M_i \to 0$ be short exact sequences of R-modules, $i \geq 1$ which fit into maps of short exact sequences*

$$\begin{array}{ccccccccc} 0 & \to & K_i & \to & L_i & \to & M_i & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & K_{i+1} & \to & L_{i+1} & \to & M_{i+1} & \to & 0 \end{array}$$

If for every i there exists a $c = c(i) \geq i$ such that $\mathrm{Im}(K_c \to K_i) = \mathrm{Im}(K_j \to K_i)$ for all $j \geq c$, then the sequence

$$0 \to \lim K_i \to \lim L_i \to \lim M_i \to 0$$

is exact.

Proof. This is a special case of the more general Lemma 86.4. $\square$

88. Mittag-Leffler modules

0599 A Mittag-Leffler module is (very roughly) a module which can be written as a directed limit whose dual is a Mittag-Leffler system. To be able to give a precise definition we need to do a bit of work.

059A **Definition 88.1.** Let (M_i, f_{ij}) be a directed system of R-modules. We say that (M_i, f_{ij}) is a *Mittag-Leffler directed system of modules* if each M_i is an R-module of finite presentation and if for every R-module N, the inverse system

$$(\mathrm{Hom}_R(M_i, N), \mathrm{Hom}_R(f_{ij}, N))$$

is Mittag-Leffler.

We are going to characterize those R-modules that are colimits of Mittag-Leffler directed systems of modules.

059B **Definition 88.2.** Let $f : M \to N$ and $g : M \to M'$ be maps of R-modules. Then we say g *dominates* f if for any R-module Q, we have $\mathrm{Ker}(f \otimes_R \mathrm{id}_Q) \subset \mathrm{Ker}(g \otimes_R \mathrm{id}_Q)$.

It is enough to check this condition for finitely presented modules.

059C **Lemma 88.3.** *Let $f : M \to N$ and $g : M \to M'$ be maps of R-modules. Then g dominates f if and only if for any finitely presented R-module Q, we have $\mathrm{Ker}(f \otimes_R id_Q) \subset \mathrm{Ker}(g \otimes_R id_Q)$.*

Proof. Suppose $\mathrm{Ker}(f \otimes_R \mathrm{id}_Q) \subset \mathrm{Ker}(g \otimes_R \mathrm{id}_Q)$ for all finitely presented modules Q. If Q is an arbitrary module, write $Q = \mathrm{colim}_{i \in I} Q_i$ as a colimit of a directed system of finitely presented modules Q_i. Then $\mathrm{Ker}(f \otimes_R \mathrm{id}_{Q_i}) \subset \mathrm{Ker}(g \otimes_R \mathrm{id}_{Q_i})$ for all i. Since taking directed colimits is exact and commutes with tensor product, it follows that $\mathrm{Ker}(f \otimes_R \mathrm{id}_Q) \subset \mathrm{Ker}(g \otimes_R \mathrm{id}_Q)$. $\square$

0AUM **Lemma 88.4.** *Let $f : M \to N$ and $g : M \to M'$ be maps of R-modules. Consider the pushout of f and g,*

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ M' & \xrightarrow{f'} & N' \end{array}$$

Then g dominates f if and only if f' is universally injective.

Proof. Recall that N' is $M' \oplus N$ modulo the submodule consisting of elements $(g(x), -f(x))$ for $x \in M$. From the construction of N' we have a short exact sequence

$$0 \to \mathrm{Ker}(f) \cap \mathrm{Ker}(g) \to \mathrm{Ker}(f) \to \mathrm{Ker}(f') \to 0.$$

Since tensoring commutes with taking pushouts, we have such a short exact sequence

$$0 \to \mathrm{Ker}(f \otimes \mathrm{id}_Q) \cap \mathrm{Ker}(g \otimes \mathrm{id}_Q) \to \mathrm{Ker}(f \otimes \mathrm{id}_Q) \to \mathrm{Ker}(f' \otimes \mathrm{id}_Q) \to 0$$

for every R-module Q. So f' is universally injective if and only if $\mathrm{Ker}(f \otimes \mathrm{id}_Q) \subset \mathrm{Ker}(g \otimes \mathrm{id}_Q)$ for every Q, if and only if g dominates f. $\square$

The above definition of domination is sometimes related to the usual notion of domination of maps as the following lemma shows.

059D **Lemma 88.5.** *Let $f : M \to N$ and $g : M \to M'$ be maps of R-modules. Suppose $\mathrm{Coker}(f)$ is of finite presentation. Then g dominates f if and only if g factors through f, i.e. there exists a module map $h : N \to M'$ such that $g = h \circ f$.*

Proof. Consider the pushout of f and g as in the statement of Lemma 88.4. From the construction of the pushout it follows that $\mathrm{Coker}(f') = \mathrm{Coker}(f)$, so $\mathrm{Coker}(f')$ is of finite presentation. Then by Lemma 82.4, f' is universally injective if and only if

$$0 \to M' \xrightarrow{f'} N' \to \mathrm{Coker}(f') \to 0$$

splits. This is the case if and only if there is a map $h' : N' \to M'$ such that $h' \circ f' = \mathrm{id}_{M'}$. From the universal property of the pushout, the existence of such an h' is equivalent to g factoring through f. $\square$

059E **Proposition 88.6.** *Let M be an R-module. Let (M_i, f_{ij}) be a directed system of finitely presented R-modules, indexed by I, such that $M = \mathrm{colim}\, M_i$. Let $f_i : M_i \to M$ be the canonical map. The following are equivalent:*

1. *For every finitely presented R-module P and module map $f : P \to M$, there exists a finitely presented R-module Q and a module map $g : P \to Q$ such that g and f dominate each other, i.e., $\mathrm{Ker}(f \otimes_R id_N) = \mathrm{Ker}(g \otimes_R id_N)$ for every R-module N.*
2. *For each $i \in I$, there exists $j \geq i$ such that $f_{ij} : M_i \to M_j$ dominates $f_i : M_i \to M$.*
3. *For each $i \in I$, there exists $j \geq i$ such that $f_{ij} : M_i \to M_j$ factors through $f_{ik} : M_i \to M_k$ for all $k \geq i$.*
4. *For every R-module N, the inverse system $(\mathrm{Hom}_R(M_i, N), \mathrm{Hom}_R(f_{ij}, N))$ is Mittag-Leffler.*
5. *For $N = \prod_{s \in I} M_s$, the inverse system $(\mathrm{Hom}_R(M_i, N), \mathrm{Hom}_R(f_{ij}, N))$ is Mittag-Leffler.*

Proof. First we prove the equivalence of (1) and (2). Suppose (1) holds and let $i \in I$. Corresponding to the map $f_i : M_i \to M$, we can choose $g : M_i \to Q$ as in (1). Since M_i and Q are of finite presentation, so is $\mathrm{Coker}(g)$. Then by Lemma 88.5, $f_i : M_i \to M$ factors through $g : M_i \to Q$, say $f_i = h \circ g$ for some $h : Q \to M$. Then since Q is finitely presented, h factors through $M_j \to M$ for some $j \geq i$, say

$h = f_j \circ h'$ for some $h' : Q \to M_j$. In total we have a commutative diagram

$$\begin{tikzcd} & M & \\ M_i \arrow[ur, "f_i"] \arrow[rr, "f_{ij}"] \arrow[dr, "g"'] & & M_j \arrow[ul, "f_j"'] \\ & Q \arrow[ur, "h'"'] & \end{tikzcd}$$

Thus f_{ij} dominates g. But g dominates f_i, so f_{ij} dominates f_i.

Conversely, suppose (2) holds. Let P be of finite presentation and $f : P \to M$ a module map. Then f factors through $f_i : M_i \to M$ for some $i \in I$, say $f = f_i \circ g'$ for some $g' : P \to M_i$. Choose by (2) a $j \geq i$ such that f_{ij} dominates f_i. We have a commutative diagram

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "g'"'] & M \\ M_i \arrow[ur, "f_i"] \arrow[r, "f_{ij}"'] & M_j \arrow[u, "f_j"'] \end{tikzcd}$$

From the diagram and the fact that f_{ij} dominates f_i, we find that f and $f_{ij} \circ g'$ dominate each other. Hence taking $g = f_{ij} \circ g' : P \to M_j$ works.

Next we prove (2) is equivalent to (3). Let $i \in I$. It is always true that f_i dominates f_{ik} for $k \geq i$, since f_i factors through f_{ik}. If (2) holds, choose $j \geq i$ such that f_{ij} dominates f_i. Then since domination is a transitive relation, f_{ij} dominates f_{ik} for $k \geq i$. All M_i are of finite presentation, so $\text{Coker}(f_{ik})$ is of finite presentation for $k \geq i$. By Lemma 88.5, f_{ij} factors through f_{ik} for all $k \geq i$. Thus (2) implies (3). On the other hand, if (3) holds then for any R-module N, $f_{ij} \otimes_R \text{id}_N$ factors through $f_{ik} \otimes_R \text{id}_N$ for $k \geq i$. So $\text{Ker}(f_{ik} \otimes_R \text{id}_N) \subset \text{Ker}(f_{ij} \otimes_R \text{id}_N)$ for $k \geq i$. But $\text{Ker}(f_i \otimes_R \text{id}_N : M_i \otimes_R N \to M \otimes_R N)$ is the union of $\text{Ker}(f_{ik} \otimes_R \text{id}_N)$ for $k \geq i$. Thus $\text{Ker}(f_i \otimes_R \text{id}_N) \subset \text{Ker}(f_{ij} \otimes_R \text{id}_N)$ for any R-module N, which by definition means f_{ij} dominates f_i.

It is trivial that (3) implies (4) implies (5). We show (5) implies (3). Let $N = \prod_{s \in I} M_s$. If (5) holds, then given $i \in I$ choose $j \geq i$ such that

$$\text{Im}(\text{Hom}(M_j, N) \to \text{Hom}(M_i, N)) = \text{Im}(\text{Hom}(M_k, N) \to \text{Hom}(M_i, N))$$

for all $k \geq j$. Passing the product over $s \in I$ outside of the Hom's and looking at the maps on each component of the product, this says

$$\text{Im}(\text{Hom}(M_j, M_s) \to \text{Hom}(M_i, M_s)) = \text{Im}(\text{Hom}(M_k, M_s) \to \text{Hom}(M_i, M_s))$$

for all $k \geq j$ and $s \in I$. Taking $s = j$ we have

$$\text{Im}(\text{Hom}(M_j, M_j) \to \text{Hom}(M_i, M_j)) = \text{Im}(\text{Hom}(M_k, M_j) \to \text{Hom}(M_i, M_j))$$

for all $k \geq j$. Since f_{ij} is the image of $\text{id} \in \text{Hom}(M_j, M_j)$ under $\text{Hom}(M_j, M_j) \to \text{Hom}(M_i, M_j)$, this shows that for any $k \geq j$ there is $h \in \text{Hom}(M_k, M_j)$ such that $f_{ij} = h \circ f_{ik}$. If $j \geq k$ then we can take $h = f_{kj}$. Hence (3) holds. $\square$

059F **Definition 88.7.** Let M be an R-module. We say that M is *Mittag-Leffler* if the equivalent conditions of Proposition 88.6 hold.

In particular a finitely presented module is Mittag-Leffler.

059G **Remark 88.8.** Let M be a flat R-module. By Lazard's theorem (Theorem 81.4) we can write $M = \operatorname{colim} M_i$ as the colimit of a directed system (M_i, f_{ij}) where the M_i are free finite R-modules. For M to be Mittag-Leffler, it is enough for the inverse system of duals $(\mathrm{Hom}_R(M_i, R), \mathrm{Hom}_R(f_{ij}, R))$ to be Mittag-Leffler. This follows from criterion (4) of Proposition 88.6 and the fact that for a free finite R-module F, there is a functorial isomorphism $\mathrm{Hom}_R(F, R) \otimes_R N \cong \mathrm{Hom}_R(F, N)$ for any R-module N.

05CN **Lemma 88.9.** *If R is a ring and M, N are Mittag-Leffler modules over R, then $M \otimes_R N$ is a Mittag-Leffler module.*

Proof. Write $M = \operatorname{colim}_{i \in I} M_i$ and $N = \operatorname{colim}_{j \in J} N_j$ as directed colimits of finitely presented R-modules. Denote $f_{ii'} : M_i \to M_{i'}$ and $g_{jj'} : N_j \to N_{j'}$ the transition maps. Then $M_i \otimes_R N_j$ is a finitely presented R-module (see Lemma 12.14), and $M \otimes_R N = \operatorname{colim}_{(i,j) \in I \times J} M_i \otimes_R M_j$. Pick $(i, j) \in I \times J$. By the definition of a Mittag-Leffler module we have Proposition 88.6 (3) for both systems. In other words there exist $i' \geq i$ and $j' \geq j$ such that for every choice of $i'' \geq i$ and $j'' \geq j$ there exist maps $a : M_{i''} \to M_{i'}$ and $b : M_{j''} \to M_{j'}$ such that $f_{ii'} = a \circ f_{ii''}$ and $g_{jj'} = b \circ g_{jj''}$. Then it is clear that $a \otimes b : M_{i''} \otimes_R N_{j''} \to M_{i'} \otimes_R N_{j'}$ serves the same purpose for the system $(M_i \otimes_R N_j, f_{ii'} \otimes g_{jj'})$. Thus by the characterization Proposition 88.6 (3) we conclude that $M \otimes_R N$ is Mittag-Leffler. $\square$

05CP **Lemma 88.10.** *Let R be a ring and M an R-module. Then M is Mittag-Leffler if and only if for every finite free R-module F and module map $f : F \to M$, there exists a finitely presented R-module Q and a module map $g : F \to Q$ such that g and f dominate each other, i.e.,* $\mathrm{Ker}(f \otimes_R id_N) = \mathrm{Ker}(g \otimes_R id_N)$ *for every R-module N.*

Proof. Since the condition is clear weaker than condition (1) of Proposition 88.6 we see that a Mittag-Leffler module satisfies the condition. Conversely, suppose that M satisfies the condition and that $f : P \to M$ is an R-module map from a finitely presented R-module P into M. Choose a surjection $F \to P$ where F is a finite free R-module. By assumption we can find a map $F \to Q$ where Q is a finitely presented R-module such that $F \to Q$ and $F \to M$ dominate each other. In particular, the kernel of $F \to Q$ contains the kernel of $F \to P$, hence we obtain an R-module map $g : P \to Q$ such that $F \to Q$ is equal to the composition $F \to P \to Q$. Let N be any R-module and consider the commutative diagram

$$
\begin{tikzcd}
F \otimes_R N \arrow[r] \arrow[d] & Q \otimes_R N \\
P \otimes_R N \arrow[r] \arrow[ur] & M \otimes_R N
\end{tikzcd}
$$

By assumption the kernels of $F \otimes_R N \to Q \otimes_R N$ and $F \otimes_R N \to M \otimes_R N$ are equal. Hence, as $F \otimes_R N \to P \otimes_R N$ is surjective, also the kernels of $P \otimes_R N \to Q \otimes_R N$ and $P \otimes_R N \to M \otimes_R N$ are equal. $\square$

05CQ **Lemma 88.11.** *Let $R \to S$ be a finite and finitely presented ring map. Let M be an S-module. If M is a Mittag-Leffler module over S then M is a Mittag-Leffler module over R.*

Proof. Assume M is a Mittag-Leffler module over S. Write $M = \operatorname{colim} M_i$ as a directed colimit of finitely presented S-modules M_i. As M is Mittag-Leffler over S there exists for each i an index $j \geq i$ such that for all $k \geq j$ there is a factorization $f_{ij} = h \circ f_{ik}$ (where h depends on i, the choice of j and k). Note that by Lemma 36.23 the modules M_i are also finitely presented as R-modules. Moreover, all the maps f_{ij}, f_{ik}, h are maps of R-modules. Thus we see that the system (M_i, f_{ij}) satisfies the same condition when viewed as a system of R-modules. Thus M is Mittag-Leffler as an R-module. □

05CR **Lemma 88.12.** *Let R be a ring. Let $S = R/I$ for some finitely generated ideal I. Let M be an S-module. Then M is a Mittag-Leffler module over R if and only if M is a Mittag-Leffler module over S.*

Proof. One implication follows from Lemma 88.11. To prove the other, assume M is Mittag-Leffler as an R-module. Write $M = \operatorname{colim} M_i$ as a directed colimit of finitely presented S-modules. As I is finitely generated, the ring S is finite and finitely presented as an R-algebra, hence the modules M_i are finitely presented as R-modules, see Lemma 36.23. Next, let N be any S-module. Note that for each i we have $\operatorname{Hom}_R(M_i, N) = \operatorname{Hom}_S(M_i, N)$ as $R \to S$ is surjective. Hence the condition that the inverse system $(\operatorname{Hom}_R(M_i, N))_i$ satisfies Mittag-Leffler, implies that the system $(\operatorname{Hom}_S(M_i, N))_i$ satisfies Mittag-Leffler. Thus M is Mittag-Leffler over S by definition. □

05CS **Remark 88.13.** Let $R \to S$ be a finite and finitely presented ring map. Let M be an S-module which is Mittag-Leffler as an R-module. Then it is in general not the case that M is Mittag-Leffler as an S-module. For example suppose that S is the ring of dual numbers over R, i.e., $S = R \oplus R\epsilon$ with $\epsilon^2 = 0$. Then an S-module consists of an R-module M endowed with a square zero R-linear endomorphism $\epsilon : M \to M$. Now suppose that M_0 is an R-module which is not Mittag-Leffler. Choose a presentation $F_1 \xrightarrow{u} F_0 \to M_0 \to 0$ with F_1 and F_0 free R-modules. Set $M = F_1 \oplus F_0$ with

$$\epsilon = \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix} : M \longrightarrow M.$$

Then $M/\epsilon M \cong F_1 \oplus M_0$ is not Mittag-Leffler over $R = S/\epsilon S$, hence not Mittag-Leffler over S (see Lemma 88.12). On the other hand, $M/\epsilon M = M \otimes_S S/\epsilon S$ which would be Mittag-Leffler over S if M was, see Lemma 88.9.

89. Interchanging direct products with tensor

059H Let M be an R-module and let $(Q_\alpha)_{\alpha \in A}$ be a family of R-modules. Then there is a canonical map $M \otimes_R \left(\prod_{\alpha \in A} Q_\alpha\right) \to \prod_{\alpha \in A}(M \otimes_R Q_\alpha)$ given on pure tensors by $x \otimes (q_\alpha) \mapsto (x \otimes q_\alpha)$. This map is not necessarily injective or surjective, as the following example shows.

059I **Example 89.1.** Take $R = \mathbf{Z}$, $M = \mathbf{Q}$, and consider the family $Q_n = \mathbf{Z}/n$ for $n \geq 1$. Then $\prod_n (M \otimes Q_n) = 0$. However there is an injection $\mathbf{Q} \to M \otimes (\prod_n Q_n)$ obtained by tensoring the injection $\mathbf{Z} \to \prod_n Q_n$ by M, so $M \otimes (\prod_n Q_n)$ is nonzero. Thus $M \otimes (\prod_n Q_n) \to \prod_n (M \otimes Q_n)$ is not injective.

On the other hand, take again $R = \mathbf{Z}$, $M = \mathbf{Q}$, and let $Q_n = \mathbf{Z}$ for $n \geq 1$. The image of $M \otimes (\prod_n Q_n) \to \prod_n (M \otimes Q_n) = \prod_n M$ consists precisely of sequences of the form $(a_n/m)_{n \geq 1}$ with $a_n \in \mathbf{Z}$ and m some nonzero integer. Hence the map is not surjective.

We determine below the precise conditions needed on M for the map $M \otimes_R (\prod_\alpha Q_\alpha) \to \prod_\alpha (M \otimes_R Q_\alpha)$ to be surjective, bijective, or injective for all choices of $(Q_\alpha)_{\alpha \in A}$. This is relevant because the modules for which it is injective turn out to be exactly Mittag-Leffler modules (Proposition 89.5). In what follows, if M is an R-module and A a set, we write M^A for the product $\prod_{\alpha \in A} M$.

059J **Proposition 89.2.** *Let M be an R-module. The following are equivalent:*

1. *M is finitely generated.*
2. *For every family $(Q_\alpha)_{\alpha \in A}$ of R-modules, the canonical map $M \otimes_R (\prod_\alpha Q_\alpha) \to \prod_\alpha (M \otimes_R Q_\alpha)$ is surjective.*
3. *For every R-module Q and every set A, the canonical map $M \otimes_R Q^A \to (M \otimes_R Q)^A$ is surjective.*
4. *For every set A, the canonical map $M \otimes_R R^A \to M^A$ is surjective.*

Proof. First we prove (1) implies (2). Choose a surjection $R^n \to M$ and consider the commutative diagram

$$\begin{tikzcd} R^n \otimes_R (\prod_\alpha Q_\alpha) \arrow[r, "\cong"] \arrow[d] & \prod_\alpha (R^n \otimes_R Q_\alpha) \arrow[d] \\ M \otimes_R (\prod_\alpha Q_\alpha) \arrow[r] & \prod_\alpha (M \otimes_R Q_\alpha). \end{tikzcd}$$

The top arrow is an isomorphism and the vertical arrows are surjections. We conclude that the bottom arrow is a surjection.

Obviously (2) implies (3) implies (4), so it remains to prove (4) implies (1). In fact for (1) to hold it suffices that the element $d = (x)_{x \in M}$ of M^M is in the image of the map $f : M \otimes_R R^M \to M^M$. In this case $d = \sum_{i=1}^n f(x_i \otimes a_i)$ for some $x_i \in M$ and $a_i \in R^M$. If for $x \in M$ we write $p_x : M^M \to M$ for the projection onto the x-th factor, then

$$x = p_x(d) = \sum\nolimits_{i=1}^n p_x(f(x_i \otimes a_i)) = \sum\nolimits_{i=1}^n p_x(a_i) x_i.$$

Thus $x_1, \ldots, x_n$ generate M. $\square$

059K **Proposition 89.3.** *Let M be an R-module. The following are equivalent:*

1. *M is finitely presented.*
2. *For every family $(Q_\alpha)_{\alpha \in A}$ of R-modules, the canonical map $M \otimes_R (\prod_\alpha Q_\alpha) \to \prod_\alpha (M \otimes_R Q_\alpha)$ is bijective.*
3. *For every R-module Q and every set A, the canonical map $M \otimes_R Q^A \to (M \otimes_R Q)^A$ is bijective.*
4. *For every set A, the canonical map $M \otimes_R R^A \to M^A$ is bijective.*

Proof. First we prove (1) implies (2). Choose a presentation $R^m \to R^n \to M$ and consider the commutative diagram

$$\begin{tikzcd} R^m \otimes_R (\prod_\alpha Q_\alpha) \arrow[r] \arrow[d, "\cong"] & R^m \otimes_R (\prod_\alpha Q_\alpha) \arrow[r] \arrow[d, "\cong"] & M \otimes_R (\prod_\alpha Q_\alpha) \arrow[r] \arrow[d] & 0 \\ \prod_\alpha (R^m \otimes_R Q_\alpha) \arrow[r] & \prod_\alpha (R^n \otimes_R Q_\alpha) \arrow[r] & \prod_\alpha (M \otimes_R Q_\alpha) \arrow[r] & 0. \end{tikzcd}$$

The first two vertical arrows are isomorphisms and the rows are exact. This implies that the map $M \otimes_R (\prod_\alpha Q_\alpha) \to \prod_\alpha (M \otimes_R Q_\alpha)$ is surjective and, by a diagram chase, also injective. Hence (2) holds.

Obviously (2) implies (3) implies (4), so it remains to prove (4) implies (1). From Proposition 89.2, if (4) holds we already know that M is finitely generated. So we can choose a surjection $F \to M$ where F is free and finite. Let K be the kernel. We must show K is finitely generated. For any set A, we have a commutative diagram

$$\begin{tikzcd} & K \otimes_R R^A \arrow[r] \arrow[d, "f_3"'] & F \otimes_R R^A \arrow[r] \arrow[d, "f_2"', "\cong"] & M \otimes_R R^A \arrow[r] \arrow[d, "f_1"', "\cong"] & 0 \\ 0 \arrow[r] & K^A \arrow[r] & F^A \arrow[r] & M^A \arrow[r] & 0. \end{tikzcd}$$

The map f_1 is an isomorphism by assumption, the map f_2 is a isomorphism since F is free and finite, and the rows are exact. A diagram chase shows that f_3 is surjective, hence by Proposition 89.2 we get that K is finitely generated. □

We need the following lemma for the next proposition.

059L **Lemma 89.4.** *Let M be an R-module, P a finitely presented R-module, and $f : P \to M$ a map. Let Q be an R-module and suppose $x \in \mathrm{Ker}(P \otimes Q \to M \otimes Q)$. Then there exists a finitely presented R-module P' and a map $f' : P \to P'$ such that f factors through f' and $x \in \mathrm{Ker}(P \otimes Q \to P' \otimes Q)$.*

Proof. Write M as a colimit $M = \mathrm{colim}_{i \in I} M_i$ of a directed system of finitely presented modules M_i. Since P is finitely presented, the map $f : P \to M$ factors through $M_j \to M$ for some $j \in I$. Upon tensoring by Q we have a commutative diagram

$$\begin{tikzcd} & M_j \otimes Q \arrow[dr] & \\ P \otimes Q \arrow[ur] \arrow[rr] & & M \otimes Q. \end{tikzcd}$$

The image y of x in $M_j \otimes Q$ is in the kernel of $M_j \otimes Q \to M \otimes Q$. Since $M \otimes Q = \mathrm{colim}_{i \in I}(M_i \otimes Q)$, this means y maps to 0 in $M_{j'} \otimes Q$ for some $j' \geq j$. Thus we may take $P' = M_{j'}$ and f' to be the composite $P \to M_j \to M_{j'}$. □

059M **Proposition 89.5.** *Let M be an R-module. The following are equivalent:*

1. *M is Mittag-Leffler.*
2. *For every family $(Q_\alpha)_{\alpha \in A}$ of R-modules, the canonical map $M \otimes_R (\prod_\alpha Q_\alpha) \to \prod_\alpha (M \otimes_R Q_\alpha)$ is injective.*

Proof. First we prove (1) implies (2). Suppose M is Mittag-Leffler and let x be in the kernel of $M \otimes_R (\prod_\alpha Q_\alpha) \to \prod_\alpha (M \otimes_R Q_\alpha)$. Write M as a colimit $M = \mathrm{colim}_{i \in I} M_i$ of a directed system of finitely presented modules M_i. Then $M \otimes_R (\prod_\alpha Q_\alpha)$ is the colimit of $M_i \otimes_R (\prod_\alpha Q_\alpha)$. So x is the image of an element $x_i \in M_i \otimes_R (\prod_\alpha Q_\alpha)$. We must show that x_i maps to 0 in $M_j \otimes_R (\prod_\alpha Q_\alpha)$ for some $j \geq i$. Since M is Mittag-Leffler, we may choose $j \geq i$ such that $M_i \to M_j$ and

$M_i \to M$ dominate each other. Then consider the commutative diagram

$$\begin{tikzcd} M \otimes_R (\prod_\alpha Q_\alpha) \arrow[r] & \prod_\alpha (M \otimes_R Q_\alpha) \\ M_i \otimes_R (\prod_\alpha Q_\alpha) \arrow[r, "\cong"] \arrow[u] \arrow[d] & \prod_\alpha (M_i \otimes_R Q_\alpha) \arrow[u] \arrow[d] \\ M_j \otimes_R (\prod_\alpha Q_\alpha) \arrow[r, "\cong"] & \prod_\alpha (M_j \otimes_R Q_\alpha) \end{tikzcd}$$

whose bottom two horizontal maps are isomorphisms, according to Proposition 89.3. Since x_i maps to 0 in $\prod_\alpha (M \otimes_R Q_\alpha)$, its image in $\prod_\alpha (M_i \otimes_R Q_\alpha)$ is in the kernel of the map $\prod_\alpha (M_i \otimes_R Q_\alpha) \to \prod_\alpha (M \otimes_R Q_\alpha)$. But this kernel equals the kernel of $\prod_\alpha (M_i \otimes_R Q_\alpha) \to \prod_\alpha (M_j \otimes_R Q_\alpha)$ according to the choice of j. Thus x_i maps to 0 in $\prod_\alpha (M_j \otimes_R Q_\alpha)$ and hence to 0 in $M_j \otimes_R (\prod_\alpha Q_\alpha)$.

Now suppose (2) holds. We prove M satisfies formulation (1) of being Mittag-Leffler from Proposition 88.6. Let $f : P \to M$ be a map from a finitely presented module P to M. Choose a set B of representatives of the isomorphism classes of finitely presented R-modules. Let A be the set of pairs (Q, x) where $Q \in B$ and $x \in \mathrm{Ker}(P \otimes Q \to M \otimes Q)$. For $\alpha = (Q, x) \in A$, we write Q_α for Q and x_α for x. Consider the commutative diagram

$$\begin{tikzcd} M \otimes_R (\prod_\alpha Q_\alpha) \arrow[r] & \prod_\alpha (M \otimes_R Q_\alpha) \\ P \otimes_R (\prod_\alpha Q_\alpha) \arrow[r, "\cong"] \arrow[u] & \prod_\alpha (P \otimes_R Q_\alpha) \arrow[u] \end{tikzcd}$$

The top arrow is an injection by assumption, and the bottom arrow is an isomorphism by Proposition 89.3. Let $x \in P \otimes_R (\prod_\alpha Q_\alpha)$ be the element corresponding to $(x_\alpha) \in \prod_\alpha (P \otimes_R Q_\alpha)$ under this isomorphism. Then $x \in \mathrm{Ker}(P \otimes_R (\prod_\alpha Q_\alpha) \to M \otimes_R (\prod_\alpha Q_\alpha))$ since the top arrow in the diagram is injective. By Lemma 89.4, we get a finitely presented module P' and a map $f' : P \to P'$ such that $f : P \to M$ factors through f' and $x \in \mathrm{Ker}(P \otimes_R (\prod_\alpha Q_\alpha) \to P' \otimes_R (\prod_\alpha Q_\alpha))$. We have a commutative diagram

$$\begin{tikzcd} P' \otimes_R (\prod_\alpha Q_\alpha) \arrow[r, "\cong"] & \prod_\alpha (P' \otimes_R Q_\alpha) \\ P \otimes_R (\prod_\alpha Q_\alpha) \arrow[r, "\cong"] \arrow[u] & \prod_\alpha (P \otimes_R Q_\alpha) \arrow[u] \end{tikzcd}$$

where both the top and bottom arrows are isomorphisms by Proposition 89.3. Thus since x is in the kernel of the left vertical map, (x_α) is in the kernel of the right vertical map. This means $x_\alpha \in \mathrm{Ker}(P \otimes_R Q_\alpha \to P' \otimes_R Q_\alpha)$ for every $\alpha \in A$. By the definition of A this means $\mathrm{Ker}(P \otimes_R Q \to P' \otimes_R Q) \supset \mathrm{Ker}(P \otimes_R Q \to M \otimes_R Q)$ for all finitely presented Q and, since $f : P \to M$ factors through $f' : P \to P'$, actually equality holds. By Lemma 88.3, f and f' dominate each other. $\square$

0AS6 **Lemma 89.6.** *Let M be a flat Mittag-Leffler module over R. Let F be an R-module and let $x \in F \otimes_R M$. Then there exists a smallest submodule $F' \subset F$ such that $x \in F' \otimes_R M$.*

Proof. Since M is flat we have $F' \otimes_R M \subset F \otimes_R M$ if $F' \subset F$ is a submodule, hence the statement makes sense. Let $I = \{F' \subset F \mid x \in F' \otimes_R M\}$ and for $i \in I$ denote $F_i \subset F$ the corresponding submodule. Then x maps to zero under the map

$$F \otimes_R M \longrightarrow \prod (F/F_i \otimes_R M)$$

whence by Proposition 89.5 x maps to zero under the map

$$F \otimes_R M \longrightarrow \left(\prod F/F_i\right) \otimes_R M$$

Since M is flat the kernel of this arrow is $(\bigcap F_i) \otimes_R M$ which proves the lemma. $\square$

059N **Lemma 89.7.** *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a universally exact sequence of R-modules. Then:*

1. *If M_2 is Mittag-Leffler, then M_1 is Mittag-Leffler.*
2. *If M_1 and M_3 are Mittag-Leffler, then M_2 is Mittag-Leffler.*

Proof. For any family $(Q_\alpha)_{\alpha \in A}$ of R-modules we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 \otimes_R (\prod_\alpha Q_\alpha) & \longrightarrow & M_2 \otimes_R (\prod_\alpha Q_\alpha) & \longrightarrow & M_3 \otimes_R (\prod_\alpha Q_\alpha) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \prod_\alpha (M_1 \otimes Q_\alpha) & \longrightarrow & \prod_\alpha (M_2 \otimes Q_\alpha) & \longrightarrow & \prod_\alpha (M_3 \otimes Q_\alpha) & \longrightarrow & 0 \end{array}$$

with exact rows. Thus (1) and (2) follow from Proposition 89.5. $\square$

0EGI **Lemma 89.8.** *Let $M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of R-modules. If M_1 is finitely generated and M_2 is Mittag-Leffler, then M_3 is Mittag-Leffler.*

Proof. For any family $(Q_\alpha)_{\alpha \in A}$ of R-modules, since tensor product is right exact, we have a commutative diagram

$$\begin{array}{ccccccc} M_1 \otimes_R (\prod_\alpha Q_\alpha) & \longrightarrow & M_2 \otimes_R (\prod_\alpha Q_\alpha) & \longrightarrow & M_3 \otimes_R (\prod_\alpha Q_\alpha) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \prod_\alpha (M_1 \otimes Q_\alpha) & \longrightarrow & \prod_\alpha (M_2 \otimes Q_\alpha) & \longrightarrow & \prod_\alpha (M_3 \otimes Q_\alpha) & \longrightarrow & 0 \end{array}$$

with exact rows. By Proposition 89.2 the left vertical arrow is surjective. By Proposition 89.5 the middle vertical arrow is injective. A diagram chase shows the right vertical arrow is injective. Hence M_3 is Mittag-Leffler by Proposition 89.5. $\square$

0AS7 **Lemma 89.9.** *If $M = \mathrm{colim}\, M_i$ is the colimit of a directed system of Mittag-Leffler R-modules M_i with universally injective transition maps, then M is Mittag-Leffler.*

Proof. Let $(Q_\alpha)_{\alpha \in A}$ be a family of R-modules. We have to show that $M \otimes_R (\prod Q_\alpha) \to \prod M \otimes_R Q_\alpha$ is injective and we know that $M_i \otimes_R (\prod Q_\alpha) \to \prod M_i \otimes_R Q_\alpha$ is injective for each i, see Proposition 89.5. Since $\otimes$ commutes with filtered colimits, it suffices to show that $\prod M_i \otimes_R Q_\alpha \to \prod M \otimes_R Q_\alpha$ is injective. This is clear as each of the maps $M_i \otimes_R Q_\alpha \to M \otimes_R Q_\alpha$ is injective by our assumption that the transition maps are universally injective. $\square$

059P **Lemma 89.10.** *If $M = \bigoplus_{i \in I} M_i$ is a direct sum of R-modules, then M is Mittag-Leffler if and only if each M_i is Mittag-Leffler.*

Proof. The "only if" direction follows from Lemma 89.7 (1) and the fact that a split short exact sequence is universally exact. The converse follows from Lemma 89.9 but we can also argue it directly as follows. First note that if I is finite then this follows from Lemma 89.7 (2). For general I, if all M_i are Mittag-Leffler then we prove the same of M by verifying condition (1) of Proposition 88.6. Let $f : P \to M$ be a map from a finitely presented module P. Then f factors as $P \xrightarrow{f'} \bigoplus_{i' \in I'} M_{i'} \hookrightarrow \bigoplus_{i \in I} M_i$ for some finite subset I' of I. By the finite case $\bigoplus_{i' \in I'} M_{i'}$ is Mittag-Leffler and hence there exists a finitely presented module Q and a map $g : P \to Q$ such that g and f' dominate each other. Then also g and f dominate each other. $\square$

05CT **Lemma 89.11.** *Let $R \to S$ be a ring map. Let M be an S-module. If S is Mittag-Leffler as an R-module, and M is flat and Mittag-Leffler as an S-module, then M is Mittag-Leffler as an R-module.*

Proof. We deduce this from the characterization of Proposition 89.5. Namely, suppose that Q_α is a family of R-modules. Consider the composition

$$\begin{tikzcd} M \otimes_R \prod_\alpha Q_\alpha = M \otimes_S S \otimes_R \prod_\alpha Q_\alpha \arrow[d] \\ M \otimes_S \prod_\alpha (S \otimes_R Q_\alpha) \arrow[d] \\ \prod_\alpha (M \otimes_S S \otimes_R Q_\alpha) = \prod_\alpha (M \otimes_R Q_\alpha) \end{tikzcd}$$

The first arrow is injective as M is flat over S and S is Mittag-Leffler over R and the second arrow is injective as M is Mittag-Leffler over S. Hence M is Mittag-Leffler over R. $\square$

90. Coherent rings

05CU We use the discussion on interchanging $\prod$ and $\otimes$ to determine for which rings products of flat modules are flat. It turns out that these are the so-called coherent rings. You may be more familiar with the notion of a coherent $\mathcal{O}_X$-module on a ringed space, see Modules, Section 12.

05CV **Definition 90.1.** Let R be a ring. Let M be an R-module.

(1) We say M is a *coherent module* if it is finitely generated and every finitely generated submodule of M is finitely presented over R.
(2) We say R is a *coherent ring* if it is coherent as a module over itself.

Thus a ring is coherent if and only if every finitely generated ideal is finitely presented as a module.

0EWV **Example 90.2.** A valuation ring is a coherent ring. Namely, every nonzero finitely generated ideal is principal (Lemma 50.15), hence free as a valuation ring is a domain, hence finitely presented.

The category of coherent modules is abelian.

05CW **Lemma 90.3.** *Let R be a ring.*

(1) *A finite submodule of a coherent module is coherent.*
(2) *Let $\varphi : N \to M$ be a homomorphism from a finite module to a coherent module. Then $\mathrm{Ker}(\varphi)$ is finite.*
(3) *Let $\varphi : N \to M$ be a homomorphism of coherent modules. Then $\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$ are coherent modules.*
(4) *Given a short exact sequence of R-modules $0 \to M_1 \to M_2 \to M_3 \to 0$ if two out of three are coherent so is the third.*

Proof. The first statement is immediate from the definition. During the rest of the proof we will use the results of Lemma 5.3 without further mention.

Let $\varphi : N \to M$ satisfy the assumptions of (2). Suppose that N is generated by $x_1, \ldots, x_n$. By Definition 90.1 the kernel K of the induced map $R^{\oplus n} \to M$, $e_i \mapsto \varphi(x_i)$ is of finite type. Hence $\mathrm{Ker}(\varphi)$ which is the image of the composition $K \to R^{\oplus n} \to N$ is of finite type. This proves (2).

Let $\varphi : N \to M$ satisfy the assumptions of (3). By (2) the kernel of φ is of finite type and hence by (1) it is coherent.

With the same hypotheses let us show that $\mathrm{Coker}(\varphi)$ is coherent. Since M is finite so is $\mathrm{Coker}(\varphi)$. Let $\overline{x}_i \in \mathrm{Coker}(\varphi)$. We have to show that the kernel of the associated morphism $\overline{\Psi} : R^{\oplus n} \to \mathrm{Coker}(\varphi)$ is finite. Choose $x_i \in M$ lifting $\overline{x}_i$. Choose additionally generators $y_1, \ldots, y_m$ of $\mathrm{Im}(\varphi)$. Let $\Phi : R^{\oplus m} \to \mathrm{Im}(\varphi)$ using y_j and $\Psi : R^{\oplus m} \oplus R^{\oplus n} \to M$ using y_j and x_i be the corresponding maps. Consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & R^{\oplus m} \arrow[r] \arrow[d, "\Phi"'] & R^{\oplus m} \oplus R^{\oplus n} \arrow[r] \arrow[d, "\Psi"'] & R^{\oplus n} \arrow[r] \arrow[d, "\overline{\Psi}"'] & 0 \\
0 \arrow[r] & \mathrm{Im}(\varphi) \arrow[r] & M \arrow[r] & \mathrm{Coker}(\varphi) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. By Lemma 4.1 we get an exact sequence $\mathrm{Ker}(\Psi) \to \mathrm{Ker}(\overline{\Psi}) \to 0$. Since $\mathrm{Ker}(\Psi)$ is a finite R-module, we see that $\mathrm{Ker}(\overline{\Psi})$ is finite.

Statement (4) follows from (3).

Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of R-modules. It suffices to prove that if M_1 and M_3 are coherent so is M_2. By Lemma 5.3 we see that M_2 is finite. Let $x_1, \ldots, x_n$ be finitely many elements of M_2. We have to show that the module of relations K between them is finite. Consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \bigoplus_{i=1}^n R \arrow[r] \arrow[d] & \bigoplus_{i=1}^n R \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_1 \arrow[r] & M_2 \arrow[r] & M_3 \arrow[r] & 0
\end{tikzcd}$$

with obvious notation. By the snake lemma we get an exact sequence $0 \to K \to K_3 \to M_1$ where K_3 is the module of relations among the images of the x_i in M_3. Since M_3 is coherent we see that K_3 is a finite module. Since M_1 is coherent we see that the image I of $K_3 \to M_1$ is coherent. Hence K is the kernel of the map $K_3 \to I$ between a finite module and a coherent module and hence finite by (2). ☐

05CX **Lemma 90.4.** *Let R be a ring. If R is coherent, then a module is coherent if and only if it is finitely presented.*

Proof. It is clear that a coherent module is finitely presented (over any ring). Conversely, if R is coherent, then $R^{\oplus n}$ is coherent and so is the cokernel of any map $R^{\oplus m} \to R^{\oplus n}$, see Lemma 90.3. ☐

05CY **Lemma 90.5.** *A Noetherian ring is a coherent ring.*

Proof. By Lemma 31.4 any finite R-module is finitely presented. In particular any ideal of R is finitely presented. ☐

05CZ **Proposition 90.6.** *Let R be a ring. The following are equivalent* This is [Cha60, Theorem 2.1].

1. *R is coherent,*
2. *any product of flat R-modules is flat, and*
3. *for every set A the module R^A is flat.*

Proof. Assume R coherent, and let Q_α, $\alpha \in A$ be a set of flat R-modules. We have to show that $I \otimes_R \prod_\alpha Q_\alpha \to \prod Q_\alpha$ is injective for every finitely generated ideal I of R, see Lemma 39.5. Since R is coherent I is an R-module of finite presentation. Hence $I \otimes_R \prod_\alpha Q_\alpha = \prod I \otimes_R Q_\alpha$ by Proposition 89.3. The desired injectivity follows as $I \otimes_R Q_\alpha \to Q_\alpha$ is injective by flatness of Q_α.

The implication (2) $\Rightarrow$ (3) is trivial.

Assume that the R-module R^A is flat for every set A. Let I be a finitely generated ideal in R. Then $I \otimes_R R^A \to R^A$ is injective by assumption. By Proposition 89.2 and the finiteness of I the image is equal to I^A. Hence $I \otimes_R R^A = I^A$ for every set A and we conclude that I is finitely presented by Proposition 89.3. ☐

91. Examples and non-examples of Mittag-Leffler modules

059Q We end this section with some examples and non-examples of Mittag-Leffler modules.

059R **Example 91.1.** Mittag-Leffler modules.

1. Any finitely presented module is Mittag-Leffler. This follows, for instance, from Proposition 88.6 (1). In general, it is true that a finitely generated module is Mittag-Leffler if and only it is finitely presented. This follows from Propositions 89.2, 89.3, and 89.5.
2. A free module is Mittag-Leffler since it satisfies condition (1) of Proposition 88.6.
3. By the previous example together with Lemma 89.10, projective modules are Mittag-Leffler.

We also want to add to our list of examples power series rings over a Noetherian ring R. This will be a consequence the following lemma.

059S **Lemma 91.2.** *Let M be a flat R-module. The following are equivalent*

1. *M is Mittag-Leffler, and*
2. *if F is a finite free R-module and $x \in F \otimes_R M$, then there exists a smallest submodule F' of F such that $x \in F' \otimes_R M$.*

Proof. The implication (1) $\Rightarrow$ (2) is a special case of Lemma 89.6. Assume (2). By Theorem 81.4 we can write M as the colimit $M = \mathrm{colim}_{i \in I} M_i$ of a directed system (M_i, f_{ij}) of finite free R-modules. By Remark 88.8, it suffices to show that the inverse system $(\mathrm{Hom}_R(M_i, R), \mathrm{Hom}_R(f_{ij}, R))$ is Mittag-Leffler. In other words, fix $i \in I$ and for $j \geq i$ let Q_j be the image of $\mathrm{Hom}_R(M_j, R) \to \mathrm{Hom}_R(M_i, R)$; we must show that the Q_j stabilize.

Since M_i is free and finite, we can make the identification $\mathrm{Hom}_R(M_i, M_j) = \mathrm{Hom}_R(M_i, R) \otimes_R M_j$ for all j. Using the fact that the M_j are free, it follows that for $j \geq i$, Q_j is the smallest submodule of $\mathrm{Hom}_R(M_i, R)$ such that $f_{ij} \in Q_j \otimes_R M_j$. Under the identification $\mathrm{Hom}_R(M_i, M) = \mathrm{Hom}_R(M_i, R) \otimes_R M$, the canonical map $f_i : M_i \to M$ is in $\mathrm{Hom}_R(M_i, R) \otimes_R M$. By the assumption on M, there exists a smallest submodule Q of $\mathrm{Hom}_R(M_i, R)$ such that $f_i \in Q \otimes_R M$. We are going to show that the Q_j stabilize to Q.

For $j \geq i$ we have a commutative diagram

$$\begin{tikzcd} Q_j \otimes_R M_j \arrow[r] \arrow[d] & \mathrm{Hom}_R(M_i, R) \otimes_R M_j \arrow[d] \\ Q_j \otimes_R M \arrow[r] & \mathrm{Hom}_R(M_i, R) \otimes_R M. \end{tikzcd}$$

Since $f_{ij} \in Q_j \otimes_R M_j$ maps to $f_i \in \mathrm{Hom}_R(M_i, R) \otimes_R M$, it follows that $f_i \in Q_j \otimes_R M$. Hence, by the choice of Q, we have $Q \subset Q_j$ for all $j \geq i$.

Since the Q_j are decreasing and $Q \subset Q_j$ for all $j \geq i$, to show that the Q_j stabilize to Q it suffices to find a $j \geq i$ such that $Q_j \subset Q$. As an element of

$$\mathrm{Hom}_R(M_i, R) \otimes_R M = \mathrm{colim}_{j \in J}(\mathrm{Hom}_R(M_i, R) \otimes_R M_j),$$

f_i is the colimit of f_{ij} for $j \geq i$, and f_i also lies in the submodule

$$\mathrm{colim}_{j \in J}(Q \otimes_R M_j) \subset \mathrm{colim}_{j \in J}(\mathrm{Hom}_R(M_i, R) \otimes_R M_j).$$

It follows that for some $j \geq i$, f_{ij} lies in $Q \otimes_R M_j$. Since Q_j is the smallest submodule of $\mathrm{Hom}_R(M_i, R)$ with $f_{ij} \in Q_j \otimes_R M_j$, we conclude $Q_j \subset Q$. □

05D0 **Lemma 91.3.** *Let R be a Noetherian ring and A a set. Then $M = R^A$ is a flat and Mittag-Leffler R-module.*

Proof. Combining Lemma 90.5 and Proposition 90.6 we see that M is flat over R. We show that M satisfies the condition of Lemma 91.2. Let F be a free finite R-module. If F' is any submodule of F then it is finitely presented since R is Noetherian. So by Proposition 89.3 we have a commutative diagram

$$\begin{tikzcd} F' \otimes_R M \arrow[r] \arrow[d, "\cong"] & F \otimes_R M \arrow[d, "\cong"] \\ (F')^A \arrow[r] & F^A \end{tikzcd}$$

by which we can identify the map $F' \otimes_R M \to F \otimes_R M$ with $(F')^A \to F^A$. Hence if $x \in F \otimes_R M$ corresponds to $(x_\alpha) \in F^A$, then the submodule of F' of F generated by the x_α is the smallest submodule of F such that $x \in F' \otimes_R M$. □

059T **Lemma 91.4.** *Let R be a Noetherian ring and n a positive integer. Then the R-module $M = R[[t_1, \ldots, t_n]]$ is flat and Mittag-Leffler.*

Proof. As an R-module, we have $M = R^A$ for a (countable) set A. Hence this lemma is a special case of Lemma 91.3. □

059U **Example 91.5.** Non Mittag-Leffler modules.

(1) By Example 89.1 and Proposition 89.5, $\mathbf{Q}$ is not a Mittag-Leffler $\mathbf{Z}$-module.
(2) We prove below (Theorem 93.3) that for a flat and countably generated module, projectivity is equivalent to being Mittag-Leffler. Thus any flat, countably generated, non-projective module M is an example of a non-Mittag-Leffler module. For such an example, see Remark 78.4.
(3) Let k be a field. Let $R = k[[x]]$. The R-module $M = \prod_{n \in \mathbf{N}} R/(x^n)$ is not Mittag-Leffler. Namely, consider the element $\xi = (\xi_1, \xi_2, \xi_3, \ldots)$ defined by $\xi_{2^m} = x^{2^{m-1}}$ and $\xi_n = 0$ else, so
$$\xi = (0, x, 0, x^2, 0, 0, 0, x^4, 0, 0, 0, 0, 0, 0, 0, x^8, \ldots)$$
Then the annihilator of ξ in $M/x^{2^m}M$ is generated $x^{2^{m-1}}$ for $m \gg 0$. But if M was Mittag-Leffler, then there would exist a finite R-module Q and an element $\xi' \in Q$ such that the annihilator of ξ' in Q/x^lQ agrees with the annihilator of ξ in M/x^lM for all $l \geq 1$, see Proposition 88.6 (1). Now you can prove there exists an integer $a \geq 0$ such that the annihilator of ξ' in Q/x^lQ is generated by either x^a or x^{l-a} for all $l \gg 0$ (depending on whether $\xi' \in Q$ is torsion or not). The combination of the above would give for all $l = 2^m >> 0$ the equality $a = l/2$ or $l - a = l/2$ which is nonsensical.
(4) The same argument shows that (x)-adic completion of $\bigoplus_{n \in \mathbf{N}} R/(x^n)$ is not Mittag-Leffler over $R = k[[x]]$ (hint: ξ is actually an element of this completion).
(5) Let $R = k[a, b]/(a^2, ab, b^2)$. Let S be the finitely presented R-algebra with presentation $S = R[t]/(at - b)$. Then as an R-module S is countably generated and indecomposable (details omitted). On the other hand, R is Artinian local, hence complete local, hence a henselian local ring, see Lemma 153.9. If S was Mittag-Leffler as an R-module, then it would be a direct sum of finite R-modules by Lemma 153.13. Thus we conclude that S is not Mittag-Leffler as an R-module.

92. Countably generated Mittag-Leffler modules

05D1 It turns out that countably generated Mittag-Leffler modules have a particularly simple structure.

059W **Lemma 92.1.** *Let M be an R-module. Write $M = \operatorname{colim}_{i \in I} M_i$ where (M_i, f_{ij}) is a directed system of finitely presented R-modules. If M is Mittag-Leffler and countably generated, then there is a directed countable subset $I' \subset I$ such that $M \cong \operatorname{colim}_{i \in I'} M_i$.*

Proof. Let $x_1, x_2, \ldots$ be a countable set of generators for M. For each x_n choose $i \in I$ such that x_n is in the image of the canonical map $f_i : M_i \to M$; let $I'_0 \subset I$ be the set of all these i. Now since M is Mittag-Leffler, for each $i \in I'_0$ we can choose $j \in I$ such that $j \geq i$ and $f_{ij} : M_i \to M_j$ factors through $f_{ik} : M_i \to M_k$ for all $k \geq i$ (condition (3) of Proposition 88.6); let I'_1 be the union of I'_0 with all of these j. Since I'_1 is a countable, we can enlarge it to a countable directed set $I'_2 \subset I$. Now we can apply the same procedure to I'_2 as we did to I'_0 to get a new countable set $I'_3 \subset I$. Then we enlarge I'_3 to a countable directed set I'_4. Continuing in this way—adding in a j as in Proposition 88.6 (3) for each $i \in I'_\ell$ if ℓ is odd and enlarging I'_ℓ to a directed set if ℓ is even—we get a sequence of subsets $I'_\ell \subset I$ for $\ell \geq 0$. The union $I' = \bigcup I'_\ell$ satisfies:

(1) I' is countable and directed;
(2) each x_n is in the image of $f_i : M_i \to M$ for some $i \in I'$;
(3) if $i \in I'$, then there is $j \in I'$ such that $j \geq i$ and $f_{ij} : M_i \to M_j$ factors through $f_{ik} : M_i \to M_k$ for all $k \in I$ with $k \geq i$. In particular $\text{Ker}(f_{ik}) \subset \text{Ker}(f_{ij})$ for $k \geq i$.

We claim that the canonical map $\text{colim}_{i \in I'} M_i \to \text{colim}_{i \in I} M_i = M$ is an isomorphism. By (2) it is surjective. For injectivity, suppose $x \in \text{colim}_{i \in I'} M_i$ maps to 0 in $\text{colim}_{i \in I} M_i$. Representing x by an element $\tilde{x} \in M_i$ for some $i \in I'$, this means that $f_{ik}(\tilde{x}) = 0$ for some $k \in I, k \geq i$. But then by (3) there is $j \in I', j \geq i$, such that $f_{ij}(\tilde{x}) = 0$. Hence $x = 0$ in $\text{colim}_{i \in I'} M_i$. $\square$

Lemma 92.1 implies that a countably generated Mittag-Leffler module M over R is the colimit of a system

$$M_1 \to M_2 \to M_3 \to M_4 \to \ldots$$

with each M_n a finitely presented R-module. To see this argue as in the proof of Lemma 86.3 to see that a countable directed set has a cofinal subset isomorphic to $(\mathbf{N}, \geq)$. Suppose $R = k[x_1, x_2, x_3, \ldots]$ and $M = R/(x_i)$. Then M is finitely generated but not finitely presented, hence not Mittag-Leffler (see Example 91.1 part (1)). But of course you can write $M = \text{colim}_n M_n$ by taking $M_n = R/(x_1, \ldots, x_n)$, hence the condition that you can write M as such a limit does not imply that M is Mittag-Leffler.

05D2 **Lemma 92.2.** *Let R be a ring. Let M be an R-module. Assume M is Mittag-Leffler and countably generated. For any R-module map $f : P \to M$ with P finitely generated there exists an endomorphism $\alpha : M \to M$ such that*

(1) *$\alpha : M \to M$ factors through a finitely presented R-module, and*
(2) *$\alpha \circ f = f$.*

Proof. Write $M = \text{colim}_{i \in I} M_i$ as a directed colimit of finitely presented R-modules with I countable, see Lemma 92.1. The transition maps are denoted f_{ij} and we use $f_i : M_i \to M$ to denote the canonical maps into M. Set $N = \prod_{s \in I} M_s$. Denote

$$M_i^* = \text{Hom}_R(M_i, N) = \prod\nolimits_{s \in I} \text{Hom}_R(M_i, M_s)$$

so that (M_i^*) is an inverse system of R-modules over I. Note that $\text{Hom}_R(M, N) = \lim M_i^*$. As M is Mittag-Leffler, we find for every $i \in I$ an index $k(i) \geq i$ such that

$$E_i := \bigcap\nolimits_{i' \geq i} \text{Im}(M_{i'}^* \to M_i^*) = \text{Im}(M_{k(i)}^* \to M_i^*)$$

Choose and fix $j \in I$ such that $\text{Im}(P \to M) \subset \text{Im}(M_j \to M)$. This is possible as P is finitely generated. Set $k = k(j)$. Let $x = (0, \ldots, 0, \text{id}_{M_k}, 0, \ldots, 0) \in M_k^*$ and note that this maps to $y = (0, \ldots, 0, f_{jk}, 0, \ldots, 0) \in M_j^*$. By our choice of k we see that $y \in E_j$. By Example 86.2 the transition maps $E_i \to E_j$ are surjective for each $i \geq j$ and $\lim E_i = \lim M_i^* = \text{Hom}_R(M, N)$. Hence Lemma 86.3 guarantees there exists an element $z \in \text{Hom}_R(M, N)$ which maps to y in $E_j \subset M_j^*$. Let z_k be the kth component of z. Then $z_k : M \to M_k$ is a homomorphism such that

$$\begin{tikzcd} M \arrow[r, "z_k"] & M_k \\ M_j \arrow[u, "f_j"] \arrow[ur, "f_{jk}"'] & \end{tikzcd}$$

commutes. Let $\alpha : M \to M$ be the composition $f_k \circ z_k : M \to M_k \to M$. Then α factors through a finitely presented module by construction and $\alpha \circ f_j = f_j$. Since the image of f is contained in the image of f_j this also implies that $\alpha \circ f = f$. □

We will see later (see Lemma 153.13) that Lemma 92.2 means that a countably generated Mittag-Leffler module over a henselian local ring is a direct sum of finitely presented modules.

93. Characterizing projective modules

059V The goal of this section is to prove that a module is projective if and only if it is flat, Mittag-Leffler, and a direct sum of countably generated modules (Theorem 93.3 below).

059X **Lemma 93.1.** *Let M be an R-module. If M is flat, Mittag-Leffler, and countably generated, then M is projective.*

Proof. By Lazard's theorem (Theorem 81.4), we can write $M = \text{colim}_{i\in I} M_i$ for a directed system of finite free R-modules (M_i, f_{ij}) indexed by a set I. By Lemma 92.1, we may assume I is countable. Now let

$$0 \to N_1 \to N_2 \to N_3 \to 0$$

be an exact sequence of R-modules. We must show that applying $\text{Hom}_R(M, -)$ preserves exactness. Since M_i is finite free,

$$0 \to \text{Hom}_R(M_i, N_1) \to \text{Hom}_R(M_i, N_2) \to \text{Hom}_R(M_i, N_3) \to 0$$

is exact for each i. Since M is Mittag-Leffler, $(\text{Hom}_R(M_i, N_1))$ is a Mittag-Leffler inverse system. So by Lemma 86.4,

$$0 \to \lim_{i\in I} \text{Hom}_R(M_i, N_1) \to \lim_{i\in I} \text{Hom}_R(M_i, N_2) \to \lim_{i\in I} \text{Hom}_R(M_i, N_3) \to 0$$

is exact. But for any R-module N there is a functorial isomorphism $\text{Hom}_R(M, N) \cong \lim_{i\in I} \text{Hom}_R(M_i, N)$, so

$$0 \to \text{Hom}_R(M, N_1) \to \text{Hom}_R(M, N_2) \to \text{Hom}_R(M, N_3) \to 0$$

is exact. □

059Y **Remark 93.2.** Lemma 93.1 does not hold without the countable generation assumption. For example, the $\mathbf{Z}$-module $M = \mathbf{Z}[[x]]$ is flat and Mittag-Leffler but not projective. It is Mittag-Leffler by Lemma 91.4. Subgroups of free abelian groups are free, hence a projective $\mathbf{Z}$-module is in fact free and so are its submodules. Thus to show M is not projective it suffices to produce a non-free submodule. Fix a prime p and consider the submodule N consisting of power series $f(x) = \sum a_i x^i$ such that for every integer $m \geq 1$, p^m divides a_i for all but finitely many i. Then $\sum a_i p^i x^i$ is in N for all $a_i \in \mathbf{Z}$, so N is uncountable. Thus if N were free it would have uncountable rank and the dimension of N/pN over $\mathbf{Z}/p$ would be uncountable. This is not true as the elements $x^i \in N/pN$ for $i \geq 0$ span N/pN.

059Z **Theorem 93.3.** *Let M be an R-module. Then M is projective if and only it satisfies:*

1. *M is flat,*
2. *M is Mittag-Leffler,*
3. *M is a direct sum of countably generated R-modules.*

Proof. First suppose M is projective. Then M is a direct summand of a free module, so M is flat and Mittag-Leffler since these properties pass to direct summands. By Kaplansky's theorem (Theorem 84.5), M satisfies (3).

Conversely, suppose M satisfies (1)-(3). Since being flat and Mittag-Leffler passes to direct summands, M is a direct sum of flat, Mittag-Leffler, countably generated R-modules. Lemma 93.1 implies M is a direct sum of projective modules. Hence M is projective. $\square$

05A0 **Lemma 93.4.** *Let $f : M \to N$ be universally injective map of R-modules. Suppose M is a direct sum of countably generated R-modules, and suppose N is flat and Mittag-Leffler. Then M is projective.*

Proof. By Lemmas 82.7 and 89.7, M is flat and Mittag-Leffler, so the conclusion follows from Theorem 93.3. $\square$

05A1 **Lemma 93.5.** *Let R be a Noetherian ring and let M be a R-module. Suppose M is a direct sum of countably generated R-modules, and suppose there is a universally injective map $M \to R[[t_1, \ldots, t_n]]$ for some n. Then M is projective.*

Proof. Follows from Lemmas 93.4 and 91.4. $\square$

94. Ascending properties of modules

05A2 All of the properties of a module in Theorem 93.3 ascend along arbitrary ring maps:

05A3 **Lemma 94.1.** *Let $R \to S$ be a ring map. Let M be an R-module. Then:*
(1) *If M is flat, then the S-module $M \otimes_R S$ is flat.*
(2) *If M is Mittag-Leffler, then the S-module $M \otimes_R S$ is Mittag-Leffler.*
(3) *If M is a direct sum of countably generated R-modules, then the S-module $M \otimes_R S$ is a direct sum of countably generated S-modules.*
(4) *If M is projective, then the S-module $M \otimes_R S$ is projective.*

Proof. All are obvious except (2). For this, use formulation (3) of being Mittag-Leffler from Proposition 88.6 and the fact that tensoring commutes with taking colimits. $\square$

95. Descending properties of modules

05A4 We address the faithfully flat descent of the properties from Theorem 93.3 that characterize projectivity. In the presence of flatness, the property of being a Mittag-Leffler module descends:

05A5 **Lemma 95.1.** *Let $R \to S$ be a faithfully flat ring map. Let M be an R-module. If the S-module $M \otimes_R S$ is Mittag-Leffler, then M is Mittag-Leffler.* Email from Juan Pablo Acosta Lopez dated 12/20/14.

Proof. Write $M = \operatorname{colim}_{i \in I} M_i$ as a directed colimit of finitely presented R-modules M_i. Using Proposition 88.6, we see that we have to prove that for each $i \in I$ there exists $i \leq j$, $j \in I$ such that $M_i \to M_j$ dominates $M_i \to M$.

Take N the pushout

$$\begin{tikzcd} M_i \arrow[r] \arrow[d] & M_j \arrow[d] \\ M \arrow[r] & N \end{tikzcd}$$

Then the lemma is equivalent to the existence of j such that $M_j \to N$ is universally injective, see Lemma 88.4. Observe that the tensorization by S

$$\begin{tikzcd} M_i \otimes_R S \arrow[r] \arrow[d] & M_j \otimes_R S \arrow[d] \\ M \otimes_R S \arrow[r] & N \otimes_R S \end{tikzcd}$$

Is a pushout diagram. So because $M \otimes_R S = \mathrm{colim}_{i \in I} M_i \otimes_R S$ expresses $M \otimes_R S$ as a colimit of S-modules of finite presentation, and $M \otimes_R S$ is Mittag-Leffler, there exists $j \geq i$ such that $M_j \otimes_R S \to N \otimes_R S$ is universally injective. So using that $R \to S$ is faithfully flat we conclude that $M_j \to N$ is universally injective too. $\square$

0GVD **Lemma 95.2.** *Let $R \to S$ be a faithfully flat ring map. Let M be an R-module. If the S-module $M \otimes_R S$ is countably generated, then M is countably generated.*

Proof. Say $M \otimes_R S$ is generated by the elements y_i, $i = 1, 2, 3, \ldots$. Write $y_i = \sum_{j=1,\ldots,n_i} x_{ij} \otimes s_{ij}$ for some $n_i \geq 0$, $x_{ij} \in M$ and $s_{ij} \in S$. Denote $M' \subset M$ the submodule generated by the countable collection of elements x_{ij}. Then $M' \otimes_R S \to M \otimes_R S$ is surjective as the image contains the generators y_i. Since S is faithfully flat over R we conclude that $M' = M$ as desired. $\square$

At this point the faithfully flat descent of countably generated projective modules follows easily.

05A6 **Lemma 95.3.** *Let $R \to S$ be a faithfully flat ring map. Let M be an R-module. If the S-module $M \otimes_R S$ is countably generated and projective, then M is countably generated and projective.*

Proof. Follows from Lemmas 83.2, 95.1, and 95.2 and Theorem 93.3. $\square$

All that remains is to use dévissage to reduce descent of projectivity in the general case to the countably generated case. First, two simple lemmas.

05A7 **Lemma 95.4.** *Let $R \to S$ be a ring map, let M be an R-module, and let Q be a countably generated S-submodule of $M \otimes_R S$. Then there exists a countably generated R-submodule P of M such that $\mathrm{Im}(P \otimes_R S \to M \otimes_R S)$ contains Q.*

Proof. Let $y_1, y_2, \ldots$ be generators for Q and write $y_j = \sum_k x_{jk} \otimes s_{jk}$ for some $x_{jk} \in M$ and $s_{jk} \in S$. Then take P be the submodule of M generated by the x_{jk}. $\square$

05A8 **Lemma 95.5.** *Let $R \to S$ be a ring map, and let M be an R-module. Suppose $M \otimes_R S = \bigoplus_{i \in I} Q_i$ is a direct sum of countably generated S-modules Q_i. If N is a countably generated submodule of M, then there is a countably generated submodule N' of M such that $N' \supset N$ and $\mathrm{Im}(N' \otimes_R S \to M \otimes_R S) = \bigoplus_{i \in I'} Q_i$ for some subset $I' \subset I$.*

Proof. Let $N'_0 = N$. We construct by induction an increasing sequence of countably generated submodules $N'_\ell \subset M$ for $\ell = 0, 1, 2, \ldots$ such that: if I'_ℓ is the set of $i \in I$ such that the projection of $\mathrm{Im}(N'_\ell \otimes_R S \to M \otimes_R S)$ onto Q_i is nonzero, then $\mathrm{Im}(N'_{\ell+1} \otimes_R S \to M \otimes_R S)$ contains Q_i for all $i \in I'_\ell$. To construct $N'_{\ell+1}$ from N'_ℓ, let Q be the sum of (the countably many) Q_i for $i \in I'_\ell$, choose P as in Lemma 95.4, and then let $N'_{\ell+1} = N'_\ell + P$. Having constructed the N'_ℓ, just take $N' = \bigcup_\ell N'_\ell$ and $I' = \bigcup_\ell I'_\ell$. $\square$

05A9 **Theorem 95.6.** *Let $R \to S$ be a faithfully flat ring map. Let M be an R-module. If the S-module $M \otimes_R S$ is projective, then M is projective.*

Proof. We are going to construct a Kaplansky dévissage of M to show that it is a direct sum of projective modules and hence projective. By Theorem 84.5 we can write $M \otimes_R S = \bigoplus_{i \in I} Q_i$ as a direct sum of countably generated S-modules Q_i. Choose a well-ordering on M. Using transfinite recursion we are going to define an increasing family of submodules M_α of M, one for each ordinal α, such that $M_\alpha \otimes_R S$ is a direct sum of some subset of the Q_i.

For $\alpha = 0$ let $M_0 = 0$. If α is a limit ordinal and M_β has been defined for all $\beta < \alpha$, then define $M_\beta = \bigcup_{\beta < \alpha} M_\beta$. Since each $M_\beta \otimes_R S$ for $\beta < \alpha$ is a direct sum of a subset of the Q_i, the same will be true of $M_\alpha \otimes_R S$. If $\alpha + 1$ is a successor ordinal and M_α has been defined, then define $M_{\alpha+1}$ as follows. If $M_\alpha = M$, then let $M_{\alpha+1} = M$. Otherwise choose the smallest $x \in M$ (with respect to the fixed well-ordering) such that $x \notin M_\alpha$. Since S is flat over R, $(M/M_\alpha) \otimes_R S = M \otimes_R S/M_\alpha \otimes_R S$, so since $M_\alpha \otimes_R S$ is a direct sum of some Q_i, the same is true of $(M/M_\alpha) \otimes_R S$. By Lemma 95.5, we can find a countably generated R-submodule P of M/M_α containing the image of x in M/M_α and such that $P \otimes_R S$ (which equals $\mathrm{Im}(P \otimes_R S \to M \otimes_R S)$ since S is flat over R) is a direct sum of some Q_i. Since $M \otimes_R S = \bigoplus_{i \in I} Q_i$ is projective and projectivity passes to direct summands, $P \otimes_R S$ is also projective. Thus by Lemma 95.3, P is projective. Finally we define $M_{\alpha+1}$ to be the preimage of P in M, so that $M_{\alpha+1}/M_\alpha = P$ is countably generated and projective. In particular M_α is a direct summand of $M_{\alpha+1}$ since projectivity of $M_{\alpha+1}/M_\alpha$ implies the sequence $0 \to M_\alpha \to M_{\alpha+1} \to M_{\alpha+1}/M_\alpha \to 0$ splits.

Transfinite induction on M (using the fact that we constructed $M_{\alpha+1}$ to contain the smallest $x \in M$ not contained in M_α) shows that each $x \in M$ is contained in some M_α. Thus, there is some large enough ordinal S satisfying: for each $x \in M$ there is $\alpha \in S$ such that $x \in M_\alpha$. This means $(M_\alpha)_{\alpha \in S}$ satisfies property (1) of a Kaplansky dévissage of M. The other properties are clear by construction. We conclude $M = \bigoplus_{\alpha+1 \in S} M_{\alpha+1}/M_\alpha$. Since each $M_{\alpha+1}/M_\alpha$ is projective by construction, M is projective. $\square$

96. Completion

00M9 Suppose that R is a ring and I is an ideal. We define the *completion of R with respect to I* to be the limit

$$R^\wedge = \lim_n R/I^n.$$

An element of $R^\wedge$ is given by a sequence of elements $f_n \in R/I^n$ such that $f_n \equiv f_{n+1} \bmod I^n$ for all n. We will view $R^\wedge$ as an R-algebra. Similarly, if M is an R-module then we define the *completion of M with respect to I* to be the limit

$$M^\wedge = \lim_n M/I^n M.$$

An element of $M^\wedge$ is given by a sequence of elements $m_n \in M/I^n M$ such that $m_n \equiv m_{n+1} \bmod I^n M$ for all n. We will view $M^\wedge$ as an $R^\wedge$-module. From this description it is clear that there are always canonical maps

$$M \longrightarrow M^\wedge \quad \text{and} \quad M \otimes_R R^\wedge \longrightarrow M^\wedge.$$

Moreover, given a map $\varphi : M \to N$ of modules we get an induced map $\varphi^\wedge : M^\wedge \to N^\wedge$ on completions making the diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d] & N \arrow[d] \\ M^\wedge \arrow[r] & N^\wedge \end{tikzcd}$$

commute. In general completion is not an exact functor, see Examples, Section 9. Here are some initial positive results.

0315 **Lemma 96.1.** *Let R be a ring. Let $I \subset R$ be an ideal. Let $\varphi : M \to N$ be a map of R-modules.*

1. *If $M/IM \to N/IN$ is surjective, then $M^\wedge \to N^\wedge$ is surjective.*
2. *If $M \to N$ is surjective, then $M^\wedge \to N^\wedge$ is surjective.*
3. *If $0 \to K \to M \to N \to 0$ is a short exact sequence of R-modules and N is flat, then $0 \to K^\wedge \to M^\wedge \to N^\wedge \to 0$ is a short exact sequence.*
4. *The map $M \otimes_R R^\wedge \to M^\wedge$ is surjective for any finite R-module M.*

Proof. Assume $M/IM \to N/IN$ is surjective. Then the map $M/I^nM \to N/I^nN$ is surjective for each $n \geq 1$ by Nakayama's lemma. More precisely, apply Lemma 20.1 part (11) to the map $M/I^nM \to N/I^nN$ over the ring R/I^n and the nilpotent ideal I/I^n to see this. Set $K_n = \{x \in M \mid \varphi(x) \in I^nN\}$. Thus we get short exact sequences

$$0 \to K_n/I^nM \to M/I^nM \to N/I^nN \to 0$$

We claim that the canonical map $K_{n+1}/I^{n+1}M \to K_n/I^nM$ is surjective. Namely, if $x \in K_n$ write $\varphi(x) = \sum z_jn_j$ with $z_j \in I^n$, $n_j \in N$. By assumption we can write $n_j = \varphi(m_j) + \sum z_{jk}n_{jk}$ with $m_j \in M$, $z_{jk} \in I$ and $n_{jk} \in N$. Hence

$$\varphi(x - \sum z_jm_j) = \sum z_jz_{jk}n_{jk}.$$

This means that $x' = x - \sum z_jm_j \in K_{n+1}$ maps to $x \bmod I^nM$ which proves the claim. Now we may apply Lemma 87.1 to the inverse system of short exact sequences above to see (1). Part (2) is a special case of (1). If the assumptions of (3) hold, then for each n the sequence

$$0 \to K/I^nK \to M/I^nM \to N/I^nN \to 0$$

is short exact by Lemma 39.12. Hence we can directly apply Lemma 87.1 to conclude (3) is true. To see (4) choose generators $x_i \in M$, $i = 1, \ldots, n$. Then the map $R^{\oplus n} \to M$, $(a_1, \ldots, a_n) \mapsto \sum a_ix_i$ is surjective. Hence by (2) we see $(R^\wedge)^{\oplus n} \to M^\wedge$, $(a_1, \ldots, a_n) \mapsto \sum a_ix_i$ is surjective. Assertion (4) follows from this. $\square$

0317 **Definition 96.2.** Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. We say M is *I-adically complete* if the map

$$M \longrightarrow M^\wedge = \lim_n M/I^nM$$

is an isomorphism[9]. We say R is *I-adically complete* if R is I-adically complete as an R-module.

It is not true that the completion of an R-module M with respect to I is I-adically complete. For an example see Examples, Section 7. If the ideal is finitely generated, then the completion is complete.

[9]This includes the condition that $\bigcap I^nM = (0)$.

05GG **Lemma 96.3.** *Let R be a ring. Let I be a finitely generated ideal of R. Let M be an R-module. Then*

(1) *the completion $M^\wedge$ is I-adically complete, and*
(2) $I^nM^\wedge = \mathrm{Ker}(M^\wedge \to M/I^nM) = (I^nM)^\wedge$ *for all $n \geq 1$.*

In particular $R^\wedge$ is I-adically complete, $I^nR^\wedge = (I^n)^\wedge$, and $R^\wedge/I^nR^\wedge = R/I^n$.

[Mat78, Theorem 15]. The slick proof given here is from an email of Bjorn Poonen dated Nov 5, 2016.

Proof. Since I is finitely generated, I^n is finitely generated, say by $f_1, \ldots, f_r$. Applying Lemma 96.1 part (2) to the surjection $(f_1, \ldots, f_r) : M^{\oplus r} \to I^nM$ yields a surjection

$$(M^\wedge)^{\oplus r} \xrightarrow{(f_1,\ldots,f_r)} (I^nM)^\wedge = \lim_{m \geq n} I^nM/I^mM = \mathrm{Ker}(M^\wedge \to M/I^nM).$$

On the other hand, the image of $(f_1, \ldots, f_r) : (M^\wedge)^{\oplus r} \to M^\wedge$ is $I^nM^\wedge$. Thus $M^\wedge/I^nM^\wedge \simeq M/I^nM$. Taking inverse limits yields $(M^\wedge)^\wedge \simeq M^\wedge$; that is, $M^\wedge$ is I-adically complete. $\square$

0BNG **Lemma 96.4.** *Let R be a ring. Let $I \subset R$ be an ideal. Let $0 \to M \to N \to Q \to 0$ be an exact sequence of R-modules such that Q is annihilated by a power of I. Then completion produces an exact sequence $0 \to M^\wedge \to N^\wedge \to Q \to 0$.*

Proof. Say $I^cQ = 0$. Then $Q/I^nQ = Q$ for $n \geq c$. On the other hand, it is clear that $I^nM \subset M \cap I^nN \subset I^{n-c}M$ for $n \geq c$. Thus $M^\wedge = \lim M/(M \cap I^nN)$. Apply Lemma 87.1 to the system of exact sequences

$$0 \to M/(M \cap I^nN) \to N/I^nN \to Q \to 0$$

for $n \geq c$ to conclude. $\square$

0318 **Lemma 96.5.** *Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. Denote $K_n = \mathrm{Ker}(M^\wedge \to M/I^nM)$. Then $M^\wedge$ is I-adically complete if and only if K_n is equal to $I^nM^\wedge$ for all $n \geq 1$.*

Taken from an unpublished note of Lenstra and de Smit.

Proof. The module $I^nM^\wedge$ is contained in K_n. Thus for each $n \geq 1$ there is a canonical exact sequence

$$0 \to K_n/I^nM^\wedge \to M^\wedge/I^nM^\wedge \to M/I^nM \to 0.$$

As $I^nM^\wedge$ maps onto $I^nM/I^{n+1}M$ we see that $K_{n+1} + I^nM^\wedge = K_n$. Thus the inverse system $\{K_n/I^nM^\wedge\}_{n \geq 1}$ has surjective transition maps. By Lemma 87.1 we see that there is a short exact sequence

$$0 \to \lim_n K_n/I^nM^\wedge \to (M^\wedge)^\wedge \to M^\wedge \to 0$$

Hence $M^\wedge$ is complete if and only if $K_n/I^nM^\wedge = 0$ for all $n \geq 1$. $\square$

05GI **Lemma 96.6.** *Let R be a ring, let $I \subset R$ be an ideal, and let $R^\wedge = \lim R/I^n$.*

(1) *any element of $R^\wedge$ which maps to a unit of R/I is a unit,*
(2) *any element of $1 + I$ maps to an invertible element of $R^\wedge$,*
(3) *any element of $1 + IR^\wedge$ is invertible in $R^\wedge$, and*
(4) *the ideals $IR^\wedge$ and $\mathrm{Ker}(R^\wedge \to R/I)$ are contained in the Jacobson radical of $R^\wedge$.*

Proof. Let $x \in R^\wedge$ map to a unit x_1 in R/I. Then x maps to a unit x_n in R/I^n for every n by Lemma 32.4. Hence $y = (x_n^{-1}) \in \lim R/I^n = R^\wedge$ is an inverse to x. Parts (2) and (3) follow immediately from (1). Part (4) follows from (1) and Lemma 19.1. $\square$

090S **Lemma 96.7.** *Let A be a ring. Let $I = (f_1, \ldots, f_r)$ be a finitely generated ideal. If $M \to \lim M/f_i^n M$ is surjective for each i, then $M \to \lim M/I^n M$ is surjective.*

Proof. Note that $\lim M/I^n M = \lim M/(f_1^n, \ldots, f_r^n)M$ as $I^n \supset (f_1^n, \ldots, f_r^n) \supset I^{rn}$. An element ξ of $\lim M/(f_1^n, \ldots, f_r^n)M$ can be symbolically written as

$$\xi = \sum\nolimits_{n \geq 0} \sum\nolimits_i f_i^n x_{n,i}$$

with $x_{n,i} \in M$. If $M \to \lim M/f_i^n M$ is surjective, then there is an $x_i \in M$ mapping to $\sum x_{n,i} f_i^n$ in $\lim M/f_i^n M$. Then $x = \sum x_i$ maps to ξ in $\lim M/I^n M$. $\square$

090T **Lemma 96.8.** *Let A be a ring. Let $I \subset J \subset A$ be ideals. If M is J-adically complete and I is finitely generated, then M is I-adically complete.*

Proof. Assume M is J-adically complete and I is finitely generated. We have $\bigcap I^n M = 0$ because $\bigcap J^n M = 0$. By Lemma 96.7 it suffices to prove the surjectivity of $M \to \lim M/I^n M$ in case I is generated by a single element. Say $I = (f)$. Let $x_n \in M$ with $x_{n+1} - x_n \in f^n M$. We have to show there exists an $x \in M$ such that $x_n - x \in f^n M$ for all n. As $x_{n+1} - x_n \in J^n M$ and as M is J-adically complete, there exists an element $x \in M$ such that $x_n - x \in J^n M$. Replacing x_n by $x_n - x$ we may assume that $x_n \in J^n M$. To finish the proof we will show that this implies $x_n \in I^n M$. Namely, write $x_n - x_{n+1} = f^n z_n$. Then

$$x_n = f^n(z_n + f z_{n+1} + f^2 z_{n+2} + \ldots)$$

The sum $z_n + f z_{n+1} + f^2 z_{n+2} + \ldots$ converges in M as $f^c \in J^c$. The sum $f^n(z_n + f z_{n+1} + f^2 z_{n+2} + \ldots)$ converges in M to x_n because the partial sums equal $x_n - x_{n+c}$ and $x_{n+c} \in J^{n+c} M$. $\square$

0319 **Lemma 96.9.** *Let R be a ring. Let I, J be ideals of R. Assume there exist integers $c, d > 0$ such that $I^c \subset J$ and $J^d \subset I$. Then completion with respect to I agrees with completion with respect to J for any R-module. In particular an R-module M is I-adically complete if and only if it is J-adically complete.*

Proof. Consider the system of maps $M/I^n M \to M/J^{\lfloor n/d \rfloor} M$ and the system of maps $M/J^m M \to M/I^{\lfloor m/c \rfloor} M$ to get mutually inverse maps between the completions. $\square$

031A **Lemma 96.10.** *Let R be a ring. Let I be an ideal of R. Let M be an I-adically complete R-module, and let $K \subset M$ be an R-submodule. The following are equivalent*

(1) *$K = \bigcap (K + I^n M)$ and*
(2) *M/K is I-adically complete.*

Proof. Set $N = M/K$. By Lemma 96.1 the map $M = M^\wedge \to N^\wedge$ is surjective. Hence $N \to N^\wedge$ is surjective. It is easy to see that the kernel of $N \to N^\wedge$ is the module $\bigcap (K + I^n M)/K$. $\square$

031B **Lemma 96.11.** *Let R be a ring. Let I be an ideal of R. Let M be an R-module. If (a) R is I-adically complete, (b) M is a finite R-module, and (c) $\bigcap I^n M = (0)$, then M is I-adically complete.*

Proof. By Lemma 96.1 the map $M = M \otimes_R R = M \otimes_R R^\wedge \to M^\wedge$ is surjective. The kernel of this map is $\bigcap I^n M$ hence zero by assumption. Hence $M \cong M^\wedge$ and M is complete. $\square$

031D **Lemma 96.12.** *Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. Assume*

1. *R is I-adically complete,*
2. *$\bigcap_{n \geq 1} I^n M = (0)$, and*
3. *M/IM is a finite R/I-module.*

Then M is a finite R-module.

Proof. Let $x_1, \ldots, x_n \in M$ be elements whose images in M/IM generate M/IM as a R/I-module. Denote $M' \subset M$ the R-submodule generated by $x_1, \ldots, x_n$. By Lemma 96.1 the map $(M')^\wedge \to M^\wedge$ is surjective. Since $\bigcap I^n M = 0$ we see in particular that $\bigcap I^n M' = (0)$. Hence by Lemma 96.11 we see that M' is complete, and we conclude that $M' \to M^\wedge$ is surjective. Finally, the kernel of $M \to M^\wedge$ is zero since it is equal to $\bigcap I^n M = (0)$. Hence we conclude that $M \cong M' \cong M^\wedge$ is finitely generated. □

97. Completion for Noetherian rings

0BNH In this section we discuss completion with respect to ideals in Noetherian rings.

00MA **Lemma 97.1.** *Let I be an ideal of a Noetherian ring R. Denote $^\wedge$ completion with respect to I.*

1. *If $K \to N$ is an injective map of finite R-modules, then the map on completions $K^\wedge \to N^\wedge$ is injective.*
2. *If $0 \to K \to N \to M \to 0$ is a short exact sequence of finite R-modules, then $0 \to K^\wedge \to N^\wedge \to M^\wedge \to 0$ is a short exact sequence.*
3. *If M is a finite R-module, then $M^\wedge = M \otimes_R R^\wedge$.*

Proof. Setting $M = N/K$ we find that part (1) follows from part (2). Let $0 \to K \to N \to M \to 0$ be as in (2). For each n we get the short exact sequence

$$0 \to K/(I^n N \cap K) \to N/I^n N \to M/I^n M \to 0.$$

By Lemma 87.1 we obtain the exact sequence

$$0 \to \lim K/(I^n N \cap K) \to N^\wedge \to M^\wedge \to 0.$$

By the Artin-Rees Lemma 51.2 we may choose c such that $I^n K \subset I^n N \cap K \subset I^{n-c} K$ for $n \geq c$. Hence $K^\wedge = \lim K/I^n K = \lim K/(I^n N \cap K)$ and we conclude that (2) is true.

Let M be as in (3) and let $0 \to K \to R^{\oplus t} \to M \to 0$ be a presentation of M. We get a commutative diagram

$$\begin{tikzcd} & K \otimes_R R^\wedge \arrow[r] \arrow[d] & R^{\oplus t} \otimes_R R^\wedge \arrow[r] \arrow[d] & M \otimes_R R^\wedge \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K^\wedge \arrow[r] & (R^{\oplus t})^\wedge \arrow[r] & M^\wedge \arrow[r] & 0 \end{tikzcd}$$

The top row is exact, see Section 39. The bottom row is exact by part (2). By Lemma 96.1 the vertical arrows are surjective. The middle vertical arrow is an isomorphism. We conclude (3) holds by the Snake Lemma 4.1. □

00MB **Lemma 97.2.** *Let I be a ideal of a Noetherian ring R. Denote $^\wedge$ completion with respect to I.*

(1) *The ring map $R \to R^\wedge$ is flat.*
(2) *The functor $M \mapsto M^\wedge$ is exact on the category of finitely generated R-modules.*

Proof. Consider $J \otimes_R R^\wedge \to R \otimes_R R^\wedge = R^\wedge$ where J is an arbitrary ideal of R. According to Lemma 97.1 this is identified with $J^\wedge \to R^\wedge$ and $J^\wedge \to R^\wedge$ is injective. Part (1) follows from Lemma 39.5. Part (2) is a reformulation of Lemma 97.1 part (2). $\square$

00MC **Lemma 97.3.** *Let $(R, \mathfrak m)$ be a Noetherian local ring. Let $I \subset \mathfrak m$ be an ideal. Denote $R^\wedge$ the completion of R with respect to I. The ring map $R \to R^\wedge$ is faithfully flat. In particular the completion with respect to $\mathfrak m$, namely $\lim_n R/\mathfrak m^n$ is faithfully flat.*

Proof. By Lemma 97.2 it is flat. The composition $R \to R^\wedge \to R/\mathfrak m$ where the last map is the projection map $R^\wedge \to R/I$ combined with $R/I \to R/\mathfrak m$ shows that $\mathfrak m$ is in the image of $\text{Spec}(R^\wedge) \to \text{Spec}(R)$. Hence the map is faithfully flat by Lemma 39.15. $\square$

031C **Lemma 97.4.** *Let R be a Noetherian ring. Let I be an ideal of R. Let M be an R-module. Then the completion $M^\wedge$ of M with respect to I is I-adically complete, $I^n M^\wedge = (I^n M)^\wedge$, and $M^\wedge / I^n M^\wedge = M/I^n M$.*

Proof. This is a special case of Lemma 96.3 because I is a finitely generated ideal. $\square$

05GH **Lemma 97.5.** *Let I be an ideal of a ring R. Assume*
(1) *R/I is a Noetherian ring,*
(2) *I is finitely generated.*

Then the completion $R^\wedge$ of R with respect to I is a Noetherian ring complete with respect to $IR^\wedge$.

Proof. By Lemma 96.3 we see that $R^\wedge$ is I-adically complete. Hence it is also $IR^\wedge$-adically complete. Since $R^\wedge/IR^\wedge = R/I$ is Noetherian we see that after replacing R by $R^\wedge$ we may in addition to assumptions (1) and (2) assume that also R is I-adically complete.

Let $f_1, \ldots, f_t$ be generators of I. Then there is a surjection of rings $R/I[T_1, \ldots, T_t] \to \bigoplus I^n/I^{n+1}$ mapping T_i to the element $\overline{f}_i \in I/I^2$. Hence $\bigoplus I^n/I^{n+1}$ is a Noetherian ring. Let $J \subset R$ be an ideal. Consider the ideal

$$\bigoplus J \cap I^n / J \cap I^{n+1} \subset \bigoplus I^n/I^{n+1}.$$

Let $\overline{g}_1, \ldots, \overline{g}_m$ be generators of this ideal. We may choose $\overline{g}_j$ to be a homogeneous element of degree d_j and we may pick $g_j \in J \cap I^{d_j}$ mapping to $\overline{g}_j \in J \cap I^{d_j}/J \cap I^{d_j+1}$. We claim that $g_1, \ldots, g_m$ generate J.

Let $x \in J \cap I^n$. There exist $a_j \in I^{\max(0, n-d_j)}$ such that $x - \sum a_j g_j \in J \cap I^{n+1}$. The reason is that $J \cap I^n/J \cap I^{n+1}$ is equal to $\sum \overline{g}_j I^{n-d_j}/I^{n-d_j+1}$ by our choice of $g_1, \ldots, g_m$. Hence starting with $x \in J$ we can find a sequence of vectors $(a_{1,n}, \ldots, a_{m,n})_{n \geq 0}$ with $a_{j,n} \in I^{\max(0, n-d_j)}$ such that

$$x = \sum\nolimits_{n=0,\ldots,N} \sum\nolimits_{j=1,\ldots,m} a_{j,n} g_j \bmod I^{N+1}$$

Setting $A_j = \sum_{n \geq 0} a_{j,n}$ we see that $x = \sum A_j g_j$ as R is complete. Hence J is finitely generated and we win. $\square$

0316 **Lemma 97.6.** *Let R be a Noetherian ring. Let I be an ideal of R. The completion $R^\wedge$ of R with respect to I is Noetherian.*

Proof. This is a consequence of Lemma 97.5. It can also be seen directly as follows. Choose generators $f_1, \ldots, f_n$ of I. Consider the map

$$R[[x_1, \ldots, x_n]] \longrightarrow R^\wedge, \quad x_i \longmapsto f_i.$$

This is a well defined and surjective ring map (details omitted). Since $R[[x_1, \ldots, x_n]]$ is Noetherian (see Lemma 31.2) we win. $\square$

Suppose $R \to S$ is a local homomorphism of local rings $(R, \mathfrak m)$ and $(S, \mathfrak n)$. Let $S^\wedge$ be the completion of S with respect to $\mathfrak n$. In general $S^\wedge$ is not the $\mathfrak m$-adic completion of S. If $\mathfrak n^t \subset \mathfrak m S$ for some $t \geq 1$ then we do have $S^\wedge = \lim S/\mathfrak m^n S$ by Lemma 96.9. In some cases this even implies that $S^\wedge$ is finite over $R^\wedge$.

0394 **Lemma 97.7.** *Let $R \to S$ be a local homomorphism of local rings $(R, \mathfrak m)$ and $(S, \mathfrak n)$. Let $R^\wedge$, resp. $S^\wedge$ be the completion of R, resp. S with respect to $\mathfrak m$, resp. $\mathfrak n$. If $\mathfrak m$ and $\mathfrak n$ are finitely generated and $\dim_{\kappa(\mathfrak m)} S/\mathfrak m S < \infty$, then*

1. *$S^\wedge$ is equal to the $\mathfrak m$-adic completion of S, and*
2. *$S^\wedge$ is a finite $R^\wedge$-module.*

Proof. We have $\mathfrak m S \subset \mathfrak n$ because $R \to S$ is a local ring map. The assumption $\dim_{\kappa(\mathfrak m)} S/\mathfrak m S < \infty$ implies that $S/\mathfrak m S$ is an Artinian ring, see Lemma 53.2. Hence has dimension 0, see Lemma 60.5, hence $\mathfrak n = \sqrt{\mathfrak m S}$. This and the fact that $\mathfrak n$ is finitely generated implies that $\mathfrak n^t \subset \mathfrak m S$ for some $t \geq 1$. By Lemma 96.9 we see that $S^\wedge$ can be identified with the $\mathfrak m$-adic completion of S. As $\mathfrak m$ is finitely generated we see from Lemma 96.3 that $S^\wedge$ and $R^\wedge$ are $\mathfrak m$-adically complete. At this point we may apply Lemma 96.12 to $S^\wedge$ as an $R^\wedge$-module to conclude. $\square$

07N9 **Lemma 97.8.** *Let R be a Noetherian ring. Let $R \to S$ be a finite ring map. Let $\mathfrak p \subset R$ be a prime and let $\mathfrak q_1, \ldots, \mathfrak q_m$ be the primes of S lying over $\mathfrak p$ (Lemma 36.21). Then*

$$R_{\mathfrak p}^\wedge \otimes_R S = (S_{\mathfrak p})^\wedge = S_{\mathfrak q_1}^\wedge \times \ldots \times S_{\mathfrak q_m}^\wedge$$

where the $(S_{\mathfrak p})^\wedge$ is the completion with respect to $\mathfrak p$ and the local rings $R_{\mathfrak p}$ and $S_{\mathfrak q_i}$ are completed with respect to their maximal ideals.

Proof. The first equality follows from Lemma 97.1. We may replace R by the localization $R_{\mathfrak p}$ and S by $S_{\mathfrak p} = S \otimes_R R_{\mathfrak p}$. Hence we may assume that R is a local Noetherian ring and that $\mathfrak p = \mathfrak m$ is its maximal ideal. The $\mathfrak q_i S_{\mathfrak q_i}$-adic completion $S_{\mathfrak q_i}^\wedge$ is equal to the $\mathfrak m$-adic completion by Lemma 97.7. For every $n \geq 1$ prime ideals of $S/\mathfrak m^n S$ are in 1-to-1 correspondence with the maximal ideals $\mathfrak q_1, \ldots, \mathfrak q_m$ of S (by going up for S over R, see Lemma 36.22). Hence $S/\mathfrak m^n S = \prod S_{\mathfrak q_i}/\mathfrak m^n S_{\mathfrak q_i}$ by Lemma 53.6 (using for example Proposition 60.7 to see that $S/\mathfrak m^n S$ is Artinian). Hence the $\mathfrak m$-adic completion $S^\wedge$ of S is equal to $\prod S_{\mathfrak q_i}^\wedge$. Finally, we have $R^\wedge \otimes_R S = S^\wedge$ by Lemma 97.1. $\square$

05D3 **Lemma 97.9.** *Let R be a ring. Let $I \subset R$ be an ideal. Let $0 \to K \to P \to M \to 0$ be a short exact sequence of R-modules. If M is flat over R and M/IM is a projective R/I-module, then the sequence of I-adic completions*

$$0 \to K^\wedge \to P^\wedge \to M^\wedge \to 0$$

is a split exact sequence.

Proof. As M is flat, each of the sequences

$$0 \to K/I^nK \to P/I^nP \to M/I^nM \to 0$$

is short exact, see Lemma 39.12 and the sequence $0 \to K^\wedge \to P^\wedge \to M^\wedge \to 0$ is a short exact sequence, see Lemma 96.1. It suffices to show that we can find splittings $s_n : M/I^nM \to P/I^nP$ such that $s_{n+1} \bmod I^n = s_n$. We will construct these s_n by induction on n. Pick any splitting s_1, which exists as M/IM is a projective R/I-module. Assume given s_n for some $n > 0$. Set $P_{n+1} = \{x \in P \mid x \bmod I^nP \in \mathrm{Im}(s_n)\}$. The map $\pi : P_{n+1}/I^{n+1}P_{n+1} \to M/I^{n+1}M$ is surjective (details omitted). As $M/I^{n+1}M$ is projective as a R/I^{n+1}-module by Lemma 77.7 we may choose a section $t : M/I^{n+1}M \to P_{n+1}/I^{n+1}P_{n+1}$ of π. Setting s_{n+1} equal to the composition of t with the canonical map $P_{n+1}/I^{n+1}P_{n+1} \to P/I^{n+1}P$ works. □

0DYC **Lemma 97.10.** *Let A be a Noetherian ring. Let $I, J \subset A$ be ideals. If A is I-adically complete and A/I is J-adically complete, then A is J-adically complete.*

Proof. Let B be the $(I + J)$-adic completion of A. By Lemma 97.2 B/IB is the J-adic completion of A/I hence isomorphic to A/I by assumption. Moreover B is I-adically complete by Lemma 96.8. Hence B is a finite A-module by Lemma 96.12. By Nakayama's lemma (Lemma 20.1 using I is in the Jacobson radical of A by Lemma 96.6) we find that $A \to B$ is surjective. The map $A \to B$ is flat by Lemma 97.2. The image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ contains $V(I)$ and as I is contained in the Jacobson radical of A we find $A \to B$ is faithfully flat (Lemma 39.16). Thus $A \to B$ is injective. Thus A is complete with respect to $I + J$, hence a fortiori complete with respect to J. □

98. Taking limits of modules

09B7 In this section we discuss what happens when we take a limit of modules.

0G1Q **Lemma 98.1.** *Let $I \subset A$ be a finitely generated ideal of a ring. Let (M_n) be an inverse system of A-modules with $I^nM_n = 0$. Then $M = \lim M_n$ is I-adically complete.*

Proof. We have $M \to M/I^nM \to M_n$. Taking the limit we get $M \to M^\wedge \to M$. Hence M is a direct summand of $M^\wedge$. Since $M^\wedge$ is I-adically complete by Lemma 96.3, so is M. □

09B8 **Lemma 98.2.** *Let $I \subset A$ be a finitely generated ideal of a ring. Let (M_n) be an inverse system of A-modules with $M_n = M_{n+1}/I^nM_{n+1}$. Then $M/I^nM = M_n$ and M is I-adically complete.*

Proof. By Lemma 98.1 we see that M is I-adically complete. Since the transition maps are surjective, the maps $M \to M_n$ are surjective. Consider the inverse system of short exact sequences

$$0 \to N_n \to M \to M_n \to 0$$

defining N_n. Since $M_n = M_{n+1}/I^nM_{n+1}$ the map $N_{n+1} + I^nM \to N_n$ is surjective. Hence $N_{n+1}/(N_{n+1} \cap I^{n+1}M) \to N_n/(N_n \cap I^nM)$ is surjective. Taking the inverse limit of the short exact sequences

$$0 \to N_n/(N_n \cap I^nM) \to M/I^nM \to M_n \to 0$$

we obtain an exact sequence

$$0 \to \lim N_n/(N_n \cap I^n M) \to M^\wedge \to M$$

Since M is I-adically complete we conclude that $\lim N_n/(N_n \cap I^n M) = 0$ and hence by the surjectivity of the transition maps we get $N_n/(N_n \cap I^n M) = 0$ for all n. Thus $M_n = M/I^n M$ as desired. $\square$

0EKC **Lemma 98.3.** *Let A be a Noetherian graded ring. Let $I \subset A_+$ be a homogeneous ideal. Let (N_n) be an inverse system of finite graded A-modules with $N_n = N_{n+1}/I^n N_{n+1}$. Then there is a finite graded A-module N such that $N_n = N/I^n N$ as graded modules for all n.*

Proof. Pick r and homogeneous elements $x_{1,1}, \ldots, x_{1,r} \in N_1$ of degrees $d_1, \ldots, d_r$ generating N_1. Since the transition maps are surjective, we can pick a compatible system of homogeneous elements $x_{n,i} \in N_n$ lifting $x_{1,i}$. By the graded Nakayama lemma (Lemma 56.1) we see that N_n is generated by the elements $x_{n,1}, \ldots, x_{n,r}$ sitting in degrees $d_1, \ldots, d_r$. Thus for $m \leq n$ we see that $N_n \to N_n/I^m N_n$ is an isomorphism in degrees $< \min(d_i) + m$ (as $I^m N_n$ is zero in those degrees). Thus the inverse system of degree d parts

$$\ldots = N_{2+d-\min(d_i),d} = N_{1+d-\min(d_i),d} = N_{d-\min(d_i),d} \to N_{-1+d-\min(d_i),d} \to \ldots$$

stabilizes as indicated. Let N be the graded A-module whose dth graded part is this stabilization. In particular, we have the elements $x_i = \lim x_{n,i}$ in N. We claim the x_i generate N: any $x \in N_d$ is a linear combination of $x_1, \ldots, x_r$ because we can check this in $N_{d-\min(d_i),d}$ where it holds as $x_{d-\min(d_i),i}$ generate $N_{d-\min(d_i)}$. Finally, the reader checks that the surjective map $N/I^n N \to N_n$ is an isomorphism by checking to see what happens in each degree as before. Details omitted. $\square$

0EKD **Lemma 98.4.** *Let A be a graded ring. Let $I \subset A_+$ be a homogeneous ideal. Denote $A' = \lim A/I^n$. Let (G_n) be an inverse system of graded A-modules with G_n annihilated by I^n. Let M be a graded A-module and let $\varphi_n : M \to G_n$ be a compatible system of graded A-module maps. If the induced map*

$$\varphi : M \otimes_A A' \longrightarrow \lim G_n$$

is an isomorphism, then $M_d \to \lim G_{n,d}$ is an isomorphism for all $d \in \mathbf{Z}$.

Proof. By convention graded rings are in degrees ≥ 0 and graded modules may have nonzero parts of any degree, see Section 56. The map φ exists because $\lim G_n$ is a module over A' as G_n is annihilated by I^n. Another useful thing to keep in mind is that we have

$$\bigoplus\nolimits_{d \in \mathbf{Z}} \lim G_{n,d} \subset \lim G_n \subset \prod\nolimits_{d \in \mathbf{Z}} \lim G_{n,d}$$

where a subscript ${}_d$ indicates the dth graded part.

Injective. Let $x \in M_d$. If $x \mapsto 0$ in $\lim G_{n,d}$ then $x \otimes 1 = 0$ in $M \otimes_A A'$. Then we can find a finitely generated submodule $M' \subset M$ with $x \in M'$ such that $x \otimes 1$ is zero in $M' \otimes_A A'$. Say M' is generated by homogeneous elements sitting in degrees $d_1, \ldots, d_r$. Let $n = d - \min(d_i) + 1$. Since A' has a map to A/I^n and since $A \to A/I^n$ is an isomorphism in degrees $\leq n - 1$ we see that $M' \to M' \otimes_A A'$ is injective in degrees $\leq n-1$. Thus $x = 0$ as desired.

Surjective. Let $y \in \lim G_{n,d}$. Choose a finite sum $\sum x_i \otimes f'_i$ in $M \otimes_A A'$ mapping to y. We may assume x_i is homogeneous, say of degree d_i. Observe that although

A' is not a graded ring, it is a limit of the graded rings A/I^nA and moreover, in any given degree the transition maps eventually become isomorphisms (see above). This gives

$$A = \bigoplus\nolimits_{d \geq 0} A_d \subset A' \subset \prod\nolimits_{d \geq 0} A_d$$

Thus we can write

$$f'_i = \sum\nolimits_{j=0,\ldots,d-d_i-1} f_{i,j} + f_i + g'_i$$

with $f_{i,j} \in A_j$, $f_i \in A_{d-d_i}$, and $g'_i \in A'$ mapping to zero in $\prod_{j \leq d-d_i} A_j$. Now if we compute $\varphi_n(\sum_{i,j} f_{i,j}x_i) \in G_n$, then we get a sum of homogeneous elements of degree $< d$. Hence $\varphi(\sum x_i \otimes f_{i,j})$ maps to zero in $\lim G_{n,d}$. Similarly, a computation shows the element $\varphi(\sum x_i \otimes g'_i)$ maps to zero in $\prod_{d' \leq d} \lim G_{n,d'}$. Since we know that $\varphi(\sum x_i \otimes f'_i)$ is y, we conclude that $\sum f_i x_i \in M_d$ maps to y as desired. □

99. Criteria for flatness

00MD In this section we prove some important technical lemmas in the Noetherian case. We will (partially) generalize these to the non-Noetherian case in Section 128.

00ME **Lemma 99.1.** *Suppose that $R \to S$ is a local homomorphism of Noetherian local rings. Denote $\mathfrak m$ the maximal ideal of R. Let M be a flat R-module and N a finite S-module. Let $u : N \to M$ be a map of R-modules. If $\overline{u} : N/\mathfrak m N \to M/\mathfrak m M$ is injective then u is injective. In this case $M/u(N)$ is flat over R.*

Proof. First we claim that $u_n : N/\mathfrak m^n N \to M/\mathfrak m^n M$ is injective for all $n \geq 1$. We proceed by induction, the base case is that $\overline{u} = u_1$ is injective. By our assumption that M is flat over R we have a short exact sequence $0 \to M \otimes_R \mathfrak m^n/\mathfrak m^{n+1} \to M/\mathfrak m^{n+1}M \to M/\mathfrak m^n M \to 0$. Also, $M \otimes_R \mathfrak m^n/\mathfrak m^{n+1} = M/\mathfrak m M \otimes_{R/\mathfrak m} \mathfrak m^n/\mathfrak m^{n+1}$. We have a similar exact sequence $N \otimes_R \mathfrak m^n/\mathfrak m^{n+1} \to N/\mathfrak m^{n+1}N \to N/\mathfrak m^n N \to 0$ for N except we do not have the zero on the left. We also have $N \otimes_R \mathfrak m^n/\mathfrak m^{n+1} = N/\mathfrak m N \otimes_{R/\mathfrak m} \mathfrak m^n/\mathfrak m^{n+1}$. Thus the map u_{n+1} is injective as both u_n and the map $\overline{u} \otimes \text{id}_{\mathfrak m^n/\mathfrak m^{n+1}}$ are.

By Krull's intersection theorem (Lemma 51.4) applied to N over the ring S and the ideal $\mathfrak m S$ we have $\bigcap \mathfrak m^n N = 0$. Thus the injectivity of u_n for all n implies u is injective.

To show that $M/u(N)$ is flat over R, it suffices to show that $\text{Tor}_1^R(M/u(N), R/I) = 0$ for every ideal $I \subset R$, see Lemma 75.8. From the short exact sequence

$$0 \to N \xrightarrow{u} M \to M/u(N) \to 0$$

and the flatness of M we obtain an exact sequence of Tors

$$0 \to \text{Tor}_1^R(M/u(N), R/I) \to N/IN \to M/IM$$

See Lemma 75.2. Thus it suffices to show that N/IN injects into M/IM. Note that $R/I \to S/IS$ is a local homomorphism of Noetherian local rings, $N/IN \to M/IM$ is a map of R/I-modules, N/IN is finite over S/IS, and M/IM is flat over R/I and $u \bmod I : N/IN \to M/IM$ is injective modulo $\mathfrak m$. Thus we may apply the first part of the proof to $u \bmod I$ and we conclude. □

00MF **Lemma 99.2.** *Suppose that $R \to S$ is a flat and local ring homomorphism of Noetherian local rings. Denote $\mathfrak m$ the maximal ideal of R. Suppose $f \in S$ is a* *nonzerodivisor in $S/\mathfrak{m}S$. Then S/fS is flat over R, and f is a nonzerodivisor in S.*

Proof. Follows directly from Lemma 99.1. □

00MG **Lemma 99.3.** *Suppose that $R \to S$ is a flat and local ring homomorphism of Noetherian local rings. Denote $\mathfrak{m}$ the maximal ideal of R. Suppose $f_1, \ldots, f_c$ is a sequence of elements of S such that the images $\overline{f}_1, \ldots, \overline{f}_c$ form a regular sequence in $S/\mathfrak{m}S$. Then $f_1, \ldots, f_c$ is a regular sequence in S and each of the quotients $S/(f_1, \ldots, f_i)$ is flat over R.*

Proof. Induction and Lemma 99.2. □

00MH **Lemma 99.4.** *Let $R \to S$ be a local homomorphism of Noetherian local rings. Let $\mathfrak{m}$ be the maximal ideal of R. Let M be a finite S-module. Suppose that (a) $M/\mathfrak{m}M$ is a free $S/\mathfrak{m}S$-module, and (b) M is flat over R. Then M is free and S is flat over R.*

Proof. Let $\overline{x}_1, \ldots, \overline{x}_n$ be a basis for the free module $M/\mathfrak{m}M$. Choose $x_1, \ldots, x_n \in M$ with x_i mapping to $\overline{x}_i$. Let $u : S^{\oplus n} \to M$ be the map which maps the ith standard basis vector to x_i. By Lemma 99.1 we see that u is injective. On the other hand, by Nakayama's Lemma 20.1 the map is surjective. The lemma follows. □

00MI **Lemma 99.5.** *Let $R \to S$ be a local homomorphism of local Noetherian rings. Let $\mathfrak{m}$ be the maximal ideal of R. Let $0 \to F_e \to F_{e-1} \to \ldots \to F_0$ be a finite complex of finite S-modules. Assume that each F_i is R-flat, and that the complex $0 \to F_e/\mathfrak{m}F_e \to F_{e-1}/\mathfrak{m}F_{e-1} \to \ldots \to F_0/\mathfrak{m}F_0$ is exact. Then $0 \to F_e \to F_{e-1} \to \ldots \to F_0$ is exact, and moreover the module* $\text{Coker}(F_1 \to F_0)$ *is R-flat.*

Proof. By induction on e. If $e = 1$, then this is exactly Lemma 99.1. If $e > 1$, we see by Lemma 99.1 that $F_e \to F_{e-1}$ is injective and that $C = \text{Coker}(F_e \to F_{e-1})$ is a finite S-module flat over R. Hence we can apply the induction hypothesis to the complex $0 \to C \to F_{e-2} \to \ldots \to F_0$. We deduce that $C \to F_{e-2}$ is injective and the exactness of the complex follows, as well as the flatness of the cokernel of $F_1 \to F_0$. □

In the rest of this section we prove two versions of what is called the "*local criterion of flatness*". Note also the interesting Lemma 128.1 below.

00MJ **Lemma 99.6.** *Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field $\kappa = R/\mathfrak{m}$. Let M be an R-module. If $Tor_1^R(\kappa, M) = 0$, then for every finite length R-module N we have $Tor_1^R(N, M) = 0$.*

Proof. By descending induction on the length of N. If the length of N is 1, then $N \cong \kappa$ and we are done. If the length of N is more than 1, then we can fit N into a short exact sequence $0 \to N' \to N \to N'' \to 0$ where N', N'' are finite length R-modules of smaller length. The vanishing of $\text{Tor}_1^R(N, M)$ follows from the vanishing of $\text{Tor}_1^R(N', M)$ and $\text{Tor}_1^R(N'', M)$ (induction hypothesis) and the long exact sequence of Tor groups, see Lemma 75.2. □

00MK **Lemma 99.7** (Local criterion for flatness)**.** *Let $R \to S$ be a local homomorphism of local Noetherian rings. Let $\mathfrak{m}$ be the maximal ideal of R, and let $\kappa = R/\mathfrak{m}$. Let M be a finite S-module. If $Tor_1^R(\kappa, M) = 0$, then M is flat over R.*

Proof. Let $I \subset R$ be an ideal. By Lemma 39.5 it suffices to show that $I \otimes_R M \to M$ is injective. By Remark 75.9 we see that this kernel is equal to $\text{Tor}_1^R(M, R/I)$. By Lemma 99.6 we see that $J \otimes_R M \to M$ is injective for all ideals of finite colength.

Choose $n >> 0$ and consider the following short exact sequence

$$0 \to I \cap \mathfrak{m}^n \to I \oplus \mathfrak{m}^n \to I + \mathfrak{m}^n \to 0$$

This is a sub sequence of the short exact sequence $0 \to R \to R^{\oplus 2} \to R \to 0$. Thus we get the diagram

$$\begin{tikzcd}
(I \cap \mathfrak{m}^n) \otimes_R M \arrow[r] \arrow[d] & I \otimes_R M \oplus \mathfrak{m}^n \otimes_R M \arrow[r] \arrow[d] & (I + \mathfrak{m}^n) \otimes_R M \arrow[d] \\
M \arrow[r] & M \oplus M \arrow[r] & M
\end{tikzcd}$$

Note that $I + \mathfrak{m}^n$ and $\mathfrak{m}^n$ are ideals of finite colength. Thus a diagram chase shows that $\text{Ker}((I \cap \mathfrak{m}^n) \otimes_R M \to M) \to \text{Ker}(I \otimes_R M \to M)$ is surjective. We conclude in particular that $K = \text{Ker}(I \otimes_R M \to M)$ is contained in the image of $(I \cap \mathfrak{m}^n) \otimes_R M$ in $I \otimes_R M$. By Artin-Rees, Lemma 51.2 we see that K is contained in $\mathfrak{m}^{n-c}(I \otimes_R M)$ for some $c > 0$ and all $n >> 0$. Since $I \otimes_R M$ is a finite S-module (!) and since S is Noetherian, we see that this implies $K = 0$. Namely, the above implies K maps to zero in the $\mathfrak{m}S$-adic completion of $I \otimes_R M$. But the map from S to its $\mathfrak{m}S$-adic completion is faithfully flat by Lemma 97.3. Hence $K = 0$, as desired. $\square$

In the following we often encounter the conditions "M/IM is flat over R/I and $\text{Tor}_1^R(R/I, M) = 0$". The following lemma gives some consequences of these conditions (it is a generalization of Lemma 99.6).

051C **Lemma 99.8.** *Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. If M/IM is flat over R/I and $Tor_1^R(R/I, M) = 0$ then*

1. *M/I^nM is flat over R/I^n for all $n \geq 1$, and*
2. *for any module N which is annihilated by I^m for some $m \geq 0$ we have $Tor_1^R(N, M) = 0$.*

In particular, if I is nilpotent, then M is flat over R.

Proof. Assume M/IM is flat over R/I and $\text{Tor}_1^R(R/I, M) = 0$. Let N be an R/I-module. Choose a short exact sequence

$$0 \to K \to \bigoplus\nolimits_{i \in I} R/I \to N \to 0$$

By the long exact sequence of Tor and the vanishing of $\text{Tor}_1^R(R/I, M)$ we get

$$0 \to \text{Tor}_1^R(N, M) \to K \otimes_R M \to (\bigoplus\nolimits_{i \in I} R/I) \otimes_R M \to N \otimes_R M \to 0$$

But since K, $\bigoplus_{i \in I} R/I$, and N are all annihilated by I we see that

$$K \otimes_R M = K \otimes_{R/I} M/IM,$$
$$(\bigoplus\nolimits_{i \in I} R/I) \otimes_R M = (\bigoplus\nolimits_{i \in I} R/I) \otimes_{R/I} M/IM,$$
$$N \otimes_R M = N \otimes_{R/I} M/IM.$$

As M/IM is flat over R/I we conclude that

$$0 \to K \otimes_{R/I} M/IM \to (\bigoplus\nolimits_{i \in I} R/I) \otimes_{R/I} M/IM \to N \otimes_{R/} M/IM \to 0$$

is exact. Combining this with the above we conclude that $\text{Tor}_1^R(N, M) = 0$ for any R-module N annihilated by I.

In particular, if we apply this to the module I/I^2, then we conclude that the sequence
$$0 \to I^2 \otimes_R M \to I \otimes_R M \to I/I^2 \otimes_R M \to 0$$
is short exact. This implies that $I^2 \otimes_R M \to M$ is injective and it implies that $I/I^2 \otimes_{R/I} M/IM = IM/I^2M$.

Let us prove that M/I^2M is flat over R/I^2. Let $I^2 \subset J$ be an ideal. We have to show that $J/I^2 \otimes_{R/I^2} M/I^2M \to M/I^2M$ is injective, see Lemma 39.5. As M/IM is flat over R/I we know that the map $(I+J)/I \otimes_{R/I} M/IM \to M/IM$ is injective. The sequence
$$(I \cap J)/I^2 \otimes_{R/I^2} M/I^2M \to J/I^2 \otimes_{R/I^2} M/I^2M \to (I + J)/I \otimes_{R/I} M/IM \to 0$$
is exact, as you get it by tensoring the exact sequence $0 \to (I \cap J) \to J \to (I + J)/I \to 0$ by M/I^2M. Hence suffices to prove the injectivity of the map $(I \cap J)/I^2 \otimes_{R/I} M/IM \to IM/I^2M$. However, the map $(I \cap J)/I^2 \to I/I^2$ is injective and as M/IM is flat over R/I the map $(I \cap J)/I^2 \otimes_{R/I} M/IM \to I/I^2 \otimes_{R/I} M/IM$ is injective. Since we have previously seen that $I/I^2 \otimes_{R/I} M/IM = IM/I^2M$ we obtain the desired injectivity.

Hence we have proven that the assumptions imply: (a) $\text{Tor}_1^R(N, M) = 0$ for all N annihilated by I, (b) $I^2 \otimes_R M \to M$ is injective, and (c) M/I^2M is flat over R/I^2. Thus we can continue by induction to get the same results for I^n for all $n \geq 1$. □

0AS8 **Lemma 99.9.** *Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module.*

(1) *If M/IM is flat over R/I and $M \otimes_R I/I^2 \to IM/I^2M$ is injective, then M/I^2M is flat over R/I^2.*
(2) *If M/IM is flat over R/I and $M \otimes_R I^n/I^{n+1} \to I^nM/I^{n+1}M$ is injective for $n = 1, \ldots, k$, then $M/I^{k+1}M$ is flat over R/I^{k+1}.*

Proof. The first statement is a consequence of Lemma 99.8 applied with R replaced by R/I^2 and M replaced by M/I^2M using that
$$\text{Tor}_1^{R/I^2}(M/I^2M, R/I) = \text{Ker}(M \otimes_R I/I^2 \to IM/I^2M),$$
see Remark 75.9. The second statement follows in the same manner using induction on n to show that $M/I^{n+1}M$ is flat over R/I^{n+1} for $n = 1, \ldots, k$. Here we use that
$$\text{Tor}_1^{R/I^{n+1}}(M/I^{n+1}M, R/I) = \text{Ker}(M \otimes_R I^n/I^{n+1} \to I^nM/I^{n+1}M)$$
for every n. □

00ML **Lemma 99.10** (Variant of the local criterion)**.** *Let $R \to S$ be a local homomorphism of Noetherian local rings. Let $I \neq R$ be an ideal in R. Let M be a finite S-module. If $Tor_1^R(M, R/I) = 0$ and M/IM is flat over R/I, then M is flat over R.*

Proof. First proof: By Lemma 99.8 we see that $\text{Tor}_1^R(\kappa, M)$ is zero where κ is the residue field of R. Hence we see that M is flat over R by Lemma 99.7.

Second proof: Let $\mathfrak{m}$ be the maximal ideal of R. We will show that $\mathfrak{m} \otimes_R M \to M$ is injective, and then apply Lemma 99.7. Suppose that $\sum f_i \otimes x_i \in \mathfrak{m} \otimes_R M$ and that $\sum f_i x_i = 0$ in M. By the equational criterion for flatness Lemma 39.11

applied to M/IM over R/I we see there exist $\overline{a}_{ij} \in R/I$ and $\overline{y}_j \in M/IM$ such that $x_i \bmod IM = \sum_j \overline{a}_{ij}\overline{y}_j$ and $0 = \sum_i (f_i \bmod I)\overline{a}_{ij}$. Let $a_{ij} \in R$ be a lift of $\overline{a}_{ij}$ and similarly let $y_j \in M$ be a lift of $\overline{y}_j$. Then we see that

$$\begin{aligned} \sum f_i \otimes x_i &= \sum f_i \otimes x_i + \sum f_i a_{ij} \otimes y_j - \sum f_i \otimes a_{ij} y_j \\ &= \sum f_i \otimes (x_i - \sum a_{ij} y_j) + \sum (\sum f_i a_{ij}) \otimes y_j \end{aligned}$$

Since $x_i - \sum a_{ij} y_j \in IM$ and $\sum f_i a_{ij} \in I$ we see that there exists an element in $I \otimes_R M$ which maps to our given element $\sum f_i \otimes x_i$ in $\mathfrak{m} \otimes_R M$. But $I \otimes_R M \to M$ is injective by assumption (see Remark 75.9) and we win. $\square$

In particular, in the situation of Lemma 99.10, suppose that $I = (x)$ is generated by a single element x which is a nonzerodivisor in R. Then $\text{Tor}_1^R(M, R/(x)) = (0)$ if and only if x is a nonzerodivisor on M.

0523 **Lemma 99.11.** *Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal. Let M be an S-module. Assume*

1. *R is a Noetherian ring,*
2. *S is a Noetherian ring,*
3. *M is a finite S-module, and*
4. *for each $n \geq 1$ the module $M/I^n M$ is flat over R/I^n.*

Then for every $\mathfrak{q} \in V(IS)$ the localization $M_{\mathfrak{q}}$ is flat over R. In particular, if S is local and IS is contained in its maximal ideal, then M is flat over R.

Proof. We are going to use Lemma 99.10. By assumption M/IM is flat over R/I. Hence it suffices to check that $\text{Tor}_1^R(M, R/I)$ is zero on localization at $\mathfrak{q}$. By Remark 75.9 this Tor group is equal to $K = \text{Ker}(I \otimes_R M \to M)$. We know for each $n \geq 1$ that the kernel $\text{Ker}(I/I^n \otimes_{R/I^n} M/I^n M \to M/I^n M)$ is zero. Since there is a module map $I/I^n \otimes_{R/I^n} M/I^n M \to (I \otimes_R M)/I^{n-1}(I \otimes_R M)$ we conclude that $K \subset I^{n-1}(I \otimes_R M)$ for each n. By the Artin-Rees lemma, and more precisely Lemma 51.5 we conclude that $K_{\mathfrak{q}} = 0$, as desired. $\square$

00MM **Lemma 99.12.** *Let $R \to R' \to R''$ be ring maps. Let M be an R-module. Suppose that $M \otimes_R R'$ is flat over R'. Then the natural map $\text{Tor}_1^R(M, R') \otimes_{R'} R'' \to \text{Tor}_1^R(M, R'')$ is onto.*

Proof. Let $F_\bullet$ be a free resolution of M over R. The complex $F_2 \otimes_R R' \to F_1 \otimes_R R' \to F_0 \otimes_R R'$ computes $\text{Tor}_1^R(M, R')$. The complex $F_2 \otimes_R R'' \to F_1 \otimes_R R'' \to F_0 \otimes_R R''$ computes $\text{Tor}_1^R(M, R'')$. Note that $F_i \otimes_R R' \otimes_{R'} R'' = F_i \otimes_R R''$. Let $K' = \text{Ker}(F_1 \otimes_R R' \to F_0 \otimes_R R')$ and similarly $K'' = \text{Ker}(F_1 \otimes_R R'' \to F_0 \otimes_R R'')$. Thus we have an exact sequence

$$0 \to K' \to F_1 \otimes_R R' \to F_0 \otimes_R R' \to M \otimes_R R' \to 0.$$

By the assumption that $M \otimes_R R'$ is flat over R', the sequence

$$K' \otimes_{R'} R'' \to F_1 \otimes_R R'' \to F_0 \otimes_R R'' \to M \otimes_R R'' \to 0$$

is still exact. This means that $K' \otimes_{R'} R'' \to K''$ is surjective. Since $\text{Tor}_1^R(M, R')$ is a quotient of K' and $\text{Tor}_1^R(M, R'')$ is a quotient of K'' we win. $\square$

00MN **Lemma 99.13.** *Let $R \to R'$ be a ring map. Let $I \subset R$ be an ideal and $I' = IR'$. Let M be an R-module and set $M' = M \otimes_R R'$. The natural map $Tor_1^R(R'/I', M) \to Tor_1^{R'}(R'/I', M')$ is surjective.*

Proof. Let $F_2 \to F_1 \to F_0 \to M \to 0$ be a free resolution of M over R. Set $F_i' = F_i \otimes_R R'$. The sequence $F_2' \to F_1' \to F_0' \to M' \to 0$ may no longer be exact at F_1'. A free resolution of M' over R' therefore looks like

$$F_2' \oplus F_2'' \to F_1' \to F_0' \to M' \to 0$$

for a suitable free module F_2'' over R'. Next, note that $F_i \otimes_R R'/I' = F_i'/IF_i' = F_i'/I'F_i'$. So the complex $F_2'/I'F_2' \to F_1'/I'F_1' \to F_0'/I'F_0'$ computes $\text{Tor}_1^R(M, R'/I')$. On the other hand $F_i' \otimes_{R'} R'/I' = F_i'/I'F_i'$ and similarly for F_2''. Thus the complex $F_2'/I'F_2' \oplus F_2''/I'F_2'' \to F_1'/I'F_1' \to F_0'/I'F_0'$ computes $\text{Tor}_1^{R'}(M', R'/I')$. Since the vertical map on complexes

$$\begin{tikzcd}
F_2'/I'F_2' \arrow[r] \arrow[d] & F_1'/I'F_1' \arrow[r] \arrow[d] & F_0'/I'F_0' \arrow[d] \\
F_2'/I'F_2' \oplus F_2''/I'F_2'' \arrow[r] & F_1'/I'F_1' \arrow[r] & F_0'/I'F_0'
\end{tikzcd}$$

clearly induces a surjection on cohomology we win. $\square$

00MO **Lemma 99.14.** *Let*

$$\begin{tikzcd}
S \arrow[r] & S' \\
R \arrow[r] \arrow[u] & R' \arrow[u]
\end{tikzcd}$$

be a commutative diagram of local homomorphisms of local Noetherian rings. Let $I \subset R$ be a proper ideal. Let M be a finite S-module. Denote $I' = IR'$ and $M' = M \otimes_S S'$. Assume that

(1) *S' is a localization of the tensor product $S \otimes_R R'$,*
(2) *M/IM is flat over R/I,*
(3) *$Tor_1^R(M, R/I) \to Tor_1^{R'}(M', R'/I')$ is zero.*

Then M' is flat over R'.

Proof. Since S' is a localization of $S \otimes_R R'$ we see that M' is a localization of $M \otimes_R R'$. Note that by Lemma 39.7 the module $M/IM \otimes_{R/I} R'/I' = M \otimes_R R'/I'(M \otimes_R R')$ is flat over R'/I'. Hence also $M'/I'M'$ is flat over R'/I' as the localization of a flat module is flat. By Lemma 99.10 it suffices to show that $\text{Tor}_1^{R'}(M', R'/I')$ is zero. Since M' is a localization of $M \otimes_R R'$, the last assumption implies that it suffices to show that $\text{Tor}_1^R(M, R/I) \otimes_R R' \to \text{Tor}_1^{R'}(M \otimes_R R', R'/I')$ is surjective.

By Lemma 99.13 we see that $\text{Tor}_1^R(M, R'/I') \to \text{Tor}_1^{R'}(M \otimes_R R', R'/I')$ is surjective. So now it suffices to show that $\text{Tor}_1^R(M, R/I) \otimes_R R' \to \text{Tor}_1^R(M, R'/I')$ is surjective. This follows from Lemma 99.12 by looking at the ring maps $R \to R/I \to R'/I'$ and the module M. $\square$

Please compare the lemma below to Lemma 101.8 (the case of a nilpotent ideal) and Lemma 128.8 (the case of finitely presented algebras).

00MP **Lemma 99.15** (Critère de platitude par fibres; Noetherian case)**.** *Let R, S, S' be Noetherian local rings and let $R \to S \to S'$ be local ring homomorphisms. Let $\mathfrak{m} \subset R$ be the maximal ideal. Let M be an S'-module. Assume*

1. *The module M is finite over S'.*
2. *The module M is not zero.*
3. *The module $M/\mathfrak{m}M$ is a flat $S/\mathfrak{m}S$-module.*
4. *The module M is a flat R-module.*

Then S is flat over R and M is a flat S-module.

Proof. Set $I = \mathfrak{m}S \subset S$. Then we see that M/IM is a flat S/I-module because of (3). Since $\mathfrak{m} \otimes_R S' \to I \otimes_S S'$ is surjective we see that also $\mathfrak{m} \otimes_R M \to I \otimes_S M$ is surjective. Consider

$$\mathfrak{m} \otimes_R M \to I \otimes_S M \to M.$$

As M is flat over R the composition is injective and so both arrows are injective. In particular $\text{Tor}_1^S(S/I, M) = 0$ see Remark 75.9. By Lemma 99.10 we conclude that M is flat over S. Note that since $M/\mathfrak{m}_{S'}M$ is not zero by Nakayama's Lemma 20.1 we see that actually M is faithfully flat over S by Lemma 39.15 (since it forces $M/\mathfrak{m}_SM \neq 0$).

Consider the exact sequence $0 \to \mathfrak{m} \to R \to \kappa \to 0$. This gives an exact sequence $0 \to \text{Tor}_1^R(\kappa, S) \to \mathfrak{m} \otimes_R S \to I \to 0$. Since M is flat over S this gives an exact sequence $0 \to \text{Tor}_1^R(\kappa, S) \otimes_S M \to \mathfrak{m} \otimes_R M \to I \otimes_S M \to 0$. By the above this implies that $\text{Tor}_1^R(\kappa, S) \otimes_S M = 0$. Since M is faithfully flat over S this implies that $\text{Tor}_1^R(\kappa, S) = 0$ and we conclude that S is flat over R by Lemma 99.7. $\square$

100. Base change and flatness

051D Some lemmas which deal with what happens with flatness when doing a base change.

00MQ **Lemma 100.1.** *Let*

$$\begin{tikzcd}
S \arrow[r] & S' \\
R \arrow[u] \arrow[r] & R' \arrow[u]
\end{tikzcd}$$

be a commutative diagram of local homomorphisms of local rings. Assume that S' is a localization of the tensor product $S \otimes_R R'$. Let M be an S-module and set $M' = S' \otimes_S M$.

1. *If M is flat over R then M' is flat over R'.*
2. *If M' is flat over R' and $R \to R'$ is flat then M is flat over R.*

In particular we have

3. *If S is flat over R then S' is flat over R'.*
4. *If $R' \to S'$ and $R \to R'$ are flat then S is flat over R.*

Proof. Proof of (1). If M is flat over R, then $M \otimes_R R'$ is flat over R' by Lemma 39.7. If $W \subset S \otimes_R R'$ is the multiplicative subset such that $W^{-1}(S \otimes_R R') = S'$ then $M' = W^{-1}(M \otimes_R R')$. Hence M' is flat over R' as the localization of a flat module, see Lemma 39.18 part (5). This proves (1) and in particular, we see that (3) holds.

Proof of (2). Suppose that M' is flat over R' and $R \to R'$ is flat. By (3) applied to the diagram reflected in the northwest diagonal we see that $S \to S'$ is flat. Thus $S \to S'$ is faithfully flat by Lemma 39.17. We are going to use the criterion of Lemma 39.5 (3) to show that M is flat. Let $I \subset R$ be an ideal. If $I \otimes_R M \to M$ has a kernel, so does $(I \otimes_R M) \otimes_S S' \to M \otimes_S S' = M'$. Note that $I \otimes_R R' = IR'$ as $R \to R'$ is flat, and that

$$(I \otimes_R M) \otimes_S S' = (I \otimes_R R') \otimes_{R'} (M \otimes_S S') = IR' \otimes_{R'} M'.$$

From flatness of M' over R' we conclude that this maps injectively into M'. This concludes the proof of (2), and hence (4) is true as well. □

Here is yet another application of the local criterion of flatness.

0GEB **Lemma 100.2.** *Consider a commutative diagram of local rings and local homomorphisms*

$$\begin{tikzcd} S \arrow[r] & S' \\ R \arrow[u] \arrow[r] & R' \arrow[u] \end{tikzcd}$$

Let M be a finite S-module. Assume that

1. *the horizontal arrows are flat ring maps*
2. *M is flat over R,*
3. *$\mathfrak m_R R' = \mathfrak m_{R'}$,*
4. *R' and S' are Noetherian.*

Then $M' = M \otimes_S S'$ is flat over R'.

Proof. Since $\mathfrak m_R \subset R$ and $R \to R'$ is flat, we get $\mathfrak m_R \otimes_R R' = \mathfrak m_R R' = \mathfrak m_{R'}$ by assumption (3). Observe that M' is a finite S'-module which is flat over R by Lemma 39.9. Thus $\mathfrak m_R \otimes_R M' \to M'$ is injective. Then we get

$$\mathfrak m_R \otimes_R M' = \mathfrak m_R \otimes_R R' \otimes_{R'} M' = \mathfrak m_{R'} \otimes_{R'} M'$$

Thus $\mathfrak m_{R'} \otimes_{R'} M' \to M'$ is injective. This shows that $\text{Tor}_1^{R'}(\kappa_{R'}, M') = 0$ (Remark 75.9). Thus M' is flat over R' by Lemma 99.7. □

101. Flatness criteria over Artinian rings

051E We discuss some flatness criteria for modules over Artinian rings. Note that an Artinian local ring has a nilpotent maximal ideal so that the following two lemmas apply to Artinian local rings.

051F **Lemma 101.1.** *Let $(R, \mathfrak m)$ be a local ring with nilpotent maximal ideal $\mathfrak m$. Let M be a flat R-module. If A is a set and $x_\alpha \in M$, $\alpha \in A$ is a collection of elements of M, then the following are equivalent:*

1. *$\{\overline{x}_\alpha\}_{\alpha \in A}$ forms a basis for the vector space $M/\mathfrak m M$ over $R/\mathfrak m$, and*
2. *$\{x_\alpha\}_{\alpha \in A}$ forms a basis for M over R.*

Proof. The implication (2) $\Rightarrow$ (1) is immediate. Assume (1). By Nakayama's Lemma 20.1 the elements x_α generate M. Then one gets a short exact sequence

$$0 \to K \to \bigoplus\nolimits_{\alpha \in A} R \to M \to 0$$

Tensoring with $R/\mathfrak m$ and using Lemma 39.12 we obtain $K/\mathfrak m K = 0$. By Nakayama's Lemma 20.1 we conclude $K = 0$. □

051G **Lemma 101.2.** *Let R be a local ring with nilpotent maximal ideal. Let M be an R-module. The following are equivalent*

(1) *M is flat over R,*
(2) *M is a free R-module, and*
(3) *M is a projective R-module.*

Proof. Since any projective module is flat (as a direct summand of a free module) and every free module is projective, it suffices to prove that a flat module is free. Let M be a flat module. Let A be a set and let $x_\alpha \in M$, $\alpha \in A$ be elements such that $\overline{x_\alpha} \in M/\mathfrak{m}M$ forms a basis over the residue field of R. By Lemma 101.1 the x_α are a basis for M over R and we win. $\square$

051H **Lemma 101.3.** *Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. Let A be a set and let $x_\alpha \in M$, $\alpha \in A$ be a collection of elements of M. Assume*

(1) *I is nilpotent,*
(2) *$\{\overline{x}_\alpha\}_{\alpha \in A}$ forms a basis for M/IM over R/I, and*
(3) *$Tor_1^R(R/I, M) = 0$.*

Then M is free on $\{x_\alpha\}_{\alpha \in A}$ over R.

Proof. Let R, I, M, $\{x_\alpha\}_{\alpha \in A}$ be as in the lemma and satisfy assumptions (1), (2), and (3). By Nakayama's Lemma 20.1 the elements x_α generate M over R. The assumption $\text{Tor}_1^R(R/I, M) = 0$ implies that we have a short exact sequence

$$0 \to I \otimes_R M \to M \to M/IM \to 0.$$

Let $\sum f_\alpha x_\alpha = 0$ be a relation in M. By choice of x_α we see that $f_\alpha \in I$. Hence we conclude that $\sum f_\alpha \otimes x_\alpha = 0$ in $I \otimes_R M$. The map $I \otimes_R M \to I/I^2 \otimes_{R/I} M/IM$ and the fact that $\{x_\alpha\}_{\alpha \in A}$ forms a basis for M/IM implies that $f_\alpha \in I^2$! Hence we conclude that there are no relations among the images of the x_α in M/I^2M. In other words, we see that M/I^2M is free with basis the images of the x_α. Using the map $I \otimes_R M \to I/I^3 \otimes_{R/I^2} M/I^2M$ we then conclude that $f_\alpha \in I^3$! And so on. Since $I^n = 0$ for some n by assumption (1) we win. $\square$

051I **Lemma 101.4.** *Let $\varphi : R \to R'$ be a ring map. Let $I \subset R$ be an ideal. Let M be an R-module. Assume*

(1) *M/IM is flat over R/I, and*
(2) *$R' \otimes_R M$ is flat over R'.*

Set $I_2 = \varphi^{-1}(\varphi(I^2)R')$. Then M/I_2M is flat over R/I_2.

Proof. We may replace R, M, and R' by R/I_2, M/I_2M, and $R'/\varphi(I)^2R'$. Then $I^2 = 0$ and φ is injective. By Lemma 99.8 and the fact that $I^2 = 0$ it suffices to prove that $\text{Tor}_1^R(R/I, M) = K = \text{Ker}(I \otimes_R M \to M)$ is zero. Set $M' = M \otimes_R R'$ and $I' = IR'$. By assumption the map $I' \otimes_{R'} M' \to M'$ is injective. Hence K maps to zero in

$$I' \otimes_{R'} M' = I' \otimes_R M = I' \otimes_{R/I} M/IM.$$

Then $I \to I'$ is an injective map of R/I-modules. Since M/IM is flat over R/I the map

$$I \otimes_{R/I} M/IM \longrightarrow I' \otimes_{R/I} M/IM$$

is injective. This implies that K is zero in $I \otimes_R M = I \otimes_{R/I} M/IM$ as desired. $\square$

051J **Lemma 101.5.** *Let $\varphi : R \to R'$ be a ring map. Let $I \subset R$ be an ideal. Let M be an R-module. Assume*

1. *I is nilpotent,*
2. *$R \to R'$ is injective,*
3. *M/IM is flat over R/I, and*
4. *$R' \otimes_R M$ is flat over R'.*

Then M is flat over R.

Proof. Define inductively $I_1 = I$ and $I_{n+1} = \varphi^{-1}(\varphi(I_n)^2R')$ for $n \geq 1$. Note that by Lemma 101.4 we find that M/I_nM is flat over R/I_n for each $n \geq 1$. It is clear that $\varphi(I_n) \subset \varphi(I)^{2^n}R'$. Since I is nilpotent we see that $\varphi(I_n) = 0$ for some n. As φ is injective we conclude that $I_n = 0$ for some n and we win. $\square$

Here is the local Artinian version of the local criterion for flatness.

051K **Lemma 101.6.** *Let R be an Artinian local ring. Let M be an R-module. Let $I \subset R$ be a proper ideal. The following are equivalent*

1. *M is flat over R, and*
2. *M/IM is flat over R/I and $Tor_1^R(R/I, M) = 0$.*

Proof. The implication (1) $\Rightarrow$ (2) follows immediately from the definitions. Assume M/IM is flat over R/I and $\text{Tor}_1^R(R/I, M) = 0$. By Lemma 101.2 this implies that M/IM is free over R/I. Pick a set A and elements $x_\alpha \in M$ such that the images in M/IM form a basis. By Lemma 101.3 we conclude that M is free and in particular flat. $\square$

It turns out that flatness descends along injective homomorphism whose source is an Artinian ring.

051L **Lemma 101.7.** *Let $R \to S$ be a ring map. Let M be an R-module. Assume*

1. *R is Artinian*
2. *$R \to S$ is injective, and*
3. *$M \otimes_R S$ is a flat S-module.*

Then M is a flat R-module.

Proof. First proof: Let $I \subset R$ be the Jacobson radical of R. Then I is nilpotent and M/IM is flat over R/I as R/I is a product of fields, see Section 53. Hence M is flat by an application of Lemma 101.5.

Second proof: By Lemma 53.6 we may write $R = \prod R_i$ as a finite product of local Artinian rings. This induces similar product decompositions for both R and S. Hence we reduce to the case where R is local Artinian (details omitted).

Assume that $R \to S$, M are as in the lemma satisfying (1), (2), and (3) and in addition that R is local with maximal ideal $\mathfrak{m}$. Let A be a set and $x_\alpha \in A$ be elements such that $\overline{x}_\alpha$ forms a basis for $M/\mathfrak{m}M$ over $R/\mathfrak{m}$. By Nakayama's Lemma 20.1 we see that the elements x_α generate M as an R-module. Set $N = S \otimes_R M$ and $I = \mathfrak{m}S$. Then $\{1 \otimes x_\alpha\}_{\alpha \in A}$ is a family of elements of N which form a basis for N/IN. Moreover, since N is flat over S we have $\text{Tor}_1^S(S/I, N) = 0$. Thus we conclude from Lemma 101.3 that N is free on $\{1 \otimes x_\alpha\}_{\alpha \in A}$. The injectivity of $R \to S$ then guarantees that there cannot be a nontrivial relation among the x_α with coefficients in R. $\square$

Please compare the lemma below to Lemma 99.15 (the case of Noetherian local rings), Lemma 128.8 (the case of finitely presented algebras), and Lemma 128.10 (the case of locally nilpotent ideals).

06A5 **Lemma 101.8** (Critère de platitude par fibres: Nilpotent case)**.** *Let*

$$\begin{tikzcd} S \arrow[rr] & & S' \\ & R \arrow[ul] \arrow[ur] & \end{tikzcd}$$

be a commutative diagram in the category of rings. Let $I \subset R$ be a nilpotent ideal and M an S'-module. Assume

(1) *The module M/IM is a flat S/IS-module.*
(2) *The module M is a flat R-module.*

Then M is a flat S-module and $S_\mathfrak{q}$ is flat over R for every $\mathfrak{q} \subset S$ such that $M \otimes_S \kappa(\mathfrak{q})$ is nonzero.

Proof. As M is flat over R tensoring with the short exact sequence $0 \to I \to R \to R/I \to 0$ gives a short exact sequence

$$0 \to I \otimes_R M \to M \to M/IM \to 0.$$

Note that $I \otimes_R M \to IS \otimes_S M$ is surjective. Combined with the above this means both maps in

$$I \otimes_R M \to IS \otimes_S M \to M$$

are injective. Hence $\text{Tor}_1^S(IS, M) = 0$ (see Remark 75.9) and we conclude that M is a flat S-module by Lemma 99.8. To finish we need to show that $S_\mathfrak{q}$ is flat over R for any prime $\mathfrak{q} \subset S$ such that $M \otimes_S \kappa(\mathfrak{q})$ is nonzero. This follows from Lemma 39.15 and 39.10. $\square$

102. What makes a complex exact?

00MR Some of this material can be found in the paper [BE73] by Buchsbaum and Eisenbud.

00MS **Situation 102.1.** Here R is a ring, and we have a complex

$$0 \to R^{n_e} \xrightarrow{\varphi_e} R^{n_{e-1}} \xrightarrow{\varphi_{e-1}} \ldots \xrightarrow{\varphi_{i+1}} R^{n_i} \xrightarrow{\varphi_i} R^{n_{i-1}} \xrightarrow{\varphi_{i-1}} \ldots \xrightarrow{\varphi_1} R^{n_0}$$

In other words we require $\varphi_i \circ \varphi_{i+1} = 0$ for $i = 1, \ldots, e-1$.

00MT **Lemma 102.2.** *Suppose R is a ring. Let*

$$\ldots \xrightarrow{\varphi_{i+1}} R^{n_i} \xrightarrow{\varphi_i} R^{n_{i-1}} \xrightarrow{\varphi_{i-1}} \ldots$$

be a complex of finite free R-modules. Suppose that for some i some matrix coefficient of the map φ_i is invertible. Then the displayed complex is isomorphic to the direct sum of a complex

$$\ldots \to R^{n_{i+2}} \xrightarrow{\varphi_{i+2}} R^{n_{i+1}} \to R^{n_i - 1} \to R^{n_{i-1}-1} \to R^{n_{i-2}} \xrightarrow{\varphi_{i-2}} R^{n_{i-3}} \to \ldots$$

and the complex $\ldots \to 0 \to R \to R \to 0 \to \ldots$ where the map $R \to R$ is the identity map.

Proof. The assumption means, after a change of basis of R^{n_i} and $R^{n_{i-1}}$ that the first basis vector of R^{n_i} is mapped via φ_i to the first basis vector of $R^{n_{i-1}}$. Let e_j denote the jth basis vector of R^{n_i} and f_k the kth basis vector of $R^{n_{i-1}}$. Write $\varphi_i(e_j) = \sum a_{jk} f_k$. So $a_{1k} = 0$ unless $k = 1$ and $a_{11} = 1$. Change basis on R^{n_i} again by setting $e'_j = e_j - a_{j1}e_1$ for $j > 1$. After this change of coordinates we have $a_{j1} = 0$ for $j > 1$. Note the image of $R^{n_{i+1}} \to R^{n_i}$ is contained in the subspace spanned by e_j, $j > 1$. Note also that $R^{n_{i-1}} \to R^{n_{i-2}}$ has to annihilate f_1 since it is in the image. These conditions and the shape of the matrix (a_{jk}) for φ_i imply the lemma. □

In Situation 102.1 we say a complex of the form

$$0 \to \ldots \to 0 \to R \xrightarrow{1} R \to 0 \to \ldots \to 0$$

or of the form

$$0 \to \ldots \to 0 \to R$$

is *trivial*. More precisely, we say $0 \to R^{n_e} \to R^{n_{e-1}} \to \ldots \to R^{n_0}$ is trivial if either there exists an $e \geq i \geq 1$ with $n_i = n_{i-1} = 1$, $\varphi_i = \text{id}_R$, and $n_j = 0$ for $j \not\in \{i, i-1\}$ or $n_0 = 1$ and $n_i = 0$ for $i > 0$. The lemma above clearly says that any finite complex of finite free modules over a local ring is up to direct sums with trivial complexes the same as a complex all of whose maps have all matrix coefficients in the maximal ideal.

00MY **Lemma 102.3.** *In Situation 102.1. Suppose R is a local Noetherian ring with maximal ideal $\mathfrak{m}$. Assume $\mathfrak{m} \in Ass(R)$, in other words R has depth 0. Suppose that $0 \to R^{n_e} \to R^{n_{e-1}} \to \ldots \to R^{n_0}$ is exact at $R^{n_e}, \ldots, R^{n_1}$. Then the complex is isomorphic to a direct sum of trivial complexes.*

Proof. Pick $x \in R$, $x \neq 0$, with $\mathfrak{m}x = 0$. Let i be the biggest index such that $n_i > 0$. If $i = 0$, then the statement is true. If $i > 0$ denote f_1 the first basis vector of R^{n_i}. Since xf_1 is not mapped to zero by exactness of the complex we deduce that some matrix coefficient of the map $R^{n_i} \to R^{n_{i-1}}$ is not in $\mathfrak{m}$. Lemma 102.2 then allows us to decrease $n_e + \ldots + n_1$. Induction finishes the proof. □

00MU **Lemma 102.4.** *In Situation 102.1. Let R be a Artinian local ring. Suppose that $0 \to R^{n_e} \to R^{n_{e-1}} \to \ldots \to R^{n_0}$ is exact at $R^{n_e}, \ldots, R^{n_1}$. Then the complex is isomorphic to a direct sum of trivial complexes.*

Proof. This is a special case of Lemma 102.3 because an Artinian local ring has depth 0. □

Below we define the rank of a map of finite free modules. This is just one possible definition of rank. It is just the definition that works in this section; there are others that may be more convenient in other settings.

00MV **Definition 102.5.** Let R be a ring. Suppose that $\varphi : R^m \to R^n$ is a map of finite free modules.

(1) The *rank* of φ is the maximal r such that $\wedge^r \varphi : \wedge^r R^m \to \wedge^r R^n$ is nonzero.
(2) We let $I(\varphi) \subset R$ be the ideal generated by the $r \times r$ minors of the matrix of φ, where r is the rank as defined above.

The rank of $\varphi : R^m \to R^n$ is 0 if and only if $\varphi = 0$ and in this case $I(\varphi) = R$.

00MW **Lemma 102.6.** *In Situation 102.1, suppose the complex is isomorphic to a direct sum of trivial complexes. Then we have*

1. *the maps φ_i have rank $r_i = n_i - n_{i+1} + \ldots + (-1)^{e-i-1}n_{e-1} + (-1)^{e-i}n_e$,*
2. *for all i, $1 \leq i \leq e-1$ we have $rank(\varphi_{i+1}) + rank(\varphi_i) = n_i$,*
3. *each $I(\varphi_i) = R$.*

Proof. We may assume the complex is the direct sum of trivial complexes. Then for each i we can split the standard basis elements of R^{n_i} into those that map to a basis element of $R^{n_{i-1}}$ and those that are mapped to zero (and these are mapped onto by basis elements of $R^{n_{i+1}}$ if $i > 0$). Using descending induction starting with $i = e$ it is easy to prove that there are r_{i+1}-basis elements of R^{n_i} which are mapped to zero and r_i which are mapped to basis elements of $R^{n_{i-1}}$. From this the result follows. □

00MZ **Lemma 102.7.** *In Situation 102.1. Suppose R is a local ring with maximal ideal $\mathfrak{m}$. Suppose that $0 \to R^{n_e} \to R^{n_{e-1}} \to \ldots \to R^{n_0}$ is exact at $R^{n_e}, \ldots, R^{n_1}$. Let $x \in \mathfrak{m}$ be a nonzerodivisor. The complex $0 \to (R/xR)^{n_e} \to \ldots \to (R/xR)^{n_1}$ is exact at $(R/xR)^{n_e}, \ldots, (R/xR)^{n_2}$.*

Proof. Denote $F_\bullet$ the complex with terms $F_i = R^{n_i}$ and differential given by φ_i. Then we have a short exact sequence of complexes

$$0 \to F_\bullet \xrightarrow{x} F_\bullet \to F_\bullet/xF_\bullet \to 0$$

Applying the snake lemma we get a long exact sequence

$$H_i(F_\bullet) \xrightarrow{x} H_i(F_\bullet) \to H_i(F_\bullet/xF_\bullet) \to H_{i-1}(F_\bullet) \xrightarrow{x} H_{i-1}(F_\bullet)$$

The lemma follows. □

00N0 **Lemma 102.8** (Acyclicity lemma)**.** *Let R be a local Noetherian ring. Let $0 \to M_e \to M_{e-1} \to \ldots \to M_0$ be a complex of finite R-modules. Assume $depth(M_i) \geq i$. Let i be the largest index such that the complex is not exact at M_i. If $i > 0$ then $\mathrm{Ker}(M_i \to M_{i-1})/\mathrm{Im}(M_{i+1} \to M_i)$ has depth ≥ 1.* [PS73, Lemma 1.8]

Proof. Let $H = \mathrm{Ker}(M_i \to M_{i-1})/\mathrm{Im}(M_{i+1} \to M_i)$ be the cohomology group in question. We may break the complex into short exact sequences $0 \to M_e \to M_{e-1} \to K_{e-2} \to 0$, $0 \to K_j \to M_j \to K_{j-1} \to 0$, for $i + 2 \leq j \leq e - 2$, $0 \to K_{i+1} \to M_{i+1} \to B_i \to 0$, $0 \to K_i \to M_i \to M_{i-1}$, and $0 \to B_i \to K_i \to H \to 0$. We proceed up through these complexes to prove the statements about depths, repeatedly using Lemma 72.6. First of all, since $\mathrm{depth}(M_e) \geq e$, and $\mathrm{depth}(M_{e-1}) \geq e - 1$ we deduce that $\mathrm{depth}(K_{e-2}) \geq e - 1$. At this point the sequences $0 \to K_j \to M_j \to K_{j-1} \to 0$ for $i + 2 \leq j \leq e - 2$ imply similarly that $\mathrm{depth}(K_{j-1}) \geq j$ for $i + 2 \leq j \leq e - 2$. The sequence $0 \to K_{i+1} \to M_{i+1} \to B_i \to 0$ then shows that $\mathrm{depth}(B_i) \geq i + 1$. The sequence $0 \to K_i \to M_i \to M_{i-1}$ shows that $\mathrm{depth}(K_i) \geq 1$ since M_i has depth $\geq i \geq 1$ by assumption. The sequence $0 \to B_i \to K_i \to H \to 0$ then implies the result. □

00N1 **Proposition 102.9.** *In Situation 102.1, suppose R is a local Noetherian ring. The following are equivalent* [BE73, Corollary 1]

1. *$0 \to R^{n_e} \to R^{n_{e-1}} \to \ldots \to R^{n_0}$ is exact at $R^{n_e}, \ldots, R^{n_1}$, and*
2. *for all i, $1 \leq i \leq e$ the following two conditions are satisfied:*
   (a) *$rank(\varphi_i) = r_i$ where $r_i = n_i - n_{i+1} + \ldots + (-1)^{e-i-1}n_{e-1} + (-1)^{e-i}n_e$,*
   (b) *$I(\varphi_i) = R$, or $I(\varphi_i)$ contains a regular sequence of length i.*

Proof. If for some i some matrix coefficient of φ_i is not in $\mathfrak m$, then we apply Lemma 102.2. It is easy to see that the proposition for a complex and for the same complex with a trivial complex added to it are equivalent. Thus we may assume that all matrix entries of each φ_i are elements of the maximal ideal. We may also assume that $e \geq 1$.

Assume the complex is exact at $R^{n_e}, \ldots, R^{n_1}$. Let $\mathfrak q \in \text{Ass}(R)$. Note that the ring $R_\mathfrak q$ has depth 0 and that the complex remains exact after localization at $\mathfrak q$. We apply Lemmas 102.3 and 102.6 to the localized complex over $R_\mathfrak q$. We conclude that $\varphi_{i,\mathfrak q}$ has rank r_i for all i. Since $R \to \bigoplus_{\mathfrak q \in \text{Ass}(R)} R_\mathfrak q$ is injective (Lemma 63.19), we conclude that φ_i has rank r_i over R by the definition of rank as given in Definition 102.5. Therefore we see that $I(\varphi_i)_\mathfrak q = I(\varphi_{i,\mathfrak q})$ as the ranks do not change. Since all of the ideals $I(\varphi_i)_\mathfrak q$, $e \geq i \geq 1$ are equal to $R_\mathfrak q$ (by the lemmas referenced above) we conclude none of the ideals $I(\varphi_i)$ is contained in $\mathfrak q$. This implies that $I(\varphi_e)I(\varphi_{e-1}) \ldots I(\varphi_1)$ is not contained in any of the associated primes of R. By Lemma 15.2 we may choose $x \in I(\varphi_e)I(\varphi_{e-1}) \ldots I(\varphi_1)$, $x \not\in \mathfrak q$ for all $\mathfrak q \in \text{Ass}(R)$. Observe that x is a nonzerodivisor (Lemma 63.9). According to Lemma 102.7 the complex $0 \to (R/xR)^{n_e} \to \ldots \to (R/xR)^{n_1}$ is exact at $(R/xR)^{n_e}, \ldots, (R/xR)^{n_2}$. By induction on e all the ideals $I(\varphi_i)/xR$ have a regular sequence of length $i-1$. This proves that $I(\varphi_i)$ contains a regular sequence of length i.

Assume (2)(a) and (2)(b) hold. We claim that for any prime $\mathfrak p \subset R$ conditions (2)(a) and (2)(b) hold for the complex $0 \to R_\mathfrak p^{n_e} \to R_\mathfrak p^{n_{e-1}} \to \ldots \to R_\mathfrak p^{n_0}$ with maps $\varphi_{i,\mathfrak p}$ over $R_\mathfrak p$. Namely, since $I(\varphi_i)$ contains a nonzero divisor, the image of $I(\varphi_i)$ in $R_\mathfrak p$ is nonzero. This implies that the rank of $\varphi_{i,\mathfrak p}$ is the same as the rank of φ_i: the rank as defined above of a matrix φ over a ring R can only drop when passing to an R-algebra R' and this happens if and only if $I(\varphi)$ maps to zero in R'. Thus (2)(a) holds. Having said this we know that $I(\varphi_{i,\mathfrak p}) = I(\varphi_i)_\mathfrak p$ and we see that (2)(b) is preserved under localization as well. By induction on the dimension of R we may assume the complex is exact when localized at any nonmaximal prime $\mathfrak p$ of R. Thus $\text{Ker}(\varphi_i)/\text{Im}(\varphi_{i+1})$ has support contained in $\{\mathfrak m\}$ and hence if nonzero has depth 0. As $I(\varphi_i) \subset \mathfrak m$ for all i because of what was said in the first paragraph of the proof, we see that (2)(b) implies $\text{depth}(R) \geq e$. By Lemma 102.8 we see that the complex is exact at $R^{n_e}, \ldots, R^{n_1}$ concluding the proof. $\square$

0GLM **Remark 102.10.** If in Proposition 102.9 the equivalent conditions (1) and (2) are satisfied, then there exists a j such that $I(\varphi_i) = R$ if and only if $i \geq j$. As in the proof of the proposition, it suffices to see this when all the matrices have coefficients in the maximal ideal $\mathfrak m$ of R. In this case we see that $I(\varphi_j) = R$ if and only if $\varphi_j = 0$. But if $\varphi_j = 0$, then we get arbitrarily long exact complexes $0 \to R^{n_e} \to R^{n_{e-1}} \to \ldots \to R^{n_j} \to 0 \to 0 \to \ldots \to 0$ and hence by the proposition we see that $I(\varphi_i)$ for $i > j$ has to be R (since otherwise it is a proper ideal of a Noetherian local ring containing arbitrary long regular sequences which is impossible).

103. Cohen-Macaulay modules

00N2 Here we show that Cohen-Macaulay modules have good properties. We postpone using Ext groups to establish the connection with duality and so on.

00N3 **Definition 103.1.** Let R be a Noetherian local ring. Let M be a finite R-module. We say M is *Cohen-Macaulay* if $\dim(\text{Supp}(M)) = \text{depth}(M)$.

A first goal will be to establish Proposition 103.4. We do this by a (perhaps nonstandard) sequence of elementary lemmas involving almost none of the earlier results on depth. Let us introduce some notation.

Let R be a local Noetherian ring. Let M be a Cohen-Macaulay module, and let $f_1, \ldots, f_d$ be an M-regular sequence with $d = \dim(\text{Supp}(M))$. We say that $g \in \mathfrak{m}$ is *good with respect to* $(M, f_1, \ldots, f_d)$ if for all $i = 0, 1, \ldots, d-1$ we have $\dim(\text{Supp}(M) \cap V(g, f_1, \ldots, f_i)) = d - i - 1$. This is equivalent to the condition that $\dim(\text{Supp}(M/(f_1, \ldots, f_i)M) \cap V(g)) = d - i - 1$ for $i = 0, 1, \ldots, d-1$.

00N4 **Lemma 103.2.** *Notation and assumptions as above. If g is good with respect to $(M, f_1, \ldots, f_d)$, then (a) g is a nonzerodivisor on M, and (b) M/gM is Cohen-Macaulay with maximal regular sequence $f_1, \ldots, f_{d-1}$.*

Proof. We prove the lemma by induction on d. If $d = 0$, then M is finite and there is no case to which the lemma applies. If $d = 1$, then we have to show that $g : M \to M$ is injective. The kernel K has support $\{\mathfrak{m}\}$ because by assumption $\dim \text{Supp}(M) \cap V(g) = 0$. Hence K has finite length. Hence $f_1 : K \to K$ injective implies the length of the image is the length of K, and hence $f_1 K = K$, which by Nakayama's Lemma 20.1 implies $K = 0$. Also, $\dim \text{Supp}(M/gM) = 0$ and so M/gM is Cohen-Macaulay of depth 0.

Assume $d > 1$. Observe that g is good for $(M/f_1M, f_2, \ldots, f_d)$, as is easily seen from the definition. By induction, we have that (a) g is a nonzerodivisor on M/f_1M and (b) $M/(g, f_1)M$ is Cohen-Macaulay with maximal regular sequence $f_2, \ldots, f_{d-1}$. By Lemma 68.4 we see that g, f_1 is an M-regular sequence. Hence g is a nonzerodivisor on M and $f_1, \ldots, f_{d-1}$ is an M/gM-regular sequence. $\square$

00N5 **Lemma 103.3.** *Let R be a Noetherian local ring. Let M be a Cohen-Macaulay module over R. Suppose $g \in \mathfrak{m}$ is such that $\dim(\mathit{Supp}(M) \cap V(g)) = \dim(\mathit{Supp}(M)) - 1$. Then (a) g is a nonzerodivisor on M, and (b) M/gM is Cohen-Macaulay of depth one less.*

Proof. Choose a M-regular sequence $f_1, \ldots, f_d$ with $d = \dim(\text{Supp}(M))$. If g is good with respect to $(M, f_1, \ldots, f_d)$ we win by Lemma 103.2. In particular the lemma holds if $d = 1$. (The case $d = 0$ does not occur.) Assume $d > 1$. Choose an element $h \in R$ such that (i) h is good with respect to $(M, f_1, \ldots, f_d)$, and (ii) $\dim(\text{Supp}(M) \cap V(h, g)) = d - 2$. To see h exists, let $\{\mathfrak{q}_j\}$ be the (finite) set of minimal primes of the closed sets $\text{Supp}(M)$, $\text{Supp}(M) \cap V(f_1, \ldots, f_i)$, $i = 1, \ldots, d-1$, and $\text{Supp}(M) \cap V(g)$. None of these $\mathfrak{q}_j$ is equal to $\mathfrak{m}$ and hence we may find $h \in \mathfrak{m}$, $h \notin \mathfrak{q}_j$ by Lemma 15.2. It is clear that h satisfies (i) and (ii). From Lemma 103.2 we conclude that M/hM is Cohen-Macaulay. By (ii) we see that the pair $(M/hM, g)$ satisfies the induction hypothesis. Hence $M/(h, g)M$ is Cohen-Macaulay and $g : M/hM \to M/hM$ is injective. By Lemma 68.4 we see that $g : M \to M$ and $h : M/gM \to M/gM$ are injective. Combined with the fact that $M/(g, h)M$ is Cohen-Macaulay this finishes the proof. $\square$

00N6 **Proposition 103.4.** *Let R be a Noetherian local ring, with maximal ideal $\mathfrak{m}$. Let M be a Cohen-Macaulay module over R whose support has dimension d. Suppose that $g_1, \ldots, g_c$ are elements of $\mathfrak{m}$ such that $\dim(\mathit{Supp}(M/(g_1, \ldots, g_c)M)) = d - c$. Then $g_1, \ldots, g_c$ is an M-regular sequence, and can be extended to a maximal M-regular sequence.*

Proof. Let $Z = \text{Supp}(M) \subset \text{Spec}(R)$. By Lemma 60.13 in the chain $Z \supset Z \cap V(g_1) \supset \ldots \supset Z \cap V(g_1, \ldots, g_c)$ each step decreases the dimension at most by 1. Hence by assumption each step decreases the dimension by exactly 1 each time. Thus we may successively apply Lemma 103.3 to the modules $M/(g_1, \ldots, g_i)$ and the element g_{i+1}.

To extend $g_1, \ldots, g_c$ by one element if $c < d$ we simply choose an element $g_{c+1} \in \mathfrak{m}$ which is not in any of the finitely many minimal primes of $Z \cap V(g_1, \ldots, g_c)$, using Lemma 15.2. ☐

Having proved Proposition 103.4 we continue the development of standard theory.

0C6G **Lemma 103.5.** *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let M be a finite R-module. Let $x \in \mathfrak{m}$ be a nonzerodivisor on M. Then M is Cohen-Macaulay if and only if M/xM is Cohen-Macaulay.*

Proof. By Lemma 72.7 we have $\text{depth}(M/xM) = \text{depth}(M) - 1$. By Lemma 63.10 we have $\dim(\text{Supp}(M/xM)) = \dim(\text{Supp}(M)) - 1$. ☐

0AAD **Lemma 103.6.** *Let $R \to S$ be a surjective homomorphism of Noetherian local rings. Let N be a finite S-module. Then N is Cohen-Macaulay as an S-module if and only if N is Cohen-Macaulay as an R-module.*

Proof. Omitted. ☐

0BUS **Lemma 103.7.** *Let R be a Noetherian local ring. Let M be a finite Cohen-Macaulay R-module. If $\mathfrak{p} \in Ass(M)$, then $\dim(R/\mathfrak{p}) = \dim(Supp(M))$ and $\mathfrak{p}$ is a minimal prime in the support of M. In particular, M has no embedded associated primes.* [DG67, Chapter 0, Proposition 16.5.4]

Proof. By Lemma 72.9 we have $\text{depth}(M) \leq \dim(R/\mathfrak{p})$. Of course $\dim(R/\mathfrak{p}) \leq \dim(\text{Supp}(M))$ as $\mathfrak{p} \in \text{Supp}(M)$ (Lemma 63.2). Thus we have equality in both inequalities as M is Cohen-Macaulay. Then $\mathfrak{p}$ must be minimal in $\text{Supp}(M)$ otherwise we would have $\dim(R/\mathfrak{p}) < \dim(\text{Supp}(M))$. Finally, minimal primes in the support of M are equal to the minimal elements of $\text{Ass}(M)$ (Proposition 63.6) hence M has no embedded associated primes (Definition 67.1). ☐

00NF **Definition 103.8.** Let R be a Noetherian local ring. A finite module M over R is called a *maximal Cohen-Macaulay* module if $\text{depth}(M) = \dim(R)$.

In other words, a maximal Cohen-Macaulay module over a Noetherian local ring is a finite module with the largest possible depth over that ring. Equivalently, a maximal Cohen-Macaulay module over a Noetherian local ring R is a Cohen-Macaulay module of dimension equal to the dimension of the ring. In particular, if M is a Cohen-Macaulay R-module with $\text{Spec}(R) = \text{Supp}(M)$, then M is maximal Cohen-Macaulay. Thus the following two lemmas are on maximal Cohen-Macaulay modules.

0AAE **Lemma 103.9.** *Let R be a Noetherian local ring. Assume there exists a Cohen-Macaulay module M with $\text{Spec}(R) = Supp(M)$. Then any maximal chain of prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n$ has length $n = \dim(R)$.*

Proof. We will prove this by induction on $\dim(R)$. If $\dim(R) = 0$, then the statement is clear. Assume $\dim(R) > 0$. Then $n > 0$. Choose an element $x \in \mathfrak{p}_1$, with x not in any of the minimal primes of R, and in particular $x \notin \mathfrak{p}_0$. (See Lemma

15.2.) Then $\dim(R/xR) = \dim(R) - 1$ by Lemma 60.13. The module M/xM is Cohen-Macaulay over R/xR by Proposition 103.4 and Lemma 103.6. The support of M/xM is $\text{Spec}(R/xR)$ by Lemma 40.9. After replacing x by x^n for some n, we may assume that $\mathfrak p_1$ is an associated prime of M/xM, see Lemma 72.8. By Lemma 103.7 we conclude that $\mathfrak p_1/(x)$ is a minimal prime of R/xR. It follows that the chain $\mathfrak p_1/(x) \subset \ldots \subset \mathfrak p_n/(x)$ is a maximal chain of primes in R/xR. By induction we find that this chain has length $\dim(R/xR) = \dim(R) - 1$ as desired. $\square$

0AAF **Lemma 103.10.** *Suppose R is a Noetherian local ring. Assume there exists a Cohen-Macaulay module M with $\text{Spec}(R) = Supp(M)$. Then for a prime $\mathfrak p \subset R$ we have*

$$\dim(R) = \dim(R_{\mathfrak p}) + \dim(R/\mathfrak p).$$

Proof. Follows immediately from Lemma 103.9. $\square$

0AAG **Lemma 103.11.** *Suppose R is a Noetherian local ring. Let M be a Cohen-Macaulay module over R. For any prime $\mathfrak p \subset R$ the module $M_{\mathfrak p}$ is Cohen-Macaulay over $R_{\mathfrak p}$.*

Proof. We may and do assume $\mathfrak p \neq \mathfrak m$ and M not zero. Choose a maximal chain of primes $\mathfrak p = \mathfrak p_c \subset \mathfrak p_{c-1} \subset \ldots \subset \mathfrak p_1 \subset \mathfrak m$. If we prove the result for $M_{\mathfrak p_1}$ over $R_{\mathfrak p_1}$, then the lemma will follow by induction on c. Thus we may assume that there is no prime strictly between $\mathfrak p$ and $\mathfrak m$. Note that $\dim(\text{Supp}(M_{\mathfrak p})) \leq \dim(\text{Supp}(M)) - 1$ because any chain of primes in the support of $M_{\mathfrak p}$ can be extended by one more prime (namely $\mathfrak m$) in the support of M. On the other hand, we have $\text{depth}(M_{\mathfrak p}) \geq \text{depth}(M) - \dim(R/\mathfrak p) = \text{depth}(M) - 1$ by Lemma 72.10 and our choice of $\mathfrak p$. Thus $\text{depth}(M_{\mathfrak p}) \geq \dim(\text{Supp}(M_{\mathfrak p}))$ as desired (the other inequality is Lemma 72.3). $\square$

0AAH **Definition 103.12.** Let R be a Noetherian ring. Let M be a finite R-module. We say M is *Cohen-Macaulay* if $M_{\mathfrak p}$ is a Cohen-Macaulay module over $R_{\mathfrak p}$ for all primes $\mathfrak p$ of R.

By Lemma 103.11 it suffices to check this in the maximal ideals of R.

0AAI **Lemma 103.13.** *Let R be a Noetherian ring. Let M be a Cohen-Macaulay module over R. Then $M \otimes_R R[x_1, \ldots, x_n]$ is a Cohen-Macaulay module over $R[x_1, \ldots, x_n]$.*

Proof. By induction on the number of variables it suffices to prove this for $M[x] = M \otimes_R R[x]$ over $R[x]$. Let $\mathfrak m \subset R[x]$ be a maximal ideal, and let $\mathfrak p = R \cap \mathfrak m$. Let $f_1, \ldots, f_d$ be a $M_{\mathfrak p}$-regular sequence in the maximal ideal of $R_{\mathfrak p}$ of length $d = \dim(\text{Supp}(M_{\mathfrak p}))$. Note that since $R[x]$ is flat over R the localization $R[x]_{\mathfrak m}$ is flat over $R_{\mathfrak p}$. Hence, by Lemma 68.5, the sequence $f_1, \ldots, f_d$ is a $M[x]_{\mathfrak m}$-regular sequence of length d in $R[x]_{\mathfrak m}$. The quotient

$$Q = M[x]_{\mathfrak m}/(f_1, \ldots, f_d)M[x]_{\mathfrak m} = M_{\mathfrak p}/(f_1, \ldots, f_d)M_{\mathfrak p} \otimes_{R_{\mathfrak p}} R[x]_{\mathfrak m}$$

has support equal to the primes lying over $\mathfrak p$ because $R_{\mathfrak p} \to R[x]_{\mathfrak m}$ is flat and the support of $M_{\mathfrak p}/(f_1, \ldots, f_d)M_{\mathfrak p}$ is equal to $\{\mathfrak p\}$ (details omitted; hint: follows from Lemmas 40.4 and 40.5). Hence the dimension is 1. To finish the proof it suffices to find an $f \in \mathfrak m$ which is a nonzerodivisor on Q. Since $\mathfrak m$ is a maximal ideal, the field extension $\kappa(\mathfrak m)/\kappa(\mathfrak p)$ is finite (Theorem 34.1). Hence we can find $f \in \mathfrak m$ which viewed as a polynomial in x has leading coefficient not in $\mathfrak p$. Such an f acts as a nonzerodivisor on

$$M_{\mathfrak p}/(f_1, \ldots, f_d)M_{\mathfrak p} \otimes_R R[x] = \bigoplus\nolimits_{n \geq 0} M_{\mathfrak p}/(f_1, \ldots, f_d)M_{\mathfrak p} \cdot x^n$$

and hence acts as a nonzerodivisor on Q. □

104. Cohen-Macaulay rings

00N7 Most of the results of this section are special cases of the results in Section 103.

00N8 **Definition 104.1.** A Noetherian local ring R is called *Cohen-Macaulay* if it is Cohen-Macaulay as a module over itself.

Note that this is equivalent to requiring the existence of a R-regular sequence $x_1, \ldots, x_d$ of the maximal ideal such that $R/(x_1, \ldots, x_d)$ has dimension 0. We will usually just say "regular sequence" and not "R-regular sequence".

02JN **Lemma 104.2.** *Let R be a Noetherian local Cohen-Macaulay ring with maximal ideal $\mathfrak{m}$. Let $x_1, \ldots, x_c \in \mathfrak{m}$ be elements. Then*

$$x_1, \ldots, x_c \text{ is a regular sequence } \Leftrightarrow \dim(R/(x_1, \ldots, x_c)) = \dim(R) - c$$

If so $x_1, \ldots, x_c$ can be extended to a regular sequence of length $\dim(R)$ and each quotient $R/(x_1, \ldots, x_i)$ is a Cohen-Macaulay ring of dimension $\dim(R) - i$.

Proof. Special case of Proposition 103.4. □

00N9 **Lemma 104.3.** *Let R be Noetherian local. Suppose R is Cohen-Macaulay of dimension d. Any maximal chain of ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n$ has length $n = d$.*

Proof. Special case of Lemma 103.9. □

00NA **Lemma 104.4.** *Suppose R is a Noetherian local Cohen-Macaulay ring of dimension d. For any prime $\mathfrak{p} \subset R$ we have*

$$\dim(R) = \dim(R_{\mathfrak{p}}) + \dim(R/\mathfrak{p}).$$

Proof. Follows immediately from Lemma 104.3. (Also, this is a special case of Lemma 103.10.) □

00NB **Lemma 104.5.** *Suppose R is a Cohen-Macaulay local ring. For any prime $\mathfrak{p} \subset R$ the ring $R_{\mathfrak{p}}$ is Cohen-Macaulay as well.*

Proof. Special case of Lemma 103.11. □

00NC **Definition 104.6.** A Noetherian ring R is called *Cohen-Macaulay* if all its local rings are Cohen-Macaulay.

00ND **Lemma 104.7.** *Suppose R is a Noetherian Cohen-Macaulay ring. Any polynomial algebra over R is Cohen-Macaulay.*

Proof. Special case of Lemma 103.13. □

00NE **Lemma 104.8.** *Let R be a Noetherian local Cohen-Macaulay ring of dimension d. Let $0 \to K \to R^{\oplus n} \to M \to 0$ be an exact sequence of R-modules. Then either $M = 0$, or depth$(K) >$ depth(M), or depth$(K) =$ depth$(M) = d$.*

Proof. This is a special case of Lemma 72.6. □

00NG **Lemma 104.9.** *Let R be a local Noetherian Cohen-Macaulay ring of dimension d. Let M be a finite R module of depth e. There exists an exact complex*

$$0 \to K \to F_{d-e-1} \to \ldots \to F_0 \to M \to 0$$

with each F_i finite free and K maximal Cohen-Macaulay.

Proof. Immediate from the definition and Lemma 104.8. $\square$

06LC **Lemma 104.10.** *Let $\varphi : A \to B$ be a map of local rings. Assume that B is Noetherian and Cohen-Macaulay and that $\mathfrak{m}_B = \sqrt{\varphi(\mathfrak{m}_A)B}$. Then there exists a sequence of elements $f_1, \ldots, f_{\dim(B)}$ in A such that $\varphi(f_1), \ldots, \varphi(f_{\dim(B)})$ is a regular sequence in B.*

Proof. By induction on $\dim(B)$ it suffices to prove: If $\dim(B) \geq 1$, then we can find an element f of A which maps to a nonzerodivisor in B. By Lemma 104.2 it suffices to find $f \in A$ whose image in B is not contained in any of the finitely many minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ of B. By the assumption that $\mathfrak{m}_B = \sqrt{\varphi(\mathfrak{m}_A)B}$ we see that $\mathfrak{m}_A \not\subset \varphi^{-1}(\mathfrak{q}_i)$. Hence we can find f by Lemma 15.2. $\square$

105. Catenary rings

00NH Compare with Topology, Section 11.

00NI **Definition 105.1.** A ring R is said to be *catenary* if for any pair of prime ideals $\mathfrak{p} \subset \mathfrak{q}$, there exists an integer bounding the lengths of all finite chains of prime ideals $\mathfrak{p} = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_e = \mathfrak{q}$ and all maximal such chains have the same length.

02IH **Lemma 105.2.** *A ring R is catenary if and only if the topological space* $\text{Spec}(R)$ *is catenary (see Topology, Definition 11.4).*

Proof. Immediate from the definition and the characterization of irreducible closed subsets in Lemma 26.1. $\square$

In general it is not the case that a finitely generated R-algebra is catenary if R is. Thus we make the following definition.

00NL **Definition 105.3.** A Noetherian ring R is said to be *universally catenary* if every R algebra of finite type is catenary.

We restrict to Noetherian rings as it is not clear this definition is the right one for non-Noetherian rings. By Lemma 105.7 to check a Noetherian ring R is universally catenary, it suffices to check each polynomial algebra $R[x_1, \ldots, x_n]$ is catenary.

00NJ **Lemma 105.4.** *Any localization of a catenary ring is catenary. Any localization of a Noetherian universally catenary ring is universally catenary.*

Proof. Let A be a ring and let $S \subset A$ be a multiplicative subset. The description of $\text{Spec}(S^{-1}A)$ in Lemma 17.5 shows that if A is catenary, then so is $S^{-1}A$. If $S^{-1}A \to C$ is of finite type, then $C = S^{-1}B$ for some finite type ring map $A \to B$. Hence if A is Noetherian and universally catenary, then B is catenary and we see that C is catenary too. This proves the lemma. $\square$

0ECE **Lemma 105.5.** *Let A be a Noetherian universally catenary ring. Any A-algebra essentially of finite type over A is universally catenary.*

Proof. If B is a finite type A-algebra, then B is Noetherian by Lemma 31.1. Any finite type B-algebra is a finite type A-algebra and hence catenary by our assumption that A is universally catenary. Thus B is universally catenary. Any localization of B is universally catenary by Lemma 105.4 and this finishes the proof. $\square$

0AUN **Lemma 105.6.** *Let R be a ring. The following are equivalent*

1. *R is catenary,*
2. *$R_\mathfrak{p}$ is catenary for all prime ideals $\mathfrak{p}$,*
3. *$R_\mathfrak{m}$ is catenary for all maximal ideals $\mathfrak{m}$.*

Assume R is Noetherian. The following are equivalent

1. *R is universally catenary,*
2. *$R_\mathfrak{p}$ is universally catenary for all prime ideals $\mathfrak{p}$,*
3. *$R_\mathfrak{m}$ is universally catenary for all maximal ideals $\mathfrak{m}$.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Lemma 105.4 in both cases. The implication (2) $\Rightarrow$ (3) is immediate in both cases. Assume $R_\mathfrak{m}$ is catenary for all maximal ideals $\mathfrak{m}$ of R. If $\mathfrak{p} \subset \mathfrak{q}$ are primes in R, then choose a maximal ideal $\mathfrak{q} \subset \mathfrak{m}$. Chains of primes ideals between $\mathfrak{p}$ and $\mathfrak{q}$ are in 1-to-1 correspondence with chains of prime ideals between $\mathfrak{p}R_\mathfrak{m}$ and $\mathfrak{q}R_\mathfrak{m}$ hence we see R is catenary. Assume R is Noetherian and $R_\mathfrak{m}$ is universally catenary for all maximal ideals $\mathfrak{m}$ of R. Let $R \to S$ be a finite type ring map. Let $\mathfrak{q}$ be a prime ideal of S lying over the prime $\mathfrak{p} \subset R$. Choose a maximal ideal $\mathfrak{p} \subset \mathfrak{m}$ in R. Then $R_\mathfrak{p}$ is a localization of $R_\mathfrak{m}$ hence universally catenary by Lemma 105.4. Then $S_\mathfrak{p}$ is catenary as a finite type ring over $R_\mathfrak{p}$. Hence $S_\mathfrak{q}$ is catenary as a localization. Thus S is catenary by the first case treated above. ☐

00NK **Lemma 105.7.** *Any quotient of a catenary ring is catenary. Any quotient of a Noetherian universally catenary ring is universally catenary.*

Proof. Let A be a ring and let $I \subset A$ be an ideal. The description of $\text{Spec}(A/I)$ in Lemma 17.7 shows that if A is catenary, then so is A/I. The second statement is a special case of Lemma 105.5. ☐

0AUP **Lemma 105.8.** *Let R be a Noetherian ring.*

1. *R is catenary if and only if $R/\mathfrak{p}$ is catenary for every minimal prime $\mathfrak{p}$.*
2. *R is universally catenary if and only if $R/\mathfrak{p}$ is universally catenary for every minimal prime $\mathfrak{p}$.*

Proof. If $\mathfrak{a} \subset \mathfrak{b}$ is an inclusion of primes of R, then we can find a minimal prime $\mathfrak{p} \subset \mathfrak{a}$ and the first assertion is clear. We omit the proof of the second. ☐

00NM **Lemma 105.9.** *A Noetherian Cohen-Macaulay ring is universally catenary. More generally, if R is a Noetherian ring and M is a Cohen-Macaulay R-module with Supp$(M) = \text{Spec}(R)$, then R is universally catenary.*

Proof. Since a polynomial algebra over R is Cohen-Macaulay, by Lemma 104.7, it suffices to show that a Cohen-Macaulay ring is catenary. Let R be Cohen-Macaulay and $\mathfrak{p} \subset \mathfrak{q}$ primes of R. By definition $R_\mathfrak{q}$ and $R_\mathfrak{p}$ are Cohen-Macaulay. Take a maximal chain of primes $\mathfrak{p} = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n = \mathfrak{q}$. Next choose a maximal chain of primes $\mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \ldots \subset \mathfrak{q}_m = \mathfrak{p}$. By Lemma 104.3 we have $n + m = \dim(R_\mathfrak{q})$. And we have $m = \dim(R_\mathfrak{p})$ by the same lemma. Hence $n = \dim(R_\mathfrak{q}) - \dim(R_\mathfrak{p})$ is independent of choices.

To prove the more general statement, argue exactly as above but using Lemmas 103.13 and 103.9. ☐

0ECF **Lemma 105.10.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. The following are equivalent*

(1) *A is catenary, and*
(2) *$\mathfrak{p} \mapsto \dim(A/\mathfrak{p})$ is a dimension function on* $\mathrm{Spec}(A)$.

Proof. If A is catenary, then $\mathrm{Spec}(A)$ has a dimension function δ by Topology, Lemma 20.4 (and Lemma 105.2). We may assume $\delta(\mathfrak{m}) = 0$. Then we see that

$$\delta(\mathfrak{p}) = \mathrm{codim}(V(\mathfrak{m}), V(\mathfrak{p})) = \dim(A/\mathfrak{p})$$

by Topology, Lemma 20.2. In this way we see that (1) implies (2). The reverse implication follows from Topology, Lemma 20.2 as well. □

106. Regular local rings

00NN It is not that easy to show that all prime localizations of a regular local ring are regular. In fact, quite a bit of the material developed so far is geared towards a proof of this fact. See Proposition 110.5, and trace back the references.

00NO **Lemma 106.1.** *Let $(R, \mathfrak{m}, \kappa)$ be a regular local ring of dimension d. The graded ring $\bigoplus \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is isomorphic to the graded polynomial algebra $\kappa[X_1, \ldots, X_d]$.*

Proof. Let $x_1, \ldots, x_d$ be a minimal set of generators for the maximal ideal $\mathfrak{m}$, see Definition 60.10. There is a surjection $\kappa[X_1, \ldots, X_d] \to \bigoplus \mathfrak{m}^n/\mathfrak{m}^{n+1}$, which maps X_i to the class of x_i in $\mathfrak{m}/\mathfrak{m}^2$. Since $d(R) = d$ by Proposition 60.9 we know that the numerical polynomial $n \mapsto \dim_\kappa \mathfrak{m}^n/\mathfrak{m}^{n+1}$ has degree $d - 1$. By Lemma 58.10 we conclude that the surjection $\kappa[X_1, \ldots, X_d] \to \bigoplus \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is an isomorphism. □

00NP **Lemma 106.2.** *Any regular local ring is a domain.*

Proof. We will use that $\bigcap \mathfrak{m}^n = 0$ by Lemma 51.4. Let $f, g \in R$ such that $fg = 0$. Suppose that $f \in \mathfrak{m}^a$ and $g \in \mathfrak{m}^b$, with a, b maximal. Since $fg = 0 \in \mathfrak{m}^{a+b+1}$ we see from the result of Lemma 106.1 that either $f \in \mathfrak{m}^{a+1}$ or $g \in \mathfrak{m}^{b+1}$. Contradiction. □

00NQ **Lemma 106.3.** *Let R be a regular local ring and let $x_1, \ldots, x_d$ be a minimal set of generators for the maximal ideal $\mathfrak{m}$. Then $x_1, \ldots, x_d$ is a regular sequence, and each $R/(x_1, \ldots, x_c)$ is a regular local ring of dimension $d - c$. In particular R is Cohen-Macaulay.*

Proof. Note that R/x_1R is a Noetherian local ring of dimension $\geq d-1$ by Lemma 60.13 with $x_2, \ldots, x_d$ generating the maximal ideal. Hence it is a regular local ring by definition. Since R is a domain by Lemma 106.2 x_1 is a nonzerodivisor. □

00NR **Lemma 106.4.** *Let R be a regular local ring. Let $I \subset R$ be an ideal such that R/I is a regular local ring as well. Then there exists a minimal set of generators $x_1, \ldots, x_d$ for the maximal ideal $\mathfrak{m}$ of R such that $I = (x_1, \ldots, x_c)$ for some $0 \leq c \leq d$.*

Proof. Say $\dim(R) = d$ and $\dim(R/I) = d - c$. Denote $\overline{\mathfrak{m}} = \mathfrak{m}/I$ the maximal ideal of R/I. Let $\kappa = R/\mathfrak{m}$. We have

$$\dim_\kappa((I + \mathfrak{m}^2)/\mathfrak{m}^2) = \dim_\kappa(\mathfrak{m}/\mathfrak{m}^2) - \dim(\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2) = d - (d - c) = c$$

by the definition of a regular local ring. Hence we can choose $x_1, \ldots, x_c \in I$ whose images in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent and supplement with $x_{c+1}, \ldots, x_d$ to get a minimal system of generators of $\mathfrak{m}$. The induced map $R/(x_1, \ldots, x_c) \to R/I$ is a surjection between regular local rings of the same dimension (Lemma 106.3). It follows that the kernel is zero, i.e., $I = (x_1, \ldots, x_c)$. Namely, if not then we would have $\dim(R/I) < \dim(R/(x_1, \ldots, x_c))$ by Lemmas 106.2 and 60.13. $\square$

00NS **Lemma 106.5.** *Let R be a Noetherian local ring. Let $x \in \mathfrak{m}$. Let M be a finite R-module such that x is a nonzerodivisor on M and M/xM is free over R/xR. Then M is free over R.*

Proof. Let $m_1, \ldots, m_r$ be elements of M which map to a R/xR-basis of M/xM. By Nakayama's Lemma 20.1 $m_1, \ldots, m_r$ generate M. If $\sum a_i m_i = 0$ is a relation, then $a_i \in xR$ for all i. Hence $a_i = b_i x$ for some $b_i \in R$. Hence the kernel K of $R^r \to M$ satisfies $xK = K$ and hence is zero by Nakayama's lemma. $\square$

00NT **Lemma 106.6.** *Let R be a regular local ring. Any maximal Cohen-Macaulay module over R is free.*

Proof. Let M be a maximal Cohen-Macaulay module over R. Let $x \in \mathfrak{m}$ be part of a regular sequence generating $\mathfrak{m}$. Then x is a nonzerodivisor on M by Proposition 103.4, and M/xM is a maximal Cohen-Macaulay module over R/xR. By induction on $\dim(R)$ we see that M/xM is free. We win by Lemma 106.5. $\square$

00NU **Lemma 106.7.** *Suppose R is a Noetherian local ring. Let $x \in \mathfrak{m}$ be a nonzerodivisor such that R/xR is a regular local ring. Then R is a regular local ring. More generally, if $x_1, \ldots, x_r$ is a regular sequence in R such that $R/(x_1, \ldots, x_r)$ is a regular local ring, then R is a regular local ring.*

Proof. This is true because x together with the lifts of a system of minimal generators of the maximal ideal of R/xR will give $\dim(R)$ generators of $\mathfrak{m}$. Use Lemma 60.13. The last statement follows from the first and induction. $\square$

07DX **Lemma 106.8.** *Let $(R_i, \varphi_{ii'})$ be a directed system of local rings whose transition maps are local ring maps. If each R_i is a regular local ring and $R = \operatorname{colim} R_i$ is Noetherian, then R is a regular local ring.*

Proof. Let $\mathfrak{m} \subset R$ be the maximal ideal; it is the colimit of the maximal ideal $\mathfrak{m}_i \subset R_i$. We prove the lemma by induction on $d = \dim \mathfrak{m}/\mathfrak{m}^2$. If $d = 0$, then $R = R/\mathfrak{m}$ is a field and R is a regular local ring. If $d > 0$ pick an $x \in \mathfrak{m}$, $x \not\in \mathfrak{m}^2$. For some i we can find an $x_i \in \mathfrak{m}_i$ mapping to x. Note that $R/xR = \operatorname{colim}_{i' \geq i} R_{i'}/x_i R_{i'}$ is a Noetherian local ring. By Lemma 106.3 we see that $R_{i'}/x_i R_{i'}$ is a regular local ring. Hence by induction we see that R/xR is a regular local ring. Since each R_i is a domain (Lemma 106.1) we see that R is a domain. Hence x is a nonzerodivisor and we conclude that R is a regular local ring by Lemma 106.7. $\square$

107. Epimorphisms of rings

04VM In any category there is a notion of an *epimorphism*. Some of this material is taken from [Laz69] and [Maz68].

04VN **Lemma 107.1.** *Let $R \to S$ be a ring map. The following are equivalent*

1. *$R \to S$ is an epimorphism,*
2. *the two ring maps $S \to S \otimes_R S$ are equal,*
3. *either of the ring maps $S \to S \otimes_R S$ is an isomorphism, and*
4. *the ring map $S \otimes_R S \to S$ is an isomorphism.*

Proof. Omitted. $\square$

04VP **Lemma 107.2.** *The composition of two epimorphisms of rings is an epimorphism.*

Proof. Omitted. Hint: This is true in any category. ☐

04VQ **Lemma 107.3.** *If $R \to S$ is an epimorphism of rings and $R \to R'$ is any ring map, then $R' \to R' \otimes_R S$ is an epimorphism.*

Proof. Omitted. Hint: True in any category with pushouts. ☐

04VR **Lemma 107.4.** *If $A \to B \to C$ are ring maps and $A \to C$ is an epimorphism, so is $B \to C$.*

Proof. Omitted. Hint: This is true in any category. ☐

This means in particular, that if $R \to S$ is an epimorphism with image $\overline{R} \subset S$, then $\overline{R} \to S$ is an epimorphism. Hence while proving results for epimorphisms we may often assume the map is injective. The following lemma means in particular that every localization is an epimorphism.

04VS **Lemma 107.5.** *Let $R \to S$ be a ring map. The following are equivalent:*

1. *$R \to S$ is an epimorphism, and*
2. *$R_{\mathfrak p} \to S_{\mathfrak p}$ is an epimorphism for each prime $\mathfrak p$ of R.*

Proof. Since $S_{\mathfrak p} = R_{\mathfrak p} \otimes_R S$ (see Lemma 12.15) we see that (1) implies (2) by Lemma 107.3. Conversely, assume that (2) holds. Let $a, b : S \to A$ be two ring maps from S to a ring A equalizing the map $R \to S$. By assumption we see that for every prime $\mathfrak p$ of R the induced maps $a_{\mathfrak p}, b_{\mathfrak p} : S_{\mathfrak p} \to A_{\mathfrak p}$ are the same. Hence $a = b$ as $A \subset \prod_{\mathfrak p} A_{\mathfrak p}$, see Lemma 23.1. ☐

04VT **Lemma 107.6.** *Let $R \to S$ be a ring map. The following are equivalent*

1. *$R \to S$ is an epimorphism and finite, and*
2. *$R \to S$ is surjective.*

Proof. (This lemma seems to have been reproved many times in the literature, and has many different proofs.) It is clear that a surjective ring map is an epimorphism. Suppose that $R \to S$ is a finite ring map such that $S \otimes_R S \to S$ is an isomorphism. Our goal is to show that $R \to S$ is surjective. Assume S/R is not zero. The exact sequence $R \to S \to S/R \to 0$ leads to an exact sequence

$$R \otimes_R S \to S \otimes_R S \to S/R \otimes_R S \to 0.$$

Our assumption implies that the first arrow is an isomorphism, hence we conclude that $S/R \otimes_R S = 0$. Hence also $S/R \otimes_R S/R = 0$. By Lemma 5.4 there exists a surjection of R-modules $S/R \to R/I$ for some proper ideal $I \subset R$. Hence there exists a surjection $S/R \otimes_R S/R \to R/I \otimes_R R/I = R/I \neq 0$, contradiction. ☐

04VU **Lemma 107.7.** *A faithfully flat epimorphism is an isomorphism.*

Proof. This is clear from Lemma 107.1 part (3) as the map $S \to S \otimes_R S$ is the map $R \to S$ tensored with S. ☐

04VV **Lemma 107.8.** *If $k \to S$ is an epimorphism and k is a field, then $S = k$ or $S = 0$.*

Proof. This is clear from the result of Lemma 107.7 (as any nonzero algebra over k is faithfully flat), or by arguing directly that $R \to R \otimes_k R$ cannot be surjective unless $\dim_k(R) \leq 1$. ☐

04VW **Lemma 107.9.** *Let $R \to S$ be an epimorphism of rings. Then*
(1) $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ *is injective, and*
(2) *for $\mathfrak{q} \subset S$ lying over $\mathfrak{p} \subset R$ we have $\kappa(\mathfrak{p}) = \kappa(\mathfrak{q})$.*

Proof. Let $\mathfrak{p}$ be a prime of R. The fibre of the map is the spectrum of the fibre ring $S \otimes_R \kappa(\mathfrak{p})$. By Lemma 107.3 the map $\kappa(\mathfrak{p}) \to S \otimes_R \kappa(\mathfrak{p})$ is an epimorphism, and hence by Lemma 107.8 we have either $S \otimes_R \kappa(\mathfrak{p}) = 0$ or $S \otimes_R \kappa(\mathfrak{p}) = \kappa(\mathfrak{p})$ which proves (1) and (2). □

04VX **Lemma 107.10.** *Let R be a ring. Let M, N be R-modules. Let $\{x_i\}_{i \in I}$ be a set of generators of M. Let $\{y_j\}_{j \in J}$ be a set of generators of N. Let $\{m_j\}_{j \in J}$ be a family of elements of M with $m_j = 0$ for all but finitely many j. Then*

$$\sum\nolimits_{j \in J} m_j \otimes y_j = 0 \text{ in } M \otimes_R N$$

is equivalent to the following: There exist $a_{i,j} \in R$ with $a_{i,j} = 0$ for all but finitely many pairs (i, j) such that

$$m_j = \sum\nolimits_{i \in I} a_{i,j} x_i \quad \text{for all } j \in J,$$
$$0 = \sum\nolimits_{j \in J} a_{i,j} y_j \quad \text{for all } i \in I.$$

Proof. The sufficiency is immediate. Suppose that $\sum_{j \in J} m_j \otimes y_j = 0$. Consider the short exact sequence

$$0 \to K \to \bigoplus\nolimits_{j \in J} R \to N \to 0$$

where the jth basis vector of $\bigoplus_{j \in J} R$ maps to y_j. Tensor this with M to get the exact sequence

$$K \otimes_R M \to \bigoplus\nolimits_{j \in J} M \to N \otimes_R M \to 0.$$

The assumption implies that there exist elements $k_i \in K$ such that $\sum k_i \otimes x_i$ maps to the element $(m_j)_{j \in J}$ of the middle. Writing $k_i = (a_{i,j})_{j \in J}$ and we obtain what we want. □

04VY **Lemma 107.11.** *Let $\varphi : R \to S$ be a ring map. Let $g \in S$. The following are equivalent:*
(1) *$g \otimes 1 = 1 \otimes g$ in $S \otimes_R S$, and*
(2) *there exist $n \geq 0$ and elements $y_i, z_j \in S$ and $x_{i,j} \in R$ for $1 \leq i, j \leq n$ such that*
  (a) *$g = \sum_{i,j \leq n} x_{i,j} y_i z_j$,*
  (b) *for each j we have $\sum x_{i,j} y_i \in \varphi(R)$, and*
  (c) *for each i we have $\sum x_{i,j} z_j \in \varphi(R)$.*

Proof. It is clear that (2) implies (1). Conversely, suppose that $g \otimes 1 = 1 \otimes g$. Choose generators $\{s_i\}_{i \in I}$ of S as an R-module with $0, 1 \in I$ and $s_0 = 1$ and $s_1 = g$. Apply Lemma 107.10 to the relation $g \otimes s_0 + (-1) \otimes s_1 = 0$. We see that there exist $a_{i,j} \in R$ such that $g = \sum_i a_{i,0} s_i$, $-1 = \sum_i a_{i,1} s_i$, and for $j \neq 0, 1$ we have $0 = \sum_i a_{i,j} s_i$, and moreover for all i we have $\sum_j a_{i,j} s_j = 0$. Then we have

$$\sum\nolimits_{i,j \neq 0} a_{i,j} s_i s_j = -g + a_{0,0}$$

and for each $j \neq 0$ we have $\sum_{i \neq 0} a_{i,j} s_i \in R$. This proves that $-g + a_{0,0}$ can be written as in (2). It follows that g can be written as in (2). Details omitted. Hint:

Show that the set of elements of S which have an expression as in (2) form an R-subalgebra of S. $\square$

04VZ **Remark 107.12.** Let $R \to S$ be a ring map. Sometimes the set of elements $g \in S$ such that $g \otimes 1 = 1 \otimes g$ is called the *epicenter* of S. It is an R-algebra. By the construction of Lemma 107.11 we get for each g in the epicenter a matrix factorization

$$(g) = YXZ$$

with $X \in \text{Mat}(n \times n, R)$, $Y \in \text{Mat}(1 \times n, S)$, and $Z \in \text{Mat}(n \times 1, S)$. Namely, let $x_{i,j}, y_i, z_j$ be as in part (2) of the lemma. Set $X = (x_{i,j})$, let y be the row vector whose entries are the y_i and let z be the column vector whose entries are the z_j. With this notation conditions (b) and (c) of Lemma 107.11 mean exactly that $YX \in \text{Mat}(1 \times n, R)$, $XZ \in \text{Mat}(n \times 1, R)$. It turns out to be very convenient to consider the triple of matrices (X, YX, XZ). Given $n \in \mathbf{N}$ and a triple (P, U, V) we say that (P, U, V) is a *n-triple associated to g* if there exists a matrix factorization as above such that $P = X$, $U = YX$ and $V = XZ$.

04W0 **Lemma 107.13.** *Let $R \to S$ be an epimorphism of rings. Then the cardinality of S is at most the cardinality of R. In a formula: $|S| \leq |R|$.*

Proof. The condition that $R \to S$ is an epimorphism means that each $g \in S$ satisfies $g \otimes 1 = 1 \otimes g$, see Lemma 107.1. We are going to use the notation introduced in Remark 107.12. Suppose that $g, g' \in S$ and suppose that (P, U, V) is an n-triple which is associated to both g and g'. Then we claim that $g = g'$. Namely, write $(P, U, V) = (X, YX, XZ)$ for a matrix factorization $(g) = YXZ$ of g and write $(P, U, V) = (X', Y'X', X'Z')$ for a matrix factorization $(g') = Y'X'Z'$ of g'. Then we see that

$$(g) = YXZ = UZ = Y'X'Z = Y'PZ = Y'XZ = Y'V = Y'X'Z' = (g')$$

and hence $g = g'$. This implies that the cardinality of S is bounded by the number of possible triples, which has cardinality at most $\sup_{n \in \mathbf{N}} |R|^n$. If R is infinite then this is at most $|R|$, see [Kun83, Ch. I, 10.13].

If R is a finite ring then the argument above only proves that S is at worst countable. In fact in this case R is Artinian and the map $R \to S$ is surjective. We omit the proof of this case. $\square$

08YS **Lemma 107.14.** *Let $R \to S$ be an epimorphism of rings. Let N_1, N_2 be S-modules. Then $\text{Hom}_S(N_1, N_2) = \text{Hom}_R(N_1, N_2)$. In other words, the restriction functor $Mod_S \to Mod_R$ is fully faithful.*

Proof. Let $\varphi : N_1 \to N_2$ be an R-linear map. For any $x \in N_1$ consider the map $S \otimes_R S \to N_2$ defined by the rule $g \otimes g' \mapsto g\varphi(g'x)$. Since both maps $S \to S \otimes_R S$ are isomorphisms (Lemma 107.1), we conclude that $g\varphi(g'x) = gg'\varphi(x) = \varphi(gg'x)$. Thus φ is S-linear. $\square$

108. Pure ideals

04PQ The material in this section is discussed in many papers, see for example [Laz67], [Bko70], and [DM83].

04PR **Definition 108.1.** Let R be a ring. We say that $I \subset R$ is *pure* if the quotient ring R/I is flat over R.

04PS **Lemma 108.2.** *Let R be a ring. Let $I \subset R$ be an ideal. The following are equivalent:*

(1) *I is pure,*
(2) *for every ideal $J \subset R$ we have $J \cap I = IJ$,*
(3) *for every finitely generated ideal $J \subset R$ we have $J \cap I = JI$,*
(4) *for every $x \in R$ we have $(x) \cap I = xI$,*
(5) *for every $x \in I$ we have $x = yx$ for some $y \in I$,*
(6) *for every $x_1, \ldots, x_n \in I$ there exists a $y \in I$ such that $x_i = yx_i$ for all $i = 1, \ldots, n$,*
(7) *for every prime $\mathfrak{p}$ of R we have $IR_\mathfrak{p} = 0$ or $IR_\mathfrak{p} = R_\mathfrak{p}$,*
(8) *$Supp(I) = \mathrm{Spec}(R) \setminus V(I)$,*
(9) *I is the kernel of the map $R \to (1+I)^{-1}R$,*
(10) *$R/I \cong S^{-1}R$ as R-algebras for some multiplicative subset S of R, and*
(11) *$R/I \cong (1+I)^{-1}R$ as R-algebras.*

Proof. For any ideal J of R we have the short exact sequence $0 \to J \to R \to R/J \to 0$. Tensoring with R/I we get an exact sequence $J \otimes_R R/I \to R/I \to R/I + J \to 0$ and $J \otimes_R R/I = J/JI$. Thus the equivalence of (1), (2), and (3) follows from Lemma 39.5. Moreover, these imply (4).

The implication (4) $\Rightarrow$ (5) is trivial. Assume (5) and let $x_1, \ldots, x_n \in I$. Choose $y_i \in I$ such that $x_i = y_i x_i$. Let $y \in I$ be the element such that $1 - y = \prod_{i=1,\ldots,n}(1 - y_i)$. Then $x_i = yx_i$ for all $i = 1, \ldots, n$. Hence (6) holds, and it follows that (5) $\Leftrightarrow$ (6).

Assume (5). Let $x \in I$. Then $x = yx$ for some $y \in I$. Hence $x(1-y) = 0$, which shows that x maps to zero in $(1+I)^{-1}R$. Of course the kernel of the map $R \to (1+I)^{-1}R$ is always contained in I. Hence we see that (5) implies (9). Assume (9). Then for any $x \in I$ we see that $x(1-y) = 0$ for some $y \in I$. In other words, $x = yx$. We conclude that (5) is equivalent to (9).

Assume (5). Let $\mathfrak{p}$ be a prime of R. If $\mathfrak{p} \not\in V(I)$, then $IR_\mathfrak{p} = R_\mathfrak{p}$. If $\mathfrak{p} \in V(I)$, in other words, if $I \subset \mathfrak{p}$, then $x \in I$ implies $x(1-y) = 0$ for some $y \in I$, implies x maps to zero in $R_\mathfrak{p}$, i.e., $IR_\mathfrak{p} = 0$. Thus we see that (7) holds.

Assume (7). Then $(R/I)_\mathfrak{p}$ is either 0 or $R_\mathfrak{p}$ for any prime $\mathfrak{p}$ of R. Hence by Lemma 39.18 we see that (1) holds. At this point we see that all of (1) – (7) and (9) are equivalent.

As $IR_\mathfrak{p} = I_\mathfrak{p}$ we see that (7) implies (8). Finally, if (8) holds, then this means exactly that $I_\mathfrak{p}$ is the zero module if and only if $\mathfrak{p} \in V(I)$, which is clearly saying that (7) holds. Now (1) – (9) are equivalent.

Assume (1) – (9) hold. Then $R/I \subset (1+I)^{-1}R$ by (9) and the map $R/I \to (1+I)^{-1}R$ is also surjective by the description of localizations at primes afforded by (7). Hence (11) holds.

The implication (11) $\Rightarrow$ (10) is trivial. And (10) implies that (1) holds because a localization of R is flat over R, see Lemma 39.18. $\square$

04PT **Lemma 108.3.** *Let R be a ring. If $I, J \subset R$ are pure ideals, then $V(I) = V(J)$ implies $I = J$.*

Proof. For example, by property (7) of Lemma 108.2 we see that $I = \mathrm{Ker}(R \to \prod_{\mathfrak{p} \in V(I)} R_\mathfrak{p})$ can be recovered from the closed subset associated to it. $\square$

04PU **Lemma 108.4.** *Let R be a ring. The rule $I \mapsto V(I)$ determines a bijection*

$$\{I \subset R \text{ pure}\} \leftrightarrow \{Z \subset \mathrm{Spec}(R) \text{ closed and closed under generalizations}\}$$

Proof. Let I be a pure ideal. Then since $R \to R/I$ is flat, by going down generalizations lift along the map $\mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$. Hence $V(I)$ is closed under generalizations. This shows that the map is well defined. By Lemma 108.3 the map is injective. Suppose that $Z \subset \mathrm{Spec}(R)$ is closed and closed under generalizations. Let $J \subset R$ be the radical ideal such that $Z = V(J)$. Let $I = \{x \in R : x \in xJ\}$. Note that I is an ideal: if $x, y \in I$ then there exist $f, g \in J$ such that $x = xf$ and $y = yg$. Then

$$x + y = (x + y)(f + g - fg)$$

Verification left to the reader. We claim that I is pure and that $V(I) = V(J)$. If the claim is true then the map of the lemma is surjective and the lemma holds.

Note that $I \subset J$, so that $V(J) \subset V(I)$. Let $I \subset \mathfrak{p}$ be a prime. Consider the multiplicative subset $S = (R \setminus \mathfrak{p})(1 + J)$. By definition of I and $I \subset \mathfrak{p}$ we see that $0 \notin S$. Hence we can find a prime $\mathfrak{q}$ of R which is disjoint from S, see Lemmas 9.4 and 17.5. Hence $\mathfrak{q} \subset \mathfrak{p}$ and $\mathfrak{q} \cap (1 + J) = \emptyset$. This implies that $\mathfrak{q} + J$ is a proper ideal of R. Let $\mathfrak{m}$ be a maximal ideal containing $\mathfrak{q} + J$. Then we get $\mathfrak{m} \in V(J)$ and hence $\mathfrak{q} \in V(J) = Z$ as Z was assumed to be closed under generalization. This in turn implies $\mathfrak{p} \in V(J)$ as $\mathfrak{q} \subset \mathfrak{p}$. Thus we see that $V(I) = V(J)$.

Finally, since $V(I) = V(J)$ (and J radical) we see that $J = \sqrt{I}$. Pick $x \in I$, so that $x = xy$ for some $y \in J$ by definition. Then $x = xy = xy^2 = \ldots = xy^n$. Since $y^n \in I$ for some $n > 0$ we conclude that property (5) of Lemma 108.2 holds and we see that I is indeed pure. □

05KK **Lemma 108.5.** *Let R be a ring. Let $I \subset R$ be an ideal. The following are equivalent*

1. *I is pure and finitely generated,*
2. *I is generated by an idempotent,*
3. *I is pure and $V(I)$ is open, and*
4. *R/I is a projective R-module.*

Proof. If (1) holds, then $I = I \cap I = I^2$ by Lemma 108.2. Hence I is generated by an idempotent by Lemma 21.5. Thus (1) ⇒ (2). If (2) holds, then $I = (e)$ and $R = (1-e) \oplus (e)$ as an R-module hence R/I is flat and I is pure and $V(I) = D(1-e)$ is open. Thus (2) ⇒ (1) + (3). Finally, assume (3). Then $V(I)$ is open and closed, hence $V(I) = D(1 - e)$ for some idempotent e of R, see Lemma 21.3. The ideal $J = (e)$ is a pure ideal such that $V(J) = V(I)$ hence $I = J$ by Lemma 108.3. In this way we see that (3) ⇒ (2). By Lemma 78.2 we see that (4) is equivalent to the assertion that I is pure and R/I finitely presented. Moreover, R/I is finitely presented if and only if I is finitely generated, see Lemma 5.3. Hence (4) is equivalent to (1). □

We can use the above to characterize those rings for which every finite flat module is finitely presented.

052U **Lemma 108.6.** *Let R be a ring. The following are equivalent:*

1. *every $Z \subset \mathrm{Spec}(R)$ which is closed and closed under generalizations is also open, and*

(2) *any finite flat R-module is finite locally free.*

Proof. If any finite flat R-module is finite locally free then the support of R/I where I is a pure ideal is open. Hence the implication (2) $\Rightarrow$ (1) follows from Lemma 108.3.

For the converse assume that R satisfies (1). Let M be a finite flat R-module. The support $Z = \text{Supp}(M)$ of M is closed, see Lemma 40.5. On the other hand, if $\mathfrak{p} \subset \mathfrak{p}'$, then by Lemma 78.5 the module $M_{\mathfrak{p}'}$ is free, and $M_{\mathfrak{p}} = M_{\mathfrak{p}'} \otimes_{R_{\mathfrak{p}'}} R_{\mathfrak{p}}$ Hence $\mathfrak{p}' \in \text{Supp}(M) \Rightarrow \mathfrak{p} \in \text{Supp}(M)$, in other words, the support is closed under generalization. As R satisfies (1) we see that the support of M is open and closed. Suppose that M is generated by r elements $m_1, \ldots, m_r$. The modules $\wedge^i(M)$, $i = 1, \ldots, r$ are finite flat R-modules also, because $\wedge^i(M)_{\mathfrak{p}} = \wedge^i(M_{\mathfrak{p}})$ is free over $R_{\mathfrak{p}}$. Note that $\text{Supp}(\wedge^{i+1}(M)) \subset \text{Supp}(\wedge^i(M))$. Thus we see that there exists a decomposition

$$\text{Spec}(R) = U_0 \amalg U_1 \amalg \ldots \amalg U_r$$

by open and closed subsets such that the support of $\wedge^i(M)$ is $U_r \cup \ldots \cup U_i$ for all $i = 0, \ldots, r$. Let $\mathfrak{p}$ be a prime of R, and say $\mathfrak{p} \in U_i$. Note that $\wedge^i(M) \otimes_R \kappa(\mathfrak{p}) = \wedge^i(M \otimes_R \kappa(\mathfrak{p}))$. Hence, after possibly renumbering $m_1, \ldots, m_r$ we may assume that $m_1, \ldots, m_i$ generate $M \otimes_R \kappa(\mathfrak{p})$. By Nakayama's Lemma 20.1 we get a surjection

$$R_f^{\oplus i} \longrightarrow M_f, \quad (a_1, \ldots, a_i) \longmapsto \sum a_i m_i$$

for some $f \in R$, $f \notin \mathfrak{p}$. We may also assume that $D(f) \subset U_i$. This means that $\wedge^i(M_f) = \wedge^i(M)_f$ is a flat R_f module whose support is all of $\text{Spec}(R_f)$. By the above it is generated by a single element, namely $m_1 \wedge \ldots \wedge m_i$. Hence $\wedge^i(M)_f \cong R_f/J$ for some pure ideal $J \subset R_f$ with $V(J) = \text{Spec}(R_f)$. Clearly this means that $J = (0)$, see Lemma 108.3. Thus $m_1 \wedge \ldots \wedge m_i$ is a basis for $\wedge^i(M_f)$ and it follows that the displayed map is injective as well as surjective. This proves that M is finite locally free as desired. $\square$

109. Rings of finite global dimension

00O2 The following lemma is often used to compare different projective resolutions of a given module.

00O3 **Lemma 109.1** (Schanuel's lemma). *Let R be a ring. Let M be an R-module. Suppose that*

$$0 \to K \xrightarrow{c_1} P_1 \xrightarrow{p_1} M \to 0 \quad \textit{and} \quad 0 \to L \xrightarrow{c_2} P_2 \xrightarrow{p_2} M \to 0$$

are two short exact sequences, with P_i projective. Then $K \oplus P_2 \cong L \oplus P_1$. More precisely, there exist a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K \oplus P_2 \arrow[r, "(c_1, id)"'] \arrow[d] & P_1 \oplus P_2 \arrow[r, "(p_1, 0)"'] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P_1 \oplus L \arrow[r, "(id, c_2)"] & P_1 \oplus P_2 \arrow[r, "(0, p_2)"] & M \arrow[r] & 0
\end{tikzcd}$$

whose vertical arrows are isomorphisms.

Proof. Consider the module N defined by the short exact sequence $0 \to N \to P_1 \oplus P_2 \to M \to 0$, where the last map is the sum of the two maps $P_i \to M$. It is easy to see that the projection $N \to P_1$ is surjective with kernel L, and that $N \to P_2$

is surjective with kernel K. Since P_i are projective we have $N \cong K \oplus P_2 \cong L \oplus P_1$. This proves the first statement.

To prove the second statement (and to reprove the first), choose $a : P_1 \to P_2$ and $b : P_2 \to P_1$ such that $p_1 = p_2 \circ a$ and $p_2 = p_1 \circ b$. This is possible because P_1 and P_2 are projective. Then we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K \oplus P_2 \arrow[r, "{(c_1, \mathrm{id})}"'] & P_1 \oplus P_2 \arrow[r, "{(p_1, 0)}"'] & M \arrow[r] & 0 \\
0 \arrow[r] & N \arrow[r] \arrow[u] \arrow[d] & P_1 \oplus P_2 \arrow[r, "{(p_1, p_2)}"'] \arrow[u, "T"] \arrow[d, "S"'] & M \arrow[r] \arrow[u, equal] \arrow[d, equal] & 0 \\
0 \arrow[r] & P_1 \oplus L \arrow[r, "{(\mathrm{id}, c_2)}"] & P_1 \oplus P_2 \arrow[r, "{(0, p_2)}"] & M \arrow[r] & 0
\end{tikzcd}$$

with T and S given by the matrices

$$S = \begin{pmatrix} \mathrm{id} & 0 \\ a & \mathrm{id} \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} \mathrm{id} & b \\ 0 & \mathrm{id} \end{pmatrix}$$

Then S, T and the maps $N \to P_1 \oplus L$ and $N \to K \oplus P_2$ are isomorphisms as desired. □

00O4 **Definition 109.2.** Let R be a ring. Let M be an R-module. We say M has *finite projective dimension* if it has a finite length resolution by projective R-modules. The minimal length of such a resolution is called the *projective dimension* of M.

It is clear that the projective dimension of M is 0 if and only if M is a projective module. The following lemma explains to what extent the projective dimension is independent of the choice of a projective resolution.

00O5 **Lemma 109.3.** *Let R be a ring. Suppose that M is an R-module of projective dimension d. Suppose that $F_e \to F_{e-1} \to \ldots \to F_0 \to M \to 0$ is exact with F_i projective and $e \geq d - 1$. Then the kernel of $F_e \to F_{e-1}$ is projective (or the kernel of $F_0 \to M$ is projective in case $e = 0$).*

Proof. We prove this by induction on d. If $d = 0$, then M is projective. In this case there is a splitting $F_0 = \mathrm{Ker}(F_0 \to M) \oplus M$, and hence $\mathrm{Ker}(F_0 \to M)$ is projective. This finishes the proof if $e = 0$, and if $e > 0$, then replacing M by $\mathrm{Ker}(F_0 \to M)$ we decrease e.

Next assume $d > 0$. Let $0 \to P_d \to P_{d-1} \to \ldots \to P_0 \to M \to 0$ be a minimal length finite resolution with P_i projective. According to Schanuel's Lemma 109.1 we have $P_0 \oplus \mathrm{Ker}(F_0 \to M) \cong F_0 \oplus \mathrm{Ker}(P_0 \to M)$. This proves the case $d = 1$, $e = 0$, because then the right hand side is $F_0 \oplus P_1$ which is projective. Hence now we may assume $e > 0$. The module $F_0 \oplus \mathrm{Ker}(P_0 \to M)$ has the finite projective resolution

$$0 \to P_d \to P_{d-1} \to \ldots \to P_2 \to P_1 \oplus F_0 \to \mathrm{Ker}(P_0 \to M) \oplus F_0 \to 0$$

of length $d - 1$. By induction applied to the exact sequence

$$F_e \to F_{e-1} \to \ldots \to F_2 \to P_0 \oplus F_1 \to P_0 \oplus \mathrm{Ker}(F_0 \to M) \to 0$$

of length $e - 1$ we conclude $\mathrm{Ker}(F_e \to F_{e-1})$ is projective (if $e \geq 2$) or that $\mathrm{Ker}(F_1 \oplus P_0 \to F_0 \oplus P_0)$ is projective. This implies the lemma. □

0CXC **Lemma 109.4.** *Let R be a ring. Let M be an R-module. Let $d \geq 0$. The following are equivalent*

(1) *M has projective dimension $\leq d$,*
(2) *there exists a resolution $0 \to P_d \to P_{d-1} \to \ldots \to P_0 \to M \to 0$ with P_i projective,*
(3) *for some resolution $\ldots \to P_2 \to P_1 \to P_0 \to M \to 0$ with P_i projective we have $\text{Ker}(P_{d-1} \to P_{d-2})$ is projective if $d \geq 2$, or $\text{Ker}(P_0 \to M)$ is projective if $d = 1$, or M is projective if $d = 0$,*
(4) *for any resolution $\ldots \to P_2 \to P_1 \to P_0 \to M \to 0$ with P_i projective we have $\text{Ker}(P_{d-1} \to P_{d-2})$ is projective if $d \geq 2$, or $\text{Ker}(P_0 \to M)$ is projective if $d = 1$, or M is projective if $d = 0$.*

Proof. The equivalence of (1) and (2) is the definition of projective dimension, see Definition 109.2. We have (2) $\Rightarrow$ (4) by Lemma 109.3. The implications (4) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2) are immediate. $\square$

0CXD **Lemma 109.5.** *Let R be a local ring. Let M be an R-module. Let $d \geq 0$. The equivalent conditions (1) – (4) of Lemma 109.4 are also equivalent to*

(5) *there exists a resolution $0 \to P_d \to P_{d-1} \to \ldots \to P_0 \to M \to 0$ with P_i free.*

Proof. Follows from Lemma 109.4 and Theorem 85.4. $\square$

0CXE **Lemma 109.6.** *Let R be a Noetherian ring. Let M be a finite R-module. Let $d \geq 0$. The equivalent conditions (1) – (4) of Lemma 109.4 are also equivalent to*

(6) *there exists a resolution $0 \to P_d \to P_{d-1} \to \ldots \to P_0 \to M \to 0$ with P_i finite projective.*

Proof. Choose a resolution $\ldots \to F_2 \to F_1 \to F_0 \to M \to 0$ with F_i finite free (Lemma 71.1). By Lemma 109.4 we see that $P_d = \text{Ker}(F_{d-1} \to F_{d-2})$ is projective at least if $d \geq 2$. Then P_d is a finite R-module as R is Noetherian and $P_d \subset F_{d-1}$ which is finite free. Whence $0 \to P_d \to F_{d-1} \to \ldots \to F_1 \to F_0 \to M \to 0$ is the desired resolution. $\square$

0CXF **Lemma 109.7.** *Let R be a local Noetherian ring. Let M be a finite R-module. Let $d \geq 0$. The equivalent conditions (1) – (4) of Lemma 109.4, condition (5) of Lemma 109.5, and condition (6) of Lemma 109.6 are also equivalent to*

(7) *there exists a resolution $0 \to F_d \to F_{d-1} \to \ldots \to F_0 \to M \to 0$ with F_i finite free.*

Proof. This follows from Lemmas 109.4, 109.5, and 109.6 and because a finite projective module over a local ring is finite free, see Lemma 78.2. $\square$

065R **Lemma 109.8.** *Let R be a ring. Let M be an R-module. Let $n \geq 0$. The following are equivalent*

(1) *M has projective dimension $\leq n$,*
(2) *$\text{Ext}^i_R(M, N) = 0$ for all R-modules N and all $i \geq n+1$, and*
(3) *$\text{Ext}^{n+1}_R(M, N) = 0$ for all R-modules N.*

Proof. Assume (1). Choose a free resolution $F_\bullet \to M$ of M. Denote $d_e : F_e \to F_{e-1}$. By Lemma 109.3 we see that $P_e = \text{Ker}(d_e)$ is projective for $e \geq n-1$. This implies that $F_e \cong P_e \oplus P_{e-1}$ for $e \geq n$ where d_e maps the summand P_{e-1} isomorphically to P_{e-1} in F_{e-1}. Hence, for any R-module N the complex $\text{Hom}_R(F_\bullet, N)$ is split exact in degrees $\geq n+1$. Whence (2) holds. The implication (2) $\Rightarrow$ (3) is trivial.

Assume (3) holds. If $n = 0$ then M is projective by Lemma 77.2 and we see that (1) holds. If $n > 0$ choose a free R-module F and a surjection $F \to M$ with kernel K. By Lemma 71.7 and the vanishing of $\text{Ext}^i_R(F, N)$ for all $i > 0$ by part (1) we see that $\text{Ext}^n_R(K, N) = 0$ for all R-modules N. Hence by induction we see that K has projective dimension $\leq n-1$. Then M has projective dimension $\leq n$ as any finite projective resolution of K gives a projective resolution of length one more for M by adding F to the front. □

065S **Lemma 109.9.** *Let R be a ring. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R-modules.*

1. *If M has projective dimension $\leq n$ and M'' has projective dimension $\leq n+1$, then M' has projective dimension $\leq n$.*
2. *If M' and M'' have projective dimension $\leq n$ then M has projective dimension $\leq n$.*
3. *If M' has projective dimension $\leq n$ and M has projective dimension $\leq n+1$ then M'' has projective dimension $\leq n+1$.*

Proof. Combine the characterization of projective dimension in Lemma 109.8 with the long exact sequence of ext groups in Lemma 71.7. □

00O6 **Definition 109.10.** Let R be a ring. The ring R is said to have *finite global dimension* if there exists an integer n such that every R-module has a resolution by projective R-modules of length at most n. The minimal such n is then called the *global dimension* of R.

The argument in the proof of the following lemma can be found in the paper [Aus55] by Auslander.

0D1U **Lemma 109.11.** *Let R be a ring. Suppose we have a module $M = \bigcup_{e \in E} M_e$ where the M_e are submodules well-ordered by inclusion. Assume the quotients $M_e / \bigcup_{e' < e} M_{e'}$ have projective dimension $\leq n$. Then M has projective dimension $\leq n$.*

Proof. We will prove this by induction on n.

Base case: $n = 0$. Then $P_e = M_e / \bigcup_{e' < e} M_{e'}$ is projective. Thus we may choose a section $P_e \to M_e$ of the projection $M_e \to P_e$. We claim that the induced map $\psi : \bigoplus_{e \in E} P_e \to M$ is an isomorphism. Namely, if $x = \sum x_e \in \bigoplus P_e$ is nonzero, then we let e_{max} be maximal such that $x_{e_{max}}$ is nonzero and we conclude that $y = \psi(x) = \psi(\sum x_e)$ is nonzero because $y \in M_{e_{max}}$ has nonzero image $x_{e_{max}}$ in $P_{e_{max}}$. On the other hand, let $y \in M$. Then $y \in M_e$ for some e. We show that $y \in \text{Im}(\psi)$ by transfinite induction on e. Let $x_e \in P_e$ be the image of y. Then $y - \psi(x_e) \in \bigcup_{e' < e} M_{e'}$. By induction hypothesis we conclude that $y - \psi(x_e) \in \text{Im}(\psi)$ hence $y \in \text{Im}(\psi)$. Thus the claim is true and ψ is an isomorphism. We conclude that M is projective as a direct sum of projectives, see Lemma 77.4.

If $n > 0$, then for $e \in E$ we denote F_e the free R-module on the set of elements of M_e. Then we have a system of short exact sequences

$$0 \to K_e \to F_e \to M_e \to 0$$

over the well-ordered set E. Note that the transition maps $F_{e'} \to F_e$ and $K_{e'} \to K_e$ are injective too. Set $F = \bigcup F_e$ and $K = \bigcup K_e$. Then

$$0 \to K_e/\bigcup\nolimits_{e'<e} K_{e'} \to F_e/\bigcup\nolimits_{e'<e} F_{e'} \to M_e/\bigcup\nolimits_{e'<e} M_{e'} \to 0$$

is a short exact sequence of R-modules too and $F_e/\bigcup_{e'<e} F_{e'}$ is the free R-module on the set of elements in M_e which are not contained in $\bigcup_{e'<e} M_{e'}$. Hence by Lemma 109.9 we see that the projective dimension of $K_e/\bigcup_{e'<e} K_{e'}$ is at most $n-1$. By induction we conclude that K has projective dimension at most $n-1$. Whence M has projective dimension at most n and we win. □

065T **Lemma 109.12.** *Let R be a ring. The following are equivalent*

1. *R has finite global dimension $\leq n$,*
2. *every finite R-module has projective dimension $\leq n$, and*
3. *every cyclic R-module R/I has projective dimension $\leq n$.*

Proof. It is clear that (1) ⇒ (2) and (2) ⇒ (3). Assume (3). Choose a set $E \subset M$ of generators of M. Choose a well ordering on E. For $e \in E$ denote M_e the submodule of M generated by the elements $e' \in E$ with $e' \leq e$. Then $M = \bigcup_{e \in E} M_e$. Note that for each $e \in E$ the quotient

$$M_e/\bigcup\nolimits_{e'<e} M_{e'}$$

is either zero or generated by one element, hence has projective dimension $\leq n$ by (3). By Lemma 109.11 this means that M has projective dimension $\leq n$. □

00O8 **Lemma 109.13.** *Let R be a ring. Let M be an R-module. Let $S \subset R$ be a multiplicative subset.*

1. *If M has projective dimension $\leq n$, then $S^{-1}M$ has projective dimension $\leq n$ over $S^{-1}R$.*
2. *If R has finite global dimension $\leq n$, then $S^{-1}R$ has finite global dimension $\leq n$.*

Proof. Let $0 \to P_n \to P_{n-1} \to \ldots \to P_0 \to M \to 0$ be a projective resolution. As localization is exact, see Proposition 9.12, and as each $S^{-1}P_i$ is a projective $S^{-1}R$-module, see Lemma 94.1, we see that $0 \to S^{-1}P_n \to \ldots \to S^{-1}P_0 \to S^{-1}M \to 0$ is a projective resolution of $S^{-1}M$. This proves (1). Let M' be an $S^{-1}R$-module. Note that $M' = S^{-1}M'$. Hence we see that (2) follows from (1). □

110. Regular rings and global dimension

065U We can use the material on rings of finite global dimension to give another characterization of regular local rings.

00O7 **Proposition 110.1.** *Let R be a regular local ring of dimension d. Every finite R-module M of depth e has a finite free resolution*

$$0 \to F_{d-e} \to \ldots \to F_0 \to M \to 0.$$

In particular a regular local ring has global dimension $\leq d$.

Proof. The first part holds in view of Lemma 106.6 and Lemma 104.9. The last part follows from this and Lemma 109.12. □

00O9 **Lemma 110.2.** *Let R be a Noetherian ring. Then R has finite global dimension if and only if there exists an integer n such that for all maximal ideals $\mathfrak m$ of R the ring $R_{\mathfrak m}$ has global dimension $\leq n$.*

Proof. We saw, Lemma 109.13 that if R has finite global dimension n, then all the localizations $R_{\mathfrak m}$ have finite global dimension at most n. Conversely, suppose that all the $R_{\mathfrak m}$ have global dimension $\leq n$. Let M be a finite R-module. Let $0 \to K_n \to F_{n-1} \to \ldots \to F_0 \to M \to 0$ be a resolution with F_i finite free. Then K_n is a finite R-module. According to Lemma 109.3 and the assumption all the modules $K_n \otimes_R R_{\mathfrak m}$ are projective. Hence by Lemma 78.2 the module K_n is finite projective. $\square$

00OA **Lemma 110.3.** *Suppose that R is a Noetherian local ring with maximal ideal $\mathfrak m$ and residue field κ. In this case the projective dimension of κ is $\geq \dim_\kappa \mathfrak m/\mathfrak m^2$.*

Proof. Let $x_1, \ldots, x_n$ be elements of $\mathfrak m$ whose images in $\mathfrak m/\mathfrak m^2$ form a basis. Consider the *Koszul complex* on $x_1, \ldots, x_n$. This is the complex

$$0 \to \wedge^n R^n \to \wedge^{n-1} R^n \to \wedge^{n-2} R^n \to \ldots \to \wedge^i R^n \to \ldots \to R^n \to R$$

with maps given by

$$e_{j_1} \wedge \ldots \wedge e_{j_i} \longmapsto \sum_{a=1}^{i} (-1)^{i+1} x_{j_a} e_{j_1} \wedge \ldots \wedge \hat{e}_{j_a} \wedge \ldots \wedge e_{j_i}$$

It is easy to see that this is a complex $K_\bullet(R, x_\bullet)$. Note that the cokernel of the last map of $K_\bullet(R, x_\bullet)$ is κ by Lemma 20.1 part (8).

If κ has finite projective dimension d, then we can find a resolution $F_\bullet \to \kappa$ by finite free R-modules of length d (Lemma 109.7). By Lemma 102.2 we may assume all the maps in the complex $F_\bullet$ have the property that $\text{Im}(F_i \to F_{i-1}) \subset \mathfrak m F_{i-1}$, because removing a trivial summand from the resolution can at worst shorten the resolution. By Lemma 71.4 we can find a map of complexes $\alpha : K_\bullet(R, x_\bullet) \to F_\bullet$ inducing the identity on κ. We will prove by induction that the maps $\alpha_i : \wedge^i R^n = K_i(R, x_\bullet) \to F_i$ have the property that $\alpha_i \otimes \kappa : \wedge^i \kappa^n \to F_i \otimes \kappa$ are injective. This shows that $F_n \neq 0$ and hence $d \geq n$ as desired.

The result is clear for $i = 0$ because the composition $R \xrightarrow{\alpha_0} F_0 \to \kappa$ is nonzero. Note that F_0 must have rank 1 since otherwise the map $F_1 \to F_0$ whose cokernel is a single copy of κ cannot have image contained in $\mathfrak m F_0$.

Next we check the case $i = 1$ as we feel that it is instructive; the reader can skip this as the induction step will deduce the $i = 1$ case from the case $i = 0$. We saw above that $F_0 = R$ and $F_1 \to F_0 = R$ has image $\mathfrak m$. We have a commutative diagram

$$\begin{array}{ccccccc} R^n & = & K_1(R, x_\bullet) & \to & K_0(R, x_\bullet) & = & R \\ & & \downarrow & & \downarrow & & \downarrow \\ & & F_1 & \to & F_0 & = & R \end{array}$$

where the rightmost vertical arrow is given by multiplication by a unit. Hence we see that the image of the composition $R^n \to F_1 \to F_0 = R$ is also equal to $\mathfrak m$. Thus the map $R^n \otimes \kappa \to F_1 \otimes \kappa$ has to be injective since $\dim_\kappa(\mathfrak m/\mathfrak m^2) = n$.

Let $i \geq 1$ and assume injectivity of $\alpha_j \otimes \kappa$ has been proved for all $j \leq i-1$. Consider the commutative diagram

$$\begin{matrix} \wedge^i R^n & = & K_i(R, x_\bullet) & \to & K_{i-1}(R, x_\bullet) & = & \wedge^{i-1} R^n \\ & & \downarrow & & \downarrow & & \\ & & F_i & \to & F_{i-1} & & \end{matrix}$$

We know that $\wedge^{i-1}\kappa^n \to F_{i-1} \otimes \kappa$ is injective. This proves that $\wedge^{i-1}\kappa^n \otimes_\kappa \mathfrak m/\mathfrak m^2 \to F_{i-1} \otimes \mathfrak m/\mathfrak m^2$ is injective. Also, by our choice of the complex, F_i maps into $\mathfrak m F_{i-1}$, and similarly for the Koszul complex. Hence we get a commutative diagram

$$\begin{matrix} \wedge^i \kappa^n & \to & \wedge^{i-1}\kappa^n \otimes \mathfrak m/\mathfrak m^2 \\ \downarrow & & \downarrow \\ F_i \otimes \kappa & \to & F_{i-1} \otimes \mathfrak m/\mathfrak m^2 \end{matrix}$$

At this point it suffices to verify the map $\wedge^i\kappa^n \to \wedge^{i-1}\kappa^n \otimes \mathfrak m/\mathfrak m^2$ is injective, which can be done by hand. □

00OB **Lemma 110.4.** *Let R be a Noetherian local ring. Suppose that the residue field κ has finite projective dimension n over R. In this case $\dim(R) \geq n$.*

Proof. Let $F_\bullet$ be a finite resolution of κ by finite free R-modules (Lemma 109.7). By Lemma 102.2 we may assume all the maps in the complex $F_\bullet$ have to property that $\text{Im}(F_i \to F_{i-1}) \subset \mathfrak m F_{i-1}$, because removing a trivial summand from the resolution can at worst shorten the resolution. Say $F_n \neq 0$ and $F_i = 0$ for $i > n$, so that the projective dimension of κ is n. By Proposition 102.9 we see that $\text{depth}_{I(\varphi_n)}(R) \geq n$ since $I(\varphi_n)$ cannot equal R by our choice of the complex. Thus by Lemma 72.3 also $\dim(R) \geq n$. □

00OC **Proposition 110.5.** *Let $(R, \mathfrak m, \kappa)$ be a Noetherian local ring. The following are equivalent*

1. *κ has finite projective dimension as an R-module,*
2. *R has finite global dimension,*
3. *R is a regular local ring.*

Moreover, in this case the global dimension of R equals $\dim(R) = \dim_\kappa(\mathfrak m/\mathfrak m^2)$.

Proof. We have (3) ⇒ (2) by Proposition 110.1. The implication (2) ⇒ (1) is trivial. Assume (1). By Lemmas 110.3 and 110.4 we see that $\dim(R) \geq \dim_\kappa(\mathfrak m/\mathfrak m^2)$. Thus R is regular, see Definition 60.10 and the discussion preceding it. Assume the equivalent conditions (1) – (3) hold. By Proposition 110.1 the global dimension of R is at most $\dim(R)$ and by Lemma 110.3 it is at least $\dim_\kappa(\mathfrak m/\mathfrak m^2)$. Thus the stated equality holds. □

0AFS **Lemma 110.6.** *A Noetherian local ring R is a regular local ring if and only if it has finite global dimension. In this case $R_\mathfrak p$ is a regular local ring for all primes $\mathfrak p$.*

Proof. By Propositions 110.5 and 110.1 we see that a Noetherian local ring is a regular local ring if and only if it has finite global dimension. Furthermore, any localization $R_\mathfrak p$ has finite global dimension, see Lemma 109.13, and hence is a regular local ring. □

By Lemma 110.6 it makes sense to make the following definition, because it does not conflict with the earlier definition of a regular local ring.

00OD **Definition 110.7.** A Noetherian ring R is said to be *regular* if all the localizations $R_\mathfrak{p}$ at primes are regular local rings.

It is enough to require the local rings at maximal ideals to be regular. Note that this is not the same as asking R to have finite global dimension, even assuming R is Noetherian. This is because there is an example of a regular Noetherian ring which does not have finite global dimension, namely because it does not have finite dimension.

00OE **Lemma 110.8.** *Let R be a Noetherian ring. The following are equivalent:*

(1) *R has finite global dimension n,*
(2) *R is a regular ring of dimension n,*
(3) *there exists an integer n such that all the localizations $R_\mathfrak{m}$ at maximal ideals are regular of dimension $\leq n$ with equality for at least one $\mathfrak{m}$, and*
(4) *there exists an integer n such that all the localizations $R_\mathfrak{p}$ at prime ideals are regular of dimension $\leq n$ with equality for at least one $\mathfrak{p}$.*

Proof. This follows from the discussion above. More precisely, it follows by combining Definition 110.7 with Lemma 110.2 and Proposition 110.5. □

00OF **Lemma 110.9.** *Let $R \to S$ be a local homomorphism of local Noetherian rings. Assume that $R \to S$ is flat and that S is regular. Then R is regular.*

Proof. Let $\mathfrak{m} \subset R$ be the maximal ideal and let $\kappa = R/\mathfrak{m}$ be the residue field. Let $d = \dim S$. Choose any resolution $F_\bullet \to \kappa$ with each F_i a finite free R-module. Set $K_d = \text{Ker}(F_{d-1} \to F_{d-2})$. By flatness of $R \to S$ the complex $0 \to K_d \otimes_R S \to F_{d-1} \otimes_R S \to \ldots \to F_0 \otimes_R S \to \kappa \otimes_R S \to 0$ is still exact. Because the global dimension of S is d, see Proposition 110.1, we see that $K_d \otimes_R S$ is a finite free S-module (see also Lemma 109.3). By Lemma 78.6 we see that K_d is a finite free R-module. Hence κ has finite projective dimension and R is regular by Proposition 110.5. □

111. Auslander-Buchsbaum

090U The following result can be found in [AB57].

090V **Proposition 111.1.** *Let R be a Noetherian local ring. Let M be a nonzero finite R-module which has finite projective dimension $pd_R(M)$. Then we have*

$$depth(R) = pd_R(M) + depth(M)$$

Proof. We prove this by induction on $\text{depth}(M)$. The most interesting case is the case $\text{depth}(M) = 0$. In this case, let

$$0 \to R^{n_e} \to R^{n_{e-1}} \to \ldots \to R^{n_0} \to M \to 0$$

be a minimal finite free resolution, so $e = \text{pd}_R(M)$. By Lemma 102.2 we may assume all matrix coefficients of the maps in the complex are contained in the maximal ideal of R. Then on the one hand, by Proposition 102.9 we see that $\text{depth}(R) \geq e$. On the other hand, breaking the long exact sequence into short exact sequences

$$\begin{aligned} 0 \to R^{n_e} \to R^{n_{e-1}} \to K_{e-2} \to 0, \\ 0 \to K_{e-2} \to R^{n_{e-2}} \to K_{e-3} \to 0, \\ \ldots, \\ 0 \to K_0 \to R^{n_0} \to M \to 0 \end{aligned}$$

we see, using Lemma 72.6, that

$$\begin{aligned} \text{depth}(K_{e-2}) \geq \text{depth}(R) - 1, \\ \text{depth}(K_{e-3}) \geq \text{depth}(R) - 2, \\ \ldots, \\ \text{depth}(K_0) \geq \text{depth}(R) - (e-1), \\ \text{depth}(M) \geq \text{depth}(R) - e \end{aligned}$$

and since $\text{depth}(M) = 0$ we conclude $\text{depth}(R) \leq e$. This finishes the proof of the case $\text{depth}(M) = 0$.

Induction step. If $\text{depth}(M) > 0$, then we pick $x \in \mathfrak{m}$ which is a nonzerodivisor on both M and R. This is possible, because either $\text{pd}_R(M) > 0$ and $\text{depth}(R) > 0$ by the aforementioned Proposition 102.9 or $\text{pd}_R(M) = 0$ in which case M is finite free hence also $\text{depth}(R) = \text{depth}(M) > 0$. Thus $\text{depth}(R \oplus M) > 0$ by Lemma 72.6 (for example) and we can find an $x \in \mathfrak{m}$ which is a nonzerodivisor on both R and M. Let

$$0 \to R^{n_e} \to R^{n_{e-1}} \to \ldots \to R^{n_0} \to M \to 0$$

be a minimal resolution as above. An application of the snake lemma shows that

$$0 \to (R/xR)^{n_e} \to (R/xR)^{n_{e-1}} \to \ldots \to (R/xR)^{n_0} \to M/xM \to 0$$

is a minimal resolution too. Thus $\text{pd}_R(M) = \text{pd}_{R/xR}(M/xM)$. By Lemma 72.7 we have $\text{depth}(R/xR) = \text{depth}(R) - 1$ and $\text{depth}(M/xM) = \text{depth}(M) - 1$. Till now depths have all been depths as R modules, but we observe that $\text{depth}_R(M/xM) = \text{depth}_{R/xR}(M/xM)$ and similarly for R/xR. By induction hypothesis we see that the Auslander-Buchsbaum formula holds for M/xM over R/xR. Since the depths of both R/xR and M/xM have decreased by one and the projective dimension has not changed we conclude. $\square$

112. Homomorphisms and dimension

00OG This section contains a collection of easy results relating dimensions of rings when there are maps between them.

00OH **Lemma 112.1.** *Suppose $R \to S$ is a ring map satisfying either going up, see Definition 41.1, or going down see Definition 41.1. Assume in addition that* $\text{Spec}(S) \to \text{Spec}(R)$ *is surjective. Then* $\dim(R) \leq \dim(S)$.

Proof. Assume going up. Take any chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_e$ of prime ideals in R. By surjectivity we may choose a prime $\mathfrak{q}_0$ mapping to $\mathfrak{p}_0$. By going up we may extend this to a chain of length e of primes $\mathfrak{q}_i$ lying over $\mathfrak{p}_i$. Thus $\dim(S) \geq \dim(R)$. The case of going down is exactly the same. See also Topology, Lemma 19.9 for a purely topological version. $\square$

00OI **Lemma 112.2.** *Suppose that $R \to S$ is a ring map with the going up property, see Definition 41.1. If $\mathfrak q \subset S$ is a maximal ideal. Then the inverse image of $\mathfrak q$ in R is a maximal ideal too.*

Proof. Trivial. □

00OJ **Lemma 112.3.** *Suppose that $R \to S$ is a ring map such that S is integral over R. Then $\dim(R) \geq \dim(S)$, and every closed point of $\text{Spec}(S)$ maps to a closed point of $\text{Spec}(R)$.*

Proof. Immediate from Lemmas 36.20 and 112.2 and the definitions. □

00OK **Lemma 112.4.** *Suppose $R \subset S$ and S integral over R. Then $\dim(R) = \dim(S)$.*

Proof. This is a combination of Lemmas 36.22, 36.17, 112.1, and 112.3. □

00OL **Definition 112.5.** Suppose that $R \to S$ is a ring map. Let $\mathfrak q \subset S$ be a prime lying over the prime $\mathfrak p$ of R. The *local ring of the fibre at* $\mathfrak q$ is the local ring

$$S_{\mathfrak q}/\mathfrak p S_{\mathfrak q} = (S/\mathfrak p S)_{\mathfrak q} = (S \otimes_R \kappa(\mathfrak p))_{\mathfrak q}$$

00OM **Lemma 112.6.** *Let $R \to S$ be a homomorphism of Noetherian rings. Let $\mathfrak q \subset S$ be a prime lying over the prime $\mathfrak p$. Then*

$$\dim(S_{\mathfrak q}) \leq \dim(R_{\mathfrak p}) + \dim(S_{\mathfrak q}/\mathfrak p S_{\mathfrak q}).$$

Proof. We use the characterization of dimension of Proposition 60.9. Let $x_1, \ldots, x_d$ be elements of $\mathfrak p$ generating an ideal of definition of $R_{\mathfrak p}$ with $d = \dim(R_{\mathfrak p})$. Let $y_1, \ldots, y_e$ be elements of $\mathfrak q$ generating an ideal of definition of $S_{\mathfrak q}/\mathfrak p S_{\mathfrak q}$ with $e = \dim(S_{\mathfrak q}/\mathfrak p S_{\mathfrak q})$. It is clear that $S_{\mathfrak q}/(x_1, \ldots, x_d, y_1, \ldots, y_e)$ has a nilpotent maximal ideal. Hence $x_1, \ldots, x_d, y_1, \ldots, y_e$ generate an ideal of definition of $S_{\mathfrak q}$. □

00ON **Lemma 112.7.** *Let $R \to S$ be a homomorphism of Noetherian rings. Let $\mathfrak q \subset S$ be a prime lying over the prime $\mathfrak p$. Assume the going down property holds for $R \to S$ (for example if $R \to S$ is flat, see Lemma 39.19). Then*

$$\dim(S_{\mathfrak q}) = \dim(R_{\mathfrak p}) + \dim(S_{\mathfrak q}/\mathfrak p S_{\mathfrak q}).$$

Proof. By Lemma 112.6 we have an inequality $\dim(S_{\mathfrak q}) \leq \dim(R_{\mathfrak p}) + \dim(S_{\mathfrak q}/\mathfrak p S_{\mathfrak q})$. To get equality, choose a chain of primes $\mathfrak p S \subset \mathfrak q_0 \subset \mathfrak q_1 \subset \ldots \subset \mathfrak q_d = \mathfrak q$ with $d = \dim(S_{\mathfrak q}/\mathfrak p S_{\mathfrak q})$. On the other hand, choose a chain of primes $\mathfrak p_0 \subset \mathfrak p_1 \subset \ldots \subset \mathfrak p_e = \mathfrak p$ with $e = \dim(R_{\mathfrak p})$. By the going down theorem we may choose $\mathfrak q_{-1} \subset \mathfrak q_0$ lying over $\mathfrak p_{e-1}$. And then we may choose $\mathfrak q_{-2} \subset \mathfrak q_{e-1}$ lying over $\mathfrak p_{e-2}$. Inductively we keep going until we get a chain $\mathfrak q_{-e} \subset \ldots \subset \mathfrak q_d$ of length $e + d$. □

031E **Lemma 112.8.** *Let $R \to S$ be a local homomorphism of local Noetherian rings. Assume*

1. *R is regular,*
2. *$S/\mathfrak m_R S$ is regular, and*
3. *$R \to S$ is flat.*

Then S is regular.

Proof. By Lemma 112.7 we have $\dim(S) = \dim(R) + \dim(S/\mathfrak m_R S)$. Pick generators $x_1, \ldots, x_d \in \mathfrak m_R$ with $d = \dim(R)$, and pick $y_1, \ldots, y_e \in \mathfrak m_S$ which generate the maximal ideal of $S/\mathfrak m_R S$ with $e = \dim(S/\mathfrak m_R S)$. Then we see that $x_1, \ldots, x_d, y_1, \ldots, y_e$ are elements which generate the maximal ideal of S and $e + d = \dim(S)$. □

The lemma below will later be used to show that rings of finite type over a field are Cohen-Macaulay if and only if they are quasi-finite flat over a polynomial ring. It is a partial converse to Lemma 128.1.

00R5 **Lemma 112.9.** *Let $R \to S$ be a local homomorphism of Noetherian local rings. Assume R Cohen-Macaulay. If S is finite flat over R, or if S is flat over R and $\dim(S) \leq \dim(R)$, then S is Cohen-Macaulay and $\dim(R) = \dim(S)$.*

Proof. Let $x_1, \ldots, x_d \in \mathfrak{m}_R$ be a regular sequence of length $d = \dim(R)$. By Lemma 68.5 this maps to a regular sequence in S. Hence S is Cohen-Macaulay if $\dim(S) \leq d$. This is true if S is finite flat over R by Lemma 112.4. And in the second case we assumed it. □

113. The dimension formula

02II Recall the definitions of catenary (Definition 105.1) and universally catenary (Definition 105.3).

02IJ **Lemma 113.1.** *Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime of S lying over the prime $\mathfrak{p}$ of R. Assume that*

1. *R is Noetherian,*
2. *$R \to S$ is of finite type,*
3. *R, S are domains, and*
4. *$R \subset S$.*

Then we have

$$height(\mathfrak{q}) \leq height(\mathfrak{p}) + trdeg_R(S) - trdeg_{\kappa(\mathfrak{p})}\kappa(\mathfrak{q})$$

with equality if R is universally catenary.

Proof. Suppose that $R \subset S' \subset S$ is a finitely generated R-subalgebra of S. In this case set $\mathfrak{q}' = S' \cap \mathfrak{q}$. The lemma for the ring maps $R \to S'$ and $S' \to S$ implies the lemma for $R \to S$ by additivity of transcendence degree in towers of fields (Fields, Lemma 26.5). Hence we can use induction on the number of generators of S over R and reduce to the case where S is generated by one element over R.

Case I: $S = R[x]$ is a polynomial algebra over R. In this case we have $\text{trdeg}_R(S) = 1$. Also $R \to S$ is flat and hence

$$\dim(S_{\mathfrak{q}}) = \dim(R_{\mathfrak{p}}) + \dim(S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}})$$

see Lemma 112.7. Let $\mathfrak{r} = \mathfrak{p}S$. Then $\text{trdeg}_{\kappa(\mathfrak{p})}\kappa(\mathfrak{q}) = 1$ is equivalent to $\mathfrak{q} = \mathfrak{r}$, and implies that $\dim(S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}) = 0$. In the same vein $\text{trdeg}_{\kappa(\mathfrak{p})}\kappa(\mathfrak{q}) = 0$ is equivalent to having a strict inclusion $\mathfrak{r} \subset \mathfrak{q}$, which implies that $\dim(S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}) = 1$. Thus we are done with case I with equality in every instance.

Case II: $S = R[x]/\mathfrak{n}$ with $\mathfrak{n} \neq 0$. In this case we have $\text{trdeg}_R(S) = 0$. Denote $\mathfrak{q}' \subset R[x]$ the prime corresponding to $\mathfrak{q}$. Thus we have

$$S_{\mathfrak{q}} = (R[x])_{\mathfrak{q}'}/\mathfrak{n}(R[x])_{\mathfrak{q}'}$$

By the previous case we have $\dim((R[x])_{\mathfrak{q}'}) = \dim(R_{\mathfrak{p}}) + 1 - \text{trdeg}_{\kappa(\mathfrak{p})}\kappa(\mathfrak{q})$. Since $\mathfrak{n} \neq 0$ we see that the dimension of $S_{\mathfrak{q}}$ decreases by at least one, see Lemma 60.13, which proves the inequality of the lemma. To see the equality in case R is universally catenary note that $\mathfrak{n} \subset R[x]$ is a height one prime as it corresponds to a nonzero prime in $F[x]$ where F is the fraction field of R. Hence any maximal chain of primes in $S_\mathfrak{q} = R[x]_{\mathfrak{q}'}/\mathfrak{n}R[x]_{\mathfrak{q}'}$ corresponds to a maximal chain of primes with length 1 greater between $\mathfrak{q}'$ and (0) in $R[x]$. If R is universally catenary these all have the same length equal to the height of $\mathfrak{q}'$. This proves that $\dim(S_\mathfrak{q}) = \dim(R[x]_{\mathfrak{q}'}) - 1$ and this implies equality holds as desired. $\square$

The following lemma says that generically finite maps tend to be quasi-finite in codimension 1.

02MA **Lemma 113.2.** *Let $A \to B$ be a ring map. Assume*

1. *$A \subset B$ is an extension of domains,*
2. *the induced extension of fraction fields is finite,*
3. *A is Noetherian, and*
4. *$A \to B$ is of finite type.*

Let $\mathfrak{p} \subset A$ be a prime of height 1. Then there are at most finitely many primes of B lying over $\mathfrak{p}$ and they all have height 1.

Proof. By the dimension formula (Lemma 113.1) for any prime $\mathfrak{q}$ lying over $\mathfrak{p}$ we have

$$\dim(B_\mathfrak{q}) \leq \dim(A_\mathfrak{p}) - \text{trdeg}_{\kappa(\mathfrak{p})}\kappa(\mathfrak{q}).$$

As the domain $B_\mathfrak{q}$ has at least 2 prime ideals we see that $\dim(B_\mathfrak{q}) \geq 1$. We conclude that $\dim(B_\mathfrak{q}) = 1$ and that the extension $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q})$ is algebraic. Hence $\mathfrak{q}$ defines a closed point of its fibre $\text{Spec}(B \otimes_A \kappa(\mathfrak{p}))$, see Lemma 35.9. Since $B \otimes_A \kappa(\mathfrak{p})$ is a Noetherian ring the fibre $\text{Spec}(B \otimes_A \kappa(\mathfrak{p}))$ is a Noetherian topological space, see Lemma 31.5. A Noetherian topological space consisting of closed points is finite, see for example Topology, Lemma 9.2. $\square$

114. Dimension of finite type algebras over fields

00OO In this section we compute the dimension of a polynomial ring over a field. We also prove that the dimension of a finite type domain over a field is the dimension of its local rings at maximal ideals. We will establish the connection with the transcendence degree over the ground field in Section 116.

00OP **Lemma 114.1.** *Let $\mathfrak{m}$ be a maximal ideal in $k[x_1, \ldots, x_n]$. The ideal $\mathfrak{m}$ is generated by n elements. The dimension of $k[x_1, \ldots, x_n]_\mathfrak{m}$ is n. Hence $k[x_1, \ldots, x_n]_\mathfrak{m}$ is a regular local ring of dimension n.*

Proof. By the Hilbert Nullstellensatz (Theorem 34.1) we know the residue field $\kappa = \kappa(\mathfrak{m})$ is a finite extension of k. Denote $\alpha_i \in \kappa$ the image of x_i. Denote $\kappa_i = k(\alpha_1, \ldots, \alpha_i) \subset \kappa$, $i = 1, \ldots, n$ and $\kappa_0 = k$. Note that $\kappa_i = k[\alpha_1, \ldots, \alpha_i]$ by field theory. Define inductively elements $f_i \in \mathfrak{m} \cap k[x_1, \ldots, x_i]$ as follows: Let $P_i(T) \in \kappa_{i-1}[T]$ be the monic minimal polynomial of α_i over κ_{i-1}. Let $Q_i(T) \in k[x_1, \ldots, x_{i-1}][T]$ be a monic lift of $P_i(T)$ (of the same degree). Set $f_i = Q_i(x_i)$. Note that if $d_i = \deg_T(P_i) = \deg_T(Q_i) = \deg_{x_i}(f_i)$ then $d_1 d_2 \ldots d_i = [\kappa_i : k]$ by Fields, Lemmas 7.7 and 9.2.

We claim that for all $i = 0, 1, \ldots, n$ there is an isomorphism

$$\psi_i : k[x_1, \ldots, x_i]/(f_1, \ldots, f_i) \cong \kappa_i.$$

By construction the composition $k[x_1, \ldots, x_i] \to k[x_1, \ldots, x_n] \to \kappa$ is surjective onto κ_i and $f_1, \ldots, f_i$ are in the kernel. This gives a surjective homomorphism.

We prove ψ_i is injective by induction. It is clear for $i = 0$. Given the statement for i we prove it for $i+1$. The ring extension $k[x_1, \ldots, x_i]/(f_1, \ldots, f_i) \to k[x_1, \ldots, x_{i+1}]/(f_1, \ldots, f_{i+1})$ is generated by 1 element over a field and one irreducible equation. By elementary field theory $k[x_1, \ldots, x_{i+1}]/(f_1, \ldots, f_{i+1})$ is a field, and hence ψ_i is injective.

This implies that $\mathfrak{m} = (f_1, \ldots, f_n)$. Moreover, we also conclude that

$$k[x_1, \ldots, x_n]/(f_1, \ldots, f_i) \cong \kappa_i[x_{i+1}, \ldots, x_n].$$

Hence $(f_1, \ldots, f_i)$ is a prime ideal. Thus

$$(0) \subset (f_1) \subset (f_1, f_2) \subset \ldots \subset (f_1, \ldots, f_n) = \mathfrak{m}$$

is a chain of primes of length n. The lemma follows. □

00OQ **Proposition 114.2.** *A polynomial algebra in n variables over a field is a regular ring. It has global dimension n. All localizations at maximal ideals are regular local rings of dimension n.*

Proof. By Lemma 114.1 all localizations $k[x_1, \ldots, x_n]_{\mathfrak{m}}$ at maximal ideals are regular local rings of dimension n. Hence we conclude by Lemma 110.8. □

00OR **Lemma 114.3.** *Let k be a field. Let $\mathfrak{p} \subset \mathfrak{q} \subset k[x_1, \ldots, x_n]$ be a pair of primes. Any maximal chain of primes between $\mathfrak{p}$ and $\mathfrak{q}$ has length height$(\mathfrak{q})$ − height$(\mathfrak{p})$.*

Proof. By Proposition 114.2 any local ring of $k[x_1, \ldots, x_n]$ is regular. Hence all local rings are Cohen-Macaulay, see Lemma 106.3. The local rings at maximal ideals have dimension n hence every maximal chain of primes in $k[x_1, \ldots, x_n]$ has length n, see Lemma 104.3. Hence every maximal chain of primes between (0) and $\mathfrak{p}$ has length height$(\mathfrak{p})$, see Lemma 104.4 for example. Putting these together leads to the assertion of the lemma. □

00OS **Lemma 114.4.** *Let k be a field. Let S be a finite type k-algebra which is an integral domain. Then $\dim(S) = \dim(S_{\mathfrak{m}})$ for any maximal ideal $\mathfrak{m}$ of S. In words: every maximal chain of primes has length equal to the dimension of S.*

Proof. Write $S = k[x_1, \ldots, x_n]/\mathfrak{p}$. By Proposition 114.2 and Lemma 114.3 all the maximal chains of primes in S (which necessarily end with a maximal ideal) have length $n - \text{height}(\mathfrak{p})$. Thus this number is the dimension of S and of $S_{\mathfrak{m}}$ for any maximal ideal $\mathfrak{m}$ of S. □

Recall that we defined the dimension $\dim_x(X)$ of a topological space X at a point x in Topology, Definition 10.1.

00OT **Lemma 114.5.** *Let k be a field. Let S be a finite type k-algebra. Let $X = \text{Spec}(S)$. Let $\mathfrak{p} \subset S$ be a prime ideal and let $x \in X$ be the corresponding point. The following numbers are equal*

(1) $\dim_x(X)$,
(2) $\max \dim(Z)$ *where the maximum is over those irreducible components Z of X passing through x, and*
(3) $\min \dim(S_{\mathfrak{m}})$ *where the minimum is over maximal ideals $\mathfrak{m}$ with $\mathfrak{p} \subset \mathfrak{m}$.*

Proof. Let $X = \bigcup_{i \in I} Z_i$ be the decomposition of X into its irreducible components. There are finitely many of them (see Lemmas 31.3 and 31.5). Let $I' = \{i \mid x \in Z_i\}$, and let $T = \bigcup_{i \notin I'} Z_i$. Then $U = X \setminus T$ is an open subset of X containing the point x. The number (2) is $\max_{i \in I'} \dim(Z_i)$. For any open $W \subset U$ with $x \in W$ the irreducible components of W are the irreducible sets $W_i = Z_i \cap W$ for $i \in I'$ and x is contained in each of these. Note that each W_i, $i \in I'$ contains a closed point because X is Jacobson, see Section 35. Since $W_i \subset Z_i$ we have $\dim(W_i) \leq \dim(Z_i)$. The existence of a closed point implies, via Lemma 114.4, that there is a chain of irreducible closed subsets of length equal to $\dim(Z_i)$ in the open W_i. Thus $\dim(W_i) = \dim(Z_i)$ for any $i \in I'$. Hence $\dim(W)$ is equal to the number (2). This proves that (1) = (2).

Let $\mathfrak m \supset \mathfrak p$ be any maximal ideal containing $\mathfrak p$. Let $x_0 \in X$ be the corresponding point. First of all, x_0 is contained in all the irreducible components Z_i, $i \in I'$. Let $\mathfrak q_i$ denote the minimal primes of S corresponding to the irreducible components Z_i. For each i such that $x_0 \in Z_i$ (which is equivalent to $\mathfrak m \supset \mathfrak q_i$) we have a surjection

$$S_{\mathfrak m} \longrightarrow S_{\mathfrak m}/\mathfrak q_i S_{\mathfrak m} = (S/\mathfrak q_i)_{\mathfrak m}$$

Moreover, the primes $\mathfrak q_i S_{\mathfrak m}$ so obtained exhaust the minimal primes of the Noetherian local ring $S_{\mathfrak m}$, see Lemma 26.3. We conclude, using Lemma 114.4, that the dimension of $S_{\mathfrak m}$ is the maximum of the dimensions of the Z_i passing through x_0. To finish the proof of the lemma it suffices to show that we can choose x_0 such that $x_0 \in Z_i \Rightarrow i \in I'$. Because S is Jacobson (as we saw above) it is enough to show that $V(\mathfrak p) \setminus T$ (with T as above) is nonempty. And this is clear since it contains the point x (i.e. $\mathfrak p$). □

00OU **Lemma 114.6.** *Let k be a field. Let S be a finite type k-algebra. Let $X = \mathrm{Spec}(S)$. Let $\mathfrak m \subset S$ be a maximal ideal and let $x \in X$ be the associated closed point. Then $\dim_x(X) = \dim(S_{\mathfrak m})$.*

Proof. This is a special case of Lemma 114.5. □

00OV **Lemma 114.7.** *Let k be a field. Let S be a finite type k algebra. Assume that S is Cohen-Macaulay. Then $\mathrm{Spec}(S) = \coprod T_d$ is a finite disjoint union of open and closed subsets T_d with T_d equidimensional (see Topology, Definition 10.5) of dimension d. Equivalently, S is a product of rings S_d, $d = 0, \ldots, \dim(S)$ such that every maximal ideal $\mathfrak m$ of S_d has height d.*

Proof. The equivalence of the two statements follows from Lemma 24.3. Let $\mathfrak m \subset S$ be a maximal ideal. Every maximal chain of primes in $S_{\mathfrak m}$ has the same length equal to $\dim(S_{\mathfrak m})$, see Lemma 104.3. Hence, the dimension of the irreducible components passing through the point corresponding to $\mathfrak m$ all have dimension equal to $\dim(S_{\mathfrak m})$, see Lemma 114.4. Since $\mathrm{Spec}(S)$ is a Jacobson topological space the intersection of any two irreducible components of it contains a closed point if nonempty, see Lemmas 35.2 and 35.4. Thus we have shown that any two irreducible components that meet have the same dimension. The lemma follows easily from this, and the fact that $\mathrm{Spec}(S)$ has a finite number of irreducible components (see Lemmas 31.3 and 31.5). □

115. Noether normalization

00OW In this section we prove variants of the Noether normalization lemma. The key ingredient we will use is contained in the following two lemmas.

051M **Lemma 115.1.** *Let $n \in \mathbf{N}$. Let N be a finite nonempty set of multi-indices $\nu = (\nu_1, \ldots, \nu_n)$. Given $e = (e_1, \ldots, e_n)$ we set $e \cdot \nu = \sum e_i\nu_i$. Then for $e_1 \gg e_2 \gg \ldots \gg e_{n-1} \gg e_n$ we have: If $\nu, \nu' \in N$ then*

$$(e \cdot \nu = e \cdot \nu') \Leftrightarrow (\nu = \nu')$$

Proof. Say $N = \{\nu_j\}$ with $\nu_j = (\nu_{j1}, \ldots, \nu_{jn})$. Let $A_i = \max_j \nu_{ji} - \min_j \nu_{ji}$. If for each i we have $e_{i-1} > A_ie_i + A_{i+1}e_{i+1} + \ldots + A_ne_n$ then the lemma holds. For suppose that $e \cdot (\nu - \nu') = 0$. Then for $n \geq 2$,

$$e_1(\nu_1 - \nu'_1) = \sum\nolimits_{i=2}^n e_i(\nu'_i - \nu_i).$$

We may assume that $(\nu_1 - \nu'_1) \geq 0$. If $(\nu_1 - \nu'_1) > 0$, then

$$e_1(\nu_1 - \nu'_1) \geq e_1 > A_2e_2 + \ldots + A_ne_n \geq \sum\nolimits_{i=2}^n e_i|\nu'_i - \nu_i| \geq \sum\nolimits_{i=2}^n e_i(\nu'_i - \nu_i).$$

This contradiction implies that $\nu'_1 = \nu_1$. By induction, $\nu'_i = \nu_i$ for $2 \leq i \leq n$. □

051N **Lemma 115.2.** *Let R be a ring. Let $g \in R[x_1, \ldots, x_n]$ be an element which is nonconstant, i.e., $g \notin R$. For $e_1 \gg e_2 \gg \ldots \gg e_{n-1} \gg e_n = 1$ the polynomial*

$$g(x_1 + x_n^{e_1}, x_2 + x_n^{e_2}, \ldots, x_{n-1} + x_n^{e_{n-1}}, x_n) = ax_n^d + \textit{lower order terms in } x_n$$

where $d > 0$ and $a \in R$ is one of the nonzero coefficients of g.

Proof. Write $g = \sum_{\nu \in N} a_\nu x^\nu$ with $a_\nu \in R$ not zero. Here N is a finite set of multi-indices as in Lemma 115.1 and $x^\nu = x_1^{\nu_1} \ldots x_n^{\nu_n}$. Note that the leading term in

$$(x_1 + x_n^{e_1})^{\nu_1} \ldots (x_{n-1} + x_n^{e_{n-1}})^{\nu_{n-1}} x_n^{\nu_n} \quad \text{is} \quad x_n^{e_1\nu_1 + \ldots + e_{n-1}\nu_{n-1} + \nu_n}.$$

Hence the lemma follows from Lemma 115.1 which guarantees that there is exactly one nonzero term $a_\nu x^\nu$ of g which gives rise to the leading term of $g(x_1 + x_n^{e_1}, x_2 + x_n^{e_2}, \ldots, x_{n-1} + x_n^{e_{n-1}}, x_n)$, i.e., $a = a_\nu$ for the unique $\nu \in N$ such that $e \cdot \nu$ is maximal. □

00OX **Lemma 115.3.** *Let k be a field. Let $S = k[x_1, \ldots, x_n]/I$ for some proper ideal I. If $I \neq 0$, then there exist $y_1, \ldots, y_{n-1} \in k[x_1, \ldots, x_n]$ such that S is finite over $k[y_1, \ldots, y_{n-1}]$. Moreover we may choose y_i to be in the $\mathbf{Z}$-subalgebra of $k[x_1, \ldots, x_n]$ generated by $x_1, \ldots, x_n$.*

Proof. Pick $f \in I$, $f \neq 0$. It suffices to show the lemma for $k[x_1, \ldots, x_n]/(f)$ since S is a quotient of that ring. We will take $y_i = x_i - x_n^{e_i}$, $i = 1, \ldots, n-1$ for suitable integers e_i. When does this work? It suffices to show that $\overline{x_n} \in k[x_1, \ldots, x_n]/(f)$ is integral over the ring $k[y_1, \ldots, y_{n-1}]$. The equation for $\overline{x_n}$ over this ring is

$$f(y_1 + x_n^{e_1}, \ldots, y_{n-1} + x_n^{e_{n-1}}, x_n) = 0.$$

Hence we are done if we can show there exists integers e_i such that the leading coefficient with respect to x_n of the equation above is a nonzero element of k. This can be achieved for example by choosing $e_1 \gg e_2 \gg \ldots \gg e_{n-1}$, see Lemma 115.2. □

00OY **Lemma 115.4.** *Let k be a field. Let $S = k[x_1, \ldots, x_n]/I$ for some ideal I. If $I \neq (1)$, there exist $r \geq 0$, and $y_1, \ldots, y_r \in k[x_1, \ldots, x_n]$ such that (a) the map $k[y_1, \ldots, y_r] \to S$ is injective, and (b) the map $k[y_1, \ldots, y_r] \to S$ is finite. In this case the integer r is the dimension of S. Moreover we may choose y_i to be in the $\mathbf{Z}$-subalgebra of $k[x_1, \ldots, x_n]$ generated by $x_1, \ldots, x_n$.*

Proof. By induction on n, with $n = 0$ being trivial. If $I = 0$, then take $r = n$ and $y_i = x_i$. If $I \neq 0$, then choose $y_1, \ldots, y_{n-1}$ as in Lemma 115.3. Let $S' \subset S$ be the subring generated by the images of the y_i. By induction we can choose r and $z_1, \ldots, z_r \in k[y_1, \ldots, y_{n-1}]$ such that (a), (b) hold for $k[z_1, \ldots, z_r] \to S'$. Since $S' \to S$ is injective and finite we see (a), (b) hold for $k[z_1, \ldots, z_r] \to S$. The last assertion follows from Lemma 112.4. $\square$

00OZ **Lemma 115.5.** *Let k be a field. Let S be a finite type k algebra and denote $X = \mathrm{Spec}(S)$. Let $\mathfrak{q}$ be a prime of S, and let $x \in X$ be the corresponding point. There exists a $g \in S$, $g \notin \mathfrak{q}$ such that $\dim(S_g) = \dim_x(X) =: d$ and such that there exists a finite injective map $k[y_1, \ldots, y_d] \to S_g$.*

Proof. Note that by definition $\dim_x(X)$ is the minimum of the dimensions of S_g for $g \in S$, $g \notin \mathfrak{q}$, i.e., the minimum is attained. Thus the lemma follows from Lemma 115.4. $\square$

051P **Lemma 115.6.** *Let k be a field. Let $\mathfrak{q} \subset k[x_1, \ldots, x_n]$ be a prime ideal. Set $r = trdeg_k\ \kappa(\mathfrak{q})$. Then there exists a finite ring map $\varphi : k[y_1, \ldots, y_n] \to k[x_1, \ldots, x_n]$ such that $\varphi^{-1}(\mathfrak{q}) = (y_{r+1}, \ldots, y_n)$.*

Proof. By induction on n. The case $n = 0$ is clear. Assume $n > 0$. If $r = n$, then $\mathfrak{q} = (0)$ and the result is clear. Choose a nonzero $f \in \mathfrak{q}$. Of course f is nonconstant. After applying an automorphism of the form

$$k[x_1, \ldots, x_n] \longrightarrow k[x_1, \ldots, x_n], \quad x_n \mapsto x_n, \quad x_i \mapsto x_i + x_n^{e_i}\ (i < n)$$

we may assume that f is monic in x_n over $k[x_1, \ldots, x_n]$, see Lemma 115.2. Hence the ring map

$$k[y_1, \ldots, y_n] \longrightarrow k[x_1, \ldots, x_n], \quad y_n \mapsto f, \quad y_i \mapsto x_i\ (i < n)$$

is finite. Moreover $y_n \in \mathfrak{q} \cap k[y_1, \ldots, y_n]$ by construction. Thus $\mathfrak{q} \cap k[y_1, \ldots, y_n] = \mathfrak{p}k[y_1, \ldots, y_n] + (y_n)$ where $\mathfrak{p} \subset k[y_1, \ldots, y_{n-1}]$ is a prime ideal. Note that $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q})$ is finite, and hence $r = \mathrm{trdeg}_k\ \kappa(\mathfrak{p})$. Apply the induction hypothesis to the pair $(k[y_1, \ldots, y_{n-1}], \mathfrak{p})$ and we obtain a finite ring map $k[z_1, \ldots, z_{n-1}] \to k[y_1, \ldots, y_{n-1}]$ such that $\mathfrak{p} \cap k[z_1, \ldots, z_{n-1}] = (z_{r+1}, \ldots, z_{n-1})$. We extend the ring map $k[z_1, \ldots, z_{n-1}] \to k[y_1, \ldots, y_{n-1}]$ to a ring map $k[z_1, \ldots, z_n] \to k[y_1, \ldots, y_n]$ by mapping z_n to y_n. The composition of the ring maps

$$k[z_1, \ldots, z_n] \to k[y_1, \ldots, y_n] \to k[x_1, \ldots, x_n]$$

solves the problem. $\square$

07NA **Lemma 115.7.** *Let $R \to S$ be an injective finite type ring map. Assume R is a domain. Then there exists an integer d and a factorization*

$$R \to R[y_1, \ldots, y_d] \to S' \to S$$

by injective maps such that S' is finite over $R[y_1, \ldots, y_d]$ and such that $S'_f \cong S_f$ for some nonzero $f \in R$.

Proof. Pick $x_1, \ldots, x_n \in S$ which generate S over R. Let K be the fraction field of R and $S_K = S \otimes_R K$. By Lemma 115.4 we can find $y_1, \ldots, y_d \in S$ such that $K[y_1, \ldots, y_d] \to S_K$ is a finite injective map. Note that $y_i \in S$ because we may pick the y_j in the $\mathbf{Z}$-algebra generated by $x_1, \ldots, x_n$. As a finite ring map is integral (see Lemma 36.3) we can find monic $P_i \in K[y_1, \ldots, y_d][T]$ such that $P_i(x_i) = 0$ in S_K. Let $f \in R$ be a nonzero element such that $fP_i \in R[y_1, \ldots, y_d][T]$ for all

i. Then $fP_i(x_i)$ maps to zero in S_K. Hence after replacing f by another nonzero element of R we may also assume $fP_i(x_i)$ is zero in S. Set $x_i' = fx_i$ and let $S' \subset S$ be the R-subalgebra generated by $y_1, \ldots, y_d$ and $x_1', \ldots, x_n'$. Note that x_i' is integral over $R[y_1, \ldots, y_d]$ as we have $Q_i(x_i') = 0$ where $Q_i = f^{\deg_T(P_i)} P_i(T/f)$ which is a monic polynomial in T with coefficients in $R[y_1, \ldots, y_d]$ by our choice of f. Hence $R[y_1, \ldots, y_d] \subset S'$ is finite by Lemma 36.5. Since $S' \subset S$ we have $S'_f \subset S_f$ (localization is exact). On the other hand, the elements $x_i = x_i'/f$ in S'_f generate S_f over R_f and hence $S'_f \to S_f$ is surjective. Whence $S'_f \cong S_f$ and we win. $\square$

116. Dimension of finite type algebras over fields, reprise

07NB This section is a continuation of Section 114. In this section we establish the connection between dimension and transcendence degree over the ground field for finite type domains over a field.

00P0 **Lemma 116.1.** *Let k be a field. Let S be a finite type k algebra which is an integral domain. Let K be the field of fractions of S. Let $r = trdeg(K/k)$ be the transcendence degree of K over k. Then $\dim(S) = r$. Moreover, the local ring of S at every maximal ideal has dimension r.*

Proof. We may write $S = k[x_1, \ldots, x_n]/\mathfrak{p}$. By Lemma 114.3 all local rings of S at maximal ideals have the same dimension. Apply Lemma 115.4. We get a finite injective ring map

$$k[y_1, \ldots, y_d] \to S$$

with $d = \dim(S)$. Clearly, $k(y_1, \ldots, y_d) \subset K$ is a finite extension and we win. $\square$

06RP **Lemma 116.2.** *Let k be a field. Let S be a finite type k-algebra. Let $\mathfrak{q} \subset \mathfrak{q}' \subset S$ be distinct prime ideals. Then $trdeg_k\ \kappa(\mathfrak{q}') < trdeg_k\ \kappa(\mathfrak{q})$.*

Proof. By Lemma 116.1 we have $\dim V(\mathfrak{q}) = \text{trdeg}_k\ \kappa(\mathfrak{q})$ and similarly for $\mathfrak{q}'$. Hence the result follows as the strict inclusion $V(\mathfrak{q}') \subset V(\mathfrak{q})$ implies a strict inequality of dimensions. $\square$

The following lemma generalizes Lemma 114.6.

00P1 **Lemma 116.3.** *Let k be a field. Let S be a finite type k algebra. Let $X = \operatorname{Spec}(S)$. Let $\mathfrak{p} \subset S$ be a prime ideal, and let $x \in X$ be the corresponding point. Then we have*

$$\dim_x(X) = \dim(S_{\mathfrak{p}}) + trdeg_k\ \kappa(\mathfrak{p}).$$

Proof. By Lemma 116.1 we know that $r = \text{trdeg}_k\ \kappa(\mathfrak{p})$ is equal to the dimension of $V(\mathfrak{p})$. Pick any maximal chain of primes $\mathfrak{p} \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_r$ starting with $\mathfrak{p}$ in S. This has length r by Lemma 114.4. Let $\mathfrak{q}_j$, $j \in J$ be the minimal primes of S which are contained in $\mathfrak{p}$. These correspond $1-1$ to minimal primes in $S_{\mathfrak{p}}$ via the rule $\mathfrak{q}_j \mapsto \mathfrak{q}_j S_{\mathfrak{p}}$. By Lemma 114.5 we know that $\dim_x(X)$ is equal to the maximum of the dimensions of the rings $S/\mathfrak{q}_j$. For each j pick a maximal chain of primes $\mathfrak{q}_j \subset \mathfrak{p}'_1 \subset \ldots \subset \mathfrak{p}'_{s(j)} = \mathfrak{p}$. Then $\dim(S_{\mathfrak{p}}) = \max_{j \in J} s(j)$. Now, each chain

$$\mathfrak{q}_i \subset \mathfrak{p}'_1 \subset \ldots \subset \mathfrak{p}'_{s(j)} = \mathfrak{p} \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_r$$

is a maximal chain in $S/\mathfrak{q}_j$, and by what was said before we have $\dim_x(X) = \max_{j \in J} r + s(j)$. The lemma follows. $\square$

The following lemma says that the codimension of one finite type Spec in another is the difference of heights.

00P2 **Lemma 116.4.** *Let k be a field. Let $S' \to S$ be a surjection of finite type k algebras. Let $\mathfrak p \subset S$ be a prime ideal, and let $\mathfrak p'$ be the corresponding prime ideal of S'. Let $X = \text{Spec}(S)$, resp. $X' = \text{Spec}(S')$, and let $x \in X$, resp. $x' \in X'$ be the point corresponding to $\mathfrak p$, resp. $\mathfrak p'$. Then*

$$\dim_{x'} X' - \dim_x X = \mathit{height}(\mathfrak p') - \mathit{height}(\mathfrak p).$$

Proof. Immediate from Lemma 116.3. □

00P3 **Lemma 116.5.** *Let k be a field. Let S be a finite type k-algebra. Let K/k be a field extension. Then $\dim(S) = \dim(K \otimes_k S)$.*

Proof. By Lemma 115.4 there exists a finite injective map $k[y_1, \ldots, y_d] \to S$ with $d = \dim(S)$. Since K is flat over k we also get a finite injective map $K[y_1, \ldots, y_d] \to K \otimes_k S$. The result follows from Lemma 112.4. □

00P4 **Lemma 116.6.** *Let k be a field. Let S be a finite type k-algebra. Set $X = \text{Spec}(S)$. Let K/k be a field extension. Set $S_K = K \otimes_k S$, and $X_K = \text{Spec}(S_K)$. Let $\mathfrak q \subset S$ be a prime corresponding to $x \in X$ and let $\mathfrak q_K \subset S_K$ be a prime corresponding to $x_K \in X_K$ lying over $\mathfrak q$. Then $\dim_x X = \dim_{x_K} X_K$.*

Proof. Choose a presentation $S = k[x_1, \ldots, x_n]/I$. This gives a presentation $K \otimes_k S = K[x_1, \ldots, x_n]/(K \otimes_k I)$. Let $\mathfrak q'_K \subset K[x_1, \ldots, x_n]$, resp. $\mathfrak q' \subset k[x_1, \ldots, x_n]$ be the corresponding primes. Consider the following commutative diagram of Noetherian local rings

$$\begin{tikzcd} K[x_1, \ldots, x_n]_{\mathfrak q'_K} \arrow[r] & (K \otimes_k S)_{\mathfrak q_K} \\ k[x_1, \ldots, x_n]_{\mathfrak q'} \arrow[r] \arrow[u] & S_{\mathfrak q} \arrow[u] \end{tikzcd}$$

Both vertical arrows are flat because they are localizations of the flat ring maps $S \to S_K$ and $k[x_1, \ldots, x_n] \to K[x_1, \ldots, x_n]$. Moreover, the vertical arrows have the same fibre rings. Hence, we see from Lemma 112.7 that $\text{height}(\mathfrak q') - \text{height}(\mathfrak q) = \text{height}(\mathfrak q'_K) - \text{height}(\mathfrak q_K)$. Denote $x' \in X' = \text{Spec}(k[x_1, \ldots, x_n])$ and $x'_K \in X'_K = \text{Spec}(K[x_1, \ldots, x_n])$ the points corresponding to $\mathfrak q'$ and $\mathfrak q'_K$. By Lemma 116.4 and what we showed above we have

$$\begin{aligned} n - \dim_x X &= \dim_{x'} X' - \dim_x X \\ &= \text{height}(\mathfrak q') - \text{height}(\mathfrak q) \\ &= \text{height}(\mathfrak q'_K) - \text{height}(\mathfrak q_K) \\ &= \dim_{x'_K} X'_K - \dim_{x_K} X_K \\ &= n - \dim_{x_K} X_K \end{aligned}$$

and the lemma follows. □

0CWE **Lemma 116.7.** *Let k be a field. Let S be a finite type k-algebra. Let K/k be a field extension. Set $S_K = K \otimes_k S$. Let $\mathfrak q \subset S$ be a prime and let $\mathfrak q_K \subset S_K$ be a prime lying over $\mathfrak q$. Then*

$$\dim(S_K \otimes_S \kappa(\mathfrak q))_{\mathfrak q_K} = \dim(S_K)_{\mathfrak q_K} - \dim S_{\mathfrak q} = \mathit{trdeg}_k \kappa(\mathfrak q) - \mathit{trdeg}_K \kappa(\mathfrak q_K)$$

Moreover, given $\mathfrak q$ we can always choose $\mathfrak q_K$ such that the number above is zero.

Proof. Observe that $S_{\mathfrak q} \to (S_K)_{\mathfrak q_K}$ is a flat local homomorphism of local Noetherian rings with special fibre $(S_K \otimes_S \kappa(\mathfrak q))_{\mathfrak q_K}$. Hence the first equality by Lemma 112.7. The second equality follows from the fact that we have $\dim_x X = \dim_{x_K} X_K$ with notation as in Lemma 116.6 and we have $\dim_x X = \dim S_{\mathfrak q} + \text{trdeg}_k \kappa(\mathfrak q)$ by Lemma 116.3 and similarly for $\dim_{x_K} X_K$. If we choose $\mathfrak q_K$ minimal over $\mathfrak q S_K$, then the dimension of the fibre ring will be zero. $\square$

117. Dimension of graded algebras over a field

00P5 Here is a basic result.

00P6 **Lemma 117.1.** *Let k be a field. Let S be a graded k-algebra generated over k by finitely many elements of degree 1. Assume $S_0 = k$. Let $P(T) \in \mathbf{Q}[T]$ be the polynomial such that $\dim(S_d) = P(d)$ for all $d \gg 0$. See Proposition 58.7. Then*

1. *The irrelevant ideal S_+ is a maximal ideal $\mathfrak m$.*
2. *Any minimal prime of S is a homogeneous ideal and is contained in $S_+ = \mathfrak m$.*
3. *We have $\dim(S) = \deg(P) + 1 = \dim_x \text{Spec}(S)$ (with the convention that $\deg(0) = -1$) where x is the point corresponding to the maximal ideal $S_+ = \mathfrak m$.*
4. *The Hilbert function of the local ring $R = S_{\mathfrak m}$ is equal to the Hilbert function of S.*

Proof. The first statement is obvious. The second follows from Lemma 57.8. By (2) every irreducible component passes through x. Thus we have $\dim(S) = \dim_x \text{Spec}(S) = \dim(S_{\mathfrak m})$ by Lemma 114.5. Since $\mathfrak m^d/\mathfrak m^{d+1} \cong \mathfrak m^d S_{\mathfrak m}/\mathfrak m^{d+1} S_{\mathfrak m}$ we see that the Hilbert function of the local ring $S_{\mathfrak m}$ is equal to the Hilbert function of S, which is (4). We conclude the last equality of (3) by Proposition 60.9. $\square$

118. Generic flatness

051Q Basically this says that a finite type algebra over a domain becomes flat after inverting a single element of the domain. There are several versions of this result (in increasing order of strength).

051R **Lemma 118.1.** *Let $R \to S$ be a ring map. Let M be an S-module. Assume*

1. *R is Noetherian,*
2. *R is a domain,*
3. *$R \to S$ is of finite type, and*
4. *M is a finite type S-module.*

Then there exists a nonzero $f \in R$ such that M_f is a free R_f-module.

Proof. Let K be the fraction field of R. Set $S_K = K \otimes_R S$. This is an algebra of finite type over K. We will argue by induction on $d = \dim(S_K)$ (which is finite for example by Noether normalization, see Section 115). Fix $d \geq 0$. Assume we know that the lemma holds in all cases where $\dim(S_K) < d$.

Suppose given $R \to S$ and M as in the lemma with $\dim(S_K) = d$. By Lemma 62.1 there exists a filtration $0 \subset M_1 \subset M_2 \subset \ldots \subset M_n = M$ so that M_i/M_{i-1} is isomorphic to $S/\mathfrak q$ for some prime $\mathfrak q$ of S. Note that $\dim((S/\mathfrak q)_K) \leq \dim(S_K)$. Also, note that an extension of free modules is free (see basic notion 50). Thus we may assume $M = S$ and that S is a domain of finite type over R.

If $R \to S$ has a nontrivial kernel, then take a nonzero $f \in R$ in this kernel. In this case $S_f = 0$ and the lemma holds. (This is really the case $d = -1$ and the start of the induction.) Hence we may assume that $R \to S$ is a finite type extension of Noetherian domains.

Apply Lemma 115.7 and replace R by R_f (with f as in the lemma) to get a factorization

$$R \subset R[y_1, \ldots, y_d] \subset S$$

where the second extension is finite. Choose $z_1, \ldots, z_r \in S$ which form a basis for the fraction field of S over the fraction field of $R[y_1, \ldots, y_d]$. This gives a short exact sequence

$$0 \to R[y_1, \ldots, y_d]^{\oplus r} \xrightarrow{(z_1, \ldots, z_r)} S \to N \to 0$$

By construction N is a finite $R[y_1, \ldots, y_d]$-module whose support does not contain the generic point (0) of $\mathrm{Spec}(R[y_1, \ldots, y_d])$. By Lemma 40.5 there exists a nonzero $g \in R[y_1, \ldots, y_d]$ such that g annihilates N, so we may view N as a finite module over $S' = R[y_1, \ldots, y_d]/(g)$. Since $\dim(S'_K) < d$ by induction there exists a nonzero $f \in R$ such that N_f is a free R_f-module. Since $(R[y_1, \ldots, y_d])_f \cong R_f[y_1, \ldots, y_d]$ is free also we conclude by the already mentioned fact that an extension of free modules is free. □

051S **Lemma 118.2.** *Let $R \to S$ be a ring map. Let M be an S-module. Assume*

1. *R is a domain,*
2. *$R \to S$ is of finite presentation, and*
3. *M is an S-module of finite presentation.*

Then there exists a nonzero $f \in R$ such that M_f is a free R_f-module.

Proof. Write $S = R[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. For $g \in R[x_1, \ldots, x_n]$ denote $\overline{g}$ its image in S. We may write $M = S^{\oplus t}/\sum S n_i$ for some $n_i \in S^{\oplus t}$. Write $n_i = (\overline{g}_{i1}, \ldots, \overline{g}_{it})$ for some $g_{ij} \in R[x_1, \ldots, x_n]$. Let $R_0 \subset R$ be the subring generated by all the coefficients of all the elements $g_i, g_{ij} \in R[x_1, \ldots, x_n]$. Define $S_0 = R_0[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. Define $M_0 = S_0^{\oplus t}/\sum S_0 n_i$. Then R_0 is a domain of finite type over $\mathbf{Z}$ and hence Noetherian (see Lemma 31.1). Moreover via the injection $R_0 \to R$ we have $S \cong R \otimes_{R_0} S_0$ and $M \cong R \otimes_{R_0} M_0$. Applying Lemma 118.1 we obtain a nonzero $f \in R_0$ such that $(M_0)_f$ is a free $(R_0)_f$-module. Hence $M_f = R_f \otimes_{(R_0)_f} (M_0)_f$ is a free R_f-module. □

051T **Lemma 118.3.** *Let $R \to S$ be a ring map. Let M be an S-module. Assume*

1. *R is a domain,*
2. *$R \to S$ is of finite type, and*
3. *M is a finite type S-module.*

Then there exists a nonzero $f \in R$ such that

(a) *M_f and S_f are free as R_f-modules, and*
(b) *S_f is a finitely presented R_f-algebra and M_f is a finitely presented S_f-module.*

Proof. We first prove the lemma for $S = R[x_1, \ldots, x_n]$, and then we deduce the result in general.

Assume $S = R[x_1, \ldots, x_n]$. Choose elements $m_1, \ldots, m_t$ which generate M. This gives a short exact sequence

$$0 \to N \to S^{\oplus t} \xrightarrow{(m_1,\ldots,m_t)} M \to 0.$$

Denote K the fraction field of R. Denote $S_K = K \otimes_R S = K[x_1, \ldots, x_n]$, and similarly $N_K = K \otimes_R N$, $M_K = K \otimes_R M$. As $R \to K$ is flat the sequence remains exact after tensoring with K. As $S_K = K[x_1, \ldots, x_n]$ is a Noetherian ring (see Lemma 31.1) we can find finitely many elements $n'_1, \ldots, n'_s \in N_K$ which generate it. Choose $n_1, \ldots, n_r \in N$ such that $n'_i = \sum a_{ij} n_j$ for some $a_{ij} \in K$. Set

$$M' = S^{\oplus t} / \sum\nolimits_{i=1,\ldots,r} S n_i$$

By construction M' is a finitely presented S-module, and there is a surjection $M' \to M$ which induces an isomorphism $M'_K \cong M_K$. We may apply Lemma 118.2 to $R \to S$ and M' and we find an $f \in R$ such that M'_f is a free R_f-module. Thus $M'_f \to M_f$ is a surjection of modules over the domain R_f where the source is a free module and which becomes an isomorphism upon tensoring with K. Thus it is injective as $M'_f \subset M'_K$ as it is free over the domain R_f. Hence $M'_f \to M_f$ is an isomorphism and the result is proved.

For the general case, choose a surjection $R[x_1, \ldots, x_n] \to S$. Think of both S and M as finite modules over $R[x_1, \ldots, x_n]$. By the special case proved above there exists a nonzero $f \in R$ such that both S_f and M_f are free as R_f-modules and finitely presented as $R_f[x_1, \ldots, x_n]$-modules. Clearly this implies that S_f is a finitely presented R_f-algebra and that M_f is a finitely presented S_f-module. □

Let $R \to S$ be a ring map. Let M be an S-module. Consider the following condition on an element $f \in R$:

051U (118.3.1)
$$\begin{cases} S_f & \text{is of finite presentation over } R_f \\ M_f & \text{is of finite presentation as } S_f\text{-module} \\ S_f, M_f & \text{are free as } R_f\text{-modules} \end{cases}$$

We define

051V (118.3.2)
$$U(R \to S, M) = \bigcup\nolimits_{f \in R \text{ with } (118.3.1)} D(f)$$

which is an open subset of $\mathrm{Spec}(R)$.

051W **Lemma 118.4.** *Let $R \to S$ be a ring map. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of S-modules. Then*

$$U(R \to S, M_1) \cap U(R \to S, M_3) \subset U(R \to S, M_2).$$

Proof. Let $u \in U(R \to S, M_1) \cap U(R \to S, M_3)$. Choose $f_1, f_3 \in R$ such that $u \in D(f_1)$, $u \in D(f_3)$ and such that (118.3.1) holds for f_1 and M_1 and for f_3 and M_3. Then set $f = f_1 f_3$. Then $u \in D(f)$ and (118.3.1) holds for f and both M_1 and M_3. An extension of free modules is free, and an extension of finitely presented modules is finitely presented (Lemma 5.3). Hence we see that (118.3.1) holds for f and M_2. Thus $u \in U(R \to S, M_2)$ and we win. □

051X **Lemma 118.5.** *Let $R \to S$ be a ring map. Let M be an S-module. Let $f \in R$. Using the identification $\mathrm{Spec}(R_f) = D(f)$ we have $U(R_f \to S_f, M_f) = D(f) \cap U(R \to S, M)$.*

Proof. Suppose that $u \in U(R_f \to S_f, M_f)$. Then there exists an element $g \in R_f$ such that $u \in D(g)$ and such that (118.3.1) holds for the pair $((R_f)_g \to (S_f)_g, (M_f)_g)$. Write $g = a/f^n$ for some $a \in R$. Set $h = af$. Then $R_h = (R_f)_g$, $S_h = (S_f)_g$, and $M_h = (M_f)_g$. Moreover $u \in D(h)$. Hence $u \in U(R \to S, M)$. Conversely, suppose that $u \in D(f) \cap U(R \to S, M)$. Then there exists an element $g \in R$ such that $u \in D(g)$ and such that (118.3.1) holds for the pair $(R_g \to S_g, M_g)$. Then it is clear that (118.3.1) also holds for the pair $(R_{fg} \to S_{fg}, M_{fg}) = ((R_f)_g \to (S_f)_g, (M_f)_g)$. Hence $u \in U(R_f \to S_f, M_f)$ and we win. $\square$

051Y **Lemma 118.6.** *Let $R \to S$ be a ring map. Let M be an S-module. Let $U \subset \mathrm{Spec}(R)$ be a dense open. Assume there is a covering $U = \bigcup_{i \in I} D(f_i)$ of opens such that $U(R_{f_i} \to S_{f_i}, M_{f_i})$ is dense in $D(f_i)$ for each $i \in I$. Then $U(R \to S, M)$ is dense in $\mathrm{Spec}(R)$.*

Proof. In view of Lemma 118.5 this is a purely topological statement. Namely, by that lemma we see that $U(R \to S, M) \cap D(f_i)$ is dense in $D(f_i)$ for each $i \in I$. By Topology, Lemma 21.4 we see that $U(R \to S, M) \cap U$ is dense in U. Since U is dense in $\mathrm{Spec}(R)$ we conclude that $U(R \to S, M)$ is dense in $\mathrm{Spec}(R)$. $\square$

051Z **Lemma 118.7.** *Let $R \to S$ be a ring map. Let M be an S-module. Assume*

1. *$R \to S$ is of finite type,*
2. *M is a finite S-module, and*
3. *R is reduced.*

Then there exists a subset $U \subset \mathrm{Spec}(R)$ such that

1. *U is open and dense in $\mathrm{Spec}(R)$,*
2. *for every $u \in U$ there exists an $f \in R$ such that $u \in D(f) \subset U$ and such that we have*
   (a) *M_f and S_f are free over R_f,*
   (b) *S_f is a finitely presented R_f-algebra, and*
   (c) *M_f is a finitely presented S_f-module.*

Proof. Note that the lemma is equivalent to the statement that the open $U(R \to S, M)$, see Equation (118.3.2), is dense in $\mathrm{Spec}(R)$. We first prove the lemma for $S = R[x_1, \ldots, x_n]$, and then we deduce the result in general.

Proof of the case $S = R[x_1, \ldots, x_n]$ and M any finite module over S. Note that in this case $S_f = R_f[x_1, \ldots, x_n]$ is free and of finite presentation over R_f, so we do not have to worry about the conditions regarding S, only those that concern M. We will use induction on n.

There exists a finite filtration

$$0 \subset M_1 \subset M_2 \subset \ldots \subset M_t = M$$

such that $M_i/M_{i-1} \cong S/J_i$ for some ideal $J_i \subset S$, see Lemma 5.4. Since a finite intersection of dense opens is dense open, we see from Lemma 118.4 that it suffices to prove the lemma for each of the modules R/J_i. Hence we may assume that $M = S/J$ for some ideal J of $S = R[x_1, \ldots, x_n]$.

Let $I \subset R$ be the ideal generated by the coefficients of elements of J. Let $U_1 = \mathrm{Spec}(R) \setminus V(I)$ and let

$$U_2 = \mathrm{Spec}(R) \setminus \overline{U_1}.$$

Then it is clear that $U = U_1 \cup U_2$ is dense in $\mathrm{Spec}(R)$. Let $f \in R$ be an element such that either (a) $D(f) \subset U_1$ or (b) $D(f) \subset U_2$. If for any such f the lemma holds for the pair $(R_f \to R_f[x_1, \ldots, x_n], M_f)$ then by Lemma 118.6 we see that $U(R \to S, M)$ is dense in $\mathrm{Spec}(R)$. Hence we may assume either (a) $I = R$, or (b) $V(I) = \mathrm{Spec}(R)$.

In case (b) we actually have $I = 0$ as R is reduced! Hence $J = 0$ and $M = S$ and the lemma holds in this case.

In case (a) we have to do a little bit more work. Note that every element of I is actually the coefficient of a monomial of an element of J, because the set of coefficients of elements of J forms an ideal (details omitted). Hence we find an element

$$g = \sum\nolimits_{K \in E} a_K x^K \in J$$

where E is a finite set of multi-indices $K = (k_1, \ldots, k_n)$ with at least one coefficient a_{K_0} a unit in R. Actually we can find one which has a coefficient equal to 1 as $1 \in I$ in case (a). Let $m = \#\{K \in E \mid a_K \text{ is not a unit}\}$. Note that $0 \leq m \leq \#E - 1$. We will argue by induction on m.

The case $m = 0$. In this case all the coefficients a_K, $K \in E$ of g are units and $E \neq \emptyset$. If $E = \{K_0\}$ is a singleton and $K_0 = (0, \ldots, 0)$, then g is a unit and $J = S$ so the result holds for sure. (This happens in particular when $n = 0$ and it provides the base case of the induction on n.) If not $E = \{(0, \ldots, 0)\}$, then at least one K is not equal to $(0, \ldots, 0)$, i.e., $g \notin R$. At this point we employ the usual trick of Noether normalization. Namely, we consider

$$G(y_1, \ldots, y_n) = g(y_1 + y_n^{e_1}, y_2 + y_n^{e_2}, \ldots, y_{n-1} + y_n^{e_{n-1}}, y_n)$$

with $0 \ll e_{n-1} \ll e_{n-2} \ll \ldots \ll e_1$. By Lemma 115.2 it follows that $G(y_1, \ldots, y_n)$ as a polynomial in y_n looks like

$$a_K y_n^{k_n + \sum_{i=1,\ldots,n-1} e_i k_i} + \text{lower order terms in } y_n$$

As a_K is a unit we conclude that $M = R[x_1, \ldots, x_n]/J$ is finite over $R[y_1, \ldots, y_{n-1}]$. Hence $U(R \to R[x_1, \ldots, x_n], M) = U(R \to R[y_1, \ldots, y_{n-1}], M)$ and we win by induction on n.

The case $m > 0$. Pick a multi-index $K \in E$ such that a_K is not a unit. As before set $U_1 = \mathrm{Spec}(R_{a_K}) = \mathrm{Spec}(R) \setminus V(a_K)$ and set

$$U_2 = \mathrm{Spec}(R) \setminus \overline{U_1}.$$

Then it is clear that $U = U_1 \cup U_2$ is dense in $\mathrm{Spec}(R)$. Let $f \in R$ be an element such that either (a) $D(f) \subset U_1$ or (b) $D(f) \subset U_2$. If for any such f the lemma holds for the pair $(R_f \to R_f[x_1, \ldots, x_n], M_f)$ then by Lemma 118.6 we see that $U(R \to S, M)$ is dense in $\mathrm{Spec}(R)$. Hence we may assume either (a) $a_K R = R$, or (b) $V(a_K) = \mathrm{Spec}(R)$. In case (a) the number m drops, as a_K has turned into a unit. In case (b), since R is reduced, we conclude that $a_K = 0$. Hence the set E decreases so the number m drops as well. In both cases we win by induction on m.

At this point we have proven the lemma in case $S = R[x_1, \ldots, x_n]$. Assume that $(R \to S, M)$ is an arbitrary pair satisfying the conditions of the lemma. Choose a surjection $R[x_1, \ldots, x_n] \to S$. Observe that, with the notation introduced in (118.3.2), we have

$$U(R \to S, M) = U(R \to R[x_1, \ldots, x_n], S) \cap U(R \to R[x_1, \ldots, x_n], M)$$

Hence as we've just finished proving the right two opens are dense also the open on the left is dense. $\square$

119. Around Krull-Akizuki

00P7 One application of Krull-Akizuki is to show that there are plenty of discrete valuation rings. More generally in this section we show how to construct discrete valuation rings dominating Noetherian local rings.

First we show how to dominate a Noetherian local domain by a 1-dimensional Noetherian local domain by blowing up the maximal ideal.

00P8 **Lemma 119.1.** *Let R be a local Noetherian domain with fraction field K. Assume R is not a field. Then there exist $R \subset R' \subset K$ with*

1. *R' local Noetherian of dimension 1,*
2. *$R \to R'$ a local ring map, i.e., R' dominates R, and*
3. *$R \to R'$ essentially of finite type.*

Proof. Choose any valuation ring $A \subset K$ dominating R (which exist by Lemma 50.2). Denote v the corresponding valuation. Let $x_1, \ldots, x_r$ be a minimal set of generators of the maximal ideal $\mathfrak m$ of R. We may and do assume that $v(x_r) = \min\{v(x_1), \ldots, v(x_r)\}$. Consider the ring

$$S = R[x_1/x_r, x_2/x_r, \ldots, x_{r-1}/x_r] \subset K.$$

Note that $\mathfrak mS = x_rS$ is a principal ideal. Note that $S \subset A$ and that $v(x_r) > 0$, hence we see that $x_rS \neq S$. Choose a minimal prime $\mathfrak q$ over x_rS. Then $\text{height}(\mathfrak q) = 1$ by Lemma 60.11 and $\mathfrak q$ lies over $\mathfrak m$. Hence we see that $R' = S_{\mathfrak q}$ is a solution. $\square$

0BHZ **Lemma 119.2** (Kollár). *Let $(R, \mathfrak m)$ be a local Noetherian ring. Then exactly one of the following holds:*

This is taken from a forthcoming paper by János Kollár entitled "Variants of normality for Noetherian schemes".

1. *$(R, \mathfrak m)$ is Artinian,*
2. *$(R, \mathfrak m)$ is regular of dimension 1,*
3. *$\text{depth}(R) \geq 2$, or*
4. *there exists a finite ring map $R \to R'$ which is not an isomorphism whose kernel and cokernel are annihilated by a power of $\mathfrak m$ such that $\mathfrak m$ is not an associated prime of R' and $R' \neq 0$.*

Proof. Observe that $(R, \mathfrak m)$ is not Artinian if and only if $V(\mathfrak m) \subset \text{Spec}(R)$ is nowhere dense. See Proposition 60.7. We assume this from now on.

Let $J \subset R$ be the largest ideal killed by a power of $\mathfrak m$. If $J \neq 0$ then $R \to R/J$ shows that $(R, \mathfrak m)$ is as in (4).

Otherwise $J = 0$. In particular $\mathfrak m$ is not an associated prime of R and we see that there is a nonzerodivisor $x \in \mathfrak m$ by Lemma 63.18. If $\mathfrak m$ is not an associated prime of R/xR then $\text{depth}(R) \geq 2$ by the same lemma. Thus we are left with the case when there is an $y \in R$, $y \notin xR$ such that $y\mathfrak m \subset xR$.

If $y\mathfrak m \subset x\mathfrak m$ then we can consider the map $\varphi : \mathfrak m \to \mathfrak m$, $f \mapsto yf/x$ (well defined as x is a nonzerodivisor). By the determinantal trick of Lemma 16.2 there exists a monic polynomial P with coefficients in R such that $P(\varphi) = 0$. We conclude that $P(y/x) = 0$ in R_x. Let $R' \subset R_x$ be the ring generated by R and y/x. Then $R \subset R'$ and R'/R is a finite R-module annihilated by a power of $\mathfrak{m}$. Thus R is as in (4).

Otherwise there is a $t \in \mathfrak{m}$ such that $yt = ux$ for some unit u of R. After replacing t by $u^{-1}t$ we get $yt = x$. In particular y is a nonzerodivisor. For any $t' \in \mathfrak{m}$ we have $yt' = xs$ for some $s \in R$. Thus $y(t' - st) = xs - xs = 0$. Since y is not a zero-divisor this implies that $t' = ts$ and so $\mathfrak{m} = (t)$. Thus $(R, \mathfrak{m})$ is regular of dimension 1. □

00P9 **Lemma 119.3.** *Let R be a local ring with maximal ideal $\mathfrak{m}$. Assume R is Noetherian, has dimension 1, and that $\dim(\mathfrak{m}/\mathfrak{m}^2) > 1$. Then there exists a ring map $R \to R'$ such that*

1. *$R \to R'$ is finite,*
2. *$R \to R'$ is not an isomorphism,*
3. *the kernel and cokernel of $R \to R'$ are annihilated by a power of $\mathfrak{m}$, and*
4. *$\mathfrak{m}$ is not an associated prime of R'.*

Proof. This follows from Lemma 119.2 and the fact that R is not Artinian, not regular, and does not have depth ≥ 2 (the last part because the depth does not exceed the dimension by Lemma 72.3). □

00PA **Example 119.4.** Consider the Noetherian local ring

$$R = k[[x, y]]/(y^2)$$

It has dimension 1 and it is Cohen-Macaulay. An example of an extension as in Lemma 119.3 is the extension

$$k[[x, y]]/(y^2) \subset k[[x, z]]/(z^2), \quad y \mapsto xz$$

in other words it is gotten by adjoining y/x to R. The effect of repeating the construction $n > 1$ times is to adjoin the element y/x^n.

00PB **Example 119.5.** Let k be a field of characteristic $p > 0$ such that k has infinite degree over its subfield k^p of pth powers. For example $k = \mathbf{F}_p(t_1, t_2, t_3, \ldots)$. Consider the ring

$$A = \left\{ \sum a_i x^i \in k[[x]] \text{ such that } [k^p(a_0, a_1, a_2, \ldots) : k^p] < \infty \right\}$$

Then A is a discrete valuation ring and its completion is $A^\wedge = k[[x]]$. Note that the induced extension of fraction fields of $A \subset k[[x]]$ is infinite purely inseparable. Choose any $f \in k[[x]]$, $f \notin A$. Let $R = A[f] \subset k[[x]]$. Then R is a Noetherian local domain of dimension 1 whose completion $R^\wedge$ is nonreduced (think!).

00PC **Remark 119.6.** Suppose that R is a 1-dimensional semi-local Noetherian domain. If there is a maximal ideal $\mathfrak{m} \subset R$ such that $R_\mathfrak{m}$ is not regular, then we may apply Lemma 119.3 to $(R, \mathfrak{m})$ to get a finite ring extension $R \subset R_1$. (For example one can do this so that $\text{Spec}(R_1) \to \text{Spec}(R)$ is the blowup of $\text{Spec}(R)$ in the ideal $\mathfrak{m}$.) Of course R_1 is a 1-dimensional semi-local Noetherian domain with the same fraction field as R. If R_1 is not a regular semi-local ring, then we may repeat the construction to get $R_1 \subset R_2$. Thus we get a sequence

$$R \subset R_1 \subset R_2 \subset R_3 \subset \ldots$$

of finite ring extensions which may stop if R_n is regular for some n. Resolution of singularities would be the claim that eventually R_n is indeed regular. In reality this is not the case. Namely, there exists a characteristic 0 Noetherian local domain A of dimension 1 whose completion is nonreduced, see [FR70, Proposition 3.1] or our Examples, Section 16. For an example in characteristic $p > 0$ see Example 119.5. Since the construction of blowing up commutes with completion it is easy to see the sequence never stabilizes. See [Ben73] for a discussion (mostly in positive characteristic). On the other hand, if the completion of R in all of its maximal ideals is reduced, then the procedure stops (insert future reference here).

00PD **Lemma 119.7.** *Let A be a ring. The following are equivalent.*

1. *The ring A is a discrete valuation ring.*
2. *The ring A is a valuation ring and Noetherian but not a field.*
3. *The ring A is a regular local ring of dimension 1.*
4. *The ring A is a Noetherian local domain with maximal ideal $\mathfrak{m}$ generated by a single nonzero element.*
5. *The ring A is a Noetherian local normal domain of dimension 1.*

In this case if π is a generator of the maximal ideal of A, then every element of A can be uniquely written as $u\pi^n$, where $u \in A$ is a unit.

Proof. The equivalence of (1) and (2) is Lemma 50.18. Moreover, in the proof of Lemma 50.18 we saw that if A is a discrete valuation ring, then A is a PID, hence (3). Note that a regular local ring is a domain (see Lemma 106.2). Using this the equivalence of (3) and (4) follows from dimension theory, see Section 60.

Assume (3) and let π be a generator of the maximal ideal $\mathfrak{m}$. For all $n \geq 0$ we have $\dim_{A/\mathfrak{m}} \mathfrak{m}^n/\mathfrak{m}^{n+1} = 1$ because it is generated by π^n (and it cannot be zero). In particular $\mathfrak{m}^n = (\pi^n)$ and the graded ring $\bigoplus \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is isomorphic to the polynomial ring $A/\mathfrak{m}[T]$. For $x \in A \setminus \{0\}$ define $v(x) = \max\{n \mid x \in \mathfrak{m}^n\}$. In other words $x = u\pi^{v(x)}$ with $u \in A^*$. By the remarks above we have $v(xy) = v(x) + v(y)$ for all $x, y \in A \setminus \{0\}$. We extend this to the field of fractions K of A by setting $v(a/b) = v(a) - v(b)$ (well defined by multiplicativity shown above). Then it is clear that A is the set of elements of K which have valuation ≥ 0. Hence we see that A is a valuation ring by Lemma 50.16.

A valuation ring is a normal domain by Lemma 50.10. Hence we see that the equivalent conditions (1) – (3) imply (5). Assume (5). Suppose that $\mathfrak{m}$ cannot be generated by 1 element to get a contradiction. Then Lemma 119.3 implies there is a finite ring map $A \to A'$ which is an isomorphism after inverting any nonzero element of $\mathfrak{m}$ but not an isomorphism. In particular we may identify A' with a subset of the fraction field of A. Since $A \to A'$ is finite it is integral (see Lemma 36.3). Since A is normal we get $A = A'$ a contradiction. $\square$

09DZ **Definition 119.8.** Let A be a discrete valuation ring. A *uniformizer* is an element $\pi \in A$ which generates the maximal ideal of A.

By Lemma 119.7 any two uniformizers of a discrete valuation ring are associates.

00PE **Lemma 119.9.** *Let R be a domain with fraction field K. Let M be an R-submodule of $K^{\oplus r}$. Assume R is local Noetherian of dimension 1. For any nonzero $x \in R$ we have $\text{length}_R(R/xR) < \infty$ and*

$$\text{length}_R(M/xM) \leq r \cdot \text{length}_R(R/xR).$$

Proof. If x is a unit then the result is true. Hence we may assume $x \in \mathfrak{m}$ the maximal ideal of R. Since x is not zero and R is a domain we have $\dim(R/xR) = 0$, and hence R/xR has finite length. Consider $M \subset K^{\oplus r}$ as in the lemma. We may assume that the elements of M generate $K^{\oplus r}$ as a K-vector space after replacing $K^{\oplus r}$ by a smaller subspace if necessary.

Suppose first that M is a finite R-module. In that case we can clear denominators and assume $M \subset R^{\oplus r}$. Since M generates $K^{\oplus r}$ as a vectors space we see that $R^{\oplus r}/M$ has finite length. In particular there exists an integer $c \geq 0$ such that $x^c R^{\oplus r} \subset M$. Note that $M \supset xM \supset x^2M \supset \ldots$ is a sequence of modules with successive quotients each isomorphic to M/xM. Hence we see that

$$n\text{length}_R(M/xM) = \text{length}_R(M/x^nM).$$

The same argument for $M = R^{\oplus r}$ shows that

$$n\text{length}_R(R^{\oplus r}/xR^{\oplus r}) = \text{length}_R(R^{\oplus r}/x^nR^{\oplus r}).$$

By our choice of c above we see that x^nM is sandwiched between $x^nR^{\oplus r}$ and $x^{n+c}R^{\oplus r}$. This easily gives that

$$r(n + c)\text{length}_R(R/xR) \geq n\text{length}_R(M/xM) \geq r(n - c)\text{length}_R(R/xR)$$

Hence in the finite case we actually get the result of the lemma with equality.

Suppose now that M is not finite. Suppose that the length of M/xM is $\geq k$ for some natural number k. Then we can find

$$0 \subset N_0 \subset N_1 \subset N_2 \subset \ldots N_k \subset M/xM$$

with $N_i \neq N_{i+1}$ for $i = 0, \ldots k - 1$. Choose an element $m_i \in M$ whose congruence class mod xM falls into N_i but not into N_{i-1} for $i = 1, \ldots, k$. Consider the finite R-module $M' = Rm_1 + \ldots + Rm_k \subset M$. Let $N'_i \subset M'/xM'$ be the inverse image of N_i. It is clear that $N'_i \neq N'_{i+1}$ by our choice of m_i. Hence we see that $\text{length}_R(M'/xM') \geq k$. By the finite case we conclude $k \leq r\text{length}_R(R/xR)$ as desired. $\square$

Here is a first application.

031F **Lemma 119.10.** *Let $R \to S$ be a homomorphism of domains inducing an injection of fraction fields $K \subset L$. If R is Noetherian local of dimension 1 and $[L : K] < \infty$ then*

1. *each prime ideal $\mathfrak{n}_i$ of S lying over the maximal ideal $\mathfrak{m}$ of R is maximal,*
2. *there are finitely many of these, and*
3. *$[\kappa(\mathfrak{n}_i) : \kappa(\mathfrak{m})] < \infty$ for each i.*

Proof. Pick $x \in \mathfrak{m}$ nonzero. Apply Lemma 119.9 to the submodule $S \subset L \cong K^{\oplus n}$ where $n = [L : K]$. Thus the ring S/xS has finite length over R. It follows that $S/\mathfrak{m}S$ has finite length over $\kappa(\mathfrak{m})$. In other words, $\dim_{\kappa(\mathfrak{m})} S/\mathfrak{m}S$ is finite (Lemma 52.6). Thus $S/\mathfrak{m}S$ is Artinian (Lemma 53.2). The structural results on Artinian rings implies parts (1) and (2), see for example Lemma 53.6. Part (3) is implied by the finiteness established above. $\square$

00PF **Lemma 119.11.** *Let R be a domain with fraction field K. Let M be an R-submodule of $K^{\oplus r}$. Assume R is Noetherian of dimension 1. For any nonzero $x \in R$ we have length$_R(M/xM) < \infty$.*

Proof. Since R has dimension 1 we see that x is contained in finitely many primes $\mathfrak m_i$, $i = 1, \ldots, n$, each maximal. Since R is Noetherian we see that R/xR is Artinian and $R/xR = \prod_{i=1,\ldots,n}(R/xR)_{\mathfrak m_i}$ by Proposition 60.7 and Lemma 53.6. Hence M/xM similarly decomposes as the product $M/xM = \prod (M/xM)_{\mathfrak m_i}$ of its localizations at the $\mathfrak m_i$. By Lemma 119.9 applied to $M_{\mathfrak m_i}$ over $R_{\mathfrak m_i}$ we see each $M_{\mathfrak m_i}/xM_{\mathfrak m_i} = (M/xM)_{\mathfrak m_i}$ has finite length over $R_{\mathfrak m_i}$. Thus M/xM has finite length over R as the above implies M/xM has a finite filtration by R-submodules whose successive quotients are isomorphic to the residue fields $\kappa(\mathfrak m_i)$. □

00PG **Lemma 119.12** (Krull-Akizuki)**.** *Let R be a domain with fraction field K. Let L/K be a finite extension of fields. Assume R is Noetherian and $\dim(R) = 1$. In this case any ring A with $R \subset A \subset L$ is Noetherian.*

Proof. To begin we may assume that L is the fraction field of A by replacing L by the fraction field of A if necessary. Let $I \subset A$ be a nonzero ideal. Clearly I generates L as a K-vector space. Hence we see that $I \cap R \neq (0)$. Pick any nonzero $x \in I \cap R$. Then we get $I/xA \subset A/xA$. By Lemma 119.11 the R-module A/xA has finite length as an R-module. Hence I/xA has finite length as an R-module. Hence I is finitely generated as an ideal in A. □

00PH **Lemma 119.13.** *Let R be a Noetherian local domain with fraction field K. Assume that R is not a field. Let L/K be a finitely generated field extension. Then there exists discrete valuation ring A with fraction field L which dominates R.*

Proof. If L is not finite over K choose a transcendence basis $x_1, \ldots, x_r$ of L over K and replace R by $R[x_1, \ldots, x_r]$ localized at the maximal ideal generated by $\mathfrak m_R$ and $x_1, \ldots, x_r$. Thus we may assume $K \subset L$ finite.

By Lemma 119.1 we may assume $\dim(R) = 1$.

Let $A \subset L$ be the integral closure of R in L. By Lemma 119.12 this is Noetherian. By Lemma 36.17 there is a prime ideal $\mathfrak q \subset A$ lying over the maximal ideal of R. By Lemma 119.7 the ring $A_{\mathfrak q}$ is a discrete valuation ring dominating R as desired. □

120. Factorization

034O Here are some notions and relations between them that are typically taught in a first year course on algebra at the undergraduate level.

034P **Definition 120.1.** Let R be a domain.

1. Elements $x, y \in R$ are called *associates* if there exists a unit $u \in R^*$ such that $x = uy$.
2. An element $x \in R$ is called *irreducible* if it is nonzero, not a unit and whenever $x = yz$, $y, z \in R$, then y is either a unit or an associate of x.
3. An element $x \in R$ is called *prime* if the ideal generated by x is a prime ideal.

034Q **Lemma 120.2.** *Let R be a domain. Let $x, y \in R$. Then x, y are associates if and only if $(x) = (y)$.*

Proof. If $x = uy$ for some unit $u \in R$, then $(x) \subset (y)$ and $y = u^{-1}x$ so also $(y) \subset (x)$. Conversely, suppose that $(x) = (y)$. Then $x = fy$ and $y = gx$ for some $f, g \in A$. Then $x = fgx$ and since R is a domain $fg = 1$. Thus x and y are associates. □

034R **Lemma 120.3.** *Let R be a domain. Consider the following conditions:*

(1) *The ring R satisfies the ascending chain condition for principal ideals.*
(2) *Every nonzero, nonunit element $a \in R$ has a factorization $a = b_1 \ldots b_k$ with each b_i an irreducible element of R.*

Then (1) implies (2).

Proof. Let x be a nonzero element, not a unit, which does not have a factorization into irreducibles. Set $x_1 = x$. We can write $x = yz$ where neither y nor z is irreducible or a unit. Then either y does not have a factorization into irreducibles, in which case we set $x_2 = y$, or z does not have a factorization into irreducibles, in which case we set $x_2 = z$. Continuing in this fashion we find a sequence

$$x_1 | x_2 | x_3 | \ldots$$

of elements of R with x_n/x_{n+1} not a unit. This gives a strictly increasing sequence of principal ideals $(x_1) \subset (x_2) \subset (x_3) \subset \ldots$ thereby finishing the proof. □

034S **Definition 120.4.** A *unique factorization domain*, abbreviated *UFD*, is a domain R such that if $x \in R$ is a nonzero, nonunit, then x has a factorization into irreducibles, and if

$$x = a_1 \ldots a_m = b_1 \ldots b_n$$

are factorizations into irreducibles then $n = m$ and there exists a permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that a_i and $b_{\sigma(i)}$ are associates.

034T **Lemma 120.5.** *Let R be a domain. Assume every nonzero, nonunit factors into irreducibles. Then R is a UFD if and only if every irreducible element is prime.*

Proof. Assume R is a UFD and let $x \in R$ be an irreducible element. Say $ab \in (x)$, i.e., $ab = cx$. Choose factorizations $a = a_1 \ldots a_n$, $b = b_1 \ldots b_m$, and $c = c_1 \ldots c_r$. By uniqueness of the factorization

$$a_1 \ldots a_n b_1 \ldots b_m = c_1 \ldots c_r x$$

we find that x is an associate of one of the elements $a_1, \ldots, b_m$. In other words, either $a \in (x)$ or $b \in (x)$ and we conclude that x is prime.

Assume every irreducible element is prime. We have to prove that factorization into irreducibles is unique up to permutation and taking associates. Say $a_1 \ldots a_m = b_1 \ldots b_n$ with a_i and b_j irreducible. Since a_1 is prime, we see that $b_j \in (a_1)$ for some j. After renumbering we may assume $b_1 \in (a_1)$. Then $b_1 = a_1 u$ and since b_1 is irreducible we see that u is a unit. Hence a_1 and b_1 are associates and $a_2 \ldots a_n = ub_2 \ldots b_m$. By induction on $n + m$ we see that $n = m$ and a_i associate to $b_{\sigma(i)}$ for $i = 2, \ldots, n$ as desired. □

0AFT **Lemma 120.6.** *Let R be a Noetherian domain. Then R is a UFD if and only if every height 1 prime ideal is principal.*

Proof. Assume R is a UFD and let $\mathfrak{p}$ be a height 1 prime ideal. Take $x \in \mathfrak{p}$ nonzero and let $x = a_1 \ldots a_n$ be a factorization into irreducibles. Since $\mathfrak{p}$ is prime we see that $a_i \in \mathfrak{p}$ for some i. By Lemma 120.5 the ideal (a_i) is prime. Since $\mathfrak{p}$ has height 1 we conclude that $(a_i) = \mathfrak{p}$.

Assume every height 1 prime is principal. Since R is Noetherian every nonzero nonunit element x has a factorization into irreducibles, see Lemma 120.3. It suffices to prove that an irreducible element x is prime, see Lemma 120.5. Let $(x) \subset \mathfrak{p}$ be a prime minimal over (x). Then $\mathfrak p$ has height 1 by Lemma 60.11. By assumption $\mathfrak p = (y)$. Hence $x = yz$ and z is a unit as x is irreducible. Thus $(x) = (y)$ and we see that x is prime. $\square$

0AFU **Lemma 120.7** (Nagata's criterion for factoriality)**.** *Let A be a domain. Let $S \subset A$ be a multiplicative subset generated by prime elements. Let $x \in A$ be irreducible. Then* [Nag57, Lemma 2]

(1) *the image of x in $S^{-1}A$ is irreducible or a unit, and*
(2) *x is prime if and only if the image of x in $S^{-1}A$ is a unit or a prime element in $S^{-1}A$.*

Moreover, then A is a UFD if and only if every element of A has a factorization into irreducibles and $S^{-1}A$ is a UFD.

Proof. Say $x = \alpha\beta$ for $\alpha, \beta \in S^{-1}A$. Then $\alpha = a/s$ and $\beta = b/s'$ for $a, b \in A$, $s, s' \in S$. Thus we get $ss'x = ab$. By assumption we can write $ss' = p_1 \ldots p_r$ for some prime elements p_i. For each i the element p_i divides either a or b. Dividing we find a factorization $x = a'b'$ and $a = s''a'$, $b = s'''b'$ for some $s'', s''' \in S$. As x is irreducible, either a' or b' is a unit. Tracing back we find that either α or β is a unit. This proves (1).

Suppose x is prime. Then $A/(x)$ is a domain. Hence $S^{-1}A/xS^{-1}A = S^{-1}(A/(x))$ is a domain or zero. Thus x maps to a prime element or a unit.

Suppose that the image of x in $S^{-1}A$ is a unit. Then $yx = s$ for some $s \in S$ and $y \in A$. By assumption $s = p_1 \ldots p_r$ with p_i a prime element. For each i either p_i divides y or p_i divides x. In the second case p_i and x are associates (as x is irreducible) and we are done. But if the first case happens for all $i = 1, \ldots, r$, then x is a unit which is a contradiction.

Suppose that the image of x in $S^{-1}A$ is a prime element. Assume $a, b \in A$ and $ab \in (x)$. Then $sa = xy$ or $sb = xy$ for some $s \in S$ and $y \in A$. Say the first case happens. By assumption $s = p_1 \ldots p_r$ with p_i a prime element. For each i either p_i divides y or p_i divides x. In the second case p_i and x are associates (as x is irreducible) and we are done. If the first case happens for all $i = 1, \ldots, r$, then $a \in (x)$ as desired. This completes the proof of (2).

The final statement of the lemma follows from (1) and (2) and Lemma 120.5. $\square$

0BUD **Lemma 120.8.** *A UFD satisfies the ascending chain condition for principal ideals.*

Proof. Consider an ascending chain $(a_1) \subset (a_2) \subset (a_3) \subset \ldots$ of principal ideals in R. Write $a_1 = p_1^{e_1} \ldots p_r^{e_r}$ with p_i prime. Then we see that a_n is an associate of $p_1^{c_1} \ldots p_r^{c_r}$ for some $0 \leq c_i \leq e_i$. Since there are only finitely many possibilities we conclude. $\square$

0BUE **Lemma 120.9.** *Let R be a domain. Assume R has the ascending chain condition for principal ideals. Then the same property holds for a polynomial ring over R.*

Proof. Consider an ascending chain $(f_1) \subset (f_2) \subset (f_3) \subset \ldots$ of principal ideals in $R[x]$. Since f_{n+1} divides f_n we see that the degrees decrease in the sequence. Thus f_n has fixed degree $d \geq 0$ for all $n \gg 0$. Let a_n be the leading coefficient of f_n. The condition $f_n \in (f_{n+1})$ implies that a_{n+1} divides a_n for all n. By our assumption on R we see that a_{n+1} and a_n are associates for all n large enough (Lemma 120.2).

Thus for large n we see that $f_n = uf_{n+1}$ where $u \in R$ (for reasons of degree) is a unit (as a_n and a_{n+1} are associates). $\square$

0BC1 **Lemma 120.10.** *A polynomial ring over a UFD is a UFD. In particular, if k is a field, then $k[x_1, \ldots, x_n]$ is a UFD.*

Proof. Let R be a UFD. Then R satisfies the ascending chain condition for principal ideals (Lemma 120.8), hence $R[x]$ satisfies the ascending chain condition for principal ideals (Lemma 120.9), and hence every element of $R[x]$ has a factorization into irreducibles (Lemma 120.3). Let $S \subset R$ be the multiplicative subset generated by prime elements. Since every nonunit of R is a product of prime elements we see that $K = S^{-1}R$ is the fraction field of R. Observe that every prime element of R maps to a prime element of $R[x]$ and that $S^{-1}(R[x]) = S^{-1}R[x] = K[x]$ is a UFD (and even a PID). Thus we may apply Lemma 120.7 to conclude. $\square$

0AFV **Lemma 120.11.** *A unique factorization domain is normal.*

Proof. Let R be a UFD. Let x be an element of the fraction field of R which is integral over R. Say $x^d - a_1x^{d-1} - \ldots - a_d = 0$ with $a_i \in R$. We can write $x = up_1^{e_1} \ldots p_r^{e_r}$ with u a unit, $e_i \in \mathbf{Z}$, and $p_1, \ldots, p_r$ irreducible elements which are not associates. To prove the lemma we have to show $e_i \geq 0$. If not, say $e_1 < 0$, then for $N \gg 0$ we get

$$u^d p_2^{de_2+N} \ldots p_r^{de_r+N} = p_1^{-de_1} p_2^N \ldots p_r^N (\sum\nolimits_{i=1,\ldots,d} a_i x^{d-i}) \in (p_1)$$

which contradicts uniqueness of factorization in R. $\square$

034U **Definition 120.12.** A *principal ideal domain*, abbreviated *PID*, is a domain R such that every ideal is a principal ideal.

034V **Lemma 120.13.** *A principal ideal domain is a unique factorization domain.*

Proof. As a PID is Noetherian this follows from Lemma 120.6. $\square$

034W **Definition 120.14.** A *Dedekind domain* is a domain R such that every nonzero ideal $I \subset R$ can be written as a product

$$I = \mathfrak{p}_1 \ldots \mathfrak{p}_r$$

of nonzero prime ideals uniquely up to permutation of the $\mathfrak{p}_i$.

0AUQ **Lemma 120.15.** *A PID is a Dedekind domain.*

Proof. Let R be a PID. Since every nonzero ideal of R is principal, and R is a UFD (Lemma 120.13), this follows from the fact that every irreducible element in R is prime (Lemma 120.5) so that factorizations of elements turn into factorizations into primes. $\square$

09ME **Lemma 120.16.** *Let A be a ring. Let I and J be nonzero ideals of A such that $IJ = (f)$ for some nonzerodivisor $f \in A$. Then I and J are finitely generated ideals and finitely locally free of rank 1 as A-modules.*

Proof. It suffices to show that I and J are finite locally free A-modules of rank 1, see Lemma 78.2. To do this, write $f = \sum_{i=1,\ldots,n} x_iy_i$ with $x_i \in I$ and $y_i \in J$. We can also write $x_iy_i = a_if$ for some $a_i \in A$. Since f is a nonzerodivisor we see that $\sum a_i = 1$. Thus it suffices to show that each I_{a_i} and J_{a_i} is free of rank 1 over A_{a_i}. After replacing A by A_{a_i} we conclude that $f = xy$ for some $x \in I$ and $y \in J$. Note that both x and y are nonzerodivisors. We claim that $I = (x)$ and $J = (y)$ which finishes the proof. Namely, if $x' \in I$, then $x'y = af = axy$ for some $a \in A$. Hence $x' = ax$ and we win. $\square$

034X **Lemma 120.17.** *Let R be a ring. The following are equivalent*

1. *R is a Dedekind domain,*
2. *R is a Noetherian domain, and for every maximal ideal $\mathfrak{m}$ the local ring $R_\mathfrak{m}$ is a discrete valuation ring, and*
3. *R is a Noetherian, normal domain, and $\dim(R) \leq 1$.*

Proof. Assume (1). The argument is nontrivial because we did not assume that R was Noetherian in our definition of a Dedekind domain. Let $\mathfrak{p} \subset R$ be a prime ideal. Observe that $\mathfrak{p} \neq \mathfrak{p}^2$ by uniqueness of the factorizations in the definition. Pick $x \in \mathfrak{p}$ with $x \notin \mathfrak{p}^2$. Let $y \in \mathfrak{p}$ be a second element (for example $y = 0$). Write $(x, y) = \mathfrak{p}_1 \ldots \mathfrak{p}_r$. Since $(x, y) \subset \mathfrak{p}$ at least one of the primes $\mathfrak{p}_i$ is contained in $\mathfrak{p}$. But as $x \notin \mathfrak{p}^2$ there is at most one. Thus exactly one of $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ is contained in $\mathfrak{p}$, say $\mathfrak{p}_1 \subset \mathfrak{p}$. We conclude that $(x, y)R_\mathfrak{p} = \mathfrak{p}_1 R_\mathfrak{p}$ is prime for every choice of y. We claim that $(x)R_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}$. Namely, pick $y \in \mathfrak{p}$. By the above applied with y^2 we see that $(x, y^2)R_\mathfrak{p}$ is prime. Hence $y \in (x, y^2)R_\mathfrak{p}$, i.e., $y = ax + by^2$ in $R_\mathfrak{p}$. Thus $(1 - by)y = ax \in (x)R_\mathfrak{p}$, i.e., $y \in (x)R_\mathfrak{p}$ as desired.

Writing $(x) = \mathfrak{p}_1 \ldots \mathfrak{p}_r$ anew with $\mathfrak{p}_1 \subset \mathfrak{p}$ we conclude that $\mathfrak{p}_1 R_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}$, i.e., $\mathfrak{p}_1 = \mathfrak{p}$. Moreover, $\mathfrak{p}_1 = \mathfrak{p}$ is a finitely generated ideal of R by Lemma 120.16. We conclude that R is Noetherian by Lemma 28.10. Moreover, it follows that $R_\mathfrak{m}$ is a discrete valuation ring for every prime ideal $\mathfrak{p}$, see Lemma 119.7.

The equivalence of (2) and (3) follows from Lemmas 37.10 and 119.7. Assume (2) and (3) are satisfied. Let $I \subset R$ be an ideal. We will construct a factorization of I. If I is prime, then there is nothing to prove. If not, pick $I \subset \mathfrak{p}$ with $\mathfrak{p} \subset R$ maximal. Let $J = \{x \in R \mid x\mathfrak{p} \subset I\}$. We claim $J\mathfrak{p} = I$. It suffices to check this after localization at the maximal ideals $\mathfrak{m}$ of R (the formation of J commutes with localization and we use Lemma 23.1). Then either $\mathfrak{p}R_\mathfrak{m} = R_\mathfrak{m}$ and the result is clear, or $\mathfrak{p}R_\mathfrak{m} = \mathfrak{m}R_\mathfrak{m}$. In the last case $\mathfrak{p}R_\mathfrak{m} = (\pi)$ and the case where $\mathfrak{p}$ is principal is immediate. By Noetherian induction the ideal J has a factorization and we obtain the desired factorization of I. We omit the proof of uniqueness of the factorization. $\square$

The following is a variant of the Krull-Akizuki lemma.

09IG **Lemma 120.18.** *Let A be a Noetherian domain of dimension* 1 *with fraction field K. Let L/K be a finite extension. Let B be the integral closure of A in L. Then B is a Dedekind domain and* $\text{Spec}(B) \to \text{Spec}(A)$ *is surjective, has finite fibres, and induces finite residue field extensions.*

Proof. By Krull-Akizuki (Lemma 119.12) the ring B is Noetherian. By Lemma 112.4 $\dim(B) = 1$. Thus B is a Dedekind domain by Lemma 120.17. Surjectivity of the map on spectra follows from Lemma 36.17. The last two statements follow from Lemma 119.10. $\square$

121. Orders of vanishing

02MB

02MC **Lemma 121.1.** *Let R be a semi-local Noetherian ring of dimension 1. If $a, b \in R$ are nonzerodivisors then*

$$length_R(R/(ab)) = length_R(R/(a)) + length_R(R/(b))$$

and these lengths are finite.

Proof. We saw the finiteness in Lemma 119.11. Additivity holds since there is a short exact sequence $0 \to R/(a) \to R/(ab) \to R/(b) \to 0$ where the first map is given by multiplication by b. (Use length is additive, see Lemma 52.3.) □

02MD **Definition 121.2.** Suppose that K is a field, and $R \subset K$ is a local[10] Noetherian subring of dimension 1 with fraction field K. In this case we define the *order of vanishing along R*

$$\text{ord}_R : K^* \longrightarrow \mathbf{Z}$$

by the rule

$$\text{ord}_R(x) = \text{length}_R(R/(x))$$

if $x \in R$ and we set $\text{ord}_R(x/y) = \text{ord}_R(x) - \text{ord}_R(y)$ for $x, y \in R$ both nonzero.

We can use the order of vanishing to compare lattices in a vector space. Here is the definition.

02ME **Definition 121.3.** Let R be a Noetherian local domain of dimension 1 with fraction field K. Let V be a finite dimensional K-vector space. A *lattice in V* is a finite R-submodule $M \subset V$ such that $V = K \otimes_R M$.

The condition $V = K \otimes_R M$ signifies that M contains a basis for the vector space K. We remark that in many places in the literature the notion of a lattice may be defined only in case the ring R is a discrete valuation ring. If R is a discrete valuation ring then any lattice is a free R-module, and this may not be the case in general.

02MF **Lemma 121.4.** *Let R be a Noetherian local domain of dimension 1 with fraction field K. Let V be a finite dimensional K-vector space.*

1. *If M is a lattice in V and $M \subset M' \subset V$ is an R-submodule of V containing M then the following are equivalent*
   (a) *M' is a lattice,*
   (b) *$length_R(M'/M)$ is finite, and*
   (c) *M' is finitely generated.*
2. *If M is a lattice in V and $M' \subset M$ is an R-submodule of M then M' is a lattice if and only if $length_R(M/M')$ is finite.*
3. *If M, M' are lattices in V, then so are $M \cap M'$ and $M + M'$.*
4. *If $M \subset M' \subset M'' \subset V$ are lattices in V then*

$$length_R(M''/M) = length_R(M'/M) + length_R(M''/M').$$

[10] We could also define this when R is only semi-local but this is probably never really what you want!

(5) *If M, M', N, N' are lattices in V and $N \subset M \cap M'$, $M + M' \subset N'$, then we have*
$$\begin{aligned} & \mathit{length}_R(M/M \cap M') - \mathit{length}_R(M'/M \cap M') \\ = \quad & \mathit{length}_R(M/N) - \mathit{length}_R(M'/N) \\ = \quad & \mathit{length}_R(M + M'/M') - \mathit{length}_R(M + M'/M) \\ = \quad & \mathit{length}_R(N'/M') - \mathit{length}_R(N'/M) \end{aligned}$$

Proof. Proof of (1). Assume (1)(a). Say $y_1, \ldots, y_m$ generate M'. Then each $y_i = x_i/f_i$ for some $x_i \in M$ and nonzero $f_i \in R$. Hence we see that $f_1 \ldots f_m M' \subset M$. Since R is Noetherian local of dimension 1 we see that $\mathfrak{m}^n \subset (f_1 \ldots f_m)$ for some n (for example combine Lemmas 60.13 and Proposition 60.7 or combine Lemmas 119.9 and 52.4). In other words $\mathfrak{m}^n M' \subset M$ for some n Hence $\text{length}(M'/M) < \infty$ by Lemma 52.8, in other words (1)(b) holds. Assume (1)(b). Then M'/M is a finite R-module (see Lemma 52.2). Hence M' is a finite R-module as an extension of finite R-modules. Hence (1)(c). The implication (1)(c) $\Rightarrow$ (1)(a) follows from the remark following Definition 121.3.

Proof of (2). Suppose M is a lattice in V and $M' \subset M$ is an R-submodule. We have seen in (1) that if M' is a lattice, then $\text{length}_R(M/M') < \infty$. Conversely, assume that $\text{length}_R(M/M') < \infty$. Then M' is finitely generated as R is Noetherian and for some n we have $\mathfrak{m}^n M \subset M'$ (Lemma 52.4). Hence it follows that M' contains a basis for V, and M' is a lattice.

Proof of (3). Assume M, M' are lattices in V. Since R is Noetherian the submodule $M \cap M'$ of M is finite. As M is a lattice we can find $x_1, \ldots, x_n \in M$ which form a K-basis for V. Because M' is a lattice we can write $x_i = y_i/f_i$ with $y_i \in M'$ and $f_i \in R$. Hence $f_i x_i \in M \cap M'$. Hence $M \cap M'$ is a lattice also. The fact that $M + M'$ is a lattice follows from part (1).

Part (4) follows from additivity of lengths (Lemma 52.3) and the exact sequence
$$0 \to M'/M \to M''/M \to M''/M' \to 0$$
Part (5) follows from repeatedly applying part (4). ☐

02MG **Definition 121.5.** Let R be a Noetherian local domain of dimension 1 with fraction field K. Let V be a finite dimensional K-vector space. Let M, M' be two lattices in V. The *distance between M and M'* is the integer
$$d(M, M') = \text{length}_R(M/M \cap M') - \text{length}_R(M'/M \cap M')$$
of Lemma 121.4 part (5).

In particular, if $M' \subset M$, then $d(M, M') = \text{length}_R(M/M')$.

02MH **Lemma 121.6.** *Let R be a Noetherian local domain of dimension 1 with fraction field K. Let V be a finite dimensional K-vector space. This distance function has the property that*
$$d(M, M'') = d(M, M') + d(M', M'')$$
whenever given three lattices M, M', M'' of V. In particular we have $d(M, M') = -d(M', M)$.

Proof. Omitted. ☐

02MI **Lemma 121.7.** *Let R be a Noetherian local domain of dimension 1 with fraction field K. Let V be a finite dimensional K-vector space. Let $\varphi : V \to V$ be a K-linear isomorphism. For any lattice $M \subset V$ we have*

$$d(M, \varphi(M)) = ord_R(\det(\varphi))$$

Proof. We can see that the integer $d(M, \varphi(M))$ does not depend on the lattice M as follows. Suppose that M' is a second such lattice. Then we see that

$$\begin{aligned} d(M, \varphi(M)) &= d(M, M') + d(M', \varphi(M)) \\ &= d(M, M') + d(\varphi(M'), \varphi(M)) + d(M', \varphi(M')) \end{aligned}$$

Since φ is an isomorphism we see that $d(\varphi(M'), \varphi(M)) = d(M', M) = -d(M, M')$, and hence $d(M, \varphi(M)) = d(M', \varphi(M'))$. Moreover, both sides of the equation (of the lemma) are additive in φ, i.e.,

$$\mathrm{ord}_R(\det(\varphi \circ \psi)) = \mathrm{ord}_R(\det(\varphi)) + \mathrm{ord}_R(\det(\psi))$$

and also

$$\begin{aligned} d(M, \varphi(\psi((M))) &= d(M, \psi(M)) + d(\psi(M), \varphi(\psi(M))) \\ &= d(M, \psi(M)) + d(M, \varphi(M)) \end{aligned}$$

by the independence shown above. Hence it suffices to prove the lemma for generators of $\mathrm{GL}(V)$. Choose an isomorphism $K^{\oplus n} \cong V$. Then $\mathrm{GL}(V) = \mathrm{GL}_n(K)$ is generated by elementary matrices E. The result is clear for E equal to the identity matrix. If $E = E_{ij}(\lambda)$ with $i \neq j$, $\lambda \in K$, $\lambda \neq 0$, for example

$$E_{12}(\lambda) = \begin{pmatrix} 1 & \lambda & \ldots \\ 0 & 1 & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}$$

then with respect to a different basis we get $E_{12}(1)$. The result is clear for $E = E_{12}(1)$ by taking as lattice $R^{\oplus n} \subset K^{\oplus n}$. Finally, if $E = E_i(a)$, with $a \in K^*$ for example

$$E_1(a) = \begin{pmatrix} a & 0 & \ldots \\ 0 & 1 & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}$$

then $E_1(a)(R^{\oplus b}) = aR \oplus R^{\oplus n-1}$ and it is clear that $d(R^{\oplus n}, aR \oplus R^{\oplus n-1}) = \mathrm{ord}_R(a)$ as desired. □

02MJ **Lemma 121.8.** *Let $A \to B$ be a ring map. Assume*

(1) *A is a Noetherian local domain of dimension 1,*
(2) *$A \subset B$ is a finite extension of domains.*

Let L/K be the corresponding finite extension of fraction fields. Let $y \in L^*$ and $x = Nm_{L/K}(y)$. In this situation B is semi-local. Let $\mathfrak{m}_i$, $i = 1, \ldots, n$ be the maximal ideals of B. Then

$$ord_A(x) = \sum\nolimits_i [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m}_A)] ord_{B_{\mathfrak{m}_i}}(y)$$

where ord is defined as in Definition 121.2.

Proof. The ring B is semi-local by Lemma 113.2. Write $y = b/b'$ for some $b, b' \in B$. By the additivity of ord and multiplicativity of Nm it suffices to prove the lemma for $y = b$ or $y = b'$. In other words we may assume $y \in B$. In this case the right hand side of the formula is

$$\sum [\kappa(\mathfrak m_i) : \kappa(\mathfrak m_A)] \text{length}_{B_{\mathfrak m_i}}((B/yB)_{\mathfrak m_i})$$

By Lemma 52.12 this is equal to $\text{length}_A(B/yB)$. By Lemma 121.7 we have

$$\text{length}_A(B/yB) = d(B, yB) = \text{ord}_A(\det_K(L \xrightarrow{y} L)).$$

Since $x = \text{Nm}_{L/K}(y) = \det_K(L \xrightarrow{y} L)$ by definition the lemma is proved. $\square$

122. Quasi-finite maps

02MK Consider a ring map $R \to S$ of finite type. A map $\text{Spec}(S) \to \text{Spec}(R)$ is quasi-finite at a point if that point is isolated in its fibre. This means that the fibre is zero dimensional at that point. In this section we study the basic properties of this important but technical notion. More advanced material can be found in the next section.

00PJ **Lemma 122.1.** *Let k be a field. Let S be a finite type k algebra. Let $\mathfrak q$ be a prime of S. The following are equivalent:*

1. *$\mathfrak q$ is an isolated point of $\text{Spec}(S)$,*
2. *$S_{\mathfrak q}$ is finite over k,*
3. *there exists a $g \in S$, $g \notin \mathfrak q$ such that $D(g) = \{\mathfrak q\}$,*
4. *$\dim_{\mathfrak q} \text{Spec}(S) = 0$,*
5. *$\mathfrak q$ is a closed point of $\text{Spec}(S)$ and $\dim(S_{\mathfrak q}) = 0$, and*
6. *the field extension $\kappa(\mathfrak q)/k$ is finite and $\dim(S_{\mathfrak q}) = 0$.*

In this case $S = S_{\mathfrak q} \times S'$ for some finite type k-algebra S'. Also, the element g as in (3) has the property $S_{\mathfrak q} = S_g$.

Proof. Suppose $\mathfrak q$ is an isolated point of $\text{Spec}(S)$, i.e., $\{\mathfrak q\}$ is open in $\text{Spec}(S)$. Because $\text{Spec}(S)$ is a Jacobson space (see Lemmas 35.2 and 35.4) we see that $\mathfrak q$ is a closed point. Hence $\{\mathfrak q\}$ is open and closed in $\text{Spec}(S)$. By Lemmas 21.3 and 24.3 we may write $S = S_1 \times S_2$ with $\mathfrak q$ corresponding to the only point $\text{Spec}(S_1)$. Hence $S_1 = S_{\mathfrak q}$ is a zero dimensional ring of finite type over k. Hence it is finite over k for example by Lemma 115.4. We have proved (1) implies (2).

Suppose $S_{\mathfrak q}$ is finite over k. Then $S_{\mathfrak q}$ is Artinian local, see Lemma 53.2. So $\text{Spec}(S_{\mathfrak q}) = \{\mathfrak q S_{\mathfrak q}\}$ by Lemma 53.6. Consider the exact sequence $0 \to K \to S \to S_{\mathfrak q} \to Q \to 0$. It is clear that $K_{\mathfrak q} = Q_{\mathfrak q} = 0$. Also, K is a finite S-module as S is Noetherian and Q is a finite S-module since $S_{\mathfrak q}$ is finite over k. Hence there exists $g \in S$, $g \notin \mathfrak q$ such that $K_g = Q_g = 0$. Thus $S_{\mathfrak q} = S_g$ and $D(g) = \{\mathfrak q\}$. We have proved that (2) implies (3).

Suppose $D(g) = \{\mathfrak q\}$. Since $D(g)$ is open by construction of the topology on $\text{Spec}(S)$ we see that $\mathfrak q$ is an isolated point of $\text{Spec}(S)$. We have proved that (3) implies (1). In other words (1), (2) and (3) are equivalent.

Assume $\dim_{\mathfrak q} \text{Spec}(S) = 0$. This means that there is some open neighbourhood of $\mathfrak q$ in $\text{Spec}(S)$ which has dimension zero. Then there is an open neighbourhood of the form $D(g)$ which has dimension zero. Since S_g is Noetherian we conclude that S_g is Artinian and $D(g) = \text{Spec}(S_g)$ is a finite discrete set, see Proposition 60.7. Thus

$\mathfrak{q}$ is an isolated point of $D(g)$ and, by the equivalence of (1) and (2) above applied to $\mathfrak{q}S_g \subset S_g$, we see that $S_\mathfrak{q} = (S_g)_{\mathfrak{q}S_g}$ is finite over k. Hence (4) implies (2). It is clear that (1) implies (4). Thus (1) – (4) are all equivalent.

Lemma 114.6 gives the implication (5) $\Rightarrow$ (4). The implication (4) $\Rightarrow$ (6) follows from Lemma 116.3. The implication (6) $\Rightarrow$ (5) follows from Lemma 35.9. At this point we know (1) – (6) are equivalent.

The two statements at the end of the lemma we saw during the course of the proof of the equivalence of (1), (2) and (3) above. $\square$

00PK **Lemma 122.2.** *Let $R \to S$ be a ring map of finite type. Let $\mathfrak{q} \subset S$ be a prime lying over $\mathfrak{p} \subset R$. Let $F = \mathrm{Spec}(S \otimes_R \kappa(\mathfrak{p}))$ be the fibre of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$, see Remark 17.8. Denote $\overline{\mathfrak{q}} \in F$ the point corresponding to $\mathfrak{q}$. The following are equivalent*

(1) *$\overline{\mathfrak{q}}$ is an isolated point of F,*
(2) *$S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$ is finite over $\kappa(\mathfrak{p})$,*
(3) *there exists a $g \in S$, $g \notin \mathfrak{q}$ such that the only prime of $D(g)$ mapping to $\mathfrak{p}$ is $\mathfrak{q}$,*
(4) *$\dim_{\overline{\mathfrak{q}}}(F) = 0$,*
(5) *$\overline{\mathfrak{q}}$ is a closed point of F and $\dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) = 0$, and*
(6) *the field extension $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ is finite and $\dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) = 0$.*

Proof. Note that $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q} = (S \otimes_R \kappa(\mathfrak{p}))_{\overline{\mathfrak{q}}}$. Moreover $S \otimes_R \kappa(\mathfrak{p})$ is of finite type over $\kappa(\mathfrak{p})$. The conditions correspond exactly to the conditions of Lemma 122.1 for the $\kappa(\mathfrak{p})$-algebra $S \otimes_R \kappa(\mathfrak{p})$ and the prime $\overline{\mathfrak{q}}$, hence they are equivalent. $\square$

00PL **Definition 122.3.** Let $R \to S$ be a finite type ring map. Let $\mathfrak{q} \subset S$ be a prime.

(1) If the equivalent conditions of Lemma 122.2 are satisfied then we say $R \to S$ is *quasi-finite at* $\mathfrak{q}$.
(2) We say a ring map $A \to B$ is *quasi-finite* if it is of finite type and quasi-finite at all primes of B.

00PM **Lemma 122.4.** *Let $R \to S$ be a finite type ring map. Then $R \to S$ is quasi-finite if and only if for all primes $\mathfrak{p} \subset R$ the fibre $S \otimes_R \kappa(\mathfrak{p})$ is finite over $\kappa(\mathfrak{p})$.*

Proof. If the fibres are finite then the map is clearly quasi-finite. For the converse, note that $S \otimes_R \kappa(\mathfrak{p})$ is a $\kappa(\mathfrak{p})$-algebra of finite type and of dimension 0. Hence it is finite over $\kappa(\mathfrak{p})$ for example by Lemma 115.4. $\square$

077H **Lemma 122.5.** *Let $R \to S$ be a finite type ring map. Let $\mathfrak{q} \subset S$ be a prime lying over $\mathfrak{p} \subset R$. Let $f \in R$, $f \notin \mathfrak{p}$ and $g \in S$, $g \notin \mathfrak{q}$. Then $R \to S$ is quasi-finite at $\mathfrak{q}$ if and only if $R_f \to S_{fg}$ is quasi-finite at $\mathfrak{q}S_{fg}$.*

Proof. The fibre of $\mathrm{Spec}(S_{fg}) \to \mathrm{Spec}(R_f)$ is homeomorphic to an open subset of the fibre of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$. Hence the lemma follows from part (1) of the equivalent conditions of Lemma 122.2. $\square$

00PN **Lemma 122.6.** *Let*

$$\begin{tikzcd} S \arrow[r] & S' \\ R \arrow[u] \arrow[r] & R' \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{q} \arrow[r, dash] \arrow[d, dash] & \mathfrak{q}' \arrow[d, dash] \\ \mathfrak{p} \arrow[r, dash] & \mathfrak{p}' \end{tikzcd}$$

be a commutative diagram of rings with primes as indicated. Assume $R \to S$ of finite type, and $S \otimes_R R' \to S'$ surjective. If $R \to S$ is quasi-finite at $\mathfrak{q}$, then $R' \to S'$ is quasi-finite at $\mathfrak{q}'$.

Proof. Write $S \otimes_R \kappa(\mathfrak{p}) = S_1 \times S_2$ with S_1 finite over $\kappa(\mathfrak{p})$ and such that $\mathfrak{q}$ corresponds to a point of S_1 as in Lemma 122.1. This product decomposition induces a corresponding product decomposition for any $S \otimes_R \kappa(\mathfrak{p})$-algebra. In particular, we obtain $S' \otimes_{R'} \kappa(\mathfrak{p}') = S'_1 \times S'_2$. Because $S \otimes_R R' \to S'$ is surjective the canonical map $(S \otimes_R \kappa(\mathfrak{p})) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}') \to S' \otimes_{R'} \kappa(\mathfrak{p}')$ is surjective and hence $S_i \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}') \to S'_i$ is surjective. It follows that S'_1 is finite over $\kappa(\mathfrak{p}')$. The map $S' \otimes_{R'} \kappa(\mathfrak{p}') \to \kappa(\mathfrak{q}')$ factors through S'_1 (i.e. it annihilates the factor S'_2) because the map $S \otimes_R \kappa(\mathfrak{p}) \to \kappa(\mathfrak{q})$ factors through S_1 (i.e. it annihilates the factor S_2). Thus $\mathfrak{q}'$ corresponds to a point of $\mathrm{Spec}(S'_1)$ in the disjoint union decomposition of the fibre: $\mathrm{Spec}(S' \otimes_{R'} \kappa(\mathfrak{p}')) = \mathrm{Spec}(S'_1) \amalg \mathrm{Spec}(S'_2)$, see Lemma 21.2. Since S'_1 is finite over a field, it is Artinian ring, and hence $\mathrm{Spec}(S'_1)$ is a finite discrete set. (See Proposition 60.7.) We conclude $\mathfrak{q}'$ is isolated in its fibre as desired. $\square$

00PO **Lemma 122.7.** *A composition of quasi-finite ring maps is quasi-finite.*

Proof. Suppose $A \to B$ and $B \to C$ are quasi-finite ring maps. By Lemma 6.2 we see that $A \to C$ is of finite type. Let $\mathfrak{r} \subset C$ be a prime of C lying over $\mathfrak{q} \subset B$ and $\mathfrak{p} \subset A$. Since $A \to B$ and $B \to C$ are quasi-finite at $\mathfrak{q}$ and $\mathfrak{r}$ respectively, then there exist $b \in B$ and $c \in C$ such that $\mathfrak{q}$ is the only prime of $D(b)$ which maps to $\mathfrak{p}$ and similarly $\mathfrak{r}$ is the only prime of $D(c)$ which maps to $\mathfrak{q}$. If $c' \in C$ is the image of $b \in B$, then $\mathfrak{r}$ is the only prime of $D(cc')$ which maps to $\mathfrak{p}$. Therefore $A \to C$ is quasi-finite at $\mathfrak{r}$. $\square$

00PP **Lemma 122.8.** *Let $R \to S$ be a ring map of finite type. Let $R \to R'$ be any ring map. Set $S' = R' \otimes_R S$.*

(1) *The set $\{\mathfrak{q}' \mid R' \to S' \text{ quasi-finite at } \mathfrak{q}'\}$ is the inverse image of the corresponding set of $\mathrm{Spec}(S)$ under the canonical map $\mathrm{Spec}(S') \to \mathrm{Spec}(S)$.*
(2) *If $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is surjective, then $R \to S$ is quasi-finite if and only if $R' \to S'$ is quasi-finite.*
(3) *Any base change of a quasi-finite ring map is quasi-finite.*

Proof. Let $\mathfrak{p}' \subset R'$ be a prime lying over $\mathfrak{p} \subset R$. Then the fibre ring $S' \otimes_{R'} \kappa(\mathfrak{p}')$ is the base change of the fibre ring $S \otimes_R \kappa(\mathfrak{p})$ by the field extension $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{p}')$. Hence the first assertion follows from the invariance of dimension under field extension (Lemma 116.6) and Lemma 122.1. The stability of quasi-finite maps under base change follows from this and the stability of finite type property under base change. The second assertion follows since the assumption implies that given a prime $\mathfrak{q} \subset S$ we can find a prime $\mathfrak{q}' \subset S'$ lying over it. $\square$

0C6H **Lemma 122.9.** *Let $A \to B$ and $B \to C$ be ring homomorphisms such that $A \to C$ is of finite type. Let $\mathfrak{r}$ be a prime of C lying over $\mathfrak{q} \subset B$ and $\mathfrak{p} \subset A$. If $A \to C$ is quasi-finite at $\mathfrak{r}$, then $B \to C$ is quasi-finite at $\mathfrak{r}$.*

Proof. Observe that $B \to C$ is of finite type (Lemma 6.2) so that the statement makes sense. Let us use characterization (3) of Lemma 122.2. If $A \to C$ is quasi-finite at $\mathfrak{r}$, then there exists some $c \in C$ such that

$$\{\mathfrak{r}' \subset C \text{ lying over } \mathfrak{p}\} \cap D(c) = \{\mathfrak{r}\}.$$

Since the primes $\mathfrak{r}' \subset C$ lying over $\mathfrak{q}$ form a subset of the primes $\mathfrak{r}' \subset C$ lying over $\mathfrak{p}$ we conclude $B \to C$ is quasi-finite at $\mathfrak{r}$. $\square$

The following lemma is not quite about quasi-finite ring maps, but it does not seem to fit anywhere else so well.

02ML **Lemma 122.10.** *Let $R \to S$ be a ring map of finite type. Let $\mathfrak{p} \subset R$ be a minimal prime. Assume that there are at most finitely many primes of S lying over $\mathfrak{p}$. Then there exists a $g \in R$, $g \notin \mathfrak{p}$ such that the ring map $R_g \to S_g$ is finite.*

Proof. Let $x_1, \ldots, x_n$ be generators of S over R. Since $\mathfrak{p}$ is a minimal prime we have that $\mathfrak{p}R_{\mathfrak{p}}$ is a locally nilpotent ideal, see Lemma 25.1. Hence $\mathfrak{p}S_{\mathfrak{p}}$ is a locally nilpotent ideal, see Lemma 32.3. By assumption the finite type $\kappa(\mathfrak{p})$-algebra $S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}$ has finitely many primes. Hence (for example by Lemmas 61.3 and 115.4) $\kappa(\mathfrak{p}) \to S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}$ is a finite ring map. Thus we may find monic polynomials $P_i \in R_{\mathfrak{p}}[X]$ such that $P_i(x_i)$ maps to zero in $S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}$. By what we said above there exist $e_i \geq 1$ such that $P(x_i)^{e_i} = 0$ in $S_{\mathfrak{p}}$. Let $g_1 \in R$, $g_1 \notin \mathfrak{p}$ be an element such that P_i has coefficients in $R[1/g_1]$ for all i. Next, let $g_2 \in R$, $g_2 \notin \mathfrak{p}$ be an element such that $P(x_i)^{e_i} = 0$ in $S_{g_1 g_2}$. Setting $g = g_1 g_2$ we win. $\square$

123. Zariski's Main Theorem

00PI In this section our aim is to prove the algebraic version of Zariski's Main theorem. This theorem will be the basis of many further developments in the theory of schemes and morphisms of schemes later in the Stacks project.

Let $R \to S$ be a ring map of finite type. Our goal in this section is to show that the set of points of $\text{Spec}(S)$ where the map is quasi-finite is *open* (Theorem 123.12). In fact, it will turn out that there exists a finite ring map $R \to S'$ such that in some sense the quasi-finite locus of S/R is open in $\text{Spec}(S')$ (but we will not prove this in the algebra chapter since we do not develop the language of schemes here – for the case where $R \to S$ is quasi-finite see Lemma 123.14). These statements are somewhat tricky to prove and we do it by a long list of lemmas concerning integral and finite extensions of rings. This material may be found in [Ray70], and [Pes66]. We also found notes by Thierry Coquand helpful.

00PQ **Lemma 123.1.** *Let $\varphi : R \to S$ be a ring map. Suppose $t \in S$ satisfies the relation $\varphi(a_0) + \varphi(a_1)t + \ldots + \varphi(a_n)t^n = 0$. Then $\varphi(a_n)t$ is integral over R.*

Proof. Namely, multiply the equation $\varphi(a_0) + \varphi(a_1)t + \ldots + \varphi(a_n)t^n = 0$ with $\varphi(a_n)^{n-1}$ and write it as $\varphi(a_0 a_n^{n-1}) + \varphi(a_1 a_n^{n-2})(\varphi(a_n)t) + \ldots + (\varphi(a_n)t)^n = 0$. $\square$

The following lemma is in some sense the key lemma in this section.

00PT **Lemma 123.2.** *Let R be a ring. Let $\varphi : R[x] \to S$ be a ring map. Let $t \in S$. Assume that (a) t is integral over $R[x]$, and (b) there exists a monic $p \in R[x]$ such that $t\varphi(p) \in \text{Im}(\varphi)$. Then there exists a $q \in R[x]$ such that $t - \varphi(q)$ is integral over R.*

Proof. Write $t\varphi(p) = \varphi(r)$ for some $r \in R[x]$. Using euclidean division, write $r = qp + r'$ with $q, r' \in R[x]$ and $\deg(r') < \deg(p)$. We may replace t by $t - \varphi(q)$ which is still integral over $R[x]$, so that we obtain $t\varphi(p) = \varphi(r')$. In the ring S_t we may write this as $\varphi(p) - (1/t)\varphi(r') = 0$. This implies that $\varphi(x)$ gives an element of the localization S_t which is integral over $\varphi(R)[1/t] \subset S_t$. On the other hand, t is integral over the subring $\varphi(R)[\varphi(x)] \subset S$. Combined we conclude that t is integral over the subring $\varphi(R)[1/t] \subset S_t$, see Lemma 36.6. In other words there exists an equation of the form

$$t^d + \sum\nolimits_{i<d} \left(\sum\nolimits_{j=0,\ldots,n_i} \varphi(r_{i,j})/t^j\right) t^i = 0$$

in S_t with $r_{i,j} \in R$. This means that $t^{d+N} + \sum_{i<d} \sum_{j=0,\ldots,n_i} \varphi(r_{i,j}) t^{i+N-j} = 0$ in S for some N large enough. In other words t is integral over R. $\square$

00PV **Lemma 123.3.** *Let R be a ring. Let $\varphi : R[x] \to S$ be a ring map. Let $t \in S$. Assume t is integral over $R[x]$. Let $p \in R[x]$, $p = a_0 + a_1 x + \ldots + a_k x^k$ such that $t\varphi(p) \in \mathrm{Im}(\varphi)$. Then there exists a $q \in R[x]$ and $n \geq 0$ such that $\varphi(a_k)^n t - \varphi(q)$ is integral over R.*

Proof. Let R' and S' be the localization of R and S at the element a_k. Let $\varphi' : R'[x] \to S'$ be the localization of φ. Let $t' \in S'$ be the image of t. Set $p' = p/a_k \in R'[x]$. Then $t'\varphi'(p') \in \mathrm{Im}(\varphi')$ since $t\varphi(p) \in \mathrm{Im}(\varphi)$. As p' is monic, by Lemma 123.2 there exists a $q' \in R'[x]$ such that $t' - \varphi'(q')$ is integral over R'. We may choose an $n \geq 0$ and an element $q \in R[x]$ such that $a_k^n q'$ is the image of q. Then $\varphi(a_k)^n t - \varphi(q)$ is an element of S whose image in S' is integral over R'. By Lemma 36.11 there exists an $m \geq 0$ such that $\varphi(a_k)^m(\varphi(a_k)^n t - \varphi(q))$ is integral over R. Thus $\varphi(a_k)^{m+n} t - \varphi(a_k^m q)$ is integral over R as desired. $\square$

00PW **Situation 123.4.** Let R be a ring. Let $\varphi : R[x] \to S$ be finite. Let

$$J = \{g \in S \mid gS \subset \mathrm{Im}(\varphi)\}$$

be the "conductor ideal" of φ. Assume $\varphi(R) \subset S$ integrally closed in S.

00PX **Lemma 123.5.** *In Situation 123.4. Suppose $u \in S$, $a_0, \ldots, a_k \in R$, $u\varphi(a_0 + a_1 x + \ldots + a_k x^k) \in J$. Then there exists an $m \geq 0$ such that $u\varphi(a_k)^m \in J$.*

Proof. Assume that S is generated by $t_1, \ldots, t_n$ as an $R[x]$-module. In this case $J = \{g \in S \mid gt_i \in \mathrm{Im}(\varphi) \text{ for all } i\}$. Note that each element ut_i is integral over $R[x]$, see Lemma 36.3. We have $\varphi(a_0 + a_1 x + \ldots + a_k x^k) ut_i \in \mathrm{Im}(\varphi)$. By Lemma 123.3, for each i there exists an integer n_i and an element $q_i \in R[x]$ such that $\varphi(a_k^{n_i}) ut_i - \varphi(q_i)$ is integral over R. By assumption this element is in $\varphi(R)$ and hence $\varphi(a_k^{n_i}) ut_i \in \mathrm{Im}(\varphi)$. It follows that $m = \max\{n_1, \ldots, n_n\}$ works. $\square$

00PY **Lemma 123.6.** *In Situation 123.4. Suppose $u \in S$, $a_0, \ldots, a_k \in R$, $u\varphi(a_0 + a_1 x + \ldots + a_k x^k) \in \sqrt{J}$. Then $u\varphi(a_i) \in \sqrt{J}$ for all i.*

Proof. Under the assumptions of the lemma we have $u^n \varphi(a_0 + a_1 x + \ldots + a_k x^k)^n \in J$ for some $n \geq 1$. By Lemma 123.5 we deduce $u^n \varphi(a_k^{nm}) \in J$ for some $m \geq 1$. Thus $u\varphi(a_k) \in \sqrt{J}$, and so $u\varphi(a_0 + a_1 x + \ldots + a_k x^k) - u\varphi(a_k x^k) = u\varphi(a_0 + a_1 x + \ldots + a_{k-1} x^{k-1}) \in \sqrt{J}$. We win by induction on k. $\square$

This lemma suggests the following definition.

00PZ **Definition 123.7.** Given an inclusion of rings $R \subset S$ and an element $x \in S$ we say that x is *strongly transcendental over R* if whenever $u(a_0 + a_1 x + \ldots + a_k x^k) = 0$ with $u \in S$ and $a_i \in R$, then we have $ua_i = 0$ for all i.

Note that if S is a domain then this is the same as saying that x as an element of the fraction field of S is transcendental over the fraction field of R.

00Q0 **Lemma 123.8.** *Suppose $R \subset S$ is an inclusion of reduced rings and suppose that $x \in S$ is strongly transcendental over R. Let $\mathfrak q \subset S$ be a minimal prime and let $\mathfrak p = R \cap \mathfrak q$. Then the image of x in $S/\mathfrak q$ is strongly transcendental over the subring $R/\mathfrak p$.*

Proof. Suppose $u(a_0 + a_1x + \ldots + a_kx^k) \in \mathfrak q$. By Lemma 25.1 the local ring $S_{\mathfrak q}$ is a field, and hence $u(a_0 + a_1x + \ldots + a_kx^k)$ is zero in $S_{\mathfrak q}$. Thus $uu'(a_0 + a_1x + \ldots + a_kx^k) = 0$ for some $u' \in S$, $u' \not\in \mathfrak q$. Since x is strongly transcendental over R we get $uu'a_i = 0$ for all i. This in turn implies that $ua_i \in \mathfrak q$. $\square$

00Q1 **Lemma 123.9.** *Suppose $R \subset S$ is an inclusion of domains and let $x \in S$. Assume x is (strongly) transcendental over R and that S is finite over $R[x]$. Then $R \to S$ is not quasi-finite at any prime of S.*

Proof. As a first case, assume that R is normal, see Definition 37.11. By Lemma 37.14 we see that $R[x]$ is normal. Take a prime $\mathfrak q \subset S$, and set $\mathfrak p = R \cap \mathfrak q$. Assume that the extension $\kappa(\mathfrak p) \subset \kappa(\mathfrak q)$ is finite. This would be the case if $R \to S$ is quasi-finite at $\mathfrak q$. Let $\mathfrak r = R[x] \cap \mathfrak q$. Then since $\kappa(\mathfrak p) \subset \kappa(\mathfrak r) \subset \kappa(\mathfrak q)$ we see that the extension $\kappa(\mathfrak p) \subset \kappa(\mathfrak r)$ is finite too. Thus the inclusion $\mathfrak r \supset \mathfrak pR[x]$ is strict. By going down for $R[x] \subset S$, see Proposition 38.7, we find a prime $\mathfrak q' \subset \mathfrak q$, lying over the prime $\mathfrak pR[x]$. Hence the fibre $\text{Spec}(S \otimes_R \kappa(\mathfrak p))$ contains a point not equal to $\mathfrak q$, namely $\mathfrak q'$, whose closure contains $\mathfrak q$ and hence $\mathfrak q$ is not isolated in its fibre.

If R is not normal, let $R \subset R' \subset K$ be the integral closure R' of R in its field of fractions K. Let $S \subset S' \subset L$ be the subring S' of the field of fractions L of S generated by R' and S. Note that by construction the map $S \otimes_R R' \to S'$ is surjective. This implies that $R'[x] \subset S'$ is finite. Also, the map $S \subset S'$ induces a surjection on Spec, see Lemma 36.17. We conclude by Lemma 122.6 and the normal case we just discussed. $\square$

00Q2 **Lemma 123.10.** *Suppose $R \subset S$ is an inclusion of reduced rings. Assume $x \in S$ be strongly transcendental over R, and S finite over $R[x]$. Then $R \to S$ is not quasi-finite at any prime of S.*

Proof. Let $\mathfrak q \subset S$ be any prime. Choose a minimal prime $\mathfrak q' \subset \mathfrak q$. According to Lemmas 123.8 and 123.9 the extension $R/(R \cap \mathfrak q') \subset S/\mathfrak q'$ is not quasi-finite at the prime corresponding to $\mathfrak q$. By Lemma 122.6 the extension $R \to S$ is not quasi-finite at $\mathfrak q$. $\square$

00Q8 **Lemma 123.11.** *Let R be a ring. Let $S = R[x]/I$. Let $\mathfrak q \subset S$ be a prime. Assume $R \to S$ is quasi-finite at $\mathfrak q$. Let $S' \subset S$ be the integral closure of R in S. Then there exists an element $g \in S'$, $g \not\in \mathfrak q$ such that $S'_g \cong S_g$.*

Proof. Let $\mathfrak p$ be the image of $\mathfrak q$ in $\text{Spec}(R)$. There exists an $f \in I$, $f = a_nx^n + \ldots + a_0$ such that $a_i \not\in \mathfrak p$ for some i. Namely, otherwise the fibre ring $S \otimes_R \kappa(\mathfrak p)$ would be $\kappa(\mathfrak p)[x]$ and the map would not be quasi-finite at any prime lying over $\mathfrak p$. We conclude there exists a relation $b_mx^m + \ldots + b_0 = 0$ with $b_j \in S'$, $j = 0, \ldots, m$ and $b_j \not\in \mathfrak q \cap S'$ for some j. We prove the lemma by induction on m. The base case is $m = 0$ is vacuous (because the statements $b_0 = 0$ and $b_0 \not\in \mathfrak q$ are contradictory).

The case $b_m \not\in \mathfrak q$. In this case x is integral over S'_{b_m}, in fact $b_mx \in S'$ by Lemma 123.1. Hence the injective map $S'_{b_m} \to S_{b_m}$ is also surjective, i.e., an isomorphism as desired.

The case $b_m \in \mathfrak{q}$. In this case we have $b_m x \in S'$ by Lemma 123.1. Set $b'_{m-1} = b_m x + b_{m-1}$. Then

$$b'_{m-1}x^{m-1} + b_{m-2}x^{m-2} + \ldots + b_0 = 0$$

Since b'_{m-1} is congruent to b_{m-1} modulo $S' \cap \mathfrak{q}$ we see that it is still the case that one of $b'_{m-1}, b_{m-2}, \ldots, b_0$ is not in $S' \cap \mathfrak{q}$. Thus we win by induction on m. $\square$

00Q9 **Theorem 123.12** (Zariski's Main Theorem)**.** *Let R be a ring. Let $R \to S$ be a finite type R-algebra. Let $S' \subset S$ be the integral closure of R in S. Let $\mathfrak{q} \subset S$ be a prime of S. If $R \to S$ is quasi-finite at $\mathfrak{q}$ then there exists a $g \in S'$, $g \notin \mathfrak{q}$ such that $S'_g \cong S_g$.*

Proof. There exist finitely many elements $x_1, \ldots, x_n \in S$ such that S is finite over the R-sub algebra generated by $x_1, \ldots, x_n$. (For example generators of S over R.) We prove the proposition by induction on the minimal such number n.

The case $n = 0$ is trivial, because in this case $S' = S$, see Lemma 36.3.

The case $n = 1$. We may replace R by its integral closure in S (Lemma 122.9 guarantees that $R \to S$ is still quasi-finite at $\mathfrak{q}$). Thus we may assume $R \subset S$ is integrally closed in S, in other words $R = S'$. Consider the map $\varphi : R[x] \to S$, $x \mapsto x_1$. (We will see that φ is not injective below.) By assumption φ is finite. Hence we are in Situation 123.4. Let $J \subset S$ be the "conductor ideal" defined in Situation 123.4. Consider the diagram

$$\begin{tikzcd}
R[x] \arrow[r] & S \arrow[r] & S/\sqrt{J} & R/(R \cap \sqrt{J})[x] \arrow[l] \\
& R \arrow[ul] \arrow[u] \arrow[r] & R/(R \cap \sqrt{J}) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

According to Lemma 123.6 the image of x in the quotient $S/\sqrt{J}$ is strongly transcendental over $R/(R \cap \sqrt{J})$. Hence by Lemma 123.10 the ring map $R/(R \cap \sqrt{J}) \to S/\sqrt{J}$ is not quasi-finite at any prime of $S/\sqrt{J}$. By Lemma 122.6 we deduce that $\mathfrak{q}$ does not lie in $V(J) \subset \mathrm{Spec}(S)$. Thus there exists an element $s \in J$, $s \notin \mathfrak{q}$. By definition of J we may write $s = \varphi(f)$ for some polynomial $f \in R[x]$. Let $I = \mathrm{Ker}(\varphi : R[x] \to S)$. Since $\varphi(f) \in J$ we get $(R[x]/I)_f \cong S_{\varphi(f)}$. Also $s \notin \mathfrak{q}$ means that $f \notin \varphi^{-1}(\mathfrak{q})$. Thus $\varphi^{-1}(\mathfrak{q})$ is a prime of $R[x]/I$ at which $R \to R[x]/I$ is quasi-finite, see Lemma 122.5. Note that R is integrally closed in $R[x]/I$ since R is integrally closed in S. By Lemma 123.11 there exists an element $h \in R$, $h \notin R \cap \mathfrak{q}$ such that $R_h \cong (R[x]/I)_h$. Thus $(R[x]/I)_{fh} = S_{\varphi(fh)}$ is isomorphic to a principal localization $R_{h'}$ of R for some $h' \in R$, $h' \notin \mathfrak{q}$.

The case $n > 1$. Consider the subring $R' \subset S$ which is the integral closure of $R[x_1, \ldots, x_{n-1}]$ in S. By Lemma 122.9 the extension S/R' is quasi-finite at $\mathfrak{q}$. Also, note that S is finite over $R'[x_n]$. By the case $n = 1$ above, there exists a $g' \in R'$, $g' \notin \mathfrak{q}$ such that $(R')_{g'} \cong S_{g'}$. At this point we cannot apply induction to $R \to R'$ since R' may not be finite type over R. Since S is finitely generated over R we deduce in particular that $(R')_{g'}$ is finitely generated over R. Say the elements g', and $y_1/(g')^{n_1}, \ldots, y_N/(g')^{n_N}$ with $y_i \in R'$ generate $(R')_{g'}$ over R. Let R'' be the R-sub algebra of R' generated by $x_1, \ldots, x_{n-1}, y_1, \ldots, y_N, g'$. This has the property $(R'')_{g'} \cong S_{g'}$. Surjectivity because of how we chose y_i, injectivity because $R'' \subset R'$, and localization is exact. Note that R'' is finite over $R[x_1, \ldots, x_{n-1}]$ because of our choice of R', see Lemma 36.4. Let $\mathfrak{q}'' = R'' \cap \mathfrak{q}$. Since $(R'')_{\mathfrak{q}''} = S_{\mathfrak{q}}$ we see that $R \to R''$ is quasi-finite at $\mathfrak{q}''$, see Lemma 122.2. We apply our induction hypothesis to $R \to R''$, $\mathfrak{q}''$ and $x_1, \ldots, x_{n-1} \in R''$ and we find a subring $R''' \subset R''$ which is integral over R and an element $g'' \in R'''$, $g'' \not\in \mathfrak{q}''$ such that $(R''')_{g''} \cong (R'')_{g''}$. Write the image of g' in $(R'')_{g''}$ as $g'''/(g'')^n$ for some $g''' \in R'''$. Set $g = g''g''' \in R'''$. Then it is clear that $g \not\in \mathfrak{q}$ and $(R''')_g \cong S_g$. Since by construction we have $R''' \subset S'$ we also have $S'_g \cong S_g$ as desired. □

00QA **Lemma 123.13.** *Let $R \to S$ be a finite type ring map. The set of points $\mathfrak{q}$ of* $\text{Spec}(S)$ *at which S/R is quasi-finite is open in* $\text{Spec}(S)$.

Proof. Let $\mathfrak{q} \subset S$ be a point at which the ring map is quasi-finite. By Theorem 123.12 there exists an integral ring extension $R \to S'$, $S' \subset S$ and an element $g \in S'$, $g \not\in \mathfrak{q}$ such that $S'_g \cong S_g$. Since S and hence S_g are of finite type over R we may find finitely many elements $y_1, \ldots, y_N$ of S' such that $S''_g \cong S_g$ where $S'' \subset S'$ is the sub R-algebra generated by $g, y_1, \ldots, y_N$. Since S'' is finite over R (see Lemma 36.4) we see that S'' is quasi-finite over R (see Lemma 122.4). It is easy to see that this implies that S''_g is quasi-finite over R, for example because the property of being quasi-finite at a prime depends only on the local ring at the prime. Thus we see that S_g is quasi-finite over R. By the same token this implies that $R \to S$ is quasi-finite at every prime of S which lies in $D(g)$. □

00QB **Lemma 123.14.** *Let $R \to S$ be a finite type ring map. Suppose that S is quasi-finite over R. Let $S' \subset S$ be the integral closure of R in S. Then*

1. $\text{Spec}(S) \to \text{Spec}(S')$ *is a homeomorphism onto an open subset,*
2. *if $g \in S'$ and $D(g)$ is contained in the image of the map, then $S'_g \cong S_g$, and*
3. *there exists a finite R-algebra $S'' \subset S'$ such that (1) and (2) hold for the ring map $S'' \to S$.*

Proof. Because S/R is quasi-finite we may apply Theorem 123.12 to each point $\mathfrak{q}$ of $\text{Spec}(S)$. Since $\text{Spec}(S)$ is quasi-compact, see Lemma 17.10, we may choose a finite number of $g_i \in S'$, $i = 1, \ldots, n$ such that $S'_{g_i} = S_{g_i}$, and such that $g_1, \ldots, g_n$ generate the unit ideal in S (in other words the standard opens of $\text{Spec}(S)$ associated to $g_1, \ldots, g_n$ cover all of $\text{Spec}(S)$).

Suppose that $D(g) \subset \text{Spec}(S')$ is contained in the image. Then $D(g) \subset \bigcup D(g_i)$. In other words, $g_1, \ldots, g_n$ generate the unit ideal of S'_g. Note that $S'_{gg_i} \cong S_{gg_i}$ by our choice of g_i. Hence $S'_g \cong S_g$ by Lemma 23.2.

We construct a finite algebra $S'' \subset S'$ as in (3). To do this note that each $S'_{g_i} \cong S_{g_i}$ is a finite type R-algebra. For each i pick some elements $y_{ij} \in S'$ such that each S'_{g_i} is generated as R-algebra by $1/g_i$ and the elements y_{ij}. Then set S'' equal to the sub R-algebra of S' generated by all g_i and all the y_{ij}. Details omitted. □

124. Applications of Zariski's Main Theorem

03GB Here is an immediate application characterizing the finite maps of 1-dimensional semi-local rings among the quasi-finite ones as those where equality always holds in the formula of Lemma 121.8.

02MM **Lemma 124.1.** *Let $A \subset B$ be an extension of domains. Assume*

1. *A is a local Noetherian ring of dimension* 1,

(2) *$A \to B$ is of finite type, and*
(3) *the induced extension L/K of fraction fields is finite.*

Then B is semi-local. Let $x \in \mathfrak{m}_A$, $x \neq 0$. Let $\mathfrak{m}_i$, $i = 1, \ldots, n$ be the maximal ideals of B. Then

$$[L : K] ord_A(x) \geq \sum\nolimits_i [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m}_A)] ord_{B_{\mathfrak{m}_i}}(x)$$

where ord is defined as in Definition 121.2. We have equality if and only if $A \to B$ is finite.

Proof. The ring B is semi-local by Lemma 113.2. Let B' be the integral closure of A in B. By Lemma 123.14 we can find a finite A-subalgebra $C \subset B'$ such that on setting $\mathfrak{n}_i = C \cap \mathfrak{m}_i$ we have $C_{\mathfrak{n}_i} \cong B_{\mathfrak{m}_i}$ and the primes $\mathfrak{n}_1, \ldots, \mathfrak{n}_n$ are pairwise distinct. The ring C is semi-local by Lemma 113.2. Let $\mathfrak{p}_j$, $j = 1, \ldots, m$ be the other maximal ideals of C (the "missing points"). By Lemma 121.8 we have

$$\text{ord}_A(x^{[L:K]}) = \sum\nolimits_i [\kappa(\mathfrak{n}_i) : \kappa(\mathfrak{m}_A)] \text{ord}_{C_{\mathfrak{n}_i}}(x) + \sum\nolimits_j [\kappa(\mathfrak{p}_j) : \kappa(\mathfrak{m}_A)] \text{ord}_{C_{\mathfrak{p}_j}}(x)$$

hence the inequality follows. In case of equality we conclude that $m = 0$ (no "missing points"). Hence $C \subset B$ is an inclusion of semi-local rings inducing a bijection on maximal ideals and an isomorphism on all localizations at maximal ideals. So if $b \in B$, then $I = \{x \in C \mid xb \in C\}$ is an ideal of C which is not contained in any of the maximal ideals of C, and hence $I = C$, hence $b \in C$. Thus $B = C$ and B is finite over A. □

Here is a more standard application of Zariski's main theorem to the structure of local homomorphisms of local rings.

052V **Lemma 124.2.** *Let $(R, \mathfrak{m}_R) \to (S, \mathfrak{m}_S)$ be a local homomorphism of local rings. Assume*

(1) *$R \to S$ is essentially of finite type,*
(2) *$\kappa(\mathfrak{m}_R) \subset \kappa(\mathfrak{m}_S)$ is finite, and*
(3) *$\dim(S/\mathfrak{m}_R S) = 0$.*

Then S is the localization of a finite R-algebra.

Proof. Let S' be a finite type R-algebra such that $S = S'_{\mathfrak{q}'}$ for some prime $\mathfrak{q}'$ of S'. By Definition 122.3 we see that $R \to S'$ is quasi-finite at $\mathfrak{q}'$. After replacing S' by $S'_{g'}$ for some $g' \in S'$, $g' \notin \mathfrak{q}'$ we may assume that $R \to S'$ is quasi-finite, see Lemma 123.13. Then by Lemma 123.14 there exists a finite R-algebra S'' and elements $g' \in S'$, $g' \notin \mathfrak{q}'$ and $g'' \in S''$ such that $S'_{g'} \cong S''_{g''}$ as R-algebras. This proves the lemma. □

07NC **Lemma 124.3.** *Let $R \to S$ be a ring map, $\mathfrak{q}$ a prime of S lying over $\mathfrak{p}$ in R. If*

(1) *R is Noetherian,*
(2) *$R \to S$ is of finite type, and*
(3) *$R \to S$ is quasi-finite at $\mathfrak{q}$,*

then $R_{\mathfrak{p}}^\wedge \otimes_R S = S_{\mathfrak{q}}^\wedge \times B$ for some $R_{\mathfrak{p}}^\wedge$-algebra B.

Proof. There exists a finite R-algebra $S' \subset S$ and an element $g \in S'$, $g \notin \mathfrak{q}' = S' \cap \mathfrak{q}$ such that $S'_g = S_g$ and in particular $S'_{\mathfrak{q}'} = S_{\mathfrak{q}}$, see Lemma 123.14. We have

$$R_{\mathfrak{p}}^\wedge \otimes_R S' = (S'_{\mathfrak{q}'})^\wedge \times B'$$

by Lemma 97.8. Observe that under this product decomposition g maps to a pair (u, b') with $u \in (S'_{\mathfrak q'})^\wedge$ a unit because $g \not\in \mathfrak q'$. The product decomposition for $R_{\mathfrak p}^\wedge \otimes_R S'$ induces a product decomposition

$$R_{\mathfrak p}^\wedge \otimes_R S = A \times B$$

Since $S'_g = S_g$ we also have $(R_{\mathfrak p}^\wedge \otimes_R S')_g = (R_{\mathfrak p}^\wedge \otimes_R S)_g$ and since $g \mapsto (u, b')$ where u is a unit we see that $(S'_{\mathfrak q'})^\wedge = A$. Since the isomorphism $S'_{\mathfrak q'} = S_{\mathfrak q}$ determines an isomorphism on completions this also tells us that $A = S_{\mathfrak q}^\wedge$. This finishes the proof, except that we should perform the sanity check that the induced map $\phi : R_{\mathfrak p}^\wedge \otimes_R S \to A = S_{\mathfrak q}^\wedge$ is the natural one. For elements of the form $x \otimes 1$ with $x \in R_{\mathfrak p}^\wedge$ this is clear as the natural map $R_{\mathfrak p}^\wedge \to S_{\mathfrak q}^\wedge$ factors through $(S'_{\mathfrak q'})^\wedge$. For elements of the form $1 \otimes y$ with $y \in S$ we can argue that for some $n \geq 1$ the element $g^n y$ is the image of some $y' \in S'$. Thus $\phi(1 \otimes g^n y)$ is the image of y' by the composition $S' \to (S'_{\mathfrak q'})^\wedge \to S_{\mathfrak q}^\wedge$ which is equal to the image of $g^n y$ by the map $S \to S_{\mathfrak q}^\wedge$. Since g maps to a unit this also implies that $\phi(1 \otimes y)$ has the correct value, i.e., the image of y by $S \to S_{\mathfrak q}^\wedge$. □

125. Dimension of fibres

00QC We study the behaviour of dimensions of fibres, using Zariski's main theorem. Recall that we defined the dimension $\dim_x(X)$ of a topological space X at a point x in Topology, Definition 10.1.

00QD **Definition 125.1.** Suppose that $R \to S$ is of finite type, and let $\mathfrak q \subset S$ be a prime lying over a prime $\mathfrak p$ of R. We define the *relative dimension of S/R at $\mathfrak q$*, denoted $\dim_{\mathfrak q}(S/R)$, to be the dimension of $\text{Spec}(S \otimes_R \kappa(\mathfrak p))$ at the point corresponding to $\mathfrak q$. We let $\dim(S/R)$ be the supremum of $\dim_{\mathfrak q}(S/R)$ over all $\mathfrak q$. This is called the *relative dimension of S/R.*

In particular, $R \to S$ is quasi-finite at $\mathfrak q$ if and only if $\dim_{\mathfrak q}(S/R) = 0$. The following lemma is more or less a reformulation of Zariski's Main Theorem.

00QE **Lemma 125.2.** *Let $R \to S$ be a finite type ring map. Let $\mathfrak q \subset S$ be a prime. Suppose that $\dim_{\mathfrak q}(S/R) = n$. There exists a $g \in S$, $g \not\in \mathfrak q$ such that S_g is quasi-finite over a polynomial algebra $R[t_1, \ldots, t_n]$.*

Proof. The ring $\overline{S} = S \otimes_R \kappa(\mathfrak p)$ is of finite type over $\kappa(\mathfrak p)$. Let $\overline{\mathfrak q}$ be the prime of $\overline{S}$ corresponding to $\mathfrak q$. By definition of the dimension of a topological space at a point there exists an open $U \subset \text{Spec}(\overline{S})$ with $\overline{q} \in U$ and $\dim(U) = n$. Since the topology on $\text{Spec}(\overline{S})$ is induced from the topology on $\text{Spec}(S)$ (see Remark 17.8), we can find a $g \in S$, $g \not\in \mathfrak q$ with image $\overline{g} \in \overline{S}$ such that $D(\overline{g}) \subset U$. Thus after replacing S by S_g we see that $\dim(\overline{S}) = n$.

Next, choose generators $x_1, \ldots, x_N$ for S as an R-algebra. By Lemma 115.4 there exist elements $y_1, \ldots, y_n$ in the $\mathbf{Z}$-subalgebra of S generated by $x_1, \ldots, x_N$ such that the map $R[t_1, \ldots, t_n] \to S$, $t_i \mapsto y_i$ has the property that $\kappa(\mathfrak p)[t_1 \ldots, t_n] \to \overline{S}$ is finite. In particular, S is quasi-finite over $R[t_1, \ldots, t_n]$ at $\mathfrak q$. Hence, by Lemma 123.13 we may replace S by S_g for some $g \in S$, $g \not\in \mathfrak q$ such that $R[t_1, \ldots, t_n] \to S$ is quasi-finite. □

0520 **Lemma 125.3.** *Let $R \to S$ be a ring map. Let $\mathfrak q \subset S$ be a prime lying over the prime $\mathfrak p$ of R. Assume*

(1) *$R \to S$ is of finite type,*
(2) *$\dim_{\mathfrak q}(S/R) = n$, and*
(3) *$trdeg_{\kappa(\mathfrak p)}\kappa(\mathfrak q) = r$.*

Then there exist $f \in R$, $f \notin \mathfrak p$, $g \in S$, $g \notin \mathfrak q$ and a quasi-finite ring map

$$\varphi : R_f[x_1, \ldots, x_n] \longrightarrow S_g$$

such that $\varphi^{-1}(\mathfrak q S_g) = (\mathfrak p, x_{r+1}, \ldots, x_n)R_f[x_{r+1}, \ldots, x_n]$

Proof. After replacing S by a principal localization we may assume there exists a quasi-finite ring map $\varphi : R[t_1, \ldots, t_n] \to S$, see Lemma 125.2. Set $\mathfrak q' = \varphi^{-1}(\mathfrak q)$. Let $\overline{\mathfrak q}' \subset \kappa(\mathfrak p)[t_1, \ldots, t_n]$ be the prime corresponding to $\mathfrak q'$. By Lemma 115.6 there exists a finite ring map $\kappa(\mathfrak p)[x_1, \ldots, x_n] \to \kappa(\mathfrak p)[t_1, \ldots, t_n]$ such that the inverse image of $\overline{\mathfrak q}'$ is $(x_{r+1}, \ldots, x_n)$. Let $\overline{h}_i \in \kappa(\mathfrak p)[t_1, \ldots, t_n]$ be the image of x_i. We can find an element $f \in R$, $f \notin \mathfrak p$ and $h_i \in R_f[t_1, \ldots, t_n]$ which map to $\overline{h}_i$ in $\kappa(\mathfrak p)[t_1, \ldots, t_n]$. Then the ring map

$$R_f[x_1, \ldots, x_n] \longrightarrow R_f[t_1, \ldots, t_n]$$

becomes finite after tensoring with $\kappa(\mathfrak p)$. In particular, $R_f[t_1, \ldots, t_n]$ is quasi-finite over $R_f[x_1, \ldots, x_n]$ at the prime $\mathfrak q' R_f[t_1, \ldots, t_n]$. Hence, by Lemma 123.13 there exists a $g \in R_f[t_1, \ldots, t_n]$, $g \notin \mathfrak q' R_f[t_1, \ldots, t_n]$ such that $R_f[x_1, \ldots, x_n] \to R_f[t_1, \ldots, t_n, 1/g]$ is quasi-finite. Thus we see that the composition

$$R_f[x_1, \ldots, x_n] \longrightarrow R_f[t_1, \ldots, t_n, 1/g] \longrightarrow S_{\varphi(g)}$$

is quasi-finite and we win. $\square$

00QF **Lemma 125.4.** *Let $R \to S$ be a finite type ring map. Let $\mathfrak q \subset S$ be a prime lying over $\mathfrak p \subset R$. If $R \to S$ is quasi-finite at $\mathfrak q$, then $\dim(S_{\mathfrak q}) \leq \dim(R_{\mathfrak p})$.*

Proof. If $R_{\mathfrak p}$ is Noetherian (and hence $S_{\mathfrak q}$ Noetherian since it is essentially of finite type over $R_{\mathfrak p}$) then this follows immediately from Lemma 112.6 and the definitions. In the general case, let S' be the integral closure of $R_{\mathfrak p}$ in $S_{\mathfrak p}$. By Zariski's Main Theorem 123.12 we have $S_{\mathfrak q} = S'_{\mathfrak q'}$ for some $\mathfrak q' \subset S'$ lying over $\mathfrak q$. By Lemma 112.3 we have $\dim(S') \leq \dim(R_{\mathfrak p})$ and hence a fortiori $\dim(S_{\mathfrak q}) = \dim(S'_{\mathfrak q'}) \leq \dim(R_{\mathfrak p})$. $\square$

00QG **Lemma 125.5.** *Let k be a field. Let S be a finite type k-algebra. Suppose there is a quasi-finite k-algebra map $k[t_1, \ldots, t_n] \subset S$. Then $\dim(S) \leq n$.*

Proof. By Lemma 114.1 the dimension of any local ring of $k[t_1, \ldots, t_n]$ is at most n. Thus the result follows from Lemma 125.4. $\square$

00QH **Lemma 125.6.** *Let $R \to S$ be a finite type ring map. Let $\mathfrak q \subset S$ be a prime. Suppose that $\dim_{\mathfrak q}(S/R) = n$. There exists an open neighbourhood V of $\mathfrak q$ in $\mathrm{Spec}(S)$ such that $\dim_{\mathfrak q'}(S/R) \leq n$ for all $\mathfrak q' \in V$.*

Proof. By Lemma 125.2 we see that we may assume that S is quasi-finite over a polynomial algebra $R[t_1, \ldots, t_n]$. Considering the fibres, we reduce to Lemma 125.5. $\square$

In other words, the lemma says that the set of points where the fibre has dimension $\leq n$ is open in $\mathrm{Spec}(S)$. The next lemma says that formation of this open commutes with base change. If the ring map is of finite presentation then this set is quasi-compact open (see below).

00QI **Lemma 125.7.** *Let $R \to S$ be a finite type ring map. Let $R \to R'$ be any ring map. Set $S' = R' \otimes_R S$ and denote $f : \mathrm{Spec}(S') \to \mathrm{Spec}(S)$ the associated map on spectra. Let $n \geq 0$. The inverse image $f^{-1}(\{\mathfrak q \in \mathrm{Spec}(S) \mid \dim_{\mathfrak q}(S/R) \leq n\})$ is equal to $\{\mathfrak q' \in \mathrm{Spec}(S') \mid \dim_{\mathfrak q'}(S'/R') \leq n\}$.*

Proof. The condition is formulated in terms of dimensions of fibre rings which are of finite type over a field. Combined with Lemma 116.6 this yields the lemma. □

00QJ **Lemma 125.8.** *Let $R \to S$ be a ring homomorphism of finite presentation. Let $n \geq 0$. The set*

$$V_n = \{\mathfrak q \in \mathrm{Spec}(S) \mid \dim_{\mathfrak q}(S/R) \leq n\}$$

is a quasi-compact open subset of $\mathrm{Spec}(S)$.

Proof. It is open by Lemma 125.6. Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ be a presentation of S. Let R_0 be the $\mathbf{Z}$-subalgebra of R generated by the coefficients of the polynomials f_i. Let $S_0 = R_0[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Then $S = R \otimes_{R_0} S_0$. By Lemma 125.7 V_n is the inverse image of an open $V_{0,n}$ under the quasi-compact continuous map $\mathrm{Spec}(S) \to \mathrm{Spec}(S_0)$. Since S_0 is Noetherian we see that $V_{0,n}$ is quasi-compact. □

00QK **Lemma 125.9.** *Let R be a valuation ring with residue field k and field of fractions K. Let S be a domain containing R such that S is of finite type over R. If $S \otimes_R k$ is not the zero ring then*

$$\dim(S \otimes_R k) = \dim(S \otimes_R K)$$

In fact, $\mathrm{Spec}(S \otimes_R k)$ *is equidimensional.*

Proof. It suffices to show that $\dim_{\mathfrak q}(S/k)$ is equal to $\dim(S \otimes_R K)$ for every prime $\mathfrak q$ of S containing $\mathfrak m_R S$. Pick such a prime. By Lemma 125.6 the inequality $\dim_{\mathfrak q}(S/k) \geq \dim(S \otimes_R K)$ holds. Set $n = \dim_{\mathfrak q}(S/k)$. By Lemma 125.2 after replacing S by S_g for some $g \in S$, $g \notin \mathfrak q$ there exists a quasi-finite ring map $R[t_1, \ldots, t_n] \to S$. If $\dim(S \otimes_R K) < n$, then $K[t_1, \ldots, t_n] \to S \otimes_R K$ has a nonzero kernel. Say $f = \sum a_I t_1^{i_1} \ldots t_n^{i_n}$. After dividing f by a nonzero coefficient of f with minimal valuation, we may assume $f \in R[t_1, \ldots, t_n]$ and some a_I does not map to zero in k. Hence the ring map $k[t_1, \ldots, t_n] \to S \otimes_R k$ has a nonzero kernel which implies that $\dim(S \otimes_R k) < n$. Contradiction. □

126. Algebras and modules of finite presentation

05N4 In this section we discuss some standard results where the key feature is that the assumption involves a finite type or finite presentation assumption.

00QP **Lemma 126.1.** *Let $R \to S$ be a ring map. Let $R \to R'$ be a faithfully flat ring map. Set $S' = R' \otimes_R S$. Then $R \to S$ is of finite type if and only if $R' \to S'$ is of finite type.*

Proof. It is clear that if $R \to S$ is of finite type then $R' \to S'$ is of finite type. Assume that $R' \to S'$ is of finite type. Say $y_1, \ldots, y_m$ generate S' over R'. Write $y_j = \sum_i a_{ij} \otimes x_{ji}$ for some $a_{ij} \in R'$ and $x_{ji} \in S$. Let $A \subset S$ be the R-subalgebra generated by the x_{ij}. By flatness we have $A' := R' \otimes_R A \subset S'$, and by construction $y_j \in A'$. Hence $A' = S'$. By faithful flatness $A = S$. □

00QQ **Lemma 126.2.** *Let $R \to S$ be a ring map. Let $R \to R'$ be a faithfully flat ring map. Set $S' = R' \otimes_R S$. Then $R \to S$ is of finite presentation if and only if $R' \to S'$ is of finite presentation.*

Proof. It is clear that if $R \to S$ is of finite presentation then $R' \to S'$ is of finite presentation. Assume that $R' \to S'$ is of finite presentation. By Lemma 126.1 we see that $R \to S$ is of finite type. Write $S = R[x_1, \ldots, x_n]/I$. By flatness $S' = R'[x_1, \ldots, x_n]/R' \otimes I$. Say $g_1, \ldots, g_m$ generate $R' \otimes I$ over $R'[x_1, \ldots, x_n]$. Write $g_j = \sum_i a_{ij} \otimes f_{ji}$ for some $a_{ij} \in R'$ and $f_{ji} \in I$. Let $J \subset I$ be the ideal generated by the f_{ij}. By flatness we have $R' \otimes_R J \subset R' \otimes_R I$, and both are ideals over $R'[x_1, \ldots, x_n]$. By construction $g_j \in R' \otimes_R J$. Hence $R' \otimes_R J = R' \otimes_R I$. By faithful flatness $J = I$. $\square$

05N5 **Lemma 126.3.** *Let R be a ring. Let $I \subset R$ be an ideal. Let $S \subset R$ be a multiplicative subset. Set $R' = S^{-1}(R/I) = S^{-1}R/S^{-1}I$.*

1. *For any finite R'-module M' there exists a finite R-module M such that $S^{-1}(M/IM) \cong M'$.*
2. *For any finitely presented R'-module M' there exists a finitely presented R-module M such that $S^{-1}(M/IM) \cong M'$.*

Proof. Proof of (1). Choose a short exact sequence $0 \to K' \to (R')^{\oplus n} \to M' \to 0$. Let $K \subset R^{\oplus n}$ be the inverse image of K' under the map $R^{\oplus n} \to (R')^{\oplus n}$. Then $M = R^{\oplus n}/K$ works.

Proof of (2). Choose a presentation $(R')^{\oplus m} \to (R')^{\oplus n} \to M' \to 0$. Suppose that the first map is given by the matrix $A' = (a'_{ij})$ and the second map is determined by generators $x'_i \in M'$, $i = 1, \ldots, n$. As $R' = S^{-1}(R/I)$ we can choose $s \in S$ and a matrix $A = (a_{ij})$ with coefficients in R such that $a'_{ij} = a_{ij}/s \bmod S^{-1}I$. Let M be the finitely presented R-module with presentation $R^{\oplus m} \to R^{\oplus n} \to M \to 0$ where the first map is given by the matrix A and the second map is determined by generators $x_i \in M$, $i = 1, \ldots, n$. Then the map $M \to M'$, $x_i \mapsto x'_i$ induces an isomorphism $S^{-1}(M/IM) \cong M'$. $\square$

05N6 **Lemma 126.4.** *Let R be a ring. Let $S \subset R$ be a multiplicative subset. Let M be an R-module.*

1. *If $S^{-1}M$ is a finite $S^{-1}R$-module then there exists a finite R-module M' and a map $M' \to M$ which induces an isomorphism $S^{-1}M' \to S^{-1}M$.*
2. *If $S^{-1}M$ is a finitely presented $S^{-1}R$-module then there exists an R-module M' of finite presentation and a map $M' \to M$ which induces an isomorphism $S^{-1}M' \to S^{-1}M$.*

Proof. Proof of (1). Let $x_1, \ldots, x_n \in M$ be elements which generate $S^{-1}M$ as an $S^{-1}R$-module. Let M' be the R-submodule of M generated by $x_1, \ldots, x_n$.

Proof of (2). Let $x_1, \ldots, x_n \in M$ be elements which generate $S^{-1}M$ as an $S^{-1}R$-module. Let $K = \text{Ker}(R^{\oplus n} \to M)$ where the map is given by the rule $(a_1, \ldots, a_n) \mapsto \sum a_i x_i$. By Lemma 5.3 we see that $S^{-1}K$ is a finite $S^{-1}R$-module. By (1) we can find a finite submodule $K' \subset K$ with $S^{-1}K' = S^{-1}K$. Take $M' = \text{Coker}(K' \to R^{\oplus n})$. $\square$

05GJ **Lemma 126.5.** *Let R be a ring. Let $\mathfrak{p} \subset R$ be a prime ideal. Let M be an R-module.*

(1) *If $M_{\mathfrak p}$ is a finite $R_{\mathfrak p}$-module then there exists a finite R-module M' and a map $M' \to M$ which induces an isomorphism $M'_{\mathfrak p} \to M_{\mathfrak p}$.*
(2) *If $M_{\mathfrak p}$ is a finitely presented $R_{\mathfrak p}$-module then there exists an R-module M' of finite presentation and a map $M' \to M$ which induces an isomorphism $M'_{\mathfrak p} \to M_{\mathfrak p}$.*

Proof. This is a special case of Lemma 126.4 □

00QR **Lemma 126.6.** *Let $\varphi : R \to S$ be a ring map. Let $\mathfrak q \subset S$ be a prime lying over $\mathfrak p \subset R$. Assume*

(1) *S is of finite presentation over R,*
(2) *φ induces an isomorphism $R_{\mathfrak p} \cong S_{\mathfrak q}$.*

Then there exist $f \in R$, $f \notin \mathfrak p$ and an R_f-algebra C such that $S_f \cong R_f \times C$ as R_f-algebras.

Proof. Write $S = R[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. Let $a_i \in R_{\mathfrak p}$ be an element mapping to the image of x_i in $S_{\mathfrak q}$. Write $a_i = b_i/f$ for some $f \in R$, $f \notin \mathfrak p$. After replacing R by R_f and x_i by $x_i - a_i$ we may assume that $S = R[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$ such that x_i maps to zero in $S_{\mathfrak q}$. Then if c_j denotes the constant term of g_j we conclude that c_j maps to zero in $R_{\mathfrak p}$. After another replacement of R we may assume that the constant coefficients c_j of the g_j are zero. Thus we obtain an R-algebra map $S \to R$, $x_i \mapsto 0$ whose kernel is the ideal $(x_1, \ldots, x_n)$.

Note that $\mathfrak q = \mathfrak p S + (x_1, \ldots, x_n)$. Write $g_j = \sum a_{ji} x_i + h.o.t.$. Since $S_{\mathfrak q} = R_{\mathfrak p}$ we have $\mathfrak p \otimes \kappa(\mathfrak p) = \mathfrak q \otimes \kappa(\mathfrak q)$. It follows that $m \times n$ matrix $A = (a_{ji})$ defines a surjective map $\kappa(\mathfrak p)^{\oplus m} \to \kappa(\mathfrak p)^{\oplus n}$. Thus after inverting some element of R not in $\mathfrak p$ we may assume there are $b_{ij} \in R$ such that $\sum b_{ij} g_j = x_i + h.o.t.$. We conclude that $(x_1, \ldots, x_n) = (x_1, \ldots, x_n)^2$ in S. It follows from Lemma 21.5 that $(x_1, \ldots, x_n)$ is generated by an idempotent e. Setting $C = eS$ finishes the proof. □

00QS **Lemma 126.7.** *Let R be a ring. Let S, S' be of finite presentation over R. Let $\mathfrak q \subset S$ and $\mathfrak q' \subset S'$ be primes. If $S_{\mathfrak q} \cong S'_{\mathfrak q'}$ as R-algebras, then there exist $g \in S$, $g \notin \mathfrak q$ and $g' \in S'$, $g' \notin \mathfrak q'$ such that $S_g \cong S'_{g'}$ as R-algebras.*

Proof. Let $\psi : S_{\mathfrak q} \to S'_{\mathfrak q'}$ be the isomorphism of the hypothesis of the lemma. Write $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$ and $S' = R[y_1, \ldots, y_m]/J$. For each $i = 1, \ldots, n$ choose a fraction h_i/g_i with $h_i, g_i \in R[y_1, \ldots, y_m]$ and $g_i \bmod J$ not in $\mathfrak q'$ which represents the image of x_i under ψ. After replacing S' by $S'_{g_1 \ldots g_n}$ and $R[y_1, \ldots, y_m, y_{m+1}]$ (mapping y_{m+1} to $1/(g_1 \ldots g_n)$) we may assume that $\psi(x_i)$ is the image of some $h_i \in R[y_1, \ldots, y_m]$. Consider the elements $f_j(h_1, \ldots, h_n) \in R[y_1, \ldots, y_m]$. Since ψ kills each f_j we see that there exists a $g \in R[y_1, \ldots, y_m]$, $g \bmod J \notin \mathfrak q'$ such that $g f_j(h_1, \ldots, h_n) \in J$ for each $j = 1, \ldots, r$. After replacing S' by S'_g and $R[y_1, \ldots, y_m, y_{m+1}]$ as before we may assume that $f_j(h_1, \ldots, h_n) \in J$. Thus we obtain a ring map $S \to S'$, $x_i \mapsto h_i$ which induces ψ on local rings. By Lemma 6.2 the map $S \to S'$ is of finite presentation. By Lemma 126.6 we may assume that $S' = S \times C$. Thus localizing S' at the idempotent corresponding to the factor C we obtain the result. □

0G8U **Lemma 126.8.** *Let R be a ring. Let $I \subset R$ be a nilpotent ideal. Let S be an R-algebra such that $R/I \to S/IS$ is of finite type. Then $R \to S$ is of finite type.*

Proof. Choose $s_1, \ldots, s_n \in S$ whose images in S/IS generate S/IS as an algebra over R/I. By Lemma 20.1 part (11) we see that the R-algebra map $R[x_1, \ldots, x_n \to S$, $x_i \mapsto s_i$ is surjective and we conclude. $\square$

07RD **Lemma 126.9.** *Let R be a ring. Let $I \subset R$ be a locally nilpotent ideal. Let $S \to S'$ be an R-algebra map such that $S \to S'/IS'$ is surjective and such that S' is of finite type over R. Then $S \to S'$ is surjective.*

Proof. Write $S' = R[x_1, \ldots, x_m]/K$ for some ideal K. By assumption there exist $g_j = x_j + \sum \delta_{j,J} x^J \in R[x_1, \ldots, x_n]$ with $\delta_{j,J} \in I$ and with $g_j \bmod K \in \mathrm{Im}(S \to S')$. Hence it suffices to show that $g_1, \ldots, g_m$ generate $R[x_1, \ldots, x_n]$. Let $R_0 \subset R$ be a finitely generated $\mathbf{Z}$-subalgebra of R containing at least the $\delta_{j,J}$. Then $R_0 \cap I$ is a nilpotent ideal (by Lemma 32.5). It follows that $R_0[x_1, \ldots, x_n]$ is generated by $g_1, \ldots, g_m$ (because $x_j \mapsto g_j$ defines an automorphism of $R_0[x_1, \ldots, x_m]$; details omitted). Since R is the union of the subrings R_0 we win. $\square$

087P **Lemma 126.10.** *Let R be a ring. Let $I \subset R$ be an ideal. Let $S \to S'$ be an R-algebra map. Let $IS \subset \mathfrak{q} \subset S$ be a prime ideal. Assume that*

1. *$S \to S'$ is surjective,*
2. *$S_\mathfrak{q}/IS_\mathfrak{q} \to S'_\mathfrak{q}/IS'_\mathfrak{q}$ is an isomorphism,*
3. *S is of finite type over R,*
4. *S' of finite presentation over R, and*
5. *$S'_\mathfrak{q}$ is flat over R.*

Then $S_g \to S'_g$ is an isomorphism for some $g \in S$, $g \notin \mathfrak{q}$.

Proof. Let $J = \mathrm{Ker}(S \to S')$. By Lemma 6.2 J is a finitely generated ideal. Since $S'_\mathfrak{q}$ is flat over R we see that $J_\mathfrak{q}/IJ_\mathfrak{q} \subset S_\mathfrak{q}/IS_\mathfrak{q}$ (apply Lemma 39.12 to $0 \to J \to S \to S' \to 0$). By assumption (2) we see that $J_\mathfrak{q}/IJ_\mathfrak{q}$ is zero. By Nakayama's lemma (Lemma 20.1) we see that there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $J_g = 0$. Hence $S_g \cong S'_g$ as desired. $\square$

07RE **Lemma 126.11.** *Let R be a ring. Let $I \subset R$ be an ideal. Let $S \to S'$ be an R-algebra map. Assume that*

1. *I is locally nilpotent,*
2. *$S/IS \to S'/IS'$ is an isomorphism,*
3. *S is of finite type over R,*
4. *S' of finite presentation over R, and*
5. *S' is flat over R.*

Then $S \to S'$ is an isomorphism.

Proof. By Lemma 126.9 the map $S \to S'$ is surjective. As I is locally nilpotent, so are the ideals IS and IS' (Lemma 32.3). Hence every prime ideal $\mathfrak{q}$ of S contains IS and (trivially) $S_\mathfrak{q}/IS_\mathfrak{q} \cong S'_\mathfrak{q}/IS'_\mathfrak{q}$. Thus Lemma 126.10 applies and we see that $S_\mathfrak{q} \to S'_\mathfrak{q}$ is an isomorphism for every prime $\mathfrak{q} \subset S$. It follows that $S \to S'$ is injective for example by Lemma 23.1. $\square$

127. Colimits and maps of finite presentation

00QL In this section we prove some preliminary lemmas which will eventually help us prove result using absolute Noetherian reduction. In Categories, Section 19 we discuss filtered colimits in general. Here is an example of this very general notion.

0BUF **Lemma 127.1.** *Let $R \to A$ be a ring map. Consider the category $\mathcal{I}$ of all diagrams of R-algebra maps $A' \to A$ with A' finitely presented over R. Then $\mathcal{I}$ is filtered, and the colimit of the A' over $\mathcal{I}$ is isomorphic to A.*

Proof. The category[11] $\mathcal{I}$ is nonempty as $R \to R$ is an object of it. Consider a pair of objects $A' \to A$, $A'' \to A$ of $\mathcal{I}$. Then $A' \otimes_R A'' \to A$ is in $\mathcal{I}$ (use Lemmas 6.2 and 14.2). The ring maps $A' \to A' \otimes_R A''$ and $A'' \to A' \otimes_R A''$ define arrows in $\mathcal{I}$ thereby proving the second defining property of a filtered category, see Categories, Definition 19.1. Finally, suppose that we have two morphisms $\sigma, \tau : A' \to A''$ in $\mathcal{I}$. If $x_1, \ldots, x_r \in A'$ are generators of A' as an R-algebra, then we can consider $A''' = A''/(\sigma(x_i) - \tau(x_i))$. This is a finitely presented R-algebra and the given R-algebra map $A'' \to A$ factors through the surjection $\nu : A'' \to A'''$. Thus ν is a morphism in $\mathcal{I}$ equalizing σ and τ as desired.

The fact that our index category is cofiltered means that we may compute the value of $B = \operatorname{colim}_{A' \to A} A'$ in the category of sets (some details omitted; compare with the discussion in Categories, Section 19). To see that $B \to A$ is surjective, for every $a \in A$ we can use $R[x] \to A$, $x \mapsto a$ to see that a is in the image of $B \to A$. Conversely, if $b \in B$ is mapped to zero in A, then we can find $A' \to A$ in $\mathcal{I}$ and $a' \in A'$ which maps to b. Then $A'/(a') \to A$ is in $\mathcal{I}$ as well and the map $A' \to B$ factors as $A' \to A'/(a') \to B$ which shows that $b = 0$ as desired. □

Often it is easier to think about colimits over preordered sets. Let $(\Lambda, \geq)$ a preordered set. A system of rings over Λ is given by a ring R_λ for every $\lambda \in \Lambda$, and a morphism $R_\lambda \to R_\mu$ whenever $\lambda \leq \mu$. These morphisms have to satisfy the rule that $R_\lambda \to R_\mu \to R_\nu$ is equal to the map $R_\lambda \to R_\nu$ for all $\lambda \leq \mu \leq \nu$. See Categories, Section 21. We will often assume that $(I, \leq)$ is *directed*, which means that Λ is nonempty and given $\lambda, \mu \in \Lambda$ there exists a $\nu \in \Lambda$ with $\lambda \leq \nu$ and $\mu \leq \nu$. Recall that the colimit $\operatorname{colim}_\lambda R_\lambda$ is sometimes called a "direct limit" in this case (but we will not use this terminology).

Note that Categories, Lemma 21.5 tells us that colimits over filtered index categories are the same thing as colimits over directed sets.

00QN **Lemma 127.2.** *Let $R \to A$ be a ring map. There exists a directed system A_λ of R-algebras of finite presentation such that $A = \operatorname{colim}_\lambda A_\lambda$. If A is of finite type over R we may arrange it so that all the transition maps in the system of A_λ are surjective.*

Proof. The first proof is that this follows from Lemma 127.1 and Categories, Lemma 21.5.

Second proof. Compare with the proof of Lemma 11.3. Consider any finite subset $S \subset A$, and any finite collection of polynomial relations E among the elements of S. So each $s \in S$ corresponds to $x_s \in A$ and each $e \in E$ consists of a polynomial $f_e \in R[X_s; s \in S]$ such that $f_e(x_s) = 0$. Let $A_{S,E} = R[X_s; s \in S]/(f_e; e \in E)$ which is a finitely presented R-algebra. There are canonical maps $A_{S,E} \to A$. If $S \subset S'$ and if the elements of E correspond, via the map $R[X_s; s \in S] \to R[X_s; s \in S']$, to a subset of E', then there is an obvious map $A_{S,E} \to A_{S',E'}$ commuting with the maps to A. Thus, setting Λ equal the set of pairs (S, E) with ordering by inclusion

[11] To avoid set theoretical difficulties we consider only $A' \to A$ such that A' is a quotient of $R[x_1, x_2, x_3, \ldots]$.

as above, we get a directed partially ordered set. It is clear that the colimit of this directed system is A.

For the last statement, suppose $A = R[x_1, \ldots, x_n]/I$. In this case, consider the subset $\Lambda' \subset \Lambda$ consisting of those systems (S, E) above with $S = \{x_1, \ldots, x_n\}$. It is easy to see that still $A = \mathrm{colim}_{\lambda' \in \Lambda'} A_{\lambda'}$. Moreover, the transition maps are clearly surjective. $\square$

It turns out that we can characterize ring maps of finite presentation as follows. This in some sense says that the algebras of finite presentation are the "compact" objects in the category of R-algebras.

00QO **Lemma 127.3.** *Let $\varphi : R \to S$ be a ring map. The following are equivalent*

(1) *φ is of finite presentation,*
(2) *for every directed system A_λ of R-algebras the map*
$$\mathrm{colim}_\lambda \mathrm{Hom}_R(S, A_\lambda) \longrightarrow \mathrm{Hom}_R(S, \mathrm{colim}_\lambda A_\lambda)$$
is bijective, and
(3) *for every directed system A_λ of R-algebras the map*
$$\mathrm{colim}_\lambda \mathrm{Hom}_R(S, A_\lambda) \longrightarrow \mathrm{Hom}_R(S, \mathrm{colim}_\lambda A_\lambda)$$
is surjective.

Proof. Assume (1) and write $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Let $A = \mathrm{colim}\, A_\lambda$. Observe that an R-algebra homomorphism $S \to A$ or $S \to A_\lambda$ is determined by the images of $x_1, \ldots, x_n$. Hence it is clear that $\mathrm{colim}_\lambda \mathrm{Hom}_R(S, A_\lambda) \to \mathrm{Hom}_R(S, A)$ is injective. To see that it is surjective, let $\chi : S \to A$ be an R-algebra homomorphism. Then each x_i maps to some element in the image of some A_{λ_i}. We may pick $\mu \geq \lambda_i$, $i = 1, \ldots, n$ and assume $\chi(x_i)$ is the image of $y_i \in A_\mu$ for $i = 1, \ldots, n$. Consider $z_j = f_j(y_1, \ldots, y_n) \in A_\mu$. Since χ is a homomorphism the image of z_j in $A = \mathrm{colim}_\lambda A_\lambda$ is zero. Hence there exists a $\mu_j \geq \mu$ such that z_j maps to zero in A_{μ_j}. Pick $\nu \geq \mu_j$, $j = 1, \ldots, m$. Then the images of $z_1, \ldots, z_m$ are zero in A_ν. This exactly means that the y_i map to elements $y'_i \in A_\nu$ which satisfy the relations $f_j(y'_1, \ldots, y'_n) = 0$. Thus we obtain a ring map $S \to A_\nu$. This shows that (1) implies (2).

It is clear that (2) implies (3). Assume (3). By Lemma 127.2 we may write $S = \mathrm{colim}_\lambda S_\lambda$ with S_λ of finite presentation over R. Then the identity map factors as
$$S \to S_\lambda \to S$$
for some λ. This implies that S is finitely presented over S_λ by Lemma 6.2 part (4) applied to $S \to S_\lambda \to S$. Applying part (2) of the same lemma to $R \to S_\lambda \to S$ we conclude that S is of finite presentation over R. $\square$

Using the basic material above we can give a criterion of when an algebra A is a filtered colimit of given type of algebra as follows.

07C3 **Lemma 127.4.** *Let $R \to \Lambda$ be a ring map. Let $\mathcal{E}$ be a set of R-algebras such that each $A \in \mathcal{E}$ is of finite presentation over R. Then the following two statements are equivalent*

(1) *Λ is a filtered colimit of elements of $\mathcal{E}$, and*
(2) *for any R algebra map $A \to \Lambda$ with A of finite presentation over R we can find a factorization $A \to B \to \Lambda$ with $B \in \mathcal{E}$.*

Proof. Suppose that $\mathcal{I} \to \mathcal{E}$, $i \mapsto A_i$ is a filtered diagram such that $\Lambda = \operatorname{colim}_i A_i$. Let $A \to \Lambda$ be an R-algebra map with A of finite presentation over R. Then we get a factorization $A \to A_i \to \Lambda$ by applying Lemma 127.3. Thus (1) implies (2).

Consider the category $\mathcal{I}$ of Lemma 127.1. By Categories, Lemma 19.3 the full subcategory $\mathcal{J}$ consisting of those $A \to \Lambda$ with $A \in \mathcal{E}$ is cofinal in $\mathcal{I}$ and is a filtered category. Then Λ is also the colimit over $\mathcal{J}$ by Categories, Lemma 17.2. □

But more is true. Namely, given $R = \operatorname{colim}_\lambda R_\lambda$ we see that the category of finitely presented R-modules is equivalent to the limit of the category of finitely presented R_λ-modules. Similarly for the categories of finitely presented R-algebras.

05LI **Lemma 127.5.** *Let A be a ring and let M, N be A-modules. Suppose that $R = \operatorname{colim}_{i \in I} R_i$ is a directed colimit of A-algebras.*

1. *If M is a finite A-module, and $u, u' : M \to N$ are A-module maps such that $u \otimes 1 = u' \otimes 1 : M \otimes_A R \to N \otimes_A R$ then for some i we have $u \otimes 1 = u' \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$.*
2. *If N is a finite A-module and $u : M \to N$ is an A-module map such that $u \otimes 1 : M \otimes_A R \to N \otimes_A R$ is surjective, then for some i the map $u \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$ is surjective.*
3. *If N is a finitely presented A-module, and $v : N \otimes_A R \to M \otimes_A R$ is an R-module map, then there exists an i and an R_i-module map $v_i : N \otimes_A R_i \to M \otimes_A R_i$ such that $v = v_i \otimes 1$.*
4. *If M is a finite A-module, N is a finitely presented A-module, and $u : M \to N$ is an A-module map such that $u \otimes 1 : M \otimes_A R \to N \otimes_A R$ is an isomorphism, then for some i the map $u \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$ is an isomorphism.*

Proof. To prove (1) assume u is as in (1) and let $x_1, \ldots, x_m \in M$ be generators. Since $N \otimes_A R = \operatorname{colim}_i N \otimes_A R_i$ we may pick an $i \in I$ such that $u(x_j) \otimes 1 = u'(x_j) \otimes 1$ in $M \otimes_A R_i$, $j = 1, \ldots, m$. For such an i we have $u \otimes 1 = u' \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$.

To prove (2) assume $u \otimes 1$ surjective and let $y_1, \ldots, y_m \in N$ be generators. Since $N \otimes_A R = \operatorname{colim}_i N \otimes_A R_i$ we may pick an $i \in I$ and $z_j \in M \otimes_A R_i$, $j = 1, \ldots, m$ whose images in $N \otimes_A R$ equal $y_j \otimes 1$. For such an i the map $u \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$ is surjective.

To prove (3) let $y_1, \ldots, y_m \in N$ be generators. Let $K = \operatorname{Ker}(A^{\oplus m} \to N)$ where the map is given by the rule $(a_1, \ldots, a_m) \mapsto \sum a_j x_j$. Let $k_1, \ldots, k_t$ be generators for K. Say $k_s = (k_{s1}, \ldots, k_{sm})$. Since $M \otimes_A R = \operatorname{colim}_i M \otimes_A R_i$ we may pick an $i \in I$ and $z_j \in M \otimes_A R_i$, $j = 1, \ldots, m$ whose images in $M \otimes_A R$ equal $v(y_j \otimes 1)$. We want to use the z_j to define the map $v_i : N \otimes_A R_i \to M \otimes_A R_i$. Since $K \otimes_A R_i \to R_i^{\oplus m} \to N \otimes_A R_i \to 0$ is a presentation, it suffices to check that $\xi_s = \sum_j k_{sj} z_j$ is zero in $M \otimes_A R_i$ for each $s = 1, \ldots, t$. This may not be the case, but since the image of ξ_s in $M \otimes_A R$ is zero we see that it will be the case after increasing i a bit.

To prove (4) assume $u \otimes 1$ is an isomorphism, that M is finite, and that N is finitely presented. Let $v : N \otimes_A R \to M \otimes_A R$ be an inverse to $u \otimes 1$. Apply part (3) to get a map $v_i : N \otimes_A R_i \to M \otimes_A R_i$ for some i. Apply part (1) to see that, after increasing i we have $v_i \circ (u \otimes 1) = \operatorname{id}_{M \otimes_R R_i}$ and $(u \otimes 1) \circ v_i = \operatorname{id}_{N \otimes_R R_i}$. □

05N7 **Lemma 127.6.** *Suppose that $R = \mathrm{colim}_{\lambda\in\Lambda} R_\lambda$ is a directed colimit of rings. Then the category of finitely presented R-modules is the colimit of the categories of finitely presented R_λ-modules. More precisely*

1. *Given a finitely presented R-module M there exists a $\lambda \in \Lambda$ and a finitely presented R_λ-module M_λ such that $M \cong M_\lambda \otimes_{R_\lambda} R$.*
2. *Given a $\lambda \in \Lambda$, finitely presented R_λ-modules M_λ, N_λ, and an R-module map $\varphi : M_\lambda \otimes_{R_\lambda} R \to N_\lambda \otimes_{R_\lambda} R$, then there exists a $\mu \geq \lambda$ and an R_μ-module map $\varphi_\mu : M_\lambda \otimes_{R_\lambda} R_\mu \to N_\lambda \otimes_{R_\lambda} R_\mu$ such that $\varphi = \varphi_\mu \otimes 1_R$.*
3. *Given a $\lambda \in \Lambda$, finitely presented R_λ-modules M_λ, N_λ, and R-module maps $\varphi_\lambda, \psi_\lambda : M_\lambda \to N_\lambda$ such that $\varphi \otimes 1_R = \psi \otimes 1_R$, then $\varphi \otimes 1_{R_\mu} = \psi \otimes 1_{R_\mu}$ for some $\mu \geq \lambda$.*

Proof. To prove (1) choose a presentation $R^{\oplus m} \to R^{\oplus n} \to M \to 0$. Suppose that the first map is given by the matrix $A = (a_{ij})$. We can choose a $\lambda \in \Lambda$ and a matrix $A_\lambda = (a_{\lambda,ij})$ with coefficients in R_λ which maps to A in R. Then we simply let M_λ be the R_λ-module with presentation $R_\lambda^{\oplus m} \to R_\lambda^{\oplus n} \to M_\lambda \to 0$ where the first arrow is given by A_λ.

Parts (2) and (3) follow from Lemma 127.5. $\square$

05N8 **Lemma 127.7.** *Let A be a ring and let B, C be A-algebras. Suppose that $R = \mathrm{colim}_{i\in I} R_i$ is a directed colimit of A-algebras.*

1. *If B is a finite type A-algebra, and $u, u' : B \to C$ are A-algebra maps such that $u \otimes 1 = u' \otimes 1 : B \otimes_A R \to C \otimes_A R$ then for some i we have $u \otimes 1 = u' \otimes 1 : B \otimes_A R_i \to C \otimes_A R_i$.*
2. *If C is a finite type A-algebra and $u : B \to C$ is an A-algebra map such that $u \otimes 1 : B \otimes_A R \to C \otimes_A R$ is surjective, then for some i the map $u \otimes 1 : B \otimes_A R_i \to C \otimes_A R_i$ is surjective.*
3. *If C is of finite presentation over A and $v : C \otimes_A R \to B \otimes_A R$ is an R-algebra map, then there exists an i and an R_i-algebra map $v_i : C \otimes_A R_i \to B \otimes_A R_i$ such that $v = v_i \otimes 1$.*
4. *If B is a finite type A-algebra, C is a finitely presented A-algebra, and $u \otimes 1 : B \otimes_A R \to C \otimes_A R$ is an isomorphism, then for some i the map $u \otimes 1 : B \otimes_A R_i \to C \otimes_A R_i$ is an isomorphism.*

Proof. To prove (1) assume u is as in (1) and let $x_1, \ldots, x_m \in B$ be generators. Since $B \otimes_A R = \mathrm{colim}_i B \otimes_A R_i$ we may pick an $i \in I$ such that $u(x_j) \otimes 1 = u'(x_j) \otimes 1$ in $B \otimes_A R_i$, $j = 1, \ldots, m$. For such an i we have $u \otimes 1 = u' \otimes 1 : B \otimes_A R_i \to C \otimes_A R_i$.

To prove (2) assume $u \otimes 1$ surjective and let $y_1, \ldots, y_m \in C$ be generators. Since $B \otimes_A R = \mathrm{colim}_i B \otimes_A R_i$ we may pick an $i \in I$ and $z_j \in B \otimes_A R_i$, $j = 1, \ldots, m$ whose images in $C \otimes_A R$ equal $y_j \otimes 1$. For such an i the map $u \otimes 1 : B \otimes_A R_i \to C \otimes_A R_i$ is surjective.

To prove (3) let $c_1, \ldots, c_m \in C$ be generators. Let $K = \mathrm{Ker}(A[x_1, \ldots, x_m] \to N)$ where the map is given by the rule $x_j \mapsto \sum c_j$. Let $f_1, \ldots, f_t$ be generators for K as an ideal in $A[x_1, \ldots, x_m]$. We think of $f_j = f_j(x_1, \ldots, x_m)$ as a polynomial. Since $B \otimes_A R = \mathrm{colim}_i B \otimes_A R_i$ we may pick an $i \in I$ and $z_j \in B \otimes_A R_i$, $j = 1, \ldots, m$ whose images in $B \otimes_A R$ equal $v(c_j \otimes 1)$. We want to use the z_j to define a map $v_i : C \otimes_A R_i \to B \otimes_A R_i$. Since $K \otimes_A R_i \to R_i[x_1, \ldots, x_m] \to C \otimes_A R_i \to 0$ is a presentation, it suffices to check that $\xi_s = f_j(z_1, \ldots, z_m)$ is zero in $B \otimes_A R_i$ for each $s = 1, \ldots, t$. This may not be the case, but since the image of ξ_s in $B \otimes_A R$ is zero we see that it will be the case after increasing i a bit.

To prove (4) assume $u \otimes 1$ is an isomorphism, that B is a finite type A-algebra, and that C is a finitely presented A-algebra. Let $v : B \otimes_A R \to C \otimes_A R$ be an inverse to $u \otimes 1$. Let $v_i : C \otimes_A R_i \to B \otimes_A R_i$ be as in part (3). Apply part (1) to see that, after increasing i we have $v_i \circ (u \otimes 1) = \text{id}_{B \otimes_R R_i}$ and $(u \otimes 1) \circ v_i = \text{id}_{C \otimes_R R_i}$. □

05N9 **Lemma 127.8.** *Suppose that $R = \text{colim}_{\lambda \in \Lambda} R_\lambda$ is a directed colimit of rings. Then the category of finitely presented R-algebras is the colimit of the categories of finitely presented R_λ-algebras. More precisely*

1. *Given a finitely presented R-algebra A there exists a $\lambda \in \Lambda$ and a finitely presented R_λ-algebra A_λ such that $A \cong A_\lambda \otimes_{R_\lambda} R$.*
2. *Given a $\lambda \in \Lambda$, finitely presented R_λ-algebras A_λ, B_λ, and an R-algebra map $\varphi : A_\lambda \otimes_{R_\lambda} R \to B_\lambda \otimes_{R_\lambda} R$, then there exists a $\mu \geq \lambda$ and an R_μ-algebra map $\varphi_\mu : A_\lambda \otimes_{R_\lambda} R_\mu \to B_\lambda \otimes_{R_\lambda} R_\mu$ such that $\varphi = \varphi_\mu \otimes 1_R$.*
3. *Given a $\lambda \in \Lambda$, finitely presented R_λ-algebras A_λ, B_λ, and R_λ-algebra maps $\varphi_\lambda, \psi_\lambda : A_\lambda \to B_\lambda$ such that $\varphi \otimes 1_R = \psi \otimes 1_R$, then $\varphi \otimes 1_{R_\mu} = \psi \otimes 1_{R_\mu}$ for some $\mu \geq \lambda$.*

Proof. To prove (1) choose a presentation $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. We can choose a $\lambda \in \Lambda$ and elements $f_{\lambda,j} \in R_\lambda[x_1, \ldots, x_n]$ mapping to $f_j \in R[x_1, \ldots, x_n]$. Then we simply let $A_\lambda = R_\lambda[x_1, \ldots, x_n]/(f_{\lambda,1}, \ldots, f_{\lambda,m})$.

Parts (2) and (3) follow from Lemma 127.7. □

00QT **Lemma 127.9.** *Suppose $R \to S$ is a local homomorphism of local rings. There exists a directed set $(\Lambda, \leq)$, and a system of local homomorphisms $R_\lambda \to S_\lambda$ of local rings such that*

1. *The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$.*
2. *Each R_λ is essentially of finite type over $\mathbf{Z}$.*
3. *Each S_λ is essentially of finite type over R_λ.*

Proof. Denote $\varphi : R \to S$ the ring map. Let $\mathfrak{m} \subset R$ be the maximal ideal of R and let $\mathfrak{n} \subset S$ be the maximal ideal of S. Let

$$\Lambda = \{(A, B) \mid A \subset R, B \subset S, \#A < \infty, \#B < \infty, \varphi(A) \subset B\}.$$

As partial ordering we take the inclusion relation. For each $\lambda = (A, B) \in \Lambda$ we let R'_λ be the sub $\mathbf{Z}$-algebra generated by $a \in A$, and we let S'_λ be the sub $\mathbf{Z}$-algebra generated by b, $b \in B$. Let R_λ be the localization of R'_λ at the prime ideal $R'_\lambda \cap \mathfrak{m}$ and let S_λ be the localization of S'_λ at the prime ideal $S'_\lambda \cap \mathfrak{n}$. In a picture

$$\begin{tikzcd} B \arrow[r] & S'_\lambda \arrow[r] & S_\lambda \arrow[r] & S \\ A \arrow[r] \arrow[u] & R'_\lambda \arrow[r] \arrow[u] & R_\lambda \arrow[r] \arrow[u] & R \arrow[u] \end{tikzcd}.$$

The transition maps are clear. We leave the proofs of the other assertions to the reader. □

00QU **Lemma 127.10.** *Suppose $R \to S$ is a local homomorphism of local rings. Assume that S is essentially of finite type over R. Then there exists a directed set $(\Lambda, \leq)$, and a system of local homomorphisms $R_\lambda \to S_\lambda$ of local rings such that*

(1) *The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$.*
(2) *Each R_λ is essentially of finite type over $\mathbf{Z}$.*
(3) *Each S_λ is essentially of finite type over R_λ.*
(4) *For each $\lambda \leq \mu$ the map $S_\lambda \otimes_{R_\lambda} R_\mu \to S_\mu$ presents S_μ as the localization of a quotient of $S_\lambda \otimes_{R_\lambda} R_\mu$.*

Proof. Denote $\varphi : R \to S$ the ring map. Let $\mathfrak{m} \subset R$ be the maximal ideal of R and let $\mathfrak{n} \subset S$ be the maximal ideal of S. Let $x_1, \ldots, x_n \in S$ be elements such that S is a localization of the sub R-algebra of S generated by $x_1, \ldots, x_n$. In other words, S is a quotient of a localization of the polynomial ring $R[x_1, \ldots, x_n]$.

Let $\Lambda = \{A \subset R \mid \#A < \infty\}$ be the set of finite subsets of R. As partial ordering we take the inclusion relation. For each $\lambda = A \in \Lambda$ we let R'_λ be the sub $\mathbf{Z}$-algebra generated by $a \in A$, and we let S'_λ be the sub $\mathbf{Z}$-algebra generated by $\varphi(a)$, $a \in A$ and the elements $x_1, \ldots, x_n$. Let R_λ be the localization of R'_λ at the prime ideal $R'_\lambda \cap \mathfrak{m}$ and let S_λ be the localization of S'_λ at the prime ideal $S'_\lambda \cap \mathfrak{n}$. In a picture

$$\begin{tikzcd}
\varphi(A) \amalg \{x_i\} \arrow[r] & S'_\lambda \arrow[r] & S_\lambda \arrow[r] & S \\
A \arrow[r] \arrow[u] & R'_\lambda \arrow[r] \arrow[u] & R_\lambda \arrow[r] \arrow[u] & R \arrow[u]
\end{tikzcd}$$

It is clear that if $A \subset B$ corresponds to $\lambda \leq \mu$ in Λ, then there are canonical maps $R_\lambda \to R_\mu$, and $S_\lambda \to S_\mu$ and we obtain a system over the directed set Λ.

The assertion that $R = \operatorname{colim} R_\lambda$ is clear because all the maps $R_\lambda \to R$ are injective and any element of R eventually is in the image. The same argument works for $S = \operatorname{colim} S_\lambda$. Assertions (2), (3) are true by construction. The final assertion holds because clearly the maps $S'_\lambda \otimes_{R'_\lambda} R'_\mu \to S'_\mu$ are surjective. $\square$

00QV **Lemma 127.11.** *Suppose $R \to S$ is a local homomorphism of local rings. Assume that S is essentially of finite presentation over R. Then there exists a directed set $(\Lambda, \leq)$, and a system of local homomorphism $R_\lambda \to S_\lambda$ of local rings such that*

(1) *The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$.*
(2) *Each R_λ is essentially of finite type over $\mathbf{Z}$.*
(3) *Each S_λ is essentially of finite type over R_λ.*
(4) *For each $\lambda \leq \mu$ the map $S_\lambda \otimes_{R_\lambda} R_\mu \to S_\mu$ presents S_μ as the localization of $S_\lambda \otimes_{R_\lambda} R_\mu$ at a prime ideal.*

Proof. By assumption we may choose an isomorphism $\Phi : (R[x_1, \ldots, x_n]/I)_\mathfrak{q} \to S$ where $I \subset R[x_1, \ldots, x_n]$ is a finitely generated ideal, and $\mathfrak{q} \subset R[x_1, \ldots, x_n]/I$ is a prime. (Note that $R \cap \mathfrak{q}$ is equal to the maximal ideal $\mathfrak{m}$ of R.) We also choose generators $f_1, \ldots, f_m \in I$ for the ideal I. Write R in any way as a colimit $R = \operatorname{colim} R_\lambda$ over a directed set $(\Lambda, \leq)$, with each R_λ local and essentially of finite type over $\mathbf{Z}$. There exists some $\lambda_0 \in \Lambda$ such that f_j is the image of some $f_{j,\lambda_0} \in R_{\lambda_0}[x_1, \ldots, x_n]$. For all $\lambda \geq \lambda_0$ denote $f_{j,\lambda} \in R_\lambda[x_1, \ldots, x_n]$ the image of f_{j,λ_0}. Thus we obtain a system of ring maps

$$R_\lambda[x_1, \ldots, x_n]/(f_{1,\lambda}, \ldots, f_{m,\lambda}) \to R[x_1, \ldots, x_n]/(f_1, \ldots, f_m) \to S$$

Set $\mathfrak{q}_\lambda$ the inverse image of $\mathfrak{q}$. Set $S_\lambda = (R_\lambda[x_1, \ldots, x_n]/(f_{1,\lambda}, \ldots, f_{m,\lambda}))_{\mathfrak{q}_\lambda}$. We leave it to the reader to see that this works. $\square$

00QW **Remark 127.12.** Suppose that $R \to S$ is a local homomorphism of local rings, which is essentially of finite presentation. Take any system $(\Lambda, \leq)$, $R_\lambda \to S_\lambda$ with the properties listed in Lemma 127.10. What may happen is that this is the "wrong" system, namely, it may happen that property (4) of Lemma 127.11 is not satisfied. Here is an example. Let k be a field. Consider the ring

$$R = k[[z, y_1, y_2, \ldots]]/(y_i^2 - zy_{i+1}).$$

Set $S = R/zR$. As system take $\Lambda = \mathbf{N}$ and $R_n = k[[z, y_1, \ldots, y_n]]/(\{y_i^2 - zy_{i+1}\}_{i \leq n-1})$ and $S_n = R_n/(z, y_n^2)$. All the maps $S_n \otimes_{R_n} R_{n+1} \to S_{n+1}$ are not localizations (i.e., isomorphisms in this case) since $1 \otimes y_{n+1}^2$ maps to zero. If we take instead $S'_n = R_n/zR_n$ then the maps $S'_n \otimes_{R_n} R_{n+1} \to S'_{n+1}$ are isomorphisms. The moral of this remark is that we do have to be a little careful in choosing the systems.

00QX **Lemma 127.13.** *Suppose $R \to S$ is a local homomorphism of local rings. Assume that S is essentially of finite presentation over R. Let M be a finitely presented S-module. Then there exists a directed set $(\Lambda, \leq)$, and a system of local homomorphisms $R_\lambda \to S_\lambda$ of local rings together with S_λ-modules M_λ, such that*

1. *The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$. The colimit of the system M_λ is M.*
2. *Each R_λ is essentially of finite type over $\mathbf{Z}$.*
3. *Each S_λ is essentially of finite type over R_λ.*
4. *Each M_λ is finite over S_λ.*
5. *For each $\lambda \leq \mu$ the map $S_\lambda \otimes_{R_\lambda} R_\mu \to S_\mu$ presents S_μ as the localization of $S_\lambda \otimes_{R_\lambda} R_\mu$ at a prime ideal.*
6. *For each $\lambda \leq \mu$ the map $M_\lambda \otimes_{S_\lambda} S_\mu \to M_\mu$ is an isomorphism.*

Proof. As in the proof of Lemma 127.11 we may first write $R = \operatorname{colim} R_\lambda$ as a directed colimit of local $\mathbf{Z}$-algebras which are essentially of finite type. Next, we may assume that for some $\lambda_1 \in \Lambda$ there exist $f_{j,\lambda_1} \in R_{\lambda_1}[x_1, \ldots, x_n]$ such that

$$S = \operatorname{colim}_{\lambda \geq \lambda_1} S_\lambda, \text{ with } S_\lambda = (R_\lambda[x_1, \ldots, x_n]/(f_{1,\lambda}, \ldots, f_{m,\lambda}))_{\mathfrak{q}_\lambda}$$

Choose a presentation

$$S^{\oplus s} \to S^{\oplus t} \to M \to 0$$

of M over S. Let $A \in \text{Mat}(t \times s, S)$ be the matrix of the presentation. For some $\lambda_2 \in \Lambda$, $\lambda_2 \geq \lambda_1$ we can find a matrix $A_{\lambda_2} \in \text{Mat}(t \times s, S_{\lambda_2})$ which maps to A. For all $\lambda \geq \lambda_2$ we let $M_\lambda = \text{Coker}(S_\lambda^{\oplus s} \xrightarrow{A_\lambda} S_\lambda^{\oplus t})$. We leave it to the reader to see that this works. $\square$

00QY **Lemma 127.14.** *Suppose $R \to S$ is a ring map. Then there exists a directed set $(\Lambda, \leq)$, and a system of ring maps $R_\lambda \to S_\lambda$ such that*

1. *The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$.*
2. *Each R_λ is of finite type over $\mathbf{Z}$.*
3. *Each S_λ is of finite type over R_λ.*

Proof. This is the non-local version of Lemma 127.9. Proof is similar and left to the reader. $\square$

0BTG **Lemma 127.15.** *Suppose $R \to S$ is a ring map. Assume that S is integral over R. Then there exists a directed set $(\Lambda, \leq)$, and a system of ring maps $R_\lambda \to S_\lambda$ such that*

(1) *The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$.*
(2) *Each R_λ is of finite type over $\mathbf{Z}$.*
(3) *Each S_λ is of finite over R_λ.*

Proof. Consider the set Λ of pairs (E, F) where $E \subset R$ is a finite subset, $F \subset S$ is a finite subset, and every element $f \in F$ is the root of a monic $P(X) \in R[X]$ whose coefficients are in E. Say $(E, F) \leq (E', F')$ if $E \subset E'$ and $F \subset F'$. Given $\lambda = (E, F) \in \Lambda$ set $R_\lambda \subset R$ equal to the $\mathbf{Z}$-subalgebra of R generated by E and $S_\lambda \subset S$ equal to the $\mathbf{Z}$-subalgebra generated by F and the image of E in S. It is clear that $R = \operatorname{colim} R_\lambda$. We have $S = \operatorname{colim} S_\lambda$ as every element of S is integral over S. The ring maps $R_\lambda \to S_\lambda$ are finite by Lemma 36.5 and the fact that S_λ is generated over R_λ by the elements of F which are integral over R_λ by our condition on the pairs (E, F). The lemma follows. $\square$

00QZ **Lemma 127.16.** *Suppose $R \to S$ is a ring map. Assume that S is of finite type over R. Then there exists a directed set $(\Lambda, \leq)$, and a system of ring maps $R_\lambda \to S_\lambda$ such that*

(1) *The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$.*
(2) *Each R_λ is of finite type over $\mathbf{Z}$.*
(3) *Each S_λ is of finite type over R_λ.*
(4) *For each $\lambda \leq \mu$ the map $S_\lambda \otimes_{R_\lambda} R_\mu \to S_\mu$ presents S_μ as a quotient of $S_\lambda \otimes_{R_\lambda} R_\mu$.*

Proof. This is the non-local version of Lemma 127.10. Proof is similar and left to the reader. $\square$

00R0 **Lemma 127.17.** *Suppose $R \to S$ is a ring map. Assume that S is of finite presentation over R. Then there exists a directed set $(\Lambda, \leq)$, and a system of ring maps $R_\lambda \to S_\lambda$ such that*

(1) *The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$.*
(2) *Each R_λ is of finite type over $\mathbf{Z}$.*
(3) *Each S_λ is of finite type over R_λ.*
(4) *For each $\lambda \leq \mu$ the map $S_\lambda \otimes_{R_\lambda} R_\mu \to S_\mu$ is an isomorphism.*

Proof. This is the non-local version of Lemma 127.11. Proof is similar and left to the reader. $\square$

00R1 **Lemma 127.18.** *Suppose $R \to S$ is a ring map. Assume that S is of finite presentation over R. Let M be a finitely presented S-module. Then there exists a directed set $(\Lambda, \leq)$, and a system of ring maps $R_\lambda \to S_\lambda$ together with S_λ-modules M_λ, such that*

(1) *The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$. The colimit of the system M_λ is M.*
(2) *Each R_λ is of finite type over $\mathbf{Z}$.*
(3) *Each S_λ is of finite type over R_λ.*
(4) *Each M_λ is finite over S_λ.*
(5) *For each $\lambda \leq \mu$ the map $S_\lambda \otimes_{R_\lambda} R_\mu \to S_\mu$ is an isomorphism.*
(6) *For each $\lambda \leq \mu$ the map $M_\lambda \otimes_{S_\lambda} S_\mu \to M_\mu$ is an isomorphism.*

In particular, for every $\lambda \in \Lambda$ we have

$$M = M_\lambda \otimes_{S_\lambda} S = M_\lambda \otimes_{R_\lambda} R.$$

Proof. This is the non-local version of Lemma 127.13. Proof is similar and left to the reader. ☐

128. More flatness criteria

00R3 The following lemma is often used in algebraic geometry to show that a finite morphism from a normal surface to a smooth surface is flat. It is a partial converse to Lemma 112.9 because an injective finite local ring map certainly satisfies condition (3).

00R4 **Lemma 128.1.** *Let $R \to S$ be a local homomorphism of Noetherian local rings. Assume*

(1) *R is regular,*
(2) *S Cohen-Macaulay,*
(3) *$\dim(S) = \dim(R) + \dim(S/\mathfrak{m}_R S)$.*

Then $R \to S$ is flat.

Proof. By induction on $\dim(R)$. The case $\dim(R) = 0$ is trivial, because then R is a field. Assume $\dim(R) > 0$. By (3) this implies that $\dim(S) > 0$. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ be the minimal primes of S. Note that $\mathfrak{q}_i \not\supset \mathfrak{m}_R S$ since

$$\dim(S/\mathfrak{q}_i) = \dim(S) > \dim(S/\mathfrak{m}_R S)$$

the first equality by Lemma 104.3 and the inequality by (3). Thus $\mathfrak{p}_i = R \cap \mathfrak{q}_i$ is not equal to $\mathfrak{m}_R$. Pick $x \in \mathfrak{m}$, $x \notin \mathfrak{m}^2$, and $x \notin \mathfrak{p}_i$, see Lemma 15.2. Hence we see that x is not contained in any of the minimal primes of S. Hence x is a nonzerodivisor on S by (2), see Lemma 104.2 and S/xS is Cohen-Macaulay with $\dim(S/xS) = \dim(S) - 1$. By (1) and Lemma 106.3 the ring R/xR is regular with $\dim(R/xR) = \dim(R) - 1$. By induction we see that $R/xR \to S/xS$ is flat. Hence we conclude by Lemma 99.10 and the remark following it. ☐

07DY **Lemma 128.2.** *Let $R \to S$ be a homomorphism of Noetherian local rings. Assume that R is a regular local ring and that a regular system of parameters maps to a regular sequence in S. Then $R \to S$ is flat.*

Proof. Suppose that $x_1, \ldots, x_d$ are a system of parameters of R which map to a regular sequence in S. Note that $S/(x_1, \ldots, x_d)S$ is flat over $R/(x_1, \ldots, x_d)$ as the latter is a field. Then x_d is a nonzerodivisor in $S/(x_1, \ldots, x_{d-1})S$ hence $S/(x_1, \ldots, x_{d-1})S$ is flat over $R/(x_1, \ldots, x_{d-1})$ by the local criterion of flatness (see Lemma 99.10 and remarks following). Then x_{d-1} is a nonzerodivisor in $S/(x_1, \ldots, x_{d-2})S$ hence $S/(x_1, \ldots, x_{d-2})S$ is flat over $R/(x_1, \ldots, x_{d-2})$ by the local criterion of flatness (see Lemma 99.10 and remarks following). Continue till one reaches the conclusion that S is flat over R. ☐

The following lemma is the key to proving that results for finitely presented modules over finitely presented rings over a base ring follow from the corresponding results for finite modules in the Noetherian case.

00R6 **Lemma 128.3.** *Let $R \to S$, M, Λ, $R_\lambda \to S_\lambda$, M_λ be as in Lemma 127.13. Assume that M is flat over R. Then for some $\lambda \in \Lambda$ the module M_λ is flat over R_λ.*

Proof. Pick some $\lambda \in \Lambda$ and consider

$$\mathrm{Tor}_1^{R_\lambda}(M_\lambda, R_\lambda/\mathfrak{m}_\lambda) = \mathrm{Ker}(\mathfrak{m}_\lambda \otimes_{R_\lambda} M_\lambda \to M_\lambda).$$

See Remark 75.9. The right hand side shows that this is a finitely generated S_λ-module (because S_λ is Noetherian and the modules in question are finite). Let $\xi_1, \ldots, \xi_n$ be generators. Because M is flat over R we have that $0 = \text{Ker}(\mathfrak{m}_\lambda R \otimes_R M \to M)$. Since $\otimes$ commutes with colimits we see there exists a $\lambda' \geq \lambda$ such that each ξ_i maps to zero in $\mathfrak{m}_\lambda R_{\lambda'} \otimes_{R_{\lambda'}} M_{\lambda'}$. Hence we see that

$$\text{Tor}_1^{R_\lambda}(M_\lambda, R_\lambda/\mathfrak{m}_\lambda) \longrightarrow \text{Tor}_1^{R_{\lambda'}}(M_{\lambda'}, R_{\lambda'}/\mathfrak{m}_\lambda R_{\lambda'})$$

is zero. Note that $M_\lambda \otimes_{R_\lambda} R_\lambda/\mathfrak{m}_\lambda$ is flat over $R_\lambda/\mathfrak{m}_\lambda$ because this last ring is a field. Hence we may apply Lemma 99.14 to get that $M_{\lambda'}$ is flat over $R_{\lambda'}$. ☐

Using the lemma above we can start to reprove the results of Section 99 in the non-Noetherian case.

046Y **Lemma 128.4.** *Suppose that $R \to S$ is a local homomorphism of local rings. Denote $\mathfrak{m}$ the maximal ideal of R. Let $u : M \to N$ be a map of S-modules. Assume*

1. *S is essentially of finite presentation over R,*
2. *M, N are finitely presented over S,*
3. *N is flat over R, and*
4. *$\overline{u} : M/\mathfrak{m}M \to N/\mathfrak{m}N$ is injective.*

Then u is injective, and $N/u(M)$ is flat over R.

Proof. By Lemma 127.13 and its proof we can find a system $R_\lambda \to S_\lambda$ of local ring maps together with maps of S_λ-modules $u_\lambda : M_\lambda \to N_\lambda$ satisfying the conclusions (1) – (6) for both N and M of that lemma and such that the colimit of the maps u_λ is u. By Lemma 128.3 we may assume that N_λ is flat over R_λ for all sufficiently large λ. Denote $\mathfrak{m}_\lambda \subset R_\lambda$ the maximal ideal and $\kappa_\lambda = R_\lambda/\mathfrak{m}_\lambda$, resp. $\kappa = R/\mathfrak{m}$ the residue fields.

Consider the map

$$\Psi_\lambda : M_\lambda/\mathfrak{m}_\lambda M_\lambda \otimes_{\kappa_\lambda} \kappa \longrightarrow M/\mathfrak{m}M.$$

Since $S_\lambda/\mathfrak{m}_\lambda S_\lambda$ is essentially of finite type over the field κ_λ we see that the tensor product $S_\lambda/\mathfrak{m}_\lambda S_\lambda \otimes_{\kappa_\lambda} \kappa$ is essentially of finite type over κ. Hence it is a Noetherian ring and we conclude the kernel of Ψ_λ is finitely generated. Since $M/\mathfrak{m}M$ is the colimit of the system $M_\lambda/\mathfrak{m}_\lambda M_\lambda$ and κ is the colimit of the fields κ_λ there exists a $\lambda' > \lambda$ such that the kernel of Ψ_λ is generated by the kernel of

$$\Psi_{\lambda,\lambda'} : M_\lambda/\mathfrak{m}_\lambda M_\lambda \otimes_{\kappa_\lambda} \kappa_{\lambda'} \longrightarrow M_{\lambda'}/\mathfrak{m}_{\lambda'} M_{\lambda'}.$$

By construction there exists a multiplicative subset $W \subset S_\lambda \otimes_{R_\lambda} R_{\lambda'}$ such that $S_{\lambda'} = W^{-1}(S_\lambda \otimes_{R_\lambda} R_{\lambda'})$ and

$$W^{-1}(M_\lambda/\mathfrak{m}_\lambda M_\lambda \otimes_{\kappa_\lambda} \kappa_{\lambda'}) = M_{\lambda'}/\mathfrak{m}_{\lambda'} M_{\lambda'}.$$

Now suppose that x is an element of the kernel of

$$\Psi_{\lambda'} : M_{\lambda'}/\mathfrak{m}_{\lambda'} M_{\lambda'} \otimes_{\kappa_{\lambda'}} \kappa \longrightarrow M/\mathfrak{m}M.$$

Then for some $w \in W$ we have $wx \in M_\lambda/\mathfrak{m}_\lambda M_\lambda \otimes \kappa$. Hence $wx \in \text{Ker}(\Psi_\lambda)$. Hence wx is a linear combination of elements in the kernel of $\Psi_{\lambda,\lambda'}$. Hence $wx = 0$ in $M_{\lambda'}/\mathfrak{m}_{\lambda'} M_{\lambda'} \otimes_{\kappa_{\lambda'}} \kappa$, hence $x = 0$ because w is invertible in $S_{\lambda'}$. We conclude that the kernel of $\Psi_{\lambda'}$ is zero for all sufficiently large λ'!

By the result of the preceding paragraph we may assume that the kernel of Ψ_λ is zero for all λ sufficiently large, which implies that the map $M_\lambda/\mathfrak{m}_\lambda M_\lambda \to M/\mathfrak{m}M$ is injective. Combined with $\overline{u}$ being injective this formally implies that also $\overline{u_\lambda}$:

$M_\lambda/\mathfrak m_\lambda M_\lambda \to N_\lambda/\mathfrak m_\lambda N_\lambda$ is injective. By Lemma 99.1 we conclude that (for all sufficiently large λ) the map u_λ is injective and that $N_\lambda/u_\lambda(M_\lambda)$ is flat over R_λ. The lemma follows. □

046Z **Lemma 128.5.** *Suppose that $R \to S$ is a local ring homomorphism of local rings. Denote $\mathfrak m$ the maximal ideal of R. Suppose*

1. *S is essentially of finite presentation over R,*
2. *S is flat over R, and*
3. *$f \in S$ is a nonzerodivisor in $S/\mathfrak m S$.*

Then S/fS is flat over R, and f is a nonzerodivisor in S.

Proof. Follows directly from Lemma 128.4. □

0470 **Lemma 128.6.** *Suppose that $R \to S$ is a local ring homomorphism of local rings. Denote $\mathfrak m$ the maximal ideal of R. Suppose*

1. *$R \to S$ is essentially of finite presentation,*
2. *$R \to S$ is flat, and*
3. *$f_1, \ldots, f_c$ is a sequence of elements of S such that the images $\overline{f}_1, \ldots, \overline{f}_c$ form a regular sequence in $S/\mathfrak m S$.*

Then $f_1, \ldots, f_c$ is a regular sequence in S and each of the quotients $S/(f_1, \ldots, f_i)$ is flat over R.

Proof. Induction and Lemma 128.5. □

Here is the version of the local criterion of flatness for the case of local ring maps which are locally of finite presentation.

0471 **Lemma 128.7.** *Let $R \to S$ be a local homomorphism of local rings. Let $I \neq R$ be an ideal in R. Let M be an S-module. Assume*

1. *S is essentially of finite presentation over R,*
2. *M is of finite presentation over S,*
3. *$Tor_1^R(M, R/I) = 0$, and*
4. *M/IM is flat over R/I.*

Then M is flat over R.

Proof. Let Λ, $R_\lambda \to S_\lambda$, M_λ be as in Lemma 127.13. Denote $I_\lambda \subset R_\lambda$ the inverse image of I. In this case the system $R/I \to S/IS$, M/IM, $R_\lambda \to S_\lambda/I_\lambda S_\lambda$, and $M_\lambda/I_\lambda M_\lambda$ satisfies the conclusions of Lemma 127.13 as well. Hence by Lemma 128.3 we may assume (after shrinking the index set Λ) that $M_\lambda/I_\lambda M_\lambda$ is flat for all λ. Pick some λ and consider

$$\mathrm{Tor}_1^{R_\lambda}(M_\lambda, R_\lambda/I_\lambda) = \mathrm{Ker}(I_\lambda \otimes_{R_\lambda} M_\lambda \to M_\lambda).$$

See Remark 75.9. The right hand side shows that this is a finitely generated S_λ-module (because S_λ is Noetherian and the modules in question are finite). Let $\xi_1, \ldots, \xi_n$ be generators. Because $\mathrm{Tor}_1^R(M, R/I) = 0$ and since $\otimes$ commutes with colimits we see there exists a $\lambda' \geq \lambda$ such that each ξ_i maps to zero in

$\mathrm{Tor}_1^{R_{\lambda'}}(M_{\lambda'}, R_{\lambda'}/I_{\lambda'})$. The composition of the maps

$$\begin{tikzcd} R_{\lambda'} \otimes_{R_\lambda} \mathrm{Tor}_1^{R_\lambda}(M_\lambda, R_\lambda/I_\lambda) \arrow[d, "\text{surjective by Lemma 99.12}"] \\ \mathrm{Tor}_1^{R_\lambda}(M_\lambda, R_{\lambda'}/I_\lambda R_{\lambda'}) \arrow[d, "\text{surjective up to localization by Lemma 99.13}"] \\ \mathrm{Tor}_1^{R_{\lambda'}}(M_{\lambda'}, R_{\lambda'}/I_\lambda R_{\lambda'}) \arrow[d, "\text{surjective by Lemma 99.12}"] \\ \mathrm{Tor}_1^{R_{\lambda'}}(M_{\lambda'}, R_{\lambda'}/I_{\lambda'}). \end{tikzcd}$$

is surjective up to a localization by the reasons indicated. The localization is necessary since $M_{\lambda'}$ is not equal to $M_\lambda \otimes_{R_\lambda} R_{\lambda'}$. Namely, it is equal to $M_\lambda \otimes_{S_\lambda} S_{\lambda'}$ and $S_{\lambda'}$ is the localization of $S_\lambda \otimes_{R_\lambda} R_{\lambda'}$ whence the statement up to a localization (or tensoring with $S_{\lambda'}$). Note that Lemma 99.12 applies to the first and third arrows because $M_\lambda/I_\lambda M_\lambda$ is flat over R_λ/I_λ and because $M_{\lambda'}/I_\lambda M_{\lambda'}$ is flat over $R_{\lambda'}/I_\lambda R_{\lambda'}$ as it is a base change of the flat module $M_\lambda/I_\lambda M_\lambda$. The composition maps the generators ξ_i to zero as we explained above. We finally conclude that $\mathrm{Tor}_1^{R_{\lambda'}}(M_{\lambda'}, R_{\lambda'}/I_{\lambda'})$ is zero. This implies that $M_{\lambda'}$ is flat over $R_{\lambda'}$ by Lemma 99.10. $\square$

Please compare the lemma below to Lemma 99.15 (the case of Noetherian local rings) and Lemma 101.8 (the case of a nilpotent ideal in the base).

00R7 **Lemma 128.8** (Critère de platitude par fibres). *Let R, S, S' be local rings and let $R \to S \to S'$ be local ring homomorphisms. Let M be an S'-module. Let $\mathfrak{m} \subset R$ be the maximal ideal. Assume*

1. *The ring maps $R \to S$ and $R \to S'$ are essentially of finite presentation.*
2. *The module M is of finite presentation over S'.*
3. *The module M is not zero.*
4. *The module $M/\mathfrak{m}M$ is a flat $S/\mathfrak{m}S$-module.*
5. *The module M is a flat R-module.*

Then S is flat over R and M is a flat S-module.

Proof. As in the proof of Lemma 127.11 we may first write $R = \mathrm{colim}\, R_\lambda$ as a directed colimit of local $\mathbf{Z}$-algebras which are essentially of finite type. Denote $\mathfrak{p}_\lambda$ the maximal ideal of R_λ. Next, we may assume that for some $\lambda_1 \in \Lambda$ there exist $f_{j,\lambda_1} \in R_{\lambda_1}[x_1, \ldots, x_n]$ such that

$$S = \mathrm{colim}_{\lambda \geq \lambda_1} S_\lambda, \text{ with } S_\lambda = (R_\lambda[x_1, \ldots, x_n]/(f_{1,\lambda}, \ldots, f_{u,\lambda}))_{\mathfrak{q}_\lambda}$$

For some $\lambda_2 \in \Lambda$, $\lambda_2 \geq \lambda_1$ there exist $g_{j,\lambda_2} \in R_{\lambda_2}[x_1, \ldots, x_n, y_1, \ldots, y_m]$ with images $\overline{g}_{j,\lambda_2} \in S_{\lambda_2}[y_1, \ldots, y_m]$ such that

$$S' = \mathrm{colim}_{\lambda \geq \lambda_2} S'_\lambda, \text{ with } S'_\lambda = (S_\lambda[y_1, \ldots, y_m]/(\overline{g}_{1,\lambda}, \ldots, \overline{g}_{v,\lambda}))_{\overline{\mathfrak{q}}'_\lambda}$$

Note that this also implies that

$$S'_\lambda = (R_\lambda[x_1, \ldots, x_n, y_1, \ldots, y_m]/(g_{1,\lambda}, \ldots, g_{v,\lambda}))_{\mathfrak{q}'_\lambda}$$

Choose a presentation

$$(S')^{\oplus s} \to (S')^{\oplus t} \to M \to 0$$

of M over S'. Let $A \in \text{Mat}(t \times s, S')$ be the matrix of the presentation. For some $\lambda_3 \in \Lambda$, $\lambda_3 \geq \lambda_2$ we can find a matrix $A_{\lambda_3} \in \text{Mat}(t \times s, S_{\lambda_3})$ which maps to A. For all $\lambda \geq \lambda_3$ we let $M_\lambda = \text{Coker}((S'_\lambda)^{\oplus s} \xrightarrow{A_\lambda} (S'_\lambda)^{\oplus t})$.

With these choices, we have for each $\lambda_3 \leq \lambda \leq \mu$ that $S_\lambda \otimes_{R_\lambda} R_\mu \to S_\mu$ is a localization, $S'_\lambda \otimes_{S_\lambda} S_\mu \to S'_\mu$ is a localization, and the map $M_\lambda \otimes_{S'_\lambda} S'_\mu \to M_\mu$ is an isomorphism. This also implies that $S'_\lambda \otimes_{R_\lambda} R_\mu \to S'_\mu$ is a localization. Thus, since M is flat over R we see by Lemma 128.3 that for all λ big enough the module M_λ is flat over R_λ. Moreover, note that $\mathfrak m = \text{colim}\, \mathfrak p_\lambda$, $S/\mathfrak mS = \text{colim}\, S_\lambda/\mathfrak p_\lambda S_\lambda$, $S'/\mathfrak mS' = \text{colim}\, S'_\lambda/\mathfrak p_\lambda S'_\lambda$, and $M/\mathfrak mM = \text{colim}\, M_\lambda/\mathfrak p_\lambda M_\lambda$. Also, for each $\lambda_3 \leq \lambda \leq \mu$ we see (from the properties listed above) that

$$S'_\lambda/\mathfrak p_\lambda S'_\lambda \otimes_{S_\lambda/\mathfrak p_\lambda S_\lambda} S_\mu/\mathfrak p_\mu S_\mu \longrightarrow S'_\mu/\mathfrak p_\mu S'_\mu$$

is a localization, and the map

$$M_\lambda/\mathfrak p_\lambda M_\lambda \otimes_{S'_\lambda/\mathfrak p_\lambda S'_\lambda} S'_\mu/\mathfrak p_\mu S'_\mu \longrightarrow M_\mu/\mathfrak p_\mu M_\mu$$

is an isomorphism. Hence the system $(S_\lambda/\mathfrak p_\lambda S_\lambda \to S'_\lambda/\mathfrak p_\lambda S'_\lambda, M_\lambda/\mathfrak p_\lambda M_\lambda)$ is a system as in Lemma 127.13 as well. We may apply Lemma 128.3 again because $M/\mathfrak mM$ is assumed flat over $S/\mathfrak mS$ and we see that $M_\lambda/\mathfrak p_\lambda M_\lambda$ is flat over $S_\lambda/\mathfrak p_\lambda S_\lambda$ for all λ big enough. Thus for λ big enough the data $R_\lambda \to S_\lambda \to S'_\lambda, M_\lambda$ satisfies the hypotheses of Lemma 99.15. Pick such a λ. Then $S = S_\lambda \otimes_{R_\lambda} R$ is flat over R, and $M = M_\lambda \otimes_{S_\lambda} S$ is flat over S (since the base change of a flat module is flat). $\square$

The following is an easy consequence of the "critère de platitude par fibres" Lemma 128.8. For more results of this kind see More on Flatness, Section 1.

05UV **Lemma 128.9.** *Let R, S, S' be local rings and let $R \to S \to S'$ be local ring homomorphisms. Let M be an S'-module. Let $\mathfrak m \subset R$ be the maximal ideal. Assume*

1. *$R \to S'$ is essentially of finite presentation,*
2. *$R \to S$ is essentially of finite type,*
3. *M is of finite presentation over S',*
4. *M is not zero,*
5. *$M/\mathfrak mM$ is a flat $S/\mathfrak mS$-module, and*
6. *M is a flat R-module.*

Then S is essentially of finite presentation and flat over R and M is a flat S-module.

Proof. As S is essentially of finite presentation over R we can write $S = C_{\overline{\mathfrak q}}$ for some finite type R-algebra C. Write $C = R[x_1, \ldots, x_n]/I$. Denote $\mathfrak q \subset R[x_1, \ldots, x_n]$ be the prime ideal corresponding to $\overline{\mathfrak q}$. Then we see that $S = B/J$ where $B = R[x_1, \ldots, x_n]_{\mathfrak q}$ is essentially of finite presentation over R and $J = IB$. We can find $f_1, \ldots, f_k \in J$ such that the images $\overline{f}_i \in B/\mathfrak mB$ generate the image $\overline{J}$ of J in the Noetherian ring $B/\mathfrak mB$. Hence there exist finitely generated ideals $J' \subset J$ such that $B/J' \to B/J$ induces an isomorphism

$$(B/J') \otimes_R R/\mathfrak m \longrightarrow B/J \otimes_R R/\mathfrak m = S/\mathfrak mS.$$

For any J' as above we see that Lemma 128.8 applies to the ring maps

$$R \longrightarrow B/J' \longrightarrow S'$$

and the module M. Hence we conclude that B/J' is flat over R for any choice J' as above. Now, if $J' \subset J' \subset J$ are two finitely generated ideals as above, then we conclude that $B/J' \to B/J''$ is a surjective map between flat R-algebras which are essentially of finite presentation which is an isomorphism modulo $\mathfrak{m}$. Hence Lemma 128.4 implies that $B/J' = B/J''$, i.e., $J' = J''$. Clearly this means that J is finitely generated, i.e., S is essentially of finite presentation over R. Thus we may apply Lemma 128.8 to $R \to S \to S'$ and we win. ☐

0CEL **Lemma 128.10** (Critère de platitude par fibres: locally nilpotent case)**.** *Let*

$$\begin{tikzcd} S \arrow[rr] & & S' \\ & R \arrow[ul] \arrow[ur] & \end{tikzcd}$$

be a commutative diagram in the category of rings. Let $I \subset R$ be a locally nilpotent ideal and M an S'-module. Assume

1. *$R \to S$ is of finite type,*
2. *$R \to S'$ is of finite presentation,*
3. *M is a finitely presented S'-module,*
4. *M/IM is flat as a S/IS-module, and*
5. *M is flat as an R-module.*

Then M is a flat S-module and $S_{\mathfrak{q}}$ is flat and essentially of finite presentation over R for every $\mathfrak{q} \subset S$ such that $M \otimes_S \kappa(\mathfrak{q})$ is nonzero.

Proof. If $M \otimes_S \kappa(\mathfrak{q})$ is nonzero, then $S' \otimes_S \kappa(\mathfrak{q})$ is nonzero and hence there exists a prime $\mathfrak{q}' \subset S'$ lying over $\mathfrak{q}$ (Lemma 17.9). Let $\mathfrak{p} \subset R$ be the image of $\mathfrak{q}$ in $\mathrm{Spec}(R)$. Then $I \subset \mathfrak{p}$ as I is locally nilpotent hence $M/\mathfrak{p}M$ is flat over $S/\mathfrak{p}S$. Hence we may apply Lemma 128.9 to $R_{\mathfrak{p}} \to S_{\mathfrak{q}} \to S'_{\mathfrak{q}'}$ and $M_{\mathfrak{q}'}$. We conclude that $M_{\mathfrak{q}'}$ is flat over S and $S_{\mathfrak{q}}$ is flat and essentially of finite presentation over R. Since $\mathfrak{q}'$ was an arbitrary prime of S' we also see that M is flat over S (Lemma 39.18). ☐

129. Openness of the flat locus

00R8 We use Lemma 128.3 to reduce to the Noetherian case. The Noetherian case is handled using the characterization of exact complexes given in Section 102.

00R9 **Lemma 129.1.** *Let k be a field. Let S be a finite type k-algebra. Let $f_1, \ldots, f_i$ be elements of S. Assume that S is Cohen-Macaulay and equidimensional of dimension d, and that $\dim V(f_1, \ldots, f_i) \leq d - i$. Then equality holds and $f_1, \ldots, f_i$ forms a regular sequence in $S_{\mathfrak{q}}$ for every prime $\mathfrak{q}$ of $V(f_1, \ldots, f_i)$.*

Proof. If S is Cohen-Macaulay and equidimensional of dimension d, then we have $\dim(S_{\mathfrak{m}}) = d$ for all maximal ideals $\mathfrak{m}$ of S, see Lemma 114.7. By Proposition 103.4 we see that for all maximal ideals $\mathfrak{m} \in V(f_1, \ldots, f_i)$ the sequence is a regular sequence in $S_{\mathfrak{m}}$ and the local ring $S_{\mathfrak{m}}/(f_1, \ldots, f_i)$ is Cohen-Macaulay of dimension $d - i$. This actually means that $S/(f_1, \ldots, f_i)$ is Cohen-Macaulay and equidimensional of dimension $d - i$. ☐

00RA **Lemma 129.2.** *Let $R \to S$ be a finite type ring map. Let d be an integer such that all fibres $S \otimes_R \kappa(\mathfrak{p})$ are Cohen-Macaulay and equidimensional of dimension d.*

Let $f_1, \ldots, f_i$ be elements of S. The set

$$\{\mathfrak q \in V(f_1, \ldots, f_i) \mid f_1, \ldots, f_i \text{ are a regular sequence in } S_{\mathfrak q}/\mathfrak pS_{\mathfrak q} \text{ where } \mathfrak p = R \cap \mathfrak q\}$$

is open in $V(f_1, \ldots, f_i)$.

Proof. Write $\overline{S} = S/(f_1, \ldots, f_i)$. Suppose $\mathfrak q$ is an element of the set defined in the lemma, and $\mathfrak p$ is the corresponding prime of R. We will use relative dimension as defined in Definition 125.1. First, note that $d = \dim_{\mathfrak q}(S/R) = \dim(S_{\mathfrak q}/\mathfrak pS_{\mathfrak q}) + \text{trdeg}_{\kappa(\mathfrak p)}\ \kappa(\mathfrak q)$ by Lemma 116.3. Since $f_1, \ldots, f_i$ form a regular sequence in the Noetherian local ring $S_{\mathfrak q}/\mathfrak pS_{\mathfrak q}$ Lemma 60.13 tells us that $\dim(\overline{S}_{\mathfrak q}/\mathfrak p\overline{S}_{\mathfrak q}) = \dim(S_{\mathfrak q}/\mathfrak pS_{\mathfrak q}) - i$. We conclude that $\dim_{\mathfrak q}(\overline{S}/R) = \dim(\overline{S}_{\mathfrak q}/\mathfrak p\overline{S}_{\mathfrak q}) + \text{trdeg}_{\kappa(\mathfrak p)}\ \kappa(\mathfrak q) = d - i$ by Lemma 116.3. By Lemma 125.6 we have $\dim_{\mathfrak q'}(\overline{S}/R) \leq d - i$ for all $\mathfrak q' \in V(f_1, \ldots, f_i) = \text{Spec}(\overline{S})$ in a neighbourhood of $\mathfrak q$. Thus after replacing S by S_g for some $g \in S$, $g \notin \mathfrak q$ we may assume that the inequality holds for all $\mathfrak q'$. The result follows from Lemma 129.1. $\square$

00RB **Lemma 129.3.** *Let $R \to S$ be a ring map. Consider a finite homological complex of finite free S-modules:*

$$F_\bullet : 0 \to S^{n_e} \xrightarrow{\varphi_e} S^{n_{e-1}} \xrightarrow{\varphi_{e-1}} \ldots \xrightarrow{\varphi_{i+1}} S^{n_i} \xrightarrow{\varphi_i} S^{n_{i-1}} \xrightarrow{\varphi_{i-1}} \ldots \xrightarrow{\varphi_1} S^{n_0}$$

For every prime $\mathfrak q$ of S consider the complex $\overline{F}_{\bullet,\mathfrak q} = F_{\bullet,\mathfrak q} \otimes_R \kappa(\mathfrak p)$ where $\mathfrak p$ is inverse image of $\mathfrak q$ in R. Assume R is Noetherian and there exists an integer d such that $R \to S$ is finite type, flat with fibres $S \otimes_R \kappa(\mathfrak p)$ Cohen-Macaulay of dimension d. The set

$$\{\mathfrak q \in \text{Spec}(S) \mid \overline{F}_{\bullet,\mathfrak q} \text{ is exact}\}$$

is open in $\text{Spec}(S)$.

Proof. Let $\mathfrak q$ be an element of the set defined in the lemma. We are going to use Proposition 102.9 to show there exists a $g \in S$, $g \notin \mathfrak q$ such that $D(g)$ is contained in the set defined in the lemma. In other words, we are going to show that after replacing S by S_g, the set of the lemma is all of $\text{Spec}(S)$. Thus during the proof we will, finitely often, replace S by such a localization. Recall that Proposition 102.9 characterizes exactness of complexes in terms of ranks of the maps φ_i and the ideals $I(\varphi_i)$, in case the ring is local. We first address the rank condition. Set $r_i = n_i - n_{i+1} + \ldots + (-1)^{e-i}n_e$. Note that $r_i + r_{i+1} = n_i$ and note that r_i is the expected rank of φ_i (in the exact case).

By Lemma 99.5 we see that if $\overline{F}_{\bullet,\mathfrak q}$ is exact, then the localization $F_{\bullet,\mathfrak q}$ is exact. In particular the complex $F_\bullet$ becomes exact after localizing by an element $g \in S$, $g \notin \mathfrak q$. In this case Proposition 102.9 applied to all localizations of S at prime ideals implies that all $(r_i + 1) \times (r_i + 1)$-minors of φ_i are zero. Thus we see that the rank of φ_i is at most r_i.

Let $I_i \subset S$ denote the ideal generated by the $r_i \times r_i$-minors of the matrix of φ_i. By Proposition 102.9 the complex $\overline{F}_{\bullet,\mathfrak q}$ is exact if and only if for every $1 \leq i \leq e$ we have either $(I_i)_{\mathfrak q} = S_{\mathfrak q}$ or $(I_i)_{\mathfrak q}$ contains a $S_{\mathfrak q}/\mathfrak pS_{\mathfrak q}$-regular sequence of length i. Namely, by our choice of r_i above and by the bound on the ranks of the φ_i this is the only way the conditions of Proposition 102.9 can be satisfied.

If $(I_i)_{\mathfrak q} = S_{\mathfrak q}$, then after localizing S at some element $g \notin \mathfrak q$ we may assume that $I_i = S$. Clearly, this is an open condition.

If $(I_i)_\mathfrak q \neq S_\mathfrak q$, then we have a sequence $f_1, \ldots, f_i \in (I_i)_\mathfrak q$ which form a regular sequence in $S_\mathfrak q/\mathfrak pS_\mathfrak q$. Note that for any prime $\mathfrak q' \subset S$ such that $(f_1, \ldots, f_i) \not\subset \mathfrak q'$ we have $(I_i)_{\mathfrak q'} = S_{\mathfrak q'}$. Thus the result follows from Lemma 129.2. □

00RC **Theorem 129.4.** *Let R be a ring. Let $R \to S$ be a ring map of finite presentation. Let M be a finitely presented S-module. The set*

$$\{\mathfrak q \in \mathrm{Spec}(S) \mid M_\mathfrak q \textit{ is flat over } R\}$$

is open in $\mathrm{Spec}(S)$.

Proof. Let $\mathfrak q \in \mathrm{Spec}(S)$ be a prime. Let $\mathfrak p \subset R$ be the inverse image of $\mathfrak q$ in R. Note that $M_\mathfrak q$ is flat over R if and only if it is flat over $R_\mathfrak p$. Let us assume that $M_\mathfrak q$ is flat over R. We claim that there exists a $g \in S$, $g \notin \mathfrak q$ such that M_g is flat over R.

We first reduce to the case where R and S are of finite type over $\mathbf Z$. Choose a directed set Λ and a system $(R_\lambda \to S_\lambda, M_\lambda)$ as in Lemma 127.18. Set $\mathfrak p_\lambda$ equal to the inverse image of $\mathfrak p$ in R_λ. Set $\mathfrak q_\lambda$ equal to the inverse image of $\mathfrak q$ in S_λ. Then the system

$$((R_\lambda)_{\mathfrak p_\lambda}, (S_\lambda)_{\mathfrak q_\lambda}, (M_\lambda)_{\mathfrak q_\lambda})$$

is a system as in Lemma 127.13. Hence by Lemma 128.3 we see that for some λ the module M_λ is flat over R_λ at the prime $\mathfrak q_\lambda$. Suppose we can prove our claim for the system $(R_\lambda \to S_\lambda, M_\lambda, \mathfrak q_\lambda)$. In other words, suppose that we can find a $g \in S_\lambda$, $g \notin \mathfrak q_\lambda$ such that $(M_\lambda)_g$ is flat over R_λ. By Lemma 127.18 we have $M = M_\lambda \otimes_{R_\lambda} R$ and hence also $M_g = (M_\lambda)_g \otimes_{R_\lambda} R$. Thus by Lemma 39.7 we deduce the claim for the system $(R \to S, M, \mathfrak q)$.

At this point we may assume that R and S are of finite type over $\mathbf Z$. We may write S as a quotient of a polynomial ring $R[x_1, \ldots, x_n]$. Of course, we may replace S by $R[x_1, \ldots, x_n]$ and assume that S is a polynomial ring over R. In particular we see that $R \to S$ is flat and all fibres rings $S \otimes_R \kappa(\mathfrak p)$ have global dimension n.

Choose a resolution $F_\bullet$ of M over S with each F_i finite free, see Lemma 71.1. Let $K_n = \mathrm{Ker}(F_{n-1} \to F_{n-2})$. Note that $(K_n)_\mathfrak q$ is flat over R, since each F_i is flat over R and by assumption on M, see Lemma 39.13. In addition, the sequence

$$0 \to K_n/\mathfrak pK_n \to F_{n-1}/\mathfrak pF_{n-1} \to \ldots \to F_0/\mathfrak pF_0 \to M/\mathfrak pM \to 0$$

is exact upon localizing at $\mathfrak q$, because of vanishing of $\mathrm{Tor}_i^{R_\mathfrak p}(\kappa(\mathfrak p), M_\mathfrak q)$. Since the global dimension of $S_\mathfrak q/\mathfrak pS_\mathfrak q$ is n we conclude that $K_n/\mathfrak pK_n$ localized at $\mathfrak q$ is a finite free module over $S_\mathfrak q/\mathfrak pS_\mathfrak q$. By Lemma 99.4 $(K_n)_\mathfrak q$ is free over $S_\mathfrak q$. In particular, there exists a $g \in S$, $g \notin \mathfrak q$ such that $(K_n)_g$ is finite free over S_g.

By Lemma 129.3 there exists a further localization S_g such that the complex

$$0 \to K_n \to F_{n-1} \to \ldots \to F_0$$

is exact on *all fibres* of $R \to S$. By Lemma 99.5 this implies that the cokernel of $F_1 \to F_0$ is flat. This proves the theorem in the Noetherian case. □

130. Openness of Cohen-Macaulay loci

00RD In this section we characterize the Cohen-Macaulay property of finite type algebras in terms of flatness. We then use this to prove the set of points where such an algebra is Cohen-Macaulay is open.

00RE **Lemma 130.1.** *Let S be a finite type algebra over a field k. Let $\varphi : k[y_1, \ldots, y_d] \to S$ be a quasi-finite ring map. As subsets of $\text{Spec}(S)$ we have*

$$\{\mathfrak q \mid S_{\mathfrak q} \textit{ flat over } k[y_1, \ldots, y_d]\} = \{\mathfrak q \mid S_{\mathfrak q} \textit{ CM and } \dim_{\mathfrak q}(S/k) = d\}$$

For notation see Definition 125.1.

Proof. Let $\mathfrak q \subset S$ be a prime. Denote $\mathfrak p = k[y_1, \ldots, y_d] \cap \mathfrak q$. Note that always $\dim(S_{\mathfrak q}) \leq \dim(k[y_1, \ldots, y_d]_{\mathfrak p})$ by Lemma 125.4 for example. Moreover, the field extension $\kappa(\mathfrak q)/\kappa(\mathfrak p)$ is finite and hence $\text{trdeg}_k(\kappa(\mathfrak p)) = \text{trdeg}_k(\kappa(\mathfrak q))$.

Let $\mathfrak q$ be an element of the left hand side. Then Lemma 112.9 applies and we conclude that $S_{\mathfrak q}$ is Cohen-Macaulay and $\dim(S_{\mathfrak q}) = \dim(k[y_1, \ldots, y_d]_{\mathfrak p})$. Combined with the equality of transcendence degrees above and Lemma 116.3 this implies that $\dim_{\mathfrak q}(S/k) = d$. Hence $\mathfrak q$ is an element of the right hand side.

Let $\mathfrak q$ be an element of the right hand side. By the equality of transcendence degrees above, the assumption that $\dim_{\mathfrak q}(S/k) = d$ and Lemma 116.3 we conclude that $\dim(S_{\mathfrak q}) = \dim(k[y_1, \ldots, y_d]_{\mathfrak p})$. Hence Lemma 128.1 applies and we see that $\mathfrak q$ is an element of the left hand side. $\square$

00RF **Lemma 130.2.** *Let S be a finite type algebra over a field k. The set of primes $\mathfrak q$ such that $S_{\mathfrak q}$ is Cohen-Macaulay is open in S.*

This lemma is a special case of Lemma 130.5 below, so you can skip straight to the proof of that lemma if you like.

Proof. Let $\mathfrak q \subset S$ be a prime such that $S_{\mathfrak q}$ is Cohen-Macaulay. We have to show there exists a $g \in S$, $g \notin \mathfrak q$ such that the ring S_g is Cohen-Macaulay. For any $g \in S$, $g \notin \mathfrak q$ we may replace S by S_g and $\mathfrak q$ by $\mathfrak q S_g$. Combining this with Lemmas 115.5 and 116.3 we may assume that there exists a finite injective ring map $k[y_1, \ldots, y_d] \to S$ with $d = \dim(S_{\mathfrak q}) + \text{trdeg}_k(\kappa(\mathfrak q))$. Set $\mathfrak p = k[y_1, \ldots, y_d] \cap \mathfrak q$. By construction we see that $\mathfrak q$ is an element of the right hand side of the displayed equality of Lemma 130.1. Hence it is also an element of the left hand side.

By Theorem 129.4 we see that for some $g \in S$, $g \notin \mathfrak q$ the ring S_g is flat over $k[y_1, \ldots, y_d]$. Hence by the equality of Lemma 130.1 again we conclude that all local rings of S_g are Cohen-Macaulay as desired. $\square$

00RG **Lemma 130.3.** *Let k be a field. Let S be a finite type k algebra. The set of Cohen-Macaulay primes forms a dense open $U \subset \text{Spec}(S)$.*

Proof. The set is open by Lemma 130.2. It contains all minimal primes $\mathfrak q \subset S$ since the local ring at a minimal prime $S_{\mathfrak q}$ has dimension zero and hence is Cohen-Macaulay. $\square$

0GEC **Lemma 130.4.** *Let k be a field. Let S be a finite type k algebra. If $\dim(S) > 0$, then there exists an element $f \in S$ which is a nonzerodivisor and a nonunit.*

Proof. Let $I \subset S$ be the radical ideal such that $V(I) \subset \text{Spec}(S)$ is the set of primes $\mathfrak{q} \subset S$ with $S_\mathfrak{q}$ not Cohen-Macaulay. See Lemma 130.3 which also tells us that $V(I)$ is nowhere dense in $\text{Spec}(S)$. Let $\mathfrak{m} \subset S$ be a maximal ideal such that $\dim(S_\mathfrak{m}) > 0$ and $\mathfrak{m} \notin V(I)$. Such a maximal ideal exists as $\dim(S) > 0$ using the Hilbert Nullstellensatz (Theorem 34.1) and Lemma 114.5 which implies that any dense open of $\text{Spec}(S)$ has the same dimension as $\text{Spec}(S)$. Finally, let $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$ be the minimal primes of S. Choose $f \in S$ with

$$f \equiv 1 \bmod I, \quad f \in \mathfrak{m}, \quad f \notin \bigcup \mathfrak{q}_i$$

This is possible by Lemma 15.3. Namely, we have $S/(I \cap \mathfrak{m}) = S/I \times S/\mathfrak{m}$ by Lemma 15.4. Thus we can first choose $g \in S$ such that $g \equiv 1 \bmod I$ and $g \in \mathfrak{m}$. Then $g + (I \cap \mathfrak{m}) \not\subset \mathfrak{q}_i$ since $V(I \cap \mathfrak{m}) \not\supset V(\mathfrak{q}_i)$. Hence the lemma applies. Clearly f is not a unit. To show that f is a nonzerodivisor, it suffices to prove that $f : S_\mathfrak{q} \to S_\mathfrak{q}$ is injective for every prime ideal $\mathfrak{q} \subset S$. If $S_\mathfrak{q}$ is not Cohen-Macaulay, then $\mathfrak{q} \in V(I)$ and f maps to a unit of $S_\mathfrak{q}$. On the other hand, if $S_\mathfrak{q}$ is Cohen-Macaulay, then we use that $\dim(S_\mathfrak{q}/fS_\mathfrak{q}) < \dim(S_\mathfrak{q})$ by the requirement $f \notin \mathfrak{q}_i$ and we conclude that f is a nonzerodivisor in $S_\mathfrak{q}$ by Lemma 104.2. $\square$

00RH **Lemma 130.5.** *Let R be a ring. Let $R \to S$ be of finite presentation and flat. For any $d \geq 0$ the set*

$$\left\{\begin{matrix} \mathfrak{q} \in \text{Spec}(S) \textit{ such that setting } \mathfrak{p} = R \cap \mathfrak{q} \textit{ the fibre ring} \\ S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q} \textit{ is Cohen-Macaulay and } \dim_\mathfrak{q}(S/R) = d \end{matrix}\right\}$$

is open in $\text{Spec}(S)$.

Proof. Let $\mathfrak{q}$ be an element of the set indicated, with $\mathfrak{p}$ the corresponding prime of R. We have to find a $g \in S$, $g \notin \mathfrak{q}$ such that all fibre rings of $R \to S_g$ are Cohen-Macaulay. During the course of the proof we may (finitely many times) replace S by S_g for a $g \in S$, $g \notin \mathfrak{q}$. Thus by Lemma 125.2 we may assume there is a quasi-finite ring map $R[t_1, \ldots, t_d] \to S$ with $d = \dim_\mathfrak{q}(S/R)$. Let $\mathfrak{q}' = R[t_1, \ldots, t_d] \cap \mathfrak{q}$. By Lemma 130.1 we see that the ring map

$$R[t_1, \ldots, t_d]_{\mathfrak{q}'}/\mathfrak{p}R[t_1, \ldots, t_d]_{\mathfrak{q}'} \longrightarrow S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$$

is flat. Hence by the critère de platitude par fibres Lemma 128.8 we see that $R[t_1, \ldots, t_d]_{\mathfrak{q}'} \to S_\mathfrak{q}$ is flat. Hence by Theorem 129.4 we see that for some $g \in S$, $g \notin \mathfrak{q}$ the ring map $R[t_1, \ldots, t_d] \to S_g$ is flat. Replacing S by S_g we see that for every prime $\mathfrak{r} \subset S$, setting $\mathfrak{r}' = R[t_1, \ldots, t_d] \cap \mathfrak{r}$ and $\mathfrak{p}' = R \cap \mathfrak{r}$ the local ring map $R[t_1, \ldots, t_d]_{\mathfrak{r}'} \to S_\mathfrak{r}$ is flat. Hence also the base change

$$R[t_1, \ldots, t_d]_{\mathfrak{r}'}/\mathfrak{p}'R[t_1, \ldots, t_d]_{\mathfrak{r}'} \longrightarrow S_\mathfrak{r}/\mathfrak{p}'S_\mathfrak{r}$$

is flat. Hence by Lemma 130.1 applied with $k = \kappa(\mathfrak{p}')$ we see $\mathfrak{r}$ is in the set of the lemma as desired. $\square$

00RI **Lemma 130.6.** *Let R be a ring. Let $R \to S$ be flat of finite presentation. The set of primes $\mathfrak{q}$ such that the fibre ring $S_\mathfrak{q} \otimes_R \kappa(\mathfrak{p})$, with $\mathfrak{p} = R \cap \mathfrak{q}$ is Cohen-Macaulay is open and dense in every fibre of* $\text{Spec}(S) \to \text{Spec}(R)$.

Proof. The set, call it W, is open by Lemma 130.5. It is dense in the fibres because the intersection of W with a fibre is the corresponding set of the fibre to which Lemma 130.3 applies. $\square$

00RJ **Lemma 130.7.** *Let k be a field. Let S be a finite type k-algebra. Let K/k be a field extension, and set $S_K = K \otimes_k S$. Let $\mathfrak q \subset S$ be a prime of S. Let $\mathfrak q_K \subset S_K$ be a prime of S_K lying over $\mathfrak q$. Then $S_{\mathfrak q}$ is Cohen-Macaulay if and only if $(S_K)_{\mathfrak q_K}$ is Cohen-Macaulay.*

Proof. During the course of the proof we may (finitely many times) replace S by S_g for any $g \in S$, $g \not\in \mathfrak q$. Hence using Lemma 115.5 we may assume that $\dim(S) = \dim_{\mathfrak q}(S/k) =: d$ and find a finite injective map $k[x_1, \ldots, x_d] \to S$. Note that this also induces a finite injective map $K[x_1, \ldots, x_d] \to S_K$ by base change. By Lemma 116.6 we have $\dim_{\mathfrak q_K}(S_K/K) = d$. Set $\mathfrak p = k[x_1, \ldots, x_d] \cap \mathfrak q$ and $\mathfrak p_K = K[x_1, \ldots, x_d] \cap \mathfrak q_K$. Consider the following commutative diagram of Noetherian local rings

$$\begin{tikzcd} S_{\mathfrak q} \arrow[r] & (S_K)_{\mathfrak q_K} \\ k[x_1, \ldots, x_d]_{\mathfrak p} \arrow[u] \arrow[r] & K[x_1, \ldots, x_d]_{\mathfrak p_K} \arrow[u] \end{tikzcd}$$

By Lemma 130.1 we have to show that the left vertical arrow is flat if and only if the right vertical arrow is flat. Because the bottom arrow is flat this equivalence holds by Lemma 100.1. $\square$

00RK **Lemma 130.8.** *Let R be a ring. Let $R \to S$ be of finite type. Let $R \to R'$ be any ring map. Set $S' = R' \otimes_R S$. Denote $f : \text{Spec}(S') \to \text{Spec}(S)$ the map associated to the ring map $S \to S'$. Set W equal to the set of primes $\mathfrak q$ such that the fibre ring $S_{\mathfrak q} \otimes_R \kappa(\mathfrak p)$, $\mathfrak p = R \cap \mathfrak q$ is Cohen-Macaulay, and let W' denote the analogue for S'/R'. Then $W' = f^{-1}(W)$.*

Proof. Trivial from Lemma 130.7 and the definitions. $\square$

00RL **Lemma 130.9.** *Let R be a ring. Let $R \to S$ be a ring map which is (a) flat, (b) of finite presentation, (c) has Cohen-Macaulay fibres. Then we can write $S = S_0 \times \ldots \times S_n$ as a product of R-algebras S_d such that each S_d satisfies (a), (b), (c) and has all fibres equidimensional of dimension d.*

Proof. For each integer d denote $W_d \subset \text{Spec}(S)$ the set defined in Lemma 130.5. Clearly we have $\text{Spec}(S) = \coprod W_d$, and each W_d is open by the lemma we just quoted. Hence the result follows from Lemma 24.3. $\square$

131. Differentials

00RM In this section we define the module of differentials of a ring map.

00RN **Definition 131.1.** Let $\varphi : R \to S$ be a ring map and let M be an S-module. A *derivation*, or more precisely an *R-derivation* into M is a map $D : S \to M$ which is additive, annihilates elements of $\varphi(R)$, and satisfies the *Leibniz rule*: $D(ab) = aD(b) + bD(a)$.

Note that $D(ra) = rD(a)$ if $r \in R$ and $a \in S$. An equivalent definition is that an R-derivation is an R-linear map $D : S \to M$ which satisfies the Leibniz rule. The set of all R-derivations forms an S-module: Given two R-derivations D, D' the sum $D + D' : S \to M$, $a \mapsto D(a) + D'(a)$ is an R-derivation, and given an

R-derivation D and an element $c \in S$ the scalar multiple $cD : S \to M$, $a \mapsto cD(a)$ is an R-derivation. We denote this S-module

$$\mathrm{Der}_R(S, M).$$

Also, if $\alpha : M \to N$ is an S-module map, then the composition $\alpha \circ D$ is an R-derivation into N. In this way the assignment $M \mapsto \mathrm{Der}_R(S, M)$ is a covariant functor.

Consider the following map of free S-modules

$$\bigoplus\nolimits_{(a,b)\in S^2} S[(a,b)] \oplus \bigoplus\nolimits_{(f,g)\in S^2} S[(f,g)] \oplus \bigoplus\nolimits_{r\in R} S[r] \longrightarrow \bigoplus\nolimits_{a\in S} S[a]$$

defined by the rules

$$[(a,b)] \longmapsto [a+b] - [a] - [b], \quad [(f,g)] \longmapsto [fg] - f[g] - g[f], \quad [r] \longmapsto [\varphi(r)]$$

with obvious notation. Let $\Omega_{S/R}$ be the cokernel of this map. There is a map $\mathrm{d} : S \to \Omega_{S/R}$ which maps a to the class $\mathrm{d}a$ of $[a]$ in the cokernel. This is an R-derivation by the relations imposed on $\Omega_{S/R}$, in other words

$$\mathrm{d}(a+b) = \mathrm{d}a + \mathrm{d}b, \quad \mathrm{d}(fg) = f\mathrm{d}g + g\mathrm{d}f, \quad \mathrm{d}\varphi(r) = 0$$

where $a, b, f, g \in S$ and $r \in R$.

07BK **Definition 131.2.** The pair $(\Omega_{S/R}, \mathrm{d})$ is called the *module of Kähler differentials* or the *module of differentials* of S over R.

00RO **Lemma 131.3.** *The module of differentials of S over R has the following universal property. The map*

$$\mathrm{Hom}_S(\Omega_{S/R}, M) \longrightarrow Der_R(S, M), \quad \alpha \longmapsto \alpha \circ d$$

is an isomorphism of functors.

Proof. By definition an R-derivation is a rule which associates to each $a \in S$ an element $D(a) \in M$. Thus D gives rise to a map $[D] : \bigoplus S[a] \to M$. However, the conditions of being an R-derivation exactly mean that $[D]$ annihilates the image of the map in the displayed presentation of $\Omega_{S/R}$ above. $\square$

00RP **Lemma 131.4.** *Suppose that $R \to S$ is surjective. Then $\Omega_{S/R} = 0$.*

Proof. You can see this either because all R-derivations clearly have to be zero, or because the map in the presentation of $\Omega_{S/R}$ is surjective. $\square$

Suppose that

00RQ (131.4.1)
$$\begin{tikzcd} S \arrow[r, "\varphi"'] & S' \\ R \arrow[u, "\alpha"] \arrow[r, "\psi"] & R' \arrow[u, "\beta"'] \end{tikzcd}$$

is a commutative diagram of rings. In this case there is a natural map of modules of differentials fitting into the commutative diagram

$$\begin{tikzcd} \Omega_{S/R} \arrow[r] & \Omega_{S'/R'} \\ S \arrow[u, "\mathrm{d}"] \arrow[r, "\varphi"] & S' \arrow[u, "\mathrm{d}"'] \end{tikzcd}$$

To construct the map just use the obvious map between the presentations for $\Omega_{S/R}$ and $\Omega_{S'/R'}$. Namely,

$$\begin{tikzcd}
\bigoplus S'[(a',b')] \oplus \bigoplus S'[(f',g')] \oplus \bigoplus S'[r'] \arrow[r] & \bigoplus S'[a'] \\
\bigoplus S[(a,b)] \oplus \bigoplus S[(f,g)] \oplus \bigoplus S[r] \arrow[r] \arrow[u, "{\begin{matrix}[(a,b)] \mapsto [(\varphi(a),\varphi(b))] \\ [(f,g)] \mapsto [(\varphi(f),\varphi(g))] \\ [r] \mapsto [\psi(r)]\end{matrix}}"] & \bigoplus S[a] \arrow[u, "{[a] \mapsto [\varphi(a)]}"']
\end{tikzcd}$$

The result is simply that $f\mathrm{d}g \in \Omega_{S/R}$ is mapped to $\varphi(f)\mathrm{d}\varphi(g)$.

031G **Lemma 131.5.** *Let I be a directed set. Let $(R_i \to S_i, \varphi_{ii'})$ be a system of ring maps over I, see Categories, Section 21. Then we have*

$$\Omega_{S/R} = \mathrm{colim}_i\, \Omega_{S_i/R_i}.$$

where $R \to S = \mathrm{colim}(R_i \to S_i)$.

Proof. This is clear from the defining presentation of $\Omega_{S/R}$ and the functoriality of this described above. ☐

00RR **Lemma 131.6.** *In diagram (131.4.1), suppose that $S \to S'$ is surjective with kernel $I \subset S$. Then $\Omega_{S/R} \to \Omega_{S'/R'}$ is surjective with kernel generated as an S-module by the elements da, where $a \in S$ is such that $\varphi(a) \in \beta(R')$. (This includes in particular the elements $d(i)$, $i \in I$.)*

Proof. We urge the reader to find their own (hopefully different) proof of this lemma. Consider the map of presentations above. Clearly the right vertical map of free modules is surjective. Thus the map is surjective. Suppose that some element η of $\Omega_{S/R}$ maps to zero in $\Omega_{S'/R'}$. Write η as the image of $\sum s_i[a_i]$ for some $s_i, a_i \in S$. Then we see that $\sum \varphi(s_i)[\varphi(a_i)]$ is the image of an element

$$\theta = \sum s'_j[a'_j, b'_j] + \sum s'_k[f'_k, g'_k] + \sum s'_l[r'_l]$$

in the upper left corner of the diagram. Since φ is surjective, the terms $s'_j[a'_j, b'_j]$ and $s'_k[f'_k, g'_k]$ are in the image of elements in the lower right corner. Thus, modifying η and θ by substracting the images of these elements, we may assume $\theta = \sum s'_l[r'_l]$. In other words, we see $\sum \varphi(s_i)[\varphi(a_i)]$ is of the form $\sum s'_l[\beta(r'_l)]$. Pick $a' \in S'$. Next, we may assume that we have some $a' \in S'$ such that $a' = \varphi(a_i)$ for all i and $a' = \beta(r'_l)$ for all l. This is clear from the direct sum decomposition of the upper right corner of the diagram. Choose $a \in S$ with $\varphi(a) = a'$. Then we can write $a_i = a + x_i$ for some $x_i \in I$. Thus we may assume that all a_i are equal to a by using the relations that are allowed. But then we may assume our element is of the form $s[a]$. We still know that $\varphi(s)[a'] = \sum \varphi(s'_l)[\beta(r'_l)]$. Hence either $\varphi(s) = 0$ and we're done, or $a' = \varphi(a)$ is in the image of β and we're done as well. ☐

00RS **Lemma 131.7.** *Let $A \to B \to C$ be ring maps. Then there is a canonical exact sequence*

$$C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

of C-modules.

Proof. We get a diagram (131.4.1) by putting $R = A$, $S = C$, $R' = B$, and $S' = C$. By Lemma 131.6 the map $\Omega_{C/A} \to \Omega_{C/B}$ is surjective, and the kernel is generated by the elements $\mathrm{d}(c)$, where $c \in C$ is in the image of $B \to C$. The lemma follows. $\square$

00RT **Lemma 131.8.** *Let $\varphi : A \to B$ be a ring map.*

1. *If $S \subset A$ is a multiplicative subset mapping to invertible elements of B, then $\Omega_{B/A} = \Omega_{B/S^{-1}A}$.*
2. *If $S \subset B$ is a multiplicative subset then $S^{-1}\Omega_{B/A} = \Omega_{S^{-1}B/A}$.*

Proof. To show the equality of (1) it is enough to show that any A-derivation $D : B \to M$ annihilates the elements $\varphi(s)^{-1}$. This is clear from the Leibniz rule applied to $1 = \varphi(s)\varphi(s)^{-1}$. To show (2) note that there is an obvious map $S^{-1}\Omega_{B/A} \to \Omega_{S^{-1}B/A}$. To show it is an isomorphism it is enough to show that there is a A-derivation d' of $S^{-1}B$ into $S^{-1}\Omega_{B/A}$. To define it we simply set $\mathrm{d}'(b/s) = (1/s)\mathrm{d}b - (1/s^2)b\mathrm{d}s$. Details omitted. $\square$

00RU **Lemma 131.9.** *In diagram (131.4.1), suppose that $S \to S'$ is surjective with kernel $I \subset S$, and assume that $R' = R$. Then there is a canonical exact sequence of S'-modules*

$$I/I^2 \longrightarrow \Omega_{S/R} \otimes_S S' \longrightarrow \Omega_{S'/R} \longrightarrow 0$$

The leftmost map is characterized by the rule that $f \in I$ maps to $df \otimes 1$.

Proof. The middle term is $\Omega_{S/R} \otimes_S S/I$. For $f \in I$ denote $\overline{f}$ the image of f in I/I^2. To show that the map $\overline{f} \mapsto \mathrm{d}f \otimes 1$ is well defined we just have to check that $\mathrm{d}f_1f_2 \otimes 1 = 0$ if $f_1, f_2 \in I$. And this is clear from the Leibniz rule $\mathrm{d}f_1f_2 \otimes 1 = (f_1\mathrm{d}f_2 + f_2\mathrm{d}f_1) \otimes 1 = \mathrm{d}f_2 \otimes f_1 + \mathrm{d}f_1 \otimes f_2 = 0$. A similar computation show this map is $S' = S/I$-linear.

The map $\Omega_{S/R} \otimes_S S' \to \Omega_{S'/R}$ is the canonical S'-linear map associated to the S-linear map $\Omega_{S/R} \to \Omega_{S'/R}$. It is surjective because $\Omega_{S/R} \to \Omega_{S'/R}$ is surjective by Lemma 131.6.

The composite of the two maps is zero because $\mathrm{d}f$ maps to zero in $\Omega_{S'/R}$ for $f \in I$. Note that exactness just says that the kernel of $\Omega_{S/R} \to \Omega_{S'/R}$ is generated as an S-submodule by the submodule $I\Omega_{S/R}$ together with the elements $\mathrm{d}f$, with $f \in I$. We know by Lemma 131.6 that this kernel is generated by the elements $\mathrm{d}(a)$ where $\varphi(a) = \beta(r)$ for some $r \in R$. But then $a = \alpha(r) + a - \alpha(r)$, so $\mathrm{d}(a) = \mathrm{d}(a - \alpha(r))$. And $a - \alpha(r) \in I$ since $\varphi(a - \alpha(r)) = \varphi(a) - \varphi(\alpha(r)) = \beta(r) - \beta(r) = 0$. We conclude the elements $\mathrm{d}f$ with $f \in I$ already generate the kernel as an S-module, as desired. $\square$

02HP **Lemma 131.10.** *In diagram (131.4.1), suppose that $S \to S'$ is surjective with kernel $I \subset S$, and assume that $R' = R$. Moreover, assume that there exists an R-algebra map $S' \to S$ which is a right inverse to $S \to S'$. Then the exact sequence of S'-modules of Lemma 131.9 turns into a short exact sequence*

$$0 \longrightarrow I/I^2 \longrightarrow \Omega_{S/R} \otimes_S S' \longrightarrow \Omega_{S'/R} \longrightarrow 0$$

which is even a split short exact sequence.

Proof. Let $\beta : S' \to S$ be the right inverse to the surjection $\alpha : S \to S'$, so $S = I \oplus \beta(S')$. Clearly we can use $\beta : \Omega_{S'/R} \to \Omega_{S/R}$, to get a right inverse to the map $\Omega_{S/R} \otimes_S S' \to \Omega_{S'/R}$. On the other hand, consider the map

$$D : S \longrightarrow I/I^2, \quad x \longmapsto x - \beta(\alpha(x))$$

It is easy to show that D is an R-derivation (omitted). Moreover $xD(s) = 0$ if $x \in I, s \in S$. Hence, by the universal property D induces a map $\tau : \Omega_{S/R} \otimes_S S' \to I/I^2$. We omit the verification that it is a left inverse to $\mathrm{d} : I/I^2 \to \Omega_{S/R} \otimes_S S'$. Hence we win. □

02HQ **Lemma 131.11.** *Let $R \to S$ be a ring map. Let $I \subset S$ be an ideal. Let $n \geq 1$ be an integer. Set $S' = S/I^{n+1}$. The map $\Omega_{S/R} \to \Omega_{S'/R}$ induces an isomorphism*

$$\Omega_{S/R} \otimes_S S/I^n \longrightarrow \Omega_{S'/R} \otimes_{S'} S/I^n.$$

Proof. This follows from Lemma 131.9 and the fact that $\mathrm{d}(I^{n+1}) \subset I^n \Omega_{S/R}$ by the Leibniz rule for d. □

00RV **Lemma 131.12.** *Suppose that we have ring maps $R \to R'$ and $R \to S$. Set $S' = S \otimes_R R'$, so that we obtain a diagram (131.4.1). Then the canonical map defined above induces an isomorphism $\Omega_{S/R} \otimes_R R' = \Omega_{S'/R'}$.*

Proof. Let $\mathrm{d}' : S' = S \otimes_R R' \to \Omega_{S/R} \otimes_R R'$ denote the map $\mathrm{d}'(\sum a_i \otimes x_i) = \sum \mathrm{d}(a_i) \otimes x_i$. It exists because the map $S \times R' \to \Omega_{S/R} \otimes_R R'$, $(a, x) \mapsto \mathrm{d}a \otimes_R x$ is R-bilinear. This is an R'-derivation, as can be verified by a simple computation. We will show that $(\Omega_{S/R} \otimes_R R', \mathrm{d}')$ satisfies the universal property. Let $D : S' \to M'$ be an R' derivation into an S'-module. The composition $S \to S' \to M'$ is an R-derivation, hence we get an S-linear map $\varphi_D : \Omega_{S/R} \to M'$. We may tensor this with R' and get the map $\varphi'_D : \Omega_{S/R} \otimes_R R' \to M'$, $\varphi'_D(\eta \otimes x) = x\varphi_D(\eta)$. It is clear that $D = \varphi'_D \circ \mathrm{d}'$. □

The multiplication map $S \otimes_R S \to S$ is the R-algebra map which maps $a \otimes b$ to ab in S. It is also an S-algebra map, if we think of $S \otimes_R S$ as an S-algebra via either of the maps $S \to S \otimes_R S$.

00RW **Lemma 131.13.** *Let $R \to S$ be a ring map. Let $J = \mathrm{Ker}(S \otimes_R S \to S)$ be the kernel of the multiplication map. There is a canonical isomorphism of S-modules $\Omega_{S/R} \to J/J^2$, $a\mathrm{d}b \mapsto a \otimes b - ab \otimes 1$.*

First proof. Apply Lemma 131.10 to the commutative diagram

$$\begin{tikzcd} S \otimes_R S \arrow[r] & S \\ S \arrow[u] \arrow[r] & S \arrow[u] \end{tikzcd}$$

where the left vertical arrow is $a \mapsto a \otimes 1$. We get the exact sequence $0 \to J/J^2 \to \Omega_{S \otimes_R S/S} \otimes_{S \otimes_R S} S \to \Omega_{S/S} \to 0$. By Lemma 131.4 the term $\Omega_{S/S}$ is 0, and we obtain an isomorphism between the other two terms. We have $\Omega_{S \otimes_R S/S} = \Omega_{S/R} \otimes_S (S \otimes_R S)$ by Lemma 131.12 as $S \to S \otimes_R S$ is the base change of $R \to S$ and hence

$$\Omega_{S \otimes_R S/S} \otimes_{S \otimes_R S} S = \Omega_{S/R} \otimes_S (S \otimes_R S) \otimes_{S \otimes_R S} S = \Omega_{S/R}$$

We omit the verification that the map is given by the rule of the lemma. □

Second proof. First we show that the rule $a\mathrm{d}b \mapsto a \otimes b - ab \otimes 1$ is well defined. In order to do this we have to show that $\mathrm{d}r$ and $a\mathrm{d}b + b\mathrm{d}a - d(ab)$ map to zero. The first because $r \otimes 1 - 1 \otimes r = 0$ by definition of the tensor product. The second because

$$(a \otimes b - ab \otimes 1) + (b \otimes a - ba \otimes 1) - (1 \otimes ab - ab \otimes 1) = (a \otimes 1 - 1 \otimes a)(1 \otimes b - b \otimes 1)$$

is in J^2.

We construct a map in the other direction. We may think of $S \to S \otimes_R S$, $a \mapsto a \otimes 1$ as the base change of $R \to S$. Hence we have $\Omega_{S \otimes_R S/S} = \Omega_{S/R} \otimes_S (S \otimes_R S)$, by Lemma 131.12. At this point the sequence of Lemma 131.9 gives a map

$$J/J^2 \to \Omega_{S \otimes_R S/S} \otimes_{S \otimes_R S} S = (\Omega_{S/R} \otimes_S (S \otimes_R S)) \otimes_{S \otimes_R S} S = \Omega_{S/R}.$$

We leave it to the reader to see it is the inverse of the map above. □

00RX **Lemma 131.14.** *If $S = R[x_1, \ldots, x_n]$, then $\Omega_{S/R}$ is a finite free S-module with basis $dx_1, \ldots, dx_n$.*

Proof. We first show that $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$ generate $\Omega_{S/R}$ as an S-module. To prove this we show that $\mathrm{d}g$ can be expressed as a sum $\sum g_i \mathrm{d}x_i$ for any $g \in R[x_1, \ldots, x_n]$. We do this by induction on the (total) degree of g. It is clear if the degree of g is 0, because then $\mathrm{d}g = 0$. If the degree of g is > 0, then we may write g as $c + \sum g_i x_i$ with $c \in R$ and $\deg(g_i) < \deg(g)$. By the Leibniz rule we have $\mathrm{d}g = \sum g_i \mathrm{d}x_i + \sum x_i \mathrm{d}g_i$, and hence we win by induction.

Consider the R-derivation $\partial/\partial x_i : R[x_1, \ldots, x_n] \to R[x_1, \ldots, x_n]$. (We leave it to the reader to define this; the defining property being that $\partial/\partial x_i(x_j) = \delta_{ij}$.) By the universal property this corresponds to an S-module map $l_i : \Omega_{S/R} \to R[x_1, \ldots, x_n]$ which maps $\mathrm{d}x_i$ to 1 and $\mathrm{d}x_j$ to 0 for $j \neq i$. Thus it is clear that there are no S-linear relations among the elements $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$. □

00RY **Lemma 131.15.** *Suppose $R \to S$ is of finite presentation. Then $\Omega_{S/R}$ is a finitely presented S-module.*

Proof. Write $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Write $I = (f_1, \ldots, f_m)$. According to Lemma 131.9 there is an exact sequence of S-modules

$$I/I^2 \to \Omega_{R[x_1, \ldots, x_n]/R} \otimes_{R[x_1, \ldots, x_n]} S \to \Omega_{S/R} \to 0$$

The result follows from the fact that I/I^2 is a finite S-module (generated by the images of the f_i), and that the middle term is finite free by Lemma 131.14. □

00RZ **Lemma 131.16.** *Suppose $R \to S$ is of finite type. Then $\Omega_{S/R}$ is finitely generated S-module.*

Proof. This is very similar to, but easier than the proof of Lemma 131.15. □

132. The de Rham complex

0FKF Let $A \to B$ be a ring map. Denote $\mathrm{d} : B \to \Omega_{B/A}$ the module of differentials with its universal A-derivation constructed in Section 131. Let $\Omega^i_{B/A} = \wedge^i_B(\Omega_{B/A})$ for $i \geq 0$ be the ith exterior power as in Section 13. The *de Rham complex of B over A* is the complex

$$\Omega^0_{B/A} \to \Omega^1_{B/A} \to \Omega^2_{B/A} \to \ldots$$

constructed and described below.

The map $\mathrm{d} : \Omega^0_{B/A} \to \Omega^1_{B/A}$ is the universal derivation $\mathrm{d} : B \to \Omega_{B/A}$.

For $p \geq 1$ we claim there is a unique A-linear map $\mathrm{d} : \Omega^p_{B/A} \to \Omega^{p+1}_{B/A}$ such that

0FKG (132.0.1) $$\mathrm{d}\left(b_0 \mathrm{d}b_1 \wedge \ldots \wedge \mathrm{d}b_p\right) = \mathrm{d}b_0 \wedge \mathrm{d}b_1 \wedge \ldots \wedge \mathrm{d}b_p$$

Recall that $\Omega_{B/A}$ is generated as a B-module by the elements $\mathrm{d}b$. Thus $\Omega^p_{B/A}$ is additively generated by the element of the form $b_0\mathrm{d}b_1 \wedge \ldots \wedge \mathrm{d}b_p$ and it follows that the map $\mathrm{d} : \Omega^p_{B/A} \to \Omega^{p+1}_{B/A}$ if it exists is unique.

Construction of $\mathrm{d} : \Omega^1_{B/A} \to \Omega^2_{B/A}$. The elements $\mathrm{d}b$ freely generate $\Omega_{B/A}$ subject to the relations $\mathrm{d}a = 0$ for $a \in A$ and $\mathrm{d}(b + b') = \mathrm{d}b + \mathrm{d}b'$ and $\mathrm{d}(bb') = b\mathrm{d}b' + b'\mathrm{d}b$ for $b, b' \in B$. We will show that the rule

$$\sum b'_i \mathrm{d}b_i \longmapsto \sum \mathrm{d}b'_i \wedge \mathrm{d}b_i$$

is well defined. To do this we have to show that the elements

$$\mathrm{d}a, \quad \text{and} \quad b\mathrm{d}(b' + b'') - b\mathrm{d}b' - b\mathrm{d}b'' \quad \text{and} \quad b\mathrm{d}(b'b'') - bb'\mathrm{d}b'' - bb''\mathrm{d}b'$$

for $a \in A$ and $b, b', b'' \in B$ are mapped to zero. This is clear by direct computation using the Leibniz rule for d.

Observe that the composition $\Omega^0_{B/A} \to \Omega^1_{B/A} \to \Omega^2_{B/A}$ is zero as $\mathrm{d}(\mathrm{d}(b)) = \mathrm{d}(1\mathrm{d}b) = \mathrm{d}(1) \wedge \mathrm{d}(b) = 0 \wedge \mathrm{d}b = 0$. Here $\mathrm{d}(1) = 0$ as $1 \in B$ is in the image of $A \to B$. We will use this below.

Construction of $\mathrm{d} : \Omega^p_{B/A} \to \Omega^{p+1}_{B/A}$ for $p \geq 2$. We will show the map

$$\gamma : \Omega^1_{B/A} \otimes_A \ldots \otimes_A \Omega^1_{B/A} \longrightarrow \Omega^{p+1}_{B/A}$$

defined by the formula

$$\omega_1 \otimes \ldots \otimes \omega_p \longmapsto \sum (-1)^{i+1} \omega_1 \wedge \ldots \wedge \mathrm{d}(\omega_i) \wedge \ldots \wedge \omega_p$$

factors over the natural surjection $\Omega^1_{B/A} \otimes_A \ldots \otimes_A \Omega^1_{B/A} \to \Omega^p_{B/A}$ to give a map $\mathrm{d} : \Omega^p_{B/A} \to \Omega^{p+1}_{B/A}$. The kernel of $\Omega^1_{B/A} \otimes_A \ldots \otimes_A \Omega^1_{B/A} \to \Omega^p_{B/A}$ is additively generated by the elements $\omega_1 \otimes \ldots \otimes \omega_p$ with $\omega_i = \omega_j$ for some $i \neq j$ and by the elements $\omega_1 \otimes \ldots \otimes f\omega_i \otimes \ldots \otimes \omega_p - \omega_1 \otimes \ldots \otimes f\omega_j \otimes \ldots \otimes \omega_p$ for $f \in B$; details omitted. A direct computation shows the first type of element is mapped to 0 by γ, in other words, γ is alternating. To finish we have to show that

$$\gamma(\omega_1 \otimes \ldots \otimes f\omega_i \otimes \ldots \otimes \omega_p) = \gamma(\omega_1 \otimes \ldots \otimes f\omega_j \otimes \ldots \otimes \omega_p)$$

for $f \in B$. By A-linearity and the alternating property, it is enough to show this for $p = 2$, $i = 1$, $j = 2$, $\omega_1 = b\mathrm{d}b'$ and $\omega_2 = c\mathrm{d}c'$ for $b, b', c, c' \in B$. Thus we need to show that

$$\begin{aligned} &\mathrm{d}(fb) \wedge \mathrm{d}b' \wedge c\mathrm{d}c' - fb\mathrm{d}b' \wedge \mathrm{d}c \wedge \mathrm{d}c' \\ &= \mathrm{d}b \wedge \mathrm{d}b' \wedge fc\mathrm{d}c' - b\mathrm{d}b' \wedge \mathrm{d}(fc) \wedge \mathrm{d}c' \end{aligned}$$

in other words that

$$(c\mathrm{d}(fb) + fb\mathrm{d}c - fc\mathrm{d}b - b\mathrm{d}(fc)) \wedge \mathrm{d}b' \wedge \mathrm{d}c' = 0.$$

This follows from the Leibniz rule. Observe that the value of γ on the element $b_0\mathrm{d}b_1 \otimes \mathrm{d}b_2 \otimes \ldots \otimes \mathrm{d}b_p$ is $\mathrm{d}b_0 \wedge \mathrm{d}b_1 \wedge \ldots \wedge \mathrm{d}b_p$ and hence (132.0.1) will be satisfied for the map $\mathrm{d} : \Omega^p_{B/A} \to \Omega^{p+1}_{B/A}$ so obtained.

Finally, since $\Omega^p_{B/A}$ is additively generated by the elements $b_0\mathrm{d}b_1 \wedge \ldots \wedge \mathrm{d}b_p$ and since $\mathrm{d}(b_0\mathrm{d}b_1 \wedge \ldots \wedge \mathrm{d}b_p) = \mathrm{d}b_0 \wedge \ldots \wedge \mathrm{d}b_p$ we see in exactly the same manner that the composition $\Omega^p_{B/A} \to \Omega^{p+1}_{B/A} \to \Omega^{p+2}_{B/A}$ is zero for $p \geq 1$. Thus the de Rham complex is indeed a complex.

Given just a ring R we set $\Omega_R = \Omega_{R/\mathbf{Z}}$. This is sometimes called the absolute module of differentials of R; this makes sense: if Ω_R is the module of differentials where we only assume the Leibniz rule and not the vanishing of d1, then the Leibniz rule gives $\mathrm{d}1 = \mathrm{d}(1 \cdot 1) = 1\mathrm{d}1 + 1\mathrm{d}1 = 2\mathrm{d}1$ and hence $\mathrm{d}1 = 0$ in Ω_R. In this case the *absolute de Rham complex of R* is the corresponding complex

$$\Omega^0_R \to \Omega^1_R \to \Omega^2_R \to \ldots$$

where we set $\Omega^i_R = \Omega^i_{R/\mathbf{Z}}$ and so on.

Suppose we have a commutative diagram of rings

$$\begin{tikzcd} B \arrow[r] & B' \\ A \arrow[u] \arrow[r] & A' \arrow[u] \end{tikzcd}$$

There is a natural map of de Rham complexes

$$\Omega^\bullet_{B/A} \longrightarrow \Omega^\bullet_{B'/A'}$$

Namely, in degree 0 this is the map $B \to B'$, in degree 1 this is the map $\Omega_{B/A} \to \Omega_{B'/A'}$ constructed in Section 131, and for $p \geq 2$ it is the induced map $\Omega^p_{B/A} = \wedge^p_B(\Omega_{B/A}) \to \wedge^p_{B'}(\Omega_{B'/A'}) = \Omega^p_{B'/A'}$. The compatibility with differentials follows from the characterization of the differentials by the formula (132.0.1).

07HY **Lemma 132.1.** *Let $A \to B$ be a ring map. Let $\pi : \Omega_{B/A} \to \Omega$ be a surjective B-module map. Denote $d : B \to \Omega$ the composition of π with the universal derivation $d_{B/A} : B \to \Omega_{B/A}$. Set $\Omega^i = \wedge^i_B(\Omega)$. Assume that the kernel of π is generated, as a B-module, by elements $\omega \in \Omega_{B/A}$ such that $d_{B/A}(\omega) \in \Omega^2_{B/A}$ maps to zero in Ω^2. Then there is a de Rham complex*

$$\Omega^0 \to \Omega^1 \to \Omega^2 \to \ldots$$

whose differential is defined by the rule

$$d : \Omega^p \to \Omega^{p+1}, \quad d(f_0 df_1 \wedge \ldots \wedge df_p) = df_0 \wedge df_1 \wedge \ldots \wedge df_p$$

Proof. We will show that there exists a commutative diagram

$$\begin{tikzcd} \Omega^0_{B/A} \arrow[r, "\mathrm{d}_{B/A}"'] \arrow[d] & \Omega^1_{B/A} \arrow[r, "\mathrm{d}_{B/A}"'] \arrow[d, "\pi"'] & \Omega^2_{B/A} \arrow[r, "\mathrm{d}_{B/A}"'] \arrow[d, "\wedge^2\pi"'] & \ldots \\ \Omega^0 \arrow[r, "\mathrm{d}"] & \Omega^1 \arrow[r, "\mathrm{d}"] & \Omega^2 \arrow[r, "\mathrm{d}"] & \ldots \end{tikzcd}$$

the description of the map d will follow from the construction of the differentials $\mathrm{d}_{B/A} : \Omega^p_{B/A} \to \Omega^{p+1}_{B/A}$ of the de Rham complex of B over A given above. Since the left most vertical arrow is an isomorphism we have the first square. Because π is surjective, to get the second square it suffices to show that $\mathrm{d}_{B/A}$ maps the kernel of π into the kernel of $\wedge^2\pi$. We are given that any element of the kernel of π is of the form $\sum b_i\omega_i$ with $\pi(\omega_i) = 0$ and $\wedge^2\pi(\mathrm{d}_{B/A}(\omega_i)) = 0$. By the Leibniz rule for

$d_{B/A}$ we have $d_{B/A}(\sum b_i\omega_i) = \sum b_i d_{B/A}(\omega_i) + \sum d_{B/A}(b_i) \wedge \omega_i$. Hence this maps to zero under $\wedge^2\pi$.

For $i > 1$ we note that $\wedge^i\pi$ is surjective with kernel the image of $\text{Ker}(\pi) \wedge \Omega^{i-1}_{B/A} \to \Omega^i_{B/A}$. For $\omega_1 \in \text{Ker}(\pi)$ and $\omega_2 \in \Omega^{i-1}_{B/A}$ we have

$$d_{B/A}(\omega_1 \wedge \omega_2) = d_{B/A}(\omega_1) \wedge \omega_2 - \omega_1 \wedge d_{B/A}(\omega_2)$$

which is in the kernel of $\wedge^{i+1}\pi$ by what we just proved above. Hence we get the $(i+1)$st square in the diagram above. This concludes the proof. □

133. Finite order differential operators

09CH In this section we introduce differential operators of finite order.

09CI **Definition 133.1.** Let $R \to S$ be a ring map. Let M, N be S-modules. Let $k \geq 0$ be an integer. We inductively define a *differential operator $D : M \to N$ of order k* to be an R-linear map such that for all $g \in S$ the map $m \mapsto D(gm) - gD(m)$ is a differential operator of order $k-1$. For the base case $k = 0$ we define a differential operator of order 0 to be an S-linear map.

If $D : M \to N$ is a differential operator of order k, then for all $g \in S$ the map gD is a differential operator of order k. The sum of two differential operators of order k is another. Hence the set of all these

$$\text{Diff}^k(M, N) = \text{Diff}^k_{S/R}(M, N)$$

is an S-module. We have

$$\text{Diff}^0(M, N) \subset \text{Diff}^1(M, N) \subset \text{Diff}^2(M, N) \subset \ldots$$

09CJ **Lemma 133.2.** *Let $R \to S$ be a ring map. Let L, M, N be S-modules. If $D : L \to M$ and $D' : M \to N$ are differential operators of order k and k', then $D' \circ D$ is a differential operator of order $k + k'$.*

Proof. Let $g \in S$. Then the map which sends $x \in L$ to

$$D'(D(gx)) - gD'(D(x)) = D'(D(gx)) - D'(gD(x)) + D'(gD(x)) - gD'(D(x))$$

is a sum of two compositions of differential operators of lower order. Hence the lemma follows by induction on $k + k'$. □

09CK **Lemma 133.3.** *Let $R \to S$ be a ring map. Let M be an S-module. Let $k \geq 0$. There exists an S-module $P^k_{S/R}(M)$ and a canonical isomorphism*

$$\mathit{Diff}^k_{S/R}(M, N) = \text{Hom}_S(P^k_{S/R}(M), N)$$

functorial in the S-module N.

Proof. The existence of $P^k_{S/R}(M)$ follows from general category theoretic arguments (insert future reference here), but we will also give a construction. Set $F = \bigoplus_{m \in M} S[m]$ where $[m]$ is a symbol indicating the basis element in the summand corresponding to m. Given any differential operator $D : M \to N$ we obtain an S-linear map $L_D : F \to N$ sending $[m]$ to $D(m)$. If D has order 0, then L_D annihilates the elements

$$[m + m'] - [m] - [m'], \quad g_0[m] - [g_0 m]$$

where $g_0 \in S$ and $m, m' \in M$. If D has order 1, then L_D annihilates the elements

$$[m + m'] - [m] - [m'], \quad f[m] - [fm], \quad g_0 g_1[m] - g_0[g_1 m] - g_1[g_0 m] + [g_1 g_0 m]$$

where $f \in R$, $g_0, g_1 \in S$, and $m \in M$. If D has order k, then L_D annihilates the elements $[m + m'] - [m] - [m']$, $f[m] - [fm]$, and the elements

$$g_0 g_1 \ldots g_k[m] - \sum g_0 \ldots \hat{g}_i \ldots g_k[g_i m] + \ldots + (-1)^{k+1}[g_0 \ldots g_k m]$$

Conversely, if $L : F \to N$ is an S-linear map annihilating all the elements listed in the previous sentence, then $m \mapsto L([m])$ is a differential operator of order k. Thus we see that $P^k_{S/R}(M)$ is the quotient of F by the submodule generated by these elements. □

09CL **Definition 133.4.** Let $R \to S$ be a ring map. Let M be an S-module. The module $P^k_{S/R}(M)$ constructed in Lemma 133.3 is called the *module of principal parts of order k* of M.

Note that the inclusions

$$\text{Diff}^0(M, N) \subset \text{Diff}^1(M, N) \subset \text{Diff}^2(M, N) \subset \ldots$$

correspond via Yoneda's lemma (Categories, Lemma 3.5) to surjections

$$\ldots \to P^2_{S/R}(M) \to P^1_{S/R}(M) \to P^0_{S/R}(M) = M$$

09CM **Example 133.5.** Let $R \to S$ be a ring map and let N be an S-module. Observe that $\text{Diff}^1(S, N) = \text{Der}_R(S, N) \oplus N$. Namely, if $D : S \to N$ is a differential operator of order 1 then $\sigma_D : S \to N$ defined by $\sigma_D(g) := D(g) - gD(1)$ is an R-derivation and $D = \sigma_D + \lambda_{D(1)}$ where $\lambda_x : S \to N$ is the linear map sending g to gx. It follows that $P^1_{S/R} = \Omega_{S/R} \oplus S$ by the universal property of $\Omega_{S/R}$.

09CN **Lemma 133.6.** *Let $R \to S$ be a ring map. Let M be an S-module. There is a canonical short exact sequence*

$$0 \to \Omega_{S/R} \otimes_S M \to P^1_{S/R}(M) \to M \to 0$$

functorial in M called the sequence of principal parts.

Proof. The map $P^1_{S/R}(M) \to M$ is given above. Let N be an S-module and let $D : M \to N$ be a differential operator of order 1. For $m \in M$ the map

$$g \longmapsto D(gm) - gD(m)$$

is an R-derivation $S \to N$ by the axioms for differential operators of order 1. Thus it corresponds to a linear map $D_m : \Omega_{S/R} \to N$ determined by the rule $adb \mapsto aD(bm) - abD(m)$ (see Lemma 131.3). The map

$$\Omega_{S/R} \times M \longrightarrow N, \quad (\eta, m) \longmapsto D_m(\eta)$$

is S-bilinear (details omitted) and hence determines an S-linear map

$$\sigma_D : \Omega_{S/R} \otimes_S M \to N$$

In this way we obtain a map $\text{Diff}^1(M, N) \to \text{Hom}_S(\Omega_{S/R} \otimes_S M, N)$, $D \mapsto \sigma_D$ functorial in N. By the Yoneda lemma this corresponds a map $\Omega_{S/R} \otimes_S M \to P^1_{S/R}(M)$. It is immediate from the construction that this map is functorial in M. The sequence

$$\Omega_{S/R} \otimes_S M \to P^1_{S/R}(M) \to M \to 0$$

is exact because for every module N the sequence

$$0 \to \mathrm{Hom}_S(M, N) \to \mathrm{Diff}^1(M, N) \to \mathrm{Hom}_S(\Omega_{S/R} \otimes_S M, N)$$

is exact by inspection.

To see that $\Omega_{S/R} \otimes_S M \to P^1_{S/R}(M)$ is injective we argue as follows. Choose an exact sequence

$$0 \to M' \to F \to M \to 0$$

with F a free S-module. This induces an exact sequence

$$0 \to \mathrm{Diff}^1(M, N) \to \mathrm{Diff}^1(F, N) \to \mathrm{Diff}^1(M', N)$$

for all N. This proves that in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega_{S/R} \otimes_S M' \arrow[r] \arrow[d] & P^1_{S/R}(M') \arrow[r] \arrow[d] & M' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega_{S/R} \otimes_S F \arrow[r] \arrow[d] & P^1_{S/R}(F) \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega_{S/R} \otimes_S M \arrow[r] \arrow[d] & P^1_{S/R}(M) \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

the middle column is exact. The left column is exact by right exactness of $\Omega_{S/R} \otimes_S -$. By the snake lemma (see Section 4) it suffices to prove exactness on the left for the free module F. Using that $P^1_{S/R}(-)$ commutes with direct sums we reduce to the case $M = S$. This case is a consequence of the discussion in Example 133.5. $\square$

09CP **Remark 133.7.** Suppose given a commutative diagram of rings

$$\begin{tikzcd}
B \arrow[r] & B' \\
A \arrow[r] \arrow[u] & A' \arrow[u]
\end{tikzcd}$$

a B-module M, a B'-module M', and a B-linear map $M \to M'$. Then we get a compatible system of module maps

$$\begin{tikzcd}
\ldots \arrow[r] & P^2_{B'/A'}(M') \arrow[r] & P^1_{B'/A'}(M') \arrow[r] & P^0_{B'/A'}(M') \\
\ldots \arrow[r] & P^2_{B/A}(M) \arrow[r] \arrow[u] & P^1_{B/A}(M) \arrow[r] \arrow[u] & P^0_{B/A}(M) \arrow[u]
\end{tikzcd}$$

These maps are compatible with further composition of maps of this type. The easiest way to see this is to use the description of the modules $P^k_{B/A}(M)$ in terms of generators and relations in the proof of Lemma 133.3 but it can also be seen directly from the universal property of these modules. Moreover, these maps are compatible with the short exact sequences of Lemma 133.6.

0G34 **Lemma 133.8.** *Let $A \to B$ be a ring map. The differentials $d : \Omega^i_{B/A} \to \Omega^{i+1}_{B/A}$ are differential operators of order* 1*.*

Proof. Given $b \in B$ we have to show that $\mathrm{d} \circ b - b \circ \mathrm{d}$ is a linear operator. Thus we have to show that

$$\mathrm{d} \circ b \circ b' - b \circ \mathrm{d} \circ b' - b' \circ \mathrm{d} \circ b + b' \circ b \circ \mathrm{d} = 0$$

To see this it suffices to check this on additive generators for $\Omega^i_{B/A}$. Thus it suffices to show that

$$\mathrm{d}(bb'b_0\mathrm{d}b_1\wedge\ldots\wedge\mathrm{d}b_i)-b\mathrm{d}(b'b_0\mathrm{d}b_1\wedge\ldots\wedge\mathrm{d}b_i)-b'\mathrm{d}(bb_0\mathrm{d}b_1\wedge\ldots\wedge\mathrm{d}b_i)+bb'\mathrm{d}(b_0\mathrm{d}b_1\wedge\ldots\wedge\mathrm{d}b_i)$$

is zero. This is a pleasant calculation using the Leibniz rule which is left to the reader. □

0G35 **Lemma 133.9.** *Let $A \to B$ be a ring map. Let $g_i \in B$, $i \in I$ be a set of generators for B as an A-algebra. Let M, N be B-modules. Let $D : M \to N$ be an A-linear map. In order to show that D is a differential operator of order k it suffices to show that $D \circ g_i - g_i \circ D$ is a differential operator of order $k-1$ for $i \in I$.*

Proof. Namely, we claim that the set of elements $g \in B$ such that $D \circ g - g \circ D$ is a differential operator of order $k-1$ is an A-subalgebra of B. This follows from the relations

$$D \circ (g + g') - (g + g') \circ D = (D \circ g - g \circ D) + (D \circ g' - g' \circ D)$$

and

$$D \circ gg' - gg' \circ D = (D \circ g - g \circ D) \circ g' + g \circ (D \circ g' - g' \circ D)$$

Strictly speaking, to conclude for products we also use Lemma 133.2. □

0G36 **Lemma 133.10.** *Let $A \to B$ be a ring map. Let M, N be B-modules. Let $S \subset B$ be a multiplicative subset. Any differential operator $D : M \to N$ of order k extends uniquely to a differential operator $E : S^{-1}M \to S^{-1}N$ of order k.*

Proof. By induction on k. If $k = 0$, then D is B-linear and hence we get the extension by the functoriality of localization. Given $b \in B$ the operator $L_b : m \mapsto D(bm) - bD(m)$ has order $k-1$. Hence it has a unique extension to a differential operator $E_b : S^{-1}M \to S^{-1}N$ of order $k-1$ by induction. Moreover, a computation shows that $L_{b'b} = L_{b'} \circ b + b' \circ L_b$ hence by uniqueness we obtain $E_{b'b} = E_{b'} \circ b + b' \circ E_b$. Similarly, we obtain $E_{b'} \circ b - b \circ E_{b'} = E_b \circ b' - b' \circ E_b$. Now for $m \in M$ and $g \in S$ we set

$$E(m/g) = (1/g)(D(m) - E_g(m/g))$$

To show that this is well defined it suffices to show that for $g' \in S$ if we use the representative $g'm/g'g$ we get the same result. We compute

$$\begin{aligned}(1/g'g)(D(g'm) - E_{g'g}(g'm/gg')) &= (1/gg')(g'D(m) + E_{g'}(m) - E_{g'g}(g'm/gg')) \\ &= (1/g'g)(g'D(m) - g'E_g(m/g))\end{aligned}$$

which is the same as before. It is clear that E is R-linear as D and E_g are R-linear. Taking $g = 1$ and using that $E_1 = 0$ we see that E extends D. By Lemma 133.9 it now suffices to show that $E \circ b - b \circ E$ for $b \in B$ and $E \circ 1/g' - 1/g' \circ E$ for $g' \in S$ are differential operators of order $k-1$ in order to show that E is a differential operator of order k. For the first, choose an element m/g in $S^{-1}M$ and observe that

$$\begin{aligned}E(bm/g) - bE(m/g) &= (1/g)(D(bm) - bD(m) - E_g(bm/g) + bE_g(m/g)) \\ &= (1/g)(L_b(m) - E_b(m) + gE_b(m/g)) \\ &= E_b(m/g)\end{aligned}$$

which is a differential operator of order $k-1$. Finally, we have

$$\begin{aligned}E(m/g'g) - (1/g')E(m/g) &= (1/g'g)(D(m) - E_{g'g}(m/g'g)) - (1/g'g)(D(m) - E_g(m/g)) \\ &= -(1/g')E_{g'}(m/g'g)\end{aligned}$$

which also is a differential operator of order $k-1$ as the composition of linear maps (multiplication by $1/g'$ and signs) and $E_{g'}$. We omit the proof of uniqueness. □

0G37 **Lemma 133.11.** *Let $R \to A$ and $R \to B$ be ring maps. Let M and M' be A-modules. Let $D : M \to M'$ be a differential operator of order k with respect to $R \to A$. Let N be any B-module. Then the map*

$$D \otimes id_N : M \otimes_R N \to M' \otimes_R N$$

is a differential operator of order k with respect to $B \to A \otimes_R B$.

Proof. It is clear that $D' = D \otimes \text{id}_N$ is B-linear. By Lemma 133.9 it suffices to show that

$$D' \circ a \otimes 1 - a \otimes 1 \circ D' = (D \circ a - a \circ D) \otimes \text{id}_N$$

is a differential operator of order $k-1$ which follows by induction on k. □

134. The naive cotangent complex

00S0 Let $R \to S$ be a ring map. Denote $R[S]$ the polynomial ring whose variables are the elements $s \in S$. Let's denote $[s] \in R[S]$ the variable corresponding to $s \in S$. Thus $R[S]$ is a free R-module on the basis elements $[s_1] \ldots [s_n]$ where $s_1, \ldots, s_n$ ranges over all unordered sequences of elements of S. There is a canonical surjection

07BL (134.0.1) $$R[S] \longrightarrow S, \quad [s] \longmapsto s$$

whose kernel we denote $I \subset R[S]$. It is a simple observation that I is generated by the elements $[s+s'] - [s] - [s']$, $[s][s'] - [ss']$ and $[r] - r$. According to Lemma 131.9 there is a canonical map

07BM (134.0.2) $$I/I^2 \longrightarrow \Omega_{R[S]/R} \otimes_{R[S]} S$$

whose cokernel is canonically isomorphic to $\Omega_{S/R}$. Observe that the S-module $\Omega_{R[S]/R} \otimes_{R[S]} S$ is free on the generators $\text{d}[s]$.

07BN **Definition 134.1.** Let $R \to S$ be a ring map. The *naive cotangent complex* $NL_{S/R}$ is the chain complex (134.0.2)

$$NL_{S/R} = \left(I/I^2 \longrightarrow \Omega_{R[S]/R} \otimes_{R[S]} S\right)$$

with I/I^2 placed in (homological) degree 1 and $\Omega_{R[S]/R} \otimes_{R[S]} S$ placed in degree 0. We will denote $H_1(L_{S/R}) = H_1(NL_{S/R})$[12] the homology in degree 1.

[12]This module is sometimes denoted $\Gamma_{S/R}$ in the literature.

Before we continue let us say a few words about the actual cotangent complex (Cotangent, Section 3). Given a ring map $R \to S$ there exists a canonical simplicial R-algebra $P_\bullet$ whose terms are polynomial algebras and which comes equipped with a canonical homotopy equivalence

$$P_\bullet \longrightarrow S$$

The cotangent complex $L_{S/R}$ of S over R is defined as the chain complex associated to the cosimplicial module

$$\Omega_{P_\bullet/R} \otimes_{P_\bullet} S$$

The naive cotangent complex as defined above is canonically isomorphic to the truncation $\tau_{\leq 1}L_{S/R}$ (see Homology, Section 15 and Cotangent, Section 11). In particular, it is indeed the case that $H_1(NL_{S/R}) = H_1(L_{S/R})$ so our definition is compatible with the one using the cotangent complex. Moreover, $H_0(L_{S/R}) = H_0(NL_{S/R}) = \Omega_{S/R}$ as we've seen above.

Let $R \to S$ be a ring map. A *presentation of S over R* is a surjection $\alpha : P \to S$ of R-algebras where P is a polynomial algebra (on a set of variables). Often, when S is of finite type over R we will indicate this by saying: "Let $R[x_1, \ldots, x_n] \to S$ be a presentation of S/R", or "Let $0 \to I \to R[x_1, \ldots, x_n] \to S \to 0$ be a presentation of S/R" if we want to indicate that I is the kernel of the presentation. Note that the map $R[S] \to S$ used to define the naive cotangent complex is an example of a presentation.

Note that for every presentation α we obtain a two term chain complex of S-modules

$$NL(\alpha) : I/I^2 \longrightarrow \Omega_{P/R} \otimes_P S.$$

Here the term I/I^2 is placed in degree 1 and the term $\Omega_{P/R} \otimes S$ is placed in degree 0. The class of $f \in I$ in I/I^2 is mapped to $\text{d}f \otimes 1$ in $\Omega_{P/R} \otimes S$. The cokernel of this complex is canonically $\Omega_{S/R}$, see Lemma 131.9. We call the complex $NL(\alpha)$ the *naive cotangent complex associated to the presentation $\alpha : P \to S$ of S/R*. Note that if $P = R[S]$ with its canonical surjection onto S, then we recover $NL_{S/R}$. If $P = R[x_1, \ldots, x_n]$ then will sometimes use the notation $I/I^2 \to \bigoplus_{i=1,\ldots,n} S\text{d}x_i$ to denote this complex.

Suppose we are given a commutative diagram

06RQ (134.1.1)
$$\begin{tikzcd} S \arrow[r, "\phi"'] & S' \\ R \arrow[u] \arrow[r] & R' \arrow[u] \end{tikzcd}$$

of rings. Let $\alpha : P \to S$ be a presentation of S over R and let $\alpha' : P' \to S'$ be a presentation of S' over R'. A *morphism of presentations from $\alpha : P \to S$ to $\alpha' : P' \to S'$* is defined to be an R-algebra map

$$\varphi : P \to P'$$

such that $\phi \circ \alpha = \alpha' \circ \varphi$. Note that in this case $\varphi(I) \subset I'$, where $I = \text{Ker}(\alpha)$ and $I' = \text{Ker}(\alpha')$. Thus φ induces a map of S-modules $I/I^2 \to I'/(I')^2$ and by functoriality of differentials also an S-module map $\Omega_{P/R} \otimes S \to \Omega_{P'/R'} \otimes S'$. These maps are compatible with the differentials of $NL(\alpha)$ and $NL(\alpha')$ and we obtain a map of naive cotangent complexes

$$NL(\alpha) \longrightarrow NL(\alpha').$$

It is often convenient to consider the induced map $NL(\alpha) \otimes_S S' \to NL(\alpha')$.

In the special case that $P = R[S]$ and $P' = R'[S']$ the map $\phi : S \to S'$ induces a canonical ring map $\varphi : P \to P'$ by the rule $[s] \mapsto [\phi(s)]$. Hence the construction above determines canonical(!) maps of chain complexes

$$NL_{S/R} \longrightarrow NL_{S'/R'}, \quad \text{and} \quad NL_{S/R} \otimes_S S' \longrightarrow NL_{S'/R'}$$

associated to the diagram (134.1.1). Note that this construction is compatible with composition: given a commutative diagram

$$\begin{tikzcd} S \arrow[r, "\phi"'] & S' \arrow[r, "\phi'"'] & S'' \\ R \arrow[u] \arrow[r] & R' \arrow[u] \arrow[r] & R'' \arrow[u] \end{tikzcd}$$

we see that the composition of

$$NL_{S/R} \longrightarrow NL_{S'/R'} \longrightarrow NL_{S''/R''}$$

is the map $NL_{S/R} \to NL_{S''/R''}$ given by the outer square.

It turns out that $NL(\alpha)$ is homotopy equivalent to $NL_{S/R}$ and that the maps constructed above are well defined up to homotopy (homotopies of maps of complexes are discussed in Homology, Section 13 but we also spell out the exact meaning of the statements in the lemma below in its proof).

00S1 **Lemma 134.2.** *Suppose given a diagram (134.1.1). Let $\alpha : P \to S$ and $\alpha' : P' \to S'$ be presentations.*

1. *There exists a morphism of presentations from α to α'.*
2. *Any two morphisms of presentations induce homotopic morphisms of complexes $NL(\alpha) \to NL(\alpha')$.*
3. *The construction is compatible with compositions of morphisms of presentations (see proof for exact statement).*
4. *If $R \to R'$ and $S \to S'$ are isomorphisms, then for any map φ of presentations from α to α' the induced map $NL(\alpha) \to NL(\alpha')$ is a homotopy equivalence and a quasi-isomorphism.*

In particular, comparing α to the canonical presentation (134.0.1) we conclude there is a quasi-isomorphism $NL(\alpha) \to NL_{S/R}$ well defined up to homotopy and compatible with all functorialities (up to homotopy).

Proof. Since P is a polynomial algebra over R we can write $P = R[x_a, a \in A]$ for some set A. As α' is surjective, we can choose for every $a \in A$ an element $f_a \in P'$ such that $\alpha'(f_a) = \phi(\alpha(x_a))$. Let $\varphi : P = R[x_a, a \in A] \to P'$ be the unique R-algebra map such that $\varphi(x_a) = f_a$. This gives the morphism in (1).

Let φ and φ' morphisms of presentations from α to α'. Let $I = \text{Ker}(\alpha)$ and $I' = \text{Ker}(\alpha')$. We have to construct the diagonal map h in the diagram

$$\begin{tikzcd} I/I^2 \arrow[r, "\mathrm{d}"] \arrow[d, shift right, "\varphi_1"'] \arrow[d, shift left, "\varphi_1'"] & \Omega_{P/R} \otimes_P S \arrow[dl, "h"'] \arrow[d, shift right, "\varphi_0"'] \arrow[d, shift left, "\varphi_0'"] \\ I'/(I')^2 \arrow[r, "\mathrm{d}"] & \Omega_{P'/R'} \otimes_{P'} S' \end{tikzcd}$$

where the vertical maps are induced by φ, φ' such that

$$\varphi_1 - \varphi'_1 = h \circ \mathrm{d} \quad \text{and} \quad \varphi_0 - \varphi'_0 = \mathrm{d} \circ h$$

Consider the map $\varphi - \varphi' : P \to P'$. Since both φ and φ' are compatible with α and α' we obtain $\varphi - \varphi' : P \to I'$. This implies that $\varphi, \varphi' : P \to P'$ induce the same P-module structure on $I'/(I')^2$, since $\varphi(p)i' - \varphi'(p)i' = (\varphi - \varphi')(p)i' \in (I')^2$. Also $\varphi - \varphi'$ is R-linear and

$$(\varphi - \varphi')(fg) = \varphi(f)(\varphi - \varphi')(g) + (\varphi - \varphi')(f)\varphi'(g)$$

Hence the induced map $D : P \to I'/(I')^2$ is a R-derivation. Thus we obtain a canonical map $h : \Omega_{P/R} \otimes_P S \to I'/(I')^2$ such that $D = h \circ \mathrm{d}$. A calculation (omitted) shows that h is the desired homotopy.

Suppose that we have a commutative diagram

$$\begin{tikzcd}
S \arrow[r, "\phi"'] & S' \arrow[r, "\phi'"'] & S'' \\
R \arrow[r] \arrow[u] & R' \arrow[r] \arrow[u] & R'' \arrow[u]
\end{tikzcd}$$

and that

(1) $\alpha : P \to S$,
(2) $\alpha' : P' \to S'$, and
(3) $\alpha'' : P'' \to S''$

are presentations. Suppose that

(1) $\varphi : P \to P$ is a morphism of presentations from α to α' and
(2) $\varphi' : P' \to P''$ is a morphism of presentations from α' to α''.

Then it is immediate that $\varphi' \circ \varphi : P \to P''$ is a morphism of presentations from α to α'' and that the induced map $NL(\alpha) \to NL(\alpha'')$ of naive cotangent complexes is the composition of the maps $NL(\alpha) \to NL(\alpha')$ and $NL(\alpha') \to NL(\alpha'')$ induced by φ and φ'.

In the simple case of complexes with 2 terms a quasi-isomorphism is just a map that induces an isomorphism on both the cokernel and the kernel of the maps between the terms. Note that homotopic maps of 2 term complexes (as explained above) define the same maps on kernel and cokernel. Hence if φ is a map from a presentation α of S over R to itself, then the induced map $NL(\alpha) \to NL(\alpha)$ is a quasi-isomorphism being homotopic to the identity by part (2). To prove (4) in full generality, consider a morphism φ' from α' to α which exists by (1). The compositions $NL(\alpha) \to NL(\alpha') \to NL(\alpha)$ and $NL(\alpha') \to NL(\alpha) \to NL(\alpha')$ are homotopic to the identity maps by (3), hence these maps are homotopy equivalences by definition. It follows formally that both maps $NL(\alpha) \to NL(\alpha')$ and $NL(\alpha') \to NL(\alpha)$ are quasi-isomorphisms. Some details omitted. □

08Q1 **Lemma 134.3.** *Let $A \to B$ be a polynomial algebra. Then $NL_{B/A}$ is homotopy equivalent to the chain complex $(0 \to \Omega_{B/A})$ with $\Omega_{B/A}$ in degree 0.*

Proof. Follows from Lemma 134.2 and the fact that $\mathrm{id}_B : B \to B$ is a presentation of B over A with zero kernel. □

The following lemma is part of the motivation for introducing the naive cotangent complex. The cotangent complex extends this to a genuine long exact cohomology sequence. If $B \to C$ is a local complete intersection, then one can extend the sequence with a zero on the left, see More on Algebra, Lemma 33.6.

00S2 **Lemma 134.4** (Jacobi-Zariski sequence)**.** *Let $A \to B \to C$ be ring maps. Choose a presentation $\alpha : A[x_s, s \in S] \to B$ with kernel I. Choose a presentation $\beta : B[y_t, t \in T] \to C$ with kernel J. Let $\gamma : A[x_s, y_t] \to C$ be the induced presentation of C with kernel K. Then we get a canonical commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & \Omega_{A[x_s]/A} \otimes C \arrow[r] & \Omega_{A[x_s,y_t]/A} \otimes C \arrow[r] & \Omega_{B[y_t]/B} \otimes C \arrow[r] & 0 \\ & I/I^2 \otimes C \arrow[r] \arrow[u] & K/K^2 \arrow[r] \arrow[u] & J/J^2 \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

with exact rows. We get the following exact sequence of homology groups

$$H_1(NL_{B/A} \otimes_B C) \to H_1(L_{C/A}) \to H_1(L_{C/B}) \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

of C-modules extending the sequence of Lemma 131.7. If $Tor_1^B(\Omega_{B/A}, C) = 0$, then $H_1(NL_{B/A} \otimes_B C) = H_1(L_{B/A}) \otimes_B C$.

Proof. The precise definition of the maps is omitted. The exactness of the top row follows as the $\mathrm{d}x_s$, $\mathrm{d}y_t$ form a basis for the middle module. The map γ factors

$$A[x_s, y_t] \to B[y_t] \to C$$

with surjective first arrow and second arrow equal to β. Thus we see that $K \to J$ is surjective. Moreover, the kernel of the first displayed arrow is $IA[x_s, y_t]$. Hence $I/I^2 \otimes C$ surjects onto the kernel of $K/K^2 \to J/J^2$. Finally, we can use Lemma 134.2 to identify the terms as homology groups of the naive cotangent complexes. The final assertion follows as the degree 0 term of the complex $NL_{B/A}$ is a free B-module. $\square$

07VC **Remark 134.5.** Let $A \to B$ and $\phi : B \to C$ be ring maps. Then the composition $NL_{B/A} \to NL_{C/A} \to NL_{C/B}$ is homotopy equivalent to zero. Namely, this composition is the functoriality of the naive cotangent complex for the square

$$\begin{tikzcd} B \arrow[r, "\phi"'] & C \\ A \arrow[r] \arrow[u] & B \arrow[u] \end{tikzcd}$$

Write $J = \mathrm{Ker}(B[C] \to C)$. An explicit homotopy is given by the map $\Omega_{A[B]/A} \otimes_A B \to J/J^2$ which maps the basis element $\mathrm{d}[b]$ to the class of $[\phi(b)] - b$ in J/J^2.

07BP **Lemma 134.6.** *Let $A \to B$ be a surjective ring map with kernel I. Then $NL_{B/A}$ is homotopy equivalent to the chain complex $(I/I^2 \to 0)$ with I/I^2 in degree 1. In particular $H_1(L_{B/A}) = I/I^2$.*

Proof. Follows from Lemma 134.2 and the fact that $A \to B$ is a presentation of B over A. $\square$

065V **Lemma 134.7.** *Let $A \to B \to C$ be ring maps. Assume $A \to C$ is surjective (so also $B \to C$ is). Denote $I = \mathrm{Ker}(A \to C)$ and $J = \mathrm{Ker}(B \to C)$. Then the sequence*

$$I/I^2 \to J/J^2 \to \Omega_{B/A} \otimes_B B/J \to 0$$

is exact.

Proof. Follows from Lemma 134.4 and the description of the naive cotangent complexes $NL_{C/B}$ and $NL_{C/A}$ in Lemma 134.6. $\square$

00S4 **Lemma 134.8** (Flat base change)**.** *Let $R \to S$ be a ring map. Let $\alpha : P \to S$ be a presentation. Let $R \to R'$ be a flat ring map. Let $\alpha' : P \otimes_R R' \to S' = S \otimes_R R'$ be the induced presentation. Then $NL(\alpha) \otimes_R R' = NL(\alpha) \otimes_S S' = NL(\alpha')$. In particular, the canonical map*

$$NL_{S/R} \otimes_S S' \longrightarrow NL_{S \otimes_R R'/R'}$$

is a homotopy equivalence if $R \to R'$ is flat.

Proof. This is true because $\mathrm{Ker}(\alpha') = R' \otimes_R \mathrm{Ker}(\alpha)$ since $R \to R'$ is flat. $\square$

07BQ **Lemma 134.9.** *Let $R_i \to S_i$ be a system of ring maps over the directed set I. Set $R = \mathrm{colim}\, R_i$ and $S = \mathrm{colim}\, S_i$. Then $NL_{S/R} = \mathrm{colim}\, NL_{S_i/R_i}$.*

Proof. Recall that $NL_{S/R}$ is the complex $I/I^2 \to \bigoplus_{s \in S} S\mathrm{d}[s]$ where $I \subset R[S]$ is the kernel of the canonical presentation $R[S] \to S$. Now it is clear that $R[S] = \mathrm{colim}\, R_i[S_i]$ and similarly that $I = \mathrm{colim}\, I_i$ where $I_i = \mathrm{Ker}(R_i[S_i] \to S_i)$. Hence the lemma is clear. $\square$

07BR **Lemma 134.10.** *If $S \subset A$ is a multiplicative subset of A, then $NL_{S^{-1}A/A}$ is homotopy equivalent to the zero complex.*

Proof. Since $A \to S^{-1}A$ is flat we see that $NL_{S^{-1}A/A} \otimes_A S^{-1}A \to NL_{S^{-1}A/S^{-1}A}$ is a homotopy equivalence by flat base change (Lemma 134.8). Since the source of the arrow is isomorphic to $NL_{S^{-1}A/A}$ and the target of the arrow is zero (by Lemma 134.6) we win. $\square$

07BS **Lemma 134.11.** *Let $S \subset A$ is a multiplicative subset of A. Let $S^{-1}A \to B$ be a ring map. Then $NL_{B/A} \to NL_{B/S^{-1}A}$ is a homotopy equivalence.*

Proof. Choose a presentation $\alpha : P \to B$ of B over A. Then $\beta : S^{-1}P \to B$ is a presentation of B over $S^{-1}A$. A direct computation shows that we have $NL(\alpha) = NL(\beta)$ which proves the lemma as the naive cotangent complex is well defined up to homotopy by Lemma 134.2. $\square$

08JZ **Lemma 134.12.** *Let $A \to B$ be a ring map. Let $g \in B$. Suppose $\alpha : P \to B$ is a presentation with kernel I. Then a presentation of B_g over A is the map*

$$\beta : P[x] \longrightarrow B_g$$

extending α and sending x to $1/g$. The kernel J of β is generated by I and the element $fx - 1$ where $f \in P$ is an element mapped to $g \in B$ by α. In this situation we have

1. $J/J^2 = (I/I^2)_g \oplus B_g(fx - 1)$,
2. $\Omega_{P[x]/A} \otimes_{P[x]} B_g = \Omega_{P/A} \otimes_P B_g \oplus B_g dx$,
3. $NL(\beta) \cong NL(\alpha) \otimes_B B_g \oplus (B_g \xrightarrow{g} B_g)$

Hence the canonical map $NL_{B/A} \otimes_B B_g \to NL_{B_g/A}$ is a homotopy equivalence.

Proof. Since $P[x]/(I, fx-1) = B[x]/(gx-1) = B_g$ we get the statement about I and $fx-1$ generating J. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega_{P/A} \otimes B_g \arrow[r] & \Omega_{P[x]/A} \otimes B_g \arrow[r] & \Omega_{B[x]/B} \otimes B_g \arrow[r] & 0 \\
& (I/I^2)_g \arrow[r] \arrow[u] & J/J^2 \arrow[r] \arrow[u] & (gx-1)/(gx-1)^2 \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

with exact rows of Lemma 134.4. The B_g-module $\Omega_{B[x]/B} \otimes B_g$ is free of rank 1 on $\mathrm{d}x$. The element $\mathrm{d}x$ in the B_g-module $\Omega_{P[x]/A} \otimes B_g$ provides a splitting for the top row. The element $gx - 1 \in (gx-1)/(gx-1)^2$ is mapped to $g\mathrm{d}x$ in $\Omega_{B[x]/B} \otimes B_g$ and hence $(gx-1)/(gx-1)^2$ is free of rank 1 over B_g. (This can also be seen by arguing that $gx - 1$ is a nonzerodivisor in $B[x]$ because it is a polynomial with invertible constant term and any nonzerodivisor gives a quasi-regular sequence of length 1 by Lemma 69.2.)

Let us prove $(I/I^2)_g \to J/J^2$ injective. Consider the P-algebra map

$$\pi : P[x] \to (P/I^2)_f = P_f/I_f^2$$

sending x to $1/f$. Since J is generated by I and $fx-1$ we see that $\pi(J) \subset (I/I^2)_f = (I/I^2)_g$. Since this is an ideal of square zero we see that $\pi(J^2) = 0$. If $a \in I$ maps to an element of J^2 in J, then $\pi(a) = 0$, which implies that a maps to zero in I_f/I_f^2. This proves the desired injectivity.

Thus we have a short exact sequence of two term complexes

$$0 \to NL(\alpha) \otimes_B B_g \to NL(\beta) \to (B_g \xrightarrow{g} B_g) \to 0$$

Such a short exact sequence can always be split in the category of complexes. In our particular case we can take as splittings

$$J/J^2 = (I/I^2)_g \oplus B_g(fx-1) \quad \text{and} \quad \Omega_{P[x]/A} \otimes B_g = \Omega_{P/A} \otimes B_g \oplus B_g(g^{-2}\mathrm{d}f + \mathrm{d}x)$$

This works because $\mathrm{d}(fx-1) = x\mathrm{d}f + f\mathrm{d}x = g(g^{-2}\mathrm{d}f + \mathrm{d}x)$ in $\Omega_{P[x]/A} \otimes B_g$. □

00S7 **Lemma 134.13.** *Let $A \to B$ be a ring map. Let $S \subset B$ be a multiplicative subset. The canonical map $NL_{B/A} \otimes_B S^{-1}B \to NL_{S^{-1}B/A}$ is a quasi-isomorphism.*

Proof. We have $S^{-1}B = \mathrm{colim}_{g \in S} B_g$ where we think of S as a directed set (ordering by divisibility), see Lemma 9.9. By Lemma 134.12 each of the maps $NL_{B/A} \otimes_B B_g \to NL_{B_g/A}$ are quasi-isomorphisms. The lemma follows from Lemma 134.9. □

00S3 **Lemma 134.14.** *Let R be a ring. Let $A_1 \to A_0$, and $B_1 \to B_0$ be two term complexes. Suppose that there exist morphisms of complexes $\varphi : A_\bullet \to B_\bullet$ and $\psi : B_\bullet \to A_\bullet$ such that $\varphi \circ \psi$ and $\psi \circ \varphi$ are homotopic to the identity maps. Then $A_1 \oplus B_0 \cong B_1 \oplus A_0$ as R-modules.*

Proof. Choose a map $h : A_0 \to A_1$ such that

$$\mathrm{id}_{A_1} - \psi_1 \circ \varphi_1 = h \circ d_A \text{ and } \mathrm{id}_{A_0} - \psi_0 \circ \varphi_0 = d_A \circ h.$$

Similarly, choose a map $h' : B_0 \to B_1$ such that

$$\mathrm{id}_{B_1} - \varphi_1 \circ \psi_1 = h' \circ d_B \text{ and } \mathrm{id}_{B_0} - \varphi_0 \circ \psi_0 = d_B \circ h'.$$

A trivial computation shows that

$$\begin{pmatrix} \mathrm{id}_{A_1} & -h' \circ \psi_1 + h \circ \psi_0 \\ 0 & \mathrm{id}_{B_0} \end{pmatrix} = \begin{pmatrix} \psi_1 & h \\ -d_B & \varphi_0 \end{pmatrix} \begin{pmatrix} \varphi_1 & -h' \\ d_A & \psi_0 \end{pmatrix}$$

This shows that both matrices on the right hand side are invertible and proves the lemma. □

00S5 **Lemma 134.15.** *Let $R \to S$ be a ring map of finite type. For any presentations $\alpha : R[x_1, \ldots, x_n] \to S$, and $\beta : R[y_1, \ldots, y_m] \to S$ we have*

$$I/I^2 \oplus S^{\oplus m} \cong J/J^2 \oplus S^{\oplus n}$$

as S-modules where $I = \mathrm{Ker}(\alpha)$ and $J = \mathrm{Ker}(\beta)$.

Proof. See Lemmas 134.2 and 134.14. □

00S6 **Lemma 134.16.** *Let $R \to S$ be a ring map of finite type. Let $g \in S$. For any presentations $\alpha : R[x_1, \ldots, x_n] \to S$, and $\beta : R[y_1, \ldots, y_m] \to S_g$ we have*

$$(I/I^2)_g \oplus S_g^{\oplus m} \cong J/J^2 \oplus S_g^{\oplus n}$$

as S_g-modules where $I = \mathrm{Ker}(\alpha)$ and $J = \mathrm{Ker}(\beta)$.

Proof. By Lemma 134.15, we see that it suffices to prove this for a single choice of α and β. Thus we may take β the presentation of Lemma 134.12 and the result is clear. □

135. Local complete intersections

00S8 The property of being a local complete intersection is an intrinsic property of a Noetherian local ring. This will be discussed in Divided Power Algebra, Section 8. However, for the moment we just define this property for finite type algebras over a field.

00S9 **Definition 135.1.** Let k be a field. Let S be a finite type k-algebra.

(1) We say that S is a *global complete intersection over k* if there exists a presentation $S = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ such that $\dim(S) = n - c$.
(2) We say that S is a *local complete intersection over k* if there exists a covering $\mathrm{Spec}(S) = \bigcup D(g_i)$ such that each of the rings S_{g_i} is a global complete intersection over k.

We will also use the convention that the zero ring is a global complete intersection over k.

Suppose S is a global complete intersection $S = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ as in Definition 135.1. For a maximal ideal $\mathfrak{m} \subset k[x_1, \ldots, x_n]$ we have $\dim(k[x_1, \ldots, x_n]_\mathfrak{m}) = n$ (Lemma 114.1). If $(f_1, \ldots, f_c) \subset \mathfrak{m}$, then we conclude that $\dim(S_\mathfrak{m}) \geq n - c$ by Lemma 60.13. Since $\dim(S) = n - c$ by Definition 135.1 we conclude that $\dim(S_\mathfrak{m}) = n - c$ for all maximal ideals of S and that $\mathrm{Spec}(S)$ is equidimensional (Topology, Definition 10.5) of dimension $n - c$, see Lemma 114.5. We will often use this without further mention.

00SA **Lemma 135.2.** *Let k be a field. Let S be a finite type k-algebra. Let $g \in S$.*

(1) *If S is a global complete intersection so is S_g.*
(2) *If S is a local complete intersection so is S_g.*

Proof. The second statement follows immediately from the first. Proof of the first statement. If S_g is the zero ring, then it is true. Assume S_g is nonzero. Write $S = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ with $n - c = \dim(S)$ as in Definition 135.1. By the remarks following the definition S is equidimensional of dimension $n - c$, so $\dim(S_g) = n - c$ as well. Let $g' \in k[x_1, \ldots, x_n]$ be an element whose residue class corresponds to g. Then $S_g = k[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, x_{n+1}g' - 1)$ as desired. □

00SB **Lemma 135.3.** *Let k be a field. Let S be a finite type k-algebra. If S is a local complete intersection, then S is a Cohen-Macaulay ring.*

Proof. Choose a maximal prime $\mathfrak{m}$ of S. We have to show that $S_\mathfrak{m}$ is Cohen-Macaulay. By assumption we may assume $S = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ with $\dim(S) = n - c$. Let $\mathfrak{m}' \subset k[x_1, \ldots, x_n]$ be the maximal ideal corresponding to $\mathfrak{m}$. According to Proposition 114.2 the local ring $k[x_1, \ldots, x_n]_{\mathfrak{m}'}$ is regular local of dimension n. In particular it is Cohen-Macaulay by Lemma 106.3. By Lemma 60.13 applied c times the local ring $S_\mathfrak{m} = k[x_1, \ldots, x_n]_{\mathfrak{m}'}/(f_1, \ldots, f_c)$ has dimension $\geq n - c$. By assumption $\dim(S_\mathfrak{m}) \leq n - c$. Thus we get equality. This implies that $f_1, \ldots, f_c$ is a regular sequence in $k[x_1, \ldots, x_n]_{\mathfrak{m}'}$ and that $S_\mathfrak{m}$ is Cohen-Macaulay, see Proposition 103.4. □

The following is the technical key to the rest of the material in this section. An important feature of this lemma is that we may choose any presentation for the ring S, but that condition (1) does not depend on this choice.

00SC **Lemma 135.4.** *Let k be a field. Let S be a finite type k-algebra. Let $\mathfrak{q}$ be a prime of S. Choose any presentation $S = k[x_1, \ldots, x_n]/I$. Let $\mathfrak{q}'$ be the prime of $k[x_1, \ldots, x_n]$ corresponding to $\mathfrak{q}$. Set $c = height(\mathfrak{q}') - height(\mathfrak{q})$, in other words $\dim_\mathfrak{q}(S) = n - c$ (see Lemma 116.4). The following are equivalent*

1. *There exists a $g \in S$, $g \notin \mathfrak{q}$ such that S_g is a global complete intersection over k.*
2. *The ideal $I_{\mathfrak{q}'} \subset k[x_1, \ldots, x_n]_{\mathfrak{q}'}$ can be generated by c elements.*
3. *The conormal module $(I/I^2)_\mathfrak{q}$ can be generated by c elements over $S_\mathfrak{q}$.*
4. *The conormal module $(I/I^2)_\mathfrak{q}$ is a free $S_\mathfrak{q}$-module of rank c.*
5. *The ideal $I_{\mathfrak{q}'}$ can be generated by a regular sequence in the regular local ring $k[x_1, \ldots, x_n]_{\mathfrak{q}'}$.*

In this case any c elements of $I_{\mathfrak{q}'}$ which generate $I_{\mathfrak{q}'}/\mathfrak{q}' I_{\mathfrak{q}'}$ form a regular sequence in the local ring $k[x_1, \ldots, x_n]_{\mathfrak{q}'}$.

Proof. Set $R = k[x_1, \ldots, x_n]_{\mathfrak{q}'}$. This is a Cohen-Macaulay local ring of dimension height($\mathfrak{q}'$), see for example Lemma 135.3. Moreover, $\overline{R} = R/IR = R/I_{\mathfrak{q}'} = S_\mathfrak{q}$ is a quotient of dimension height($\mathfrak{q}$). Let $f_1, \ldots, f_c \in I_{\mathfrak{q}'}$ be elements which generate $(I/I^2)_\mathfrak{q}$. By Lemma 20.1 we see that $f_1, \ldots, f_c$ generate $I_{\mathfrak{q}'}$. Since the dimensions work out, we conclude by Proposition 103.4 that $f_1, \ldots, f_c$ is a regular sequence in R. By Lemma 69.2 we see that $(I/I^2)_\mathfrak{q}$ is free. These arguments show that (2), (3), (4) are equivalent and that they imply the last statement of the lemma, and therefore they imply (5).

If (5) holds, say $I_{\mathfrak{q}'}$ is generated by a regular sequence of length e, then height($\mathfrak{q}$) = $\dim(S_\mathfrak{q}) = \dim(k[x_1, \ldots, x_n]_{\mathfrak{q}'}) - e$ = height($\mathfrak{q}'$)$ - e$ by dimension theory, see Section 60. We conclude that $e = c$. Thus (5) implies (2).

We continue with the notation introduced in the first paragraph. For each f_i we may find $d_i \in k[x_1, \ldots, x_n]$, $d_i \not\in \mathfrak q'$ such that $f'_i = d_i f_i \in k[x_1, \ldots, x_n]$. Then it is still true that $I_{\mathfrak q'} = (f'_1, \ldots, f'_c)R$. Hence there exists a $g' \in k[x_1, \ldots, x_n]$, $g' \not\in \mathfrak q'$ such that $I_{g'} = (f'_1, \ldots, f'_c)$. Moreover, pick $g'' \in k[x_1, \ldots, x_n]$, $g'' \not\in \mathfrak q'$ such that $\dim(S_{g''}) = \dim_{\mathfrak q} \mathrm{Spec}(S)$. By Lemma 116.4 this dimension is equal to $n - c$. Finally, set g equal to the image of $g'g''$ in S. Then we see that

$$S_g \cong k[x_1, \ldots, x_n, x_{n+1}]/(f'_1, \ldots, f'_c, x_{n+1}g'g'' - 1)$$

and by our choice of g'' this ring has dimension $n - c$. Therefore it is a global complete intersection. Thus each of (2), (3), and (4) implies (1).

Assume (1). Let $S_g \cong k[y_1, \ldots, y_m]/(f_1, \ldots, f_t)$ be a presentation of S_g as a global complete intersection. Write $J = (f_1, \ldots, f_t)$. Let $\mathfrak q'' \subset k[y_1, \ldots, y_m]$ be the prime corresponding to $\mathfrak q S_g$. Note that $t = m - \dim(S_g) = \text{height}(\mathfrak q'') - \text{height}(\mathfrak q)$, see Lemma 116.4 for the last equality. As seen in the proof of Lemma 135.3 (and also above) the elements $f_1, \ldots, f_t$ form a regular sequence in the local ring $k[y_1, \ldots, y_m]_{\mathfrak q''}$. By Lemma 69.2 we see that $(J/J^2)_{\mathfrak q}$ is free of rank t. By Lemma 134.16 we have

$$J/J^2 \oplus S_g^n \cong (I/I^2)_g \oplus S_g^m$$

Thus $(I/I^2)_{\mathfrak q}$ is free of rank $t + n - m = m - \dim(S_g) + n - m = n - \dim(S_g) = \text{height}(\mathfrak q') - \text{height}(\mathfrak q) = c$. Thus we obtain (4). $\square$

The result of Lemma 135.4 suggests the following definition.

00SD **Definition 135.5.** Let k be a field. Let S be a local k-algebra essentially of finite type over k. We say S is a *complete intersection (over k)* if there exists a local k-algebra R and elements $f_1, \ldots, f_c \in \mathfrak m_R$ such that

(1) R is essentially of finite type over k,
(2) R is a regular local ring,
(3) $f_1, \ldots, f_c$ form a regular sequence in R, and
(4) $S \cong R/(f_1, \ldots, f_c)$ as k-algebras.

By the Cohen structure theorem (see Theorem 160.8) any complete Noetherian local ring may be written as the quotient of some regular complete local ring. Hence we may use the definition above to define the notion of a complete intersection ring for any complete Noetherian local ring. We will discuss this in Divided Power Algebra, Section 8. In the meantime the following lemma shows that such a definition makes sense.

00SE **Lemma 135.6.** *Let $A \to B \to C$ be surjective local ring homomorphisms. Assume A and B are regular local rings. The following are equivalent*

(1) $\mathrm{Ker}(A \to C)$ *is generated by a regular sequence,*
(2) $\mathrm{Ker}(A \to C)$ *is generated by* $\dim(A) - \dim(C)$ *elements,*
(3) $\mathrm{Ker}(B \to C)$ *is generated by a regular sequence, and*
(4) $\mathrm{Ker}(B \to C)$ *is generated by* $\dim(B) - \dim(C)$ *elements.*

Proof. A regular local ring is Cohen-Macaulay, see Lemma 106.3. Hence the equivalences (1) $\Leftrightarrow$ (2) and (3) $\Leftrightarrow$ (4), see Proposition 103.4. By Lemma 106.4 the ideal $\mathrm{Ker}(A \to B)$ can be generated by $\dim(A) - \dim(B)$ elements. Hence we see that (4) implies (2).

It remains to show that (1) implies (4). We do this by induction on $\dim(A) - \dim(B)$. The case $\dim(A) - \dim(B) = 0$ is trivial. Assume $\dim(A) > \dim(B)$. Write $I = \mathrm{Ker}(A \to C)$ and $J = \mathrm{Ker}(A \to B)$. Note that $J \subset I$. Our assumption is that the minimal number of generators of I is $\dim(A) - \dim(C)$. Let $\mathfrak m \subset A$ be the maximal ideal. Consider the maps

$$J/\mathfrak m J \to I/\mathfrak m I \to \mathfrak m/\mathfrak m^2$$

By Lemma 106.4 and its proof the composition is injective. Take any element $x \in J$ which is not zero in $J/\mathfrak m J$. By the above and Nakayama's lemma x is an element of a minimal set of generators of I. Hence we may replace A by A/xA and I by I/xA which decreases both $\dim(A)$ and the minimal number of generators of I by 1. Thus we win. □

00SF **Lemma 135.7.** *Let k be a field. Let S be a local k-algebra essentially of finite type over k. The following are equivalent:*

1. *S is a complete intersection over k,*
2. *for any surjection $R \to S$ with R a regular local ring essentially of finite presentation over k the ideal $\mathrm{Ker}(R \to S)$ can be generated by a regular sequence,*
3. *for some surjection $R \to S$ with R a regular local ring essentially of finite presentation over k the ideal $\mathrm{Ker}(R \to S)$ can be generated by $\dim(R) - \dim(S)$ elements,*
4. *there exists a global complete intersection A over k and a prime $\mathfrak a$ of A such that $S \cong A_{\mathfrak a}$, and*
5. *there exists a local complete intersection A over k and a prime $\mathfrak a$ of A such that $S \cong A_{\mathfrak a}$.*

Proof. It is clear that (2) implies (1) and (1) implies (3). It is also clear that (4) implies (5). Let us show that (3) implies (4). Thus we assume there exists a surjection $R \to S$ with R a regular local ring essentially of finite presentation over k such that the ideal $\mathrm{Ker}(R \to S)$ can be generated by $\dim(R) - \dim(S)$ elements. We may write $R = (k[x_1, \ldots, x_n]/J)_{\mathfrak q}$ for some $J \subset k[x_1, \ldots, x_n]$ and some prime $\mathfrak q \subset k[x_1, \ldots, x_n]$ with $J \subset \mathfrak q$. Let $I \subset k[x_1, \ldots, x_n]$ be the kernel of the map $k[x_1, \ldots, x_n] \to S$ so that $S \cong (k[x_1, \ldots, x_n]/I)_{\mathfrak q}$. By assumption $(I/J)_{\mathfrak q}$ is generated by $\dim(R) - \dim(S)$ elements. We conclude that $I_{\mathfrak q}$ can be generated by $\dim(k[x_1, \ldots, x_n]_{\mathfrak q}) - \dim(S)$ elements by Lemma 135.6. From Lemma 135.4 we see that for some $g \in k[x_1, \ldots, x_n]$, $g \notin \mathfrak q$ the algebra $(k[x_1, \ldots, x_n]/I)_g$ is a global complete intersection and S is isomorphic to a local ring of it.

To finish the proof of the lemma we have to show that (5) implies (2). Assume (5) and let $\pi : R \to S$ be a surjection with R a regular local k-algebra essentially of finite type over k. By assumption we have $S = A_{\mathfrak a}$ for some local complete intersection A over k. Choose a presentation $R = (k[y_1, \ldots, y_m]/J)_{\mathfrak q}$ with $J \subset \mathfrak q \subset k[y_1, \ldots, y_m]$. We may and do assume that J is the kernel of the map $k[y_1, \ldots, y_m] \to R$. Let $I \subset k[y_1, \ldots, y_m]$ be the kernel of the map $k[y_1, \ldots, y_m] \to S = A_{\mathfrak a}$. Then $J \subset I$ and $(I/J)_{\mathfrak q}$ is the kernel of the surjection $\pi : R \to S$. So $S = (k[y_1, \ldots, y_m]/I)_{\mathfrak q}$.

By Lemma 126.7 we see that there exist $g \in A$, $g \notin \mathfrak a$ and $g' \in k[y_1, \ldots, y_m]$, $g' \notin \mathfrak q$ such that $A_g \cong (k[y_1, \ldots, y_m]/I)_{g'}$. After replacing A by A_g and $k[y_1, \ldots, y_m]$ by $k[y_1, \ldots, y_{m+1}]$ we may assume that $A \cong k[y_1, \ldots, y_m]/I$. Consider the surjective maps of local rings

$$k[y_1, \ldots, y_m]_{\mathfrak q} \to R \to S.$$

We have to show that the kernel of $R \to S$ is generated by a regular sequence. By Lemma 135.4 we know that $k[y_1, \ldots, y_m]_{\mathfrak q} \to A_{\mathfrak a} = S$ has this property (as A is a local complete intersection over k). We win by Lemma 135.6. $\square$

00SG **Lemma 135.8.** *Let k be a field. Let S be a finite type k-algebra. Let $\mathfrak q$ be a prime of S. The following are equivalent:*

1. *The local ring $S_{\mathfrak q}$ is a complete intersection ring (Definition 135.5).*
2. *There exists a $g \in S$, $g \notin \mathfrak q$ such that S_g is a local complete intersection over k.*
3. *There exists a $g \in S$, $g \notin \mathfrak q$ such that S_g is a global complete intersection over k.*
4. *For any presentation $S = k[x_1, \ldots, x_n]/I$ with $\mathfrak q' \subset k[x_1, \ldots, x_n]$ corresponding to $\mathfrak q$ any of the equivalent conditions (1) – (5) of Lemma 135.4 hold.*

Proof. This is a combination of Lemmas 135.4 and 135.7 and the definitions. $\square$

00SH **Lemma 135.9.** *Let k be a field. Let S be a finite type k-algebra. The following are equivalent:*

1. *The ring S is a local complete intersection over k.*
2. *All local rings of S are complete intersection rings over k.*
3. *All localizations of S at maximal ideals are complete intersection rings over k.*

Proof. This follows from Lemma 135.8, the fact that $\mathrm{Spec}(S)$ is quasi-compact and the definitions. $\square$

The following lemma says that being a complete intersection is preserved under change of base field (in a strong sense).

00SI **Lemma 135.10.** *Let K/k be a field extension. Let S be a finite type algebra over k. Let $\mathfrak q_K$ be a prime of $S_K = K \otimes_k S$ and let $\mathfrak q$ be the corresponding prime of S. Then $S_{\mathfrak q}$ is a complete intersection over k (Definition 135.5) if and only if $(S_K)_{\mathfrak q_K}$ is a complete intersection over K.*

Proof. Choose a presentation $S = k[x_1, \ldots, x_n]/I$. This gives a presentation $S_K = K[x_1, \ldots, x_n]/I_K$ where $I_K = K \otimes_k I$. Let $\mathfrak q'_K \subset K[x_1, \ldots, x_n]$, resp. $\mathfrak q' \subset k[x_1, \ldots, x_n]$ be the corresponding prime. We will show that the equivalent conditions of Lemma 135.4 hold for the pair $(S = k[x_1, \ldots, x_n]/I, \mathfrak q)$ if and only if they hold for the pair $(S_K = K[x_1, \ldots, x_n]/I_K, \mathfrak q_K)$. The lemma will follow from this (see Lemma 135.8).

By Lemma 116.6 we have $\dim_{\mathfrak q} S = \dim_{\mathfrak q_K} S_K$. Hence the integer c occurring in Lemma 135.4 is the same for the pair $(S = k[x_1, \ldots, x_n]/I, \mathfrak q)$ as for the pair $(S_K = K[x_1, \ldots, x_n]/I_K, \mathfrak q_K)$. On the other hand we have

$$\begin{aligned} I \otimes_{k[x_1,\ldots,x_n]} \kappa(\mathfrak q') \otimes_{\kappa(\mathfrak q')} \kappa(\mathfrak q'_K) &= I \otimes_{k[x_1,\ldots,x_n]} \kappa(\mathfrak q'_K) \\ &= I \otimes_{k[x_1,\ldots,x_n]} K[x_1, \ldots, x_n] \otimes_{K[x_1,\ldots,x_n]} \kappa(\mathfrak q'_K) \\ &= (K \otimes_k I) \otimes_{K[x_1,\ldots,x_n]} \kappa(\mathfrak q'_K) \\ &= I_K \otimes_{K[x_1,\ldots,x_n]} \kappa(\mathfrak q'_K). \end{aligned}$$

Therefore, $\dim_{\kappa(\mathfrak q')} I \otimes_{k[x_1,\ldots,x_n]} \kappa(\mathfrak q') = \dim_{\kappa(\mathfrak q'_K)} I_K \otimes_{K[x_1,\ldots,x_n]} \kappa(\mathfrak q'_K)$. Thus it follows from Nakayama's Lemma 20.1 that the minimal number of generators of $I_{\mathfrak q'}$ is the same as the minimal number of generators of $(I_K)_{\mathfrak q'_K}$. Thus the lemma follows from characterization (2) of Lemma 135.4. $\square$

00SJ **Lemma 135.11.** *Let $k \to K$ be a field extension. Let S be a finite type k-algebra. Then S is a local complete intersection over k if and only if $S \otimes_k K$ is a local complete intersection over K.*

Proof. This follows from a combination of Lemmas 135.9 and 135.10. But we also give a different proof here (based on the same principles).

Set $S' = S \otimes_k K$. Let $\alpha : k[x_1, \ldots, x_n] \to S$ be a presentation with kernel I. Let $\alpha' : K[x_1, \ldots, x_n] \to S'$ be the induced presentation with kernel I'.

Suppose that S is a local complete intersection. Pick a prime $\mathfrak q \subset S'$. Denote $\mathfrak q'$ the corresponding prime of $K[x_1, \ldots, x_n]$, $\mathfrak p$ the corresponding prime of S, and $\mathfrak p'$ the corresponding prime of $k[x_1, \ldots, x_n]$. Consider the following diagram of Noetherian local rings

$$\begin{tikzcd} S'_{\mathfrak q} & K[x_1, \ldots, x_n]_{\mathfrak q'} \arrow[l] \\ S_{\mathfrak p} \arrow[u] & k[x_1, \ldots, x_n]_{\mathfrak p'} \arrow[l] \arrow[u] \end{tikzcd}$$

By Lemma 135.4 we know that $S_{\mathfrak p}$ is cut out by some regular sequence $f_1, \ldots, f_c$ in $k[x_1, \ldots, x_n]_{\mathfrak p'}$. Since the right vertical arrow is flat we see that the images of $f_1, \ldots, f_c$ form a regular sequence in $K[x_1, \ldots, x_n]_{\mathfrak q'}$. Because tensoring with K over k is an exact functor we have $S'_{\mathfrak q} = K[x_1, \ldots, x_n]_{\mathfrak q'}/(f_1, \ldots, f_c)$. Hence by Lemma 135.4 again we see that S' is a local complete intersection in a neighbourhood of $\mathfrak q$. Since $\mathfrak q$ was arbitrary we see that S' is a local complete intersection over K.

Suppose that S' is a local complete intersection. Pick a maximal ideal $\mathfrak m$ of S. Let $\mathfrak m'$ denote the corresponding maximal ideal of $k[x_1, \ldots, x_n]$. Denote $\kappa = \kappa(\mathfrak m)$ the residue field. By Remark 17.8 the primes of S' lying over $\mathfrak m$ correspond to primes in $K \otimes_k \kappa$. By the Hilbert-Nullstellensatz Theorem 34.1 we have $[\kappa : k] < \infty$. Hence $K \otimes_k \kappa$ is finite nonzero over K. Hence $K \otimes_k \kappa$ has a finite number > 0 of primes which are all maximal, each of which has a residue field finite over K (see Section 53). Hence there are finitely many > 0 prime ideals $\mathfrak n \subset S'$ lying over $\mathfrak m$, each of which is maximal and has a residue field which is finite over K. Pick one, say $\mathfrak n \subset S'$, and let $\mathfrak n' \subset K[x_1, \ldots, x_n]$ denote the corresponding prime ideal of $K[x_1, \ldots, x_n]$. Note that since $V(\mathfrak m S')$ is finite, we see that $\mathfrak n$ is an isolated closed point of it, and we deduce that $\mathfrak m S'_{\mathfrak n}$ is an ideal of definition of $S'_{\mathfrak n}$. This implies that $\dim(S_{\mathfrak m}) = \dim(S'_{\mathfrak n})$ for example by Lemma 112.7. (This can also be seen using Lemma 116.6.) Consider the corresponding diagram of Noetherian local rings

$$\begin{tikzcd} S'_{\mathfrak n} & K[x_1, \ldots, x_n]_{\mathfrak n'} \arrow[l] \\ S_{\mathfrak m} \arrow[u] & k[x_1, \ldots, x_n]_{\mathfrak m'} \arrow[l] \arrow[u] \end{tikzcd}$$

According to Lemma 134.8 we have $NL(\alpha) \otimes_S S' = NL(\alpha')$, in particular $I'/(I')^2 = I/I^2 \otimes_S S'$. Thus $(I/I^2)_{\mathfrak m} \otimes_{S_{\mathfrak m}} \kappa$ and $(I'/(I')^2)_{\mathfrak n} \otimes_{S'_{\mathfrak n}} \kappa(\mathfrak n)$ have the same dimension.

Since $(I'/(I')^2)_{\mathfrak n}$ is free of rank $n - \dim S'_{\mathfrak n}$ we deduce that $(I/I^2)_{\mathfrak m}$ can be generated by $n - \dim S'_{\mathfrak n} = n - \dim S_{\mathfrak m}$ elements. By Lemma 135.4 we see that S is a local complete intersection in a neighbourhood of $\mathfrak m$. Since $\mathfrak m$ was any maximal ideal we conclude that S is a local complete intersection. $\square$

We end with a lemma which we will later use to prove that given ring maps $T \to A \to B$ where B is syntomic over T, and B is syntomic over A, then A is syntomic over T.

02JP **Lemma 135.12.** *Let*

$$\begin{tikzcd} B & S \arrow[l] \\ A \arrow[u] & R \arrow[l] \arrow[u] \end{tikzcd}$$

be a commutative square of local rings. Assume

1. *R and $\overline{S} = S/\mathfrak m_R S$ are regular local rings,*
2. *$A = R/I$ and $B = S/J$ for some ideals I, J,*
3. *$J \subset S$ and $\overline{J} = J/\mathfrak m_R \cap J \subset \overline{S}$ are generated by regular sequences, and*
4. *$A \to B$ and $R \to S$ are flat.*

Then I is generated by a regular sequence.

Proof. Set $\overline{B} = B/\mathfrak m_R B = B/\mathfrak m_A B$ so that $\overline{B} = \overline{S}/\overline{J}$. Let $f_1, \ldots, f_{\overline{c}} \in J$ be elements such that $\overline{f}_1, \ldots, \overline{f}_{\overline{c}} \in \overline{J}$ form a regular sequence generating $\overline{J}$. Note that $\overline{c} = \dim(\overline{S}) - \dim(\overline{B})$, see Lemma 135.6. By Lemma 99.3 the ring $S/(f_1, \ldots, f_{\overline{c}})$ is flat over R. Hence $S/(f_1, \ldots, f_{\overline{c}}) + IS$ is flat over A. The map $S/(f_1, \ldots, f_{\overline{c}}) + IS \to B$ is therefore a surjection of finite S/IS-modules flat over A which is an isomorphism modulo $\mathfrak m_A$, and hence an isomorphism by Lemma 99.1. In other words, $J = (f_1, \ldots, f_{\overline{c}}) + IS$.

By Lemma 135.6 again the ideal J is generated by a regular sequence of $c = \dim(S) - \dim(B)$ elements. Hence $J/\mathfrak m_S J$ is a vector space of dimension c. By the description of J above there exist $g_1, \ldots, g_{c-\overline{c}} \in I$ such that J is generated by $f_1, \ldots, f_{\overline{c}}, g_1, \ldots, g_{c-\overline{c}}$ (use Nakayama's Lemma 20.1). Consider the ring $A' = R/(g_1, \ldots, g_{c-\overline{c}})$ and the surjection $A' \to A$. We see from the above that $B = S/(f_1, \ldots, f_{\overline{c}}, g_1, \ldots, g_{c-\overline{c}})$ is flat over A' (as $S/(f_1, \ldots, f_{\overline{c}})$ is flat over R). Hence $A' \to B$ is injective (as it is faithfully flat, see Lemma 39.17). Since this map factors through A we get $A' = A$. Note that $\dim(B) = \dim(A) + \dim(\overline{B})$, and $\dim(S) = \dim(R) + \dim(\overline{S})$, see Lemma 112.7. Hence $c - \overline{c} = \dim(R) - \dim(A)$ by elementary algebra. Thus $I = (g_1, \ldots, g_{c-\overline{c}})$ is generated by a regular sequence according to Lemma 135.6. $\square$

136. Syntomic morphisms

00SK Syntomic ring maps are flat finitely presented ring maps all of whose fibers are local complete intersections. We discuss general local complete intersection ring maps in More on Algebra, Section 33.

00SL **Definition 136.1.** A ring map $R \to S$ is called *syntomic*, or we say S is a *flat local complete intersection over R* if it is flat, of finite presentation, and if all of its fibre rings $S \otimes_R \kappa(\mathfrak p)$ are local complete intersections, see Definition 135.1.

Clearly, an algebra over a field is syntomic over the field if and only if it is a local complete intersection. Here is a pleasing feature of this definition.

00SM **Lemma 136.2.** *Let $R \to S$ be a ring map. Let $R \to R'$ be a faithfully flat ring map. Set $S' = R' \otimes_R S$. Then $R \to S$ is syntomic if and only if $R' \to S'$ is syntomic.*

Proof. By Lemma 126.2 and Lemma 39.8 this holds for the property of being flat and for the property of being of finite presentation. The map $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is surjective, see Lemma 39.16. Thus it suffices to show given primes $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p} \subset R$ that $S \otimes_R \kappa(\mathfrak{p})$ is a local complete intersection if and only if $S' \otimes_{R'} \kappa(\mathfrak{p}')$ is a local complete intersection. Note that $S' \otimes_{R'} \kappa(\mathfrak{p}') = S \otimes_R \kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}')$. Thus Lemma 135.11 applies. $\square$

00SN **Lemma 136.3.** *Any base change of a syntomic map is syntomic.*

Proof. This is true for being flat, for being of finite presentation, and for having local complete intersections as fibres by Lemmas 39.7, 6.2 and 135.11. $\square$

00SO **Lemma 136.4.** *Let $R \to S$ be a ring map. Suppose we have $g_1, \ldots g_m \in S$ which generate the unit ideal such that each $R \to S_{g_i}$ is syntomic. Then $R \to S$ is syntomic.*

Proof. This is true for being flat and for being of finite presentation by Lemmas 39.18 and 23.3. The property of having fibre rings which are local complete intersections is local on S by its very definition, see Definition 135.1. $\square$

00SP **Definition 136.5.** Let $R \to S$ be a ring map. We say that $R \to S$ is a *relative global complete intersection* if there exists a presentation $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ and every nonempty fibre of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ has dimension $n - c$. We will say "let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ be a relative global complete intersection" to indicate this situation.

The following lemma is occasionally useful to find global presentations.

07CF **Lemma 136.6.** *Let S be a finitely presented R-algebra which has a presentation $S = R[x_1, \ldots, x_n]/I$ such that I/I^2 is free over S. Then S has a presentation $S = R[y_1, \ldots, y_m]/(f_1, \ldots, f_c)$ such that $(f_1, \ldots, f_c)/(f_1, \ldots, f_c)^2$ is free with basis given by the classes of $f_1, \ldots, f_c$.*

Proof. Note that I is a finitely generated ideal by Lemma 6.3. Let $f_1, \ldots, f_c \in I$ be elements which map to a basis of I/I^2. By Nakayama's lemma (Lemma 20.1) there exists a $g \in 1 + I$ such that

$$g \cdot I \subset (f_1, \ldots, f_c)$$

and $I_g \cong (f_1, \ldots, f_c)_g$. Hence we see that

$$S \cong R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)[1/g] \cong R[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, gx_{n+1} - 1)$$

as desired. It follows that $f_1, \ldots, f_c, gx_{n+1} - 1$ form a basis for $(f_1, \ldots, f_c, gx_{n+1} - 1)/(f_1, \ldots, f_c, gx_{n+1} - 1)^2$ for example by applying Lemma 134.12. $\square$

00SQ **Example 136.7.** Let $n, m \geq 1$ be integers. Consider the ring map

$$\begin{array}{rcl} R = \mathbf{Z}[a_1, \ldots, a_{n+m}] & \longrightarrow & S = \mathbf{Z}[b_1, \ldots, b_n, c_1, \ldots, c_m] \\ a_1 & \longmapsto & b_1 + c_1 \\ a_2 & \longmapsto & b_2 + b_1c_1 + c_2 \\ \ldots & \ldots & \ldots \\ a_{n+m} & \longmapsto & b_nc_m \end{array}$$

In other words, this is the unique ring map of polynomial rings as indicated such that the polynomial factorization

$$x^{n+m} + a_1x^{n+m-1} + \ldots + a_{n+m} = (x^n + b_1x^{n-1} + \ldots + b_n)(x^m + c_1x^{m-1} + \ldots + c_m)$$

holds. Note that S is generated by $n+m$ elements over R (namely, b_i, c_j) and that there are $n + m$ equations (namely $a_k = a_k(b_i, c_j)$). In order to show that S is a relative global complete intersection over R it suffices to prove that all fibres have dimension 0.

To prove this, let $R \to k$ be a ring map into a field k. Say a_i maps to $\alpha_i \in k$. Consider the fibre ring $S_k = k \otimes_R S$. Let $k \to K$ be a field extension. A k-algebra map of $S_k \to K$ is the same thing as finding $\beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_m \in K$ such that

$$x^{n+m} + \alpha_1x^{n+m-1} + \ldots + \alpha_{n+m} = (x^n + \beta_1x^{n-1} + \ldots + \beta_n)(x^m + \gamma_1x^{m-1} + \ldots + \gamma_m).$$

Hence we see there are at most finitely many choices of such $n+m$-tuples in K. This proves that all fibres have finitely many closed points (use Hilbert's Nullstellensatz to see they all correspond to solutions in $\overline{k}$ for example) and hence that $R \to S$ is a relative global complete intersection.

Another way to argue this is to show $\mathbf{Z}[a_1, \ldots, a_{n+m}] \to \mathbf{Z}[b_1, \ldots, b_n, c_1, \ldots, c_m]$ is actually also a *finite* ring map. Namely, by Lemma 38.5 each of b_i, c_j is integral over R, and hence $R \to S$ is finite by Lemma 36.4.

00SR **Example 136.8.** Consider the ring map

$$\begin{array}{rcl} R = \mathbf{Z}[a_1, \ldots, a_n] & \longrightarrow & S = \mathbf{Z}[\alpha_1, \ldots, \alpha_n] \\ a_1 & \longmapsto & \alpha_1 + \ldots + \alpha_n \\ \ldots & \ldots & \ldots \\ a_n & \longmapsto & \alpha_1 \ldots \alpha_n \end{array}$$

In other words this is the unique ring map of polynomial rings as indicated such that

$$x^n + a_1x^{n-1} + \ldots + a_n = \prod\nolimits_{i=1}^n (x + \alpha_i)$$

holds in $\mathbf{Z}[\alpha_i, x]$. Another way to say this is that a_i maps to the ith elementary symmetric function in $\alpha_1, \ldots, \alpha_n$. Note that S is generated by n elements over R subject to n equations. Hence to show that S is a relative global complete intersection over R we have to show that the fibre rings $S \otimes_R \kappa(\mathfrak{p})$ have dimension 0. This follows as in Example 136.7 because the ring map $\mathbf{Z}[a_1, \ldots, a_n] \to \mathbf{Z}[\alpha_1, \ldots, \alpha_n]$ is actually *finite* since each $\alpha_i \in S$ satisfies the monic equation $x^n - a_1x^{n-1} + \ldots + (-1)^na_n$ over R.

03HS **Lemma 136.9.** *Suppose that A is a ring, and $P(x) = x^n + b_1x^{n-1} + \ldots + b_n \in A[x]$ is a monic polynomial over A. Then there exists a syntomic, finite locally free,* *faithfully flat ring extension $A \subset A'$ such that $P(x) = \prod_{i=1,\ldots,n}(x - \beta_i)$ for certain $\beta_i \in A'$.*

Proof. Take $A' = A \otimes_R S$, where R and S are as in Example 136.8, where $R \to A$ maps a_i to b_i, and let $\beta_i = -1 \otimes \alpha_i$. $\square$

00SS **Lemma 136.10.** *Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ be a relative global complete intersection (Definition 136.5)*

1. *For any $R \to R'$ the base change $R' \otimes_R S = R'[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a relative global complete intersection.*
2. *For any $g \in S$ which is the image of $h \in R[x_1, \ldots, x_n]$ the ring $S_g = R[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, hx_{n+1} - 1)$ is a relative global complete intersection.*
3. *If $R \to S$ factors as $R \to R_f \to S$ for some $f \in R$. Then the ring $S = R_f[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a relative global complete intersection over R_f.*

Proof. By Lemma 116.5 the fibres of a base change have the same dimension as the fibres of the original map. Moreover $R' \otimes_R R[x_1, \ldots, x_n]/(f_1, \ldots, f_c) = R'[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. Thus (1) follows. The proof of (2) is that the localization at one element can be described as $S_g \cong S[x_{n+1}]/(gx_{n+1} - 1)$. Assertion (3) follows from (1) since under the assumptions of (3) we have $R_f \otimes_R S \cong S$. $\square$

00ST **Lemma 136.11.** *Let R be a ring. Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. We will find $h \in R[x_1, \ldots, x_n]$ which maps to $g \in S$ such that*

$$S_g = R[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, hx_{n+1} - 1)$$

is a relative global complete intersection with a presentation as in Definition 136.5 in each of the following cases:

1. *Let $I \subset R$ be an ideal. If the fibres of $\mathrm{Spec}(S/IS) \to \mathrm{Spec}(R/I)$ have dimension $n - c$, then we can find (h, g) as above such that g maps to $1 \in S/IS$.*
2. *Let $\mathfrak{p} \subset R$ be a prime. If $\dim(S \otimes_R \kappa(\mathfrak{p})) = n - c$, then we can find (h, g) as above such that g maps to a unit of $S \otimes_R \kappa(\mathfrak{p})$.*
3. *Let $\mathfrak{q} \subset S$ be a prime lying over $\mathfrak{p} \subset R$. If $\dim_{\mathfrak{q}}(S/R) = n - c$, then we can find (h, g) as above such that $g \notin \mathfrak{q}$.*

Proof. Ad (1). By Lemma 125.6 there exists an open subset $W \subset \mathrm{Spec}(S)$ containing $V(IS)$ such that all fibres of $W \to \mathrm{Spec}(R)$ have dimension $\leq n - c$. Say $W = \mathrm{Spec}(S) \setminus V(J)$. Then $V(J) \cap V(IS) = \emptyset$ hence we can find a $g \in J$ which maps to $1 \in S/IS$. Let $h \in R[x_1, \ldots, x_n]$ be any preimage of g.

Ad (2). By Lemma 125.6 there exists an open subset $W \subset \mathrm{Spec}(S)$ containing $\mathrm{Spec}(S \otimes_R \kappa(\mathfrak{p}))$ such that all fibres of $W \to \mathrm{Spec}(R)$ have dimension $\leq n - c$. Say $W = \mathrm{Spec}(S) \setminus V(J)$. Then $V(J \cdot S \otimes_R \kappa(\mathfrak{p})) = \emptyset$. Hence we can find a $g \in J$ which maps to a unit in $S \otimes_R \kappa(\mathfrak{p})$ (details omitted). Let $h \in R[x_1, \ldots, x_n]$ be any preimage of g.

Ad (3). By Lemma 125.6 there exists a $g \in S$, $g \notin \mathfrak{q}$ such that all nonempty fibres of $R \to S_g$ have dimension $\leq n - c$. Let $h \in R[x_1, \ldots, x_n]$ be any element that maps to g. $\square$

The following lemma says we can do absolute Noetherian approximation for relative global complete intersections.

00SU **Lemma 136.12.** *Let R be a ring. Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ be a relative global complete intersection (Definition 136.5). There exist a finite type $\mathbf{Z}$-subalgebra $R_0 \subset R$ such that $f_i \in R_0[x_1, \ldots, x_n]$ and such that*

$$S_0 = R_0[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$$

is a relative global complete intersection.

Proof. Let $R_0 \subset R$ be the $\mathbf{Z}$-algebra of R generated by all the coefficients of the polynomials $f_1, \ldots, f_c$. Let $S_0 = R_0[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. Clearly, $S = R \otimes_{R_0} S_0$. Pick a prime $\mathfrak{q} \subset S$ and denote $\mathfrak{p} \subset R$, $\mathfrak{q}_0 \subset S_0$, and $\mathfrak{p}_0 \subset R_0$ the primes it lies over. Because $\dim(S \otimes_R \kappa(\mathfrak{p})) = n - c$ we also have $\dim(S_0 \otimes_{R_0} \kappa(\mathfrak{p}_0)) = n - c$, see Lemma 116.5. By Lemma 125.6 there exists a $g \in S_0$, $g \notin \mathfrak{q}_0$ such that all nonempty fibres of $R_0 \to (S_0)_g$ have dimension $\leq n - c$. As $\mathfrak{q}$ was arbitrary and $\text{Spec}(S)$ quasi-compact, we can find finitely many $g_1, \ldots, g_m \in S_0$ such that (a) for $j = 1, \ldots, m$ the nonempty fibres of $R_0 \to (S_0)_{g_j}$ have dimension $\leq n - c$ and (b) the image of $\text{Spec}(S) \to \text{Spec}(S_0)$ is contained in $D(g_1) \cup \ldots \cup D(g_m)$. In other words, the images of $g_1, \ldots, g_m$ in $S = R \otimes_{R_0} S_0$ generate the unit ideal. After increasing R_0 we may assume that $g_1, \ldots, g_m$ generate the unit ideal in S_0. By (a) the nonempty fibres of $R_0 \to S_0$ all have dimension $\leq n - c$ and we conclude. $\square$

00SV **Lemma 136.13.** *Let R be a ring. Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ be a relative global complete intersection (Definition 136.5). For every prime $\mathfrak{q}$ of S, let $\mathfrak{q}'$ denote the corresponding prime of $R[x_1, \ldots, x_n]$. Then*

1. *$f_1, \ldots, f_c$ is a regular sequence in the local ring $R[x_1, \ldots, x_n]_{\mathfrak{q}'}$,*
2. *each of the rings $R[x_1, \ldots, x_n]_{\mathfrak{q}'}/(f_1, \ldots, f_i)$ is flat over R, and*
3. *the S-module $(f_1, \ldots, f_c)/(f_1, \ldots, f_c)^2$ is free with basis given by the elements $f_i \bmod (f_1, \ldots, f_c)^2$.*

Proof. By Lemma 69.2 part (3) follows from part (1).

Assume R is Noetherian. Let $\mathfrak{p} = R \cap \mathfrak{q}'$. By Lemma 135.4 for example we see that $f_1, \ldots, f_c$ form a regular sequence in the local ring $R[x_1, \ldots, x_n]_{\mathfrak{q}'} \otimes_R \kappa(\mathfrak{p})$. Moreover, the local ring $R[x_1, \ldots, x_n]_{\mathfrak{q}'}$ is flat over $R_{\mathfrak{p}}$. Since R, and hence $R[x_1, \ldots, x_n]_{\mathfrak{q}'}$ is Noetherian we see from Lemma 99.3 that (1) and (2) hold.

Let R be general. Write $R = \text{colim}_{\lambda \in \Lambda} R_\lambda$ as the filtered colimit of finite type $\mathbf{Z}$-subalgebras (compare with Section 127). We may assume that $f_1, \ldots, f_c \in R_\lambda[x_1, \ldots, x_n]$ for all λ. Let $R_0 \subset R$ be as in Lemma 136.12. Then we may assume $R_0 \subset R_\lambda$ for all λ. It follows that $S_\lambda = R_\lambda[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a relative global complete intersection (as base change of S_0 via $R_0 \to R_\lambda$, see Lemma 136.10). Denote $\mathfrak{p}_\lambda$, $\mathfrak{q}_\lambda$, $\mathfrak{q}'_\lambda$ the prime of R_λ, S_λ, $R_\lambda[x_1, \ldots, x_n]$ induced by $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{q}'$. With this notation, we have (1) and (2) for each λ. Since

$$R[x_1, \ldots, x_n]_{\mathfrak{q}'}/(f_1, \ldots, f_i) = \text{colim}\, R_\lambda[x_1, \ldots, x_n]_{\mathfrak{q}'_\lambda}/(f_1, \ldots, f_i)$$

we deduce flatness in (2) over R from Lemma 39.6. Since we have

$$\begin{aligned} & R[x_1, \ldots, x_n]_{\mathfrak{q}'}/(f_1, \ldots, f_i) \xrightarrow{f_{i+1}} R[x_1, \ldots, x_n]_{\mathfrak{q}'}/(f_1, \ldots, f_i) \\ = \text{colim} & \left(R_\lambda[x_1, \ldots, x_n]_{\mathfrak{q}'_\lambda}/(f_1, \ldots, f_i) \xrightarrow{f_{i+1}} R_\lambda[x_1, \ldots, x_n]_{\mathfrak{q}'_\lambda}/(f_1, \ldots, f_i) \right) \end{aligned}$$

and since filtered colimits are exact (Lemma 8.8) we conclude that we have (1). $\square$

00SW **Lemma 136.14.** *A relative global complete intersection is syntomic, i.e., flat.*

Proof. Let $R \to S$ be a relative global complete intersection. The fibres are global complete intersections, and S is of finite presentation over R. Thus the only thing to prove is that $R \to S$ is flat. This is true by (2) of Lemma 136.13. $\square$

00SY **Lemma 136.15.** *Let $R \to S$ be a ring map. Let $\mathfrak q \subset S$ be a prime lying over the prime $\mathfrak p$ of R. The following are equivalent:*

1. *There exists an element $g \in S$, $g \notin \mathfrak q$ such that $R \to S_g$ is syntomic.*
2. *There exists an element $g \in S$, $g \notin \mathfrak q$ such that S_g is a relative global complete intersection over R.*
3. *There exists an element $g \in S$, $g \notin \mathfrak q$, such that $R \to S_g$ is of finite presentation, the local ring map $R_{\mathfrak p} \to S_{\mathfrak q}$ is flat, and the local ring $S_{\mathfrak q}/\mathfrak p S_{\mathfrak q}$ is a complete intersection ring over $\kappa(\mathfrak p)$ (see Definition 135.5).*

Proof. The implication (1) $\Rightarrow$ (3) is Lemma 135.8. The implication (2) $\Rightarrow$ (1) is Lemma 136.14. It remains to show that (3) implies (2).

Assume (3). After replacing S by S_g for some $g \in S$, $g \notin \mathfrak q$ we may assume S is finitely presented over R. Choose a presentation $S = R[x_1, \ldots, x_n]/I$. Let $\mathfrak q' \subset R[x_1, \ldots, x_n]$ be the prime corresponding to $\mathfrak q$. Write $\kappa(\mathfrak p) = k$. Note that $S \otimes_R k = k[x_1, \ldots, x_n]/\overline{I}$ where $\overline{I} \subset k[x_1, \ldots, x_n]$ is the ideal generated by the image of I. Let $\overline{\mathfrak q}' \subset k[x_1, \ldots, x_n]$ be the prime ideal generated by the image of $\mathfrak q'$. By Lemma 135.8 the equivalent conditions of Lemma 135.4 hold for $\overline{I}$ and $\overline{\mathfrak q}'$. Say the dimension of $\overline{I}_{\overline{\mathfrak q}'}/\overline{\mathfrak q}'\overline{I}_{\overline{\mathfrak q}'}$ over $\kappa(\overline{\mathfrak q}')$ is c. Pick $f_1, \ldots, f_c \in I$ mapping to a basis of this vector space. The images $\overline{f}_j \in \overline{I}$ generate $\overline{I}_{\overline{\mathfrak q}'}$ (by Lemma 135.4). Set $S' = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. Let J be the kernel of the surjection $S' \to S$. Since S is of finite presentation J is a finitely generated ideal (Lemma 6.2). Consider the short exact sequence

$$0 \to J \to S' \to S \to 0$$

As $S_{\mathfrak q}$ is flat over R we see that $J_{\mathfrak q'} \otimes_R k \to S'_{\mathfrak q'} \otimes_R k$ is injective (Lemma 39.12). However, by construction $S'_{\mathfrak q'} \otimes_R k$ maps isomorphically to $S_{\mathfrak q} \otimes_R k$. Hence we conclude that $J_{\mathfrak q'} \otimes_R k = J_{\mathfrak q'}/\mathfrak p J_{\mathfrak q'} = 0$. By Nakayama's lemma (Lemma 20.1) we conclude that there exists a $g \in R[x_1, \ldots, x_n]$, $g \notin \mathfrak q'$ such that $J_g = 0$. In other words $S'_g \cong S_g$. After further localizing we see that S' (and hence S) becomes a relative global complete intersection by Lemma 136.11 as desired. $\square$

07BT **Lemma 136.16.** *Let R be a ring. Let $S = R[x_1, \ldots, x_n]/I$ for some finitely generated ideal I. If $g \in S$ is such that S_g is syntomic over R, then $(I/I^2)_g$ is a finite projective S_g-module.*

Proof. By Lemma 136.15 there exist finitely many elements $g_1, \ldots, g_m \in S$ which generate the unit ideal in S_g such that each S_{gg_j} is a relative global complete intersection over R. Since it suffices to prove that $(I/I^2)_{gg_j}$ is finite projective, see Lemma 78.2, we may assume that S_g is a relative global complete intersection. In this case the result follows from Lemmas 134.16 and 136.13. $\square$

00SZ **Lemma 136.17.** *Let $R \to S$, $S \to S'$ be ring maps.*

1. *If $R \to S$ and $S \to S'$ are syntomic, then $R \to S'$ is syntomic.*

(2) *If $R \to S$ and $S \to S'$ are relative global complete intersections, then $R \to S'$ is a relative global complete intersection.*

Proof. Proof of (2). Say $R \to S$ and $S \to S'$ are relative global complete intersections and we have presentations $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ and $S' = S[y_1, \ldots, y_m]/(h_1, \ldots, h_d)$ as in Definition 136.5. Then

$$S' \cong R[x_1, \ldots, x_n, y_1, \ldots, y_m]/(f_1, \ldots, f_c, h'_1, \ldots, h'_d)$$

for some lifts $h'_j \in R[x_1, \ldots, x_n, y_1, \ldots, y_m]$ of the h_j. Hence it suffices to bound the dimensions of the fibre rings. Thus we may assume $R = k$ is a field. In this case we see that we have a ring, namely S, which is of finite type over k and equidimensional of dimension $n - c$, and a finite type ring map $S \to S'$ all of whose nonempty fibre rings are equidimensional of dimension $m - d$. Then, by Lemma 112.6 for example applied to localizations at maximal ideals of S', we see that $\dim(S') \leq n - c + m - d$ as desired.

We will reduce part (1) to part (2). Assume $R \to S$ and $S \to S'$ are syntomic. Let $\mathfrak{q}' \subset S$ be a prime ideal lying over $\mathfrak{q} \subset S$. By Lemma 136.15 there exists a $g' \in S'$, $g' \notin \mathfrak{q}'$ such that $S \to S'_{g'}$ is a relative global complete intersection. Similarly, we find $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is a relative global complete intersection. By Lemma 136.10 the ring map $S_g \to S_{gg'}$ is a relative global complete intersection. By part (2) we see that $R \to S_{gg'}$ is a relative global complete intersection and $gg' \notin \mathfrak{q}'$. Since $\mathfrak{q}'$ was arbitrary combining Lemmas 136.15 and 136.4 we see that $R \to S'$ is syntomic (this also uses that the spectrum of S' is quasi-compact, see Lemma 17.10). □

The following lemma will be improved later, see Smoothing Ring Maps, Proposition 3.2.

00T0 **Lemma 136.18.** *Let R be a ring and let $I \subset R$ be an ideal. Let $R/I \to \overline{S}$ be a syntomic map. Then there exists elements $\overline{g}_i \in \overline{S}$ which generate the unit ideal of $\overline{S}$ such that each $\overline{S}_{g_i} \cong S_i/IS_i$ for some relative global complete intersection S_i over R.*

Proof. By Lemma 136.15 we find a collection of elements $\overline{g}_i \in \overline{S}$ which generate the unit ideal of $\overline{S}$ such that each $\overline{S}_{g_i}$ is a relative global complete intersection over R/I. Hence we may assume that $\overline{S}$ is a relative global complete intersection. Write $\overline{S} = (R/I)[x_1, \ldots, x_n]/(\overline{f}_1, \ldots, \overline{f}_c)$ as in Definition 136.5. Choose $f_1, \ldots, f_c \in R[x_1, \ldots, x_n]$ lifting $\overline{f}_1, \ldots, \overline{f}_c$. Set $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. Note that $S/IS \cong \overline{S}$. By Lemma 136.11 we can find $g \in S$ mapping to 1 in $\overline{S}$ such that S_g is a relative global complete intersection over R. Since $\overline{S} \cong S_g/IS_g$ this finishes the proof. □

137. Smooth ring maps

00T1 Let us motivate the definition of a smooth ring map by an example. Suppose R is a ring and $S = R[x, y]/(f)$ for some nonzero $f \in R[x, y]$. In this case there is an exact sequence

$$S \to S\mathrm{d}x \oplus S\mathrm{d}y \to \Omega_{S/R} \to 0$$

where the first arrow maps 1 to $\frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y$ see Section 134. We conclude that $\Omega_{S/R}$ is locally free of rank 1 if the partial derivatives of f generate the unit ideal in S. In this case S is smooth of relative dimension 1 over R. But it can happen that $\Omega_{S/R}$ is locally free of rank 2 namely if both partial derivatives of f are zero. For example if for a prime p we have $p = 0$ in R and $f = x^p + y^p$ then this happens. Here $R \to S$ is a relative global complete intersection of relative dimension 1 which is not smooth. Hence, in order to check that a ring map is smooth it is not sufficient to check whether the module of differentials is free. The correct condition is the following.

00T2 **Definition 137.1.** A ring map $R \to S$ is *smooth* if it is of finite presentation and the naive cotangent complex $NL_{S/R}$ is quasi-isomorphic to a finite projective S-module placed in degree 0.

In particular, if $R \to S$ is smooth then the module $\Omega_{S/R}$ is a finite projective S-module. Moreover, by Lemma 137.2 the naive cotangent complex of any presentation has the same structure. Thus, for a surjection $\alpha : R[x_1, \ldots, x_n] \to S$ with kernel I the map

$$I/I^2 \longrightarrow \Omega_{R[x_1,\ldots,x_n]/R} \otimes_{R[x_1,\ldots,x_n]} S$$

is a split injection. In other words $\bigoplus_{i=1}^n S\mathrm{d}x_i \cong I/I^2 \oplus \Omega_{S/R}$ as S-modules. This implies that I/I^2 is a finite projective S-module too!

05GK **Lemma 137.2.** *Let $R \to S$ be a ring map of finite presentation. If for some presentation α of S over R the naive cotangent complex $NL(\alpha)$ is quasi-isomorphic to a finite projective S-module placed in degree 0, then this holds for any presentation.*

Proof. Immediate from Lemma 134.2. □

00T3 **Lemma 137.3.** *Let $R \to S$ be a smooth ring map. Any localization S_g is smooth over R. If $f \in R$ maps to an invertible element of S, then $R_f \to S$ is smooth.*

Proof. By Lemma 134.13 the naive cotangent complex for S_g over R is the base change of the naive cotangent complex of S over R. The assumption is that the naive cotangent complex of S/R is $\Omega_{S/R}$ and that this is a finite projective S-module. Hence so is its base change. Thus S_g is smooth over R.

The second assertion follows in the same way from Lemma 134.11. □

00T4 **Lemma 137.4.** *Let $R \to S$ be a smooth ring map. Let $R \to R'$ be any ring map. Then the base change $R' \to S' = R' \otimes_R S$ is smooth.*

Proof. Let $\alpha : R[x_1, \ldots, x_n] \to S$ be a presentation with kernel I. Let $\alpha' : R'[x_1, \ldots, x_n] \to R' \otimes_R S$ be the induced presentation. Let $I' = \mathrm{Ker}(\alpha')$. Since $0 \to I \to R[x_1, \ldots, x_n] \to S \to 0$ is exact, the sequence $R' \otimes_R I \to R'[x_1, \ldots, x_n] \to R' \otimes_R S \to 0$ is exact. Thus $R' \otimes_R I \to I'$ is surjective. By Definition 137.1 there is a short exact sequence

$$0 \to I/I^2 \to \Omega_{R[x_1,\ldots,x_n]/R} \otimes_{R[x_1,\ldots,x_n]} S \to \Omega_{S/R} \to 0$$

and the S-module $\Omega_{S/R}$ is finite projective. In particular I/I^2 is a direct summand of $\Omega_{R[x_1,\ldots,x_n]/R} \otimes_{R[x_1,\ldots,x_n]} S$. Consider the commutative diagram

$$\begin{tikzcd}
R' \otimes_R (I/I^2) \arrow[r] \arrow[d] & R' \otimes_R (\Omega_{R[x_1,\ldots,x_n]/R} \otimes_{R[x_1,\ldots,x_n]} S) \arrow[d] \\
I'/(I')^2 \arrow[r] & \Omega_{R'[x_1,\ldots,x_n]/R'} \otimes_{R'[x_1,\ldots,x_n]} (R' \otimes_R S)
\end{tikzcd}$$

Since the right vertical map is an isomorphism we see that the left vertical map is injective and surjective by what was said above. Thus we conclude that $NL(\alpha')$ is quasi-isomorphic to $\Omega_{S'/R'} \cong S' \otimes_S \Omega_{S/R}$. And this is finite projective since it is the base change of a finite projective module. □

00T5 **Lemma 137.5.** *Let k be a field. Let S be a smooth k-algebra. Then S is a local complete intersection.*

Proof. By Lemmas 137.4 and 135.11 it suffices to prove this when k is algebraically closed. Choose a presentation $\alpha : k[x_1, \ldots, x_n] \to S$ with kernel I. Let $\mathfrak m$ be a maximal ideal of S, and let $\mathfrak m' \supset I$ be the corresponding maximal ideal of $k[x_1, \ldots, x_n]$. We will show that condition (5) of Lemma 135.4 holds (with $\mathfrak m$ instead of $\mathfrak q$). We may write $\mathfrak m' = (x_1 - a_1, \ldots, x_n - a_n)$ for some $a_i \in k$, because k is algebraically closed, see Theorem 34.1. By our assumption that $k \to S$ is smooth the S-module map $\mathrm{d} : I/I^2 \to \bigoplus_{i=1}^n S\mathrm{d}x_i$ is a split injection. Hence the corresponding map $I/\mathfrak m' I \to \bigoplus \kappa(\mathfrak m')\mathrm{d}x_i$ is injective. Say $\dim_{\kappa(\mathfrak m')}(I/\mathfrak m' I) = c$ and pick $f_1, \ldots, f_c \in I$ which map to a $\kappa(\mathfrak m')$-basis of $I/\mathfrak m' I$. By Nakayama's Lemma 20.1 we see that $f_1, \ldots, f_c$ generate $I_{\mathfrak m'}$ over $k[x_1, \ldots, x_n]_{\mathfrak m'}$. Consider the commutative diagram

$$\begin{tikzcd}
I \arrow[r] \arrow[d] & I/I^2 \arrow[r] \arrow[d] & I/\mathfrak m' I \arrow[d] \\
\Omega_{k[x_1, \ldots, x_n]/k} \arrow[r] & \bigoplus S\mathrm{d}x_i \arrow[r, "\mathrm{d}x_i \mapsto x_i - a_i"] & \mathfrak m'/(\mathfrak m')^2
\end{tikzcd}$$

(proof commutativity omitted). The middle vertical map is the one defining the naive cotangent complex of α. Note that the right lower horizontal arrow induces an isomorphism $\bigoplus \kappa(\mathfrak m')\mathrm{d}x_i \to \mathfrak m'/(\mathfrak m')^2$. Hence our generators $f_1, \ldots, f_c$ of $I_{\mathfrak m'}$ map to a collection of elements in $k[x_1, \ldots, x_n]_{\mathfrak m'}$ whose classes in $\mathfrak m'/(\mathfrak m')^2$ are linearly independent over $\kappa(\mathfrak m')$. Therefore they form a regular sequence in the ring $k[x_1, \ldots, x_n]_{\mathfrak m'}$ by Lemma 106.3. This verifies condition (5) of Lemma 135.4 hence S_g is a global complete intersection over k for some $g \in S$, $g \notin \mathfrak m$. As this works for any maximal ideal of S we conclude that S is a local complete intersection over k. □

00T6 **Definition 137.6.** Let R be a ring. Given integers $n \geq c \geq 0$ and $f_1, \ldots, f_c \in R[x_1, \ldots, x_n]$ we say

$$S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$$

is a *standard smooth algebra over R* if the polynomial

$$g = \det \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_2/\partial x_1 & \ldots & \partial f_c/\partial x_1 \\ \partial f_1/\partial x_2 & \partial f_2/\partial x_2 & \ldots & \partial f_c/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_c & \partial f_2/\partial x_c & \ldots & \partial f_c/\partial x_c \end{pmatrix}$$

maps to an invertible element in S.

00T7 **Lemma 137.7.** *Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c) = R[x_1, \ldots, x_n]/I$ be a standard smooth algebra. Then*

1. *the ring map $R \to S$ is smooth,*
2. *the S-module $\Omega_{S/R}$ is free on $dx_{c+1}, \ldots, dx_n$,*
3. *the S-module I/I^2 is free on the classes of $f_1, \ldots, f_c$,*
4. *for any $g \in S$ the ring map $R \to S_g$ is standard smooth,*

(5) *for any ring map $R \to R'$ the base change $R' \to R' \otimes_R S$ is standard smooth,*
(6) *if $f \in R$ maps to an invertible element in S, then $R_f \to S$ is standard smooth, and*
(7) *the ring S is a relative global complete intersection over R.*

Proof. Consider the naive cotangent complex of the given presentation

$$(f_1, \ldots, f_c)/(f_1, \ldots, f_c)^2 \longrightarrow \bigoplus\nolimits_{i=1}^{n} S\mathrm{d}x_i$$

Let us compose this map with the projection onto the first c direct summands of the direct sum. According to the definition of a standard smooth algebra the classes $f_i \bmod (f_1, \ldots, f_c)^2$ map to a basis of $\bigoplus_{i=1}^{c} S\mathrm{d}x_i$. We conclude that $(f_1, \ldots, f_c)/(f_1, \ldots, f_c)^2$ is free of rank c with a basis given by the elements $f_i \bmod (f_1, \ldots, f_c)^2$, and that the homology in degree 0, i.e., $\Omega_{S/R}$, of the naive cotangent complex is a free S-module with basis the images of $\mathrm{d}x_{c+j}$, $j = 1, \ldots, n-c$. In particular, this proves $R \to S$ is smooth.

The proofs of (4) and (6) are omitted. But see the example below and the proof of Lemma 136.10.

Let $\varphi : R \to R'$ be any ring map. Denote $S' = R'[x_1, \ldots, x_n]/(f_1^{\varphi}, \ldots, f_c^{\varphi})$ where f^{φ} is the polynomial obtained from $f \in R[x_1, \ldots, x_n]$ by applying φ to all the coefficients. Then $S' \cong R' \otimes_R S$. Moreover, the determinant of Definition 137.6 for S'/R' is equal to g^{φ}. Its image in S' is therefore the image of g via $R[x_1, \ldots, x_n] \to S \to S'$ and hence invertible. This proves (5).

To prove (7) it suffices to show that $S \otimes_R \kappa(\mathfrak{p})$ has dimension $n - c$ for every prime $\mathfrak{p} \subset R$. By (5) it suffices to prove that any standard smooth algebra $k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ over a field k has dimension $n - c$. We already know that $k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a local complete intersection by Lemma 137.5. Hence, since I/I^2 is free of rank c we see that $k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ has dimension $n-c$, by Lemma 135.4 for example. $\square$

00T8 **Example 137.8.** Let R be a ring. Let $f_1, \ldots, f_c \in R[x_1, \ldots, x_n]$. Let

$$h = \det \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_2/\partial x_1 & \ldots & \partial f_c/\partial x_1 \\ \partial f_1/\partial x_2 & \partial f_2/\partial x_2 & \ldots & \partial f_c/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_c & \partial f_2/\partial x_c & \ldots & \partial f_c/\partial x_c \end{pmatrix}.$$

Set $S = R[x_1, \ldots, x_{n+1}]/(f_1, \ldots, f_c, x_{n+1}h - 1)$. This is an example of a standard smooth algebra, except that the presentation is wrong and the variables should be in the following order: $x_1, \ldots, x_c, x_{n+1}, x_{c+1}, \ldots, x_n$.

00T9 **Lemma 137.9.** *A composition of standard smooth ring maps is standard smooth.*

Proof. Suppose that $R \to S$ and $S \to S'$ are standard smooth. We choose presentations $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ and $S' = S[y_1, \ldots, y_m]/(g_1, \ldots, g_d)$. Choose elements $g_j' \in R[x_1, \ldots, x_n, y_1, \ldots, y_m]$ mapping to the g_j. In this way we see $S' = R[x_1, \ldots, x_n, y_1, \ldots, y_m]/(f_1, \ldots, f_c, g_1', \ldots, g_d')$. To show that S' is standard smooth it suffices to verify that the determinant

$$\det \begin{pmatrix} \partial f_1/\partial x_1 & \ldots & \partial f_c/\partial x_1 & \partial g_1/\partial x_1 & \ldots & \partial g_d/\partial x_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_c & \ldots & \partial f_c/\partial x_c & \partial g_1/\partial x_c & \ldots & \partial g_d/\partial x_c \\ 0 & \ldots & 0 & \partial g_1/\partial y_1 & \ldots & \partial g_d/\partial y_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \partial g_1/\partial y_d & \ldots & \partial g_d/\partial y_d \end{pmatrix}$$

is invertible in S'. This is clear since it is the product of the two determinants which were assumed to be invertible by hypothesis. ☐

00TA **Lemma 137.10.** *Let $R \to S$ be a smooth ring map. There exists an open covering of* $\mathrm{Spec}(S)$ *by standard opens $D(g)$ such that each S_g is standard smooth over R. In particular $R \to S$ is syntomic.*

Proof. Choose a presentation $\alpha : R[x_1, \ldots, x_n] \to S$ with kernel $I = (f_1, \ldots, f_m)$. For every subset $E \subset \{1, \ldots, m\}$ consider the open subset U_E where the classes $f_e, e \in E$ freely generate the finite projective S-module I/I^2, see Lemma 79.3. We may cover $\mathrm{Spec}(S)$ by standard opens $D(g)$ each completely contained in one of the opens U_E. For such a g we look at the presentation

$$\beta : R[x_1, \ldots, x_n, x_{n+1}] \longrightarrow S_g$$

mapping x_{n+1} to $1/g$. Setting $J = \mathrm{Ker}(\beta)$ we use Lemma 134.12 to see that $J/J^2 \cong (I/I^2)_g \oplus S_g$ is free. We may and do replace S by S_g. Then using Lemma 136.6 we may assume we have a presentation $\alpha : R[x_1, \ldots, x_n] \to S$ with kernel $I = (f_1, \ldots, f_c)$ such that I/I^2 is free on the classes of $f_1, \ldots, f_c$.

Using the presentation α obtained at the end of the previous paragraph, we more or less repeat this argument with the basis elements $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$ of $\Omega_{R[x_1,\ldots,x_n]/R}$. Namely, for any subset $E \subset \{1, \ldots, n\}$ of cardinality c we may consider the open subset U_E of $\mathrm{Spec}(S)$ where the differential of $NL(\alpha)$ composed with the projection

$$S^{\oplus c} \cong I/I^2 \longrightarrow \Omega_{R[x_1,\ldots,x_n]/R} \otimes_{R[x_1,\ldots,x_n]} S \longrightarrow \bigoplus\nolimits_{i \in E} S\mathrm{d}x_i$$

is an isomorphism. Again we may find a covering of $\mathrm{Spec}(S)$ by (finitely many) standard opens $D(g)$ such that each $D(g)$ is completely contained in one of the opens U_E. By renumbering, we may assume $E = \{1, \ldots, c\}$. For a g with $D(g) \subset U_E$ we look at the presentation

$$\beta : R[x_1, \ldots, x_n, x_{n+1}] \to S_g$$

mapping x_{n+1} to $1/g$. Setting $J = \mathrm{Ker}(\beta)$ we conclude from Lemma 134.12 that $J = (f_1, \ldots, f_c, fx_{n+1} - 1)$ where $\alpha(f) = g$ and that the composition

$$J/J^2 \longrightarrow \Omega_{R[x_1,\ldots,x_{n+1}]/R} \otimes_{R[x_1,\ldots,x_{n+1}]} S_g \longrightarrow \bigoplus\nolimits_{i=1}^{c} S_g\mathrm{d}x_i \oplus S_g\mathrm{d}x_{n+1}$$

is an isomorphism. Reordering the coordinates as $x_1, \ldots, x_c, x_{n+1}, x_{c+1}, \ldots, x_n$ we conclude that S_g is standard smooth over R as desired.

This finishes the proof as standard smooth algebras are syntomic (Lemmas 137.7 and 136.14) and being syntomic over R is local on S (Lemma 136.4). ☐

00TB **Definition 137.11.** Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime of S. We say $R \to S$ is *smooth at* $\mathfrak{q}$ if there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is smooth.

For ring maps of finite presentation we can characterize this as follows.

07BU **Lemma 137.12.** *Let $R \to S$ be of finite presentation. Let $\mathfrak{q}$ be a prime of S. The following are equivalent*

(1) *$R \to S$ is smooth at $\mathfrak{q}$,*
(2) *$H_1(L_{S/R})_{\mathfrak{q}} = 0$ and $\Omega_{S/R,\mathfrak{q}}$ is a finite free $S_{\mathfrak{q}}$-module,*
(3) *$H_1(L_{S/R})_{\mathfrak{q}} = 0$ and $\Omega_{S/R,\mathfrak{q}}$ is a projective $S_{\mathfrak{q}}$-module, and*
(4) *$H_1(L_{S/R})_{\mathfrak{q}} = 0$ and $\Omega_{S/R,\mathfrak{q}}$ is a flat $S_{\mathfrak{q}}$-module.*

Proof. We will use without further mention that formation of the naive cotangent complex commutes with localization, see Section 134, especially Lemma 134.13. Note that $\Omega_{S/R}$ is a finitely presented S-module, see Lemma 131.15. Hence (2), (3), and (4) are equivalent by Lemma 78.2. It is clear that (1) implies the equivalent conditions (2), (3), and (4). Assume (2) holds. Writing $S_{\mathfrak{q}}$ as the colimit of principal localizations we see from Lemma 127.6 that we can find a $g \in S$, $g \notin \mathfrak{q}$ such that $(\Omega_{S/R})_g$ is finite free. Choose a presentation $\alpha : R[x_1, \ldots, x_n] \to S$ with kernel I. We may work with $NL(\alpha)$ instead of $NL_{S/R}$, see Lemma 134.2. The surjection

$$\Omega_{R[x_1,\ldots,x_n]/R} \otimes_R S \to \Omega_{S/R} \to 0$$

has a right inverse after inverting g because $(\Omega_{S/R})_g$ is projective. Hence the image of $\mathrm{d} : (I/I^2)_g \to \Omega_{R[x_1,\ldots,x_n]/R} \otimes_R S_g$ is a direct summand and this map has a right inverse too. We conclude that $H_1(L_{S/R})_g$ is a quotient of $(I/I^2)_g$. In particular $H_1(L_{S/R})_g$ is a finite S_g-module. Thus the vanishing of $H_1(L_{S/R})_{\mathfrak{q}}$ implies the vanishing of $H_1(L_{S/R})_{gg'}$ for some $g' \in S$, $g' \notin \mathfrak{q}$. Then $R \to S_{gg'}$ is smooth by definition. $\square$

00TC **Lemma 137.13.** *Let $R \to S$ be a ring map. Then $R \to S$ is smooth if and only if $R \to S$ is smooth at every prime $\mathfrak{q}$ of S.*

Proof. The direct implication is trivial. Suppose that $R \to S$ is smooth at every prime $\mathfrak{q}$ of S. Since $\text{Spec}(S)$ is quasi-compact, see Lemma 17.10, there exists a finite covering $\text{Spec}(S) = \bigcup D(g_i)$ such that each S_{g_i} is smooth. By Lemma 23.3 this implies that S is of finite presentation over R. According to Lemma 134.13 we see that $NL_{S/R} \otimes_S S_{g_i}$ is quasi-isomorphic to a finite projective S_{g_i}-module. By Lemma 78.2 this implies that $NL_{S/R}$ is quasi-isomorphic to a finite projective S-module. $\square$

00TD **Lemma 137.14.** *A composition of smooth ring maps is smooth.*

Proof. You can prove this in many different ways. One way is to use the snake lemma (Lemma 4.1), the Jacobi-Zariski sequence (Lemma 134.4), combined with the characterization of projective modules as being direct summands of free modules (Lemma 77.2). Another proof can be obtained by combining Lemmas 137.10, 137.9 and 137.13. $\square$

0GIF **Lemma 137.15.** *Let R be a ring. Let $S = S' \times S''$ be a product of R-algebras. Then S is smooth over R if and only if both S' and S'' are smooth over R.*

Proof. Omitted. Hints: By Lemma 137.13 we can check smoothness one prime at a time. Since $\text{Spec}(S)$ is the disjoint union of $\text{Spec}(S')$ and $\text{Spec}(S'')$ by Lemma 21.2 we find that smoothness of $R \to S$ at $\mathfrak{q}$ corresponds to either smoothness of $R \to S'$ at the corresponding prime or smoothness of $R \to S''$ at the corresponding prime. $\square$

00TE **Lemma 137.16.** *Let R be a ring. Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ be a relative global complete intersection. Let $\mathfrak{q} \subset S$ be a prime. Then $R \to S$ is smooth at $\mathfrak{q}$ if and only if there exists a subset $I \subset \{1, \ldots, n\}$ of cardinality c such that the polynomial*

$$g_I = \det(\partial f_j/\partial x_i)_{j=1,\ldots,c,\ i\in I}.$$

does not map to an element of $\mathfrak{q}$.

Proof. By Lemma 136.13 we see that the naive cotangent complex associated to the given presentation of S is the complex

$$\bigoplus\nolimits_{j=1}^{c} S \cdot f_j \longrightarrow \bigoplus\nolimits_{i=1}^{n} S \cdot \mathrm{d}x_i, \quad f_j \longmapsto \sum \frac{\partial f_j}{\partial x_i}\mathrm{d}x_i.$$

The maximal minors of the matrix giving the map are exactly the polynomials g_I.

Assume g_I maps to $g \in S$, with $g \notin \mathfrak{q}$. Then the algebra S_g is smooth over R. Namely, its naive cotangent complex is quasi-isomorphic to the complex above localized at g, see Lemma 134.13. And by construction it is quasi-isomorphic to a free rank $n - c$ module in degree 0.

Conversely, suppose that all g_I end up in $\mathfrak{q}$. In this case the complex above tensored with $\kappa(\mathfrak{q})$ does not have maximal rank, and hence there is no localization by an element $g \in S$, $g \notin \mathfrak{q}$ where this map becomes a split injection. By Lemma 134.13 again there is no such localization which is smooth over R. □

00TF **Lemma 137.17.** *Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime lying over the prime $\mathfrak{p}$ of R. Assume*

1. *there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is of finite presentation,*
2. *the local ring homomorphism $R_{\mathfrak{p}} \to S_{\mathfrak{q}}$ is flat,*
3. *the fibre $S \otimes_R \kappa(\mathfrak{p})$ is smooth over $\kappa(\mathfrak{p})$ at the prime corresponding to $\mathfrak{q}$.*

Then $R \to S$ is smooth at $\mathfrak{q}$.

Proof. By Lemmas 136.15 and 137.5 we see that there exists a $g \in S$ such that S_g is a relative global complete intersection. Replacing S by S_g we may assume $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a relative global complete intersection. For any subset $I \subset \{1, \ldots, n\}$ of cardinality c consider the polynomial $g_I = \det(\partial f_j/\partial x_i)_{j=1,\ldots,c,i\in I}$ of Lemma 137.16. Note that the image $\overline{g}_I$ of g_I in the polynomial ring $\kappa(\mathfrak{p})[x_1, \ldots, x_n]$ is the determinant of the partial derivatives of the images $\overline{f}_j$ of the f_j in the ring $\kappa(\mathfrak{p})[x_1, \ldots, x_n]$. Thus the lemma follows by applying Lemma 137.16 both to $R \to S$ and to $\kappa(\mathfrak{p}) \to S \otimes_R \kappa(\mathfrak{p})$. □

Note that the sets U, V in the following lemma are open by definition.

00TG **Lemma 137.18.** *Let $R \to S$ be a ring map of finite presentation. Let $R \to R'$ be a flat ring map. Denote $S' = R' \otimes_R S$ the base change. Let $U \subset \mathrm{Spec}(S)$ be the set of primes at which $R \to S$ is smooth. Let $V \subset \mathrm{Spec}(S')$ the set of primes at which $R' \to S'$ is smooth. Then V is the inverse image of U under the map $f : \mathrm{Spec}(S') \to \mathrm{Spec}(S)$.*

Proof. By Lemma 134.8 we see that $NL_{S/R} \otimes_S S'$ is homotopy equivalent to $NL_{S'/R'}$. This already implies that $f^{-1}(U) \subset V$.

Let $\mathfrak{q}' \subset S'$ be a prime lying over $\mathfrak{q} \subset S$. Assume $\mathfrak{q}' \in V$. We have to show that $\mathfrak{q} \in U$. Since $S \to S'$ is flat, we see that $S_{\mathfrak{q}} \to S'_{\mathfrak{q}'}$ is faithfully flat (Lemma 39.17).

Thus the vanishing of $H_1(L_{S'/R'})_{\mathfrak q'}$ implies the vanishing of $H_1(L_{S/R})_{\mathfrak q}$. By Lemma 78.6 applied to the $S_{\mathfrak q}$-module $(\Omega_{S/R})_{\mathfrak q}$ and the map $S_{\mathfrak q} \to S'_{\mathfrak q'}$ we see that $(\Omega_{S/R})_{\mathfrak q}$ is projective. Hence $R \to S$ is smooth at $\mathfrak q$ by Lemma 137.12. $\square$

02UQ **Lemma 137.19.** *Let K/k be a field extension. Let S be a finite type algebra over k. Let $\mathfrak q_K$ be a prime of $S_K = K \otimes_k S$ and let $\mathfrak q$ be the corresponding prime of S. Then S is smooth over k at $\mathfrak q$ if and only if S_K is smooth at $\mathfrak q_K$ over K.*

Proof. This is a special case of Lemma 137.18. $\square$

04B1 **Lemma 137.20.** *Let R be a ring and let $I \subset R$ be an ideal. Let $R/I \to \overline{S}$ be a smooth ring map. Then there exists elements $\overline{g}_i \in \overline{S}$ which generate the unit ideal of $\overline{S}$ such that each $\overline{S}_{g_i} \cong S_i/IS_i$ for some (standard) smooth ring S_i over R.*

Proof. By Lemma 137.10 we find a collection of elements $\overline{g}_i \in \overline{S}$ which generate the unit ideal of $\overline{S}$ such that each $\overline{S}_{g_i}$ is standard smooth over R/I. Hence we may assume that $\overline{S}$ is standard smooth over R/I. Write $\overline{S} = (R/I)[x_1, \ldots, x_n]/(\overline{f}_1, \ldots, \overline{f}_c)$ as in Definition 137.6. Choose $f_1, \ldots, f_c \in R[x_1, \ldots, x_n]$ lifting $\overline{f}_1, \ldots, \overline{f}_c$. Set $S = R[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, x_{n+1}\Delta - 1)$ where $\Delta = \det(\frac{\partial f_j}{\partial x_i})_{i,j=1,\ldots,c}$ as in Example 137.8. This proves the lemma. $\square$

138. Formally smooth maps

00TH In this section we define formally smooth ring maps. It will turn out that a ring map of finite presentation is formally smooth if and only if it is smooth, see Proposition 138.13.

00TI **Definition 138.1.** Let $R \to S$ be a ring map. We say S is *formally smooth over R* if for every commutative solid diagram

$$\begin{tikzcd} S \arrow[r] \arrow[rd, dashed] & A/I \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

where $I \subset A$ is an ideal of square zero, a dotted arrow exists which makes the diagram commute.

00TJ **Lemma 138.2.** *Let $R \to S$ be a formally smooth ring map. Let $R \to R'$ be any ring map. Then the base change $S' = R' \otimes_R S$ is formally smooth over R'.*

Proof. Let a solid diagram

$$\begin{tikzcd} S \arrow[r] \arrow[rrd, dashed] & R' \otimes_R S \arrow[r] \arrow[rd, dashed] & A/I \\ R \arrow[u] \arrow[r] & R' \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

as in Definition 138.1 be given. By assumption the longer dotted arrow exists. By the universal property of tensor product we obtain the shorter dotted arrow. $\square$

031H **Lemma 138.3.** *A composition of formally smooth ring maps is formally smooth.*

Proof. Omitted. (Hint: This is completely formal, and follows from considering a suitable diagram.) $\square$

00TK **Lemma 138.4.** *A polynomial ring over R is formally smooth over R.*

Proof. Suppose we have a diagram as in Definition 138.1 with $S = R[x_j; j \in J]$. Then there exists a dotted arrow simply by choosing lifts $a_j \in A$ of the elements in A/I to which the elements x_j map to under the top horizontal arrow. □

00TL **Lemma 138.5.** *Let $R \to S$ be a ring map. Let $P \to S$ be a surjective R-algebra map from a polynomial ring P onto S. Denote $J \subset P$ the kernel. Then $R \to S$ is formally smooth if and only if there exists an R-algebra map $\sigma : S \to P/J^2$ which is a right inverse to the surjection $P/J^2 \to S$.*

Proof. Assume $R \to S$ is formally smooth. Consider the commutative diagram

$$\begin{tikzcd} S \arrow[r] \arrow[rd, dashed] & P/J \\ R \arrow[u] \arrow[r] & P/J^2 \arrow[u] \end{tikzcd}$$

By assumption the dotted arrow exists. This proves that σ exists.

Conversely, suppose we have a σ as in the lemma. Let a solid diagram

$$\begin{tikzcd} S \arrow[r] \arrow[rd, dashed] & A/I \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

as in Definition 138.1 be given. Because P is formally smooth by Lemma 138.4, there exists an R-algebra homomorphism $\psi : P \to A$ which lifts the map $P \to S \to A/I$. Clearly $\psi(J) \subset I$ and since $I^2 = 0$ we conclude that $\psi(J^2) = 0$. Hence ψ factors as $\overline{\psi} : P/J^2 \to A$. The desired dotted arrow is the composition $\overline{\psi} \circ \sigma : S \to A$. □

00TM **Remark 138.6.** Lemma 138.5 holds more generally whenever P is formally smooth over R.

031I **Lemma 138.7.** *Let $R \to S$ be a ring map. Let $P \to S$ be a surjective R-algebra map from a polynomial ring P onto S. Denote $J \subset P$ the kernel. Then $R \to S$ is formally smooth if and only if the sequence*

$$0 \to J/J^2 \to \Omega_{P/R} \otimes_P S \to \Omega_{S/R} \to 0$$

of Lemma 131.9 is a split exact sequence.

Proof. Assume S is formally smooth over R. By Lemma 138.5 this means there exists an R-algebra map $S \to P/J^2$ which is a right inverse to the canonical map $P/J^2 \to S$. By Lemma 131.11 we have $\Omega_{P/R} \otimes_P S = \Omega_{(P/J^2)/R} \otimes_{P/J^2} S$. By Lemma 131.10 the sequence is split.

Assume the exact sequence of the lemma is split exact. Choose a splitting $\sigma : \Omega_{S/R} \to \Omega_{P/R} \otimes_R S$. For each $\lambda \in S$ choose $x_\lambda \in P$ which maps to λ. Next, for each $\lambda \in S$ choose $f_\lambda \in J$ such that

$$\mathrm{d}f_\lambda = \mathrm{d}x_\lambda - \sigma(\mathrm{d}\lambda)$$

in the middle term of the exact sequence. We claim that $s : \lambda \mapsto x_\lambda - f_\lambda \mod J^2$ is an R-algebra homomorphism $s : S \to P/J^2$. To prove this we will repeatedly use that if $h \in J$ and $\mathrm{d}h = 0$ in $\Omega_{P/R} \otimes_R S$, then $h \in J^2$. Let $\lambda, \mu \in S$. Then $\sigma(\mathrm{d}\lambda + \mathrm{d}\mu - \mathrm{d}(\lambda + \mu)) = 0$. This implies

$$\mathrm{d}(x_\lambda + x_\mu - x_{\lambda+\mu} - f_\lambda - f_\mu + f_{\lambda+\mu}) = 0$$

which means that $x_\lambda + x_\mu - x_{\lambda+\mu} - f_\lambda - f_\mu + f_{\lambda+\mu} \in J^2$, which in turn means that $s(\lambda) + s(\mu) = s(\lambda + \mu)$. Similarly, we have $\sigma(\lambda \mathrm{d}\mu + \mu \mathrm{d}\lambda - \mathrm{d}\lambda\mu) = 0$ which implies that

$$\mu(\mathrm{d}x_\lambda - \mathrm{d}f_\lambda) + \lambda(\mathrm{d}x_\mu - \mathrm{d}f_\mu) - \mathrm{d}x_{\lambda\mu} + \mathrm{d}f_{\lambda\mu} = 0$$

in the middle term of the exact sequence. Moreover we have

$$\mathrm{d}(x_\lambda x_\mu) = x_\lambda \mathrm{d}x_\mu + x_\mu \mathrm{d}x_\lambda = \lambda \mathrm{d}x_\mu + \mu \mathrm{d}x_\lambda$$

in the middle term again. Combined these equations mean that $x_\lambda x_\mu - x_{\lambda\mu} - \mu f_\lambda - \lambda f_\mu + f_{\lambda\mu} \in J^2$, hence $(x_\lambda - f_\lambda)(x_\mu - f_\mu) - (x_{\lambda\mu} - f_{\lambda\mu}) \in J^2$ as $f_\lambda f_\mu \in J^2$, which means that $s(\lambda)s(\mu) = s(\lambda\mu)$. If $\lambda \in R$, then $\mathrm{d}\lambda = 0$ and we see that $\mathrm{d}f_\lambda = \mathrm{d}x_\lambda$, hence $\lambda - x_\lambda + f_\lambda \in J^2$ and hence $s(\lambda) = \lambda$ as desired. At this point we can apply Lemma 138.5 to conclude that S/R is formally smooth. ☐

031J **Proposition 138.8.** *Let $R \to S$ be a ring map. Consider a formally smooth R-algebra P and a surjection $P \to S$ with kernel J. The following are equivalent*

1. *S is formally smooth over R,*
2. *for some $P \to S$ as above there exists a section to $P/J^2 \to S$,*
3. *for all $P \to S$ as above there exists a section to $P/J^2 \to S$,*
4. *for some $P \to S$ as above the sequence $0 \to J/J^2 \to \Omega_{P/R} \otimes S \to \Omega_{S/R} \to 0$ is split exact,*
5. *for all $P \to S$ as above the sequence $0 \to J/J^2 \to \Omega_{P/R} \otimes S \to \Omega_{S/R} \to 0$ is split exact, and*
6. *the naive cotangent complex $NL_{S/R}$ is quasi-isomorphic to a projective S-module placed in degree 0.*

Proof. It is clear that (1) implies (3) implies (2), see first part of the proof of Lemma 138.5. It is also true that (3) implies (5) implies (4) and that (2) implies (4), see first part of the proof of Lemma 138.7. Finally, Lemma 138.7 applied to the canonical surjection $R[S] \to S$ (134.0.1) shows that (1) implies (6).

Assume (4) and let's prove (6). Consider the sequence of Lemma 134.4 associated to the ring maps $R \to P \to S$. By the implication (1) $\Rightarrow$ (6) proved above we see that $NL_{P/R} \otimes_R S$ is quasi-isomorphic to $\Omega_{P/R} \otimes_P S$ placed in degree 0. Hence $H_1(NL_{P/R} \otimes_P S) = 0$. Since $P \to S$ is surjective we see that $NL_{S/P}$ is homotopy equivalent to J/J^2 placed in degree 1 (Lemma 134.6). Thus we obtain the exact sequence $0 \to H_1(L_{S/R}) \to J/J^2 \to \Omega_{P/R} \otimes_P S \to \Omega_{S/R} \to 0$. By assumption we see that $H_1(L_{S/R}) = 0$ and that $\Omega_{S/R}$ is a projective S-module. Thus (6) follows.

Finally, let's prove that (6) implies (1). The assumption means that the complex $J/J^2 \to \Omega_{P/R} \otimes S$ where $P = R[S]$ and $P \to S$ is the canonical surjection (134.0.1). Hence Lemma 138.7 shows that S is formally smooth over R. ☐

031K **Lemma 138.9.** *Let $A \to B \to C$ be ring maps. Assume $B \to C$ is formally smooth. Then the sequence*

$$0 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

of Lemma 131.7 is a split short exact sequence.

Proof. Follows from Proposition 138.8 and Lemma 134.4. $\square$

06A6 **Lemma 138.10.** *Let $A \to B \to C$ be ring maps with $A \to C$ formally smooth and $B \to C$ surjective with kernel $J \subset B$. Then the exact sequence*

$$0 \to J/J^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0$$

of Lemma 131.9 is split exact.

Proof. Follows from Proposition 138.8, Lemma 134.4, and Lemma 131.9. $\square$

06A7 **Lemma 138.11.** *Let $A \to B \to C$ be ring maps. Assume $A \to C$ is surjective (so also $B \to C$ is) and $A \to B$ formally smooth. Denote $I = \mathrm{Ker}(A \to C)$ and $J = \mathrm{Ker}(B \to C)$. Then the sequence*

$$0 \to I/I^2 \to J/J^2 \to \Omega_{B/A} \otimes_B B/J \to 0$$

of Lemma 134.7 is split exact.

Proof. Since $A \to B$ is formally smooth there exists a ring map $\sigma : B \to A/I^2$ whose composition with $A \to B$ equals the quotient map $A \to A/I^2$. Then σ induces a map $J/J^2 \to I/I^2$ which is inverse to the map $I/I^2 \to J/J^2$. $\square$

031L **Lemma 138.12.** *Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal. Assume*

1. *$I^2 = 0$,*
2. *$R \to S$ is flat, and*
3. *$R/I \to S/IS$ is formally smooth.*

Then $R \to S$ is formally smooth.

Proof. Assume (1), (2) and (3). Let $P = R[\{x_t\}_{t \in T}] \to S$ be a surjection of R-algebras with kernel J. Thus $0 \to J \to P \to S \to 0$ is a short exact sequence of flat R-modules. This implies that $I \otimes_R S = IS$, $I \otimes_R P = IP$ and $I \otimes_R J = IJ$ as well as $J \cap IP = IJ$. We will use throughout the proof that

$$\Omega_{(S/IS)/(R/I)} = \Omega_{S/R} \otimes_S (S/IS) = \Omega_{S/R} \otimes_R R/I = \Omega_{S/R}/I\Omega_{S/R}$$

and similarly for P (see Lemma 131.12). By Lemma 138.7 the sequence

031M (138.12.1) $$0 \to J/(IJ + J^2) \to \Omega_{P/R} \otimes_P S/IS \to \Omega_{S/R} \otimes_S S/IS \to 0$$

is split exact. Of course the middle term is $\bigoplus_{t \in T} S/IS \mathrm{d}x_t$. Choose a splitting $\sigma : \Omega_{P/R} \otimes_P S/IS \to J/(IJ + J^2)$. For each $t \in T$ choose an element $f_t \in J$ which maps to $\sigma(\mathrm{d}x_t)$ in $J/(IJ + J^2)$. This determines a unique S-module map

$$\tilde{\sigma} : \Omega_{P/R} \otimes_R S = \bigoplus S\mathrm{d}x_t \longrightarrow J/J^2$$

with the property that $\tilde{\sigma}(\mathrm{d}x_t) = f_t$. As σ is a section to d the difference

$$\Delta = \mathrm{id}_{J/J^2} - \tilde{\sigma} \circ \mathrm{d}$$

is a self map $J/J^2 \to J/J^2$ whose image is contained in $(IJ + J^2)/J^2$. In particular $\Delta((IJ + J^2)/J^2) = 0$ because $I^2 = 0$. This means that Δ factors as

$$J/J^2 \to J/(IJ + J^2) \xrightarrow{\overline{\Delta}} (IJ + J^2)/J^2 \to J/J^2$$

where $\overline{\Delta}$ is a S/IS-module map. Using again that the sequence (138.12.1) is split, we can find a S/IS-module map $\overline{\delta} : \Omega_{P/R} \otimes_P S/IS \to (IJ + J^2)/J^2$ such that $\overline{\delta} \circ d$ is equal to $\overline{\Delta}$. In the same manner as above the map $\overline{\delta}$ determines an S-module map $\delta : \Omega_{P/R} \otimes_P S \to J/J^2$. After replacing $\tilde{\sigma}$ by $\tilde{\sigma} + \delta$ a simple computation shows that $\Delta = 0$. In other words $\tilde{\sigma}$ is a section of $J/J^2 \to \Omega_{P/R} \otimes_P S$. By Lemma 138.7 we conclude that $R \to S$ is formally smooth. $\square$

00TN **Proposition 138.13.** *Let $R \to S$ be a ring map. The following are equivalent*

1. *$R \to S$ is of finite presentation and formally smooth,*
2. *$R \to S$ is smooth.*

Proof. Follows from Proposition 138.8 and Definition 137.1. (Note that $\Omega_{S/R}$ is a finitely presented S-module if $R \to S$ is of finite presentation, see Lemma 131.15.) $\square$

00TP **Lemma 138.14.** *Let $R \to S$ be a smooth ring map. Then there exists a subring $R_0 \subset R$ of finite type over $\mathbf{Z}$ and a smooth ring map $R_0 \to S_0$ such that $S \cong R \otimes_{R_0} S_0$.*

Proof. We are going to use that smooth is equivalent to finite presentation and formally smooth, see Proposition 138.13. Write $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and denote $I = (f_1, \ldots, f_m)$. Choose a right inverse $\sigma : S \to R[x_1, \ldots, x_n]/I^2$ to the projection to S as in Lemma 138.5. Choose $h_i \in R[x_1, \ldots, x_n]$ such that $\sigma(x_i \bmod I) = h_i \bmod I^2$. The fact that σ is an R-algebra homomorphism $R[x_1, \ldots, x_n]/I \to R[x_1, \ldots, x_n]/I^2$ is equivalent to the condition that

$$f_j(h_1, \ldots, h_n) = \sum\nolimits_{j_1 j_2} a_{j_1 j_2} f_{j_1} f_{j_2}$$

for certain $a_{kl} \in R[x_1, \ldots, x_n]$. Let $R_0 \subset R$ be the subring generated over $\mathbf{Z}$ by all the coefficients of the polynomials f_j, h_i, a_{kl}. Set $S_0 = R_0[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, with $I_0 = (f_1, \ldots, f_m)$. Let $\sigma_0 : S_0 \to R_0[x_1, \ldots, x_n]/I_0^2$ defined by the rule $x_i \mapsto h_i \bmod I_0^2$; this works since the a_{lk} are defined over R_0 and satisfy the same relations. Thus by Lemma 138.5 the ring S_0 is formally smooth over R_0. $\square$

0CAQ **Lemma 138.15.** *Let $A = \operatorname{colim} A_i$ be a filtered colimit of rings. Let $A \to B$ be a smooth ring map. There exists an i and a smooth ring map $A_i \to B_i$ such that $B = B_i \otimes_{A_i} A$.*

Proof. Follows from Lemma 138.14 since $R_0 \to A$ will factor through A_i for some i by Lemma 127.3. $\square$

06CM **Lemma 138.16.** *Let $R \to S$ be a ring map. Let $R \to R'$ be a faithfully flat ring map. Set $S' = S \otimes_R R'$. Then $R \to S$ is formally smooth if and only if $R' \to S'$ is formally smooth.*

Proof. If $R \to S$ is formally smooth, then $R' \to S'$ is formally smooth by Lemma 138.2. To prove the converse, assume $R' \to S'$ is formally smooth. Note that $N \otimes_R R' = N \otimes_S S'$ for any S-module N. In particular $S \to S'$ is faithfully flat also. Choose a polynomial ring $P = R[\{x_i\}_{i \in I}]$ and a surjection of R-algebras $P \to S$ with kernel J. Note that $P' = P \otimes_R R'$ is a polynomial algebra over R'. Since $R \to R'$ is flat the kernel J' of the surjection $P' \to S'$ is $J \otimes_R R'$. Hence the split exact sequence (see Lemma 138.7)

$$0 \to J'/(J')^2 \to \Omega_{P'/R'} \otimes_{P'} S' \to \Omega_{S'/R'} \to 0$$

is the base change via $S \to S'$ of the corresponding sequence

$$J/J^2 \to \Omega_{P/R} \otimes_P S \to \Omega_{S/R} \to 0$$

see Lemma 131.9. As $S \to S'$ is faithfully flat we conclude two things: (1) this sequence (without $'$) is exact too, and (2) $\Omega_{S/R}$ is a projective S-module. Namely, $\Omega_{S'/R'}$ is projective as a direct sum of the free module $\Omega_{P'/R'} \otimes_{P'} S'$ and $\Omega_{S/R} \otimes_S S' = \Omega_{S'/R'}$ by what we said above. Thus (2) follows by descent of projectivity through faithfully flat ring maps, see Theorem 95.6. Hence the sequence $0 \to J/J^2 \to \Omega_{P/R} \otimes_P S \to \Omega_{S/R} \to 0$ is exact also and we win by applying Lemma 138.7 once more. $\square$

It turns out that smooth ring maps satisfy the following strong lifting property.

07K4 **Lemma 138.17.** *Let $R \to S$ be a smooth ring map. Given a commutative solid diagram*

$$\begin{tikzcd} S \arrow[r] \arrow[rd, dashed] & A/I \\ R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

where $I \subset A$ is a locally nilpotent ideal, a dotted arrow exists which makes the diagram commute.

Proof. By Lemma 138.14 we can extend the diagram to a commutative diagram

$$\begin{tikzcd} S_0 \arrow[r] & S \arrow[r] \arrow[rd, dashed] & A/I \\ R_0 \arrow[u] \arrow[r] & R \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

with $R_0 \to S_0$ smooth, R_0 of finite type over $\mathbf{Z}$, and $S = S_0 \otimes_{R_0} R$. Let $x_1, \ldots, x_n \in S_0$ be generators of S_0 over R_0. Let $a_1, \ldots, a_n$ be elements of A which map to the same elements in A/I as the elements $x_1, \ldots, x_n$. Denote $A_0 \subset A$ the subring generated by the image of R_0 and the elements $a_1, \ldots, a_n$. Set $I_0 = A_0 \cap I$. Then $A_0/I_0 \subset A/I$ and $S_0 \to A/I$ maps into A_0/I_0. Thus it suffices to find the dotted arrow in the diagram

$$\begin{tikzcd} S_0 \arrow[r] \arrow[rd, dashed] & A_0/I_0 \\ R_0 \arrow[u] \arrow[r] & A_0 \arrow[u] \end{tikzcd}$$

The ring A_0 is of finite type over $\mathbf{Z}$ by construction. Hence A_0 is Noetherian, whence I_0 is nilpotent, see Lemma 32.5. Say $I_0^n = 0$. By Proposition 138.13 we can successively lift the R_0-algebra map $S_0 \to A_0/I_0$ to $S_0 \to A_0/I_0^2$, $S_0 \to A_0/I_0^3$, ..., and finally $S_0 \to A_0/I_0^n = A_0$. $\square$

139. Smoothness and differentials

05D4 Some results on differentials and smooth ring maps.

04B2 **Lemma 139.1.** *Given ring maps $A \to B \to C$ with $B \to C$ smooth, then the sequence*

$$0 \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

of Lemma 131.7 is exact.

Proof. This follows from the more general Lemma 138.9 because a smooth ring map is formally smooth, see Proposition 138.13. But it also follows directly from Lemma 134.4 since $H_1(L_{C/B}) = 0$ is part of the definition of smoothness of $B \to C$. $\square$

06A8 **Lemma 139.2.** *Let $A \to B \to C$ be ring maps with $A \to C$ smooth and $B \to C$ surjective with kernel $J \subset B$. Then the exact sequence*

$$0 \to J/J^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0$$

of Lemma 131.9 is split exact.

Proof. This follows from the more general Lemma 138.10 because a smooth ring map is formally smooth, see Proposition 138.13. $\square$

06A9 **Lemma 139.3.** *Let $A \to B \to C$ be ring maps. Assume $A \to C$ is surjective (so also $B \to C$ is) and $A \to B$ smooth. Denote $I = \mathrm{Ker}(A \to C)$ and $J = \mathrm{Ker}(B \to C)$. Then the sequence*

$$0 \to I/I^2 \to J/J^2 \to \Omega_{B/A} \otimes_B B/J \to 0$$

of Lemma 134.7 is exact.

Proof. This follows from the more general Lemma 138.11 because a smooth ring map is formally smooth, see Proposition 138.13. $\square$

05D5 **Lemma 139.4.** *Let $\varphi : R \to S$ be a smooth ring map. Let $\sigma : S \to R$ be a left inverse to φ. Set $I = \mathrm{Ker}(\sigma)$. Then*

1. *I/I^2 is a finite locally free R-module, and*
2. *if I/I^2 is free, then $S^\wedge \cong R[[t_1, \ldots, t_d]]$ as R-algebras, where $S^\wedge$ is the I-adic completion of S.*

Proof. By Lemma 131.10 applied to $R \to S \to R$ we see that $I/I^2 = \Omega_{S/R} \otimes_{S,\sigma} R$. Since by definition of a smooth morphism the module $\Omega_{S/R}$ is finite locally free over S we deduce that (1) holds. If I/I^2 is free, then choose $f_1, \ldots, f_d \in I$ whose images in I/I^2 form an R-basis. Consider the R-algebra map defined by

$$\Psi : R[[x_1, \ldots, x_d]] \longrightarrow S^\wedge, \quad x_i \longmapsto f_i.$$

Denote $P = R[[x_1, \ldots, x_d]]$ and $J = (x_1, \ldots, x_d) \subset P$. We write $\Psi_n : P/J^n \to S/I^n$ for the induced map of quotient rings. Note that $S/I^2 = \varphi(R) \oplus I/I^2$. Thus Ψ_2 is an isomorphism. Denote $\sigma_2 : S/I^2 \to P/J^2$ the inverse of Ψ_2. We will prove by induction on n that for all $n > 2$ there exists an inverse $\sigma_n : S/I^n \to P/J^n$ of Ψ_n. Namely, as S is formally smooth over R (by Proposition 138.13) we see that in the solid diagram

$$\begin{tikzcd} S \arrow[r, dotted] \arrow[dr, "\sigma_{n-1}"'] & P/J^n \arrow[d] \\ & P/J^{n-1} \end{tikzcd}$$

of R-algebras we can fill in the dotted arrow by some R-algebra map $\tau : S \to P/J^n$ making the diagram commute. This induces an R-algebra map $\overline{\tau} : S/I^n \to P/J^n$ which is equal to σ_{n-1} modulo J^n. By construction the map Ψ_n is surjective and now $\overline{\tau} \circ \Psi_n$ is an R-algebra endomorphism of P/J^n which maps x_i to $x_i + \delta_{i,n}$ with $\delta_{i,n} \in J^{n-1}/J^n$. It follows that Ψ_n is an isomorphism and hence it has an inverse σ_n. This proves the lemma. $\square$

140. Smooth algebras over fields

00TQ Warning: The following two lemmas do not hold over nonperfect fields in general.

00TR **Lemma 140.1.** *Let k be an algebraically closed field. Let S be a finite type k-algebra. Let $\mathfrak{m} \subset S$ be a maximal ideal. Then*

$$\dim_{\kappa(\mathfrak{m})} \Omega_{S/k} \otimes_S \kappa(\mathfrak{m}) = \dim_{\kappa(\mathfrak{m})} \mathfrak{m}/\mathfrak{m}^2.$$

Proof. Consider the exact sequence

$$\mathfrak{m}/\mathfrak{m}^2 \to \Omega_{S/k} \otimes_S \kappa(\mathfrak{m}) \to \Omega_{\kappa(\mathfrak{m})/k} \to 0$$

of Lemma 131.9. We would like to show that the first map is an isomorphism. Since k is algebraically closed the composition $k \to \kappa(\mathfrak{m})$ is an isomorphism by Theorem 34.1. So the surjection $S \to \kappa(\mathfrak{m})$ splits as a map of k-algebras, and Lemma 131.10 shows that the sequence above is exact on the left. Since $\Omega_{\kappa(\mathfrak{m})/k} = 0$, we win. □

00TS **Lemma 140.2.** *Let k be an algebraically closed field. Let S be a finite type k-algebra. Let $\mathfrak{m} \subset S$ be a maximal ideal. The following are equivalent:*

1. *The ring $S_\mathfrak{m}$ is a regular local ring.*
2. *We have $\dim_{\kappa(\mathfrak{m})} \Omega_{S/k} \otimes_S \kappa(\mathfrak{m}) \leq \dim(S_\mathfrak{m})$.*
3. *We have $\dim_{\kappa(\mathfrak{m})} \Omega_{S/k} \otimes_S \kappa(\mathfrak{m}) = \dim(S_\mathfrak{m})$.*
4. *There exists a $g \in S$, $g \notin \mathfrak{m}$ such that S_g is smooth over k. In other words S/k is smooth at $\mathfrak{m}$.*

Proof. Note that (1), (2) and (3) are equivalent by Lemma 140.1 and Definition 110.7.

Assume that S is smooth at $\mathfrak{m}$. By Lemma 137.10 we see that S_g is standard smooth over k for a suitable $g \in S$, $g \notin \mathfrak{m}$. Hence by Lemma 137.7 we see that $\Omega_{S_g/k}$ is free of rank $\dim(S_g)$. Hence by Lemma 140.1 we see that $\dim(S_\mathfrak{m}) = \dim(\mathfrak{m}/\mathfrak{m}^2)$ in other words $S_\mathfrak{m}$ is regular.

Conversely, suppose that $S_\mathfrak{m}$ is regular. Let $d = \dim(S_\mathfrak{m}) = \dim \mathfrak{m}/\mathfrak{m}^2$. Choose a presentation $S = k[x_1, \ldots, x_n]/I$ such that x_i maps to an element of $\mathfrak{m}$ for all i. In other words, $\mathfrak{m}'' = (x_1, \ldots, x_n)$ is the corresponding maximal ideal of $k[x_1, \ldots, x_n]$. Note that we have a short exact sequence

$$I/\mathfrak{m}''I \to \mathfrak{m}''/(\mathfrak{m}'')^2 \to \mathfrak{m}/(\mathfrak{m})^2 \to 0$$

Pick $c = n - d$ elements $f_1, \ldots, f_c \in I$ such that their images in $\mathfrak{m}''/(\mathfrak{m}'')^2$ span the kernel of the map to $\mathfrak{m}/\mathfrak{m}^2$. This is clearly possible. Denote $J = (f_1, \ldots, f_c)$. So $J \subset I$. Denote $S' = k[x_1, \ldots, x_n]/J$ so there is a surjection $S' \to S$. Denote $\mathfrak{m}' = \mathfrak{m}''S'$ the corresponding maximal ideal of S'. Hence we have

$$\begin{tikzcd} k[x_1, \ldots, x_n] \arrow[r] & S' \arrow[r] & S \\ \mathfrak{m}'' \arrow[r] \arrow[u] & \mathfrak{m}' \arrow[r] \arrow[u] & \mathfrak{m} \arrow[u] \end{tikzcd}$$

By our choice of J the exact sequence

$$J/\mathfrak{m}''J \to \mathfrak{m}''/(\mathfrak{m}'')^2 \to \mathfrak{m}'/(\mathfrak{m}')^2 \to 0$$

shows that $\dim(\mathfrak{m}'/(\mathfrak{m}')^2) = d$. Since $S'_{\mathfrak{m}'}$ surjects onto $S_\mathfrak{m}$ we see that $\dim(S_{\mathfrak{m}'}) \geq d$. Hence by the discussion preceding Definition 60.10 we conclude that $S'_{\mathfrak{m}'}$ is regular of dimension d as well. Because S' was cut out by $c = n - d$ equations we conclude that there exists a $g' \in S'$, $g' \notin \mathfrak{m}'$ such that $S'_{g'}$ is a global complete intersection over k, see Lemma 135.4. Also the map $S'_{\mathfrak{m}'} \to S_{\mathfrak{m}}$ is a surjection of Noetherian local domains of the same dimension and hence an isomorphism. Hence $S' \to S$ is surjective with finitely generated kernel and becomes an isomorphism after localizing at $\mathfrak{m}'$. Thus we can find $g' \in S'$, $g \notin \mathfrak{m}'$ such that $S'_{g'} \to S_{g'}$ is an isomorphism. All in all we conclude that after replacing S by a principal localization we may assume that S is a global complete intersection.

At this point we may write $S = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ with $\dim S = n - c$. Recall that the naive cotangent complex of this algebra is given by

$$\bigoplus S \cdot f_j \to \bigoplus S \cdot \mathrm{d}x_i$$

see Lemma 136.13. By Lemma 137.16 in order to show that S is smooth at $\mathfrak{m}$ we have to show that one of the $c \times c$ minors g_I of the matrix "A" giving the map above does not vanish at $\mathfrak{m}$. By Lemma 140.1 the matrix A mod $\mathfrak{m}$ has rank c. Thus we win. ☐

00TT **Lemma 140.3.** *Let k be any field. Let S be a finite type k-algebra. Let $X = \mathrm{Spec}(S)$. Let $\mathfrak{q} \subset S$ be a prime corresponding to $x \in X$. The following are equivalent:*

1. *The k-algebra S is smooth at $\mathfrak{q}$ over k.*
2. *We have $\dim_{\kappa(\mathfrak{q})} \Omega_{S/k} \otimes_S \kappa(\mathfrak{q}) \leq \dim_x X$.*
3. *We have $\dim_{\kappa(\mathfrak{q})} \Omega_{S/k} \otimes_S \kappa(\mathfrak{q}) = \dim_x X$.*

Moreover, in this case the local ring $S_{\mathfrak{q}}$ is regular.

Proof. If S is smooth at $\mathfrak{q}$ over k, then there exists a $g \in S$, $g \notin \mathfrak{q}$ such that S_g is standard smooth over k, see Lemma 137.10. A standard smooth algebra over k has a module of differentials which is free of rank equal to the dimension, see Lemma 137.7 (use that a relative global complete intersection over a field has dimension equal to the number of variables minus the number of equations). Thus we see that (1) implies (3). To finish the proof of the lemma it suffices to show that (2) implies (1) and that it implies that $S_{\mathfrak{q}}$ is regular.

Assume (2). By Nakayama's Lemma 20.1 we see that $\Omega_{S/k,\mathfrak{q}}$ can be generated by $\leq \dim_x X$ elements. We may replace S by S_g for some $g \in S$, $g \notin \mathfrak{q}$ such that $\Omega_{S/k}$ is generated by at most $\dim_x X$ elements. Let K/k be an algebraically closed field extension such that there exists a k-algebra map $\psi : \kappa(\mathfrak{q}) \to K$. Consider $S_K = K \otimes_k S$. Let $\mathfrak{m} \subset S_K$ be the maximal ideal corresponding to the surjection

$$S_K = K \otimes_k S \longrightarrow K \otimes_k \kappa(\mathfrak{q}) \xrightarrow{\mathrm{id}_K \otimes \psi} K.$$

Note that $\mathfrak{m} \cap S = \mathfrak{q}$, in other words $\mathfrak{m}$ lies over $\mathfrak{q}$. By Lemma 116.6 the dimension of $X_K = \mathrm{Spec}(S_K)$ at the point corresponding to $\mathfrak{m}$ is $\dim_x X$. By Lemma 114.6 this is equal to $\dim((S_K)_{\mathfrak{m}})$. By Lemma 131.12 the module of differentials of S_K over K is the base change of $\Omega_{S/k}$, hence also generated by at most $\dim_x X = \dim((S_K)_{\mathfrak{m}})$ elements. By Lemma 140.2 we see that S_K is smooth at $\mathfrak{m}$ over K. By Lemma 137.18 this implies that S is smooth at $\mathfrak{q}$ over k. This proves (1). Moreover, we know by Lemma 140.2 that the local ring $(S_K)_{\mathfrak{m}}$ is regular. Since $S_{\mathfrak{q}} \to (S_K)_{\mathfrak{m}}$ is flat we conclude from Lemma 110.9 that $S_{\mathfrak{q}}$ is regular. ☐

The following lemma can be significantly generalized (in several different ways).

00TU **Lemma 140.4.** *Let k be a field. Let R be a Noetherian local ring containing k. Assume that the residue field $\kappa = R/\mathfrak{m}$ is a finitely generated separable extension of k. Then the map*
$$d : \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \Omega_{R/k} \otimes_R \kappa(\mathfrak{m})$$
is injective.

Proof. We may replace R by $R/\mathfrak{m}^2$. Hence we may assume that $\mathfrak{m}^2 = 0$. By assumption we may write $\kappa = k(\overline{x}_1, \ldots, \overline{x}_r, \overline{y})$ where $\overline{x}_1, \ldots, \overline{x}_r$ is a transcendence basis of κ over k and $\overline{y}$ is separable algebraic over $k(\overline{x}_1, \ldots, \overline{x}_r)$. Say its minimal equation is $P(\overline{y}) = 0$ with $P(T) = T^d + \sum_{i<d} a_i T^i$, with $a_i \in k(\overline{x}_1, \ldots, \overline{x}_r)$ and $P'(\overline{y}) \neq 0$. Choose any lifts $x_i \in R$ of the elements $\overline{x}_i \in \kappa$. This gives a commutative diagram

$$\begin{tikzcd} R \arrow[r] & \kappa \\ & k(\overline{x}_1, \ldots, \overline{x}_r) \arrow[u] \arrow[ul, "\varphi"] \end{tikzcd}$$

of k-algebras. We want to extend the left upwards arrow φ to a k-algebra map from κ to R. To do this choose any $y \in R$ lifting $\overline{y}$. To see that it defines a k-algebra map defined on $\kappa \cong k(\overline{x}_1, \ldots, \overline{x}_r)[T]/(P)$ all we have to show is that we may choose y such that $P^\varphi(y) = 0$. If not then we compute for $\delta \in \mathfrak{m}$ that
$$P(y + \delta) = P(y) + P'(y)\delta$$
because $\mathfrak{m}^2 = 0$. Since $P'(y)\delta = P'(\overline{y})\delta$ we see that we can adjust our choice as desired. This shows that $R \cong \kappa \oplus \mathfrak{m}$ as k-algebras! From a direct computation of $\Omega_{\kappa \oplus \mathfrak{m}/k}$ the lemma follows. $\square$

00TV **Lemma 140.5.** *Let k be a field. Let S be a finite type k-algebra. Let $\mathfrak{q} \subset S$ be a prime. Assume $\kappa(\mathfrak{q})$ is separable over k. The following are equivalent:*

(1) *The algebra S is smooth at $\mathfrak{q}$ over k.*
(2) *The ring $S_\mathfrak{q}$ is regular.*

Proof. Denote $R = S_\mathfrak{q}$ and denote its maximal by $\mathfrak{m}$ and its residue field κ. By Lemma 140.4 and 131.9 we see that there is a short exact sequence
$$0 \to \mathfrak{m}/\mathfrak{m}^2 \to \Omega_{R/k} \otimes_R \kappa \to \Omega_{\kappa/k} \to 0$$
Note that $\Omega_{R/k} = \Omega_{S/k,\mathfrak{q}}$, see Lemma 131.8. Moreover, since κ is separable over k we have $\dim_\kappa \Omega_{\kappa/k} = \text{trdeg}_k(\kappa)$. Hence we get
$$\dim_\kappa \Omega_{R/k} \otimes_R \kappa = \dim_\kappa \mathfrak{m}/\mathfrak{m}^2 + \text{trdeg}_k(\kappa) \geq \dim R + \text{trdeg}_k(\kappa) = \dim_\mathfrak{q} S$$
(see Lemma 116.3 for the last equality) with equality if and only if R is regular. Thus we win by applying Lemma 140.3. $\square$

00TW **Lemma 140.6.** *Let $R \to S$ be a $\mathbf{Q}$-algebra map. Let $f \in S$ be such that $\Omega_{S/R} = S df \oplus C$ for some S-submodule C. Then*

(1) *f is not nilpotent, and*
(2) *if S is a Noetherian local ring, then f is a nonzerodivisor in S.*

Proof. For $a \in S$ write $\mathrm{d}(a) = \theta(a)\mathrm{d}f + c(a)$ for some $\theta(a) \in S$ and $c(a) \in C$. Consider the R-derivation $S \to S$, $a \mapsto \theta(a)$. Note that $\theta(f) = 1$.

If $f^n = 0$ with $n > 1$ minimal, then $0 = \theta(f^n) = nf^{n-1}$ contradicting the minimality of n. We conclude that f is not nilpotent.

Suppose $fa = 0$. If f is a unit then $a = 0$ and we win. Assume f is not a unit. Then $0 = \theta(fa) = f\theta(a) + a$ by the Leibniz rule and hence $a \in (f)$. By induction suppose we have shown $fa = 0 \Rightarrow a \in (f^n)$. Then writing $a = f^n b$ we get $0 = \theta(f^{n+1}b) = (n+1)f^n b + f^{n+1}\theta(b)$. Hence $a = f^n b = -f^{n+1}\theta(b)/(n+1) \in (f^{n+1})$. Since in the Noetherian local ring S we have $\bigcap(f^n) = 0$, see Lemma 51.4 we win. □

The following is probably quite useless in applications.

00TX **Lemma 140.7.** *Let k be a field of characteristic 0. Let S be a finite type k-algebra. Let $\mathfrak{q} \subset S$ be a prime. The following are equivalent:*

1. *The algebra S is smooth at $\mathfrak{q}$ over k.*
2. *The $S_\mathfrak{q}$-module $\Omega_{S/k,\mathfrak{q}}$ is (finite) free.*
3. *The ring $S_\mathfrak{q}$ is regular.*

Proof. In characteristic zero any field extension is separable and hence the equivalence of (1) and (3) follows from Lemma 140.5. Also (1) implies (2) by definition of smooth algebras. Assume that $\Omega_{S/k,\mathfrak{q}}$ is free over $S_\mathfrak{q}$. We are going to use the notation and observations made in the proof of Lemma 140.5. So $R = S_\mathfrak{q}$ with maximal ideal $\mathfrak{m}$ and residue field κ. Our goal is to prove R is regular.

If $\mathfrak{m}/\mathfrak{m}^2 = 0$, then $\mathfrak{m} = 0$ and $R \cong \kappa$. Hence R is regular and we win.

If $\mathfrak{m}/\mathfrak{m}^2 \neq 0$, then choose any $f \in \mathfrak{m}$ whose image in $\mathfrak{m}/\mathfrak{m}^2$ is not zero. By Lemma 140.4 we see that $\mathrm{d}f$ has nonzero image in $\Omega_{R/k}/\mathfrak{m}\Omega_{R/k}$. By assumption $\Omega_{R/k} = \Omega_{S/k,\mathfrak{q}}$ is finite free and hence by Nakayama's Lemma 20.1 we see that $\mathrm{d}f$ generates a direct summand. We apply Lemma 140.6 to deduce that f is a nonzerodivisor in R. Furthermore, by Lemma 131.9 we get an exact sequence

$$(f)/(f^2) \to \Omega_{R/k} \otimes_R R/fR \to \Omega_{(R/fR)/k} \to 0$$

This implies that $\Omega_{(R/fR)/k}$ is finite free as well. Hence by induction we see that R/fR is a regular local ring. Since $f \in \mathfrak{m}$ was a nonzerodivisor we conclude that R is regular, see Lemma 106.7. □

00TY **Example 140.8.** Lemma 140.7 does not hold in characteristic $p > 0$. The standard examples are the ring maps

$$\mathbf{F}_p \longrightarrow \mathbf{F}_p[x]/(x^p)$$

whose module of differentials is free but is clearly not smooth, and the ring map ($p > 2$)

$$\mathbf{F}_p(t) \to \mathbf{F}_p(t)[x,y]/(x^p + y^2 + \alpha)$$

which is not smooth at the prime $\mathfrak{q} = (y, x^p + \alpha)$ but is regular.

Using the material above we can characterize smoothness at the generic point in terms of field extensions.

07ND **Lemma 140.9.** *Let $R \to S$ be an injective finite type ring map with R and S domains. Then $R \to S$ is smooth at $\mathfrak{q} = (0)$ if and only if the induced extension L/K of fraction fields is separable.*

Proof. Assume $R \to S$ is smooth at (0). We may replace S by S_g for some nonzero $g \in S$ and assume that $R \to S$ is smooth. Then $K \to S \otimes_R K$ is smooth (Lemma 137.4). Moreover, for any field extension K'/K the ring map $K' \to S \otimes_R K'$ is smooth as well. Hence $S \otimes_R K'$ is a regular ring by Lemma 140.3, in particular reduced. It follows that $S \otimes_R K$ is a geometrically reduced over K. Hence L is geometrically reduced over K, see Lemma 43.3. Hence L/K is separable by Lemma 44.1.

Conversely, assume that L/K is separable. We may assume $R \to S$ is of finite presentation, see Lemma 30.1. It suffices to prove that $K \to S \otimes_R K$ is smooth at (0), see Lemma 137.18. This follows from Lemma 140.5, the fact that a field is a regular ring, and the assumption that L/K is separable. □

141. Smooth ring maps in the Noetherian case

02HR

02HS **Definition 141.1.** Let $\varphi : B' \to B$ be a ring map. We say φ is a *small extension* if B' and B are local Artinian rings, φ is surjective and $I = \mathrm{Ker}(\varphi)$ has length 1 as a B'-module.

Clearly this means that $I^2 = 0$ and that $I = (x)$ for some $x \in B'$ such that $\mathfrak{m}'x = 0$ where $\mathfrak{m}' \subset B'$ is the maximal ideal.

02HT **Lemma 141.2.** *Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime ideal of S lying over $\mathfrak{p} \subset R$. Assume R is Noetherian and $R \to S$ of finite type. The following are equivalent:*

1. *$R \to S$ is smooth at $\mathfrak{q}$,*
2. *for every surjection of local R-algebras $(B', \mathfrak{m}') \to (B, \mathfrak{m})$ with $\mathrm{Ker}(B' \to B)$ having square zero and every solid commutative diagram*

$$\begin{tikzcd} S \arrow[r] \arrow[rd, dashed] & B \\ R \arrow[u] \arrow[r] & B' \arrow[u] \end{tikzcd}$$

   such that $\mathfrak{q} = S \cap \mathfrak{m}$ there exists a dotted arrow making the diagram commute,

3. *same as in (2) but with $B' \to B$ ranging over small extensions, and*
4. *same as in (2) but with $B' \to B$ ranging over small extensions such that in addition $S \to B$ induces an isomorphism $\kappa(\mathfrak{q}) \cong \kappa(\mathfrak{m})$.*

Proof. Assume (1). This means there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is smooth. By Proposition 138.13 we know that $R \to S_g$ is formally smooth. Note that given any diagram as in (2) the map $S \to B$ factors automatically through $S_\mathfrak{q}$ and a fortiori through S_g. The formal smoothness of S_g over R gives us a morphism $S_g \to B'$ fitting into a similar diagram with S_g at the upper left corner. Composing with $S \to S_g$ gives the desired arrow. In other words, we have shown that (1) implies (2).

Clearly (2) implies (3) and (3) implies (4).

Assume (4). We are going to show that (1) holds, thereby finishing the proof of the lemma. Choose a presentation $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. This is possible as S is of finite type over R and therefore of finite presentation (see Lemma 31.4). Set $I = (f_1, \ldots, f_m)$. Consider the naive cotangent complex

$$\mathrm{d} : I/I^2 \longrightarrow \bigoplus\nolimits_{j=1}^{m} S\mathrm{d}x_j$$

of this presentation (see Section 134). It suffices to show that when we localize this complex at $\mathfrak q$ then the map becomes a split injection, see Lemma 137.12. Denote $S' = R[x_1, \ldots, x_n]/I^2$. By Lemma 131.11 we have

$$S \otimes_{S'} \Omega_{S'/R} = S \otimes_{R[x_1,\ldots,x_n]} \Omega_{R[x_1,\ldots,x_n]/R} = \bigoplus\nolimits_{j=1}^{m} S\mathrm{d}x_j.$$

Thus the map

$$\mathrm{d} : I/I^2 \longrightarrow S \otimes_{S'} \Omega_{S'/R}$$

is the same as the map in the naive cotangent complex above. In particular the truth of the assertion we are trying to prove depends only on the three rings $R \to S' \to S$. Let $\mathfrak q' \subset R[x_1, \ldots, x_n]$ be the prime ideal corresponding to $\mathfrak q$. Since localization commutes with taking modules of differentials (Lemma 131.8) we see that it suffices to show that the map

02HU (141.2.1)
$$\mathrm{d} : I_{\mathfrak q'}/I_{\mathfrak q'}^2 \longrightarrow S_{\mathfrak q} \otimes_{S'_{\mathfrak q'}} \Omega_{S'_{\mathfrak q'}/R}$$

coming from $R \to S'_{\mathfrak q'} \to S_{\mathfrak q}$ is a split injection.

Let $N \in \mathbf{N}$ be an integer. Consider the ring

$$B'_N = S'_{\mathfrak q'}/(\mathfrak q')^N S'_{\mathfrak q'} = (S'/(\mathfrak q')^N S')_{\mathfrak q'}$$

and its quotient $B_N = B'_N/IB'_N$. Note that $B_N \cong S_{\mathfrak q}/\mathfrak q^N S_{\mathfrak q}$. Observe that B'_N is an Artinian local ring since it is the quotient of a local Noetherian ring by a power of its maximal ideal. Consider a filtration of the kernel I_N of $B'_N \to B_N$ by B'_N-submodules

$$0 \subset J_{N,1} \subset J_{N,2} \subset \ldots \subset J_{N,n(N)} = I_N$$

such that each successive quotient $J_{N,i}/J_{N,i-1}$ has length 1. (As B'_N is Artinian such a filtration exists.) This gives a sequence of small extensions

$$B'_N \to B'_N/J_{N,1} \to B'_N/J_{N,2} \to \ldots \to B'_N/J_{N,n(N)} = B'_N/I_N = B_N = S_{\mathfrak q}/\mathfrak q^N S_{\mathfrak q}$$

Applying condition (4) successively to these small extensions starting with the map $S \to B_N$ we see there exists a commutative diagram

$$\begin{tikzcd} S \arrow[r] \arrow[dr] & B_N \\ R \arrow[u] \arrow[r] & B'_N \arrow[u] \end{tikzcd}$$

Clearly the ring map $S \to B'_N$ factors as $S \to S_{\mathfrak q} \to B'_N$ where $S_{\mathfrak q} \to B'_N$ is a local homomorphism of local rings. Moreover, since the maximal ideal of B'_N to the Nth power is zero we conclude that $S_{\mathfrak q} \to B'_N$ factors through $S_{\mathfrak q}/(\mathfrak q)^N S_{\mathfrak q} = B_N$. In other words we have shown that for all $N \in \mathbf{N}$ the surjection of R-algebras $B'_N \to B_N$ has a splitting.

Consider the presentation

$$I_N \to B_N \otimes_{B'_N} \Omega_{B'_N/R} \to \Omega_{B_N/R} \to 0$$

coming from the surjection $B'_N \to B_N$ with kernel I_N (see Lemma 131.9). By the above the R-algebra map $B'_N \to B_N$ has a right inverse. Hence by Lemma 131.10 we see that the sequence above is split exact! Thus for every N the map

$$I_N \longrightarrow B_N \otimes_{B'_N} \Omega_{B'_N/R}$$

is a split injection. The rest of the proof is gotten by unwinding what this means exactly. Note that

$$I_N = I_{\mathfrak q'}/(I_{\mathfrak q'}^2 + (\mathfrak q')^N \cap I_{\mathfrak q'})$$

By Artin-Rees (Lemma 51.2) we find a $c \geq 0$ such that

$$S_{\mathfrak q}/\mathfrak q^{N-c}S_{\mathfrak q} \otimes_{S_{\mathfrak q}} I_N = S_{\mathfrak q}/\mathfrak q^{N-c}S_{\mathfrak q} \otimes_{S_{\mathfrak q}} I_{\mathfrak q'}/I_{\mathfrak q'}^2$$

for all $N \geq c$ (these tensor product are just a fancy way of dividing by $\mathfrak q^{N-c}$). We may of course assume $c \geq 1$. By Lemma 131.11 we see that

$$S'_{\mathfrak q'}/(\mathfrak q')^{N-c}S'_{\mathfrak q'} \otimes_{S'_{\mathfrak q'}} \Omega_{B'_N/R} = S'_{\mathfrak q'}/(\mathfrak q')^{N-c}S'_{\mathfrak q'} \otimes_{S'_{\mathfrak q'}} \Omega_{S'_{\mathfrak q'}/R}$$

we can further tensor this by $B_N = S_{\mathfrak q}/\mathfrak q^N$ to see that

$$S_{\mathfrak q}/\mathfrak q^{N-c}S_{\mathfrak q} \otimes_{S'_{\mathfrak q'}} \Omega_{B'_N/R} = S_{\mathfrak q}/\mathfrak q^{N-c}S_{\mathfrak q} \otimes_{S'_{\mathfrak q'}} \Omega_{S'_{\mathfrak q'}/R}.$$

Since a split injection remains a split injection after tensoring with anything we see that

$$S_{\mathfrak q}/\mathfrak q^{N-c}S_{\mathfrak q} \otimes_{S_{\mathfrak q}} (141.2.1) = S_{\mathfrak q}/\mathfrak q^{N-c}S_{\mathfrak q} \otimes_{S_{\mathfrak q}/\mathfrak q^N S_{\mathfrak q}} (I_N \longrightarrow B_N \otimes_{B'_N} \Omega_{B'_N/R})$$

is a split injection for all $N \geq c$. By Lemma 74.1 we see that (141.2.1) is a split injection. This finishes the proof. □

142. Overview of results on smooth ring maps

00TZ Here is a list of results on smooth ring maps that we proved in the preceding sections. For more precise statements and definitions please consult the references given.

(1) A ring map $R \to S$ is smooth if it is of finite presentation and the naive cotangent complex of S/R is quasi-isomorphic to a finite projective S-module in degree 0, see Definition 137.1.
(2) If S is smooth over R, then $\Omega_{S/R}$ is a finite projective S-module, see discussion following Definition 137.1.
(3) The property of being smooth is local on S, see Lemma 137.13.
(4) The property of being smooth is stable under base change, see Lemma 137.4.
(5) The property of being smooth is stable under composition, see Lemma 137.14.
(6) A smooth ring map is syntomic, in particular flat, see Lemma 137.10.
(7) A finitely presented, flat ring map with smooth fibre rings is smooth, see Lemma 137.17.
(8) A finitely presented ring map $R \to S$ is smooth if and only if it is formally smooth, see Proposition 138.13.
(9) If $R \to S$ is a finite type ring map with R Noetherian then to check that $R \to S$ is smooth it suffices to check the lifting property of formal smoothness along small extensions of Artinian local rings, see Lemma 141.2.

(10) A smooth ring map $R \to S$ is the base change of a smooth ring map $R_0 \to S_0$ with R_0 of finite type over $\mathbf{Z}$, see Lemma 138.14.
(11) Formation of the set of points where a ring map is smooth commutes with flat base change, see Lemma 137.18.
(12) If S is of finite type over an algebraically closed field k, and $\mathfrak{m} \subset S$ a maximal ideal, then the following are equivalent
    (a) S is smooth over k in a neighbourhood of $\mathfrak{m}$,
    (b) $S_{\mathfrak{m}}$ is a regular local ring,
    (c) $\dim(S_{\mathfrak{m}}) = \dim_{\kappa(m)} \Omega_{S/k} \otimes_S \kappa(\mathfrak{m})$.

    see Lemma 140.2.
(13) If S is of finite type over a field k, and $\mathfrak{q} \subset S$ a prime ideal, then the following are equivalent
    (a) S is smooth over k in a neighbourhood of $\mathfrak{q}$,
    (b) $\dim_{\mathfrak{q}}(S/k) = \dim_{\kappa(\mathfrak{q})} \Omega_{S/k} \otimes_S \kappa(\mathfrak{q})$.

    see Lemma 140.3.
(14) If S is smooth over a field, then all its local rings are regular, see Lemma 140.3.
(15) If S is of finite type over a field k, $\mathfrak{q} \subset S$ a prime ideal, the field extension $\kappa(\mathfrak{q})/k$ is separable and $S_{\mathfrak{q}}$ is regular, then S is smooth over k at $\mathfrak{q}$, see Lemma 140.5.
(16) If S is of finite type over a field k, if k has characteristic 0, if $\mathfrak{q} \subset S$ a prime ideal, and if $\Omega_{S/k,\mathfrak{q}}$ is free, then S is smooth over k at $\mathfrak{q}$, see Lemma 140.7.

Some of these results were proved using the notion of a standard smooth ring map, see Definition 137.6. This is the analogue of what a relative global complete intersection map is for the case of syntomic morphisms. It is also the easiest way to make examples.

143. Étale ring maps

00U0 An étale ring map is a smooth ring map whose relative dimension is equal to zero. This is the same as the following slightly more direct definition.

00U1 **Definition 143.1.** Let $R \to S$ be a ring map. We say $R \to S$ is *étale* if it is of finite presentation and the naive cotangent complex $NL_{S/R}$ is quasi-isomorphic to zero. Given a prime $\mathfrak{q}$ of S we say that $R \to S$ is *étale at* $\mathfrak{q}$ if there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is étale.

In particular we see that $\Omega_{S/R} = 0$ if S is étale over R. If $R \to S$ is smooth, then $R \to S$ is étale if and only if $\Omega_{S/R} = 0$. From our results on smooth ring maps we automatically get a whole host of results for étale maps. We summarize these in Lemma 143.3 below. But before we do so we prove that *any* étale ring map is standard smooth.

00U9 **Lemma 143.2.** *Any étale ring map is standard smooth. More precisely, if $R \to S$ is étale, then there exists a presentation $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ such that the image of $\det(\partial f_j/\partial x_i)$ is invertible in S.*

Proof. Let $R \to S$ be étale. Choose a presentation $S = R[x_1, \ldots, x_n]/I$. As $R \to S$ is étale we know that

$$\mathrm{d} : I/I^2 \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} S\mathrm{d}x_i$$

is an isomorphism, in particular I/I^2 is a free S-module. Thus by Lemma 136.6 we may assume (after possibly changing the presentation), that $I = (f_1, \ldots, f_c)$ such that the classes f_i mod I^2 form a basis of I/I^2. It follows immediately from the fact that the displayed map above is an isomorphism that $c = n$ and that $\det(\partial f_j/\partial x_i)$ is invertible in S. □

00U2 **Lemma 143.3.** *Results on étale ring maps.*

(1) *The ring map $R \to R_f$ is étale for any ring R and any $f \in R$.*
(2) *Compositions of étale ring maps are étale.*
(3) *A base change of an étale ring map is étale.*
(4) *The property of being étale is local: Given a ring map $R \to S$ and elements $g_1, \ldots, g_m \in S$ which generate the unit ideal such that $R \to S_{g_j}$ is étale for $j = 1, \ldots, m$ then $R \to S$ is étale.*
(5) *Given $R \to S$ of finite presentation, and a flat ring map $R \to R'$, set $S' = R' \otimes_R S$. The set of primes where $R' \to S'$ is étale is the inverse image via $\mathrm{Spec}(S') \to \mathrm{Spec}(S)$ of the set of primes where $R \to S$ is étale.*
(6) *An étale ring map is syntomic, in particular flat.*
(7) *If S is finite type over a field k, then S is étale over k if and only if $\Omega_{S/k} = 0$.*
(8) *Any étale ring map $R \to S$ is the base change of an étale ring map $R_0 \to S_0$ with R_0 of finite type over $\mathbf{Z}$.*
(9) *Let $A = \mathrm{colim}\, A_i$ be a filtered colimit of rings. Let $A \to B$ be an étale ring map. Then there exists an étale ring map $A_i \to B_i$ for some i such that $B \cong A \otimes_{A_i} B_i$.*
(10) *Let A be a ring. Let S be a multiplicative subset of A. Let $S^{-1}A \to B'$ be étale. Then there exists an étale ring map $A \to B$ such that $B' \cong S^{-1}B$.*
(11) *Let A be a ring. Let $B = B' \times B''$ be a product of A-algebras. Then B is étale over A if and only if both B' and B'' are étale over A.*

Proof. In each case we use the corresponding result for smooth ring maps with a small argument added to show that $\Omega_{S/R}$ is zero.

Proof of (1). The ring map $R \to R_f$ is smooth and $\Omega_{R_f/R} = 0$.

Proof of (2). The composition $A \to C$ of smooth maps $A \to B$ and $B \to C$ is smooth, see Lemma 137.14. By Lemma 131.7 we see that $\Omega_{C/A}$ is zero as both $\Omega_{C/B}$ and $\Omega_{B/A}$ are zero.

Proof of (3). Let $R \to S$ be étale and $R \to R'$ be arbitrary. Then $R' \to S' = R' \otimes_R S$ is smooth, see Lemma 137.4. Since $\Omega_{S'/R'} = S' \otimes_S \Omega_{S/R}$ by Lemma 131.12 we conclude that $\Omega_{S'/R'} = 0$. Hence $R' \to S'$ is étale.

Proof of (4). Assume the hypotheses of (4). By Lemma 137.13 we see that $R \to S$ is smooth. We are also given that $\Omega_{S_{g_i}/R} = (\Omega_{S/R})_{g_i} = 0$ for all i. Then $\Omega_{S/R} = 0$, see Lemma 23.2.

Proof of (5). The result for smooth maps is Lemma 137.18. In the proof of that lemma we used that $NL_{S/R} \otimes_S S'$ is homotopy equivalent to $NL_{S'/R'}$. This reduces us to showing that if M is a finitely presented S-module the set of primes $\mathfrak{q}'$ of S' such that $(M \otimes_S S')_{\mathfrak{q}'} = 0$ is the inverse image of the set of primes $\mathfrak{q}$ of S such that $M_{\mathfrak{q}} = 0$. This follows from Lemma 40.6.

Proof of (6). Follows directly from the corresponding result for smooth ring maps (Lemma 137.10).

Proof of (7). Follows from Lemma 140.3 and the definitions.

Proof of (8). Lemma 138.14 gives the result for smooth ring maps. The resulting smooth ring map $R_0 \to S_0$ satisfies the hypotheses of Lemma 130.9, and hence we may replace S_0 by the factor of relative dimension 0 over R_0.

Proof of (9). Follows from (8) since $R_0 \to A$ will factor through A_i for some i by Lemma 127.3.

Proof of (10). Follows from (9), (1), and (2) since $S^{-1}A$ is a filtered colimit of principal localizations of A.

Proof of (11). Use Lemma 137.15 to see the result for smoothness and then use that $\Omega_{B/A}$ is zero if and only if both $\Omega_{B'/A}$ and $\Omega_{B''/A}$ are zero. □

Next we work out in more detail what it means to be étale over a field.

00U3 **Lemma 143.4.** *Let k be a field. A ring map $k \to S$ is étale if and only if S is isomorphic as a k-algebra to a finite product of finite separable extensions of k.*

Proof. We are going to use without further mention: if $S = S_1 \times \ldots \times S_n$ is a finite product of k-algebras, then S is étale over k if and only if each S_i is étale over k. See Lemma 143.3 part (11).

If k'/k is a finite separable field extension then we can write $k' = k(\alpha) \cong k[x]/(f)$. Here f is the minimal polynomial of the element α. Since k' is separable over k we have $\gcd(f, f') = 1$. This implies that $\mathrm{d} : k' \cdot f \to k' \cdot \mathrm{d}x$ is an isomorphism. Hence $k \to k'$ is étale. Thus if S is a finite product of finite separable extension of k, then S is étale over k.

Conversely, suppose that $k \to S$ is étale. Then S is smooth over k and $\Omega_{S/k} = 0$. By Lemma 140.3 we see that $\dim_{\mathfrak m} \mathrm{Spec}(S) = 0$ for every maximal ideal $\mathfrak m$ of S. Thus $\dim(S) = 0$. By Proposition 60.7 we find that S is a finite product of Artinian local rings. By the already used Lemma 140.3 these local rings are fields. Hence we may assume $S = k'$ is a field. By the Hilbert Nullstellensatz (Theorem 34.1) we see that the extension k'/k is finite. The smoothness of $k \to k'$ implies by Lemma 140.9 that k'/k is a separable extension and the proof is complete. □

00U4 **Lemma 143.5.** *Let $R \to S$ be a ring map. Let $\mathfrak q \subset S$ be a prime lying over $\mathfrak p$ in R. If S/R is étale at $\mathfrak q$ then*

1. *we have $\mathfrak p S_{\mathfrak q} = \mathfrak q S_{\mathfrak q}$ is the maximal ideal of the local ring $S_{\mathfrak q}$, and*
2. *the field extension $\kappa(\mathfrak q)/\kappa(\mathfrak p)$ is finite separable.*

Proof. First we may replace S by S_g for some $g \in S$, $g \notin \mathfrak q$ and assume that $R \to S$ is étale. Then the lemma follows from Lemma 143.4 by unwinding the fact that $S \otimes_R \kappa(\mathfrak p)$ is étale over $\kappa(\mathfrak p)$. □

00U5 **Lemma 143.6.** *An étale ring map is quasi-finite.*

Proof. Let $R \to S$ be an étale ring map. By definition $R \to S$ is of finite type. For any prime $\mathfrak p \subset R$ the fibre ring $S \otimes_R \kappa(\mathfrak p)$ is étale over $\kappa(\mathfrak p)$ and hence a finite products of fields finite separable over $\kappa(\mathfrak p)$, in particular finite over $\kappa(\mathfrak p)$. Thus $R \to S$ is quasi-finite by Lemma 122.4. □

00U6 **Lemma 143.7.** *Let $R \to S$ be a ring map. Let $\mathfrak q$ be a prime of S lying over a prime $\mathfrak p$ of R. If*

1. *$R \to S$ is of finite presentation,*
2. *$R_{\mathfrak p} \to S_{\mathfrak q}$ is flat*
3. *$\mathfrak p S_{\mathfrak q}$ is the maximal ideal of the local ring $S_{\mathfrak q}$, and*
4. *the field extension $\kappa(\mathfrak q)/\kappa(\mathfrak p)$ is finite separable,*

then $R \to S$ is étale at $\mathfrak q$.

Proof. Apply Lemma 122.2 to find a $g \in S$, $g \notin \mathfrak q$ such that $\mathfrak q$ is the only prime of S_g lying over $\mathfrak p$. We may and do replace S by S_g. Then $S \otimes_R \kappa(\mathfrak p)$ has a unique prime, hence is a local ring, hence is equal to $S_{\mathfrak q}/\mathfrak p S_{\mathfrak q} \cong \kappa(\mathfrak q)$. By Lemma 137.17 there exists a $g \in S$, $g \notin \mathfrak q$ such that $R \to S_g$ is smooth. Replace S by S_g again we may assume that $R \to S$ is smooth. By Lemma 137.10 we may even assume that $R \to S$ is standard smooth, say $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. Since $S \otimes_R \kappa(\mathfrak p) = \kappa(\mathfrak q)$ has dimension 0 we conclude that $n = c$, i.e., $R \to S$ is étale. $\square$

Here is a completely new phenomenon.

00U7 **Lemma 143.8.** *Let $R \to S$ and $R \to S'$ be étale. Then any R-algebra map $S' \to S$ is étale.*

Proof. First of all we note that $S' \to S$ is of finite presentation by Lemma 6.2. Let $\mathfrak q \subset S$ be a prime ideal lying over the primes $\mathfrak q' \subset S'$ and $\mathfrak p \subset R$. By Lemma 143.5 the ring map $S'_{\mathfrak q'}/\mathfrak p S'_{\mathfrak q'} \to S_{\mathfrak q}/\mathfrak p S_{\mathfrak q}$ is a map finite separable extensions of $\kappa(\mathfrak p)$. In particular it is flat. Hence by Lemma 128.8 we see that $S'_{\mathfrak q'} \to S_{\mathfrak q}$ is flat. Thus $S' \to S$ is flat. Moreover, the above also shows that $\mathfrak q' S_{\mathfrak q}$ is the maximal ideal of $S_{\mathfrak q}$ and that the residue field extension of $S'_{\mathfrak q'} \to S_{\mathfrak q}$ is finite separable. Hence from Lemma 143.7 we conclude that $S' \to S$ is étale at $\mathfrak q$. Since being étale is local (see Lemma 143.3) we win. $\square$

00U8 **Lemma 143.9.** *Let $\varphi : R \to S$ be a ring map. If $R \to S$ is surjective, flat and finitely presented then there exist an idempotent $e \in R$ such that $S = R_e$.*

First proof. Let I be the kernel of φ. We have that I is finitely generated by Lemma 6.3 since φ is of finite presentation. Moreover, since S is flat over R, tensoring the exact sequence $0 \to I \to R \to S \to 0$ over R with S gives $I/I^2 = 0$. Now we conclude by Lemma 21.5. $\square$

Second proof. Since $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is a homeomorphism onto a closed subset (see Lemma 17.7) and is open (see Proposition 41.8) we see that the image is $D(e)$ for some idempotent $e \in R$ (see Lemma 21.3). Thus $R_e \to S$ induces a bijection on spectra. Now this map induces an isomorphism on all local rings for example by Lemmas 78.5 and 20.1. Then it follows that $R_e \to S$ is also injective, for example see Lemma 23.1. $\square$

04D1 **Lemma 143.10.** *Let R be a ring and let $I \subset R$ be an ideal. Let $R/I \to \overline{S}$ be an étale ring map. Then there exists an étale ring map $R \to S$ such that $\overline{S} \cong S/IS$ as R/I-algebras.*

Proof. By Lemma 143.2 we can write $\overline{S} = (R/I)[x_1, \ldots, x_n]/(\overline{f}_1, \ldots, \overline{f}_n)$ as in Definition 137.6 with $\overline{\Delta} = \det(\frac{\partial \overline{f}_i}{\partial x_j})_{i,j=1,\ldots,n}$ invertible in $\overline{S}$. Just take some lifts f_i and set $S = R[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_n, x_{n+1}\Delta - 1)$ where $\Delta = \det(\frac{\partial f_i}{\partial x_j})_{i,j=1,\ldots,n}$ as in Example 137.8. This proves the lemma. $\square$

05YT **Lemma 143.11.** *Consider a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] & B' \arrow[r] & B \arrow[r] & 0 \\ 0 \arrow[r] & I \arrow[r] \arrow[u] & A' \arrow[r] \arrow[u] & A \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

with exact rows where $B' \to B$ and $A' \to A$ are surjective ring maps whose kernels are ideals of square zero. If $A \to B$ is étale, and $J = I \otimes_A B$, then $A' \to B'$ is étale.

Proof. By Lemma 143.10 there exists an étale ring map $A' \to C$ such that $C/IC = B$. Then $A' \to C$ is formally smooth (by Proposition 138.13) hence we get an A'-algebra map $\varphi : C \to B'$. Since $A' \to C$ is flat we have $I \otimes_A B = I \otimes_A C/IC = IC$. Hence the assumption that $J = I \otimes_A B$ implies that φ induces an isomorphism $IC \to J$ and an isomorphism $C/IC \to B'/IB'$, whence φ is an isomorphism. $\square$

00UA **Example 143.12.** Let $n, m \geq 1$ be integers. Consider the ring map

$$\begin{array}{rcl} R = \mathbf{Z}[a_1, \ldots, a_{n+m}] & \longrightarrow & S = \mathbf{Z}[b_1, \ldots, b_n, c_1, \ldots, c_m] \\ a_1 & \longmapsto & b_1 + c_1 \\ a_2 & \longmapsto & b_2 + b_1c_1 + c_2 \\ \ldots & \ldots & \ldots \\ a_{n+m} & \longmapsto & b_nc_m \end{array}$$

of Example 136.7. Write symbolically

$$S = R[b_1, \ldots, c_m]/(\{a_k(b_i, c_j) - a_k\}_{k=1,\ldots,n+m})$$

where for example $a_1(b_i, c_j) = b_1 + c_1$. The matrix of partial derivatives is

$$\begin{pmatrix} 1 & c_1 & \ldots & c_m & 0 & \ldots & \ldots & 0 \\ 0 & 1 & c_1 & \ldots & c_m & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 1 & c_1 & c_2 & \ldots & c_m \\ 1 & b_1 & \ldots & b_{n-1} & b_n & 0 & \ldots & 0 \\ 0 & 1 & b_1 & \ldots & b_{n-1} & b_n & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & 0 & 1 & b_1 & \ldots & b_n \end{pmatrix}$$

The determinant Δ of this matrix is better known as the *resultant* of the polynomials $g = x^n + b_1x^{n-1} + \ldots + b_n$ and $h = x^m + c_1x^{m-1} + \ldots + c_m$, and the matrix above is known as the *Sylvester matrix* associated to g, h. In a formula $\Delta = \text{Res}_x(g, h)$. The Sylvester matrix is the transpose of the matrix of the linear map

$$\begin{array}{rcl} S[x]_{<m} \oplus S[x]_{<n} & \longrightarrow & S[x]_{<n+m} \\ a \oplus b & \longmapsto & ag + bh \end{array}$$

Let $\mathfrak{q} \subset S$ be any prime. By the above the following are equivalent:

(1) $R \to S$ is étale at $\mathfrak{q}$,
(2) $\Delta = \text{Res}_x(g, h) \notin \mathfrak{q}$,

(3) the images $\overline{g}, \overline{h} \in \kappa(\mathfrak q)[x]$ of the polynomials g, h are relatively prime in $\kappa(\mathfrak q)[x]$.

The equivalence of (2) and (3) holds because the image of the Sylvester matrix in $\text{Mat}(n + m, \kappa(\mathfrak q))$ has a kernel if and only if the polynomials $\overline{g}, \overline{h}$ have a factor in common. We conclude that the ring map

$$R \longrightarrow S[\frac{1}{\Delta}] = S[\frac{1}{\text{Res}_x(g, h)}]$$

is étale.

00UH **Lemma 143.13.** *Let R be a ring. Let $f \in R[x]$ be a monic polynomial. Let $\mathfrak p$ be a prime of R. Let $f \bmod \mathfrak p = \overline{g}\overline{h}$ be a factorization of the image of f in $\kappa(\mathfrak p)[x]$. If $\gcd(\overline{g}, \overline{h}) = 1$, then there exist*

(1) *an étale ring map $R \to R'$,*
(2) *a prime $\mathfrak p' \subset R'$ lying over $\mathfrak p$, and*
(3) *a factorization $f = gh$ in $R'[x]$*

such that

(1) *$\kappa(\mathfrak p) = \kappa(\mathfrak p')$,*
(2) *$\overline{g} = g \bmod \mathfrak p'$, $\overline{h} = h \bmod \mathfrak p'$, and*
(3) *the polynomials g, h generate the unit ideal in $R'[x]$.*

Proof. Suppose $\overline{g} = \overline{b}_0 x^n + \overline{b}_1 x^{n-1} + \ldots + \overline{b}_n$, and $\overline{h} = \overline{c}_0 x^m + \overline{c}_1 x^{m-1} + \ldots + \overline{c}_m$ with $\overline{b}_0, \overline{c}_0 \in \kappa(\mathfrak p)$ nonzero. After localizing R at some element of R not contained in $\mathfrak p$ we may assume $\overline{b}_0$ is the image of an invertible element $b_0 \in R$. Replacing $\overline{g}$ by $\overline{g}/b_0$ and $\overline{h}$ by $b_0\overline{h}$ we reduce to the case where $\overline{g}$, $\overline{h}$ are monic (verification omitted). Say $\overline{g} = x^n + \overline{b}_1 x^{n-1} + \ldots + \overline{b}_n$, and $\overline{h} = x^m + \overline{c}_1 x^{m-1} + \ldots + \overline{c}_m$. Write $f = x^{n+m} + a_1 x^{n-1} + \ldots + a_{n+m}$. Consider the fibre product

$$R' = R \otimes_{\mathbf{Z}[a_1, \ldots, a_{n+m}]} \mathbf{Z}[b_1, \ldots, b_n, c_1, \ldots, c_m]$$

where the map $\mathbf{Z}[a_k] \to \mathbf{Z}[b_i, c_j]$ is as in Examples 136.7 and 143.12. By construction there is an R-algebra map

$$R' = R \otimes_{\mathbf{Z}[a_1, \ldots, a_{n+m}]} \mathbf{Z}[b_1, \ldots, b_n, c_1, \ldots, c_m] \longrightarrow \kappa(\mathfrak p)$$

which maps b_i to $\overline{b}_i$ and c_j to $\overline{c}_j$. Denote $\mathfrak p' \subset R'$ the kernel of this map. Since by assumption the polynomials $\overline{g}, \overline{h}$ are relatively prime we see that the element $\Delta = \text{Res}_x(g, h) \in \mathbf{Z}[b_i, c_j]$ (see Example 143.12) does not map to zero in $\kappa(\mathfrak p)$ under the displayed map. We conclude that $R \to R'$ is étale at $\mathfrak p'$. In fact a solution to the problem posed in the lemma is the ring map $R \to R'[1/\Delta]$ and the prime $\mathfrak p' R'[1/\Delta]$. Because $\text{Res}_x(f, g)$ is invertible in this ring the Sylvester matrix is invertible over $R'[1/\Delta]$ and hence $1 = ag + bh$ for some $a, b \in R'[1/\Delta][x]$ see Example 143.12. $\square$

144. Local structure of étale ring maps

0G1A Lemma 143.2 tells us that it does not really make sense to define a standard étale morphism to be a standard smooth morphism of relative dimension 0. As a model for an étale morphism we take the example given by a finite separable extension k'/k of fields. Namely, we can always find an element $\alpha \in k'$ such that $k' = k(\alpha)$ and such that the minimal polynomial $f(x) \in k[x]$ of α has derivative f' which is relatively prime to f.

00UB **Definition 144.1.** Let R be a ring. Let $g, f \in R[x]$. Assume that f is monic and the derivative f' is invertible in the localization $R[x]_g/(f)$. In this case the ring map $R \to R[x]_g/(f)$ is said to be *standard étale*.

00UC **Lemma 144.2.** *Let $R \to R[x]_g/(f)$ be standard étale.*

(1) *The ring map $R \to R[x]_g/(f)$ is étale.*
(2) *For any ring map $R \to R'$ the base change $R' \to R'[x]_g/(f)$ of the standard étale ring map $R \to R[x]_g/(f)$ is standard étale.*
(3) *Any principal localization of $R[x]_g/(f)$ is standard étale over R.*
(4) *A composition of standard étale maps is* **not** *standard étale in general.*

Proof. Omitted. Here is an example for (4). The ring map $\mathbf{F}_2 \to \mathbf{F}_{2^2}$ is standard étale. The ring map $\mathbf{F}_{2^2} \to \mathbf{F}_{2^2} \times \mathbf{F}_{2^2} \times \mathbf{F}_{2^2} \times \mathbf{F}_{2^2}$ is standard étale. But the ring map $\mathbf{F}_2 \to \mathbf{F}_{2^2} \times \mathbf{F}_{2^2} \times \mathbf{F}_{2^2} \times \mathbf{F}_{2^2}$ is not standard étale. $\square$

Standard étale morphisms are a convenient way to produce étale maps. Here is an example.

00UD **Lemma 144.3.** *Let R be a ring. Let $\mathfrak{p}$ be a prime of R. Let $L/\kappa(\mathfrak{p})$ be a finite separable field extension. There exists an étale ring map $R \to R'$ together with a prime $\mathfrak{p}'$ lying over $\mathfrak{p}$ such that the field extension $\kappa(\mathfrak{p}')/\kappa(\mathfrak{p})$ is isomorphic to $\kappa(\mathfrak{p}) \subset L$.*

Proof. By the theorem of the primitive element we may write $L = \kappa(\mathfrak{p})[\alpha]$. Let $\overline{f} \in \kappa(\mathfrak{p})[x]$ denote the minimal polynomial for α (in particular this is monic). After replacing α by $c\alpha$ for some $c \in R$, $c \notin \mathfrak{p}$ we may assume all the coefficients of $\overline{f}$ are in the image of $R \to \kappa(\mathfrak{p})$ (verification omitted). Thus we can find a monic polynomial $f \in R[x]$ which maps to $\overline{f}$ in $\kappa(\mathfrak{p})[x]$. Since $\kappa(\mathfrak{p}) \subset L$ is separable, we see that $\gcd(\overline{f}, \overline{f}') = 1$. Hence there is an element $\gamma \in L$ such that $\overline{f}'(\alpha)\gamma = 1$. Thus we get a R-algebra map

$$\begin{array}{rcl} R[x, 1/f']/(f) & \longrightarrow & L \\ x & \longmapsto & \alpha \\ 1/f' & \longmapsto & \gamma \end{array}$$

The left hand side is a standard étale algebra R' over R and the kernel of the ring map gives the desired prime. $\square$

00UE **Proposition 144.4.** *Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime. If $R \to S$ is étale at $\mathfrak{q}$, then there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is standard étale.*

Proof. The following proof is a little roundabout and there may be ways to shorten it.

Step 1. By Definition 143.1 there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is étale. Thus we may assume that S is étale over R.

Step 2. By Lemma 143.3 there exists an étale ring map $R_0 \to S_0$ with R_0 of finite type over $\mathbf{Z}$, and a ring map $R_0 \to R$ such that $R = R \otimes_{R_0} S_0$. Denote $\mathfrak{q}_0$ the prime of S_0 corresponding to $\mathfrak{q}$. If we show the result for $(R_0 \to S_0, \mathfrak{q}_0)$ then the result follows for $(R \to S, \mathfrak{q})$ by base change. Hence we may assume that R is Noetherian.

Step 3. Note that $R \to S$ is quasi-finite by Lemma 143.6. By Lemma 123.14 there exists a finite ring map $R \to S'$, an R-algebra map $S' \to S$, an element $g' \in S'$

such that $g' \not\in \mathfrak q$ such that $S' \to S$ induces an isomorphism $S'_{g'} \cong S_{g'}$. (Note that of course S' is not étale over R in general.) Thus we may assume that (a) R is Noetherian, (b) $R \to S$ is finite and (c) $R \to S$ is étale at $\mathfrak q$ (but no longer necessarily étale at all primes).

Step 4. Let $\mathfrak p \subset R$ be the prime corresponding to $\mathfrak q$. Consider the fibre ring $S \otimes_R \kappa(\mathfrak p)$. This is a finite algebra over $\kappa(\mathfrak p)$. Hence it is Artinian (see Lemma 53.2) and so a finite product of local rings

$$S \otimes_R \kappa(\mathfrak p) = \prod\nolimits_{i=1}^n A_i$$

see Proposition 60.7. One of the factors, say A_1, is the local ring $S_{\mathfrak q}/\mathfrak pS_{\mathfrak q}$ which is isomorphic to $\kappa(\mathfrak q)$, see Lemma 143.5. The other factors correspond to the other primes, say $\mathfrak q_2, \ldots, \mathfrak q_n$ of S lying over $\mathfrak p$.

Step 5. We may choose a nonzero element $\alpha \in \kappa(\mathfrak q)$ which generates the finite separable field extension $\kappa(\mathfrak q)/\kappa(\mathfrak p)$ (so even if the field extension is trivial we do not allow $\alpha = 0$). Note that for any $\lambda \in \kappa(\mathfrak p)^*$ the element $\lambda\alpha$ also generates $\kappa(\mathfrak q)$ over $\kappa(\mathfrak p)$. Consider the element

$$\overline{t} = (\alpha, 0, \ldots, 0) \in \prod\nolimits_{i=1}^n A_i = S \otimes_R \kappa(\mathfrak p).$$

After possibly replacing α by $\lambda\alpha$ as above we may assume that $\overline{t}$ is the image of $t \in S$. Let $I \subset R[x]$ be the kernel of the R-algebra map $R[x] \to S$ which maps x to t. Set $S' = R[x]/I$, so $S' \subset S$. Here is a diagram

$$\begin{tikzcd} R[x] \arrow[r] & S' \arrow[r] & S \\ R \arrow[u] \arrow[ru] \arrow[rru] & & \end{tikzcd}$$

By construction the primes $\mathfrak q_j$, $j \geq 2$ of S all lie over the prime $(\mathfrak p, x)$ of $R[x]$, whereas the prime $\mathfrak q$ lies over a different prime of $R[x]$ because $\alpha \neq 0$.

Step 6. Denote $\mathfrak q' \subset S'$ the prime of S' corresponding to $\mathfrak q$. By the above $\mathfrak q$ is the only prime of S lying over $\mathfrak q'$. Thus we see that $S_{\mathfrak q} = S_{\mathfrak q'}$, see Lemma 41.11 (we have going up for $S' \to S$ by Lemma 36.22 since $S' \to S$ is finite as $R \to S$ is finite). It follows that $S'_{\mathfrak q'} \to S_{\mathfrak q}$ is finite and injective as the localization of the finite injective ring map $S' \to S$. Consider the maps of local rings

$$R_{\mathfrak p} \to S'_{\mathfrak q'} \to S_{\mathfrak q}$$

The second map is finite and injective. We have $S_{\mathfrak q}/\mathfrak pS_{\mathfrak q} = \kappa(\mathfrak q)$, see Lemma 143.5. Hence a fortiori $S_{\mathfrak q}/\mathfrak q'S_{\mathfrak q} = \kappa(\mathfrak q)$. Since

$$\kappa(\mathfrak p) \subset \kappa(\mathfrak q') \subset \kappa(\mathfrak q)$$

and since α is in the image of $\kappa(\mathfrak q')$ in $\kappa(\mathfrak q)$ we conclude that $\kappa(\mathfrak q') = \kappa(\mathfrak q)$. Hence by Nakayama's Lemma 20.1 applied to the $S'_{\mathfrak q'}$-module map $S'_{\mathfrak q'} \to S_{\mathfrak q}$, the map $S'_{\mathfrak q'} \to S_{\mathfrak q}$ is surjective. In other words, $S'_{\mathfrak q'} \cong S_{\mathfrak q}$.

Step 7. By Lemma 126.7 there exist $g \in S$, $g \not\in \mathfrak q$ and $g' \in S'$, $g' \not\in \mathfrak q'$ such that $S'_{g'} \cong S_g$. As R is Noetherian the ring S' is finite over R because it is an R-submodule of the finite R-module S. Hence after replacing S by S' we may assume that (a) R is Noetherian, (b) S finite over R, (c) S is étale over R at $\mathfrak q$, and (d) $S = R[x]/I$.

Step 8. Consider the ring $S \otimes_R \kappa(\mathfrak p) = \kappa(\mathfrak p)[x]/\overline{I}$ where $\overline{I} = I \cdot \kappa(\mathfrak p)[x]$ is the ideal generated by I in $\kappa(\mathfrak p)[x]$. As $\kappa(\mathfrak p)[x]$ is a PID we know that $\overline{I} = (\overline{h})$ for some monic $\overline{h} \in \kappa(\mathfrak p)[x]$. After replacing $\overline{h}$ by $\lambda \cdot \overline{h}$ for some $\lambda \in \kappa(\mathfrak p)$ we may assume that $\overline{h}$ is the image of some $h \in I \subset R[x]$. (The problem is that we do not know if we may choose h monic.) Also, as in Step 4 we know that $S \otimes_R \kappa(\mathfrak p) = A_1 \times \ldots \times A_n$ with $A_1 = \kappa(\mathfrak q)$ a finite separable extension of $\kappa(\mathfrak p)$ and $A_2, \ldots, A_n$ local. This implies that

$$\overline{h} = \overline{h}_1 \overline{h}_2^{e_2} \ldots \overline{h}_n^{e_n}$$

for certain pairwise coprime irreducible monic polynomials $\overline{h}_i \in \kappa(\mathfrak p)[x]$ and certain $e_2, \ldots, e_n \geq 1$. Here the numbering is chosen so that $A_i = \kappa(\mathfrak p)[x]/(\overline{h}_i^{e_i})$ as $\kappa(\mathfrak p)[x]$-algebras. Note that $\overline{h}_1$ is the minimal polynomial of $\alpha \in \kappa(\mathfrak q)$ and hence is a separable polynomial (its derivative is prime to itself).

Step 9. Let $m \in I$ be a monic element; such an element exists because the ring extension $R \to R[x]/I$ is finite hence integral. Denote $\overline{m}$ the image in $\kappa(\mathfrak p)[x]$. We may factor

$$\overline{m} = \overline{k}\overline{h}_1^{d_1} \overline{h}_2^{d_2} \ldots \overline{h}_n^{d_n}$$

for some $d_1 \geq 1$, $d_j \geq e_j$, $j = 2, \ldots, n$ and $\overline{k} \in \kappa(\mathfrak p)[x]$ prime to all the $\overline{h}_i$. Set $f = m^l + h$ where $l \deg(m) > \deg(h)$, and $l \geq 2$. Then f is monic as a polynomial over R. Also, the image $\overline{f}$ of f in $\kappa(\mathfrak p)[x]$ factors as

$$\overline{f} = \overline{h}_1 \overline{h}_2^{e_2} \ldots \overline{h}_n^{e_n} + \overline{k}^l \overline{h}_1^{ld_1} \overline{h}_2^{ld_2} \ldots \overline{h}_n^{ld_n} = \overline{h}_1 (\overline{h}_2^{e_2} \ldots \overline{h}_n^{e_n} + \overline{k}^l \overline{h}_1^{ld_1 - 1} \overline{h}_2^{ld_2} \ldots \overline{h}_n^{ld_n}) = \overline{h}_1 \overline{w}$$

with $\overline{w}$ a polynomial relatively prime to $\overline{h}_1$. Set $g = f'$ (the derivative with respect to x).

Step 10. The ring map $R[x] \to S = R[x]/I$ has the properties: (1) it maps f to zero, and (2) it maps g to an element of $S \setminus \mathfrak q$. The first assertion is clear since f is an element of I. For the second assertion we just have to show that g does not map to zero in $\kappa(\mathfrak q) = \kappa(\mathfrak p)[x]/(\overline{h}_1)$. The image of g in $\kappa(\mathfrak p)[x]$ is the derivative of $\overline{f}$. Thus (2) is clear because

$$\overline{g} = \frac{\mathrm{d}\overline{f}}{\mathrm{d}x} = \overline{w}\frac{\mathrm{d}\overline{h}_1}{\mathrm{d}x} + \overline{h}_1 \frac{\mathrm{d}\overline{w}}{\mathrm{d}x},$$

$\overline{w}$ is prime to $\overline{h}_1$ and $\overline{h}_1$ is separable.

Step 11. We conclude that $\varphi : R[x]/(f) \to S$ is a surjective ring map, $R[x]_g/(f)$ is étale over R (because it is standard étale, see Lemma 144.2) and $\varphi(g) \notin \mathfrak q$. Pick an element $g' \in R[x]/(f)$ such that also $\varphi(g') \notin \mathfrak q$ and $S_{\varphi(g')}$ is étale over R (which exists since S is étale over R at $\mathfrak q$). Then the ring map $R[x]_{gg'}/(f) \to S_{\varphi(gg')}$ is a surjective map of étale algebras over R. Hence it is étale by Lemma 143.8. Hence it is a localization by Lemma 143.9. Thus a localization of S at an element not in $\mathfrak q$ is isomorphic to a localization of a standard étale algebra over R which is what we wanted to show. $\square$

The following two lemmas say that the étale topology is coarser than the topology generated by Zariski coverings and finite flat morphisms. They should be skipped on a first reading.

00UF **Lemma 144.5.** *Let $R \to S$ be a standard étale morphism. There exists a ring map $R \to S'$ with the following properties*

(1) $R \to S'$ *is finite, finitely presented, and flat (in other words S' is finite projective as an R-module),*
(2) $\text{Spec}(S') \to \text{Spec}(R)$ *is surjective,*
(3) *for every prime $\mathfrak{q} \subset S$, lying over $\mathfrak{p} \subset R$ and every prime $\mathfrak{q}' \subset S'$ lying over $\mathfrak{p}$ there exists a $g' \in S'$, $g' \notin \mathfrak{q}'$ such that the ring map $R \to S'_{g'}$ factors through a map $\varphi : S \to S'_{g'}$ with $\varphi^{-1}(\mathfrak{q}'S'_{g'}) = \mathfrak{q}$.*

Proof. Let $S = R[x]_g/(f)$ be a presentation of S as in Definition 144.1. Write $f = x^n + a_1x^{n-1} + \ldots + a_n$ with $a_i \in R$. By Lemma 136.9 there exists a finite locally free and faithfully flat ring map $R \to S'$ such that $f = \prod(x - \alpha_i)$ for certain $\alpha_i \in S'$. Hence $R \to S'$ satisfies conditions (1), (2). Let $\mathfrak{q} \subset R[x]/(f)$ be a prime ideal with $g \notin \mathfrak{q}$ (i.e., it corresponds to a prime of S). Let $\mathfrak{p} = R \cap \mathfrak{q}$ and let $\mathfrak{q}' \subset S'$ be a prime lying over $\mathfrak{p}$. Note that there are n maps of R-algebras

$$\begin{matrix} \varphi_i : R[x]/(f) & \longrightarrow & S' \\ x & \longmapsto & \alpha_i \end{matrix}$$

To finish the proof we have to show that for some i we have (a) the image of $\varphi_i(g)$ in $\kappa(\mathfrak{q}')$ is not zero, and (b) $\varphi_i^{-1}(\mathfrak{q}') = \mathfrak{q}$. Because then we can just take $g' = \varphi_i(g)$, and $\varphi = \varphi_i$ for that i.

Let $\overline{f}$ denote the image of f in $\kappa(\mathfrak{p})[x]$. Note that as a point of $\text{Spec}(\kappa(\mathfrak{p})[x]/(\overline{f}))$ the prime $\mathfrak{q}$ corresponds to an irreducible factor f_1 of $\overline{f}$. Moreover, $g \notin \mathfrak{q}$ means that f_1 does not divide the image $\overline{g}$ of g in $\kappa(\mathfrak{p})[x]$. Denote $\overline{\alpha}_1, \ldots, \overline{\alpha}_n$ the images of $\alpha_1, \ldots, \alpha_n$ in $\kappa(\mathfrak{q}')$. Note that the polynomial $\overline{f}$ splits completely in $\kappa(\mathfrak{q}')[x]$, namely

$$\overline{f} = \prod\nolimits_i (x - \overline{\alpha}_i)$$

Moreover $\varphi_i(g)$ reduces to $\overline{g}(\overline{\alpha}_i)$. It follows we may pick i such that $f_1(\overline{\alpha}_i) = 0$ and $\overline{g}(\overline{\alpha}_i) \neq 0$. For this i properties (a) and (b) hold. Some details omitted. $\square$

00UG **Lemma 144.6.** *Let $R \to S$ be a ring map. Assume that*
(1) $R \to S$ *is étale, and*
(2) $\text{Spec}(S) \to \text{Spec}(R)$ *is surjective.*

Then there exists a ring map $R \to S'$ such that
(1) $R \to S'$ *is finite, finitely presented, and flat (in other words it is finite projective as an R-module),*
(2) $\text{Spec}(S') \to \text{Spec}(R)$ *is surjective,*
(3) *for every prime $\mathfrak{q}' \subset S'$ there exists a $g' \in S'$, $g' \notin \mathfrak{q}'$ such that the ring map $R \to S'_{g'}$ factors as $R \to S \to S'_{g'}$.*

Proof. By Proposition 144.4 and the quasi-compactness of $\text{Spec}(S)$ (see Lemma 17.10) we can find $g_1, \ldots, g_n \in S$ generating the unit ideal of S such that each $R \to S_{g_i}$ is standard étale. If we prove the lemma for the ring map $R \to \prod_{i=1,\ldots,n} S_{g_i}$ then the lemma follows for the ring map $R \to S$. Hence we may assume that $S = \prod_{i=1,\ldots,n} S_i$ is a finite product of standard étale morphisms.

For each i choose a ring map $R \to S'_i$ as in Lemma 144.5 adapted to the standard étale morphism $R \to S_i$. Set $S' = S'_1 \otimes_R \ldots \otimes_R S'_n$; we will use the R-algebra maps $S'_i \to S'$ without further mention below. We claim this works. Properties (1) and (2) are immediate. For property (3) suppose that $\mathfrak{q}' \subset S'$ is a prime. Denote $\mathfrak{p}$ its image in $\text{Spec}(R)$. Choose $i \in \{1, \ldots, n\}$ such that $\mathfrak{p}$ is in the image of $\text{Spec}(S_i) \to \text{Spec}(R)$; this is possible by assumption. Set $\mathfrak{q}'_i \subset S'_i$ the image of $\mathfrak{q}'$ in the spectrum of S'_i. By construction of S'_i there exists a $g'_i \in S'_i$ such that $R \to (S'_i)_{g'_i}$ factors as $R \to S_i \to (S'_i)_{g'_i}$. Hence also $R \to S'_{g'_i}$ factors as

$$R \to S_i \to (S'_i)_{g'_i} \to S'_{g'_i}$$

as desired. $\square$

145. Étale local structure of quasi-finite ring maps

0G1B The following lemmas say roughly that after an étale extension a quasi-finite ring map becomes finite. To help interpret the results recall that the locus where a finite type ring map is quasi-finite is open (see Lemma 123.13) and that formation of this locus commutes with arbitrary base change (see Lemma 122.8).

00UI **Lemma 145.1.** *Let $R \to S' \to S$ be ring maps. Let $\mathfrak{p} \subset R$ be a prime. Let $g \in S'$ be an element. Assume*

1. *$R \to S'$ is integral,*
2. *$R \to S$ is finite type,*
3. *$S'_g \cong S_g$, and*
4. *g invertible in $S' \otimes_R \kappa(\mathfrak{p})$.*

Then there exists a $f \in R$, $f \notin \mathfrak{p}$ such that $R_f \to S_f$ is finite.

Proof. By assumption the image T of $V(g) \subset \text{Spec}(S')$ under the morphism $\text{Spec}(S') \to \text{Spec}(R)$ does not contain $\mathfrak{p}$. By Section 41 especially, Lemma 41.6 we see T is closed. Pick $f \in R$, $f \notin \mathfrak{p}$ such that $T \cap D(f) = \emptyset$. Then we see that g becomes invertible in S'_f. Hence $S'_f \cong S_f$. Thus S_f is both of finite type and integral over R_f, hence finite. $\square$

00UJ **Lemma 145.2.** *Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime lying over the prime $\mathfrak{p} \subset R$. Assume $R \to S$ finite type and quasi-finite at $\mathfrak{q}$. Then there exists*

1. *an étale ring map $R \to R'$,*
2. *a prime $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p}$,*
3. *a product decomposition*

$$R' \otimes_R S = A \times B$$

with the following properties

1. *$\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}')$,*
2. *$R' \to A$ is finite,*
3. *A has exactly one prime $\mathfrak{r}$ lying over $\mathfrak{p}'$, and*
4. *$\mathfrak{r}$ lies over $\mathfrak{q}$.*

Proof. Let $S' \subset S$ be the integral closure of R in S. Let $\mathfrak{q}' = S' \cap \mathfrak{q}$. By Zariski's Main Theorem 123.12 there exists a $g \in S'$, $g \notin \mathfrak{q}'$ such that $S'_g \cong S_g$. Consider the fibre rings $F = S \otimes_R \kappa(\mathfrak{p})$ and $F' = S' \otimes_R \kappa(\mathfrak{p})$. Denote $\overline{\mathfrak{q}}'$ the prime of F' corresponding to $\mathfrak{q}'$. Since F' is integral over $\kappa(\mathfrak{p})$ we see that $\overline{\mathfrak{q}}'$ is a closed point of $\text{Spec}(F')$, see Lemma 36.19. Note that $\mathfrak{q}$ defines an isolated closed point $\overline{\mathfrak{q}}$ of $\text{Spec}(F)$ (see Definition 122.3). Since $S'_g \cong S_g$ we have $F'_g \cong F_g$, so $\overline{\mathfrak{q}}$ and $\overline{\mathfrak{q}}'$ have isomorphic open neighbourhoods in $\text{Spec}(F)$ and $\text{Spec}(F')$. We conclude the set $\{\overline{\mathfrak{q}}'\} \subset \text{Spec}(F')$ is open. Combined with $\mathfrak{q}'$ being closed (shown above) we conclude that $\overline{\mathfrak{q}}'$ defines an isolated closed point of $\text{Spec}(F')$ as well.

An additional small remark is that under the map $\mathrm{Spec}(F) \to \mathrm{Spec}(F')$ the point $\overline{\mathfrak q}$ is the only point mapping to $\overline{\mathfrak q}'$. This follows from the discussion above.

By Lemma 24.3 we may write $F' = F'_1 \times F'_2$ with $\mathrm{Spec}(F'_1) = \{\overline{\mathfrak q}'\}$. Since $F' = S' \otimes_R \kappa(\mathfrak p)$, there exists an $s' \in S'$ which maps to the element $(r, 0) \in F'_1 \times F'_2 = F'$ for some $r \in R$, $r \not\in \mathfrak p$. In fact, what we will use about s' is that it is an element of S', not contained in $\mathfrak q'$, and contained in any other prime lying over $\mathfrak p$.

Let $f(x) \in R[x]$ be a monic polynomial such that $f(s') = 0$. Denote $\overline{f} \in \kappa(\mathfrak p)[x]$ the image. We can factor it as $\overline{f} = x^e\overline{h}$ where $\overline{h}(0) \neq 0$. By Lemma 143.13 we can find an étale ring extension $R \to R'$, a prime $\mathfrak p'$ lying over $\mathfrak p$, and a factorization $f = hi$ in $R'[x]$ such that $\kappa(\mathfrak p) = \kappa(\mathfrak p')$, $\overline{h} = h \bmod \mathfrak p'$, $x^e = i \bmod \mathfrak p'$, and we can write $ah + bi = 1$ in $R'[x]$ (for suitable a, b).

Consider the elements $h(s'), i(s') \in R' \otimes_R S'$. By construction we have $h(s')i(s') = f(s') = 0$. On the other hand they generate the unit ideal since $a(s')h(s') + b(s')i(s') = 1$. Thus we see that $R' \otimes_R S'$ is the product of the localizations at these elements:

$$R' \otimes_R S' = (R' \otimes_R S')_{i(s')} \times (R' \otimes_R S')_{h(s')} = S'_1 \times S'_2$$

Moreover this product decomposition is compatible with the product decomposition we found for the fibre ring F'; this comes from our choices of s', i, h which guarantee that $\overline{\mathfrak q}'$ is the only prime of F' which does not contain the image of $i(s')$ in F'. Here we use that the fibre ring of $R' \otimes_R S'$ over R' at $\mathfrak p'$ is the same as F' due to the fact that $\kappa(\mathfrak p) = \kappa(\mathfrak p')$. It follows that S'_1 has exactly one prime, say $\mathfrak r'$, lying over $\mathfrak p'$ and that this prime lies over $\mathfrak q$. Hence the element $g \in S'$ maps to an element of S'_1 not contained in $\mathfrak r'$.

The base change $R' \otimes_R S$ inherits a similar product decomposition

$$R' \otimes_R S = (R' \otimes_R S)_{i(s')} \times (R' \otimes_R S)_{h(s')} = S_1 \times S_2$$

It follows from the above that S_1 has exactly one prime, say $\mathfrak r$, lying over $\mathfrak p'$ (consider the fibre ring as above), and that this prime lies over $\mathfrak q$.

Now we may apply Lemma 145.1 to the ring maps $R' \to S'_1 \to S_1$, the prime $\mathfrak p'$ and the element g to see that after replacing R' by a principal localization we can assume that S_1 is finite over R' as desired. ☐

00UK **Lemma 145.3.** *Let $R \to S$ be a ring map. Let $\mathfrak p \subset R$ be a prime. Assume $R \to S$ finite type. Then there exists*

1. *an étale ring map $R \to R'$,*
2. *a prime $\mathfrak p' \subset R'$ lying over $\mathfrak p$,*
3. *a product decomposition*

$$R' \otimes_R S = A_1 \times \ldots \times A_n \times B$$

with the following properties

1. *we have $\kappa(\mathfrak p) = \kappa(\mathfrak p')$,*
2. *each A_i is finite over R',*
3. *each A_i has exactly one prime $\mathfrak r_i$ lying over $\mathfrak p'$, and*
4. *$R' \to B$ not quasi-finite at any prime lying over $\mathfrak p'$.*

Proof. Denote $F = S \otimes_R \kappa(\mathfrak p)$ the fibre ring of S/R at the prime $\mathfrak p$. As F is of finite type over $\kappa(\mathfrak p)$ it is Noetherian and hence $\text{Spec}(F)$ has finitely many isolated closed points. If there are no isolated closed points, i.e., no primes $\mathfrak q$ of S over $\mathfrak p$ such that S/R is quasi-finite at $\mathfrak q$, then the lemma holds. If there exists at least one such prime $\mathfrak q$, then we may apply Lemma 145.2. This gives a diagram

$$\begin{tikzcd} S \arrow[r] & R' \otimes_R S \arrow[r, equal] & A_1 \times B' \\ R \arrow[u] \arrow[r] & R' \arrow[u] \arrow[ur] & \end{tikzcd}$$

as in said lemma. Since the residue fields at $\mathfrak p$ and $\mathfrak p'$ are the same, the fibre rings of S/R and $(A_1 \times B')/R'$ are the same. Hence, by induction on the number of isolated closed points of the fibre we may assume that the lemma holds for $R' \to B'$ and $\mathfrak p'$. Thus we get an étale ring map $R' \to R''$, a prime $\mathfrak p'' \subset R''$ and a decomposition

$$R'' \otimes_{R'} B' = A_2 \times \ldots \times A_n \times B$$

We omit the verification that the ring map $R \to R''$, the prime $\mathfrak p''$ and the resulting decomposition

$$R'' \otimes_R S = (R'' \otimes_{R'} A_1) \times A_2 \times \ldots \times A_n \times B$$

is a solution to the problem posed in the lemma. □

00UL **Lemma 145.4.** *Let $R \to S$ be a ring map. Let $\mathfrak p \subset R$ be a prime. Assume $R \to S$ finite type. Then there exists*

1. *an étale ring map $R \to R'$,*
2. *a prime $\mathfrak p' \subset R'$ lying over $\mathfrak p$,*
3. *a product decomposition*

$$R' \otimes_R S = A_1 \times \ldots \times A_n \times B$$

with the following properties

1. *each A_i is finite over R',*
2. *each A_i has exactly one prime $\mathfrak r_i$ lying over $\mathfrak p'$,*
3. *the finite field extensions $\kappa(\mathfrak r_i)/\kappa(\mathfrak p')$ are purely inseparable, and*
4. *$R' \to B$ not quasi-finite at any prime lying over $\mathfrak p'$.*

Proof. The strategy of the proof is to make two étale ring extensions: first we control the residue fields, then we apply Lemma 145.3.

Denote $F = S \otimes_R \kappa(\mathfrak p)$ the fibre ring of S/R at the prime $\mathfrak p$. As in the proof of Lemma 145.3 there are finitely may primes, say $\mathfrak q_1, \ldots, \mathfrak q_n$ of S lying over R at which the ring map $R \to S$ is quasi-finite. Let $\kappa(\mathfrak p) \subset L_i \subset \kappa(\mathfrak q_i)$ be the subfield such that $\kappa(\mathfrak p) \subset L_i$ is separable, and the field extension $\kappa(\mathfrak q_i)/L_i$ is purely inseparable. Let $L/\kappa(\mathfrak p)$ be a finite Galois extension into which L_i embeds for $i = 1, \ldots, n$. By Lemma 144.3 we can find an étale ring extension $R \to R'$ together with a prime $\mathfrak p'$ lying over $\mathfrak p$ such that the field extension $\kappa(\mathfrak p')/\kappa(\mathfrak p)$ is isomorphic to $\kappa(\mathfrak p) \subset L$. Thus the fibre ring of $R' \otimes_R S$ at $\mathfrak p'$ is isomorphic to $F \otimes_{\kappa(\mathfrak p)} L$. The primes lying over $\mathfrak q_i$ correspond to primes of $\kappa(\mathfrak q_i) \otimes_{\kappa(\mathfrak p)} L$ which is a product of fields purely inseparable over L by our choice of L and elementary field theory. These are also the only primes over $\mathfrak p'$ at which $R' \to R' \otimes_R S$ is quasi-finite, by Lemma 122.8. Hence after replacing R by R', $\mathfrak p$ by $\mathfrak p'$, and S by $R' \otimes_R S$ we may assume that for all primes $\mathfrak{q}$ lying over $\mathfrak{p}$ for which S/R is quasi-finite the field extensions $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ are purely inseparable.

Next apply Lemma 145.3. The result is what we want since the field extensions do not change under this étale ring extension. □

146. Local homomorphisms

053J Some lemmas which don't have a natural section to go into. The first lemma says that

0GSD **Lemma 146.1.** *Let $(R, \mathfrak{m}_R) \to (S, \mathfrak{m}_S)$ be a local homomorphism of local rings. Assume S is the localization of an étale ring extension of R and that $\kappa(\mathfrak{m}_R) \to \kappa(\mathfrak{m}_S)$ is an isomorphism. Then there exists an $t \in \mathfrak{m}_R$ such that $R/t^nR \to S/t^nS$ is an isomorphsm for all $n \geq 1$.* [Lin82, Lemma on page 321], [Ces22, Lemma 4.1.5]

Proof. Write $S = T_\mathfrak{q}$ for some étale R-algebra T and prime ideal $\mathfrak{q} \subset T$ lying over $\mathfrak{m}_R$. By Proposition 144.4 we may assume $R \to T$ is standard étale. Write $T = R[x]_g/(f)$ as in Definition 144.1. By our assumption on residue fields, we may choose $a \in R$ such that x and a have the same image in $\kappa(\mathfrak{q}) = \kappa(\mathfrak{m}_S) = \kappa(\mathfrak{m}_R)$. Then after replacing x by $x - a$ we may assume that $\mathfrak{q}$ is generated by x and $\mathfrak{m}_R$ in T. In particular $t = f(0) \in \mathfrak{m}_R$. We will show that $t = f(0)$ works.

Write $f = x^d + \sum_{i=1,\ldots,d-1} a_i x^i + t$. Since $R \to T$ is standard étale we find that a_1 is a unit in R: the derivative of f is invertible in T in particular is not contained in $\mathfrak{q}$. Let $h = a_1 + a_2 x + \ldots + a_{d-1}x^{d-2} + x^{d-1} \in R[x]$ so that $f = t + xh$ in $R[x]$. We see that $h \notin \mathfrak{q}$ and hence we may replace T by $R[x]_{hg}/(f)$. After this replacement we see that

$$T/tT = (R/tR)[x]_{hg}/(f) = (R/tR)[x]_{hg}/(xh) = (R/tR)[x]_{hg}/(x)$$

is a quotient of R/tR. By Lemma 126.9 we conclude that $R/t^nR \to T/t^nT$ is surjective for all $n \geq 1$. On the other hand, we know that the flat local ring map $R/t^nR \to S/t^nS$ factors through $R/t^nR \to T/t^nT$ for all n, hence these maps are also injective (a flat local homomorphism of local rings is faithfully flat and hence injective, see Lemmas 39.17 and 82.11). As S is the localization of T we see that S/t^nS is the localization of $T/t^nT = R/t^nR$ at a prime lying over the maximal ideal, but this ring is already local and the proof is complete. □

053K **Lemma 146.2.** *Let $(R, \mathfrak{m}_R) \to (S, \mathfrak{m}_S)$ be a local homomorphism of local rings. Assume S is the localization of an étale ring extension of R. Then there exists a finite, finitely presented, faithfully flat ring map $R \to S'$ such that for every maximal ideal $\mathfrak{m}'$ of S' there is a factorization*

$$R \to S \to S'_{\mathfrak{m}'}.$$

of the ring map $R \to S'_{\mathfrak{m}'}$.

Proof. Write $S = T_\mathfrak{q}$ for some étale R-algebra T. By Proposition 144.4 we may assume T is standard étale. Apply Lemma 144.5 to the ring map $R \to T$ to get $R \to S'$. Then in particular for every maximal ideal $\mathfrak{m}'$ of S' we get a factorization $\varphi : T \to S'_{g'}$ for some $g' \notin \mathfrak{m}'$ such that $\mathfrak{q} = \varphi^{-1}(\mathfrak{m}'S'_{g'})$. Thus φ induces the desired local ring map $S \to S'_{\mathfrak{m}'}$. □

147. Integral closure and smooth base change

03GC

03GD **Lemma 147.1.** *Let R be a ring. Let $f \in R[x]$ be a monic polynomial. Let $R \to B$ be a ring map. If $h \in B[x]/(f)$ is integral over R, then the element $f'h$ can be written as $f'h = \sum_i b_i x^i$ with $b_i \in B$ integral over R.*

Proof. Say $h^e + r_1 h^{e-1} + \ldots + r_e = 0$ in the ring $B[x]/(f)$ with $r_i \in R$. There exists a finite free ring extension $B \subset B'$ such that $f = (x - \alpha_1) \ldots (x - \alpha_d)$ for some $\alpha_i \in B'$, see Lemma 136.9. Note that each α_i is integral over R. We may represent $h = h_0 + h_1 x + \ldots + h_{d-1} x^{d-1}$ with $h_i \in B$. Then it is a universal fact that

$$f'h \equiv \sum\nolimits_{i=1,\ldots,d} h(\alpha_i)(x - \alpha_1) \ldots \widehat{(x - \alpha_i)} \ldots (x - \alpha_d)$$

as elements of $B[x]/(f)$. You prove this by evaluating both sides at the points α_i over the ring $B_{univ} = \mathbf{Z}[\alpha_i, h_j]$ (some details omitted). By our assumption that h satisfies $h^e + r_1 h^{e-1} + \ldots + r_e = 0$ in the ring $B[x]/(f)$ we see that

$$h(\alpha_i)^e + r_1 h(\alpha_i)^{e-1} + \ldots + r_e = 0$$

in B'. Hence $h(\alpha_i)$ is integral over R. Using the formula above we see that $f'h \equiv \sum_{j=0,\ldots,d-1} b'_j x^j$ in $B'[x]/(f)$ with $b'_j \in B'$ integral over R. However, since $f'h \in B[x]/(f)$ and since $1, x, \ldots, x^{d-1}$ is a B'-basis for $B'[x]/(f)$ we see that $b'_j \in B$ as desired. $\square$

03GE **Lemma 147.2.** *Let $R \to S$ be an étale ring map. Let $R \to B$ be any ring map. Let $A \subset B$ be the integral closure of R in B. Let $A' \subset S \otimes_R B$ be the integral closure of S in $S \otimes_R B$. Then the canonical map $S \otimes_R A \to A'$ is an isomorphism.*

Proof. The map $S \otimes_R A \to A'$ is injective because $A \subset B$ and $R \to S$ is flat. We are going to use repeatedly that taking integral closure commutes with localization, see Lemma 36.11. Hence we may localize on S, by Lemma 23.2 (the criterion for checking whether an S-module map is an isomorphism). Thus we may assume that $S = R[x]_g/(f) = (R[x]/(f))_g$ is standard étale over R, see Proposition 144.4. Applying localization one more time we see that A' is $(A'')_g$ where A'' is the integral closure of $R[x]/(f)$ in $B[x]/(f)$. Suppose that $a \in A''$. It suffices to show that a is in $S \otimes_R A$. By Lemma 147.1 we see that $f'a = \sum a_i x^i$ with $a_i \in A$. Since f' is invertible in $B[x]_g/(f)$ (by definition of a standard étale ring map) we conclude that $a \in S \otimes_R A$ as desired. $\square$

03GF **Example 147.3.** Let p be a prime number. The ring extension

$$R = \mathbf{Z}[1/p] \subset R' = \mathbf{Z}[1/p][x]/(x^{p-1} + \ldots + x + 1)$$

has the following property: For $d < p$ there exist elements $\alpha_0, \ldots, \alpha_{d-1} \in R'$ such that

$$\prod\nolimits_{0 \leq i < j < d} (\alpha_i - \alpha_j)$$

is a unit in R'. Namely, take α_i equal to the class of x^i in R' for $i = 0, \ldots, p-1$. Then we have

$$T^p - 1 = \prod\nolimits_{i=0,\ldots,p-1} (T - \alpha_i)$$

in $R'[T]$. Namely, the ring $\mathbf{Q}[x]/(x^{p-1} + \ldots + x + 1)$ is a field because the cyclotomic polynomial $x^{p-1} + \ldots + x + 1$ is irreducible over $\mathbf{Q}$ and the α_i are pairwise distinct roots of $T^p - 1$, whence the equality. Taking derivatives on both sides and substituting $T = \alpha_i$ we obtain

$$p\alpha_i^{p-1} = (\alpha_i - \alpha_1)\ldots(\widehat{\alpha_i - \alpha_i})\ldots(\alpha_i - \alpha_1)$$

and we see this is invertible in R'.

03GG **Lemma 147.4.** *Let $R \to S$ be a smooth ring map. Let $R \to B$ be any ring map. Let $A \subset B$ be the integral closure of R in B. Let $A' \subset S \otimes_R B$ be the integral closure of S in $S \otimes_R B$. Then the canonical map $S \otimes_R A \to A'$ is an isomorphism.*

Proof. Arguing as in the proof of Lemma 147.2 we may localize on S. Hence we may assume that $R \to S$ is a standard smooth ring map, see Lemma 137.10. By definition of a standard smooth ring map we see that S is étale over a polynomial ring $R[x_1, \ldots, x_n]$. Since we have seen the result in the case of an étale ring extension (Lemma 147.2) this reduces us to the case where $S = R[x]$. Thus we have to show

$$f = \sum b_i x^i \text{ integral over } R[x] \Leftrightarrow \text{each } b_i \text{ integral over } R.$$

The implication from right to left holds because the set of elements in $B[x]$ integral over $R[x]$ is a ring (Lemma 36.7) and contains x.

Suppose that $f \in B[x]$ is integral over $R[x]$, and assume that $f = \sum_{i<d} b_i x^i$ has degree $< d$. Since integral closure and localization commute, it suffices to show there exist distinct primes p, q such that each b_i is integral both over $R[1/p]$ and over $R[1/q]$. Hence, we can find a finite free ring extension $R \subset R'$ such that R' contains $\alpha_1, \ldots, \alpha_d$ with the property that $\prod_{i<j}(\alpha_i - \alpha_j)$ is a unit in R', see Example 147.3. In this case we have the universal equality

$$f = \sum_i f(\alpha_i)\frac{(x-\alpha_1)\ldots(\widehat{x-\alpha_i})\ldots(x-\alpha_d)}{(\alpha_i-\alpha_1)\ldots(\widehat{\alpha_i-\alpha_i})\ldots(\alpha_i-\alpha_d)}.$$

OK, and the elements $f(\alpha_i)$ are integral over R' since $(R' \otimes_R B)[x] \to R' \otimes_R B$, $h \mapsto h(\alpha_i)$ is a ring map. Hence we see that the coefficients of f in $(R' \otimes_R B)[x]$ are integral over R'. Since R' is finite over R (hence integral over R) we see that they are integral over R also, as desired. $\square$

0CBF **Lemma 147.5.** *Let $R \to S$ and $R \to B$ be ring maps. Let $A \subset B$ be the integral closure of R in B. Let $A' \subset S \otimes_R B$ be the integral closure of S in $S \otimes_R B$. If S is a filtered colimit of smooth R-algebras, then the canonical map $S \otimes_R A \to A'$ is an isomorphism.*

Proof. This follows from the straightforward fact that taking tensor products and taking integral closures commutes with filtered colimits and Lemma 147.4. $\square$

148. Formally unramified maps

00UM It turns out to be logically more efficient to define the notion of a formally unramified map before introducing the notion of a formally étale one.

00UN **Definition 148.1.** Let $R \to S$ be a ring map. We say S is *formally unramified over* R if for every commutative solid diagram

$$\begin{tikzcd} S \arrow[r] \arrow[rd, dashed] & A/I \\ R \arrow[r] \arrow[u] & A \arrow[u] \end{tikzcd}$$

where $I \subset A$ is an ideal of square zero, there exists at most one dotted arrow making the diagram commute.

00UO **Lemma 148.2.** *Let $R \to S$ be a ring map. The following are equivalent:*

(1) *$R \to S$ is formally unramified,*
(2) *the module of differentials $\Omega_{S/R}$ is zero.*

Proof. Let $J = \text{Ker}(S \otimes_R S \to S)$ be the kernel of the multiplication map. Let $A_{univ} = S \otimes_R S/J^2$. Recall that $I_{univ} = J/J^2$ is isomorphic to $\Omega_{S/R}$, see Lemma 131.13. Moreover, the two R-algebra maps $\sigma_1, \sigma_2 : S \to A_{univ}$, $\sigma_1(s) = s \otimes 1 \bmod J^2$, and $\sigma_2(s) = 1 \otimes s \bmod J^2$ differ by the universal derivation $\text{d} : S \to \Omega_{S/R} = I_{univ}$.

Assume $R \to S$ formally unramified. Then we see that $\sigma_1 = \sigma_2$. Hence $\text{d}(s) = 0$ for all $s \in S$. Hence $\Omega_{S/R} = 0$.

Assume that $\Omega_{S/R} = 0$. Let $A, I, R \to A, S \to A/I$ be a solid diagram as in Definition 148.1. Let $\tau_1, \tau_2 : S \to A$ be two dotted arrows making the diagram commute. Consider the R-algebra map $A_{univ} \to A$ defined by the rule $s_1 \otimes s_2 \mapsto \tau_1(s_1)\tau_2(s_2)$. We omit the verification that this is well defined. Since $A_{univ} \cong S$ as $I_{univ} = \Omega_{S/R} = 0$ we conclude that $\tau_1 = \tau_2$. ☐

04E8 **Lemma 148.3.** *Let $R \to S$ be a ring map. The following are equivalent:*

(1) *$R \to S$ is formally unramified,*
(2) *$R \to S_{\mathfrak q}$ is formally unramified for all primes $\mathfrak q$ of S, and*
(3) *$R_{\mathfrak p} \to S_{\mathfrak q}$ is formally unramified for all primes $\mathfrak q$ of S with $\mathfrak p = R \cap \mathfrak q$.*

Proof. We have seen in Lemma 148.2 that (1) is equivalent to $\Omega_{S/R} = 0$. Similarly, by Lemma 131.8 we see that (2) and (3) are equivalent to $(\Omega_{S/R})_{\mathfrak q} = 0$ for all $\mathfrak q$. Hence the equivalence follows from Lemma 23.1. ☐

04E9 **Lemma 148.4.** *Let $A \to B$ be a formally unramified ring map.*

(1) *For $S \subset A$ a multiplicative subset, $S^{-1}A \to S^{-1}B$ is formally unramified.*
(2) *For $S \subset B$ a multiplicative subset, $A \to S^{-1}B$ is formally unramified.*

Proof. Follows from Lemma 148.3. (You can also deduce it from Lemma 148.2 combined with Lemma 131.8.) ☐

07QE **Lemma 148.5.** *Let R be a ring. Let I be a directed set. Let $(S_i, \varphi_{ii'})$ be a system of R-algebras over I. If each $R \to S_i$ is formally unramified, then $S = \text{colim}_{i \in I} S_i$ is formally unramified over R*

Proof. Consider a diagram as in Definition 148.1. By assumption there exists at most one R-algebra map $S_i \to A$ lifting the compositions $S_i \to S \to A/I$. Since every element of S is in the image of one of the maps $S_i \to S$ we see that there is at most one map $S \to A$ fitting into the diagram. ☐

149. Conormal modules and universal thickenings

04EA It turns out that one can define the first infinitesimal neighbourhood not just for a closed immersion of schemes, but already for any formally unramified morphism. This is based on the following algebraic fact.

04EB **Lemma 149.1.** *Let $R \to S$ be a formally unramified ring map. There exists a surjection of R-algebras $S' \to S$ whose kernel is an ideal of square zero with the following universal property: Given any commutative diagram*

$$\begin{tikzcd} S \arrow[r, "a"'] & A/I \\ R \arrow[u] \arrow[r, "b"] & A \arrow[u] \end{tikzcd}$$

where $I \subset A$ is an ideal of square zero, there is a unique R-algebra map $a' : S' \to A$ such that $S' \to A \to A/I$ is equal to $S' \to S \to A/I$.

Proof. Choose a set of generators $z_i \in S$, $i \in I$ for S as an R-algebra. Let $P = R[\{x_i\}_{i \in I}]$ denote the polynomial ring on generators x_i, $i \in I$. Consider the R-algebra map $P \to S$ which maps x_i to z_i. Let $J = \text{Ker}(P \to S)$. Consider the map

$$\mathrm{d} : J/J^2 \longrightarrow \Omega_{P/R} \otimes_P S$$

see Lemma 131.9. This is surjective since $\Omega_{S/R} = 0$ by assumption, see Lemma 148.2. Note that $\Omega_{P/R}$ is free on $\mathrm{d}x_i$, and hence the module $\Omega_{P/R} \otimes_P S$ is free over S. Thus we may choose a splitting of the surjection above and write

$$J/J^2 = K \oplus \Omega_{P/R} \otimes_P S$$

Let $J^2 \subset J' \subset J$ be the ideal of P such that J'/J^2 is the second summand in the decomposition above. Set $S' = P/J'$. We obtain a short exact sequence

$$0 \to J/J' \to S' \to S \to 0$$

and we see that $J/J' \cong K$ is a square zero ideal in S'. Hence

$$\begin{tikzcd} S \arrow[r, "1"'] & S \\ R \arrow[u] \arrow[r] & S' \arrow[u] \end{tikzcd}$$

is a diagram as above. In fact we claim that this is an initial object in the category of diagrams. Namely, let $(I \subset A, a, b)$ be an arbitrary diagram. We may choose an R-algebra map $\beta : P \to A$ such that

$$\begin{tikzcd} S \arrow[r, "1"'] & S \arrow[r, "a"'] & A/I \\ R \arrow[u] \arrow[r] \arrow[rr, bend right, "b"'] & P \arrow[u] \arrow[r, "\beta"] & A \arrow[u] \end{tikzcd}$$

is commutative. Now it may not be the case that $\beta(J') = 0$, in other words it may not be true that β factors through $S' = P/J'$. But what is clear is that $\beta(J') \subset I$ and since $\beta(J) \subset I$ and $I^2 = 0$ we have $\beta(J^2) = 0$. Thus the "obstruction" to finding a morphism from $(J/J' \subset S', 1, R \to S')$ to $(I \subset A, a, b)$ is the corresponding S-linear map $\overline{\beta} : J'/J^2 \to I$. The choice in picking β lies in the choice of $\beta(x_i)$. A different choice of β, say β', is gotten by taking $\beta'(x_i) = \beta(x_i) + \delta_i$ with $\delta_i \in I$. In this case, for $g \in J'$, we obtain

$$\beta'(g) = \beta(g) + \sum\nolimits_i \delta_i \frac{\partial g}{\partial x_i}.$$

Since the map $\mathrm{d}|_{J'/J^2} : J'/J^2 \to \Omega_{P/R} \otimes_P S$ given by $g \mapsto \frac{\partial g}{\partial x_i} \mathrm{d}x_i$ is an isomorphism by construction, we see that there is a unique choice of $\delta_i \in I$ such that $\beta'(g) = 0$ for all $g \in J'$. (Namely, δ_i is $-\overline{\beta}(g)$ where $g \in J'/J^2$ is the unique element with $\frac{\partial g}{\partial x_j} = 1$ if $i = j$ and 0 else.) The uniqueness of the solution implies the uniqueness required in the lemma. $\square$

In the situation of Lemma 149.1 the R-algebra map $S' \to S$ is unique up to unique isomorphism.

04EC **Definition 149.2.** Let $R \to S$ be a formally unramified ring map.

1. The *universal first order thickening* of S over R is the surjection of R-algebras $S' \to S$ of Lemma 149.1.
2. The *conormal module* of $R \to S$ is the kernel I of the universal first order thickening $S' \to S$, seen as an S-module.

We often denote the conormal module $C_{S/R}$ in this situation.

04ED **Lemma 149.3.** *Let $I \subset R$ be an ideal of a ring. The universal first order thickening of R/I over R is the surjection $R/I^2 \to R/I$. The conormal module of R/I over R is $C_{(R/I)/R} = I/I^2$.*

Proof. Omitted. $\square$

04EE **Lemma 149.4.** *Let $A \to B$ be a formally unramified ring map. Let $\varphi : B' \to B$ be the universal first order thickening of B over A.*

1. *Let $S \subset A$ be a multiplicative subset. Then $S^{-1}B' \to S^{-1}B$ is the universal first order thickening of $S^{-1}B$ over $S^{-1}A$. In particular $S^{-1}C_{B/A} = C_{S^{-1}B/S^{-1}A}$.*
2. *Let $S \subset B$ be a multiplicative subset. Then $S' = \varphi^{-1}(S)$ is a multiplicative subset in B' and $(S')^{-1}B' \to S^{-1}B$ is the universal first order thickening of $S^{-1}B$ over A. In particular $S^{-1}C_{B/A} = C_{S^{-1}B/A}$.*

Note that the lemma makes sense by Lemma 148.4.

Proof. With notation and assumptions as in (1). Let $(S^{-1}B)' \to S^{-1}B$ be the universal first order thickening of $S^{-1}B$ over $S^{-1}A$. Note that $S^{-1}B' \to S^{-1}B$ is a surjection of $S^{-1}A$-algebras whose kernel has square zero. Hence by definition we obtain a map $(S^{-1}B)' \to S^{-1}B'$ compatible with the maps towards $S^{-1}B$. Consider any commutative diagram

$$\begin{tikzcd} B \arrow[r] & S^{-1}B \arrow[r] & D/I \\ A \arrow[u] \arrow[r] & S^{-1}A \arrow[u] \arrow[r] & D \arrow[u] \end{tikzcd}$$

where $I \subset D$ is an ideal of square zero. Since B' is the universal first order thickening of B over A we obtain an A-algebra map $B' \to D$. But it is clear that the image of S in D is mapped to invertible elements of D, and hence we obtain a compatible map $S^{-1}B' \to D$. Applying this to $D = (S^{-1}B)'$ we see that we get a map $S^{-1}B' \to (S^{-1}B)'$. We omit the verification that this map is inverse to the map described above.

With notation and assumptions as in (2). Let $(S^{-1}B)' \to S^{-1}B$ be the universal first order thickening of $S^{-1}B$ over A. Note that $(S')^{-1}B' \to S^{-1}B$ is a surjection of A-algebras whose kernel has square zero. Hence by definition we obtain a map $(S^{-1}B)' \to (S')^{-1}B'$ compatible with the maps towards $S^{-1}B$. Consider any commutative diagram

$$\begin{tikzcd} B \arrow[r] & S^{-1}B \arrow[r] & D/I \\ A \arrow[r] \arrow[u] & A \arrow[r] \arrow[u] & D \arrow[u] \end{tikzcd}$$

where $I \subset D$ is an ideal of square zero. Since B' is the universal first order thickening of B over A we obtain an A-algebra map $B' \to D$. But it is clear that the image of S' in D is mapped to invertible elements of D, and hence we obtain a compatible map $(S')^{-1}B' \to D$. Applying this to $D = (S^{-1}B)'$ we see that we get a map $(S')^{-1}B' \to (S^{-1}B)'$. We omit the verification that this map is inverse to the map described above. □

04EF **Lemma 149.5.** *Let $R \to A \to B$ be ring maps. Assume $A \to B$ formally unramified. Let $B' \to B$ be the universal first order thickening of B over A. Then B' is formally unramified over A, and the canonical map $\Omega_{A/R} \otimes_A B \to \Omega_{B'/R} \otimes_{B'} B$ is an isomorphism.*

Proof. We are going to use the construction of B' from the proof of Lemma 149.1 although in principle it should be possible to deduce these results formally from the definition. Namely, we choose a presentation $B = P/J$, where $P = A[x_i]$ is a polynomial ring over A. Next, we choose elements $f_i \in J$ such that $\mathrm{d}f_i = \mathrm{d}x_i \otimes 1$ in $\Omega_{P/A} \otimes_P B$. Having made these choices we have $B' = P/J'$ with $J' = (f_i) + J^2$, see proof of Lemma 149.1.

Consider the canonical exact sequence

$$J'/(J')^2 \to \Omega_{P/A} \otimes_P B' \to \Omega_{B'/A} \to 0$$

see Lemma 131.9. By construction the classes of the $f_i \in J'$ map to elements of the module $\Omega_{P/A} \otimes_P B'$ which generate it modulo J'/J^2 by construction. Since J'/J^2 is a nilpotent ideal, we see that these elements generate the module altogether (by Nakayama's Lemma 20.1). This proves that $\Omega_{B'/A} = 0$ and hence that B' is formally unramified over A, see Lemma 148.2.

Since P is a polynomial ring over A we have $\Omega_{P/R} = \Omega_{A/R} \otimes_A P \oplus \bigoplus P\mathrm{d}x_i$. We are going to use this decomposition. Consider the following exact sequence

$$J'/(J')^2 \to \Omega_{P/R} \otimes_P B' \to \Omega_{B'/R} \to 0$$

see Lemma 131.9. We may tensor this with B and obtain the exact sequence

$$J'/(J')^2 \otimes_{B'} B \to \Omega_{P/R} \otimes_P B \to \Omega_{B'/R} \otimes_{B'} B \to 0$$

If we remember that $J' = (f_i) + J^2$ then we see that the first arrow annihilates the submodule $J^2/(J')^2$. In terms of the direct sum decomposition $\Omega_{P/R} \otimes_P B = \Omega_{A/R} \otimes_A B \oplus \bigoplus B\mathrm{d}x_i$ given we see that the submodule $(f_i)/(J')^2 \otimes_{B'} B$ maps isomorphically onto the summand $\bigoplus B\mathrm{d}x_i$. Hence what is left of this exact sequence is an isomorphism $\Omega_{A/R} \otimes_A B \to \Omega_{B'/R} \otimes_{B'} B$ as desired. □

150. Formally étale maps

00UP

00UQ **Definition 150.1.** Let $R \to S$ be a ring map. We say S is *formally étale over R* if for every commutative solid diagram

$$\begin{tikzcd} S \arrow[r] \arrow[rd, dashed] & A/I \\ R \arrow[r] \arrow[u] & A \arrow[u] \end{tikzcd}$$

where $I \subset A$ is an ideal of square zero, there exists a unique dotted arrow making the diagram commute.

Clearly a ring map is formally étale if and only if it is both formally smooth and formally unramified.

00UR **Lemma 150.2.** *Let $R \to S$ be a ring map of finite presentation. The following are equivalent:*

(1) *$R \to S$ is formally étale,*
(2) *$R \to S$ is étale.*

Proof. Assume that $R \to S$ is formally étale. Then $R \to S$ is smooth by Proposition 138.13. By Lemma 148.2 we have $\Omega_{S/R} = 0$. Hence $R \to S$ is étale by definition.

Assume that $R \to S$ is étale. Then $R \to S$ is formally smooth by Proposition 138.13. By Lemma 148.2 it is formally unramified. Hence $R \to S$ is formally étale. $\square$

031N **Lemma 150.3.** *Let R be a ring. Let I be a directed set. Let $(S_i, \varphi_{ii'})$ be a system of R-algebras over I. If each $R \to S_i$ is formally étale, then $S = \mathrm{colim}_{i \in I} S_i$ is formally étale over R*

Proof. Consider a diagram as in Definition 150.1. By assumption we get unique R-algebra maps $S_i \to A$ lifting the compositions $S_i \to S \to A/I$. Hence these are compatible with the transition maps $\varphi_{ii'}$ and define a lift $S \to A$. This proves existence. The uniqueness is clear by restricting to each S_i. $\square$

04EG **Lemma 150.4.** *Let R be a ring. Let $S \subset R$ be any multiplicative subset. Then the ring map $R \to S^{-1}R$ is formally étale.*

Proof. Let $I \subset A$ be an ideal of square zero. What we are saying here is that given a ring map $\varphi : R \to A$ such that $\varphi(f) \mod I$ is invertible for all $f \in S$ we have also that $\varphi(f)$ is invertible in A for all $f \in S$. This is true because A^* is the inverse image of $(A/I)^*$ under the canonical map $A \to A/I$. $\square$

151. Unramified ring maps

00US The definition of a G-unramified ring map is the one from EGA. The definition of an unramified ring map is the one from [Ray70].

00UT **Definition 151.1.** Let $R \to S$ be a ring map.

(1) We say $R \to S$ is *unramified* if $R \to S$ is of finite type and $\Omega_{S/R} = 0$.

(2) We say $R \to S$ is *G-unramified* if $R \to S$ is of finite presentation and $\Omega_{S/R} = 0$.
(3) Given a prime $\mathfrak{q}$ of S we say that S is *unramified at* $\mathfrak{q}$ if there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is unramified.
(4) Given a prime $\mathfrak{q}$ of S we say that S is *G-unramified at* $\mathfrak{q}$ if there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is G-unramified.

Of course a G-unramified map is unramified.

00UU **Lemma 151.2.** *Let $R \to S$ be a ring map. The following are equivalent*

(1) *$R \to S$ is formally unramified and of finite type, and*
(2) *$R \to S$ is unramified.*

Moreover, also the following are equivalent

(1) *$R \to S$ is formally unramified and of finite presentation, and*
(2) *$R \to S$ is G-unramified.*

Proof. Follows from Lemma 148.2 and the definitions. ☐

00UV **Lemma 151.3.** *Properties of unramified and G-unramified ring maps.*

(1) *The base change of an unramified ring map is unramified. The base change of a G-unramified ring map is G-unramified.*
(2) *The composition of unramified ring maps is unramified. The composition of G-unramified ring maps is G-unramified.*
(3) *Any principal localization $R \to R_f$ is G-unramified and unramified.*
(4) *If $I \subset R$ is an ideal, then $R \to R/I$ is unramified. If $I \subset R$ is a finitely generated ideal, then $R \to R/I$ is G-unramified.*
(5) *An étale ring map is G-unramified and unramified.*
(6) *If $R \to S$ is of finite type (resp. finite presentation), $\mathfrak{q} \subset S$ is a prime and $(\Omega_{S/R})_{\mathfrak{q}} = 0$, then $R \to S$ is unramified (resp. G-unramified) at $\mathfrak{q}$.*
(7) *If $R \to S$ is of finite type (resp. finite presentation), $\mathfrak{q} \subset S$ is a prime and $\Omega_{S/R} \otimes_S \kappa(\mathfrak{q}) = 0$, then $R \to S$ is unramified (resp. G-unramified) at $\mathfrak{q}$.*
(8) *If $R \to S$ is of finite type (resp. finite presentation), $\mathfrak{q} \subset S$ is a prime lying over $\mathfrak{p} \subset R$ and $(\Omega_{S \otimes_R \kappa(\mathfrak{p})/\kappa(\mathfrak{p})})_{\mathfrak{q}} = 0$, then $R \to S$ is unramified (resp. G-unramified) at $\mathfrak{q}$.*
(9) *If $R \to S$ is of finite type (resp. presentation), $\mathfrak{q} \subset S$ is a prime lying over $\mathfrak{p} \subset R$ and $(\Omega_{S \otimes_R \kappa(\mathfrak{p})/\kappa(\mathfrak{p})}) \otimes_{S \otimes_R \kappa(\mathfrak{p})} \kappa(\mathfrak{q}) = 0$, then $R \to S$ is unramified (resp. G-unramified) at $\mathfrak{q}$.*
(10) *If $R \to S$ is a ring map, $g_1, \ldots, g_m \in S$ generate the unit ideal and $R \to S_{g_j}$ is unramified (resp. G-unramified) for $j = 1, \ldots, m$, then $R \to S$ is unramified (resp. G-unramified).*
(11) *If $R \to S$ is a ring map which is unramified (resp. G-unramified) at every prime of S, then $R \to S$ is unramified (resp. G-unramified).*
(12) *If $R \to S$ is G-unramified, then there exists a finite type $\mathbf{Z}$-algebra R_0 and a G-unramified ring map $R_0 \to S_0$ and a ring map $R_0 \to R$ such that $S = R \otimes_{R_0} S_0$.*
(13) *If $R \to S$ is unramified, then there exists a finite type $\mathbf{Z}$-algebra R_0 and an unramified ring map $R_0 \to S_0$ and a ring map $R_0 \to R$ such that S is a quotient of $R \otimes_{R_0} S_0$.*

Proof. We prove each point, in order.

Ad (1). Follows from Lemmas 131.12 and 14.2.

Ad (2). Follows from Lemmas 131.7 and 14.2.

Ad (3). Follows by direct computation of $\Omega_{R_f/R}$ which we omit.

Ad (4). We have $\Omega_{(R/I)/R} = 0$, see Lemma 131.4, and the ring map $R \to R/I$ is of finite type. If I is a finitely generated ideal then $R \to R/I$ is of finite presentation.

Ad (5). See discussion following Definition 143.1.

Ad (6). In this case $\Omega_{S/R}$ is a finite S-module (see Lemma 131.16) and hence there exists a $g \in S$, $g \notin \mathfrak q$ such that $(\Omega_{S/R})_g = 0$. By Lemma 131.8 this means that $\Omega_{S_g/R} = 0$ and hence $R \to S_g$ is unramified as desired.

Ad (7). Use Nakayama's lemma (Lemma 20.1) to see that the condition is equivalent to the condition of (6).

Ad (8) and (9). These are equivalent in the same manner that (6) and (7) are equivalent. Moreover $\Omega_{S \otimes_R \kappa(\mathfrak p)/\kappa(\mathfrak p)} = \Omega_{S/R} \otimes_S (S \otimes_R \kappa(\mathfrak p))$ by Lemma 131.12. Hence we see that (9) is equivalent to (7) since the $\kappa(\mathfrak q)$ vector spaces in both are canonically isomorphic.

Ad (10). Follows from Lemmas 23.2 and 131.8.

Ad (11). Follows from (6) and (7) and the fact that the spectrum of S is quasi-compact.

Ad (12). Write $S = R[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. As $\Omega_{S/R} = 0$ we can write

$$\mathrm{d}x_i = \sum h_{ij}\mathrm{d}g_j + \sum a_{ijk}g_j\mathrm{d}x_k$$

in $\Omega_{R[x_1,\ldots,x_n]/R}$ for some $h_{ij}, a_{ijk} \in R[x_1, \ldots, x_n]$. Choose a finitely generated $\mathbf{Z}$-subalgebra $R_0 \subset R$ containing all the coefficients of the polynomials g_i, h_{ij}, a_{ijk}. Set $S_0 = R_0[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. This works.

Ad (13). Write $S = R[x_1, \ldots, x_n]/I$. As $\Omega_{S/R} = 0$ we can write

$$\mathrm{d}x_i = \sum h_{ij}\mathrm{d}g_{ij} + \sum g'_{ik}\mathrm{d}x_k$$

in $\Omega_{R[x_1,\ldots,x_n]/R}$ for some $h_{ij} \in R[x_1, \ldots, x_n]$ and $g_{ij}, g'_{ik} \in I$. Choose a finitely generated $\mathbf{Z}$-subalgebra $R_0 \subset R$ containing all the coefficients of the polynomials g_{ij}, h_{ij}, g'_{ik}. Set $S_0 = R_0[x_1, \ldots, x_n]/(g_{ij}, g'_{ik})$. This works. □

02FL **Lemma 151.4.** *Let $R \to S$ be a ring map. If $R \to S$ is unramified, then there exists an idempotent $e \in S \otimes_R S$ such that $S \otimes_R S \to S$ is isomorphic to $S \otimes_R S \to (S \otimes_R S)_e$.*

Proof. Let $J = \mathrm{Ker}(S \otimes_R S \to S)$. By assumption $J/J^2 = 0$, see Lemma 131.13. Since S is of finite type over R we see that J is finitely generated, namely by $x_i \otimes 1 - 1 \otimes x_i$, where x_i generate S over R. We win by Lemma 21.5. □

00UW **Lemma 151.5.** *Let $R \to S$ be a ring map. Let $\mathfrak q \subset S$ be a prime lying over $\mathfrak p$ in R. If S/R is unramified at $\mathfrak q$ then*

1. *we have $\mathfrak p S_{\mathfrak q} = \mathfrak q S_{\mathfrak q}$ is the maximal ideal of the local ring $S_{\mathfrak q}$, and*
2. *the field extension $\kappa(\mathfrak q)/\kappa(\mathfrak p)$ is finite separable.*

Proof. We may first replace S by S_g for some $g \in S$, $g \notin \mathfrak q$ and assume that $R \to S$ is unramified. The base change $S \otimes_R \kappa(\mathfrak p)$ is unramified over $\kappa(\mathfrak p)$ by Lemma 151.3. By Lemma 140.3 it is smooth hence étale over $\kappa(\mathfrak p)$. Hence we see that $S \otimes_R \kappa(\mathfrak p) = (R \setminus \mathfrak p)^{-1}S/\mathfrak pS$ is a product of finite separable field extensions of $\kappa(\mathfrak p)$ by Lemma 143.4. This implies the lemma. ☐

02UR **Lemma 151.6.** *Let $R \to S$ be a finite type ring map. Let $\mathfrak q$ be a prime of S. If $R \to S$ is unramified at $\mathfrak q$ then $R \to S$ is quasi-finite at $\mathfrak q$. In particular, an unramified ring map is quasi-finite.*

Proof. An unramified ring map is of finite type. Thus it is clear that the second statement follows from the first. To see the first statement apply the characterization of Lemma 122.2 part (2) using Lemma 151.5. ☐

02FM **Lemma 151.7.** *Let $R \to S$ be a ring map. Let $\mathfrak q$ be a prime of S lying over a prime $\mathfrak p$ of R. If*

(1) *$R \to S$ is of finite type,*
(2) *$\mathfrak pS_{\mathfrak q}$ is the maximal ideal of the local ring $S_{\mathfrak q}$, and*
(3) *the field extension $\kappa(\mathfrak q)/\kappa(\mathfrak p)$ is finite separable,*

then $R \to S$ is unramified at $\mathfrak q$.

Proof. By Lemma 151.3 (8) it suffices to show that $\Omega_{S\otimes_R\kappa(\mathfrak p)/\kappa(\mathfrak p)}$ is zero when localized at $\mathfrak q$. Hence we may replace S by $S \otimes_R \kappa(\mathfrak p)$ and R by $\kappa(\mathfrak p)$. In other words, we may assume that $R = k$ is a field and S is a finite type k-algebra. In this case the hypotheses imply that $S_{\mathfrak q} \cong \kappa(\mathfrak q)$. Thus $(\Omega_{S/k})_{\mathfrak q} = \Omega_{S_{\mathfrak q}/k} = \Omega_{\kappa(\mathfrak q)/k}$ is zero as desired (the first equality is Lemma 131.8). ☐

08WD **Lemma 151.8.** *Let $R \to S$ be a ring map. The following are equivalent*

(1) *$R \to S$ is étale,*
(2) *$R \to S$ is flat and G-unramified, and*
(3) *$R \to S$ is flat, unramified, and of finite presentation.*

Proof. Parts (2) and (3) are equivalent by definition. The implication (1) ⇒ (3) follows from the fact that étale ring maps are of finite presentation, Lemma 143.3 (flatness of étale maps), and Lemma 151.3 (étale maps are unramified). Conversely, the characterization of étale ring maps in Lemma 143.7 and the structure of unramified ring maps in Lemma 151.5 shows that (3) implies (1). (This uses that $R \to S$ is étale if $R \to S$ is étale at every prime $\mathfrak q \subset S$, see Lemma 143.3.) ☐

0G1C **Lemma 151.9.** *Let k be a field. Let*

$$\varphi : k[x_1, \ldots, x_n] \to A, \quad x_i \longmapsto a_i$$

be a finite type ring map. Then φ is étale if and only if we have the following two conditions: (a) the local rings of A at maximal ideals have dimension n, and (b) the elements $d(a_1), \ldots, d(a_n)$ generate $\Omega_{A/k}$ as an A-module.

Proof. Assume (a) and (b). Condition (b) implies that $\Omega_{A/k[x_1,\ldots,x_n]} = 0$ and hence φ is unramified. Thus it suffices to prove that φ is flat, see Lemma 151.8. Let $\mathfrak m \subset A$ be a maximal ideal. Set $X = \mathrm{Spec}(A)$ and denote $x \in X$ the closed point corresponding to $\mathfrak m$. Then $\dim(A_{\mathfrak m})$ is $\dim_x X$, see Lemma 114.6. Thus by Lemma 140.3 we see that if (a) and (b) hold, then $A_{\mathfrak m}$ is a regular local ring for every maximal ideal $\mathfrak m$. Then $k[x_1, \ldots, x_n]_{\varphi^{-1}(\mathfrak m)} \to A_{\mathfrak m}$ is flat by Lemma 128.1

(and the fact that a regular local ring is CM, see Lemma 106.3). Thus φ is flat by Lemma 39.18.

Assume φ is étale. Then $\Omega_{A/k[x_1,\ldots,x_n]} = 0$ and hence (b) holds. On the other hand, étale ring maps are flat (Lemma 143.3) and quasi-finite (Lemma 143.6). Hence for every maximal ideal $\mathfrak{m}$ of A we my apply Lemma 112.7 to $k[x_1,\ldots,x_n]_{\varphi^{-1}(\mathfrak{m})} \to A_{\mathfrak{m}}$ to see that $\dim(A_{\mathfrak{m}}) = n$ and hence (a) holds. $\square$

152. Local structure of unramified ring maps

0G1D An unramified morphism is locally (in a suitable sense) the composition of a closed immersion and an étale morphism. The algebraic underpinnings of this fact are discussed in this section.

0395 **Proposition 152.1.** *Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime. If $R \to S$ is unramified at $\mathfrak{q}$, then there exist*

1. *a $g \in S$, $g \notin \mathfrak{q}$,*
2. *a standard étale ring map $R \to S'$, and*
3. *a surjective R-algebra map $S' \to S_g$.*

Proof. This proof is the "same" as the proof of Proposition 144.4. The proof is a little roundabout and there may be ways to shorten it.

Step 1. By Definition 151.1 there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is unramified. Thus we may assume that S is unramified over R.

Step 2. By Lemma 151.3 there exists an unramified ring map $R_0 \to S_0$ with R_0 of finite type over $\mathbf{Z}$, and a ring map $R_0 \to R$ such that S is a quotient of $R \otimes_{R_0} S_0$. Denote $\mathfrak{q}_0$ the prime of S_0 corresponding to $\mathfrak{q}$. If we show the result for $(R_0 \to S_0, \mathfrak{q}_0)$ then the result follows for $(R \to S, \mathfrak{q})$ by base change. Hence we may assume that R is Noetherian.

Step 3. Note that $R \to S$ is quasi-finite by Lemma 151.6. By Lemma 123.14 there exists a finite ring map $R \to S'$, an R-algebra map $S' \to S$, an element $g' \in S'$ such that $g' \notin \mathfrak{q}$ such that $S' \to S$ induces an isomorphism $S'_{g'} \cong S_{g'}$. (Note that S' may not be unramified over R.) Thus we may assume that (a) R is Noetherian, (b) $R \to S$ is finite and (c) $R \to S$ is unramified at $\mathfrak{q}$ (but no longer necessarily unramified at all primes).

Step 4. Let $\mathfrak{p} \subset R$ be the prime corresponding to $\mathfrak{q}$. Consider the fibre ring $S \otimes_R \kappa(\mathfrak{p})$. This is a finite algebra over $\kappa(\mathfrak{p})$. Hence it is Artinian (see Lemma 53.2) and so a finite product of local rings

$$S \otimes_R \kappa(\mathfrak{p}) = \prod\nolimits_{i=1}^n A_i$$

see Proposition 60.7. One of the factors, say A_1, is the local ring $S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}$ which is isomorphic to $\kappa(\mathfrak{q})$, see Lemma 151.5. The other factors correspond to the other primes, say $\mathfrak{q}_2, \ldots, \mathfrak{q}_n$ of S lying over $\mathfrak{p}$.

Step 5. We may choose a nonzero element $\alpha \in \kappa(\mathfrak{q})$ which generates the finite separable field extension $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ (so even if the field extension is trivial we do not allow $\alpha = 0$). Note that for any $\lambda \in \kappa(\mathfrak{p})^*$ the element $\lambda\alpha$ also generates $\kappa(\mathfrak{q})$ over $\kappa(\mathfrak{p})$. Consider the element

$$\bar{t} = (\alpha, 0, \ldots, 0) \in \prod\nolimits_{i=1}^n A_i = S \otimes_R \kappa(\mathfrak{p}).$$

After possibly replacing α by $\lambda\alpha$ as above we may assume that $\bar{t}$ is the image of $t \in S$. Let $I \subset R[x]$ be the kernel of the R-algebra map $R[x] \to S$ which maps x to t. Set $S' = R[x]/I$, so $S' \subset S$. Here is a diagram

$$\begin{tikzcd} R[x] \arrow[r] & S' \arrow[r] & S \\ R \arrow[u] \arrow[ur] \arrow[urr] & & \end{tikzcd}$$

By construction the primes $\mathfrak{q}_j$, $j \geq 2$ of S all lie over the prime $(\mathfrak{p}, x)$ of $R[x]$, whereas the prime $\mathfrak{q}$ lies over a different prime of $R[x]$ because $\alpha \neq 0$.

Step 6. Denote $\mathfrak{q}' \subset S'$ the prime of S' corresponding to $\mathfrak{q}$. By the above $\mathfrak{q}$ is the only prime of S lying over $\mathfrak{q}'$. Thus we see that $S_\mathfrak{q} = S_{\mathfrak{q}'}$, see Lemma 41.11 (we have going up for $S' \to S$ by Lemma 36.22 since $S' \to S$ is finite as $R \to S$ is finite). It follows that $S'_{\mathfrak{q}'} \to S_\mathfrak{q}$ is finite and injective as the localization of the finite injective ring map $S' \to S$. Consider the maps of local rings

$$R_\mathfrak{p} \to S'_{\mathfrak{q}'} \to S_\mathfrak{q}$$

The second map is finite and injective. We have $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q} = \kappa(\mathfrak{q})$, see Lemma 151.5. Hence a fortiori $S_\mathfrak{q}/\mathfrak{q}'S_\mathfrak{q} = \kappa(\mathfrak{q})$. Since

$$\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q}') \subset \kappa(\mathfrak{q})$$

and since α is in the image of $\kappa(\mathfrak{q}')$ in $\kappa(\mathfrak{q})$ we conclude that $\kappa(\mathfrak{q}') = \kappa(\mathfrak{q})$. Hence by Nakayama's Lemma 20.1 applied to the $S'_{\mathfrak{q}'}$-module map $S'_{\mathfrak{q}'} \to S_\mathfrak{q}$, the map $S'_{\mathfrak{q}'} \to S_\mathfrak{q}$ is surjective. In other words, $S'_{\mathfrak{q}'} \cong S_\mathfrak{q}$.

Step 7. By Lemma 126.7 there exist $g \in S$, $g \notin \mathfrak{q}$ and $g' \in S'$, $g' \notin \mathfrak{q}'$ such that $S'_{g'} \cong S_g$. As R is Noetherian the ring S' is finite over R because it is an R-submodule of the finite R-module S. Hence after replacing S by S' we may assume that (a) R is Noetherian, (b) S finite over R, (c) S is unramified over R at $\mathfrak{q}$, and (d) $S = R[x]/I$.

Step 8. Consider the ring $S \otimes_R \kappa(\mathfrak{p}) = \kappa(\mathfrak{p})[x]/\overline{I}$ where $\overline{I} = I \cdot \kappa(\mathfrak{p})[x]$ is the ideal generated by I in $\kappa(\mathfrak{p})[x]$. As $\kappa(\mathfrak{p})[x]$ is a PID we know that $\overline{I} = (\overline{h})$ for some monic $\overline{h} \in \kappa(\mathfrak{p})$. After replacing $\overline{h}$ by $\lambda \cdot \overline{h}$ for some $\lambda \in \kappa(\mathfrak{p})$ we may assume that $\overline{h}$ is the image of some $h \in R[x]$. (The problem is that we do not know if we may choose h monic.) Also, as in Step 4 we know that $S \otimes_R \kappa(\mathfrak{p}) = A_1 \times \ldots \times A_n$ with $A_1 = \kappa(\mathfrak{q})$ a finite separable extension of $\kappa(\mathfrak{p})$ and $A_2, \ldots, A_n$ local. This implies that

$$\overline{h} = \overline{h}_1 \overline{h}_2^{e_2} \ldots \overline{h}_n^{e_n}$$

for certain pairwise coprime irreducible monic polynomials $\overline{h}_i \in \kappa(\mathfrak{p})[x]$ and certain $e_2, \ldots, e_n \geq 1$. Here the numbering is chosen so that $A_i = \kappa(\mathfrak{p})[x]/(\overline{h}_i^{e_i})$ as $\kappa(\mathfrak{p})[x]$-algebras. Note that $\overline{h}_1$ is the minimal polynomial of $\alpha \in \kappa(\mathfrak{q})$ and hence is a separable polynomial (its derivative is prime to itself).

Step 9. Let $m \in I$ be a monic element; such an element exists because the ring extension $R \to R[x]/I$ is finite hence integral. Denote $\overline{m}$ the image in $\kappa(\mathfrak{p})[x]$. We may factor

$$\overline{m} = \overline{k}\overline{h}_1^{d_1} \overline{h}_2^{d_2} \ldots \overline{h}_n^{d_n}$$

for some $d_1 \geq 1$, $d_j \geq e_j$, $j = 2, \ldots, n$ and $\overline{k} \in \kappa(\mathfrak{p})[x]$ prime to all the $\overline{h}_i$. Set $f = m^l + h$ where $l \deg(m) > \deg(h)$, and $l \geq 2$. Then f is monic as a polynomial over R. Also, the image $\overline{f}$ of f in $\kappa(\mathfrak{p})[x]$ factors as

$$\overline{f} = \overline{h}_1\overline{h}_2^{e_2} \ldots \overline{h}_n^{e_n} + \overline{k}^l\overline{h}_1^{ld_1}\overline{h}_2^{ld_2} \ldots \overline{h}_n^{ld_n} = \overline{h}_1(\overline{h}_2^{e_2} \ldots \overline{h}_n^{e_n} + \overline{k}^l\overline{h}_1^{ld_1-1}\overline{h}_2^{ld_2} \ldots \overline{h}_n^{ld_n}) = \overline{h}_1\overline{w}$$

with $\overline{w}$ a polynomial relatively prime to $\overline{h}_1$. Set $g = f'$ (the derivative with respect to x).

Step 10. The ring map $R[x] \to S = R[x]/I$ has the properties: (1) it maps f to zero, and (2) it maps g to an element of $S \setminus \mathfrak{q}$. The first assertion is clear since f is an element of I. For the second assertion we just have to show that g does not map to zero in $\kappa(\mathfrak{q}) = \kappa(\mathfrak{p})[x]/(\overline{h}_1)$. The image of g in $\kappa(\mathfrak{p})[x]$ is the derivative of $\overline{f}$. Thus (2) is clear because

$$\overline{g} = \frac{\mathrm{d}\overline{f}}{\mathrm{d}x} = \overline{w}\frac{\mathrm{d}\overline{h}_1}{\mathrm{d}x} + \overline{h}_1\frac{\mathrm{d}\overline{w}}{\mathrm{d}x},$$

$\overline{w}$ is prime to $\overline{h}_1$ and $\overline{h}_1$ is separable.

Step 11. We conclude that $\varphi : R[x]/(f) \to S$ is a surjective ring map, $R[x]_g/(f)$ is étale over R (because it is standard étale, see Lemma 144.2) and $\varphi(g) \notin \mathfrak{q}$. Thus the map $(R[x]/(f))_g \to S_{\varphi(g)}$ is the desired surjection. □

00UX **Lemma 152.2.** *Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime of S lying over $\mathfrak{p} \subset R$. Assume that $R \to S$ is of finite type and unramified at $\mathfrak{q}$. Then there exist*

1. *an étale ring map $R \to R'$,*
2. *a prime $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p}$.*
3. *a product decomposition*
$$R' \otimes_R S = A \times B$$

with the following properties

1. *$R' \to A$ is surjective, and*
2. *$\mathfrak{p}'A$ is a prime of A lying over $\mathfrak{p}'$ and over $\mathfrak{q}$.*

Proof. We may replace $(R \to S, \mathfrak{p}, \mathfrak{q})$ with any base change $(R' \to R' \otimes_R S, \mathfrak{p}', \mathfrak{q}')$ by an étale ring map $R \to R'$ with a prime $\mathfrak{p}'$ lying over $\mathfrak{p}$, and a choice of $\mathfrak{q}'$ lying over both $\mathfrak{q}$ and $\mathfrak{p}'$. Note also that given $R \to R'$ and $\mathfrak{p}'$ a suitable $\mathfrak{q}'$ can always be found.

The assumption that $R \to S$ is of finite type means that we may apply Lemma 145.4. Thus we may assume that $S = A_1 \times \ldots \times A_n \times B$, that each $R \to A_i$ is finite with exactly one prime $\mathfrak{r}_i$ lying over $\mathfrak{p}$ such that $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{r}_i)$ is purely inseparable and that $R \to B$ is not quasi-finite at any prime lying over $\mathfrak{p}$. Then clearly $\mathfrak{q} = \mathfrak{r}_i$ for some i, since an unramified morphism is quasi-finite (see Lemma 151.6). Say $\mathfrak{q} = \mathfrak{r}_1$. By Lemma 151.5 we see that $\kappa(\mathfrak{r}_1)/\kappa(\mathfrak{p})$ is separable hence the trivial field extension, and that $\mathfrak{p}(A_1)_{\mathfrak{r}_1}$ is the maximal ideal. Also, by Lemma 41.11 (which applies to $R \to A_1$ because a finite ring map satisfies going up by Lemma 36.22) we have $(A_1)_{\mathfrak{r}_1} = (A_1)_{\mathfrak{p}}$. It follows from Nakayama's Lemma 20.1 that the map of local rings $R_{\mathfrak{p}} \to (A_1)_{\mathfrak{p}} = (A_1)_{\mathfrak{r}_1}$ is surjective. Since A_1 is finite over R we see that there exists a $f \in R$, $f \notin \mathfrak{p}$ such that $R_f \to (A_1)_f$ is surjective. After replacing R by R_f we win. □

00UY **Lemma 152.3.** *Let $R \to S$ be a ring map. Let $\mathfrak p$ be a prime of R. If $R \to S$ is unramified then there exist*

(1) *an étale ring map $R \to R'$,*
(2) *a prime $\mathfrak p' \subset R'$ lying over $\mathfrak p$.*
(3) *a product decomposition*
$$R' \otimes_R S = A_1 \times \ldots \times A_n \times B$$

with the following properties

(1) *$R' \to A_i$ is surjective,*
(2) *$\mathfrak p' A_i$ is a prime of A_i lying over $\mathfrak p'$, and*
(3) *there is no prime of B lying over $\mathfrak p'$.*

Proof. We may apply Lemma 145.4. Thus, after an étale base change, we may assume that $S = A_1 \times \ldots \times A_n \times B$, that each $R \to A_i$ is finite with exactly one prime $\mathfrak r_i$ lying over $\mathfrak p$ such that $\kappa(\mathfrak p) \subset \kappa(\mathfrak r_i)$ is purely inseparable, and that $R \to B$ is not quasi-finite at any prime lying over $\mathfrak p$. Since $R \to S$ is quasi-finite (see Lemma 151.6) we see there is no prime of B lying over $\mathfrak p$. By Lemma 151.5 we see that $\kappa(\mathfrak r_i)/\kappa(\mathfrak p)$ is separable hence the trivial field extension, and that $\mathfrak p(A_i)_{\mathfrak r_i}$ is the maximal ideal. Also, by Lemma 41.11 (which applies to $R \to A_i$ because a finite ring map satisfies going up by Lemma 36.22) we have $(A_i)_{\mathfrak r_i} = (A_i)_{\mathfrak p}$. It follows from Nakayama's Lemma 20.1 that the map of local rings $R_{\mathfrak p} \to (A_i)_{\mathfrak p} = (A_i)_{\mathfrak r_i}$ is surjective. Since A_i is finite over R we see that there exists a $f \in R$, $f \notin \mathfrak p$ such that $R_f \to (A_i)_f$ is surjective. After replacing R by R_f we win. $\square$

153. Henselian local rings

04GE In this section we discuss a bit the notion of a henselian local ring. Let $(R, \mathfrak m, \kappa)$ be a local ring. For $a \in R$ we denote $\overline{a}$ the image of a in κ. For a polynomial $f \in R[T]$ we often denote $\overline{f}$ the image of f in $\kappa[T]$. Given a polynomial $f \in R[T]$ we denote f' the derivative of f with respect to T. Note that $\overline{f}' = \overline{f'}$.

04GF **Definition 153.1.** Let $(R, \mathfrak m, \kappa)$ be a local ring.

(1) We say R is *henselian* if for every monic $f \in R[T]$ and every root $a_0 \in \kappa$ of $\overline{f}$ such that $\overline{f'}(a_0) \neq 0$ there exists an $a \in R$ such that $f(a) = 0$ and $a_0 = \overline{a}$.
(2) We say R is *strictly henselian* if R is henselian and its residue field is separably algebraically closed.

Note that the condition $\overline{f'}(a_0) \neq 0$ is equivalent to the condition that a_0 is a simple root of the polynomial $\overline{f}$. In fact, it implies that the lift $a \in R$, if it exists, is unique.

06RR **Lemma 153.2.** *Let $(R, \mathfrak m, \kappa)$ be a local ring. Let $f \in R[T]$. Let $a, b \in R$ such that $f(a) = f(b) = 0$, $a = b \bmod \mathfrak m$, and $f'(a) \notin \mathfrak m$. Then $a = b$.*

Proof. Write $f(x+y) - f(x) = f'(x)y + g(x, y)y^2$ in $R[x, y]$ (this is possible as one sees by expanding $f(x + y)$; details omitted). Then we see that $0 = f(b) - f(a) = f(a + (b-a)) - f(a) = f'(a)(b-a) + c(b-a)^2$ for some $c \in R$. By assumption $f'(a)$ is a unit in R. Hence $(b - a)(1 + f'(a)^{-1}c(b - a)) = 0$. By assumption $b - a \in \mathfrak m$, hence $1 + f'(a)^{-1}c(b - a)$ is a unit in R. Hence $b - a = 0$ in R. $\square$

Here is the characterization of henselian local rings.

04GG **Lemma 153.3.** *Let $(R, \mathfrak m, \kappa)$ be a local ring. The following are equivalent*

(1) *R is henselian,*
(2) *for every $f \in R[T]$ and every root $a_0 \in \kappa$ of $\overline{f}$ such that $\overline{f'}(a_0) \neq 0$ there exists an $a \in R$ such that $f(a) = 0$ and $a_0 = \overline{a}$,*
(3) *for any monic $f \in R[T]$ and any factorization $\overline{f} = g_0 h_0$ with $\gcd(g_0, h_0) = 1$ there exists a factorization $f = gh$ in $R[T]$ such that $g_0 = \overline{g}$ and $h_0 = \overline{h}$,*
(4) *for any monic $f \in R[T]$ and any factorization $\overline{f} = g_0 h_0$ with $\gcd(g_0, h_0) = 1$ there exists a factorization $f = gh$ in $R[T]$ such that $g_0 = \overline{g}$ and $h_0 = \overline{h}$ and moreover $\deg_T(g) = \deg_T(g_0)$,*
(5) *for any $f \in R[T]$ and any factorization $\overline{f} = g_0 h_0$ with $\gcd(g_0, h_0) = 1$ there exists a factorization $f = gh$ in $R[T]$ such that $g_0 = \overline{g}$ and $h_0 = \overline{h}$,*
(6) *for any $f \in R[T]$ and any factorization $\overline{f} = g_0 h_0$ with $\gcd(g_0, h_0) = 1$ there exists a factorization $f = gh$ in $R[T]$ such that $g_0 = \overline{g}$ and $h_0 = \overline{h}$ and moreover $\deg_T(g) = \deg_T(g_0)$,*
(7) *for any étale ring map $R \to S$ and prime $\mathfrak{q}$ of S lying over $\mathfrak{m}$ with $\kappa = \kappa(\mathfrak{q})$ there exists a section $\tau : S \to R$ of $R \to S$,*
(8) *for any étale ring map $R \to S$ and prime $\mathfrak{q}$ of S lying over $\mathfrak{m}$ with $\kappa = \kappa(\mathfrak{q})$ there exists a section $\tau : S \to R$ of $R \to S$ with $\mathfrak{q} = \tau^{-1}(\mathfrak{m})$,*
(9) *any finite R-algebra is a product of local rings,*
(10) *any finite R-algebra is a finite product of local rings,*
(11) *any finite type R-algebra S can be written as $A \times B$ with $R \to A$ finite and $R \to B$ not quasi-finite at any prime lying over $\mathfrak{m}$,*
(12) *any finite type R-algebra S can be written as $A \times B$ with $R \to A$ finite such that each irreducible component of $\mathrm{Spec}(B \otimes_R \kappa)$ has dimension ≥ 1, and*
(13) *any quasi-finite R-algebra S can be written as $S = A \times B$ with $R \to A$ finite such that $B \otimes_R \kappa = 0$.*

Proof. Here is a list of the easier implications:

(1) 2⇒1 because in (2) we consider all polynomials and in (1) only monic ones,
(2) 5⇒3 because in (5) we consider all polynomials and in (3) only monic ones,
(3) 6⇒4 because in (6) we consider all polynomials and in (4) only monic ones,
(4) 4⇒3 is obvious,
(5) 6⇒5 is obvious,
(6) 8⇒7 is obvious,
(7) 10⇒9 is obvious,
(8) 11⇔12 by definition of being quasi-finite at a prime,
(9) 11⇒13 by definition of being quasi-finite,

Proof of 1⇒8. Assume (1). Let $R \to S$ be étale, and let $\mathfrak{q} \subset S$ be a prime ideal such that $\kappa(\mathfrak{q}) \cong \kappa$. By Proposition 144.4 we can find a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is standard étale. After replacing S by S_g we may assume that $S = R[t]_g/(f)$ is standard étale. Since the prime $\mathfrak{q}$ has residue field κ it corresponds to a root a_0 of $\overline{f}$ which is not a root of $\overline{g}$. By definition of a standard étale algebra this also means that $\overline{f'}(a_0) \neq 0$. Since also f is monic by definition of a standard étale algebra again we may use that R is henselian to conclude that there exists an $a \in R$ with $a_0 = \overline{a}$ such that $f(a) = 0$. This implies that $g(a)$ is a unit of R and we obtain the desired map $\tau : S = R[t]_g/(f) \to R$ by the rule $t \mapsto a$. By construction $\tau^{-1}(\mathfrak{q}) = \mathfrak{m}$. This proves (8) holds.

Proof of 7⇒8. (This is really unimportant and should be skipped.) Assume (7) holds and assume $R \to S$ is étale. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ be the other primes of S lying over

$\mathfrak{m}$. Then we can find a $g \in S$, $g \notin \mathfrak{q}$ and $g \in \mathfrak{q}_i$ for $i = 1, \ldots, r$. Namely, we can argue that $\bigcap_{i=1}^r \mathfrak{q}_i \not\subset \mathfrak{q}$ since otherwise $\mathfrak{q}_i \subset \mathfrak{q}$ for some i, but this cannot happen as the fiber of an étale morphism is discrete (use Lemma 143.4 for example). Apply (7) to the étale ring map $R \to S_g$ and the prime $\mathfrak{q}S_g$. This gives a section $\tau_g : S_g \to R$ such that the composition $\tau : S \to S_g \to R$ has the property $\tau^{-1}(\mathfrak{m}) = \mathfrak{q}$. Minor details omitted.

Proof of 8⇒11. Assume (8) and let $R \to S$ be a finite type ring map. Apply Lemma 145.3. We find an étale ring map $R \to R'$ and a prime $\mathfrak{m}' \subset R'$ lying over $\mathfrak{m}$ with $\kappa = \kappa(\mathfrak{m}')$ such that $R' \otimes_R S = A' \times B'$ with A' finite over R' and B' not quasi-finite over R' at any prime lying over $\mathfrak{m}'$. Apply (8) to get a section $\tau : R' \to R$ with $\mathfrak{m} = \tau^{-1}(\mathfrak{m}')$. Then use that

$$S = (S \otimes_R R') \otimes_{R', \tau} R = (A' \times B') \otimes_{R', \tau} R = (A' \otimes_{R', \tau} R) \times (B' \otimes_{R', \tau} R)$$

which gives a decomposition as in (11).

Proof of 8⇒10. Assume (8) and let $R \to S$ be a finite ring map. Apply Lemma 145.3. We find an étale ring map $R \to R'$ and a prime $\mathfrak{m}' \subset R'$ lying over $\mathfrak{m}$ with $\kappa = \kappa(\mathfrak{m}')$ such that $R' \otimes_R S = A'_1 \times \ldots \times A'_n \times B'$ with A'_i finite over R' having exactly one prime over $\mathfrak{m}'$ and B' not quasi-finite over R' at any prime lying over $\mathfrak{m}'$. Apply (8) to get a section $\tau : R' \to R$ with $\mathfrak{m}' = \tau^{-1}(\mathfrak{m})$. Then we obtain

$$\begin{aligned} S &= (S \otimes_R R') \otimes_{R', \tau} R \\ &= (A'_1 \times \ldots \times A'_n \times B') \otimes_{R', \tau} R \\ &= (A'_1 \otimes_{R', \tau} R) \times \ldots \times (A'_1 \otimes_{R', \tau} R) \times (B' \otimes_{R', \tau} R) \\ &= A_1 \times \ldots \times A_n \times B \end{aligned}$$

The factor B is finite over R but $R \to B$ is not quasi-finite at any prime lying over $\mathfrak{m}$. Hence $B = 0$. The factors A_i are finite R-algebras having exactly one prime lying over $\mathfrak{m}$, hence they are local rings. This proves that S is a finite product of local rings.

Proof of 9⇒10. This holds because if S is finite over the local ring R, then it has at most finitely many maximal ideals. Namely, by going up for $R \to S$ the maximal ideals of S all lie over $\mathfrak{m}$, and $S/\mathfrak{m}S$ is Artinian hence has finitely many primes.

Proof of 10⇒1. Assume (10). Let $f \in R[T]$ be a monic polynomial and $a_0 \in \kappa$ a simple root of $\overline{f}$. Then $S = R[T]/(f)$ is a finite R-algebra. Applying (10) we get $S = A_1 \times \ldots \times A_r$ is a finite product of local R-algebras. In particular we see that $S/\mathfrak{m}S = \prod A_i/\mathfrak{m}A_i$ is the decomposition of $\kappa[T]/(\overline{f})$ as a product of local rings. This means that one of the factors, say $A_1/\mathfrak{m}A_1$ is the quotient $\kappa[T]/(\overline{f}) \to \kappa[T]/(T - a_0)$. Since A_1 is a summand of the finite free R-module S it is a finite free R-module itself. As $A_1/\mathfrak{m}A_1$ is a κ-vector space of dimension 1 we see that $A_1 \cong R$ as an R-module. Clearly this means that $R \to A_1$ is an isomorphism. Let $a \in R$ be the image of T under the map $R[T] \to S \to A_1 \to R$. Then $f(a) = 0$ and $\overline{a} = a_0$ as desired.

Proof of 13⇒1. Assume (13). Let $f \in R[T]$ be a monic polynomial and $a_0 \in \kappa$ a simple root of $\overline{f}$. Then $S_1 = R[T]/(f)$ is a finite R-algebra. Let $g \in R[T]$ be any element such that $\overline{g} = \overline{f}/(T - a_0)$. Then $S = (S_1)_g$ is a quasi-finite R-algebra such that $S \otimes_R \kappa \cong \kappa[T]_{\overline{g}}/(\overline{f}) \cong \kappa[T]/(T - a_0) \cong \kappa$. Applying (13) to S we get $S = A \times B$ with A finite over R and $B \otimes_R \kappa = 0$. In particular we see that $\kappa \cong S/\mathfrak{m}S = A/\mathfrak{m}A$.

Since A is a summand of the flat R-algebra S we see that it is finite flat, hence free over R. As $A/\mathfrak{m}A$ is a κ-vector space of dimension 1 we see that $A \cong R$ as an R-module. Clearly this means that $R \to A$ is an isomorphism. Let $a \in R$ be the image of T under the map $R[T] \to S \to A \to R$. Then $f(a) = 0$ and $\overline{a} = a_0$ as desired.

Proof of 8⇒2. Assume (8). Let $f \in R[T]$ be any polynomial and let $a_0 \in \kappa$ be a simple root. Then the algebra $S = R[T]_{f'}/(f)$ is étale over R. Let $\mathfrak{q} \subset S$ be the prime generated by $\mathfrak{m}$ and $T - b$ where $b \in R$ is any element such that $\overline{b} = a_0$. Apply (8) to S and $\mathfrak{q}$ to get $\tau : S \to R$. Then the image $\tau(T) = a \in R$ works in (2).

At this point we see that (1), (2), (7), (8), (9), (10), (11), (12), (13) are all equivalent. The weakest assertion of (3), (4), (5) and (6) is (3) and the strongest is (6). Hence we still have to prove that (3) implies (1) and (1) implies (6).

Proof of 3⇒1. Assume (3). Let $f \in R[T]$ be monic and let $a_0 \in \kappa$ be a simple root of $\overline{f}$. This gives a factorization $\overline{f} = (T - a_0)h_0$ with $h_0(a_0) \neq 0$, so $\gcd(T - a_0, h_0) = 1$. Apply (3) to get a factorization $f = gh$ with $\overline{g} = T - a_0$ and $\overline{h} = h_0$. Set $S = R[T]/(f)$ which is a finite free R-algebra. We will write g, h also for the images of g and h in S. Then $gS + hS = S$ by Nakayama's Lemma 20.1 as the equality holds modulo $\mathfrak{m}$. Since $gh = f = 0$ in S this also implies that $gS \cap hS = 0$. Hence by the Chinese Remainder theorem we obtain $S = S/(g) \times S/(h)$. This implies that $A = S/(g)$ is a summand of a finite free R-module, hence finite free. Moreover, the rank of A is 1 as $A/\mathfrak{m}A = \kappa[T]/(T - a_0)$. Thus the map $R \to A$ is an isomorphism. Setting $a \in R$ equal to the image of T under the maps $R[T] \to S \to A \to R$ gives an element of R with $f(a) = 0$ and $\overline{a} = a_0$.

Proof of 1⇒6. Assume (1) or equivalently all of (1), (2), (7), (8), (9), (10), (11), (12), (13). Let $f \in R[T]$ be a polynomial. Suppose that $\overline{f} = g_0 h_0$ is a factorization with $\gcd(g_0, h_0) = 1$. We may and do assume that g_0 is monic. Consider $S = R[T]/(f)$. Because we have the factorization we see that the coefficients of f generate the unit ideal in R. This implies that S has finite fibres over R, hence is quasi-finite over R. It also implies that S is flat over R by Lemma 128.5. Combining (13) and (10) we may write $S = A_1 \times \ldots \times A_n \times B$ where each A_i is local and finite over R, and $B \otimes_R \kappa = 0$. After reordering the factors $A_1, \ldots, A_n$ we may assume that

$$\kappa[T]/(g_0) = A_1/\mathfrak{m}A_1 \times \ldots \times A_r/\mathfrak{m}A_r,\ \kappa[T]/(h_0) = A_{r+1}/\mathfrak{m}A_{r+1} \times \ldots \times A_n/\mathfrak{m}A_n$$

as quotients of $\kappa[T]$. The finite flat R-algebra $A = A_1 \times \ldots \times A_r$ is free as an R-module, see Lemma 78.5. Its rank is $\deg_T(g_0)$. Let $g \in R[T]$ be the characteristic polynomial of the R-linear operator $T : A \to A$. Then g is a monic polynomial of degree $\deg_T(g) = \deg_T(g_0)$ and moreover $\overline{g} = g_0$. By Cayley-Hamilton (Lemma 16.1) we see that $g(T_A) = 0$ where T_A indicates the image of T in A. Hence we obtain a well defined surjective map $R[T]/(g) \to A$ which is an isomorphism by Nakayama's Lemma 20.1. The map $R[T] \to A$ factors through $R[T]/(f)$ by construction hence we may write $f = gh$ for some h. This finishes the proof. ☐

04GH **Lemma 153.4.** *Let $(R, \mathfrak{m}, \kappa)$ be a henselian local ring.*

(1) *If $R \to S$ is a finite ring map then S is a finite product of henselian local rings each finite over R.*

(2) *If $R \to S$ is a finite ring map and S is local, then S is a henselian local ring and $R \to S$ is a (finite) local ring map.*
(3) *If $R \to S$ is a finite type ring map, and $\mathfrak{q}$ is a prime of S lying over $\mathfrak{m}$ at which $R \to S$ is quasi-finite, then $S_\mathfrak{q}$ is henselian and finite over R.*
(4) *If $R \to S$ is quasi-finite then $S_\mathfrak{q}$ is henselian and finite over R for every prime $\mathfrak{q}$ lying over $\mathfrak{m}$.*

Proof. Part (2) implies part (1) since S as in part (1) is a finite product of its localizations at the primes lying over $\mathfrak{m}$ by Lemma 153.3 part (10). Part (2) also follows from Lemma 153.3 part (10) since any finite S-algebra is also a finite R-algebra (of course any finite ring map between local rings is local).

Let $R \to S$ and $\mathfrak{q}$ be as in (3). Write $S = A \times B$ with A finite over R and B not quasi-finite over R at any prime lying over $\mathfrak{m}$, see Lemma 153.3 part (11). Hence $S_\mathfrak{q}$ is a localization of A at a maximal ideal and we deduce (3) from (1). Part (4) follows from part (3). □

04GJ **Lemma 153.5.** *Let $(R, \mathfrak{m}, \kappa)$ be a henselian local ring. Any finite type R-algebra S can be written as $S = A_1 \times \ldots \times A_n \times B$ with A_i local and finite over R and $R \to B$ not quasi-finite at any prime of B lying over $\mathfrak{m}$.*

Proof. This is a combination of parts (11) and (10) of Lemma 153.3. □

06DD **Lemma 153.6.** *Let $(R, \mathfrak{m}, \kappa)$ be a strictly henselian local ring. Any finite type R-algebra S can be written as $S = A_1 \times \ldots \times A_n \times B$ with A_i local and finite over R and $\kappa \subset \kappa(\mathfrak{m}_{A_i})$ finite purely inseparable and $R \to B$ not quasi-finite at any prime of B lying over $\mathfrak{m}$.*

Proof. First write $S = A_1 \times \ldots \times A_n \times B$ as in Lemma 153.5. The field extension $\kappa(\mathfrak{m}_{A_i})/\kappa$ is finite and κ is separably algebraically closed, hence it is finite purely inseparable. □

04GK **Lemma 153.7.** *Let $(R, \mathfrak{m}, \kappa)$ be a henselian local ring. The category of finite étale ring extensions $R \to S$ is equivalent to the category of finite étale algebras $\kappa \to \overline{S}$ via the functor $S \mapsto S/\mathfrak{m}S$.*

Proof. Denote $\mathcal{C} \to \mathcal{D}$ the functor of categories of the statement. Suppose that $R \to S$ is finite étale. Then we may write

$$S = A_1 \times \ldots \times A_n$$

with A_i local and finite étale over S, use either Lemma 153.5 or Lemma 153.3 part (10). In particular $A_i/\mathfrak{m}A_i$ is a finite separable field extension of κ, see Lemma 143.5. Thus we see that every object of $\mathcal{C}$ and $\mathcal{D}$ decomposes canonically into irreducible pieces which correspond via the given functor. Next, suppose that S_1, S_2 are finite étale over R such that $\kappa_1 = S_1/\mathfrak{m}S_1$ and $\kappa_2 = S_2/\mathfrak{m}S_2$ are fields (finite separable over κ). Then $S_1 \otimes_R S_2$ is finite étale over R and we may write

$$S_1 \otimes_R S_2 = A_1 \times \ldots \times A_n$$

as before. Then we see that $\mathrm{Hom}_R(S_1, S_2)$ is identified with the set of indices $i \in \{1, \ldots, n\}$ such that $S_2 \to A_i$ is an isomorphism. To see this use that given any R-algebra map $\varphi : S_1 \to S_2$ the map $\varphi \times 1 : S_1 \otimes_R S_2 \to S_2$ is surjective, and hence is equal to projection onto one of the factors A_i. But in exactly the same way we see that $\mathrm{Hom}_\kappa(\kappa_1, \kappa_2)$ is identified with the set of indices $i \in \{1, \ldots, n\}$

such that $\kappa_2 \to A_i/\mathfrak{m}A_i$ is an isomorphism. By the discussion above these sets of indices match, and we conclude that our functor is fully faithful. Finally, let κ'/κ be a finite separable field extension. By Lemma 144.3 there exists an étale ring map $R \to S$ and a prime $\mathfrak{q}$ of S lying over $\mathfrak{m}$ such that $\kappa \subset \kappa(\mathfrak{q})$ is isomorphic to the given extension. By part (1) we may write $S = A_1 \times \ldots \times A_n \times B$. Since $R \to S$ is quasi-finite we see that there exists no prime of B over $\mathfrak{m}$. Hence $S_\mathfrak{q}$ is equal to A_i for some i. Hence $R \to A_i$ is finite étale and produces the given residue field extension. Thus the functor is essentially surjective and we win. □

04GL **Lemma 153.8.** *Let $(R, \mathfrak{m}, \kappa)$ be a strictly henselian local ring. Let $R \to S$ be an unramified ring map. Then*

$$S = A_1 \times \ldots \times A_n \times B$$

with each $R \to A_i$ surjective and no prime of B lying over $\mathfrak{m}$.

Proof. First write $S = A_1 \times \ldots \times A_n \times B$ as in Lemma 153.5. Now we see that $R \to A_i$ is finite unramified and A_i local. Hence the maximal ideal of A_i is $\mathfrak{m}A_i$ and its residue field $A_i/\mathfrak{m}A_i$ is a finite separable extension of κ, see Lemma 151.5. However, the condition that R is strictly henselian means that κ is separably algebraically closed, so $\kappa = A_i/\mathfrak{m}A_i$. By Nakayama's Lemma 20.1 we conclude that $R \to A_i$ is surjective as desired. □

04GM **Lemma 153.9.** *Let $(R, \mathfrak{m}, \kappa)$ be a complete local ring, see Definition 160.1. Then R is henselian.*

Proof. Let $f \in R[T]$ be monic. Denote $f_n \in R/\mathfrak{m}^{n+1}[T]$ the image. Denote f'_n the derivative of f_n with respect to T. Let $a_0 \in \kappa$ be a simple root of f_0. We lift this to a solution of f over R inductively as follows: Suppose given $a_n \in R/\mathfrak{m}^{n+1}$ such that $a_n \bmod \mathfrak{m} = a_0$ and $f_n(a_n) = 0$. Pick any element $b \in R/\mathfrak{m}^{n+2}$ such that $a_n = b \bmod \mathfrak{m}^{n+1}$. Then $f_{n+1}(b) \in \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2}$. Set

$$a_{n+1} = b - f_{n+1}(b)/f'_{n+1}(b)$$

(Newton's method). This makes sense as $f'_{n+1}(b) \in R/\mathfrak{m}^{n+1}$ is invertible by the condition on a_0. Then we compute $f_{n+1}(a_{n+1}) = f_{n+1}(b) - f_{n+1}(b) = 0$ in $R/\mathfrak{m}^{n+2}$. Since the system of elements $a_n \in R/\mathfrak{m}^{n+1}$ so constructed is compatible we get an element $a \in \lim R/\mathfrak{m}^n = R$ (here we use that R is complete). Moreover, $f(a) = 0$ since it maps to zero in each $R/\mathfrak{m}^n$. Finally $\overline{a} = a_0$ and we win. □

06RS **Lemma 153.10.** *Let $(R, \mathfrak{m})$ be a local ring of dimension 0. Then R is henselian.*

Proof. Let $R \to S$ be a finite ring map. By Lemma 153.3 it suffices to show that S is a product of local rings. By Lemma 36.21 S has finitely many primes $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ which all lie over $\mathfrak{m}$. There are no inclusions among these primes, see Lemma 36.20, hence they are all maximal. Every element of $\mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_r$ is nilpotent by Lemma 17.2. It follows S is the product of the localizations of S at the primes $\mathfrak{m}_i$ by Lemma 53.5. □

The following lemma will be the key to the uniqueness and functorial properties of henselization and strict henselization.

08HQ **Lemma 153.11.** *Let $R \to S$ be a ring map with S henselian local. Given*

(1) *an étale ring map $R \to A$,*
(2) *a prime $\mathfrak{q}$ of A lying over $\mathfrak{p} = R \cap \mathfrak{m}_S$,*

(3) *a $\kappa(\mathfrak p)$-algebra map $\tau : \kappa(\mathfrak q) \to S/\mathfrak m_S$,*

then there exists a unique homomorphism of R-algebras $f : A \to S$ such that $\mathfrak q = f^{-1}(\mathfrak m_S)$ and $f \bmod \mathfrak q = \tau$.

Proof. Consider $A \otimes_R S$. This is an étale algebra over S, see Lemma 143.3. Moreover, the kernel

$$\mathfrak q' = \mathrm{Ker}(A \otimes_R S \to \kappa(\mathfrak q) \otimes_{\kappa(\mathfrak p)} \kappa(\mathfrak m_S) \to \kappa(\mathfrak m_S))$$

of the map using the map given in (3) is a prime ideal lying over $\mathfrak m_S$ with residue field equal to the residue field of S. Hence by Lemma 153.3 there exists a unique splitting $\tau : A \otimes_R S \to S$ with $\tau^{-1}(\mathfrak m_S) = \mathfrak q'$. Set f equal to the composition $A \to A \otimes_R S \to S$. □

04GX **Lemma 153.12.** *Let $\varphi : R \to S$ be a local homomorphism of strictly henselian local rings. Let $P_1, \ldots, P_n \in R[x_1, \ldots, x_n]$ be polynomials such that $R[x_1, \ldots, x_n]/(P_1, \ldots, P_n)$ is étale over R. Then the map*

$$R^n \longrightarrow S^n, \quad (h_1, \ldots, h_n) \longmapsto (\varphi(h_1), \ldots, \varphi(h_n))$$

induces a bijection between

$$\{(r_1, \ldots, r_n) \in R^n \mid P_i(r_1, \ldots, r_n) = 0,\ i = 1, \ldots, n\}$$

and

$$\{(s_1, \ldots, s_n) \in S^n \mid P'_i(s_1, \ldots, s_n) = 0,\ i = 1, \ldots, n\}$$

where $P'_i \in S[x_1, \ldots, x_n]$ are the images of the P_i under φ.

Proof. The first solution set is canonically isomorphic to the set

$$\mathrm{Hom}_R(R[x_1, \ldots, x_n]/(P_1, \ldots, P_n), R).$$

As R is henselian the map $R \to R/\mathfrak m_R$ induces a bijection between this set and the set of solutions in the residue field $R/\mathfrak m_R$, see Lemma 153.3. The same is true for S. Now since $R[x_1, \ldots, x_n]/(P_1, \ldots, P_n)$ is étale over R and $R/\mathfrak m_R$ is separably algebraically closed we see that $R/\mathfrak m_R[x_1, \ldots, x_n]/(\overline{P_1}, \ldots, \overline{P_n})$ is a finite product of copies of $R/\mathfrak m_R$. Hence the tensor product

$$R/\mathfrak m_R[x_1, \ldots, x_n]/(\overline{P_1}, \ldots, \overline{P_n}) \otimes_{R/\mathfrak m_R} S/\mathfrak m_S = S/\mathfrak m_S[x_1, \ldots, x_n]/(\overline{P'_1}, \ldots, \overline{P'_n})$$

is also a finite product of copies of $S/\mathfrak m_S$ with the same index set. This proves the lemma. □

05D6 **Lemma 153.13.** *Let R be a henselian local ring. Any countably generated Mittag-Leffler module over R is a direct sum of finitely presented R-modules.*

Proof. Let M be a countably generated and Mittag-Leffler R-module. We claim that for any element $x \in M$ there exists a direct sum decomposition $M = N \oplus K$ with $x \in N$, the module N finitely presented, and K Mittag-Leffler.

Suppose the claim is true. Choose generators $x_1, x_2, x_3, \ldots$ of M. By the claim we can inductively find direct sum decompositions

$$M = N_1 \oplus N_2 \oplus \ldots \oplus N_n \oplus K_n$$

with N_i finitely presented, $x_1, \ldots, x_n \in N_1 \oplus \ldots \oplus N_n$, and K_n Mittag-Leffler. Repeating ad infinitum we see that $M = \bigoplus N_i$.

We still have to prove the claim. Let $x \in M$. By Lemma 92.2 there exists an endomorphism $\alpha : M \to M$ such that α factors through a finitely presented module, and $\alpha(x) = x$. Say α factors as

$$M \xrightarrow{\pi} P \xrightarrow{i} M$$

Set $a = \pi \circ \alpha \circ i : P \to P$, so $i \circ a \circ \pi = \alpha^3$. By Lemma 16.2 there exists a monic polynomial $P \in R[T]$ such that $P(a) = 0$. Note that this implies formally that $\alpha^2 P(\alpha) = 0$. Hence we may think of M as a module over $R[T]/(T^2P)$. Assume that $x \neq 0$. Then $\alpha(x) = x$ implies that $0 = \alpha^2 P(\alpha)x = P(1)x$ hence $P(1) = 0$ in R/I where $I = \{r \in R \mid rx = 0\}$ is the annihilator of x. As $x \neq 0$ we see $I \subset \mathfrak m_R$, hence 1 is a root of $\overline{P} = P \bmod \mathfrak m_R \in R/\mathfrak m_R[T]$. As R is henselian we can find a factorization

$$T^2P = (T^2Q_1)Q_2$$

for some $Q_1, Q_2 \in R[T]$ with $Q_2 = (T-1)^e \bmod \mathfrak m_R R[T]$ and $Q_1(1) \neq 0 \bmod \mathfrak m_R$, see Lemma 153.3. Let $N = \mathrm{Im}(\alpha^2 Q_1(\alpha) : M \to M)$ and $K = \mathrm{Im}(Q_2(\alpha) : M \to M)$. As T^2Q_1 and Q_2 generate the unit ideal of $R[T]$ we get a direct sum decomposition $M = N \oplus K$. Moreover, Q_2 acts as zero on N and T^2Q_1 acts as zero on K. Note that N is a quotient of P hence is finitely generated. Also $x \in N$ because $\alpha^2 Q_1(\alpha)x = Q_1(1)x$ and $Q_1(1)$ is a unit in R. By Lemma 89.10 the modules N and K are Mittag-Leffler. Finally, the finitely generated module N is finitely presented as a finitely generated Mittag-Leffler module is finitely presented, see Example 91.1 part (1). ☐

154. Filtered colimits of étale ring maps

0BSG This section is a precursor to the section on ind-étale ring maps (Pro-étale Cohomology, Section 7). The material will also be useful to prove uniqueness properties of the henselization and strict henselization of a local ring.

0BSH **Lemma 154.1.** *Let $R \to A$ and $R \to R'$ be ring maps. If A is a filtered colimit of étale ring maps, then so is $R' \to R' \otimes_R A$.*

Proof. This is true because colimits commute with tensor products and étale ring maps are preserved under base change (Lemma 143.3). ☐

0BSI **Lemma 154.2.** *Let $A \to B \to C$ be ring maps. If $A \to B$ is a filtered colimit of étale ring maps and $B \to C$ is a filtered colimit of étale ring maps, then $A \to C$ is a filtered colimit of étale ring maps.*

Proof. We will use the criterion of Lemma 127.4. Let $A \to P \to C$ be a factorization of $A \to C$ with P of finite presentation over A. Write $B = \mathrm{colim}_{i \in I} B_i$ where I is a directed set and where B_i is an étale A-algebra. Write $C = \mathrm{colim}_{j \in J} C_j$ where J is a directed set and where C_j is an étale B-algebra. We can factor $P \to C$ as $P \to C_j \to C$ for some j by Lemma 127.3. By Lemma 143.3 we can find an $i \in I$ and an étale ring map $B_i \to C'_j$ such that $C_j = B \otimes_{B_i} C'_j$. Then $C_j = \mathrm{colim}_{i' \geq i} B_{i'} \otimes_{B_i} C'_j$ and again we see that $P \to C_j$ factors as $P \to B_{i'} \otimes_{B_i} C'_j \to C$. As $A \to C' = B_{i'} \otimes_{B_i} C'_j$ is étale as compositions and tensor products of étale ring maps are étale. Hence we have factored $P \to C$ as $P \to C' \to C$ with C' étale over A and the criterion of Lemma 127.4 applies. ☐

0BSJ **Lemma 154.3.** *Let R be a ring. Let $A = \operatorname{colim} A_i$ be a filtered colimit of R-algebras such that each A_i is a filtered colimit of étale R-algebras. Then A is a filtered colimit of étale R-algebras.*

Proof. Write $A_i = \operatorname{colim}_{j \in J_i} A_j$ where J_i is a directed set and A_j is an étale R-algebra. For each $i \leq i'$ and $j \in J_i$ there exists an $j' \in J_{i'}$ and an R-algebra map $\varphi_{jj'} : A_j \to A_{j'}$ making the diagram

$$\begin{tikzcd} A_i \arrow[r] & A_{i'} \\ A_j \arrow[u] \arrow[r, "\varphi_{jj'}"] & A_{j'} \arrow[u] \end{tikzcd}$$

commute. This is true because $R \to A_j$ is of finite presentation so that Lemma 127.3 applies. Let $\mathcal{J}$ be the category with objects $\coprod_{i \in I} J_i$ and morphisms triples $(j, j', \varphi_{jj'})$ as above (and obvious composition law). Then $\mathcal{J}$ is a filtered category and $A = \operatorname{colim}_{\mathcal{J}} A_j$. Details omitted. $\square$

0GIM **Lemma 154.4.** *Let I be a directed set. Let $i \mapsto (R_i \to A_i)$ be a system of arrows of rings over I. Set $R = \operatorname{colim} R_i$ and $A = \operatorname{colim} A_i$. If each A_i is a filtered colimit of étale R_i-algebras, then A is a filtered colimit of étale R-algebras.*

Proof. This is true because $A = A \otimes_R R = \operatorname{colim} A_i \otimes_{R_i} R$ and hence we can apply Lemma 154.3 because $R \to A_i \otimes_{R_i} R$ is a filtered colimit of étale ring maps by Lemma 154.1. $\square$

08HS **Lemma 154.5.** *Let R be a ring. Let $A \to B$ be an R-algebra homomorphism. If A and B are filtered colimits of étale R-algebras, then B is a filtered colimit of étale A-algebras.*

Proof. Write $A = \operatorname{colim} A_i$ and $B = \operatorname{colim} B_j$ as filtered colimits with A_i and B_j étale over R. For each i we can find a j such that $A_i \to B$ factors through B_j, see Lemma 127.3. The factorization $A_i \to B_j$ is étale by Lemma 143.8. Since $A \to A \otimes_{A_i} B_j$ is étale (Lemma 143.3) it suffices to prove that $B = \operatorname{colim} A \otimes_{A_i} B_j$ where the colimit is over pairs (i, j) and factorizations $A_i \to B_j \to B$ of $A_i \to B$ (this is a directed system; details omitted). This is clear because colimits commute with tensor products and hence $\operatorname{colim} A \otimes_{A_i} B_j = A \otimes_A B = B$. $\square$

08HR **Lemma 154.6.** *Let $R \to S$ be a ring map with S henselian local. Given*

(1) *an R-algebra A which is a filtered colimit of étale R-algebras,*
(2) *a prime $\mathfrak{q}$ of A lying over $\mathfrak{p} = R \cap \mathfrak{m}_S$,*
(3) *a $\kappa(\mathfrak{p})$-algebra map $\tau : \kappa(\mathfrak{q}) \to S/\mathfrak{m}_S$,*

then there exists a unique homomorphism of R-algebras $f : A \to S$ such that $\mathfrak{q} = f^{-1}(\mathfrak{m}_S)$ and $f \bmod \mathfrak{q} = \tau$.

Proof. Write $A = \operatorname{colim} A_i$ as a filtered colimit of étale R-algebras. Set $\mathfrak{q}_i = A_i \cap \mathfrak{q}$. We obtain $f_i : A_i \to S$ by applying Lemma 153.11. Set $f = \operatorname{colim} f_i$. $\square$

08HT **Lemma 154.7.** *Let R be a ring. Given a commutative diagram of ring maps*

$$\begin{tikzcd} S \arrow[r] & K \\ R \arrow[u] \arrow[r] & S' \arrow[u] \end{tikzcd}$$

where S, S' are henselian local, S, S' are filtered colimits of étale R-algebras, K is a field and the arrows $S \to K$ and $S' \to K$ identify K with the residue field of both S and S'. Then there exists an unique R-algebra isomorphism $S \to S'$ compatible with the maps to K.

Proof. Follows immediately from Lemma 154.6. □

The following lemma is not strictly speaking about colimits of étale ring maps.

04GI **Lemma 154.8.** *A filtered colimit of (strictly) henselian local rings along local homomorphisms is (strictly) henselian.*

Proof. Categories, Lemma 21.5 says that this is really just a question about a colimit of (strictly) henselian local rings over a directed set. Let $(R_i, \varphi_{ii'})$ be such a system with each $\varphi_{ii'}$ local. Then $R = \text{colim}_i R_i$ is local, and its residue field κ is $\text{colim}\, \kappa_i$ (argument omitted). It is easy to see that $\text{colim}\, \kappa_i$ is separably algebraically closed if each κ_i is so; thus it suffices to prove R is henselian if each R_i is henselian. Suppose that $f \in R[T]$ is monic and that $a_0 \in \kappa$ is a simple root of $\overline{f}$. Then for some large enough i there exists an $f_i \in R_i[T]$ mapping to f and an $a_{0,i} \in \kappa_i$ mapping to a_0. Since $\overline{f_i}(a_{0,i}) \in \kappa_i$, resp. $\overline{f_i'}(a_{0,i}) \in \kappa_i$ maps to $0 = \overline{f}(a_0) \in \kappa$, resp. $0 \neq \overline{f'}(a_0) \in \kappa$ we conclude that $a_{0,i}$ is a simple root of $\overline{f_i}$. As R_i is henselian we can find $a_i \in R_i$ such that $f_i(a_i) = 0$ and $a_{0,i} = \overline{a_i}$. Then the image $a \in R$ of a_i is the desired solution. Thus R is henselian. □

155. Henselization and strict henselization

0BSK In this section we construct the henselization. We encourage the reader to keep in mind the uniqueness already proved in Lemma 154.7 and the functorial behaviour pointed out in Lemma 154.6 while reading this material.

04GN **Lemma 155.1.** *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. There exists a local ring map $R \to R^h$ with the following properties*

1. *R^h is henselian,*
2. *R^h is a filtered colimit of étale R-algebras,*
3. *$\mathfrak{m}R^h$ is the maximal ideal of R^h, and*
4. *$\kappa = R^h/\mathfrak{m}R^h$.*

Proof. Consider the category of pairs $(S, \mathfrak{q})$ where $R \to S$ is an étale ring map, and $\mathfrak{q}$ is a prime of S lying over $\mathfrak{m}$ with $\kappa = \kappa(\mathfrak{q})$. A morphism of pairs $(S, \mathfrak{q}) \to (S', \mathfrak{q}')$ is given by an R-algebra map $\varphi : S \to S'$ such that $\varphi^{-1}(\mathfrak{q}') = \mathfrak{q}$. We set

$$R^h = \text{colim}_{(S,\mathfrak{q})} S.$$

Let us show that the category of pairs is filtered, see Categories, Definition 19.1. The category contains the pair $(R, \mathfrak{m})$ and hence is not empty, which proves part (1) of Categories, Definition 19.1. For any pair $(S, \mathfrak{q})$ the prime ideal $\mathfrak{q}$ is maximal with residue field κ since the composition $\kappa \to S/\mathfrak{q} \to \kappa(\mathfrak{q})$ is an isomorphism. Suppose that $(S, \mathfrak{q})$ and $(S', \mathfrak{q}')$ are two objects. Set $S'' = S \otimes_R S'$ and $\mathfrak{q}'' = \mathfrak{q}S'' + \mathfrak{q}'S''$. Then $S''/\mathfrak{q}'' = S/\mathfrak{q} \otimes_R S'/\mathfrak{q}' = \kappa$ by what we said above. Moreover, $R \to S''$ is étale by Lemma 143.3. This proves part (2) of Categories, Definition 19.1. Next, suppose that $\varphi, \psi : (S, \mathfrak{q}) \to (S', \mathfrak{q}')$ are two morphisms of pairs. Then φ, ψ, and $S' \otimes_R S' \to S'$ are étale ring maps by Lemma 143.8. Consider

$$S'' = (S' \otimes_{\varphi, S, \psi} S') \otimes_{S' \otimes_R S'} S'$$

with prime ideal

$$\mathfrak{q}'' = (\mathfrak{q}' \otimes S' + S' \otimes \mathfrak{q}') \otimes S' + (S' \otimes_{\varphi, S, \psi} S') \otimes \mathfrak{q}'$$

Arguing as above (base change of étale maps is étale, composition of étale maps is étale) we see that S'' is étale over R. Moreover, the canonical map $S' \to S''$ (using the right most factor for example) equalizes φ and ψ. This proves part (3) of Categories, Definition 19.1. Hence we conclude that R^h consists of triples $(S, \mathfrak{q}, f)$ with $f \in S$, and two such triples $(S, \mathfrak{q}, f)$, $(S', \mathfrak{q}', f')$ define the same element of R^h if and only if there exists a pair $(S'', \mathfrak{q}'')$ and morphisms of pairs $\varphi : (S, \mathfrak{q}) \to (S'', \mathfrak{q}'')$ and $\varphi' : (S', \mathfrak{q}') \to (S'', \mathfrak{q}'')$ such that $\varphi(f) = \varphi'(f')$.

Suppose that $x \in R^h$. Represent x by a triple $(S, \mathfrak{q}, f)$. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ be the other primes of S lying over $\mathfrak{m}$. Then $\mathfrak{q} \not\subset \mathfrak{q}_i$ as we have seen above that $\mathfrak{q}$ is maximal. Thus, since $\mathfrak{q}$ is a prime ideal, we can find a $g \in S$, $g \notin \mathfrak{q}$ and $g \in \mathfrak{q}_i$ for $i = 1, \ldots, r$. Consider the morphism of pairs $(S, \mathfrak{q}) \to (S_g, \mathfrak{q}S_g)$. In this way we see that we may always assume that x is given by a triple $(S, \mathfrak{q}, f)$ where $\mathfrak{q}$ is the only prime of S lying over $\mathfrak{m}$, i.e., $\sqrt{\mathfrak{m}S} = \mathfrak{q}$. But since $R \to S$ is étale, we have $\mathfrak{m}S_{\mathfrak{q}} = \mathfrak{q}S_{\mathfrak{q}}$, see Lemma 143.5. Hence we actually get that $\mathfrak{m}S = \mathfrak{q}$.

Suppose that $x \not\in \mathfrak{m}R^h$. Represent x by a triple $(S, \mathfrak{q}, f)$ with $\mathfrak{m}S = \mathfrak{q}$. Then $f \not\in \mathfrak{m}S$, i.e., $f \not\in \mathfrak{q}$. Hence $(S, \mathfrak{q}) \to (S_f, \mathfrak{q}S_f)$ is a morphism of pairs such that the image of f becomes invertible. Hence x is invertible with inverse represented by the triple $(S_f, \mathfrak{q}S_f, 1/f)$. We conclude that R^h is a local ring with maximal ideal $\mathfrak{m}R^h$. The residue field is κ since we can define $R^h/\mathfrak{m}R^h \to \kappa$ by mapping a triple $(S, \mathfrak{q}, f)$ to the residue class of f modulo $\mathfrak{q}$.

We still have to show that R^h is henselian. Namely, suppose that $P \in R^h[T]$ is a monic polynomial and $a_0 \in \kappa$ is a simple root of the reduction $\overline{P} \in \kappa[T]$. Then we can find a pair $(S, \mathfrak{q})$ such that P is the image of a monic polynomial $Q \in S[T]$. Since $S \to R^h$ induces an isomorphism of residue fields we see that $S' = S[T]/(Q)$ has a prime ideal $\mathfrak{q}' = (\mathfrak{q}, T - a_0)$ at which $S \to S'$ is standard étale. Moreover, $\kappa = \kappa(\mathfrak{q}')$. Pick $g \in S'$, $g \notin \mathfrak{q}'$ such that $S'' = S'_g$ is étale over S. Then $(S, \mathfrak{q}) \to (S'', \mathfrak{q}'S'')$ is a morphism of pairs. Now that triple $(S'', \mathfrak{q}'S'', \text{class of } T)$ determines an element $a \in R^h$ with the properties $P(a) = 0$, and $\overline{a} = a_0$ as desired. $\square$

04GP **Lemma 155.2.** *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Let $\kappa \subset \kappa^{sep}$ be a separable algebraic closure. There exists a commutative diagram*

$$\begin{tikzcd}
\kappa \arrow[r] & \kappa \arrow[r] & \kappa^{sep} \\
R \arrow[r] \arrow[u] & R^h \arrow[r] \arrow[u] & R^{sh} \arrow[u]
\end{tikzcd}$$

with the following properties

(1) *the map $R^h \to R^{sh}$ is local*
(2) *R^{sh} is strictly henselian,*
(3) *R^{sh} is a filtered colimit of étale R-algebras,*
(4) *$\mathfrak{m}R^{sh}$ is the maximal ideal of R^{sh}, and*
(5) *$\kappa^{sep} = R^{sh}/\mathfrak{m}R^{sh}$.*

Proof. This is proved by exactly the same proof as used for Lemma 155.1. The only difference is that, instead of pairs, one uses triples $(S, \mathfrak{q}, \alpha)$ where $R \to S$ étale,

$\mathfrak{q}$ is a prime of S lying over $\mathfrak{m}$, and $\alpha : \kappa(\mathfrak{q}) \to \kappa^{sep}$ is an embedding of extensions of κ. □

04GQ **Definition 155.3.** Let $(R, \mathfrak{m}, \kappa)$ be a local ring.

(1) The local ring map $R \to R^h$ constructed in Lemma 155.1 is called the *henselization* of R.
(2) Given a separable algebraic closure $\kappa \subset \kappa^{sep}$ the local ring map $R \to R^{sh}$ constructed in Lemma 155.2 is called the *strict henselization of R with respect to $\kappa \subset \kappa^{sep}$*.
(3) A local ring map $R \to R^{sh}$ is called a *strict henselization* of R if it is isomorphic to one of the local ring maps constructed in Lemma 155.2

The maps $R \to R^h \to R^{sh}$ are flat local ring homomorphisms. By Lemma 154.7 the R-algebras R^h and R^{sh} are well defined up to unique isomorphism by the conditions that they are henselian local, filtered colimits of étale R-algebras with residue field κ and κ^{sep}. In the rest of this section we mostly just discuss functoriality of the (strict) henselizations. We will discuss more intricate results concerning the relationship between R and its henselization in More on Algebra, Section 45.

0BSL **Remark 155.4.** We can also construct R^{sh} from R^h. Namely, for any finite separable subextension $\kappa^{sep}/\kappa'/\kappa$ there exists a unique (up to unique isomorphism) finite étale local ring extension $R^h \subset R^h(\kappa')$ whose residue field extension reproduces the given extension, see Lemma 153.7. Hence we can set

$$R^{sh} = \bigcup\nolimits_{\kappa \subset \kappa' \subset \kappa^{sep}} R^h(\kappa')$$

The arrows in this system, compatible with the arrows on the level of residue fields, exist by Lemma 153.7. This will produce a henselian local ring by Lemma 154.8 since each of the rings $R^h(\kappa')$ is henselian by Lemma 153.4. By construction the residue field extension induced by $R^h \to R^{sh}$ is the field extension κ^{sep}/κ. Hence R^{sh} so constructed is strictly henselian. By Lemma 154.2 the R-algebra R^{sh} is a colimit of étale R-algebras. Hence the uniqueness of Lemma 154.7 shows that R^{sh} is the strict henselization.

04GR **Lemma 155.5.** *Let $R \to S$ be a local map of local rings. Let $S \to S^h$ be the henselization. Let $R \to A$ be an étale ring map and let $\mathfrak{q}$ be a prime of A lying over $\mathfrak{m}_R$ such that $R/\mathfrak{m}_R \cong \kappa(\mathfrak{q})$. Then there exists a unique morphism of rings $f : A \to S^h$ fitting into the commutative diagram*

$$\begin{tikzcd} A \arrow[r, "f"'] & S^h \\ R \arrow[u] \arrow[r] & S \arrow[u] \end{tikzcd}$$

such that $f^{-1}(\mathfrak{m}_{S^h}) = \mathfrak{q}$.

Proof. This is a special case of Lemma 153.11. □

04GS **Lemma 155.6.** *Let $R \to S$ be a local map of local rings. Let $R \to R^h$ and $S \to S^h$ be the henselizations. There exists a unique local ring map $R^h \to S^h$ fitting into* *the commutative diagram*

$$\begin{tikzcd} R^h \arrow[r, "f"'] & S^h \\ R \arrow[u] \arrow[r] & S \arrow[u] \end{tikzcd}$$

Proof. Follows immediately from Lemma 154.6. $\square$

Here is a slightly different construction of the henselization.

04GV **Lemma 155.7.** *Let R be a ring. Let $\mathfrak p \subset R$ be a prime ideal. Consider the category of pairs $(S, \mathfrak q)$ where $R \to S$ is étale and $\mathfrak q$ is a prime lying over $\mathfrak p$ such that $\kappa(\mathfrak p) = \kappa(\mathfrak q)$. This category is filtered and*

$$(R_{\mathfrak p})^h = \mathrm{colim}_{(S,\mathfrak q)} S = \mathrm{colim}_{(S,\mathfrak q)} S_{\mathfrak q}$$

canonically.

Proof. A morphism of pairs $(S, \mathfrak q) \to (S', \mathfrak q')$ is given by an R-algebra map $\varphi : S \to S'$ such that $\varphi^{-1}(\mathfrak q') = \mathfrak q$. Let us show that the category of pairs is filtered, see Categories, Definition 19.1. The category contains the pair $(R, \mathfrak p)$ and hence is not empty, which proves part (1) of Categories, Definition 19.1. Suppose that $(S, \mathfrak q)$ and $(S', \mathfrak q')$ are two pairs. Note that $\mathfrak q$, resp. $\mathfrak q'$ correspond to primes of the fibre rings $S \otimes \kappa(\mathfrak p)$, resp. $S' \otimes \kappa(\mathfrak p)$ with residue fields $\kappa(\mathfrak p)$, hence they correspond to maximal ideals of $S \otimes \kappa(\mathfrak p)$, resp. $S' \otimes \kappa(\mathfrak p)$. Set $S'' = S \otimes_R S'$. By the above there exists a unique prime $\mathfrak q'' \subset S''$ lying over $\mathfrak q$ and over $\mathfrak q'$ whose residue field is $\kappa(\mathfrak p)$. The ring map $R \to S''$ is étale by Lemma 143.3. This proves part (2) of Categories, Definition 19.1. Next, suppose that $\varphi, \psi : (S, \mathfrak q) \to (S', \mathfrak q')$ are two morphisms of pairs. Then φ, ψ, and $S' \otimes_R S' \to S'$ are étale ring maps by Lemma 143.8. Consider

$$S'' = (S' \otimes_{\varphi, S, \psi} S') \otimes_{S' \otimes_R S'} S'$$

Arguing as above (base change of étale maps is étale, composition of étale maps is étale) we see that S'' is étale over R. The fibre ring of S'' over $\mathfrak p$ is

$$F'' = (F' \otimes_{\varphi, F, \psi} F') \otimes_{F' \otimes_{\kappa(\mathfrak p)} F'} F'$$

where F', F are the fibre rings of S' and S. Since φ and ψ are morphisms of pairs the map $F' \to \kappa(\mathfrak p)$ corresponding to $\mathfrak p'$ extends to a map $F'' \to \kappa(\mathfrak p)$ and in turn corresponds to a prime ideal $\mathfrak q'' \subset S''$ whose residue field is $\kappa(\mathfrak p)$. The canonical map $S' \to S''$ (using the right most factor for example) is a morphism of pairs $(S', \mathfrak q') \to (S'', \mathfrak q'')$ which equalizes φ and ψ. This proves part (3) of Categories, Definition 19.1. Hence we conclude that the category is filtered.

Recall that in the proof of Lemma 155.1 we constructed $(R_{\mathfrak p})^h$ as the corresponding colimit but starting with $R_{\mathfrak p}$ and its maximal ideal $\mathfrak pR_{\mathfrak p}$. Now, given any pair $(S, \mathfrak q)$ for $(R, \mathfrak p)$ we obtain a pair $(S_{\mathfrak p}, \mathfrak qS_{\mathfrak p})$ for $(R_{\mathfrak p}, \mathfrak pR_{\mathfrak p})$. Moreover, in this situation

$$S_{\mathfrak p} = \mathrm{colim}_{f \in R, f \not\in \mathfrak p} S_f.$$

Hence in order to show the equalities of the lemma, it suffices to show that any pair $(S_{loc}, \mathfrak q_{loc})$ for $(R_{\mathfrak p}, \mathfrak pR_{\mathfrak p})$ is of the form $(S_{\mathfrak p}, \mathfrak qS_{\mathfrak p})$ for some pair $(S, \mathfrak q)$ over $(R, \mathfrak p)$ (some details omitted). This follows from Lemma 143.3. $\square$

08HU **Lemma 155.8.** *Let $R \to S$ be a ring map. Let $\mathfrak q \subset S$ be a prime lying over $\mathfrak p \subset R$. Let $R \to R^h$ and $S \to S^h$ be the henselizations of $R_\mathfrak p$ and $S_\mathfrak q$. The local ring map $R^h \to S^h$ of Lemma 155.6 identifies S^h with the henselization of $R^h \otimes_R S$ at the unique prime lying over $\mathfrak m^h$ and $\mathfrak q$.*

Proof. By Lemma 155.7 we see that R^h, resp. S^h are filtered colimits of étale R, resp. S-algebras. Hence we see that $R^h \otimes_R S$ is a filtered colimit of étale S-algebras A_i (Lemma 143.3). By Lemma 154.5 we see that S^h is a filtered colimit of étale $R^h \otimes_R S$-algebras. Since moreover S^h is a henselian local ring with residue field equal to $\kappa(\mathfrak q)$, the statement follows from the uniqueness result of Lemma 154.7. $\square$

04GT **Lemma 155.9.** *Let $\varphi : R \to S$ be a local map of local rings. Let $S/\mathfrak m_S \subset \kappa^{sep}$ be a separable algebraic closure. Let $S \to S^{sh}$ be the strict henselization of S with respect to $S/\mathfrak m_S \subset \kappa^{sep}$. Let $R \to A$ be an étale ring map and let $\mathfrak q$ be a prime of A lying over $\mathfrak m_R$. Given any commutative diagram*

$$\begin{tikzcd} \kappa(\mathfrak q) \arrow[r, "\phi"'] & \kappa^{sep} \\ R/\mathfrak m_R \arrow[u] \arrow[r, "\varphi"] & S/\mathfrak m_S \arrow[u] \end{tikzcd}$$

there exists a unique morphism of rings $f : A \to S^{sh}$ fitting into the commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"'] & S^{sh} \\ R \arrow[u] \arrow[r, "\varphi"] & S \arrow[u] \end{tikzcd}$$

such that $f^{-1}(\mathfrak m_{S^h}) = \mathfrak q$ and the induced map $\kappa(\mathfrak q) \to \kappa^{sep}$ is the given one.

Proof. This is a special case of Lemma 153.11. $\square$

04GU **Lemma 155.10.** *Let $R \to S$ be a local map of local rings. Choose separable algebraic closures $R/\mathfrak m_R \subset \kappa_1^{sep}$ and $S/\mathfrak m_S \subset \kappa_2^{sep}$. Let $R \to R^{sh}$ and $S \to S^{sh}$ be the corresponding strict henselizations. Given any commutative diagram*

$$\begin{tikzcd} \kappa_1^{sep} \arrow[r, "\phi"'] & \kappa_2^{sep} \\ R/\mathfrak m_R \arrow[u] \arrow[r, "\varphi"] & S/\mathfrak m_S \arrow[u] \end{tikzcd}$$

There exists a unique local ring map $R^{sh} \to S^{sh}$ fitting into the commutative diagram

$$\begin{tikzcd} R^{sh} \arrow[r, "f"'] & S^{sh} \\ R \arrow[u] \arrow[r] & S \arrow[u] \end{tikzcd}$$

and inducing ϕ on the residue fields of R^{sh} and S^{sh}.

Proof. Follows immediately from Lemma 154.6. $\square$

04GW **Lemma 155.11.** *Let R be a ring. Let $\mathfrak p \subset R$ be a prime ideal. Let $\kappa(\mathfrak p) \subset \kappa^{sep}$ be a separable algebraic closure. Consider the category of triples $(S, \mathfrak q, \phi)$ where $R \to S$ is étale, $\mathfrak q$ is a prime lying over $\mathfrak p$, and $\phi : \kappa(\mathfrak q) \to \kappa^{sep}$ is a $\kappa(\mathfrak p)$-algebra map. This category is filtered and*

$$(R_{\mathfrak p})^{sh} = \mathrm{colim}_{(S,\mathfrak q,\phi)} S = \mathrm{colim}_{(S,\mathfrak q,\phi)} S_{\mathfrak q}$$

canonically.

Proof. A morphism of triples $(S, \mathfrak q, \phi) \to (S', \mathfrak q', \phi')$ is given by an R-algebra map $\varphi : S \to S'$ such that $\varphi^{-1}(\mathfrak q') = \mathfrak q$ and such that $\phi' \circ \varphi = \phi$. Let us show that the category of pairs is filtered, see Categories, Definition 19.1. The category contains the triple $(R, \mathfrak p, \kappa(\mathfrak p) \subset \kappa^{sep})$ and hence is not empty, which proves part (1) of Categories, Definition 19.1. Suppose that $(S, \mathfrak q, \phi)$ and $(S', \mathfrak q', \phi')$ are two triples. Note that $\mathfrak q$, resp. $\mathfrak q'$ correspond to primes of the fibre rings $S \otimes \kappa(\mathfrak p)$, resp. $S' \otimes \kappa(\mathfrak p)$ with residue fields finite separable over $\kappa(\mathfrak p)$ and ϕ, resp. ϕ' correspond to maps into κ^{sep}. Hence this data corresponds to $\kappa(\mathfrak p)$-algebra maps

$$\phi : S \otimes_R \kappa(\mathfrak p) \longrightarrow \kappa^{sep}, \quad \phi' : S' \otimes_R \kappa(\mathfrak p) \longrightarrow \kappa^{sep}.$$

Set $S'' = S \otimes_R S'$. Combining the maps the above we get a unique $\kappa(\mathfrak p)$-algebra map

$$\phi'' = \phi \otimes \phi' : S'' \otimes_R \kappa(\mathfrak p) \longrightarrow \kappa^{sep}$$

whose kernel corresponds to a prime $\mathfrak q'' \subset S''$ lying over $\mathfrak q$ and over $\mathfrak q'$, and whose residue field maps via ϕ'' to the compositum of $\phi(\kappa(\mathfrak q))$ and $\phi'(\kappa(\mathfrak q'))$ in κ^{sep}. The ring map $R \to S''$ is étale by Lemma 143.3. Hence $(S'', \mathfrak q'', \phi'')$ is a triple dominating both $(S, \mathfrak q, \phi)$ and $(S', \mathfrak q', \phi')$. This proves part (2) of Categories, Definition 19.1. Next, suppose that $\varphi, \psi : (S, \mathfrak q, \phi) \to (S', \mathfrak q', \phi')$ are two morphisms of pairs. Then φ, ψ, and $S' \otimes_R S' \to S'$ are étale ring maps by Lemma 143.8. Consider

$$S'' = (S' \otimes_{\varphi, S, \psi} S') \otimes_{S' \otimes_R S'} S'$$

Arguing as above (base change of étale maps is étale, composition of étale maps is étale) we see that S'' is étale over R. The fibre ring of S'' over $\mathfrak p$ is

$$F'' = (F' \otimes_{\varphi, F, \psi} F') \otimes_{F' \otimes_{\kappa(\mathfrak p)} F'} F'$$

where F', F are the fibre rings of S' and S. Since φ and ψ are morphisms of triples the map $\phi' : F' \to \kappa^{sep}$ extends to a map $\phi'' : F'' \to \kappa^{sep}$ which in turn corresponds to a prime ideal $\mathfrak q'' \subset S''$. The canonical map $S' \to S''$ (using the right most factor for example) is a morphism of triples $(S', \mathfrak q', \phi') \to (S'', \mathfrak q'', \phi'')$ which equalizes φ and ψ. This proves part (3) of Categories, Definition 19.1. Hence we conclude that the category is filtered.

We still have to show that the colimit R_{colim} of the system is equal to the strict henselization of $R_{\mathfrak p}$ with respect to κ^{sep}. To see this note that the system of triples $(S, \mathfrak q, \phi)$ contains as a subsystem the pairs $(S, \mathfrak q)$ of Lemma 155.7. Hence R_{colim} contains $R_{\mathfrak p}^h$ by the result of that lemma. Moreover, it is clear that $R_{\mathfrak p}^h \subset R_{colim}$ is a directed colimit of étale ring extensions. It follows that R_{colim} is henselian by Lemmas 153.4 and 154.8. Finally, by Lemma 144.3 we see that the residue field of R_{colim} is equal to κ^{sep}. Hence we conclude that R_{colim} is strictly henselian and hence equals the strict henselization of $R_{\mathfrak p}$ as desired. Some details omitted. $\square$

08HV **Lemma 155.12.** *Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime lying over $\mathfrak{p} \subset R$. Choose separable algebraic closures $\kappa(\mathfrak{p}) \subset \kappa_1^{sep}$ and $\kappa(\mathfrak{q}) \subset \kappa_2^{sep}$. Let R^{sh} and S^{sh} be the corresponding strict henselizations of $R_\mathfrak{p}$ and $S_\mathfrak{q}$. Given any commutative diagram*

$$\begin{tikzcd} \kappa_1^{sep} \arrow[r, "\phi"'] & \kappa_2^{sep} \\ \kappa(\mathfrak{p}) \arrow[u] \arrow[r, "\varphi"] & \kappa(\mathfrak{q}) \arrow[u] \end{tikzcd}$$

The local ring map $R^{sh} \to S^{sh}$ of Lemma 155.10 identifies S^{sh} with the strict henselization of $R^{sh} \otimes_R S$ at a prime lying over $\mathfrak{q}$ and the maximal ideal $\mathfrak{m}^{sh} \subset R^{sh}$.

Proof. The proof is identical to the proof of Lemma 155.8 except that it uses Lemma 155.11 instead of Lemma 155.7. □

0C2Z **Lemma 155.13.** *Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime lying over $\mathfrak{p} \subset R$ such that $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{q})$ is an isomorphism. Choose a separable algebraic closure κ^{sep} of $\kappa(\mathfrak{p}) = \kappa(\mathfrak{q})$. Then*

$$(S_\mathfrak{q})^{sh} = (S_\mathfrak{q})^h \otimes_{(R_\mathfrak{p})^h} (R_\mathfrak{p})^{sh}$$

Proof. This follows from the alternative construction of the strict henselization of a local ring in Remark 155.4 and the fact that the residue fields are equal. Some details omitted. □

156. Henselization and quasi-finite ring maps

0GIN In this section we prove some results concerning the functorial maps between (strict) henselizations for quasi-finite ring maps.

05WP **Lemma 156.1.** *Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime of S lying over $\mathfrak{p}$ in R. Assume $R \to S$ is quasi-finite at $\mathfrak{q}$. The commutative diagram*

$$\begin{tikzcd} R_\mathfrak{p}^h \arrow[r] & S_\mathfrak{q}^h \\ R_\mathfrak{p} \arrow[u] \arrow[r] & S_\mathfrak{q} \arrow[u] \end{tikzcd}$$

of Lemma 155.6 identifies $S_\mathfrak{q}^h$ with the localization of $R_\mathfrak{p}^h \otimes_{R_\mathfrak{p}} S_\mathfrak{q}$ at the prime generated by $\mathfrak{q}$. Moreover, the ring map $R_\mathfrak{p}^h \to S_\mathfrak{q}^h$ is finite.

Proof. Note that $R_\mathfrak{p}^h \otimes_R S$ is quasi-finite over $R_\mathfrak{p}^h$ at the prime ideal corresponding to $\mathfrak{q}$, see Lemma 122.6. Hence the localization S' of $R_\mathfrak{p}^h \otimes_{R_\mathfrak{p}} S_\mathfrak{q}$ is henselian and finite over $R_\mathfrak{p}^h$, see Lemma 153.4. As a localization S' is a filtered colimit of étale $R_\mathfrak{p}^h \otimes_{R_\mathfrak{p}} S_\mathfrak{q}$-algebras. By Lemma 155.8 we see that $S_\mathfrak{q}^h$ is the henselization of $R_\mathfrak{p}^h \otimes_{R_\mathfrak{p}} S_\mathfrak{q}$. Thus $S' = S_\mathfrak{q}^h$ by the uniqueness result of Lemma 154.7. □

05WQ **Lemma 156.2.** *Let R be a local ring with henselization R^h. Let $I \subset \mathfrak{m}_R$. Then R^h/IR^h is the henselization of R/I.*

Proof. This is a special case of Lemma 156.1. □

05WR **Lemma 156.3.** *Let $R \to S$ be a ring map. Let $\mathfrak q$ be a prime of S lying over $\mathfrak p$ in R. Assume $R \to S$ is quasi-finite at $\mathfrak q$. Let $\kappa_2^{sep}/\kappa(\mathfrak q)$ be a separable algebraic closure and denote $\kappa_1^{sep} \subset \kappa_2^{sep}$ the subfield of elements separable algebraic over $\kappa(\mathfrak q)$ (Fields, Lemma 14.6). The commutative diagram*

$$\begin{tikzcd} R_{\mathfrak p}^{sh} \arrow[r] & S_{\mathfrak q}^{sh} \\ R_{\mathfrak p} \arrow[u] \arrow[r] & S_{\mathfrak q} \arrow[u] \end{tikzcd}$$

of Lemma 155.10 identifies $S_{\mathfrak q}^{sh}$ with the localization of $R_{\mathfrak p}^{sh} \otimes_{R_{\mathfrak p}} S_{\mathfrak q}$ at the prime ideal which is the kernel of the map

$$R_{\mathfrak p}^{sh} \otimes_{R_{\mathfrak p}} S_{\mathfrak q} \longrightarrow \kappa_1^{sep} \otimes_{\kappa(\mathfrak p)} \kappa(\mathfrak q) \longrightarrow \kappa_2^{sep}$$

Moreover, the ring map $R_{\mathfrak p}^{sh} \to S_{\mathfrak q}^{sh}$ is a finite local homomorphism of local rings whose residue field extension is the extension $\kappa_2^{sep}/\kappa_1^{sep}$ which is both finite and purely inseparable.

Proof. Since $R \to S$ is quasi-finite at $\mathfrak q$ we see that the extension $\kappa(\mathfrak q)/\kappa(\mathfrak p)$ is finite, see Definition 122.3 and Lemma 122.2. Hence κ_1^{sep} is a separable algebraic closure of $\kappa(\mathfrak p)$ (small detail omitted). In particular Lemma 155.10 does really apply. Next, the compositum of $\kappa(\mathfrak p)$ and κ_1^{sep} in κ_2^{sep} is separably algebraically closed and hence equal to κ_2^{sep}. We conclude that $\kappa_2^{sep}/\kappa_1^{sep}$ is finite. By construction the extension $\kappa_2^{sep}/\kappa_1^{sep}$ is purely inseparable. The ring map $R_{\mathfrak p}^{sh} \to S_{\mathfrak q}^{sh}$ is indeed local and induces the residue field extension $\kappa_2^{sep}/\kappa_1^{sep}$ which is indeed finite purely inseparable.

Note that $R_{\mathfrak p}^{sh} \otimes_R S$ is quasi-finite over $R_{\mathfrak p}^{sh}$ at the prime ideal $\mathfrak q'$ given in the statement of the lemma, see Lemma 122.6. Hence the localization S' of $R_{\mathfrak p}^{sh} \otimes_{R_{\mathfrak p}} S_{\mathfrak q}$ at $\mathfrak q'$ is henselian and finite over $R_{\mathfrak p}^{sh}$, see Lemma 153.4. Note that the residue field of S' is κ_2^{sep} as the map $\kappa_1^{sep} \otimes_{\kappa(\mathfrak p)} \kappa(\mathfrak q) \to \kappa_2^{sep}$ is surjective by the discussion in the previous paragraph. Furthermore, as a localization S' is a filtered colimit of étale $R_{\mathfrak p}^{sh} \otimes_{R_{\mathfrak p}} S_{\mathfrak q}$-algebras. By Lemma 155.12 we see that $S_{\mathfrak q}^{sh}$ is the strict henselization of $R_{\mathfrak p}^{sh} \otimes_{R_{\mathfrak p}} S_{\mathfrak q}$ at $\mathfrak q'$. Thus $S' = S_{\mathfrak q}^{sh}$ by the uniqueness result of Lemma 154.7. $\square$

05WS **Lemma 156.4.** *Let R be a local ring with strict henselization R^{sh}. Let $I \subset \mathfrak m_R$. Then R^{sh}/IR^{sh} is a strict henselization of R/I.*

Proof. This is a special case of Lemma 156.3. $\square$

092Y **Lemma 156.5.** *Let $A \to B$ and $A \to C$ be local homomorphisms of local rings. If $A \to C$ is integral and either $\kappa(\mathfrak m_C)/\kappa(\mathfrak m_A)$ or $\kappa(\mathfrak m_B)/\kappa(\mathfrak m_A)$ is purely inseparable, then $D = B \otimes_A C$ is a local ring and $B \to D$ and $C \to D$ are local.*

Proof. Any maximal ideal of D lies over the maximal ideal of B by going up for the integral ring map $B \to D$ (Lemma 36.22). Now $D/\mathfrak m_B D = \kappa(\mathfrak m_B) \otimes_A C = \kappa(\mathfrak m_B) \otimes_{\kappa(\mathfrak m_A)} C/\mathfrak m_A C$. The spectrum of $C/\mathfrak m_A C$ consists of a single point, namely $\mathfrak m_C$. Thus the spectrum of $D/\mathfrak m_B D$ is the same as the spectrum of $\kappa(\mathfrak m_B) \otimes_{\kappa(\mathfrak m_A)} \kappa(\mathfrak m_C)$ which is a single point by our assumption that either $\kappa(\mathfrak m_C)/\kappa(\mathfrak m_A)$ or $\kappa(\mathfrak m_B)/\kappa(\mathfrak m_A)$ is purely inseparable. This proves that D is local and that the ring maps $B \to D$ and $C \to D$ are local. $\square$

0GIP **Lemma 156.6.** *Let $A \to B$ and $A \to C$ be ring maps. Let κ be a separably algebraically closed field and let $B \otimes_A C \to \kappa$ be a ring homomorphism. Denote*

$$\begin{array}{ccc} B^{sh} & \longrightarrow & (B \otimes_A C)^{sh} \\ \uparrow & & \uparrow \\ A^{sh} & \longrightarrow & C^{sh} \end{array}$$

the corresponding maps of strict henselizations (see proof). If

1. *$A \to B$ is quasi-finite at the prime $\mathfrak{p}_B = \mathrm{Ker}(B \to \kappa)$, or*
2. *B is a filtered colimit of quasi-finite A-algebras, or*
3. *$B_{\mathfrak{p}_B}$ is a filtered colimit of quasi-finite algebras over $A_{\mathfrak{p}_A}$, or*
4. *B is integral over A,*

then $B^{sh} \otimes_{A^{sh}} C^{sh} \to (B \otimes_A C)^{sh}$ is an isomorphism.

Proof. Write $D = B \otimes_A C$. Denote $\mathfrak{p}_A = \mathrm{Ker}(A \to \kappa)$ and similarly for $\mathfrak{p}_B$, $\mathfrak{p}_C$, and $\mathfrak{p}_D$. Denote $\kappa_A \subset \kappa$ the separable algebraic closure of $\kappa(\mathfrak{p}_A)$ in κ and similarly for κ_B, κ_C, and κ_D. Denote A^{sh} the strict henselization of $A_{\mathfrak{p}_A}$ constructed using the separable algebraic closure $\kappa_A/\kappa(\mathfrak{p}_A)$. Similarly for B^{sh}, C^{sh}, and D^{sh}. We obtain the commutative diagram of the lemma from the functoriality of Lemma 155.10.

Consider the map

$$c : B^{sh} \otimes_{A^{sh}} C^{sh} \to D^{sh} = (B \otimes_A C)^{sh}$$

we obtain from the commutative diagram. If $A \to B$ is quasi-finite at $\mathfrak{p}_B = \mathrm{Ker}(B \to \kappa)$, then the ring map $C \to D$ is quasi-finite at $\mathfrak{p}_D$ by Lemma 122.6. Hence by Lemma 156.3 (and Lemma 36.13) the ring map c is a homomorphism of finite C^{sh}-algebras and

$$B^{sh} = (B \otimes_A A^{sh})_{\mathfrak{q}} \quad \text{and} \quad D^{sh} = (D \otimes_C C^{sh})_{\mathfrak{r}} = (B \otimes_A C^{sh})_{\mathfrak{r}}$$

for some primes $\mathfrak{q}$ and $\mathfrak{r}$. Since

$$B^{sh} \otimes_{A^{sh}} C^{sh} = (B \otimes_A A^{sh})_{\mathfrak{q}} \otimes_{A^{sh}} C^{sh} = \text{a localization of } B \otimes_A C^{sh}$$

we conclude that source and target of c are both localizations of $B \otimes_A C^{sh}$ (compatibly with the map). Hence it suffices to show that $B^{sh} \otimes_{A^{sh}} C^{sh}$ is local (small detail omitted). This follows from Lemma 156.5 and the fact that $A^{sh} \to B^{sh}$ is finite with purely inseparable residue field extension by the already used Lemma 156.3. This proves case (1) of the lemma.

In case (2) write $B = \mathrm{colim}\, B_i$ as a filtered colimit of quasi-finite A-algebras. We correspondingly get $D = \mathrm{colim}\, D_i$ with $D_i = B_i \otimes_A C$. Observe that $B^{sh} = \mathrm{colim}\, B_i^{sh}$. Namely, the ring $\mathrm{colim}\, B_i^{sh}$ is a strictly henselian local ring by Lemma 154.8. Also $\mathrm{colim}\, B_i^{sh}$ is a filtered colimit of étale B-algebras by Lemma 154.4. Finally, the residue field of $\mathrm{colim}\, B_i^{sh}$ is a separable algebraic closure of $\kappa(\mathfrak{p}_B)$ (details omitted). Hence we conclude that $B^{sh} = \mathrm{colim}\, B_i^{sh}$, see discussion following Definition 155.3. Similarly, we have $D^{sh} = \mathrm{colim}\, D_i^{sh}$. Then we conclude by case (1) because

$$D^{sh} = \mathrm{colim}\, D_i^{sh} = \mathrm{colim}\, B_i^{sh} \otimes_{A^{sh}} C^{sh} = B^{sh} \otimes_{A^{sh}} C^{sh}$$

since filtered colimit commute with tensor products.

Case (3). We may replace A, B, C by their localizations at $\mathfrak{p}_A$, $\mathfrak{p}_B$, and $\mathfrak{p}_C$. Thus (3) follows from (2).

Since an integral ring map is a filtered colimit of finite ring maps, we see that (4) follows from (2) as well. □

157. Serre's criterion for normality

031O We introduce the following properties of Noetherian rings.

031P **Definition 157.1.** Let R be a Noetherian ring. Let $k \geq 0$ be an integer.

(1) We say R has property (R_k) if for every prime $\mathfrak{p}$ of height $\leq k$ the local ring $R_\mathfrak{p}$ is regular. We also say that R is *regular in codimension* $\leq k$.
(2) We say R has property (S_k) if for every prime $\mathfrak{p}$ the local ring $R_\mathfrak{p}$ has depth at least $\min\{k, \dim(R_\mathfrak{p})\}$.
(3) Let M be a finite R-module. We say M has property (S_k) if for every prime $\mathfrak{p}$ the module $M_\mathfrak{p}$ has depth at least $\min\{k, \dim(\text{Supp}(M_\mathfrak{p}))\}$.

Any Noetherian ring has property (S_0) and so does any finite module over it. Our convention that the depth of the zero module is ∞ (see Section 72) and the dimension of the empty set is $-\infty$ (see Topology, Section 10) guarantees that the zero module has property (S_k) for all k.

031Q **Lemma 157.2.** *Let R be a Noetherian ring. Let M be a finite R-module. The following are equivalent:*

(1) *M has no embedded associated prime, and*
(2) *M has property (S_1).*

Proof. Let $\mathfrak{p}$ be an embedded associated prime of M. Then there exists another associated prime $\mathfrak{q}$ of M such that $\mathfrak{p} \supset \mathfrak{q}$. In particular this implies that $\dim(\text{Supp}(M_\mathfrak{p})) \geq 1$ (since $\mathfrak{q}$ is in the support as well). On the other hand $\mathfrak{p}R_\mathfrak{p}$ is associated to $M_\mathfrak{p}$ (Lemma 63.15) and hence $\text{depth}(M_\mathfrak{p}) = 0$ (see Lemma 63.18). In other words (S_1) does not hold. Conversely, if (S_1) does not hold then there exists a prime $\mathfrak{p}$ such that $\dim(\text{Supp}(M_\mathfrak{p})) \geq 1$ and $\text{depth}(M_\mathfrak{p}) = 0$. Then we see (arguing backwards using the lemmas cited above) that $\mathfrak{p}$ is an embedded associated prime. □

031R **Lemma 157.3.** *Let R be a Noetherian ring. The following are equivalent:*

(1) *R is reduced, and*
(2) *R has properties (R_0) and (S_1).*

Proof. Suppose that R is reduced. Then $R_\mathfrak{p}$ is a field for every minimal prime $\mathfrak{p}$ of R, according to Lemma 25.1. Hence we have (R_0). Let $\mathfrak{p}$ be a prime of height ≥ 1. Then $A = R_\mathfrak{p}$ is a reduced local ring of dimension ≥ 1. Hence its maximal ideal $\mathfrak{m}$ is not an associated prime since this would mean there exists a $x \in \mathfrak{m}$ with annihilator $\mathfrak{m}$ so $x^2 = 0$. Hence the depth of $A = R_\mathfrak{p}$ is at least one, by Lemma 63.9. This shows that (S_1) holds.

Conversely, assume that R satisfies (R_0) and (S_1). If $\mathfrak{p}$ is a minimal prime of R, then $R_\mathfrak{p}$ is a field by (R_0), and hence is reduced. If $\mathfrak{p}$ is not minimal, then we see that $R_\mathfrak{p}$ has depth ≥ 1 by (S_1) and we conclude there exists an element $t \in \mathfrak{p}R_\mathfrak{p}$ such that $R_\mathfrak{p} \to R_\mathfrak{p}[1/t]$ is injective. This implies that $R_\mathfrak{p}$ is a subring of localizations of R at primes of smaller height. Thus by induction on the height we conclude that R is reduced. □

031S **Lemma 157.4** (Serre's criterion for normality)**.** *Let R be a Noetherian ring. The following are equivalent:* [DG67, IV, Theorem 5.8.6]

(1) *R is a normal ring, and*
(2) *R has properties (R_1) and (S_2).*

Proof. Proof of (1) $\Rightarrow$ (2). Assume R is normal, i.e., all localizations $R_\mathfrak{p}$ at primes are normal domains. In particular we see that R has (R_0) and (S_1) by Lemma 157.3. Hence it suffices to show that a local Noetherian normal domain R of dimension d has depth $\geq \min(2, d)$ and is regular if $d = 1$. The assertion if $d = 1$ follows from Lemma 119.7.

Let R be a local Noetherian normal domain with maximal ideal $\mathfrak{m}$ and dimension $d \geq 2$. Apply Lemma 119.2 to R. It is clear that R does not fall into cases (1) or (2) of the lemma. Let $R \to R'$ as in (4) of the lemma. Since R is a domain we have $R \subset R'$. Since $\mathfrak{m}$ is not an associated prime of R' there exists an $x \in \mathfrak{m}$ which is a nonzerodivisor on R'. Then $R_x = R'_x$ so R and R' are domains with the same fraction field. But finiteness of $R \subset R'$ implies every element of R' is integral over R (Lemma 36.3) and we conclude that $R = R'$ as R is normal. This means (4) does not happen. Thus we get the remaining possibility (3), i.e., $\text{depth}(R) \geq 2$ as desired.

Proof of (2) $\Rightarrow$ (1). Assume R satisfies (R_1) and (S_2). By Lemma 157.3 we conclude that R is reduced. Hence it suffices to show that if R is a reduced local Noetherian ring of dimension d satisfying (S_2) and (R_1) then R is a normal domain. If $d = 0$, the result is clear. If $d = 1$, then the result follows from Lemma 119.7.

Let R be a reduced local Noetherian ring with maximal ideal $\mathfrak{m}$ and dimension $d \geq 2$ which satisfies (R_1) and (S_2). By Lemma 37.16 it suffices to show that R is integrally closed in its total ring of fractions $Q(R)$. Pick $x \in Q(R)$ which is integral over R. Then $R' = R[x]$ is a finite ring extension of R (Lemma 36.5). Because $\dim(R_\mathfrak{p}) < d$ for every nonmaximal prime $\mathfrak{p} \subset R$ we have $R_\mathfrak{p} = R'_\mathfrak{p}$ by induction. Hence the support of R'/R is $\{\mathfrak{m}\}$. It follows that R'/R is annihilated by a power of $\mathfrak{m}$ (Lemma 62.4). By Lemma 119.2 this contradicts the assumption that the depth of R is $\geq 2 = \min(2, d)$ and the proof is complete. $\square$

0567 **Lemma 157.5.** *A regular ring is normal.*

Proof. Let R be a regular ring. By Lemma 157.4 it suffices to prove that R is (R_1) and (S_2). As a regular local ring is Cohen-Macaulay, see Lemma 106.3, it is clear that R is (S_2). Property (R_1) is immediate. $\square$

031T **Lemma 157.6.** *Let R be a Noetherian normal domain with fraction field K. Then*

(1) *for any nonzero $a \in R$ the quotient R/aR has no embedded primes, and all its associated primes have height* 1
(2)
$$R = \bigcap\nolimits_{height(\mathfrak{p})=1} R_\mathfrak{p}$$
(3) *For any nonzero $x \in K$ the quotient $R/(R \cap xR)$ has no embedded primes, and all its associates primes have height* 1*.*

Proof. By Lemma 157.4 we see that R has (S_2). Hence for any nonzero element $a \in R$ we see that R/aR has (S_1) (use Lemma 72.6 for example) Hence R/aR has no embedded primes (Lemma 157.2). We conclude the associated primes of R/aR

are exactly the minimal primes $\mathfrak{p}$ over (a), which have height 1 as a is not zero (Lemma 60.11). This proves (1).

Thus, given $b \in R$ we have $b \in aR$ if and only if $b \in aR_{\mathfrak{p}}$ for every minimal prime $\mathfrak{p}$ over (a) (see Lemma 63.19). These primes all have height 1 as seen above so $b/a \in R$ if and only if $b/a \in R_{\mathfrak{p}}$ for all height 1 primes. Hence (2) holds.

For (3) write $x = a/b$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the minimal primes over (ab). These all have height 1 by the above. Then we see that $R \cap xR = \bigcap_{i=1,\ldots,r}(R \cap xR_{\mathfrak{p}_i})$ by part (2) of the lemma. Hence $R/(R \cap xR)$ is a submodule of $\bigoplus R/(R \cap xR_{\mathfrak{p}_i})$. As $R_{\mathfrak{p}_i}$ is a discrete valuation ring (by property (R_1) for the Noetherian normal domain R, see Lemma 157.4) we have $xR_{\mathfrak{p}_i} = \mathfrak{p}_i^{e_i}R_{\mathfrak{p}_i}$ for some $e_i \in \mathbf{Z}$. Hence the direct sum is equal to $\bigoplus_{e_i>0} R/\mathfrak{p}_i^{(e_i)}$, see Definition 64.1. By Lemma 64.2 the only associated prime of the module $R/\mathfrak{p}^{(n)}$ is $\mathfrak{p}$. Hence the set of associate primes of $R/(R \cap xR)$ is a subset of $\{\mathfrak{p}_i\}$ and there are no inclusion relations among them. This proves (3). $\square$

158. Formal smoothness of fields

031U In this section we show that field extensions are formally smooth if and only if they are separable. However, we first prove finitely generated field extensions are separable algebraic if and only if they are formally unramified.

090W **Lemma 158.1.** *Let K/k be a finitely generated field extension. The following are equivalent*

1. *K is a finite separable field extension of k,*
2. *$\Omega_{K/k} = 0$,*
3. *K is formally unramified over k,*
4. *K is unramified over k,*
5. *K is formally étale over k,*
6. *K is étale over k.*

Proof. The equivalence of (2) and (3) is Lemma 148.2. By Lemma 143.4 we see that (1) is equivalent to (6). Property (6) implies (5) and (4) which both in turn imply (3) (Lemmas 150.2, 151.3, and 151.2). Thus it suffices to show that (2) implies (1). Choose a finitely generated k-subalgebra $A \subset K$ such that K is the fraction field of the domain A. Set $S = A \setminus \{0\}$. Since $0 = \Omega_{K/k} = S^{-1}\Omega_{A/k}$ (Lemma 131.8) and since $\Omega_{A/k}$ is finitely generated (Lemma 131.16), we can replace A by a localization A_f to reduce to the case that $\Omega_{A/k} = 0$ (details omitted). Then A is unramified over k, hence K/k is finite separable for example by Lemma 151.5 applied with $\mathfrak{q} = (0)$. $\square$

031W **Lemma 158.2.** *Let k be a perfect field of characteristic $p > 0$. Let K/k be an extension. Let $a \in K$. Then $da = 0$ in $\Omega_{K/k}$ if and only if a is a pth power.*

Proof. By Lemma 131.5 we see that there exists a subfield $k \subset L \subset K$ such that L/k is a finitely generated field extension and such that $\mathrm{d}a$ is zero in $\Omega_{L/k}$. Hence we may assume that K is a finitely generated field extension of k.

Choose a transcendence basis $x_1, \ldots, x_r \in K$ such that K is finite separable over $k(x_1, \ldots, x_r)$. This is possible by the definitions, see Definitions 45.1 and 42.1. We remark that the result holds for the purely transcendental subfield $k(x_1, \ldots, x_r) \subset K$. Namely,

$$\Omega_{k(x_1,\ldots,x_r)/k} = \bigoplus\nolimits_{i=1}^r k(x_1, \ldots, x_r)\mathrm{d}x_i$$

and any rational function all of whose partial derivatives are zero is a pth power. Moreover, we also have

$$\Omega_{K/k} = \bigoplus\nolimits_{i=1}^r K\mathrm{d}x_i$$

since $k(x_1, \ldots, x_r) \subset K$ is finite separable (computation omitted). Suppose $a \in K$ is an element such that $\mathrm{d}a = 0$ in the module of differentials. By our choice of x_i we see that the minimal polynomial $P(T) \in k(x_1, \ldots, x_r)[T]$ of a is separable. Write

$$P(T) = T^d + \sum\nolimits_{i=1}^d a_i T^{d-i}$$

and hence

$$0 = \mathrm{d}P(a) = \sum\nolimits_{i=1}^d a^{d-i}\mathrm{d}a_i$$

in $\Omega_{K/k}$. By the description of $\Omega_{K/k}$ above and the fact that P was the minimal polynomial of a, we see that this implies $\mathrm{d}a_i = 0$. Hence $a_i = b_i^p$ for each i. Therefore by Fields, Lemma 28.2 we see that a is a pth power. □

07DZ **Lemma 158.3.** *Let k be a field of characteristic $p > 0$. Let $a_1, \ldots, a_n \in k$ be elements such that $\mathrm{d}a_1, \ldots, \mathrm{d}a_n$ are linearly independent in $\Omega_{k/\mathbf{F}_p}$. Then the field extension $k(a_1^{1/p}, \ldots, a_n^{1/p})$ has degree p^n over k.*

Proof. By induction on n. If $n = 1$ the result is Lemma 158.2. For the induction step, suppose that $k(a_1^{1/p}, \ldots, a_{n-1}^{1/p})$ has degree p^{n-1} over k. We have to show that a_n does not map to a pth power in $k(a_1^{1/p}, \ldots, a_{n-1}^{1/p})$. If it does then we can write

$$\begin{aligned} a_n &= \left(\sum\nolimits_{I=(i_1,\ldots,i_{n-1}),\ 0\leq i_j\leq p-1} \lambda_I a_1^{i_1/p} \ldots a_{n-1}^{i_{n-1}/p}\right)^p \\ &= \sum\nolimits_{I=(i_1,\ldots,i_{n-1}),\ 0\leq i_j\leq p-1} \lambda_I^p a_1^{i_1} \ldots a_{n-1}^{i_{n-1}} \end{aligned}$$

Applying d we see that $\mathrm{d}a_n$ is linearly dependent on $\mathrm{d}a_i$, $i < n$. This is a contradiction. □

031X **Lemma 158.4.** *Let k be a field of characteristic $p > 0$. The following are equivalent:*

1. *the field extension K/k is separable (see Definition 42.1), and*
2. *the map $K \otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{K/\mathbf{F}_p}$ is injective.*

Proof. Write K as a directed colimit $K = \mathrm{colim}_i K_i$ of finitely generated field extensions K_i/k. By definition K is separable if and only if each K_i is separable over k, and by Lemma 131.5 we see that $K \otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{K/\mathbf{F}_p}$ is injective if and only if each $K_i \otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{K_i/\mathbf{F}_p}$ is injective. Hence we may assume that K/k is a finitely generated field extension.

Assume K/k is a finitely generated field extension which is separable. Choose $x_1, \ldots, x_{r+1} \in K$ as in Lemma 42.3. In this case there exists an irreducible polynomial $G(X_1, \ldots, X_{r+1}) \in k[X_1, \ldots, X_{r+1}]$ such that $G(x_1, \ldots, x_{r+1}) = 0$ and such that $\partial G/\partial X_{r+1}$ is not identically zero. Moreover K is the field of fractions of the domain. $S = K[X_1, \ldots, X_{r+1}]/(G)$. Write

$$G = \sum a_I X^I, \quad X^I = X_1^{i_1} \ldots X_{r+1}^{i_{r+1}}.$$

Using the presentation of S above we see that

$$\Omega_{S/\mathbf{F}_p} = \frac{S \otimes_k \Omega_k \oplus \bigoplus_{i=1,\ldots,r+1} S\mathrm{d}X_i}{\langle \sum X^I \mathrm{d}a_I + \sum \partial G/\partial X_i \mathrm{d}X_i \rangle}$$

Since $\Omega_{K/\mathbf{F}_p}$ is the localization of the S-module $\Omega_{S/\mathbf{F}_p}$ (see Lemma 131.8) we conclude that

$$\Omega_{K/\mathbf{F}_p} = \frac{K \otimes_k \Omega_k \oplus \bigoplus_{i=1,\ldots,r+1} K\mathrm{d}X_i}{\langle \sum X^I \mathrm{d}a_I + \sum \partial G/\partial X_i \mathrm{d}X_i \rangle}$$

Now, since the polynomial $\partial G/\partial X_{r+1}$ is not identically zero we conclude that the map $K \otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{S/\mathbf{F}_p}$ is injective as desired.

Assume K/k is a finitely generated field extension and that $K \otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{K/\mathbf{F}_p}$ is injective. (This part of the proof is the same as the argument proving Lemma 44.1.) Let $x_1, \ldots, x_r$ be a transcendence basis of K over k such that the degree of inseparability of the finite extension $k(x_1, \ldots, x_r) \subset K$ is minimal. If K is separable over $k(x_1, \ldots, x_r)$ then we win. Assume this is not the case to get a contradiction. Then there exists an element $\alpha \in K$ which is not separable over $k(x_1, \ldots, x_r)$. Let $P(T) \in k(x_1, \ldots, x_r)[T]$ be its minimal polynomial. Because α is not separable actually P is a polynomial in T^p. Clear denominators to get an irreducible polynomial

$$G(X_1, \ldots, X_r, T) = \sum a_{I,i} X^I T^i \in k[X_1, \ldots, X_r, T]$$

such that $G(x_1, \ldots, x_r, \alpha) = 0$ in L. Note that this means $k[X_1, \ldots, X_r, T]/(G) \subset L$. We may assume that for some pair (I_0, i_0) the coefficient $a_{I_0,i_0} = 1$. We claim that $\mathrm{d}G/\mathrm{d}X_i$ is not identically zero for at least one i. Namely, if this is not the case, then G is actually a polynomial in $X_1^p, \ldots, X_r^p, T^p$. Then this means that

$$\sum\nolimits_{(I,i) \neq (I_0,i_0)} x^I \alpha^i \mathrm{d}a_{I,i}$$

is zero in $\Omega_{K/\mathbf{F}_p}$. Note that there is no k-linear relation among the elements

$$\{x^I \alpha^i \mid a_{I,i} \neq 0 \text{ and } (I,i) \neq (I_0, i_0)\}$$

of K. Hence the assumption that $K \otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{K/\mathbf{F}_p}$ is injective this implies that $\mathrm{d}a_{I,i} = 0$ in $\Omega_{k/\mathbf{F}_p}$ for all (I,i). By Lemma 158.2 we see that each $a_{I,i}$ is a pth power, which implies that G is a pth power contradicting the irreducibility of G. Thus, after renumbering, we may assume that $\mathrm{d}G/\mathrm{d}X_1$ is not zero. Then we see that x_1 is separably algebraic over $k(x_2, \ldots, x_r, \alpha)$, and that $x_2, \ldots, x_r, \alpha$ is a transcendence basis of L over k. This means that the degree of inseparability of the finite extension $k(x_2, \ldots, x_r, \alpha) \subset L$ is less than the degree of inseparability of the finite extension $k(x_1, \ldots, x_r) \subset L$, which is a contradiction. □

031Y **Lemma 158.5.** *Let K/k be an extension of fields. If K is formally smooth over k, then K is a separable extension of k.*

Proof. Assume K is formally smooth over k. By Lemma 138.9 we see that $K \otimes_k \Omega_{k/\mathbf{Z}} \to \Omega_{K/\mathbf{Z}}$ is injective. Hence K is separable over k by Lemma 158.4. □

031Z **Lemma 158.6.** *Let K/k be an extension of fields. Then K is formally smooth over k if and only if $H_1(L_{K/k}) = 0$.*

Proof. This follows from Proposition 138.8 and the fact that a vector spaces is free (hence projective). □

0320 **Lemma 158.7.** *Let K/k be an extension of fields.*

(1) *If K is purely transcendental over k, then K is formally smooth over k.*
(2) *If K is separable algebraic over k, then K is formally smooth over k.*
(3) *If K is separable over k, then K is formally smooth over k.*

Proof. For (1) write $K = k(x_j; j \in J)$. Suppose that A is a k-algebra, and $I \subset A$ is an ideal of square zero. Let $\varphi : K \to A/I$ be a k-algebra map. Let $a_j \in A$ be an element such that $a_j \mod I = \varphi(x_j)$. Then it is easy to see that there is a unique k-algebra map $K \to A$ which maps x_j to a_j and which reduces to φ mod I. Hence $k \subset K$ is formally smooth.

In case (2) we see that $k \subset K$ is a colimit of étale ring extensions. An étale ring map is formally étale (Lemma 150.2). Hence this case follows from Lemma 150.3 and the trivial observation that a formally étale ring map is formally smooth.

In case (3), write $K = \operatorname{colim} K_i$ as the filtered colimit of its finitely generated sub k-extensions. By Definition 42.1 each K_i is separable algebraic over a purely transcendental extension of k. Hence K_i/k is formally smooth by cases (1) and (2) and Lemma 138.3. Thus $H_1(L_{K_i/k}) = 0$ by Lemma 158.6. Hence $H_1(L_{K/k}) = 0$ by Lemma 134.9. Hence K/k is formally smooth by Lemma 158.6 again. □

0321 **Lemma 158.8.** *Let k be a field.*

(1) *If the characteristic of k is zero, then any extension field of k is formally smooth over k.*
(2) *If the characteristic of k is $p > 0$, then K/k is formally smooth if and only if it is a separable field extension.*

Proof. Combine Lemmas 158.5 and 158.7. □

Here we put together all the different characterizations of separable field extensions.

0322 **Proposition 158.9.** *Let K/k be a field extension. If the characteristic of k is zero then*

(1) *K is separable over k,*
(2) *K is geometrically reduced over k,*
(3) *K is formally smooth over k,*
(4) *$H_1(L_{K/k}) = 0$, and*
(5) *the map $K \otimes_k \Omega_{k/\mathbf{Z}} \to \Omega_{K/\mathbf{Z}}$ is injective.*

If the characteristic of k is $p > 0$, then the following are equivalent:

(1) *K is separable over k,*
(2) *the ring $K \otimes_k k^{1/p}$ is reduced,*
(3) *K is geometrically reduced over k,*
(4) *the map $K \otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{K/\mathbf{F}_p}$ is injective,*
(5) *$H_1(L_{K/k}) = 0$, and*
(6) *K is formally smooth over k.*

Proof. This is a combination of Lemmas 44.1, 158.8 158.5, and 158.4. □

Here is yet another characterization of finitely generated separable field extensions.

037X **Lemma 158.10.** *Let K/k be a finitely generated field extension. Then K is separable over k if and only if K is the localization of a smooth k-algebra.*

Proof. Choose a finite type k-algebra R which is a domain whose fraction field is K. Lemma 140.9 says that $k \to R$ is smooth at (0) if and only if K/k is separable. This proves the lemma. $\square$

07BV **Lemma 158.11.** *Let K/k be a field extension. Then K is a filtered colimit of global complete intersection algebras over k. If K/k is separable, then K is a filtered colimit of smooth algebras over k.*

Proof. Suppose that $E \subset K$ is a finite subset. It suffices to show that there exists a k subalgebra $A \subset K$ which contains E and which is a global complete intersection (resp. smooth) over k. The separable/smooth case follows from Lemma 158.10. In general let $L \subset K$ be the subfield generated by E. Pick a transcendence basis $x_1, \ldots, x_d \in L$ over k. The extension $L/k(x_1, \ldots, x_d)$ is finite. Say $L = k(x_1, \ldots, x_d)[y_1, \ldots, y_r]$. Pick inductively polynomials $P_i \in k(x_1, \ldots, x_d)[Y_1, \ldots, Y_r]$ such that $P_i = P_i(Y_1, \ldots, Y_i)$ is monic in Y_i over $k(x_1, \ldots, x_d)[Y_1, \ldots, Y_{i-1}]$ and maps to the minimum polynomial of y_i in $k(x_1, \ldots, x_d)[y_1, \ldots, y_{i-1}][Y_i]$. Then it is clear that $P_1, \ldots, P_r$ is a regular sequence in $k(x_1, \ldots, x_r)[Y_1, \ldots, Y_r]$ and that $L = k(x_1, \ldots, x_r)[Y_1, \ldots, Y_r]/(P_1, \ldots, P_r)$. If $h \in k[x_1, \ldots, x_d]$ is a polynomial such that $P_i \in k[x_1, \ldots, x_d, 1/h, Y_1, \ldots, Y_r]$, then we see that $P_1, \ldots, P_r$ is a regular sequence in $k[x_1, \ldots, x_d, 1/h, Y_1, \ldots, Y_r]$ and $A = k[x_1, \ldots, x_d, 1/h, Y_1, \ldots, Y_r]/(P_1, \ldots, P_r)$ is a global complete intersection. After adjusting our choice of h we may assume $E \subset A$ and we win. $\square$

159. Constructing flat ring maps

03C2 The following lemma is occasionally useful.

03C3 **Lemma 159.1.** *Let $(R, \mathfrak{m}, k)$ be a local ring. Let K/k be a field extension. There exists a local ring $(R', \mathfrak{m}', k')$, a flat local ring map $R \to R'$ such that $\mathfrak{m}' = \mathfrak{m}R'$ and such that k' is isomorphic to K as an extension of k.*

Proof. Suppose that $k' = k(\alpha)$ is a monogenic extension of k. Then k' is the residue field of a flat local extension $R \subset R'$ as in the lemma. Namely, if α is transcendental over k, then we let R' be the localization of $R[x]$ at the prime $\mathfrak{m}R[x]$. If α is algebraic with minimal polynomial $T^d + \sum \overline{\lambda}_i T^{d-i}$, then we let $R' = R[T]/(T^d + \sum \lambda_i T^{d-i})$.

Consider the collection of triples $(k', R \to R', \phi)$, where $k \subset k' \subset K$ is a subfield, $R \to R'$ is a local ring map as in the lemma, and $\phi : R' \to k'$ induces an isomorphism $R'/\mathfrak{m}R' \cong k'$ of k-extensions. These form a "big" category $\mathcal{C}$ with morphisms $(k_1, R_1, \phi_1) \to (k_2, R_2, \phi_2)$ given by ring maps $\psi : R_1 \to R_2$ such that

$$\begin{tikzcd} R_1 \arrow[r, "\phi_1"'] \arrow[d, "\psi"'] & k_1 \arrow[r] & K \arrow[d, equal] \\ R_2 \arrow[r, "\phi_2"] & k_2 \arrow[r] & K \end{tikzcd}$$

commutes. This implies that $k_1 \subset k_2$.

Suppose that I is a directed set, and $((R_i, k_i, \phi_i), \psi_{ii'})$ is a system over I, see Categories, Section 21. In this case we can consider

$$R' = \operatorname{colim}_{i \in I} R_i$$

This is a local ring with maximal ideal $\mathfrak{m}R'$, and residue field $k' = \bigcup_{i \in I} k_i$. Moreover, the ring map $R \to R'$ is flat as it is a colimit of flat maps (and tensor products commute with directed colimits). Hence we see that (R', k', ϕ') is an "upper bound" for the system.

An almost trivial application of Zorn's Lemma would finish the proof if $\mathcal{C}$ was a set, but it isn't. (Actually, you can make this work by finding a reasonable bound on the cardinals of the local rings occurring.) To get around this problem we choose a well ordering on K. For $x \in K$ we let $K(x)$ be the subfield of K generated by all elements of K which are $\leq x$. By transfinite recursion on $x \in K$ we will produce ring maps $R \subset R(x)$ as in the lemma with residue field extension $K(x)/k$. Moreover, by construction we will have that $R(x)$ will contain $R(y)$ for all $y \leq x$. Namely, if x has a predecessor x', then $K(x) = K(x')[x]$ and hence we can let $R(x') \subset R(x)$ be the local ring extension constructed in the first paragraph of the proof. If x does not have a predecessor, then we first set $R'(x) = \operatorname{colim}_{x' < x} R(x')$ as in the third paragraph of the proof. The residue field of $R'(x)$ is $K'(x) = \bigcup_{x' < x} K(x')$. Since $K(x) = K'(x)[x]$ we see that we can use the construction of the first paragraph of the proof to produce $R'(x) \subset R(x)$. This finishes the proof of the lemma. $\square$

09E0 **Lemma 159.2.** *Let $(R, \mathfrak{m}, k)$ be a local ring. If $k \subset K$ is a separable algebraic extension, then there exists a directed set I and a system of finite étale extensions $R \subset R_i$, $i \in I$ of local rings such that $R' = \operatorname{colim} R_i$ has residue field K (as extension of k).*

Proof. Let $R \subset R'$ be the extension constructed in the proof of Lemma 159.1. By construction $R' = \operatorname{colim}_{\alpha \in A} R_\alpha$ where A is a well-ordered set and the transition maps $R_\alpha \to R_{\alpha+1}$ are finite étale and $R_\alpha = \operatorname{colim}_{\beta < \alpha} R_\beta$ if α is not a successor. We will prove the result by transfinite induction.

Suppose the result holds for R_α, i.e., $R_\alpha = \operatorname{colim} R_i$ with R_i finite étale over R. Since $R_\alpha \to R_{\alpha+1}$ is finite étale there exists an i and a finite étale extension $R_i \to R_{i,1}$ such that $R_{\alpha+1} = R_\alpha \otimes_{R_i} R_{i,1}$. Thus $R_{\alpha+1} = \operatorname{colim}_{i' \geq i} R_{i'} \otimes_{R_i} R_{i,1}$ and the result holds for $\alpha + 1$. Suppose α is not a successor and the result holds for R_β for all $\beta < \alpha$. Since every finite subset $E \subset R_\alpha$ is contained in R_β for some $\beta < \alpha$ and we see that E is contained in a finite étale subextension by assumption. Thus the result holds for R_α. $\square$

07NE **Lemma 159.3.** *Let R be a ring. Let $\mathfrak{p} \subset R$ be a prime and let $L/\kappa(\mathfrak{p})$ be a finite extension of fields. Then there exists a finite free ring map $R \to S$ such that $\mathfrak{q} = \mathfrak{p}S$ is prime and $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ is isomorphic to the given extension $L/\kappa(\mathfrak{p})$.*

Proof. By induction of the degree of $\kappa(\mathfrak{p}) \subset L$. If the degree is 1, then we take $R = S$. In general, if there exists a sub extension $\kappa(\mathfrak{p}) \subset L' \subset L$ then we win by induction on the degree (by first constructing $R \subset S'$ corresponding to $L'/\kappa(\mathfrak{p})$ and then construction $S' \subset S$ corresponding to L/L'). Thus we may assume that $L \supset \kappa(\mathfrak{p})$ is generated by a single element $\alpha \in L$. Let $X^d + \sum_{i<d} a_i X^i$ be the minimal polynomial of α over $\kappa(\mathfrak{p})$, so $a_i \in \kappa(\mathfrak{p})$. We may write a_i as the image of f_i/g for some $f_i, g \in R$ and $g \notin \mathfrak{p}$. After replacing α by $g\alpha$ (and correspondingly

replacing a_i by $g^{d-i}a_i$) we may assume that a_i is the image of some $f_i \in R$. Then we simply take $S = R[x]/(x^d + \sum f_i x^i)$. □

0GIL **Lemma 159.4.** *Let A be a ring. Let $\kappa = \max(|A|, \aleph_0)$. Then every flat A-algebra B is the filtered colimit of its flat A-subalgebras $B' \subset B$ of cardinality $|B'| \leq \kappa$. (Observe that B' is faithfully flat over A if B is faithfully flat over A.)*

Proof. If B has cardinality $\leq \kappa$ then this is true. Let $E \subset B$ be an A-subalgebra with $|E| \leq \kappa$. We will show that E is contained in a flat A-subalgebra B' with $|B'| \leq \kappa$. The lemma follows because (a) every finite subset of B is contained in an A-subalgebra of cardinality at most κ and (b) every pair of A-subalgebras of B of cardinality at most κ is contained in an A-subalgebra of cardinality at most κ. Details omitted.

We will inductively construct a sequence of A-subalgebras

$$E = E_0 \subset E_1 \subset E_2 \subset \ldots$$

each having cardinality $\leq \kappa$ and we will show that $B' = \bigcup E_k$ is flat over A to finish the proof.

The construction is as follows. Set $E_0 = E$. Given E_k for $k \geq 0$ we consider the set S_k of relations between elements of E_k with coefficients in A. Thus an element $s \in S_k$ is given by an integer $n \geq 1$ and $a_1, \ldots, a_n \in A$, and $e_1, \ldots, e_n \in E_k$ such that $\sum a_i e_i = 0$ in E_k. The flatness of $A \to B$ implies by Lemma 39.11 that for every $s = (n, a_1, \ldots, a_n, e_1, \ldots, e_n) \in S_k$ we may choose

$$(m_s, b_{s,1}, \ldots, b_{s,m_s}, a_{s,11}, \ldots, a_{s,nm_s})$$

where $m_s \geq 0$ is an integer, $b_{s,j} \in B$, $a_{s,ij} \in A$, and

$$e_i = \sum\nolimits_j a_{s,ij} b_{s,j}, \forall i, \quad \text{and} \quad 0 = \sum\nolimits_i a_i a_{s,ij}, \forall j.$$

Given these choicse, we let $E_{k+1} \subset B$ be the A-subalgebra generated by

(1) E_k and
(2) the elements $b_{s,1}, \ldots, b_{s,m_s}$ for every $s \in S_k$.

Some set theory (omitted) shows that E_{k+1} has at most cardinality κ (this uses that we inductively know $|E_k| \leq \kappa$ and consequently the cardinality of S_k is also at most κ).

To show that $B' = \bigcup E_k$ is flat over A we consider a relation $\sum_{i=1,\ldots,n} a_i b'_i = 0$ in B' with coefficients in A. Choose k large enough so that $b'_i \in E_k$ for $i = 1, \ldots, n$. Then $(n, a_1, \ldots, a_n, b'_1, \ldots, b'_n) \in S_k$ and hence we see that the relation is trivial in E_{k+1} and a fortiori in B'. Thus $A \to B'$ is flat by Lemma 39.11. □

160. The Cohen structure theorem

0323 Here is a fundamental notion in commutative algebra.

0324 **Definition 160.1.** Let $(R, \mathfrak{m})$ be a local ring. We say R is a *complete local ring* if the canonical map

$$R \longrightarrow \lim\nolimits_n R/\mathfrak{m}^n$$

to the completion of R with respect to $\mathfrak{m}$ is an isomorphism[13].

Note that an Artinian local ring R is a complete local ring because $\mathfrak{m}_R^n = 0$ for some $n > 0$. In this section we mostly focus on Noetherian complete local rings.

0325 **Lemma 160.2.** *Let R be a Noetherian complete local ring. Any quotient of R is also a Noetherian complete local ring. Given a finite ring map $R \to S$, then S is a product of Noetherian complete local rings.*

Proof. The ring S is Noetherian by Lemma 31.1. As an R-module S is complete by Lemma 97.1. Hence S is the product of the completions at its maximal ideals by Lemma 97.8. □

032B **Lemma 160.3.** *Let $(R, \mathfrak{m})$ be a complete local ring. If $\mathfrak{m}$ is a finitely generated ideal then R is Noetherian.*

Proof. See Lemma 97.5. □

0326 **Definition 160.4.** Let $(R, \mathfrak{m})$ be a complete local ring. A subring $\Lambda \subset R$ is called a *coefficient ring* if the following conditions hold:

(1) Λ is a complete local ring with maximal ideal $\Lambda \cap \mathfrak{m}$,
(2) the residue field of Λ maps isomorphically to the residue field of R, and
(3) $\Lambda \cap \mathfrak{m} = p\Lambda$, where p is the characteristic of the residue field of R.

Let us make some remarks on this definition. We split the discussion into the following cases:

(1) The local ring R contains a field. This happens if either $\mathbf{Q} \subset R$, or $pR = 0$ where p is the characteristic of $R/\mathfrak{m}$. In this case a coefficient ring Λ is a field contained in R which maps isomorphically to $R/\mathfrak{m}$.
(2) The characteristic of $R/\mathfrak{m}$ is $p > 0$ but no power of p is zero in R. In this case Λ is a complete discrete valuation ring with uniformizer p and residue field $R/\mathfrak{m}$.
(3) The characteristic of $R/\mathfrak{m}$ is $p > 0$, and for some $n > 1$ we have $p^{n-1} \neq 0$, $p^n = 0$ in R. In this case Λ is an Artinian local ring whose maximal ideal is generated by p and which has residue field $R/\mathfrak{m}$.

The complete discrete valuation rings with uniformizer p above play a special role and we baptize them as follows.

0327 **Definition 160.5.** A *Cohen ring* is a complete discrete valuation ring with uniformizer p a prime number.

0328 **Lemma 160.6.** *Let p be a prime number. Let k be a field of characteristic p. There exists a Cohen ring Λ with $\Lambda/p\Lambda \cong k$.*

Proof. First note that the p-adic integers $\mathbf{Z}_p$ form a Cohen ring for $\mathbf{F}_p$. Let k be an arbitrary field of characteristic p. Let $\mathbf{Z}_p \to R$ be a flat local ring map such that $\mathfrak{m}_R = pR$ and $R/pR = k$, see Lemma 159.1. By Lemma 97.5 the completion $\Lambda = R^\wedge$ is Noetherian. It is a complete Noetherian local ring with maximal ideal (p) as $\Lambda/p\Lambda = R/pR$ is a field (use Lemma 96.3). Since $\mathbf{Z}_p \to R \to \Lambda$ is flat (by

[13]This includes the condition that $\bigcap \mathfrak{m}^n = (0)$; in some texts this may be indicated by saying that R is complete and separated. Warning: It can happen that the completion $\lim_n R/\mathfrak{m}^n$ of a local ring is non-complete, see Examples, Lemma 7.1. This does not happen when $\mathfrak{m}$ is finitely generated, see Lemma 96.3 in which case the completion is Noetherian, see Lemma 97.5.

Lemma 97.2) we see that p is a nonzerodivisor in Λ. Hence Λ has dimension ≥ 1 (Lemma 60.13) and we conclude that Λ is regular of dimension 1, i.e., a discrete valuation ring by Lemma 119.7. We conclude Λ is a Cohen ring for k. □

0329 **Lemma 160.7.** *Let $p > 0$ be a prime. Let Λ be a Cohen ring with residue field of characteristic p. For every $n \geq 1$ the ring map*

$$\mathbf{Z}/p^n\mathbf{Z} \to \Lambda/p^n\Lambda$$

is formally smooth.

Proof. If $n = 1$, this follows from Proposition 158.9. For general n we argue by induction on n. Namely, if $\mathbf{Z}/p^n\mathbf{Z} \to \Lambda/p^n\Lambda$ is formally smooth, then we can apply Lemma 138.12 to the ring map $\mathbf{Z}/p^{n+1}\mathbf{Z} \to \Lambda/p^{n+1}\Lambda$ and the ideal $I = (p^n) \subset \mathbf{Z}/p^{n+1}\mathbf{Z}$. □

032A **Theorem 160.8** (Cohen structure theorem)**.** *Let $(R, \mathfrak{m})$ be a complete local ring.*

1. *R has a coefficient ring (see Definition 160.4),*
2. *if $\mathfrak{m}$ is a finitely generated ideal, then R is isomorphic to a quotient*

$$\Lambda[[x_1, \ldots, x_n]]/I$$

where Λ is either a field or a Cohen ring.

Proof. Let us prove a coefficient ring exists. First we prove this in case the characteristic of the residue field κ is zero. Namely, in this case we will prove by induction on $n > 0$ that there exists a section

$$\varphi_n : \kappa \longrightarrow R/\mathfrak{m}^n$$

to the canonical map $R/\mathfrak{m}^n \to \kappa = R/\mathfrak{m}$. This is trivial for $n = 1$. If $n > 1$, let φ_{n-1} be given. The field extension $\kappa/\mathbf{Q}$ is formally smooth by Proposition 158.9. Hence we can find the dotted arrow in the following diagram

$$\begin{tikzcd}
R/\mathfrak{m}^{n-1} & R/\mathfrak{m}^n \arrow[l] \\
\kappa \arrow[u, "\varphi_{n-1}"] \arrow[ur, dotted] & \mathbf{Q} \arrow[l] \arrow[u]
\end{tikzcd}$$

This proves the induction step. Putting these maps together

$$\lim_n\ \varphi_n : \kappa \longrightarrow R = \lim_n\ R/\mathfrak{m}^n$$

gives a map whose image is the desired coefficient ring.

Next, we prove the existence of a coefficient ring in the case where the characteristic of the residue field κ is $p > 0$. Namely, choose a Cohen ring Λ with $\kappa = \Lambda/p\Lambda$, see Lemma 160.6. In this case we will prove by induction on $n > 0$ that there exists a map

$$\varphi_n : \Lambda/p^n\Lambda \longrightarrow R/\mathfrak{m}^n$$

whose composition with the reduction map $R/\mathfrak{m}^n \to \kappa$ produces the given isomorphism $\Lambda/p\Lambda = \kappa$. This is trivial for $n = 1$. If $n > 1$, let φ_{n-1} be given. The ring map $\mathbf{Z}/p^n\mathbf{Z} \to \Lambda/p^n\Lambda$ is formally smooth by Lemma 160.7. Hence we can find the dotted arrow in the following diagram

$$\begin{tikzcd} R/\mathfrak{m}^{n-1} & R/\mathfrak{m}^n \arrow[l] \\ \Lambda/p^n\Lambda \arrow[u, "\varphi_{n-1}"] \arrow[ur, dotted] & \mathbf{Z}/p^n\mathbf{Z} \arrow[l] \arrow[u] \end{tikzcd}$$

This proves the induction step. Putting these maps together

$$\lim_n\ \varphi_n : \Lambda = \lim_n\ \Lambda/p^n\Lambda \longrightarrow R = \lim_n\ R/\mathfrak{m}^n$$

gives a map whose image is the desired coefficient ring.

The final statement of the theorem follows readily. Namely, if $y_1, \ldots, y_n$ are generators of the ideal $\mathfrak{m}$, then we can use the map $\Lambda \to R$ just constructed to get a map

$$\Lambda[[x_1, \ldots, x_n]] \longrightarrow R, \quad x_i \longmapsto y_i.$$

Since both sides are $(x_1, \ldots, x_n)$-adically complete this map is surjective by Lemma 96.1 as it is surjective modulo $(x_1, \ldots, x_n)$ by construction. ☐

032C **Remark 160.9.** If k is a field then the power series ring $k[[X_1, \ldots, X_d]]$ is a Noetherian complete local regular ring of dimension d. If Λ is a Cohen ring then $\Lambda[[X_1, \ldots, X_d]]$ is a complete local Noetherian regular ring of dimension $d + 1$. Hence the Cohen structure theorem implies that any Noetherian complete local ring is a quotient of a regular local ring. In particular we see that a Noetherian complete local ring is universally catenary, see Lemma 105.9 and Lemma 106.3.

0C0S **Lemma 160.10.** *Let $(R, \mathfrak{m})$ be a Noetherian complete local ring. Assume R is regular.*

(1) *If R contains either $\mathbf{F}_p$ or $\mathbf{Q}$, then R is isomorphic to a power series ring over its residue field.*
(2) *If k is a field and $k \to R$ is a ring map inducing an isomorphism $k \to R/\mathfrak{m}$, then R is isomorphic as a k-algebra to a power series ring over k.*

Proof. In case (1), by the Cohen structure theorem (Theorem 160.8) there exists a coefficient ring which must be a field mapping isomorphically to the residue field. Thus it suffices to prove (2). In case (2) we pick $f_1, \ldots, f_d \in \mathfrak{m}$ which map to a basis of $\mathfrak{m}/\mathfrak{m}^2$ and we consider the continuous k-algebra map $k[[x_1, \ldots, x_d]] \to R$ sending x_i to f_i. As both source and target are $(x_1, \ldots, x_d)$-adically complete, this map is surjective by Lemma 96.1. On the other hand, it has to be injective because otherwise the dimension of R would be $< d$ by Lemma 60.13. ☐

032D **Lemma 160.11.** *Let $(R, \mathfrak{m})$ be a Noetherian complete local domain. Then there exists a $R_0 \subset R$ with the following properties*

(1) *R_0 is a regular complete local ring,*
(2) *$R_0 \subset R$ is finite and induces an isomorphism on residue fields,*
(3) *R_0 is either isomorphic to $k[[X_1, \ldots, X_d]]$ where k is a field or $\Lambda[[X_1, \ldots, X_d]]$ where Λ is a Cohen ring.*

Proof. Let Λ be a coefficient ring of R. Since R is a domain we see that either Λ is a field or Λ is a Cohen ring.

Case I: $\Lambda = k$ is a field. Let $d = \dim(R)$. Choose $x_1, \ldots, x_d \in \mathfrak{m}$ which generate an ideal of definition $I \subset R$. (See Section 60.) By Lemma 96.9 we see that R is I-adically complete as well. Consider the map $R_0 = k[[X_1, \ldots, X_d]] \to R$ which maps X_i to x_i. Note that R_0 is complete with respect to the ideal $I_0 = (X_1, \ldots, X_d)$, and that $R/I_0R \cong R/IR$ is finite over $k = R_0/I_0$ (because $\dim(R/I) = 0$, see Section 60.) Hence we conclude that $R_0 \to R$ is finite by Lemma 96.12. Since $\dim(R) = \dim(R_0)$ this implies that $R_0 \to R$ is injective (see Lemma 112.3), and the lemma is proved.

Case II: Λ is a Cohen ring. Let $d + 1 = \dim(R)$. Let $p > 0$ be the characteristic of the residue field k. As R is a domain we see that p is a nonzerodivisor in R. Hence $\dim(R/pR) = d$, see Lemma 60.13. Choose $x_1, \ldots, x_d \in R$ which generate an ideal of definition in R/pR. Then $I = (p, x_1, \ldots, x_d)$ is an ideal of definition of R. By Lemma 96.9 we see that R is I-adically complete as well. Consider the map $R_0 = \Lambda[[X_1, \ldots, X_d]] \to R$ which maps X_i to x_i. Note that R_0 is complete with respect to the ideal $I_0 = (p, X_1, \ldots, X_d)$, and that $R/I_0R \cong R/IR$ is finite over $k = R_0/I_0$ (because $\dim(R/I) = 0$, see Section 60.) Hence we conclude that $R_0 \to R$ is finite by Lemma 96.12. Since $\dim(R) = \dim(R_0)$ this implies that $R_0 \to R$ is injective (see Lemma 112.3), and the lemma is proved. □

161. Japanese rings

0BI1 In this section we begin to discuss finiteness of integral closure.

032F **Definition 161.1.** Let R be a domain with field of fractions K. [DG67, Chapter 0, Definition 23.1.1]

(1) We say R is *N-1* if the integral closure of R in K is a finite R-module.
(2) We say R is *N-2* or *Japanese* if for any finite extension L/K of fields the integral closure of R in L is finite over R.

The main interest in these notions is for Noetherian rings, but here is a non-Noetherian example.

0350 **Example 161.2.** Let k be a field. The domain $R = k[x_1, x_2, x_3, \ldots]$ is N-2, but not Noetherian. The reason is the following. Suppose that $R \subset L$ and the field L is a finite extension of the fraction field of R. Then there exists an integer n such that L comes from a finite extension $L_0/k(x_1, \ldots, x_n)$ by adjoining the (transcendental) elements x_{n+1}, x_{n+2}, etc. Let S_0 be the integral closure of $k[x_1, \ldots, x_n]$ in L_0. By Proposition 162.16 below it is true that S_0 is finite over $k[x_1, \ldots, x_n]$. Moreover, the integral closure of R in L is $S = S_0[x_{n+1}, x_{n+2}, \ldots]$ (use Lemma 37.8) and hence finite over R. The same argument works for $R = \mathbf{Z}[x_1, x_2, x_3, \ldots]$.

032G **Lemma 161.3.** *Let R be a domain. If R is N-1 then so is any localization of R. Same for N-2.*

Proof. These statements hold because taking integral closure commutes with localization, see Lemma 36.11. □

032H **Lemma 161.4.** *Let R be a domain. Let $f_1, \ldots, f_n \in R$ generate the unit ideal. If each domain R_{f_i} is N-1 then so is R. Same for N-2.*

Proof. Assume R_{f_i} is N-2 (or N-1). Let L be a finite extension of the fraction field of R (equal to the fraction field in the N-1 case). Let S be the integral closure of R in L. By Lemma 36.11 we see that S_{f_i} is the integral closure of R_{f_i} in L. Hence S_{f_i} is finite over R_{f_i} by assumption. Thus S is finite over R by Lemma 23.2. □

032I **Lemma 161.5.** *Let R be a domain. Let $R \subset S$ be a quasi-finite extension of domains (for example finite). Assume R is N-2 and Noetherian. Then S is N-2.*

Proof. Let L/K be the induced extension of fraction fields. Note that this is a finite field extension (for example by Lemma 122.2 (2) applied to the fibre $S \otimes_R K$, and the definition of a quasi-finite ring map). Let S' be the integral closure of R in S. Then S' is contained in the integral closure of R in L which is finite over R by assumption. As R is Noetherian this implies S' is finite over R. By Lemma 123.14 there exist elements $g_1, \ldots, g_n \in S'$ such that $S'_{g_i} \cong S_{g_i}$ and such that $g_1, \ldots, g_n$ generate the unit ideal in S. Hence it suffices to show that S' is N-2 by Lemmas 161.3 and 161.4. Thus we have reduced to the case where S is finite over R.

Assume $R \subset S$ with hypotheses as in the lemma and moreover that S is finite over R. Let M be a finite field extension of the fraction field of S. Then M is also a finite field extension of K and we conclude that the integral closure T of R in M is finite over R. By Lemma 36.16 we see that T is also the integral closure of S in M and we win by Lemma 36.15. $\square$

032J **Lemma 161.6.** *Let R be a Noetherian domain. If $R[z, z^{-1}]$ is N-1, then so is R.*

Proof. Let R' be the integral closure of R in its field of fractions K. Let S' be the integral closure of $R[z, z^{-1}]$ in its field of fractions. Clearly $R' \subset S'$. Since $K[z, z^{-1}]$ is a normal domain we see that $S' \subset K[z, z^{-1}]$. Suppose that $f_1, \ldots, f_n \in S'$ generate S' as $R[z, z^{-1}]$-module. Say $f_i = \sum a_{ij} z^j$ (finite sum), with $a_{ij} \in K$. For any $x \in R'$ we can write

$$x = \sum h_i f_i$$

with $h_i \in R[z, z^{-1}]$. Thus we see that R' is contained in the finite R-submodule $\sum R a_{ij} \subset K$. Since R is Noetherian we conclude that R' is a finite R-module. $\square$

032K **Lemma 161.7.** *Let R be a Noetherian domain, and let $R \subset S$ be a finite extension of domains. If S is N-1, then so is R. If S is N-2, then so is R.*

Proof. Omitted. (Hint: Integral closures of R in extension fields are contained in integral closures of S in extension fields.) $\square$

032L **Lemma 161.8.** *Let R be a Noetherian normal domain with fraction field K. Let L/K be a finite separable field extension. Then the integral closure of R in L is finite over R.*

Proof. Consider the trace pairing (Fields, Definition 20.6)

$$L \times L \longrightarrow K, \quad (x, y) \longmapsto \langle x, y \rangle := \text{Trace}_{L/K}(xy).$$

Since L/K is separable this is nondegenerate (Fields, Lemma 20.7). Moreover, if $x \in L$ is integral over R, then $\text{Trace}_{L/K}(x)$ is in R. This is true because the minimal polynomial of x over K has coefficients in R (Lemma 38.6) and because $\text{Trace}_{L/K}(x)$ is an integer multiple of one of these coefficients (Fields, Lemma 20.3). Pick $x_1, \ldots, x_n \in L$ which are integral over R and which form a K-basis of L. Then the integral closure $S \subset L$ is contained in the R-module

$$M = \{y \in L \mid \langle x_i, y \rangle \in R, \ i = 1, \ldots, n\}$$

By linear algebra we see that $M \cong R^{\oplus n}$ as an R-module. Hence $S \subset R^{\oplus n}$ is a finitely generated R-module as R is Noetherian. $\square$

03B7 **Example 161.9.** Lemma 161.8 does not work if the ring is not Noetherian. For example consider the action of $G = \{+1, -1\}$ on $A = \mathbf{C}[x_1, x_2, x_3, \ldots]$ where -1 acts by mapping x_i to $-x_i$. The invariant ring $R = A^G$ is the $\mathbf{C}$-algebra generated by all $x_i x_j$. Hence $R \subset A$ is not finite. But R is a normal domain with fraction field $K = L^G$ the G-invariants in the fraction field L of A. And clearly A is the integral closure of R in L.

The following lemma can sometimes be used as a substitute for Lemma 161.8 in case of purely inseparable extensions.

0AE0 **Lemma 161.10.** *Let R be a Noetherian normal domain with fraction field K of characteristic $p > 0$. Let $a \in K$ be an element such that there exists a derivation $D : R \to R$ with $D(a) \neq 0$. Then the integral closure of R in $L = K[x]/(x^p - a)$ is finite over R.*

Proof. After replacing x by fx and a by $f^p a$ for some $f \in R$ we may assume $a \in R$. Hence also $D(a) \in R$. We will show by induction on $i \leq p - 1$ that if

$$y = a_0 + a_1 x + \ldots + a_i x^i, \quad a_j \in K$$

is integral over R, then $D(a)^i a_j \in R$. Thus the integral closure is contained in the finite R-module with basis $D(a)^{-p+1} x^j$, $j = 0, \ldots, p-1$. Since R is Noetherian this proves the lemma.

If $i = 0$, then $y = a_0$ is integral over R if and only if $a_0 \in R$ and the statement is true. Suppose the statement holds for some $i < p - 1$ and suppose that

$$y = a_0 + a_1 x + \ldots + a_{i+1} x^{i+1}, \quad a_j \in K$$

is integral over R. Then

$$y^p = a_0^p + a_1^p a + \ldots + a_{i+1}^p a^{i+1}$$

is an element of R (as it is in K and integral over R). Applying D we obtain

$$(a_1^p + 2a_2^p a + \ldots + (i+1) a_{i+1}^p a^i) D(a)$$

is in R. Hence it follows that

$$D(a) a_1 + 2D(a) a_2 x + \ldots + (i+1) D(a) a_{i+1} x^i$$

is integral over R. By induction we find $D(a)^{i+1} a_j \in R$ for $j = 1, \ldots, i+1$. (Here we use that $1, \ldots, i+1$ are invertible.) Hence $D(a)^{i+1} a_0$ is also in R because it is the difference of y and $\sum_{j>0} D(a)^{i+1} a_j x^j$ which are integral over R (since x is integral over R as $a \in R$). $\square$

032M **Lemma 161.11.** *A Noetherian domain whose fraction field has characteristic zero is N-1 if and only if it is N-2 (i.e., Japanese).*

Proof. This is clear from Lemma 161.8 since every field extension in characteristic zero is separable. $\square$

032N **Lemma 161.12.** *Let R be a Noetherian domain with fraction field K of characteristic $p > 0$. Then R is N-2 if and only if for every finite purely inseparable extension L/K the integral closure of R in L is finite over R.*

Proof. Assume the integral closure of R in every finite purely inseparable field extension of K is finite. Let L/K be any finite extension. We have to show the integral closure of R in L is finite over R. Choose a finite normal field extension M/K containing L. As R is Noetherian it suffices to show that the integral closure of R in M is finite over R. By Fields, Lemma 27.3 there exists a subextension $M/M_{insep}/K$ such that M_{insep}/K is purely inseparable, and M/M_{insep} is separable. By assumption the integral closure R' of R in M_{insep} is finite over R. By Lemma 161.8 the integral closure R'' of R' in M is finite over R'. Then R'' is finite over R by Lemma 7.3. Since R'' is also the integral closure of R in M (see Lemma 36.16) we win. $\square$

032O **Lemma 161.13.** *Let R be a Noetherian domain. If R is N-1 then $R[x]$ is N-1. If R is N-2 then $R[x]$ is N-2.*

Proof. Assume R is N-1. Let R' be the integral closure of R which is finite over R. Hence also $R'[x]$ is finite over $R[x]$. The ring $R'[x]$ is normal (see Lemma 37.8), hence N-1. This proves the first assertion.

For the second assertion, by Lemma 161.7 it suffices to show that $R'[x]$ is N-2. In other words we may and do assume that R is a normal N-2 domain. In characteristic zero we are done by Lemma 161.11. In characteristic $p > 0$ we have to show that the integral closure of $R[x]$ is finite in any finite purely inseparable extension of $L/K(x)$ where K is the fraction field of R. There exists a finite purely inseparable field extension L'/K and $q = p^e$ such that $L \subset L'(x^{1/q})$; some details omitted. As $R[x]$ is Noetherian it suffices to show that the integral closure of $R[x]$ in $L'(x^{1/q})$ is finite over $R[x]$. And this integral closure is equal to $R'[x^{1/q}]$ with $R \subset R' \subset L'$ the integral closure of R in L'. Since R is N-2 we see that R' is finite over R and hence $R'[x^{1/q}]$ is finite over $R[x]$. $\square$

0332 **Lemma 161.14.** *Let R be a Noetherian domain. If there exists an $f \in R$ such that R_f is normal then*

$$U = \{\mathfrak p \in \mathrm{Spec}(R) \mid R_{\mathfrak p} \textit{ is normal}\}$$

is open in $\mathrm{Spec}(R)$.

Proof. It is clear that the standard open $D(f)$ is contained in U. By Serre's criterion Lemma 157.4 we see that $\mathfrak p \not\in U$ implies that for some $\mathfrak q \subset \mathfrak p$ we have either

(1) Case I: $\mathrm{depth}(R_{\mathfrak q}) < 2$ and $\dim(R_{\mathfrak q}) \geq 2$, and
(2) Case II: $R_{\mathfrak q}$ is not regular and $\dim(R_{\mathfrak q}) = 1$.

This in particular also means that $R_{\mathfrak q}$ is not normal, and hence $f \in \mathfrak q$. In case I we see that $\mathrm{depth}(R_{\mathfrak q}) = \mathrm{depth}(R_{\mathfrak q}/fR_{\mathfrak q}) + 1$. Hence such a prime $\mathfrak q$ is the same thing as an embedded associated prime of R/fR. In case II $\mathfrak q$ is an associated prime of R/fR of height 1. Thus there is a finite set E of such primes $\mathfrak q$ (see Lemma 63.5) and

$$\mathrm{Spec}(R) \setminus U = \bigcup\nolimits_{\mathfrak q \in E} V(\mathfrak q)$$

as desired. $\square$

0333 **Lemma 161.15.** *Let R be a Noetherian domain. Then R is N-1 if and only if the following two conditions hold*

(1) *there exists a nonzero $f \in R$ such that R_f is normal, and*

(2) *for every maximal ideal* $\mathfrak{m} \subset R$ *the local ring* $R_\mathfrak{m}$ *is N-1.*

Proof. First assume R is N-1. Let R' be the integral closure of R in its field of fractions K. By assumption we can find $x_1, \ldots, x_n$ in R' which generate R' as an R-module. Since $R' \subset K$ we can find $f_i \in R$ nonzero such that $f_i x_i \in R$. Then $R_f \cong R'_f$ where $f = f_1 \ldots f_n$. Hence R_f is normal and we have (1). Part (2) follows from Lemma 161.3.

Assume (1) and (2). Let K be the fraction field of R. Suppose that $R \subset R' \subset K$ is a finite extension of R contained in K. Note that $R_f = R'_f$ since R_f is already normal. Hence by Lemma 161.14 the set of primes $\mathfrak{p}' \in \mathrm{Spec}(R')$ with $R'_{\mathfrak{p}'}$ non-normal is closed in $\mathrm{Spec}(R')$. Since $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is closed the image of this set is closed in $\mathrm{Spec}(R)$. For such a ring R' denote $Z_{R'} \subset \mathrm{Spec}(R)$ this image.

Pick a maximal ideal $\mathfrak{m} \subset R$. Let $R_\mathfrak{m} \subset R'_\mathfrak{m}$ be the integral closure of the local ring in K. By assumption this is a finite ring extension. By Lemma 36.11 we can find finitely many elements $x_1, \ldots, x_n \in K$ integral over R such that $R'_\mathfrak{m}$ is generated by $x_1, \ldots, x_n$ over $R_\mathfrak{m}$. Let $R' = R[x_1, \ldots, x_n] \subset K$. With this choice it is clear that $\mathfrak{m} \notin Z_{R'}$.

As $\mathrm{Spec}(R)$ is quasi-compact, the above shows that we can find a finite collection $R \subset R'_i \subset K$ such that $\bigcap Z_{R'_i} = \emptyset$. Let R' be the subring of K generated by all of these. It is finite over R. Also $Z_{R'} = \emptyset$. Namely, every prime $\mathfrak{p}'$ lies over a prime $\mathfrak{p}'_i$ such that $(R'_i)_{\mathfrak{p}'_i}$ is normal. This implies that $R'_{\mathfrak{p}'} = (R'_i)_{\mathfrak{p}'_i}$ is normal too. Hence R' is normal, in other words R' is the integral closure of R in K. ☐

032P **Lemma 161.16** (Tate)**.** *Let R be a ring. Let $x \in R$. Assume* [DG67, Theorem 23.1.3]

(1) *R is a normal Noetherian domain,*
(2) *R/xR is a domain and N-2,*
(3) *$R \cong \lim_n R/x^nR$ is complete with respect to x.*

Then R is N-2.

Proof. We may assume $x \neq 0$ since otherwise the lemma is trivial. Let K be the fraction field of R. If the characteristic of K is zero the lemma follows from (1), see Lemma 161.11. Hence we may assume that the characteristic of K is $p > 0$, and we may apply Lemma 161.12. Thus given L/K a finite purely inseparable field extension we have to show that the integral closure S of R in L is finite over R.

Let q be a power of p such that $L^q \subset K$. By enlarging L if necessary we may assume there exists an element $y \in L$ such that $y^q = x$. Since $R \to S$ induces a homeomorphism of spectra (see Lemma 46.7) there is a unique prime ideal $\mathfrak{q} \subset S$ lying over the prime ideal $\mathfrak{p} = xR$. It is clear that

$$\mathfrak{q} = \{f \in S \mid f^q \in \mathfrak{p}\} = yS$$

since $y^q = x$. Observe that $R_\mathfrak{p}$ is a discrete valuation ring by Lemma 119.7. Then $S_\mathfrak{q}$ is Noetherian by Krull-Akizuki (Lemma 119.12). Whereupon we conclude $S_\mathfrak{q}$ is a discrete valuation ring by Lemma 119.7 once again. By Lemma 119.10 we see that $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ is a finite field extension. Hence the integral closure $S' \subset \kappa(\mathfrak{q})$ of R/xR is finite over R/xR by assumption (2). Since $S/yS \subset S'$ this implies that S/yS is finite over R. Note that S/y^nS has a finite filtration whose subquotients are the modules $y^iS/y^{i+1}S \cong S/yS$. Hence we see that each S/y^nS is finite over R. In particular S/xS is finite over R. Also, it is clear that $\bigcap x^nS = (0)$ since an element in the intersection has qth power contained in $\bigcap x^n R = (0)$ (Lemma 51.4). Thus we may apply Lemma 96.12 to conclude that S is finite over R, and we win. □

032Q **Lemma 161.17.** *Let R be a ring. If R is Noetherian, a domain, and N-2, then so is $R[[x]]$.*

Proof. Observe that $R[[x]]$ is Noetherian by Lemma 31.2. Let $R' \supset R$ be the integral closure of R in its fraction field. Because R is N-2 this is finite over R. Hence $R'[[x]]$ is finite over $R[[x]]$. By Lemma 37.9 we see that $R'[[x]]$ is a normal domain. Apply Lemma 161.16 to the element $x \in R'[[x]]$ to see that $R'[[x]]$ is N-2. Then Lemma 161.7 shows that $R[[x]]$ is N-2. □

162. Nagata rings

032E Here is the definition.

032R **Definition 162.1.** Let R be a ring.

(1) We say R is *universally Japanese* if for any finite type ring map $R \to S$ with S a domain we have that S is N-2 (i.e., Japanese).
(2) We say that R is a *Nagata ring* if R is Noetherian and for every prime ideal $\mathfrak{p}$ the ring $R/\mathfrak{p}$ is N-2.

It is clear that a Noetherian universally Japanese ring is a Nagata ring. It is our goal to show that a Nagata ring is universally Japanese. This is not obvious at all, and requires some work. But first, here is a useful lemma.

03GH **Lemma 162.2.** *Let R be a Nagata ring. Let $R \to S$ be essentially of finite type with S reduced. Then the integral closure of R in S is finite over R.*

Proof. As S is essentially of finite type over R it is Noetherian and has finitely many minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$, see Lemma 31.6. Since S is reduced we have $S \subset \prod S_{\mathfrak{q}_i}$ and each $S_{\mathfrak{q}_i} = K_i$ is a field, see Lemmas 25.4 and 25.1. It suffices to show that the integral closure A'_i of R in each K_i is finite over R. This is true because R is Noetherian and $A \subset \prod A'_i$. Let $\mathfrak{p}_i \subset R$ be the prime of R corresponding to $\mathfrak{q}_i$. As S is essentially of finite type over R we see that $K_i = S_{\mathfrak{q}_i} = \kappa(\mathfrak{q}_i)$ is a finitely generated field extension of $\kappa(\mathfrak{p}_i)$. Hence the algebraic closure L_i of $\kappa(\mathfrak{p}_i)$ in K_i is finite over $\kappa(\mathfrak{p}_i)$, see Fields, Lemma 26.11. It is clear that A'_i is the integral closure of $R/\mathfrak{p}_i$ in L_i, and hence we win by definition of a Nagata ring. □

0351 **Lemma 162.3.** *Let R be a ring. To check that R is universally Japanese it suffices to show: If $R \to S$ is of finite type, and S a domain then S is N-1.*

Proof. Namely, assume the condition of the lemma. Let $R \to S$ be a finite type ring map with S a domain. Let L be a finite extension of the fraction field of S. Then there exists a finite ring extension $S \subset S' \subset L$ such that L is the fraction field of S'. By assumption S' is N-1, and hence the integral closure S'' of S' in L is finite over S'. Thus S'' is finite over S (Lemma 7.3) and S'' is the integral closure of S in L (Lemma 36.16). We conclude that R is universally Japanese. □

032S **Lemma 162.4.** *If R is universally Japanese then any algebra essentially of finite type over R is universally Japanese.*

Proof. The case of an algebra of finite type over R is immediate from the definition. The general case follows on applying Lemma 161.3. □

032T **Lemma 162.5.** *Let R be a Nagata ring. If $R \to S$ is a quasi-finite ring map (for example finite) then S is a Nagata ring also.*

Proof. First note that S is Noetherian as R is Noetherian and a quasi-finite ring map is of finite type. Let $\mathfrak{q} \subset S$ be a prime ideal, and set $\mathfrak{p} = R \cap \mathfrak{q}$. Then $R/\mathfrak{p} \subset S/\mathfrak{q}$ is quasi-finite and hence we conclude that $S/\mathfrak{q}$ is N-2 by Lemma 161.5 as desired. □

032U **Lemma 162.6.** *A localization of a Nagata ring is a Nagata ring.*

Proof. Clear from Lemma 161.3. □

032V **Lemma 162.7.** *Let R be a ring. Let $f_1, \ldots, f_n \in R$ generate the unit ideal.*

(1) *If each R_{f_i} is universally Japanese then so is R.*
(2) *If each R_{f_i} is Nagata then so is R.*

Proof. Let $\varphi : R \to S$ be a finite type ring map so that S is a domain. Then $\varphi(f_1), \ldots, \varphi(f_n)$ generate the unit ideal in S. Hence if each $S_{f_i} = S_{\varphi(f_i)}$ is N-1 then so is S, see Lemma 161.4. This proves (1).

If each R_{f_i} is Nagata, then each R_{f_i} is Noetherian and hence R is Noetherian, see Lemma 23.2. And if $\mathfrak{p} \subset R$ is a prime, then we see each $R_{f_i}/\mathfrak{p}R_{f_i} = (R/\mathfrak{p})_{f_i}$ is N-2 and hence we conclude $R/\mathfrak{p}$ is N-2 by Lemma 161.4. This proves (2). □

032W **Lemma 162.8.** *A Noetherian complete local ring is a Nagata ring.*

Proof. Let R be a complete local Noetherian ring. Let $\mathfrak{p} \subset R$ be a prime. Then $R/\mathfrak{p}$ is also a complete local Noetherian ring, see Lemma 160.2. Hence it suffices to show that a Noetherian complete local domain R is N-2. By Lemmas 161.5 and 160.11 we reduce to the case $R = k[[X_1, \ldots, X_d]]$ where k is a field or $R = \Lambda[[X_1, \ldots, X_d]]$ where Λ is a Cohen ring.

In the case $k[[X_1, \ldots, X_d]]$ we reduce to the statement that a field is N-2 by Lemma 161.17. This is clear. In the case $\Lambda[[X_1, \ldots, X_d]]$ we reduce to the statement that a Cohen ring Λ is N-2. Applying Lemma 161.16 once more with $x = p \in \Lambda$ we reduce yet again to the case of a field. Thus we win. □

032X **Definition 162.9.** Let $(R, \mathfrak{m})$ be a Noetherian local ring. We say R is *analytically unramified* if its completion $R^\wedge = \lim_n R/\mathfrak{m}^n$ is reduced. A prime ideal $\mathfrak{p} \subset R$ is said to be *analytically unramified* if $R/\mathfrak{p}$ is analytically unramified.

At this point we know the following are true for any Noetherian local ring R: The map $R \to R^\wedge$ is a faithfully flat local ring homomorphism (Lemma 97.3). The completion $R^\wedge$ is Noetherian (Lemma 97.5) and complete (Lemma 97.4). Hence the completion $R^\wedge$ is a Nagata ring (Lemma 162.8). Moreover, we have seen in Section 160 that $R^\wedge$ is a quotient of a regular local ring (Theorem 160.8), and hence universally catenary (Remark 160.9).

032Y **Lemma 162.10.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring.*

(1) *If R is analytically unramified, then R is reduced.*
(2) *If R is analytically unramified, then each minimal prime of R is analytically unramified.*
(3) *If R is reduced with minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$, and each $\mathfrak{q}_i$ is analytically unramified, then R is analytically unramified.*

(4) *If R is analytically unramified, then the integral closure of R in its total ring of fractions $Q(R)$ is finite over R.*
(5) *If R is a domain and analytically unramified, then R is N-1.*

Proof. In this proof we will use the remarks immediately following Definition 162.9. As $R \to R^\wedge$ is a faithfully flat local ring homomorphism it is injective and (1) follows.

Let $\mathfrak q$ be a minimal prime of R, and assume R is analytically unramified. Then $\mathfrak q$ is an associated prime of R (see Proposition 63.6). Hence there exists an $f \in R$ such that $\{x \in R \mid fx = 0\} = \mathfrak q$. Note that $(R/\mathfrak q)^\wedge = R^\wedge/\mathfrak q^\wedge$, and that $\{x \in R^\wedge \mid fx = 0\} = \mathfrak q^\wedge$, because completion is exact (Lemma 97.2). If $x \in R^\wedge$ is such that $x^2 \in \mathfrak q^\wedge$, then $fx^2 = 0$ hence $(fx)^2 = 0$ hence $fx = 0$ hence $x \in \mathfrak q^\wedge$. Thus $\mathfrak q$ is analytically unramified and (2) holds.

Assume R is reduced with minimal primes $\mathfrak q_1, \ldots, \mathfrak q_t$, and each $\mathfrak q_i$ is analytically unramified. Then $R \to R/\mathfrak q_1 \times \ldots \times R/\mathfrak q_t$ is injective. Since completion is exact (see Lemma 97.2) we see that $R^\wedge \subset (R/\mathfrak q_1)^\wedge \times \ldots \times (R/\mathfrak q_t)^\wedge$. Hence (3) is clear.

Assume R is analytically unramified. Let $\mathfrak p_1, \ldots, \mathfrak p_s$ be the minimal primes of $R^\wedge$. Then we see that

$$Q(R^\wedge) = R^\wedge_{\mathfrak p_1} \times \ldots \times R^\wedge_{\mathfrak p_s}$$

with each $R^\wedge_{\mathfrak p_i}$ a field as $R^\wedge$ is reduced (see Lemma 25.4). Hence the integral closure S of $R^\wedge$ in $Q(R^\wedge)$ is equal to $S = S_1 \times \ldots \times S_s$ with S_i the integral closure of $R^\wedge/\mathfrak p_i$ in its fraction field. In particular S is finite over $R^\wedge$. Denote R' the integral closure of R in $Q(R)$. As $R \to R^\wedge$ is flat we see that $R' \otimes_R R^\wedge \subset Q(R) \otimes_R R^\wedge \subset Q(R^\wedge)$. Moreover $R' \otimes_R R^\wedge$ is integral over $R^\wedge$ (Lemma 36.13). Hence $R' \otimes_R R^\wedge \subset S$ is a $R^\wedge$-submodule. As $R^\wedge$ is Noetherian it is a finite $R^\wedge$-module. Thus we may find $f_1, \ldots, f_n \in R'$ such that $R' \otimes_R R^\wedge$ is generated by the elements $f_i \otimes 1$ as a $R^\wedge$-module. By faithful flatness we see that R' is generated by $f_1, \ldots, f_n$ as an R-module. This proves (4).

Part (5) is a special case of part (4). $\square$

032Z **Lemma 162.11.** *Let R be a Noetherian local ring. Let $\mathfrak p \subset R$ be a prime. Assume*
(1) *$R_{\mathfrak p}$ is a discrete valuation ring, and*
(2) *$\mathfrak p$ is analytically unramified.*

Then for any associated prime $\mathfrak q$ of $R^\wedge/\mathfrak pR^\wedge$ the local ring $(R^\wedge)_{\mathfrak q}$ is a discrete valuation ring.

Proof. Assumption (2) says that $R^\wedge/\mathfrak pR^\wedge$ is a reduced ring. Hence an associated prime $\mathfrak q \subset R^\wedge$ of $R^\wedge/\mathfrak pR^\wedge$ is the same thing as a minimal prime over $\mathfrak pR^\wedge$. In particular we see that the maximal ideal of $(R^\wedge)_{\mathfrak q}$ is $\mathfrak p(R^\wedge)_{\mathfrak q}$. Choose $x \in R$ such that $xR_{\mathfrak p} = \mathfrak pR_{\mathfrak p}$. By the above we see that $x \in (R^\wedge)_{\mathfrak q}$ generates the maximal ideal. As $R \to R^\wedge$ is faithfully flat we see that x is a nonzerodivisor in $(R^\wedge)_{\mathfrak q}$. Hence we win. $\square$

0330 **Lemma 162.12.** *Let $(R, \mathfrak m)$ be a Noetherian local domain. Let $x \in \mathfrak m$. Assume*
(1) *$x \neq 0$,*
(2) *R/xR has no embedded primes, and*
(3) *for each associated prime $\mathfrak p \subset R$ of R/xR we have*
    (a) *the local ring $R_{\mathfrak p}$ is regular, and*

(b) $\mathfrak p$ *is analytically unramified.*

Then R is analytically unramified.

Proof. Let $\mathfrak p_1, \ldots, \mathfrak p_t$ be the associated primes of the R-module R/xR. Since R/xR has no embedded primes we see that each $\mathfrak p_i$ has height 1, and is a minimal prime over (x). For each i, let $\mathfrak q_{i1}, \ldots, \mathfrak q_{is_i}$ be the associated primes of the $R^\wedge$-module $R^\wedge/\mathfrak p_i R^\wedge$. By Lemma 162.11 we see that $(R^\wedge)_{\mathfrak q_{ij}}$ is regular. By Lemma 65.3 we see that

$$\text{Ass}_{R^\wedge}(R^\wedge/xR^\wedge) = \bigcup\nolimits_{\mathfrak p \in \text{Ass}_R(R/xR)} \text{Ass}_{R^\wedge}(R^\wedge/\mathfrak p R^\wedge) = \{\mathfrak q_{ij}\}.$$

Let $y \in R^\wedge$ with $y^2 = 0$. As $(R^\wedge)_{\mathfrak q_{ij}}$ is regular, and hence a domain (Lemma 106.2) we see that y maps to zero in $(R^\wedge)_{\mathfrak q_{ij}}$. Hence y maps to zero in $R^\wedge/xR^\wedge$ by Lemma 63.19. Hence $y = xy'$. Since x is a nonzerodivisor (as $R \to R^\wedge$ is flat) we see that $(y')^2 = 0$. Hence we conclude that $y \in \bigcap x^n R^\wedge = (0)$ (Lemma 51.4). □

0331 **Lemma 162.13.** *Let $(R, \mathfrak m)$ be a local ring. If R is Noetherian, a domain, and Nagata, then R is analytically unramified.*

Proof. By induction on $\dim(R)$. The case $\dim(R) = 0$ is trivial. Hence we assume $\dim(R) = d$ and that the lemma holds for all Noetherian Nagata domains of dimension $< d$.

Let $R \subset S$ be the integral closure of R in the field of fractions of R. By assumption S is a finite R-module. By Lemma 162.5 we see that S is Nagata. By Lemma 112.4 we see $\dim(R) = \dim(S)$. Let $\mathfrak m_1, \ldots, \mathfrak m_t$ be the maximal ideals of S. Each of these lies over the maximal ideal $\mathfrak m$ of R. Moreover

$$(\mathfrak m_1 \cap \ldots \cap \mathfrak m_t)^n \subset \mathfrak m S$$

for sufficiently large n as $S/\mathfrak m S$ is Artinian. By Lemma 97.2 $R^\wedge \to S^\wedge$ is an injective map, and by the Chinese Remainder Lemma 15.4 combined with Lemma 96.9 we have $S^\wedge = \prod S_i^\wedge$ where $S_i^\wedge$ is the completion of S with respect to the maximal ideal $\mathfrak m_i$. Hence it suffices to show that $S_{\mathfrak m_i}$ is analytically unramified. In other words, we have reduced to the case where R is a Noetherian normal Nagata domain.

Assume R is a Noetherian, normal, local Nagata domain. Pick a nonzero $x \in \mathfrak m$ in the maximal ideal. We are going to apply Lemma 162.12. We have to check properties (1), (2), (3)(a) and (3)(b). Property (1) is clear. We have that R/xR has no embedded primes by Lemma 157.6. Thus property (2) holds. The same lemma also tells us each associated prime $\mathfrak p$ of R/xR has height 1. Hence $R_{\mathfrak p}$ is a 1-dimensional normal domain hence regular (Lemma 119.7). Thus (3)(a) holds. Finally (3)(b) holds by induction hypothesis, since $R/\mathfrak p$ is Nagata (by Lemma 162.5 or directly from the definition). Thus we conclude R is analytically unramified. □

0BI2 **Lemma 162.14.** *Let $(R, \mathfrak m)$ be a Noetherian local ring. The following are equivalent*

(1) *R is Nagata,*
(2) *for $R \to S$ finite with S a domain and $\mathfrak m' \subset S$ maximal the local ring $S_{\mathfrak m'}$ is analytically unramified,*
(3) *for $(R, \mathfrak m) \to (S, \mathfrak m')$ finite local homomorphism with S a domain, then S is analytically unramified.*

Proof. Assume R is Nagata and let $R \to S$ and $\mathfrak{m}' \subset S$ be as in (2). Then S is Nagata by Lemma 162.5. Hence the local ring $S_{\mathfrak{m}'}$ is Nagata (Lemma 162.6). Thus it is analytically unramified by Lemma 162.13. It is clear that (2) implies (3).

Assume (3) holds. Let $\mathfrak{p} \subset R$ be a prime ideal and let $L/\kappa(\mathfrak{p})$ be a finite extension of fields. To prove (1) we have to show that the integral closure of $R/\mathfrak{p}$ is finite over $R/\mathfrak{p}$. Choose $x_1, \ldots, x_n \in L$ which generate L over $\kappa(\mathfrak{p})$. For each i let $P_i(T) = T^{d_i} + a_{i,1}T^{d_i-1} + \ldots + a_{i,d_i}$ be the minimal polynomial for x_i over $\kappa(\mathfrak{p})$. After replacing x_i by $f_i x_i$ for a suitable $f_i \in R$, $f_i \notin \mathfrak{p}$ we may assume $a_{i,j} \in R/\mathfrak{p}$. In fact, after further multiplying by elements of $\mathfrak{m}$, we may assume $a_{i,j} \in \mathfrak{m}/\mathfrak{p} \subset R/\mathfrak{p}$ for all i, j. Having done this let $S = R/\mathfrak{p}[x_1, \ldots, x_n] \subset L$. Then S is finite over R, a domain, and $S/\mathfrak{m}S$ is a quotient of $R/\mathfrak{m}[T_1, \ldots, T_n]/(T_1^{d_1}, \ldots, T_n^{d_n})$. Hence S is local. By (3) S is analytically unramified and by Lemma 162.10 we find that its integral closure S' in L is finite over S. Since S' is also the integral closure of $R/\mathfrak{p}$ in L we win. □

The following proposition says in particular that an algebra of finite type over a Nagata ring is a Nagata ring.

0334 **Proposition 162.15** (Nagata)**.** *Let R be a ring. The following are equivalent:*

1. *R is a Nagata ring,*
2. *any finite type R-algebra is Nagata, and*
3. *R is universally Japanese and Noetherian.*

Proof. It is clear that a Noetherian universally Japanese ring is universally Nagata (i.e., condition (2) holds). Let R be a Nagata ring. We will show that any finitely generated R-algebra S is Nagata. This will prove the proposition.

Step 1. There exists a sequence of ring maps $R = R_0 \to R_1 \to R_2 \to \ldots \to R_n = S$ such that each $R_i \to R_{i+1}$ is generated by a single element. Hence by induction it suffices to prove S is Nagata if $S \cong R[x]/I$.

Step 2. Let $\mathfrak{q} \subset S$ be a prime of S, and let $\mathfrak{p} \subset R$ be the corresponding prime of R. We have to show that $S/\mathfrak{q}$ is N-2. Hence we have reduced to the proving the following: (*) Given a Nagata domain R and a monogenic extension $R \subset S$ of domains then S is N-2.

Step 3. Let R be a Nagata domain and $R \subset S$ a monogenic extension of domains. Let $R \subset R'$ be the integral closure of R in its fraction field. Let S' be the subring of the fraction field of S generated by R' and S. As R' is finite over R (by the Nagata property) also S' is finite over S. Since S is Noetherian it suffices to prove that S' is N-2 (Lemma 161.7). Hence we have reduced to proving the following: (**) Given a normal Nagata domain R and a monogenic extension $R \subset S$ of domains then S is N-2.

Step 4: Let R be a normal Nagata domain and let $R \subset S$ be a monogenic extension of domains. Suppose the induced extension of fraction fields of R and S is purely transcendental. In this case $S = R[x]$. By Lemma 161.13 we see that S is N-2. Hence we have reduced to proving the following: (**) Given a normal Nagata domain R and a monogenic extension $R \subset S$ of domains inducing a finite extension of fraction fields then S is N-2.

Step 5. Let R be a normal Nagata domain and let $R \subset S$ be a monogenic extension of domains inducing a finite extension of fraction fields L/K. Choose an element

$x \in S$ which generates S as an R-algebra. Let M/L be a finite extension of fields. Let R' be the integral closure of R in M. Then the integral closure S' of S in M is equal to the integral closure of $R'[x]$ in M. Also the fraction field of R' is M and $R \subset R'$ is finite (by the Nagata property of R). This implies that R' is a Nagata ring (Lemma 162.5). To show that S' is finite over S is the same as showing that S' is finite over $R'[x]$. Replace R by R' and S by $R'[x]$ to reduce to the following statement: (***) Given a normal Nagata domain R with fraction field K, and $x \in K$, the ring $S \subset K$ generated by R and x is N-1.

Step 6. Let R be a normal Nagata domain with fraction field K. Let $x = b/a \in K$. We have to show that the ring $S \subset K$ generated by R and x is N-1. Note that $S_a \cong R_a$ is normal. Hence by Lemma 161.15 it suffices to show that $S_\mathfrak{m}$ is N-1 for every maximal ideal $\mathfrak{m}$ of S.

With assumptions as in the preceding paragraph, pick such a maximal ideal and set $\mathfrak{n} = R \cap \mathfrak{m}$. The residue field extension $\kappa(\mathfrak{m})/\kappa(\mathfrak{n})$ is finite (Theorem 34.1) and generated by the image of x. Hence there exists a monic polynomial $f(X) = X^d + \sum_{i=1,\ldots,d} a_i X^{d-i}$ with $f(x) \in \mathfrak{m}$. Let K''/K be a finite extension of fields such that $f(X)$ splits completely in $K''[X]$. Let R' be the integral closure of R in K''. Let $S' \subset K''$ be the subring generated by R' and x. As R is Nagata we see R' is finite over R and Nagata (Lemma 162.5). Moreover, S' is finite over S. If for every maximal ideal $\mathfrak{m}'$ of S' the local ring $S'_{\mathfrak{m}'}$ is N-1, then $S'_\mathfrak{m}$ is N-1 by Lemma 161.15, which in turn implies that $S_\mathfrak{m}$ is N-1 by Lemma 161.7. After replacing R by R' and S by S', and $\mathfrak{m}$ by any of the maximal ideals $\mathfrak{m}'$ lying over $\mathfrak{m}$ we reach the situation where the polynomial f above split completely: $f(X) = \prod_{i=1,\ldots,d}(X - a_i)$ with $a_i \in R$. Since $f(x) \in \mathfrak{m}$ we see that $x - a_i \in \mathfrak{m}$ for some i. Finally, after replacing x by $x - a_i$ we may assume that $x \in \mathfrak{m}$.

To recapitulate: R is a normal Nagata domain with fraction field K, $x \in K$ and S is the subring of K generated by x and R, finally $\mathfrak{m} \subset S$ is a maximal ideal with $x \in \mathfrak{m}$. We have to show $S_\mathfrak{m}$ is N-1.

We will show that Lemma 162.12 applies to the local ring $S_\mathfrak{m}$ and the element x. This will imply that $S_\mathfrak{m}$ is analytically unramified, whereupon we see that it is N-1 by Lemma 162.10.

We have to check properties (1), (2), (3)(a) and (3)(b). Property (1) is trivial. Let $I = \text{Ker}(R[X] \to S)$ where $X \mapsto x$. We claim that I is generated by all linear forms $aX - b$ such that $ax = b$ in K. Clearly all these linear forms are in I. If $g = a_d X^d + \ldots a_1 X + a_0 \in I$, then we see that $a_d x$ is integral over R (Lemma 123.1) and hence $b := a_d x \in R$ as R is normal. Then $g - (a_d X - b)X^{d-1} \in I$ and we win by induction on the degree. As a consequence we see that

$$S/xS = R[X]/(X, I) = R/J$$

where

$$J = \{b \in R \mid ax = b \text{ for some } a \in R\} = xR \cap R$$

By Lemma 157.6 we see that $S/xS = R/J$ has no embedded primes as an R-module, hence as an R/J-module, hence as an S/xS-module, hence as an S-module. This proves property (2). Take such an associated prime $\mathfrak{q} \subset S$ with the property $\mathfrak{q} \subset \mathfrak{m}$ (so that it is an associated prime of $S_\mathfrak{m}/xS_\mathfrak{m}$ – it does not matter for the arguments). Then $\mathfrak{q}$ is minimal over xS and hence has height 1. By the sequence of equalities above we see that $\mathfrak{p} = R \cap \mathfrak{q}$ is an associated prime of R/J, and so has height 1 (see Lemma 157.6). Thus $R_\mathfrak{p}$ is a discrete valuation ring and therefore $R_\mathfrak{p} \subset S_\mathfrak{q}$ is an equality. This shows that $S_\mathfrak{q}$ is regular. This proves property (3)(a). Finally, $(S/\mathfrak{q})_\mathfrak{m}$ is a localization of $S/\mathfrak{q}$, which is a quotient of $S/xS = R/J$. Hence $(S/\mathfrak{q})_\mathfrak{m}$ is a localization of a quotient of the Nagata ring R, hence Nagata (Lemmas 162.5 and 162.6) and hence analytically unramified (Lemma 162.13). This shows (3)(b) holds and we are done. □

0335 **Proposition 162.16.** *The following types of rings are Nagata and in particular universally Japanese:*

1. *fields,*
2. *Noetherian complete local rings,*
3. $\mathbf{Z}$*,*
4. *Dedekind domains with fraction field of characteristic zero,*
5. *finite type ring extensions of any of the above.*

Proof. The Noetherian complete local ring case is Lemma 162.8. In the other cases you just check if $R/\mathfrak{p}$ is N-2 for every prime ideal $\mathfrak{p}$ of the ring. This is clear whenever $R/\mathfrak{p}$ is a field, i.e., $\mathfrak{p}$ is maximal. Hence for the Dedekind ring case we only need to check it when $\mathfrak{p} = (0)$. But since we assume the fraction field has characteristic zero Lemma 161.11 kicks in. □

09E1 **Example 162.17.** A discrete valuation ring is Nagata if and only if it is N-2 (because the quotient by the maximal ideal is a field and hence N-2). The discrete valuation ring A of Example 119.5 is not Nagata, i.e., it is not N-2. Namely, the finite extension $A \subset R = A[f]$ is not N-1. To see this say $f = \sum a_i x^i$. For every $n \geq 1$ set $g_n = \sum_{i<n} a_i x^i \in A$. Then $h_n = (f - g_n)/x^n$ is an element of the fraction field of R and $h_n^p \in k^p[[x]] \subset A$. Hence the integral closure R' of R contains $h_1, h_2, h_3, \ldots$. Now, if R' were finite over R and hence A, then $f = x^n h_n + g_n$ would be contained in the submodule $A + x^n R'$ for all n. By Artin-Rees this would imply $f \in A$ (Lemma 51.4), a contradiction.

09E2 **Lemma 162.18.** *Let $(A, \mathfrak{m})$ be a Noetherian local domain which is Nagata and has fraction field of characteristic p. If $a \in A$ has a pth root in $A^\wedge$, then a has a pth root in A.*

Proof. Consider the ring extension $A \subset B = A[x]/(x^p - a)$. If a does not have a pth root in A, then B is a domain whose completion isn't reduced. This contradicts our earlier results, as B is a Nagata ring (Proposition 162.15) and hence analytically unramified by Lemma 162.13. □

163. Ascending properties

0336 In this section we start proving some algebraic facts concerning the "ascent" of properties of rings. To do this for depth of rings one uses the following result on ascending depth of modules, see [DG67, IV, Proposition 6.3.1].

0338 **Lemma 163.1.** *We have* [DG67, IV, Proposition 6.3.1]

$$depth(M \otimes_R N) = depth(M) + depth(N/\mathfrak{m}_R N)$$

where $R \to S$ is a local homomorphism of local Noetherian rings, M is a finite R-module, and N is a finite S-module flat over R.

Proof. In the statement and in the proof below, we take the depth of M as an R-module, the depth of $M \otimes_R N$ as an S-module, and the depth of $N/\mathfrak{m}_R N$ as an $S/\mathfrak{m}_R S$-module. Denote n the right hand side. First assume that n is zero. Then both $\text{depth}(M) = 0$ and $\text{depth}(N/\mathfrak{m}_R N) = 0$. This means there is a $z \in M$ whose annihilator is $\mathfrak{m}_R$ and a $\overline{y} \in N/\mathfrak{m}_R N$ whose annihilator is $\mathfrak{m}_S/\mathfrak{m}_R S$. Let $y \in N$ be a lift of $\overline{y}$. Since N is flat over R the map $z : R/\mathfrak{m}_R \to M$ produces an injective map $N/\mathfrak{m}_R N \to M \otimes_R N$. Hence the annihilator of $z \otimes y$ is $\mathfrak{m}_S$. Thus $\text{depth}(M \otimes_R N) = 0$ as well.

Assume $n > 0$. If $\text{depth}(N/\mathfrak{m}_R N) > 0$, then we may choose $f \in \mathfrak{m}_S$ mapping to $\overline{f} \in S/\mathfrak{m}_R S$ which is a nonzerodivisor on $N/\mathfrak{m}_R N$. Then $\text{depth}(N/\mathfrak{m}_R N) = \text{depth}(N/(f, \mathfrak{m}_R)N) + 1$ by Lemma 72.7. According to Lemma 99.1 the element $f \in S$ is a nonzerodivisor on N and N/fN is flat over R. Hence by induction on n we have

$$\text{depth}(M \otimes_R N/fN) = \text{depth}(M) + \text{depth}(N/(f, \mathfrak{m}_R)N).$$

Because N/fN is flat over R the sequence

$$0 \to M \otimes_R N \to M \otimes_R N \to M \otimes_R N/fN \to 0$$

is exact where the first map is multiplication by f (Lemma 39.12). Hence by Lemma 72.7 we find that $\text{depth}(M \otimes_R N) = \text{depth}(M \otimes_R N/fN) + 1$ and we conclude that equality holds in the formula of the lemma.

If $n > 0$, but $\text{depth}(N/\mathfrak{m}_R N) = 0$, then we can choose $f \in \mathfrak{m}_R$ which is a nonzerodivisor on M. As N is flat over R it is also the case that f is a nonzerodivisor on $M \otimes_R N$. By induction on n again we have

$$\text{depth}(M/fM \otimes_R N) = \text{depth}(M/fM) + \text{depth}(N/\mathfrak{m}_R N).$$

In this case $\text{depth}(M \otimes_R N) = \text{depth}(M/fM \otimes_R N) + 1$ and $\text{depth}(M) = \text{depth}(M/fM) + 1$ by Lemma 72.7 and we conclude that equality holds in the formula of the lemma. ☐

0337 **Lemma 163.2.** *Suppose that $R \to S$ is a flat and local ring homomorphism of Noetherian local rings. Then*

$$\mathit{depth}(S) = \mathit{depth}(R) + \mathit{depth}(S/\mathfrak{m}_R S).$$

Proof. This is a special case of Lemma 163.1. ☐

045J **Lemma 163.3.** *Let $R \to S$ be a flat local homomorphism of local Noetherian rings. Then the following are equivalent*

1. *S is Cohen-Macaulay, and*
2. *R and $S/\mathfrak{m}_R S$ are Cohen-Macaulay.*

Proof. Follows from the definitions and Lemmas 163.2 and 112.7. ☐

0339 **Lemma 163.4.** *Let $\varphi : R \to S$ be a ring map. Assume*

1. *R is Noetherian,*
2. *S is Noetherian,*
3. *φ is flat,*
4. *the fibre rings $S \otimes_R \kappa(\mathfrak{p})$ are (S_k), and*
5. *R has property (S_k).*

Then S has property (S_k).

Proof. Let $\mathfrak{q}$ be a prime of S lying over a prime $\mathfrak{p}$ of R. By Lemma 163.2 we have

$$\text{depth}(S_\mathfrak{q}) = \text{depth}(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) + \text{depth}(R_\mathfrak{p}).$$

On the other hand, we have

$$\dim(R_\mathfrak{p}) + \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) \geq \dim(S_\mathfrak{q})$$

by Lemma 112.6. (Actually equality holds, by Lemma 112.7 but strictly speaking we do not need this.) Finally, as the fibre rings of the map are assumed (S_k) we see that $\text{depth}(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) \geq \min(k, \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}))$. Thus the lemma follows by the following string of inequalities

$$\begin{aligned} \text{depth}(S_\mathfrak{q}) &= \text{depth}(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) + \text{depth}(R_\mathfrak{p}) \\ &\geq \min(k, \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q})) + \min(k, \dim(R_\mathfrak{p})) \\ &= \min(2k, \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) + k, k + \dim(R_\mathfrak{p}), \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) + \dim(R_\mathfrak{p})) \\ &\geq \min(k, \dim(S_\mathfrak{q})) \end{aligned}$$

as desired. □

033A **Lemma 163.5.** *Let $\varphi : R \to S$ be a ring map. Assume*

1. *R is Noetherian,*
2. *S is Noetherian*
3. *φ is flat,*
4. *the fibre rings $S \otimes_R \kappa(\mathfrak{p})$ have property (R_k), and*
5. *R has property (R_k).*

Then S has property (R_k).

Proof. Let $\mathfrak{q}$ be a prime of S lying over a prime $\mathfrak{p}$ of R. Assume that $\dim(S_\mathfrak{q}) \leq k$. Since $\dim(S_\mathfrak{q}) = \dim(R_\mathfrak{p}) + \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q})$ by Lemma 112.7 we see that $\dim(R_\mathfrak{p}) \leq k$ and $\dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) \leq k$. Hence $R_\mathfrak{p}$ and $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$ are regular by assumption. It follows that $S_\mathfrak{q}$ is regular by Lemma 112.8. □

0C21 **Lemma 163.6.** *Let $\varphi : R \to S$ be a ring map. Assume*

1. *R is Noetherian,*
2. *S is Noetherian*
3. *φ is flat,*
4. *the fibre rings $S \otimes_R \kappa(\mathfrak{p})$ are reduced,*
5. *R is reduced.*

Then S is reduced.

Proof. For Noetherian rings reduced is the same as having properties (S_1) and (R_0), see Lemma 157.3. Thus we know R and the fibre rings have these properties. Hence we may apply Lemmas 163.4 and 163.5 and we see that S is (S_1) and (R_0), in other words reduced by Lemma 157.3 again. □

033B **Lemma 163.7.** *Let $\varphi : R \to S$ be a ring map. Assume*

1. *φ is smooth,*
2. *R is reduced.*

Then S is reduced.

Proof. Observe that $R \to S$ is flat with regular fibres (see the list of results on smooth ring maps in Section 142). In particular, the fibres are reduced. Thus if R is Noetherian, then S is Noetherian and we get the result from Lemma 163.6.

In the general case we may find a finitely generated $\mathbf{Z}$-subalgebra $R_0 \subset R$ and a smooth ring map $R_0 \to S_0$ such that $S \cong R \otimes_{R_0} S_0$, see remark (10) in Section 142. Now, if $x \in S$ is an element with $x^2 = 0$, then we can enlarge R_0 and assume that x comes from an element $x_0 \in S_0$. After enlarging R_0 once more we may assume that $x_0^2 = 0$ in S_0. However, since $R_0 \subset R$ is reduced we see that S_0 is reduced and hence $x_0 = 0$ as desired. $\square$

0C22 **Lemma 163.8.** *Let $\varphi : R \to S$ be a ring map. Assume*

1. *R is Noetherian,*
2. *S is Noetherian,*
3. *φ is flat,*
4. *the fibre rings $S \otimes_R \kappa(\mathfrak p)$ are normal, and*
5. *R is normal.*

Then S is normal.

Proof. For a Noetherian ring being normal is the same as having properties (S_2) and (R_1), see Lemma 157.4. Thus we know R and the fibre rings have these properties. Hence we may apply Lemmas 163.4 and 163.5 and we see that S is (S_2) and (R_1), in other words normal by Lemma 157.4 again. $\square$

033C **Lemma 163.9.** *Let $\varphi : R \to S$ be a ring map. Assume*

1. *φ is smooth,*
2. *R is normal.*

Then S is normal.

Proof. Observe that $R \to S$ is flat with regular fibres (see the list of results on smooth ring maps in Section 142). In particular, the fibres are normal. Thus if R is Noetherian, then S is Noetherian and we get the result from Lemma 163.8.

The general case. First note that R is reduced and hence S is reduced by Lemma 163.7. Let $\mathfrak q$ be a prime of S and let $\mathfrak p$ be the corresponding prime of R. Note that $R_{\mathfrak p}$ is a normal domain. We have to show that $S_{\mathfrak q}$ is a normal domain. To do this we may replace R by $R_{\mathfrak p}$ and S by $S_{\mathfrak p}$. Hence we may assume that R is a normal domain.

Assume $R \to S$ smooth, and R a normal domain. We may find a finitely generated $\mathbf{Z}$-subalgebra $R_0 \subset R$ and a smooth ring map $R_0 \to S_0$ such that $S \cong R \otimes_{R_0} S_0$, see remark (10) in Section 142. As R_0 is a Nagata domain (see Proposition 162.16) we see that its integral closure R_0' is finite over R_0. Moreover, as R is a normal domain it is clear that $R_0' \subset R$. Hence we may replace R_0 by R_0' and S_0 by $R_0' \otimes_{R_0} S_0$ and assume that R_0 is a normal Noetherian domain. By the first paragraph of the proof we conclude that S_0 is a normal ring (it need not be a domain of course). In this way we see that $R = \bigcup R_\lambda$ is the union of normal Noetherian domains and correspondingly $S = \operatorname{colim} R_\lambda \otimes_{R_0} S_0$ is the colimit of normal rings. This implies that S is a normal ring. Some details omitted. $\square$

07NF **Lemma 163.10.** *Let $\varphi : R \to S$ be a ring map. Assume*

1. *φ is smooth,*

(2) *R is a regular ring.*

Then S is regular.

Proof. This follows from Lemma 163.5 applied for all (R_k) using Lemma 140.3 to see that the hypotheses are satisfied. □

164. Descending properties

033D In this section we start proving some algebraic facts concerning the "descent" of properties of rings. It turns out that it is often "easier" to descend properties than it is to ascend them. In other words, the assumption on the ring map $R \to S$ are often weaker than the assumptions in the corresponding lemma of the preceding section. However, we warn the reader that the results on descent are often useless unless the corresponding ascent can also be shown! Here is a typical result which illustrates this phenomenon.

033E **Lemma 164.1.** *Let $R \to S$ be a ring map. Assume that*

(1) *$R \to S$ is faithfully flat, and*
(2) *S is Noetherian.*

Then R is Noetherian.

Proof. Let $I_0 \subset I_1 \subset I_2 \subset \ldots$ be a growing sequence of ideals of R. By assumption we have $I_nS = I_{n+1}S = I_{n+2}S = \ldots$ for some n. Since $R \to S$ is flat we have $I_kS = I_k \otimes_R S$. Hence, as $R \to S$ is faithfully flat we see that $I_nS = I_{n+1}S = I_{n+2}S = \ldots$ implies that $I_n = I_{n+1} = I_{n+2} = \ldots$ as desired. □

033F **Lemma 164.2.** *Let $R \to S$ be a ring map. Assume that*

(1) *$R \to S$ is faithfully flat, and*
(2) *S is reduced.*

Then R is reduced.

Proof. This is clear as $R \to S$ is injective. □

033G **Lemma 164.3.** *Let $R \to S$ be a ring map. Assume that*

(1) *$R \to S$ is faithfully flat, and*
(2) *S is a normal ring.*

Then R is a normal ring.

Proof. Since S is reduced it follows that R is reduced. Let $\mathfrak{p}$ be a prime of R. We have to show that $R_{\mathfrak{p}}$ is a normal domain. Since $S_{\mathfrak{p}}$ is faithfully over $R_{\mathfrak{p}}$ too we may assume that R is local with maximal ideal $\mathfrak{m}$. Let $\mathfrak{q}$ be a prime of S lying over $\mathfrak{m}$. Then we see that $R \to S_{\mathfrak{q}}$ is faithfully flat (Lemma 39.17). Hence we may assume S is local as well. In particular S is a normal domain. Since $R \to S$ is faithfully flat and S is a normal domain we see that R is a domain. Next, suppose that a/b is integral over R with $a, b \in R$. Then $a/b \in S$ as S is normal. Hence $a \in bS$. This means that $a : R \to R/bR$ becomes the zero map after base change to S. By faithful flatness we see that $a \in bR$, so $a/b \in R$. Hence R is normal. □

07NG **Lemma 164.4.** *Let $R \to S$ be a ring map. Assume that*

(1) *$R \to S$ is faithfully flat, and*
(2) *S is a regular ring.*

Then R is a regular ring.

Proof. We see that R is Noetherian by Lemma 164.1. Let $\mathfrak{p} \subset R$ be a prime. Choose a prime $\mathfrak{q} \subset S$ lying over $\mathfrak{p}$. Then Lemma 110.9 applies to $R_\mathfrak{p} \to S_\mathfrak{q}$ and we conclude that $R_\mathfrak{p}$ is regular. Since $\mathfrak{p}$ was arbitrary we see R is regular. $\square$

0352 **Lemma 164.5.** *Let $R \to S$ be a ring map. Assume that*

1. *$R \to S$ is faithfully flat, and*
2. *S is Noetherian and has property (S_k).*

Then R is Noetherian and has property (S_k).

Proof. We have already seen that (1) and (2) imply that R is Noetherian, see Lemma 164.1. Let $\mathfrak{p} \subset R$ be a prime ideal. Choose a prime $\mathfrak{q} \subset S$ lying over $\mathfrak{p}$ which corresponds to a minimal prime of the fibre ring $S \otimes_R \kappa(\mathfrak{p})$. Then $A = R_\mathfrak{p} \to S_\mathfrak{q} = B$ is a flat local ring homomorphism of Noetherian local rings with $\mathfrak{m}_A B$ an ideal of definition of B. Hence $\dim(A) = \dim(B)$ (Lemma 112.7) and $\text{depth}(A) = \text{depth}(B)$ (Lemma 163.2). Hence since B has (S_k) we see that A has (S_k). $\square$

0353 **Lemma 164.6.** *Let $R \to S$ be a ring map. Assume that*

1. *$R \to S$ is faithfully flat, and*
2. *S is Noetherian and has property (R_k).*

Then R is Noetherian and has property (R_k).

Proof. We have already seen that (1) and (2) imply that R is Noetherian, see Lemma 164.1. Let $\mathfrak{p} \subset R$ be a prime ideal and assume $\dim(R_\mathfrak{p}) \leq k$. Choose a prime $\mathfrak{q} \subset S$ lying over $\mathfrak{p}$ which corresponds to a minimal prime of the fibre ring $S \otimes_R \kappa(\mathfrak{p})$. Then $A = R_\mathfrak{p} \to S_\mathfrak{q} = B$ is a flat local ring homomorphism of Noetherian local rings with $\mathfrak{m}_A B$ an ideal of definition of B. Hence $\dim(A) = \dim(B)$ (Lemma 112.7). As S has (R_k) we conclude that B is a regular local ring. By Lemma 110.9 we conclude that A is regular. $\square$

0354 **Lemma 164.7.** *Let $R \to S$ be a ring map. Assume that*

1. *$R \to S$ is smooth and surjective on spectra, and*
2. *S is a Nagata ring.*

Then R is a Nagata ring.

Proof. Recall that a Nagata ring is the same thing as a Noetherian universally Japanese ring (Proposition 162.15). We have already seen that R is Noetherian in Lemma 164.1. Let $R \to A$ be a finite type ring map into a domain. According to Lemma 162.3 it suffices to check that A is N-1. It is clear that $B = A \otimes_R S$ is a finite type S-algebra and hence Nagata (Proposition 162.15). Since $A \to B$ is smooth (Lemma 137.4) we see that B is reduced (Lemma 163.7). Since B is Noetherian it has only a finite number of minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ (see Lemma 31.6). As $A \to B$ is flat each of these lies over $(0) \subset A$ (by going down, see Lemma 39.19) The total ring of fractions $Q(B)$ is the product of the $L_i = \kappa(\mathfrak{q}_i)$ (Lemmas 25.4 and 25.1). Moreover, the integral closure B' of B in $Q(B)$ is the product of the integral closures B'_i of the $B/\mathfrak{q}_i$ in the factors L_i (compare with Lemma 37.16). Since B is universally Japanese the ring extensions $B/\mathfrak{q}_i \subset B'_i$ are finite and we conclude that $B' = \prod B'_i$ is finite over B. Since $A \to B$ is flat we see that any nonzerodivisor on A maps to a nonzerodivisor on B. The corresponding map

$$Q(A) \otimes_A B = (A \setminus \{0\})^{-1} A \otimes_A B = (A \setminus \{0\})^{-1} B \to Q(B)$$

is injective (we used Lemma 12.15). Via this map A' maps into B'. This induces a map

$$A' \otimes_A B \longrightarrow B'$$

which is injective (by the above and the flatness of $A \to B$). Since B' is a finite B-module and B is Noetherian we see that $A' \otimes_A B$ is a finite B-module. Hence there exist finitely many elements $x_i \in A'$ such that the elements $x_i \otimes 1$ generate $A' \otimes_A B$ as a B-module. Finally, by faithful flatness of $A \to B$ we conclude that the x_i also generated A' as an A-module, and we win. □

0355 **Remark 164.8.** The property of being "universally catenary" does not descend; not even along étale ring maps. In Examples, Section 18 there is a construction of a finite ring map $A \to B$ with A local Noetherian and not universally catenary, B semi-local with two maximal ideals $\mathfrak{m}$, $\mathfrak{n}$ with $B_\mathfrak{m}$ and $B_\mathfrak{n}$ regular of dimension 2 and 1 respectively, and the same residue fields as that of A. Moreover, $\mathfrak{m}_A$ generates the maximal ideal in both $B_\mathfrak{m}$ and $B_\mathfrak{n}$ (so $A \to B$ is unramified as well as finite). By Lemma 152.3 there exists a local étale ring map $A \to A'$ such that $B \otimes_A A' = B_1 \times B_2$ decomposes with $A' \to B_i$ surjective. This shows that A' has two minimal primes $\mathfrak{q}_i$ with $A'/\mathfrak{q}_i \cong B_i$. Since B_i is regular local (since it is étale over either $B_\mathfrak{m}$ or $B_\mathfrak{n}$) we conclude that A' is universally catenary.

165. Geometrically normal algebras

037Y In this section we put some applications of ascent and descent of properties of rings.

037Z **Lemma 165.1.** *Let k be a field. Let A be a k-algebra. The following properties of A are equivalent:*

1. *$k' \otimes_k A$ is a normal ring for every field extension k'/k,*
2. *$k' \otimes_k A$ is a normal ring for every finitely generated field extension k'/k,*
3. *$k' \otimes_k A$ is a normal ring for every finite purely inseparable extension k'/k,*
4. *$k^{perf} \otimes_k A$ is a normal ring.*

Here normal ring is defined in Definition 37.11.

Proof. It is clear that (1) ⇒ (2) ⇒ (3) and (1) ⇒ (4).

If k'/k is a finite purely inseparable extension, then there is an embedding $k' \to k^{perf}$ of k-extensions. The ring map $k' \otimes_k A \to k^{perf} \otimes_k A$ is faithfully flat, hence $k' \otimes_k A$ is normal if $k^{perf} \otimes_k A$ is normal by Lemma 164.3. In this way we see that (4) ⇒ (3).

Assume (2) and let k'/k be any field extension. Then we can write $k' = \text{colim}_i k_i$ as a directed colimit of finitely generated field extensions. Hence we see that $k' \otimes_k A = \text{colim}_i k_i \otimes_k A$ is a directed colimit of normal rings. Thus we see that $k' \otimes_k A$ is a normal ring by Lemma 37.17. Hence (1) holds.

Assume (3) and let K/k be a finitely generated field extension. By Lemma 45.3 we can find a diagram

$$\begin{tikzcd} K \arrow[r] & K' \\ k \arrow[u] \arrow[r] & k' \arrow[u] \end{tikzcd}$$

where k'/k, K'/K are finite purely inseparable field extensions such that K'/k' is separable. By Lemma 158.10 there exists a smooth k'-algebra B such that K'

is the fraction field of B. Now we can argue as follows: Step 1: $k' \otimes_k A$ is a normal ring because we assumed (3). Step 2: $B \otimes_{k'} k' \otimes_k A$ is a normal ring as $k' \otimes_k A \to B \otimes_{k'} k' \otimes_k A$ is smooth (Lemma 137.4) and ascent of normality along smooth maps (Lemma 163.9). Step 3. $K' \otimes_{k'} k' \otimes_k A = K' \otimes_k A$ is a normal ring as it is a localization of a normal ring (Lemma 37.13). Step 4. Finally $K \otimes_k A$ is a normal ring by descent of normality along the faithfully flat ring map $K \otimes_k A \to K' \otimes_k A$ (Lemma 164.3). This proves the lemma. □

0380 **Definition 165.2.** Let k be a field. A k-algebra R is called *geometrically normal* over k if the equivalent conditions of Lemma 165.1 hold.

06DE **Lemma 165.3.** *Let k be a field. A localization of a geometrically normal k-algebra is geometrically normal.*

Proof. This is clear as being a normal ring is checked at the localizations at prime ideals. □

0C30 **Lemma 165.4.** *Let k be a field. Let K/k be a separable field extension. Then K is geometrically normal over k.*

Proof. This is true because $k^{perf} \otimes_k K$ is a field. Namely, it is reduced for example by Lemma 44.1 and it has a unique prime ideal because $K \subset k^{perf} \otimes_k K$ is a universal homeomorphism. □

06DF **Lemma 165.5.** *Let k be a field. Let A, B be k-algebras. Assume A is geometrically normal over k and B is a normal ring. Then $A \otimes_k B$ is a normal ring.*

Proof. Let $\mathfrak{r}$ be a prime ideal of $A \otimes_k B$. Denote $\mathfrak{p}$, resp. $\mathfrak{q}$ the corresponding prime of A, resp. B. Then $(A \otimes_k B)_\mathfrak{r}$ is a localization of $A_\mathfrak{p} \otimes_k B_\mathfrak{q}$. Hence it suffices to prove the result for the ring $A_\mathfrak{p} \otimes_k B_\mathfrak{q}$, see Lemma 37.13 and Lemma 165.3. Thus we may assume A and B are domains.

Assume that A and B are domains with fractions fields K and L. Note that B is the filtered colimit of its finite type normal k-sub algebras (as k is a Nagata ring, see Proposition 162.16, and hence the integral closure of a finite type k-sub algebra is still a finite type k-sub algebra by Proposition 162.15). By Lemma 37.17 we reduce to the case that B is of finite type over k.

Assume that A and B are domains with fractions fields K and L and B of finite type over k. In this case the ring $K \otimes_k B$ is of finite type over K, hence Noetherian (Lemma 31.1). In particular $K \otimes_k B$ has finitely many minimal primes (Lemma 31.6). Since $A \to A \otimes_k B$ is flat, this implies that $A \otimes_k B$ has finitely many minimal primes (by going down for flat ring maps – Lemma 39.19 – these primes all lie over $(0) \subset A$). Thus it suffices to prove that $A \otimes_k B$ is integrally closed in its total ring of fractions (Lemma 37.16).

We claim that $K \otimes_k B$ and $A \otimes_k L$ are both normal rings. If this is true then any element x of $Q(A \otimes_k B)$ which is integral over $A \otimes_k B$ is (by Lemma 37.12) contained in $K \otimes_k B \cap A \otimes_k L = A \otimes_k B$ and we're done. Since $A \otimes_K L$ is a normal ring by assumption, it suffices to prove that $K \otimes_k B$ is normal.

As A is geometrically normal over k we see K is geometrically normal over k (Lemma 165.3) hence K is geometrically reduced over k. Hence $K = \bigcup K_i$ is the union of finitely generated field extensions of k which are geometrically reduced (Lemma 43.2). Each K_i is the localization of a smooth k-algebra (Lemma 158.10).

So $K_i \otimes_k B$ is the localization of a smooth B-algebra hence normal (Lemma 163.9). Thus $K \otimes_k B$ is a normal ring (Lemma 37.17) and we win. □

0C31 **Lemma 165.6.** *Let k'/k be a separable algebraic field extension. Let A be an algebra over k'. Then A is geometrically normal over k if and only if it is geometrically normal over k'.*

Proof. Let L/k be a finite purely inseparable field extension. Then $L' = k' \otimes_k L$ is a field (see material in Fields, Section 28) and $A \otimes_k L = A \otimes_{k'} L'$. Hence if A is geometrically normal over k', then A is geometrically normal over k.

Assume A is geometrically normal over k. Let K/k' be a field extension. Then

$$K \otimes_{k'} A = (K \otimes_k A) \otimes_{(k' \otimes_k k')} k'$$

Since $k' \otimes_k k' \to k'$ is a localization by Lemma 43.8, we see that $K \otimes_{k'} A$ is a localization of a normal ring, hence normal. □

166. Geometrically regular algebras

045K Let k be a field. Let A be a Noetherian k-algebra. Let K/k be a finitely generated field extension. Then the ring $K \otimes_k A$ is Noetherian as well, see Lemma 31.8. Thus the following lemma makes sense.

0381 **Lemma 166.1.** *Let k be a field. Let A be a k-algebra. Assume A is Noetherian. The following properties of A are equivalent:*

1. *$k' \otimes_k A$ is regular for every finitely generated field extension k'/k, and*
2. *$k' \otimes_k A$ is regular for every finite purely inseparable extension k'/k.*

Here regular ring is as in Definition 110.7.

Proof. The lemma makes sense by the remarks preceding the lemma. It is clear that (1) ⇒ (2).

Assume (2) and let K/k be a finitely generated field extension. By Lemma 45.3 we can find a diagram

$$\begin{tikzcd} K \arrow[r] & K' \\ k \arrow[u] \arrow[r] & k' \arrow[u] \end{tikzcd}$$

where k'/k, K'/K are finite purely inseparable field extensions such that K'/k' is separable. By Lemma 158.10 there exists a smooth k'-algebra B such that K' is the fraction field of B. Now we can argue as follows: Step 1: $k' \otimes_k A$ is a regular ring because we assumed (2). Step 2: $B \otimes_{k'} k' \otimes_k A$ is a regular ring as $k' \otimes_k A \to B \otimes_{k'} k' \otimes_k A$ is smooth (Lemma 137.4) and ascent of regularity along smooth maps (Lemma 163.10). Step 3. $K' \otimes_{k'} k' \otimes_k A = K' \otimes_k A$ is a regular ring as it is a localization of a regular ring (immediate from the definition). Step 4. Finally $K \otimes_k A$ is a regular ring by descent of regularity along the faithfully flat ring map $K \otimes_k A \to K' \otimes_k A$ (Lemma 164.4). This proves the lemma. □

0382 **Definition 166.2.** Let k be a field. Let R be a Noetherian k-algebra. The k-algebra R is called *geometrically regular* over k if the equivalent conditions of Lemma 166.1 hold.

It is clear from the definition that $K \otimes_k R$ is a geometrically regular algebra over K for any finitely generated field extension K of k. We will see later (More on Algebra, Proposition 35.1) that it suffices to check $R \otimes_k k'$ is regular whenever $k \subset k' \subset k^{1/p}$ (finite).

07NH **Lemma 166.3.** *Let k be a field. Let $A \to B$ be a faithfully flat k-algebra map. If B is geometrically regular over k, so is A.*

Proof. Assume B is geometrically regular over k. Let k'/k be a finite, purely inseparable extension. Then $A \otimes_k k' \to B \otimes_k k'$ is faithfully flat as a base change of $A \to B$ (by Lemmas 30.3 and 39.7) and $B \otimes_k k'$ is regular by our assumption on B over k. Then $A \otimes_k k'$ is regular by Lemma 164.4. □

07QF **Lemma 166.4.** *Let k be a field. Let $A \to B$ be a smooth ring map of k-algebras. If A is geometrically regular over k, then B is geometrically regular over k.*

Proof. Let k'/k be a finitely generated field extension. Then $A \otimes_k k' \to B \otimes_k k'$ is a smooth ring map (Lemma 137.4) and $A \otimes_k k'$ is regular. Hence $B \otimes_k k'$ is regular by Lemma 163.10. □

07QG **Lemma 166.5.** *Let k be a field. Let A be an algebra over k. Let $k = \operatorname{colim} k_i$ be a directed colimit of subfields. If A is geometrically regular over each k_i, then A is geometrically regular over k.*

Proof. Let k'/k be a finite purely inseparable field extension. We can get k' by adjoining finitely many variables to k and imposing finitely many polynomial relations. Hence we see that there exists an i and a finite purely inseparable field extension k_i'/k_i such that $k_i = k \otimes_{k_i} k_i'$. Thus $A \otimes_k k' = A \otimes_{k_i} k_i'$ and the lemma is clear. □

07QH **Lemma 166.6.** *Let k'/k be a separable algebraic field extension. Let A be an algebra over k'. Then A is geometrically regular over k if and only if it is geometrically regular over k'.*

Proof. Let L/k be a finite purely inseparable field extension. Then $L' = k' \otimes_k L$ is a field (see material in Fields, Section 28) and $A \otimes_k L = A \otimes_{k'} L'$. Hence if A is geometrically regular over k', then A is geometrically regular over k.

Assume A is geometrically regular over k. Since k' is the filtered colimit of finite extensions of k we may assume by Lemma 166.5 that k'/k is finite separable. Consider the ring maps

$$k' \to A \otimes_k k' \to A.$$

Note that $A \otimes_k k'$ is geometrically regular over k' as a base change of A to k'. Note that $A \otimes_k k' \to A$ is the base change of $k' \otimes_k k' \to k'$ by the map $k' \to A$. Since k'/k is an étale extension of rings, we see that $k' \otimes_k k' \to k'$ is étale (Lemma 143.3). Hence A is geometrically regular over k' by Lemma 166.4. □

167. Geometrically Cohen-Macaulay algebras

045L This section is a bit of a misnomer, since Cohen-Macaulay algebras are automatically geometrically Cohen-Macaulay. Namely, see Lemma 130.7 and Lemma 167.2 below.

045M **Lemma 167.1.** *Let k be a field and let K/k and L/k be two field extensions such that one of them is a field extension of finite type. Then $K \otimes_k L$ is a Noetherian Cohen-Macaulay ring.*

Proof. The ring $K \otimes_k L$ is Noetherian by Lemma 31.8. Say K is a finite extension of the purely transcendental extension $k(t_1, \ldots, t_r)$. Then $k(t_1, \ldots, t_r) \otimes_k L \to K \otimes_k L$ is a finite free ring map. By Lemma 112.9 it suffices to show that $k(t_1, \ldots, t_r) \otimes_k L$ is Cohen-Macaulay. This is clear because it is a localization of the polynomial ring $L[t_1, \ldots, t_r]$. (See for example Lemma 104.7 for the fact that a polynomial ring is Cohen-Macaulay.) $\square$

045N **Lemma 167.2.** *Let k be a field. Let S be a Noetherian k-algebra. Let K/k be a finitely generated field extension, and set $S_K = K \otimes_k S$. Let $\mathfrak q \subset S$ be a prime of S. Let $\mathfrak q_K \subset S_K$ be a prime of S_K lying over $\mathfrak q$. Then $S_{\mathfrak q}$ is Cohen-Macaulay if and only if $(S_K)_{\mathfrak q_K}$ is Cohen-Macaulay.*

Proof. By Lemma 31.8 the ring S_K is Noetherian. Hence $S_{\mathfrak q} \to (S_K)_{\mathfrak q_K}$ is a flat local homomorphism of Noetherian local rings. Note that the fibre

$$(S_K)_{\mathfrak q_K}/\mathfrak q(S_K)_{\mathfrak q_K} \cong (\kappa(\mathfrak q) \otimes_k K)_{\mathfrak q'}$$

is the localization of the Cohen-Macaulay (Lemma 167.1) ring $\kappa(\mathfrak q) \otimes_k K$ at a suitable prime ideal $\mathfrak q'$. Hence the lemma follows from Lemma 163.3. $\square$

168. Colimits and maps of finite presentation, II

07RF This section is a continuation of Section 127.

We start with an application of the openness of flatness. It says that we can approximate flat modules by flat modules which is useful.

02JO **Lemma 168.1.** *Let $R \to S$ be a ring map. Let M be an S-module. Assume that*

1. *$R \to S$ is of finite presentation,*
2. *M is a finitely presented S-module, and*
3. *M is flat over R.*

In this case we have the following:

1. *There exists a finite type $\mathbf{Z}$-algebra R_0 and a finite type ring map $R_0 \to S_0$ and a finite S_0-module M_0 such that M_0 is flat over R_0, together with a ring maps $R_0 \to R$ and $S_0 \to S$ and an S_0-module map $M_0 \to M$ such that $S \cong R \otimes_{R_0} S_0$ and $M = S \otimes_{S_0} M_0$.*
2. *If $R = \mathrm{colim}_{\lambda \in \Lambda} R_\lambda$ is written as a directed colimit, then there exists a λ and a ring map $R_\lambda \to S_\lambda$ of finite presentation, and an S_λ-module M_λ of finite presentation such that M_λ is flat over R_λ and such that $S = R \otimes_{R_\lambda} S_\lambda$ and $M = S \otimes_{S_\lambda} M_\lambda$.*
3. *If*
$$(R \to S, M) = \mathrm{colim}_{\lambda \in \Lambda}(R_\lambda \to S_\lambda, M_\lambda)$$
is written as a directed colimit such that
   (a) *$R_\mu \otimes_{R_\lambda} S_\lambda \to S_\mu$ and $S_\mu \otimes_{S_\lambda} M_\lambda \to M_\mu$ are isomorphisms for $\mu \geq \lambda$,*
   (b) *$R_\lambda \to S_\lambda$ is of finite presentation,*
   (c) *M_λ is a finitely presented S_λ-module,*
then for all sufficiently large λ the module M_λ is flat over R_λ.

Proof. We first write $(R \to S, M)$ as the directed colimit of a system $(R_\lambda \to S_\lambda, M_\lambda)$ as in as in Lemma 127.18. Let $\mathfrak q \subset S$ be a prime. Let $\mathfrak p \subset R$, $\mathfrak q_\lambda \subset S_\lambda$, and $\mathfrak p_\lambda \subset R_\lambda$ the corresponding primes. As seen in the proof of Theorem 129.4

$$((R_\lambda)_{\mathfrak p_\lambda}, (S_\lambda)_{\mathfrak q_\lambda}, (M_\lambda)_{\mathfrak q_\lambda})$$

is a system as in Lemma 127.13, and hence by Lemma 128.3 we see that for some $\lambda_{\mathfrak q} \in \Lambda$ for all $\lambda \geq \lambda_{\mathfrak q}$ the module M_λ is flat over R_λ at the prime $\mathfrak q_\lambda$.

By Theorem 129.4 we get an open subset $U_\lambda \subset \text{Spec}(S_\lambda)$ such that M_λ flat over R_λ at all the primes of U_λ. Denote $V_\lambda \subset \text{Spec}(S)$ the inverse image of U_λ under the map $\text{Spec}(S) \to \text{Spec}(S_\lambda)$. The argument above shows that for every $\mathfrak q \in \text{Spec}(S)$ there exists a $\lambda_{\mathfrak q}$ such that $\mathfrak q \in V_\lambda$ for all $\lambda \geq \lambda_{\mathfrak q}$. Since $\text{Spec}(S)$ is quasi-compact we see this implies there exists a single $\lambda_0 \in \Lambda$ such that $V_{\lambda_0} = \text{Spec}(S)$.

The complement $\text{Spec}(S_{\lambda_0}) \setminus U_{\lambda_0}$ is $V(I)$ for some ideal $I \subset S_{\lambda_0}$. As $V_{\lambda_0} = \text{Spec}(S)$ we see that $IS = S$. Choose $f_1, \ldots, f_r \in I$ and $s_1, \ldots, s_n \in S$ such that $\sum f_i s_i = 1$. Since $\text{colim}\, S_\lambda = S$, after increasing λ_0 we may assume there exist $s_{i,\lambda_0} \in S_{\lambda_0}$ such that $\sum f_i s_{i,\lambda_0} = 1$. Hence for this λ_0 we have $U_{\lambda_0} = \text{Spec}(S_{\lambda_0})$. This proves (1).

Proof of (2). Let $(R_0 \to S_0, M_0)$ be as in (1) and suppose that $R = \text{colim}\, R_\lambda$. Since R_0 is a finite type $\mathbf{Z}$ algebra, there exists a λ and a map $R_0 \to R_\lambda$ such that $R_0 \to R_\lambda \to R$ is the given map $R_0 \to R$ (see Lemma 127.3). Then, part (2) follows by taking $S_\lambda = R_\lambda \otimes_{R_0} S_0$ and $M_\lambda = S_\lambda \otimes_{S_0} M_0$.

Finally, we come to the proof of (3). Let $(R_\lambda \to S_\lambda, M_\lambda)$ be as in (3). Choose $(R_0 \to S_0, M_0)$ and $R_0 \to R$ as in (1). As in the proof of (2), there exists a λ_0 and a ring map $R_0 \to R_{\lambda_0}$ such that $R_0 \to R_{\lambda_0} \to R$ is the given map $R_0 \to R$. Since S_0 is of finite presentation over R_0 and since $S = \text{colim}\, S_\lambda$ we see that for some $\lambda_1 \geq \lambda_0$ we get an R_0-algebra map $S_0 \to S_{\lambda_1}$ such that the composition $S_0 \to S_{\lambda_1} \to S$ is the given map $S_0 \to S$ (see Lemma 127.3). For all $\lambda \geq \lambda_1$ this gives maps

$$\Psi_\lambda : R_\lambda \otimes_{R_0} S_0 \longrightarrow R_\lambda \otimes_{R_{\lambda_1}} S_{\lambda_1} \cong S_\lambda$$

the last isomorphism by assumption. By construction $\text{colim}_\lambda \Psi_\lambda$ is an isomorphism. Hence Ψ_λ is an isomorphism for all λ large enough by Lemma 127.8. In the same vein, there exists a $\lambda_2 \geq \lambda_1$ and an S_0-module map $M_0 \to M_{\lambda_2}$ such that $M_0 \to M_{\lambda_2} \to M$ is the given map $M_0 \to M$ (see Lemma 127.5). For $\lambda \geq \lambda_2$ there is an induced map

$$S_\lambda \otimes_{S_0} M_0 \longrightarrow S_\lambda \otimes_{S_{\lambda_2}} M_{\lambda_2} \cong M_\lambda$$

and for λ large enough this map is an isomorphism by Lemma 127.6. This implies (3) because M_0 is flat over R_0. $\square$

034Y **Lemma 168.2.** *Let $R \to A \to B$ be ring maps. Assume $A \to B$ faithfully flat of finite presentation. Then there exists a commutative diagram*

$$\begin{tikzcd} R \arrow[r] \arrow[d, equal] & A_0 \arrow[r] \arrow[d] & B_0 \arrow[d] \\ R \arrow[r] & A \arrow[r] & B \end{tikzcd}$$

with $R \to A_0$ of finite presentation, $A_0 \to B_0$ faithfully flat of finite presentation and $B = A \otimes_{A_0} B_0$.

Proof. We first prove the lemma with R replaced $\mathbf{Z}$. By Lemma 168.1 there exists a diagram

$$\begin{tikzcd} A_0 \arrow[r] & A \\ B_0 \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

where A_0 is of finite type over $\mathbf{Z}$, B_0 is flat of finite presentation over A_0 such that $B = A \otimes_{A_0} B_0$. As $A_0 \to B_0$ is flat of finite presentation we see that the image of $\mathrm{Spec}(B_0) \to \mathrm{Spec}(A_0)$ is open, see Proposition 41.8. Hence the complement of the image is $V(I_0)$ for some ideal $I_0 \subset A_0$. As $A \to B$ is faithfully flat the map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective, see Lemma 39.16. Now we use that the base change of the image is the image of the base change. Hence $I_0 A = A$. Pick a relation $\sum f_i r_i = 1$, with $r_i \in A$, $f_i \in I_0$. Then after enlarging A_0 to contain the elements r_i (and correspondingly enlarging B_0) we see that $A_0 \to B_0$ is surjective on spectra also, i.e., faithfully flat.

Thus the lemma holds in case $R = \mathbf{Z}$. In the general case, take the solution $A_0' \to B_0'$ just obtained and set $A_0 = A_0' \otimes_{\mathbf{Z}} R$, $B_0 = B_0' \otimes_{\mathbf{Z}} R$. $\square$

07RG **Lemma 168.3.** *Let $A = \mathrm{colim}_{i \in I} A_i$ be a directed colimit of rings. Let $0 \in I$ and $\varphi_0 : B_0 \to C_0$ a map of A_0-algebras. Assume*

1. *$A \otimes_{A_0} B_0 \to A \otimes_{A_0} C_0$ is finite,*
2. *C_0 is of finite type over B_0.*

Then there exists an $i \geq 0$ such that the map $A_i \otimes_{A_0} B_0 \to A_i \otimes_{A_0} C_0$ is finite.

Proof. Let $x_1, \ldots, x_m$ be generators for C_0 over B_0. Pick monic polynomials $P_j \in A \otimes_{A_0} B_0[T]$ such that $P_j(1 \otimes x_j) = 0$ in $A \otimes_{A_0} C_0$. For some $i \geq 0$ we can find $P_{j,i} \in A_i \otimes_{A_0} B_0[T]$ mapping to P_j. Since $\otimes$ commutes with colimits we see that $P_{j,i}(1 \otimes x_j)$ is zero in $A_i \otimes_{A_0} C_0$ after possibly increasing i. Then this i works. $\square$

07RH **Lemma 168.4.** *Let $A = \mathrm{colim}_{i \in I} A_i$ be a directed colimit of rings. Let $0 \in I$ and $\varphi_0 : B_0 \to C_0$ a map of A_0-algebras. Assume*

1. *$A \otimes_{A_0} B_0 \to A \otimes_{A_0} C_0$ is surjective,*
2. *C_0 is of finite type over B_0.*

Then for some $i \geq 0$ the map $A_i \otimes_{A_0} B_0 \to A_i \otimes_{A_0} C_0$ is surjective.

Proof. Let $x_1, \ldots, x_m$ be generators for C_0 over B_0. Pick $b_j \in A \otimes_{A_0} B_0$ mapping to $1 \otimes x_j$ in $A \otimes_{A_0} C_0$. For some $i \geq 0$ we can find $b_{j,i} \in A_i \otimes_{A_0} B_0$ mapping to b_j. Then this i works. $\square$

0C4F **Lemma 168.5.** *Let $A = \mathrm{colim}_{i \in I} A_i$ be a directed colimit of rings. Let $0 \in I$ and $\varphi_0 : B_0 \to C_0$ a map of A_0-algebras. Assume*

1. *$A \otimes_{A_0} B_0 \to A \otimes_{A_0} C_0$ is unramified,*
2. *C_0 is of finite type over B_0.*

Then for some $i \geq 0$ the map $A_i \otimes_{A_0} B_0 \to A_i \otimes_{A_0} C_0$ is unramified.

Proof. Set $B_i = A_i \otimes_{A_0} B_0$, $C_i = A_i \otimes_{A_0} C_0$, $B = A \otimes_{A_0} B_0$, and $C = A \otimes_{A_0} C_0$. Let $x_1, \ldots, x_m$ be generators for C_0 over B_0. Then $\mathrm{d}x_1, \ldots, \mathrm{d}x_m$ generate Ω_{C_0/B_0} over C_0 and their images generate Ω_{C_i/B_i} over C_i (Lemmas 131.14 and 131.9). Observe that $0 = \Omega_{C/B} = \mathrm{colim}\, \Omega_{C_i/B_i}$ (Lemma 131.5). Thus there is an i such that $\mathrm{d}x_1, \ldots, \mathrm{d}x_m$ map to zero and hence $\Omega_{C_i/B_i} = 0$ as desired. $\square$

0C32 **Lemma 168.6.** *Let $A = \mathrm{colim}_{i \in I} A_i$ be a directed colimit of rings. Let $0 \in I$ and $\varphi_0 : B_0 \to C_0$ a map of A_0-algebras. Assume*

1. *$A \otimes_{A_0} B_0 \to A \otimes_{A_0} C_0$ is an isomorphism,*
2. *$B_0 \to C_0$ is of finite presentation.*

Then for some $i \geq 0$ the map $A_i \otimes_{A_0} B_0 \to A_i \otimes_{A_0} C_0$ is an isomorphism.

Proof. By Lemma 168.4 there exists an i such that $A_i \otimes_{A_0} B_0 \to A_i \otimes_{A_0} C_0$ is surjective. Since the map is of finite presentation the kernel is a finitely generated ideal. Let $g_1, \ldots, g_r \in A_i \otimes_{A_0} B_0$ generate the kernel. Then we may pick $i' \geq i$ such that g_j map to zero in $A_{i'} \otimes_{A_0} B_0$. Then $A_{i'} \otimes_{A_0} B_0 \to A_{i'} \otimes_{A_0} C_0$ is an isomorphism. □

07RI **Lemma 168.7.** *Let $A = \mathrm{colim}_{i \in I} A_i$ be a directed colimit of rings. Let $0 \in I$ and $\varphi_0 : B_0 \to C_0$ a map of A_0-algebras. Assume*

1. *$A \otimes_{A_0} B_0 \to A \otimes_{A_0} C_0$ is étale,*
2. *$B_0 \to C_0$ is of finite presentation.*

Then for some $i \geq 0$ the map $A_i \otimes_{A_0} B_0 \to A_i \otimes_{A_0} C_0$ is étale.

Proof. Write $C_0 = B_0[x_1, \ldots, x_n]/(f_{1,0}, \ldots, f_{m,0})$. Write $B_i = A_i \otimes_{A_0} B_0$ and $C_i = A_i \otimes_{A_0} C_0$. Note that $C_i = B_i[x_1, \ldots, x_n]/(f_{1,i}, \ldots, f_{m,i})$ where $f_{j,i}$ is the image of $f_{j,0}$ in the polynomial ring over B_i. Write $B = A \otimes_{A_0} B_0$ and $C = A \otimes_{A_0} C_0$. Note that $C = B[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ where f_j is the image of $f_{j,0}$ in the polynomial ring over B. The assumption is that the map

$$\mathrm{d} : (f_1, \ldots, f_m)/(f_1, \ldots, f_m)^2 \longrightarrow \bigoplus C \mathrm{d}x_k$$

is an isomorphism. Thus for sufficiently large i we can find elements

$$\xi_{k,i} \in (f_{1,i}, \ldots, f_{m,i})/(f_{1,i}, \ldots, f_{m,i})^2$$

with $\mathrm{d}\xi_{k,i} = \mathrm{d}x_k$ in $\bigoplus C_i \mathrm{d}x_k$. Moreover, on increasing i if necessary, we see that $\sum (\partial f_{j,i}/\partial x_k) \xi_{k,i} = f_{j,i} \bmod (f_{1,i}, \ldots, f_{m,i})^2$ since this is true in the limit. Then this i works. □

0C0B **Lemma 168.8.** *Let $A = \mathrm{colim}_{i \in I} A_i$ be a directed colimit of rings. Let $0 \in I$ and $\varphi_0 : B_0 \to C_0$ a map of A_0-algebras. Assume*

1. *$A \otimes_{A_0} B_0 \to A \otimes_{A_0} C_0$ is smooth,*
2. *$B_0 \to C_0$ is of finite presentation.*

Then for some $i \geq 0$ the map $A_i \otimes_{A_0} B_0 \to A_i \otimes_{A_0} C_0$ is smooth.

Proof. Write $C_0 = B_0[x_1, \ldots, x_n]/(f_{1,0}, \ldots, f_{m,0})$. Write $B_i = A_i \otimes_{A_0} B_0$ and $C_i = A_i \otimes_{A_0} C_0$. Note that $C_i = B_i[x_1, \ldots, x_n]/(f_{1,i}, \ldots, f_{m,i})$ where $f_{j,i}$ is the image of $f_{j,0}$ in the polynomial ring over B_i. Write $B = A \otimes_{A_0} B_0$ and $C = A \otimes_{A_0} C_0$. Note that $C = B[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ where f_j is the image of $f_{j,0}$ in the polynomial ring over B. The assumption is that the map

$$\mathrm{d} : (f_1, \ldots, f_m)/(f_1, \ldots, f_m)^2 \longrightarrow \bigoplus C \mathrm{d}x_k$$

is a split injection. Let $\xi_k \in (f_1, \ldots, f_m)/(f_1, \ldots, f_m)^2$ be elements such that $\sum (\partial f_j/\partial x_k) \xi_k = f_j \bmod (f_1, \ldots, f_m)^2$. Then for sufficiently large i we can find elements

$$\xi_{k,i} \in (f_{1,i}, \ldots, f_{m,i})/(f_{1,i}, \ldots, f_{m,i})^2$$

with $\sum(\partial f_{j,i}/\partial x_k)\xi_{k,i} = f_{j,i} \bmod (f_{1,i}, \ldots, f_{m,i})^2$ since this is true in the limit. Then this i works. $\square$

0C33 **Lemma 168.9.** *Let $A = \mathrm{colim}_{i \in I} A_i$ be a directed colimit of rings. Let $0 \in I$ and $\varphi_0 : B_0 \to C_0$ a map of A_0-algebras. Assume*

1. *$A \otimes_{A_0} B_0 \to A \otimes_{A_0} C_0$ is syntomic (resp. a relative global complete intersection),*
2. *C_0 is of finite presentation over B_0.*

Then there exists an $i \geq 0$ such that the map $A_i \otimes_{A_0} B_0 \to A_i \otimes_{A_0} C_0$ is syntomic (resp. a relative global complete intersection).

Proof. Assume $A \otimes_{A_0} B_0 \to A \otimes_{A_0} C_0$ is a relative global complete intersection. By Lemma 136.12 there exists a finite type $\mathbf{Z}$-algebra R, a ring map $R \to A \otimes_{A_0} B_0$, a relative global complete intersection $R \to S$, and an isomorphism

$$(A \otimes_{A_0} B_0) \otimes_R S \longrightarrow A \otimes_{A_0} C_0$$

Because R is of finite type (and hence finite presentation) over $\mathbf{Z}$, there exists an i and a map $R \to A_i \otimes_{A_0} B_0$ lifting the map $R \to A \otimes_{A_0} B_0$, see Lemma 127.3. Using the same lemma, there exists an $i' \geq i$ such that $(A_i \otimes_{A_0} B_0) \otimes_R S \to A \otimes_{A_0} C_0$ comes from a map $(A_i \otimes_{A_0} B_0) \otimes_R S \to A_{i'} \otimes_{A_0} C_0$. Thus we may assume, after replacing i by i', that the displayed map comes from an $A_i \otimes_{A_0} B_0$-algebra map

$$(A_i \otimes_{A_0} B_0) \otimes_R S \longrightarrow A_i \otimes_{A_0} C_0$$

By Lemma 168.6 after increasing i this map is an isomorphism. This finishes the proof in this case because the base change of a relative global complete intersection is a relative global complete intersection by Lemma 136.10.

Assume $A \otimes_{A_0} B_0 \to A \otimes_{A_0} C_0$ is syntomic. Then there exist elements $g_1, \ldots, g_m$ in $A \otimes_{A_0} C_0$ generating the unit ideal such that $A \otimes_{A_0} B_0 \to (A \otimes_{A_0} C_0)_{g_j}$ is a relative global complete intersection, see Lemma 136.15. We can find an i and elements $g_{i,j} \in A_i \otimes_{A_0} C_0$ mapping to g_j. After increasing i we may assume $g_{i,1}, \ldots, g_{i,m}$ generate the unit ideal of $A_i \otimes_{A_0} C_0$. The result of the previous paragraph implies that, after increasing i, we may assume the maps $A_i \otimes_{A_0} B_0 \to (A_i \otimes_{A_0} C_0)_{g_{i,j}}$ are relative global complete intersections. Then $A_i \otimes_{A_0} B_0 \to A_i \otimes_{A_0} C_0$ is syntomic by Lemma 136.4 (and the already used Lemma 136.15). $\square$

The following lemma is an application of the results above which doesn't seem to fit well anywhere else.

034Z **Lemma 168.10.** *Let $R \to S$ be a faithfully flat ring map of finite presentation. Then there exists a commutative diagram*

$$\begin{tikzcd} S \arrow[rr] & & S' \\ & R \arrow[ul] \arrow[ur] & \end{tikzcd}$$

where $R \to S'$ is quasi-finite, faithfully flat and of finite presentation.

Proof. As a first step we reduce this lemma to the case where R is of finite type over $\mathbf{Z}$. By Lemma 168.2 there exists a diagram

$$\begin{tikzcd} S_0 \arrow[r] & S \\ R_0 \arrow[u] \arrow[r] & R \arrow[u] \end{tikzcd}$$

where R_0 is of finite type over $\mathbf{Z}$, and S_0 is faithfully flat of finite presentation over R_0 such that $S = R \otimes_{R_0} S_0$. If we prove the lemma for the ring map $R_0 \to S_0$, then the lemma follows for $R \to S$ by base change, as the base change of a quasi-finite ring map is quasi-finite, see Lemma 122.8. (Of course we also use that base changes of flat maps are flat and base changes of maps of finite presentation are of finite presentation.)

Assume $R \to S$ is a faithfully flat ring map of finite presentation and that R is Noetherian (which we may assume by the preceding paragraph). Let $W \subset \text{Spec}(S)$ be the open set of Lemma 130.5. As $R \to S$ is faithfully flat the map $\text{Spec}(S) \to \text{Spec}(R)$ is surjective, see Lemma 39.16. By Lemma 130.6 the map $W \to \text{Spec}(R)$ is also surjective. Hence by replacing S with a product $S_{g_1} \times \ldots \times S_{g_m}$ we may assume $W = \text{Spec}(S)$; here we use that $\text{Spec}(R)$ is quasi-compact (Lemma 17.10), and that the map $\text{Spec}(S) \to \text{Spec}(R)$ is open (Proposition 41.8). Suppose that $\mathfrak{p} \subset R$ is a prime. Choose a prime $\mathfrak{q} \subset S$ lying over $\mathfrak{p}$ which corresponds to a maximal ideal of the fibre ring $S \otimes_R \kappa(\mathfrak{p})$. The Noetherian local ring $\overline{S}_\mathfrak{q} = S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$ is Cohen-Macaulay, say of dimension d. We may choose $f_1, \ldots, f_d$ in the maximal ideal of $S_\mathfrak{q}$ which map to a regular sequence in $\overline{S}_\mathfrak{q}$. Choose a common denominator $g \in S$, $g \notin \mathfrak{q}$ of $f_1, \ldots, f_d$, and consider the R-algebra

$$S' = S_g/(f_1, \ldots, f_d).$$

By construction there is a prime ideal $\mathfrak{q}' \subset S'$ lying over $\mathfrak{p}$ and corresponding to $\mathfrak{q}$ (via $S_g \to S'_g$). Also by construction the ring map $R \to S'$ is quasi-finite at $\mathfrak{q}$ as the local ring

$$S'_{\mathfrak{q}'}/\mathfrak{p}S'_{\mathfrak{q}'} = S_\mathfrak{q}/(f_1, \ldots, f_d) + \mathfrak{p}S_\mathfrak{q} = \overline{S}_\mathfrak{q}/(\overline{f}_1, \ldots, \overline{f}_d)$$

has dimension zero, see Lemma 122.2. Also by construction $R \to S'$ is of finite presentation. Finally, by Lemma 99.3 the local ring map $R_\mathfrak{p} \to S'_{\mathfrak{q}'}$ is flat (this is where we use that R is Noetherian). Hence, by openness of flatness (Theorem 129.4), and openness of quasi-finiteness (Lemma 123.13) we may after replacing g by gg' for a suitable $g' \in S$, $g' \notin \mathfrak{q}$ assume that $R \to S'$ is flat and quasi-finite. The image $\text{Spec}(S') \to \text{Spec}(R)$ is open and contains $\mathfrak{p}$. In other words we have shown a ring S' as in the statement of the lemma exists (except possibly the faithfulness part) whose image contains any given prime. Using one more time the quasi-compactness of $\text{Spec}(R)$ we see that a finite product of such rings does the job. $\square$

169. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves

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