

Comparison of Homotopy Perturbation Sumudu Transform method and Homotopy Decomposition method for solving nonlinear Fractional Partial Differential Equations

¹Rodrigue Batogna Gnitchogna, ²Abdon Atangana,

Abstract—We paid attention to the methodology of two integral transform methods for solving nonlinear fractional partial differential equations. On one hand, the Homotopy Perturbation Sumudu Transform Method (HPSTM) is the coupling of the Sumudu transform and the HPM using He's polynomials. On the other hand, the Homotopy Decomposition Method (HDM) is the coupling of Adomian Decomposition Method and Perturbation Method. Both methods are very powerful and efficient techniques for solving different kinds of linear and nonlinear fractional differential equations arising in different fields of science and engineering. However, the HDM has an advantage over the HPSTM which is that it solves the nonlinear problems using only the inverse operator which is basically the fractional integral. Additionally there is no need to use any other inverse transform to find the components of the series solutions as in the case of HPSTM. As a consequence the calculations involved in HDM are very simple and straightforward.

Keywords—Homotopy decomposition method, Integral transforms, nonlinear fractional differential equation, Sumudu transform.

I. INTRODUCTION

Fractional Calculus has been used to model physical and engineering processes, which are found to be best described by fractional differential equations. It is worth nothing that the standard mathematical models of integer-order derivatives, including nonlinear models, do not work adequately in many cases. In the recent years, fractional calculus has played a very important role in various fields an excellent literature of this can be found in [1-10]. However, analytical solutions of these equations are quickly difficult to find.

One can find in the literature a wide class of methods dealing with approximate solutions to problems described by

nonlinear fractional differential equations, asymptotic and perturbation methods for instance. Perturbation methods carry among others the inconvenient that approximate solutions engage series of small parameters which cause difficulties since most nonlinear problems have no small parameters at all. Even though a suitable choice of small parameters occasionally lead to ideal solution, in most cases unsuitable choices lead to serious effects in the solutions. Therefore, an analytical method which does not require a small parameter in the equation modelling the phenomenon is welcome. To deal with the pitfall presented by perturbation methods for solving nonlinear equations, we present a literature review in some new asymptotic methods aiming for the search of solitary solutions of nonlinear differential equations, nonlinear differential-difference equations, and nonlinear fractional differential equations; see in [11]. The homotopy perturbation method (HPM) was first initiated by He [12]. The HPM was also studied by many authors to present approximate and exact solution of linear and nonlinear equations arising in various scientific and technological fields [13–23]. The Adomian decomposition method (ADM) [24] and variational iteration method (VIM) [25] have also been applied to study the various physical problems. The Homotopy decomposition method (HDM) was recently proposed by [26-27] to solve the groundwater flow equation and the modified fractional KDV equation [26-27]. The Homotopy decomposition method is actually the combination of perturbation method and Adomian decomposition method. Singh et al. [28] studied solutions of linear and nonlinear partial differential equations by using the homotopy perturbation Sumudu transform method (HPSTM). The HPSTM is a combination of Sumudu transform, HPM, and He's polynomials.

II. SUMUDU TRANSFORM

The Sumudu transform, is an integral transform similar to the Laplace transform, introduced in the early 1990s by Gamage K. Watugala [29] to solve differential equations and control engineering problems. It is equivalent to the Laplace-Carson transform with the substitution $p = 1/u$. Sumudu is a Sinhala word, meaning "smooth". The Sumudu transform of a function $f(t)$, defined for all real numbers $t \geq 0$, is the function $F_s(u)$, defined by:

$$(2.1).$$

¹Department of Pure and Applied Mathematics University of Namibia Bag 13301,340 MandumeNdemufayo Ave Windhoek, Namibia, E-mail: rbatogna@yahoo.fr

²Institute for Groundwater Studies, University of the Free State, P0 Box 399, Bloemfontein, South Africa, Email address: abdonatangana@yahoo.fr

$$S(f(t)) = F_s(u) = \int_0^\infty \frac{1}{u} \exp\left[-\frac{t}{u}\right] f(t) dt$$

A. Properties of Sumudu Transform [30-33]

- The transform of a Heaviside unit step function is a Heaviside unit step function in the transformed domain.
- The transform of a Heaviside unit ramp function is a Heaviside unit ramp function in the transformed domain.
- The transform of a monomial t^n is the called monomial $S(t^n) = n! u^n$.
- If $f(t)$ is a monotonically increasing function, so is $F(u)$ and the converse is true for decreasing functions.
- The Sumudu transform can be defined for functions which are discontinuous at the origin. In that case the two branches of the function should be transformed separately.
- If $f(t)$ is C^n continuous at the origin, so is the transformation $F(u)$.
- The limit of $f(t)$ as t tends to zero is equal to the limit of $F(u)$ as u tends to zero provided both limits exist.
- The limit of $f(t)$ as t tends to infinity is equal to the limit of $F(u)$ as u tends to infinity provided both limits exist.
- Scaling of the function by a factor $c > 0$ to form the function $f(ct)$ gives a transform $F(cu)$ which is the result of scaling by the same factor.

III. BASIC DEFINITION OF FRACTIONAL CALCULUS

Definition 1 A real function $f(x), x > 0$, is said to be in the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $f(x) = x^p h(x)$, where $h(x) \in C[0, \infty)$, and it is said to be in space C_μ^m if $f^{(m)} \in C_\mu, m \in \mathbb{N}$

Definition 2 The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu, \mu \geq -1$, is defined as (3.1)

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \quad \alpha > 0, x > 0$$

$$J^0 f(x) = f(x).$$

Properties of the operator can be found in [1-4] we mention only the following:

For $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma > -1$: (3.2)

$$J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x),$$

$$J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x) J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}.$$

Lemma 1 If $m - 1 < \alpha \leq m, m \in \mathbb{N}$ and $f \in C_\mu^m, \mu \geq -1$, then

$$D^\alpha J^\alpha f(x) = f(x) \text{ and,}$$

$$J^\alpha D_0^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0 \quad (3.3)$$

Definition 3: Partial Derivatives of Fractional order

Assume now that $f(x)$ is a function of n variables $x_i, i = 1, \dots, n$, also of class C on $D \in \mathbb{R}_n$. As an extension of definition 3 we define partial derivative of order α for $f(x)$ respect to x_i the function (3.4)

$$a \partial_{x_i}^\alpha f = \frac{1}{\Gamma(m-\alpha)} \int_a^{x_i} (x_i-t)^{m-\alpha-1} \partial_{x_i}^m f(x_j) |_{x_j=t} dt$$

If it exists, where $\partial_{x_i}^m$ is the usual partial derivative of integer order m .

Definition 4: The Sumudu transform of the Caputo fractional derivative is defined as follows [30-33]: (3.5)

$$S[D_t^\alpha f(t)] = u^{-\alpha} S[f(t)] - \sum_{k=0}^{m-1} u^{-\alpha+k} f^{(k)}(0^+), \quad (m-1 < \alpha \leq m)$$

IV. SOLUTION BY (HPSTM) AND (HDM)

V.I. Basic Idea of HPSTM

We illustrate the basic idea of this method, by considering a general fractional nonlinear non-homogeneous partial differential equation with the initial condition of the form of general form:

$$D_t^\alpha U(x, t) = L(U(x, t)) + N(U(x, t)) + f(x, t), \quad \alpha > 0 \quad (4.1)$$

subject to the initial condition

$$D_0^k U(x, 0) = g_k, \quad (k = 0, \dots, n - 1), D_0^n U(x, 0) = 0 \text{ and } n = [\alpha]$$

where, D_t^α denotes without loss of generality the Caputo fraction derivative operator, f is a known function, N is the general nonlinear fractional differential operator and L represents a linear fractional differential operator.

Applying the Sumudu Transform on Both sides of equation (4.1), we obtain:

$$S[D_t^\alpha U(x, t)] = S[L(U(x, t))] + S[N(U(x, t))] + S[f(x, t)] \tag{4.2}$$

Using the property of the Sumudu transform, we have

$$S[U(x, t)] = u^\alpha S[L(U(x, t))] + u^\alpha S[N(U(x, t))] + u^\alpha S[f(x, t)] + g(x, t) \tag{4.3}$$

Now applying the Sumudu inverse on both sides of (4.3) we obtain:

$$U(x, t) = S^{-1} \left[u^\alpha S[L(U(x, t))] + u^\alpha S[N(U(x, t))] \right] + G(x, t) \tag{4.4}$$

$G(x, t)$ represents the term arising from the known function $f(x, t)$ and the initial conditions.

Now we apply the HPM: (4.5)

$$U(x, t) = \sum_{n=0}^{\infty} p^n U_n(x, t)$$

The nonlinear term can be decomposed (4.6)

$$NU(x, t) = \sum_{n=0}^{\infty} p^n \mathcal{H}_n(U)$$

using the He's polynomial $\mathcal{H}_n(U)$ [22] given as: (4.7)

$$\mathcal{H}_n(U_0, \dots, U_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[N \left(\sum_{j=0}^{\infty} p^j U_j(x, t) \right) \right], n = 0, 1, 2, \dots$$

Substituting (4.5) and (4.6) $\sum_{n=0}^{\infty} p^n U_n(x, t) = G(x, t) +$

$$p \left[S^{-1} \left[u^\alpha S \left[L \left(\sum_{n=0}^{\infty} p^n U_n(x, t) \right) \right] \right] + u^\alpha S \left[N \left(\sum_{n=0}^{\infty} p^n U_n(x, t) \right) \right] \right] \tag{4.8}$$

which is the coupling of the Sumudu transform and the HPM using He's polynomials. Comparing the coefficients of like powers of p , the following approximations are obtained.

$$\begin{aligned} p^0: U_0(x, t) &= G(x, t), \\ p^1: U_1(x, t) &= S^{-1} \left[u^\alpha S \left[L(U_0(x, t)) + H_0(U) \right] \right], \\ p^2: U_2(x, t) &= S^{-1} \left[u^\alpha S \left[L(U_1(x, t)) + H_1(U) \right] \right], \\ p^3: U_3(x, t) &= S^{-1} \left[u^\alpha S \left[L(U_2(x, t)) + H_2(U) \right] \right], \\ p^n: U_n(x, t) &= S^{-1} \left[u^\alpha S \left[L(U_{n-1}(x, t)) + H_{n-1}(U) \right] \right], \end{aligned}$$

Finally, we approximate the analytical solution $U(x, t)$ by the truncated series: (4.10)

$$U(x, t) = \lim_{N \rightarrow \infty} \sum_{n=0}^N U_n(x, t)$$

The above series solution generally converges very rapidly [33]

V.II. Basic Idea of HDM [26-27]

The method first step here is to transform the fractional partial differential equation to the fractional partial integral equation by applying the inverse operator D_t^α of on both sides of equation (4.1) to obtain: (4.11)

$$\begin{aligned} U(x, t) &= \sum_{j=1}^{n-1} \frac{g_j(x)}{\Gamma(\alpha - j + 1)} t^j \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} [L(U(x, \tau)) + N(U(x, \tau)) + f(x, \tau)] d\tau \end{aligned}$$

Or in general by putting

$$\begin{aligned} \sum_{j=1}^{n-1} \frac{f_j(x)}{\Gamma(\alpha - j + 1)} t^{\alpha-j} &= f(x, t) \text{ or } f(x, t) \\ &= \sum_{j=1}^{n-1} \frac{g_j(x)}{\Gamma(\alpha - j + 1)} t^j \end{aligned}$$

We obtain:

$$\begin{aligned} (4.12) \quad U(x, t) &= T(x, t) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} [L(U(x, \tau)) + N(U(x, \tau)) + f(x, \tau)] d\tau \end{aligned}$$

In the homotopy decomposition method, the basic assumption is that the solutions can be written as a power series in p

$$U(x, t, p) = \sum_{n=0}^{\infty} p^n U_n(x, t) \quad (4.13)$$

$$U(x, t) = \lim_{p \rightarrow 1} U(x, t, p) \quad (4.14)$$

and the nonlinear term can be decomposed as

$$NU(x, t) = \sum_{n=0}^{\infty} p^n \mathcal{H}_n(U) \quad (4.14)$$

where $p \in (0, 1]$ is an embedding parameter. $\mathcal{H}_n(U)$ [22] is the He's polynomials that can be generated by

$$\begin{aligned} \mathcal{H}_n(U_0, \dots, U_n) &= \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[N \left(\sum_{j=0}^{\infty} p^j U_j(x, t) \right) \right], n \\ &= 0, 1, 2, \dots \end{aligned} \quad (4.16)$$

The homotopy decomposition method is obtained by the graceful coupling of homotopy technique with Abel integral and is given by

$$\begin{aligned} \sum_{n=0}^{\infty} p^n U_n(x, t) - T(x, t) &= \frac{p}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \left[f(x, \tau) \right. \\ &+ L \left(\sum_{n=0}^{\infty} p^n U_n(x, \tau) \right) \\ &\left. + N \left(\sum_{n=0}^{\infty} p^n U_n(x, \tau) \right) \right] d\tau \end{aligned} \quad (4.17)$$

Comparing the terms of same powers of p gives solutions of various orders with the first term:

$$U_0(x, t) = T(x, t) \quad (4.18)$$

It is worth noting that, the term $T(x, t)$ is the Taylor series of the exact solution of equation (4.1) of order $n - 1$.

V. APPLICATIONS

In this section we solve some popular nonlinear partial differential equation with both methods.

Example 1:

Let consider the following one-dimensional fractional heat-like problem: (5.1)

$$D_t^\alpha u(x, t) = \frac{1}{2} x^2 u_{xx}(x, t), 0 < x < 1, 0 < \alpha \leq 1, t > 0$$

Subject to the boundary condition:

$$u(0, t) = 0, \quad u(1, t) = \exp [t]$$

and initial condition $u(x, 0) = x^2$

Example 2

Consider the following time-fractional derivative in x, y -plane as (5.2)

$$D_t^\alpha u(x, y, t) = \frac{1}{2} \nabla^2 u(x, y, t), 1 < \alpha \leq 2, x, y \in \mathbb{R}, t > 0$$

subject to the initial conditions (5.3)

$$u(x, y, 0) = \sin(x + y), \quad u_t(x, y, 0) = -\cos(x + y)$$

Example 3

Consider the following nonlinear time-fractional gas dynamics equations [Kilicman]

$$D_t^\alpha U + \frac{1}{2} (U^2)_x - U(1 - U) = 0, 0 < \alpha \leq 1, (5.4)$$

with the initial conditions

$$U(x, 0) = \exp [-x] \quad (5.5)$$

Example 4: Consider the following three-dimensional fractional heat-like equation

$$\partial_t^\alpha u(x, y, z, t) = x^4 y^4 z^4 + \frac{1}{36} (x^2 u_{xx} + y^2 u_{yy} + z^2 u_{zz}), 0 <$$

$$x, y, z < 1, 0 < \alpha \leq 1 \quad (5.6)$$

Subject to the initial condition:

$$u(x, y, z, 0) = 0 \quad (5.7)$$

V.I. Solution via HPSTM

Example1: Apply the steps involved in HPSTM as presented in section 4.1 to equation (5.1) we obtain the following:

$$p^0: u_0(x, t) = x^2, \quad (5.8)$$

$$p^1: u_1(x, t) = S^{-1} \left[u^\alpha S \left(\frac{1}{2} x^2 (u_0)_{xx} \right) \right] = \frac{x^2 t^\alpha}{\Gamma(\alpha+1)},$$

$$p^2: u_2(x, t) = S^{-1} \left[u^\alpha S \left(\frac{1}{2} x^2 (u_1)_{xx} \right) \right] = \frac{x^2 t^{2\alpha}}{\Gamma(2\alpha+1)}, \quad (5.9)$$

$$p^3: u_3(x, t) = S^{-1} \left[u^\alpha S \left(\frac{1}{2} x^2 (u_2)_{xx} \right) \right] = \frac{x^2 t^{3\alpha}}{\Gamma(3\alpha+1)}, \quad (5.10)$$

$$p^n: u_n(x, t) = S^{-1} \left[u^\alpha S \left(\frac{1}{2} x^2 (u_n)_{xx} \right) \right] = \frac{x^2 t^{n\alpha}}{\Gamma(n\alpha+1)},$$

Therefore the series solution is given as:

$$\begin{aligned} (5.11) \quad u(x, t) = x^2 \left[1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \dots \right. \\ \left. + \frac{t^{n\alpha}}{\Gamma(3\alpha+1)} + \dots \right] \end{aligned}$$

This equivalent to the exact solution in closed form:

$$(5.12)$$

$$u(x, t) = x^2 E_{1,\alpha}(t^\alpha)$$

where $E_{1,\alpha}()$ is the Mittag-Leffler function.

Example 2: Applying the steps involved in HPSTM as presented in section 4.1 to equation (5.2) we obtain:

$$p^0: u_0(x, y, t) = \sin(x + y) - \cos(x + y)t,$$

$$p^1: u_1(x, t) = S^{-1} \left[u^\alpha S \left(\frac{1}{2} x^2 [(u_0)_{xx} + (u_0)_{yy}] \right) \right] = -\sin(x + y) \frac{t^2}{2} + \cos(x + y) \frac{t^3}{3!},$$

$$p^2: u_2(x, t) = S^{-1} \left[u^\alpha S \left(\frac{1}{2} x^2 [(u_1)_{xx} + (u_1)_{yy}] \right) \right] = \sin(x + y) \left[-\frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^{4-\alpha}}{\Gamma(5-\alpha)} \right] + \cos(x + y) \left[-\frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^{5-\alpha}}{\Gamma(6-\alpha)} \right]$$

$$p^3: u_3(x, t) = S^{-1} \left[u^\alpha S \left(\frac{1}{2} x^2 [(u_2)_{xx} + (u_2)_{yy}] \right) \right] = \sin(x + y) \left[-\frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \frac{2t^{4-\alpha}}{\Gamma(5-\alpha)} - \frac{2t^{6-\alpha}}{\Gamma(7-\alpha)} - \frac{4^{\alpha-2} \sqrt{\pi} t^{6-2\alpha}}{(6-2\alpha)(5-2\alpha)\Gamma(3-\alpha)\Gamma(2.5-\alpha)} \right] + \cos(x + y) \left[\frac{t^3}{3!} - \frac{t^5}{5!} + \frac{t^7}{7!} + \frac{2t^{7-\alpha}}{\Gamma(7-\alpha)} + \frac{2t^{7-2\alpha}}{\Gamma(8-2\alpha)} \right],$$

Therefore the series solution is given as:

$$(5.13)$$

$$u(x, y, t) = \sin(x + y) \left[1 - \frac{3t^2}{2!} + \frac{t^4}{8} + \frac{t^6}{6!} + \frac{3t^{4-\alpha}}{\Gamma(5-\alpha)} - \frac{2t^{6-\alpha}}{\Gamma(7-\alpha)} - \frac{4^{\alpha-2} \sqrt{\pi} t^{6-2\alpha}}{(6-2\alpha)(5-2\alpha)\Gamma(3-\alpha)\Gamma(2.5-\alpha)} \right] + \cos(x + y) \left[-t + \frac{t^3}{3!} - \frac{t^5}{5!} + \frac{t^7}{7!} + \frac{3t^{7-\alpha}}{\Gamma(7-\alpha)} + \frac{t^{7-2\alpha}}{\Gamma(8-2\alpha)} \right] + \dots$$

It is important to point out that if $\alpha = 2$ the above solution takes the form: (5.14)

$$u_{N=4}(x, y, t) = \sin(x + y) \left[1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} \right] - \cos(x + y) \left[t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} \right]$$

which is the first four terms of the series expansion of the exact solution $u(x, y, t) = \sin(x + y - t)$

Example 3: Apply the steps involved in HPSTM as presented in section 4.1 to equation (5.4) Kilicman et al [33] obtained the following:

$$p^0: u_0(x, t) = \exp(-x),$$

$$p^1: u_1(x, t) = S^{-1} \left[u^\alpha S \left(\frac{1}{2} x^2 (u_0)_{xx} \right) \right] = \frac{\exp(-x)t^\alpha}{\Gamma(\alpha+1)},$$

$$p^2: u_2(x, t) = S^{-1} \left[u^\alpha S \left(\frac{1}{2} x^2 (u_1)_{xx} \right) \right] = \frac{\exp(-x)t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$p^3: u_3(x, t) = S^{-1} \left[u^\alpha S \left(\frac{1}{2} x^2 (u_2)_{xx} \right) \right] = \frac{\exp(-x)t^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$(5.15)$$

$$p^n: u_n(x, t) = S^{-1} \left[u^\alpha S \left(\frac{1}{2} x^2 (u_n)_{xx} \right) \right] = \frac{\exp(-x)t^{n\alpha}}{\Gamma(n\alpha+1)},$$

Therefore the series solution is given as:

$$(5.16)$$

$$u(x, t) = \exp(-x) \left[1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \dots + \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} + \dots \right]$$

Example 4: Applying the steps involved in HPSTM as presented in section 4.1 to equation (5.2) we obtain:

$$p^0: u_0(x, t) = x^4 y^4 z^4$$

$$p^1: u_1(x, t) = S^{-1} \left[u^\alpha S \left(\frac{1}{36} (x^2 (u_0)_{xx} + y^2 (u_0)_{yy} + z^2 (u_0)_{zz}) \right) \right] = \frac{t^\alpha x^4 y^4 z^4}{\Gamma(\alpha+1)}$$

$$p^2: u_2(x, t) = S^{-1} \left[u^\alpha S \left(\frac{1}{36} (x^2 (u_1)_{xx} + y^2 (u_1)_{yy} + z^2 (u_1)_{zz}) \right) \right] = \frac{t^{2\alpha} x^4 y^4 z^4}{\Gamma(2\alpha+1)} p^3: u_3(x, t)$$

$$= S^{-1} \left[u^\alpha S \left(\frac{1}{36} (x^2 (u_2)_{xx} + y^2 (u_2)_{yy} + z^2 (u_2)_{zz}) \right) \right] = \frac{t^{3\alpha} x^4 y^4 z^4}{\Gamma(3\alpha+1)}$$

$$(5.17)$$

$$p^n: u_n(x, t) = S^{-1} \left[u^\alpha S \left(\frac{1}{36} (x^2 (u_{n-1})_{xx} + y^2 (u_{n-1})_{yy} + z^2 (u_{n-1})_{zz}) \right) \right] = \frac{t^{n\alpha} x^4 y^4 z^4}{\Gamma(n\alpha+1)}$$

Therefore the approximate solution of equation for the first n is given below as:

$$u_N(x, y, z, t) = \sum_{n=1}^N \frac{t^{n\alpha} x^4 y^4 z^4}{\Gamma(n\alpha+1)}. (5.18)$$

V.II. Solution via HDM

Example 1: Apply the steps involved in HDM as presented in section 4.2 to equation (5.1) we obtain the following (5.19)

$$\sum_{n=0}^{\infty} p^n u_n(x, t) - x^2 = \frac{p}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \left[x^2 \sum_{n=0}^{\infty} p^n \frac{\partial^2 u_n(x, \tau)}{\partial x^2} \right] d\tau$$

Comparing the terms of the same powers of p we obtain: (5.20)

$$\begin{aligned} u_0(x, t) &= x^2 \\ u_1(x, t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \frac{x^2 \partial^2 u_0(x, \tau)}{\partial x^2} d\tau = \frac{x^2 t^{\alpha}}{\Gamma(\alpha + 1)} \\ u_2(x, t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \frac{x^2 \partial^2 u_1(x, \tau)}{\partial x^2} d\tau = \frac{x^2 t^{2\alpha}}{\Gamma(2\alpha + 1)} \\ u_3(x, t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \frac{x^2 \partial^2 u_2(x, \tau)}{\partial x^2} d\tau = \frac{x^2 t^{3\alpha}}{\Gamma(2\alpha + 1)} \\ u_n(x, t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \frac{x^2 \partial^2 u_{n-1}(x, \tau)}{\partial x^2} d\tau = \frac{x^2 t^{n\alpha}}{\Gamma(n\alpha + 1)} \end{aligned}$$

The asymptotic solution is given by

$$\begin{aligned} u_N(x, t) &= x^2 \left[\frac{t^{\alpha}}{\Gamma(\alpha + 1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} + \dots + \frac{t^{N\alpha}}{\Gamma(N\alpha + 1)} + \dots \right] \\ \lim_{N \rightarrow \infty} u_N(x, t, \alpha) &= u(x, t, \alpha) \quad (5.20) \\ \lim_{\substack{\alpha \rightarrow 1 \\ N \rightarrow \infty}} u_N(x, t, \alpha) &= x^2 \exp(t) \end{aligned}$$

This is the exact solution of (5.1) when $\alpha = 1$.

Example 2

Following the discussion presented earlier, applying the initial conditions and comparing the terms of the same power of p, integrating, we obtain the following solutions:

$$\begin{aligned} u_0(x, t) &= \sin(x + y) - \cos(x + y)t \\ u_1(x, t) &= -\sin(x + y) \frac{t^2}{2} + \cos(x + y) \frac{t^3}{3!} \\ u_2(x, t) &= \sin(x + y) \left[-\frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^{4-\alpha}}{\Gamma(5-\alpha)} \right] + \cos(x + y) \left[-\frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^{5-\alpha}}{\Gamma(6-\alpha)} \right] \quad (5.21) \\ u_3(x, t) &= \sin(x + y) \left[-\frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \frac{2t^{4-\alpha}}{\Gamma(5-\alpha)} - \frac{2t^{6-\alpha}}{\Gamma(7-\alpha)} - \frac{4^{\alpha-2} \sqrt{\pi} t^{6-2\alpha}}{(6-2\alpha)(5-2\alpha)\Gamma(3-\alpha)\Gamma(2.5-\alpha)} \right] \\ &\quad + \cos(x + y) \left[\frac{t^3}{3!} - \frac{t^5}{5!} + \frac{t^7}{7!} + \frac{2t^{7-\alpha}}{\Gamma(7-\alpha)} + \frac{2t^{7-2\alpha}}{\Gamma(8-2\alpha)} \right] \end{aligned}$$

Using the package Mathematica, in the same manner one can obtain the rest of the components. But for four terms were computed and the asymptotic solution is given by: (5. 22)

$$\begin{aligned} u(x, y, t) &= \sin(x + y) \left[1 - \frac{3t^2}{2!} + \frac{t^4}{8} + \frac{t^6}{6!} + \frac{3t^{4-\alpha}}{\Gamma(5-\alpha)} - \frac{2t^{6-\alpha}}{\Gamma(7-\alpha)} - \frac{4^{\alpha-2} \sqrt{\pi} t^{6-2\alpha}}{(6-2\alpha)(5-2\alpha)\Gamma(3-\alpha)\Gamma(2.5-\alpha)} \right] \\ &\quad + \cos(x + y) \left[-t + \frac{t^3}{3!} - \frac{t^5}{5!} + \frac{t^7}{7!} + \frac{3t^{7-\alpha}}{\Gamma(7-\alpha)} + \frac{t^{7-2\alpha}}{\Gamma(8-2\alpha)} + \dots \right] \end{aligned}$$

It is important to point out that if $\alpha = 2$ the above solution takes the form:

$$\begin{aligned} u_{N=4}(x, y, t) &= \sin(x + y) \left[1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} \right] - \cos(x + y) \left[t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} \right] \end{aligned}$$

Which are the first four terms of the series expansion of the exact solution $u(x, y, t) = \sin(x + y - t)$

Example 3:

$$\begin{aligned} p^0: u_0(x, t) &= \exp(-x), \\ p^1: u_1(x, t) &= \frac{\exp(-x)}{\Gamma(\alpha+1)} t^{\alpha}, \\ p^2: u_2(x, t) &= \frac{\exp(-x)}{\Gamma(2\alpha+1)} t^{2\alpha}, \\ p^3: u_3(x, t) &= \frac{\exp(-x)}{\Gamma(3\alpha+1)} t^{3\alpha}, \end{aligned}$$

(5.23)

∴

$$p^n: u_n(x, t) = \frac{\exp(-x)}{\Gamma(n\alpha+1)} t^{n\alpha},$$

Therefore the series solution is given as:

(5.24)

$$u(x, t) = \exp(-x) \left[1 + \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} + \dots + \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)} + \dots \right]$$

Example 4: Following carefully the steps involved in the HDM, we arrive at the following equations

$$\begin{aligned} & \sum_{n=0}^{\infty} p^n u_n(x, y, z, t) \\ &= \frac{p}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \left(x^4 y^4 z^4 \right. \\ &+ \frac{1}{36} \left(x^2 \left(\sum_{n=0}^{\infty} p^n u_n(x, y, z, t) \right)_{xx} \right. \\ &+ y^2 \left(\sum_{n=0}^{\infty} p^n u_n(x, y, z, t) \right)_{yy} \\ &+ \left. \left. z^2 \left(\sum_{n=0}^{\infty} p^n u_n(x, y, z, t) \right)_{zz} \right) \right) d\tau \end{aligned}$$

(5.25)

Now comparing the terms of the same power of p yields:

$$p^0: u_0(x, y, z, t) \tag{5.26}$$

$$p^1: u_1(x, y, z, t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} x^4 y^4 z^4 d\tau$$

⋮

$$p^n: u_n(x, y, z, t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \left(\frac{1}{36} (x^2 (u_{n-1})_{xx} + y^2 (u_{n-1})_{yy} + z^2 (u_{n-1})_{zz}) \right) d\tau, u_n(x, y, z, 0) = 0, n \geq 2$$

Thus the following components are obtained as results of the above integrals

$$u_0(x, y, z, t) = 0$$

$$u_1(x, y, z, t) = \frac{t^\alpha x^4 y^4 z^4}{\Gamma(\alpha + 1)}$$

$$u_2(x, y, z, t) = \frac{t^{2\alpha} x^4 y^4 z^4}{\Gamma(2\alpha + 1)}$$

$$u_3(x, y, z, t) = \frac{t^{3\alpha} x^4 y^4 z^4}{\Gamma(3\alpha + 1)}$$

⋮

$$u_n(x, y, z, t) = \frac{t^{n\alpha} x^4 y^4 z^4}{\Gamma(n\alpha + 1)} \tag{5.27}$$

Therefore the approximate solution of equation for the first n is given below as:

$$u_N(x, y, z, t) = \sum_{n=1}^N \frac{t^{n\alpha} x^4 y^4 z^4}{\Gamma(n\alpha + 1)}$$

Now when $N \rightarrow \infty$ we obtained the follow solution (5.28)

$$\begin{aligned} u(x, y, z, t) &= \sum_{n=0}^{\infty} \frac{t^{n\alpha} x^4 y^4 z^4}{\Gamma(n\alpha + 1)} - x^4 y^4 z^4 \\ &= x^4 y^4 z^4 (E_\alpha(t^\alpha) - 1) \end{aligned}$$

Where $E_\alpha(t^\alpha)$ is the generalized Mittag-Leffler function. Note that in the case $\alpha = 1$

$$u(x, y, z, t) = x^4 y^4 z^4 (\exp(t) - 1) \tag{5.29}$$

This is the exact solution for this case.

VI. COMPARISON OF METHODS

This section is devoted to the comparison between the two integral transform methods.

The two methods are very powerful and efficient techniques for solving different kinds of linear and nonlinear fractional differential equations arising in different fields of science and engineering. However, it can be noted that the HDM has an advantage over the HPSTM which is that it solves the nonlinear problems using only the inverse operator which is simply the fractional integral. There is no need to use any other inverse transform to find the components of the series solutions as in the case of HPSTM. In addition the calculations involved in HDM are very simple and straightforward. In conclusion, the HDM and the HPSTM may be considered as a nice refinement in existing numerical techniques and might find wide applications.

Table 1: Numerical results of equation (5.2) via mathematica

t	x	y	HPSTM and HDM $\alpha = 1.25$	HPSTM and HDM $\alpha = 1.75$	HPSTM and HDM $\alpha = 2.0$	Exact	Errors
0.2	0.	0.	0.6436	0.6660	0.6816	0.6816	0
	5	5	24	50	39	39	0
	0.	1.	0.9015	0.9294	0.9489	0.9489	0
	5	0	11	40	85	85	0
	1.	0.	0.9015	0.9294	0.9489	0.9489	
	0	5	11	40	85	85	
	1.	1.	0.9386	0.9652	0.9839	0.9839	
	0	0	76	71	86	86	
0.5	0.	0.	0.3665	0.4402	0.4794	0.4794	0.0000
	5	5	63	70	25	26	06
	0.	1.	0.6913	0.7888	0.8414	0.8414	0
	5	0	70	06	71	71	0
	1.	0.	0.6913	0.7888	0.8414	0.8414	0
	0	5	70	06	71	71	
	1.	1.	0.8469	0.9442	0.9974	0.9974	
	0	0	06	15	95	95	
0.7	0.	0.	0.0670	0.1925	0.2474	0.2474	0.0000
	5	5	24	05	02	04	02
	1.	1.	0.4215	0.6004	0.6816	0.6816	0.0000
	0	0	20	31	36	39	03
	0.	0.	0.4215	0.6004	0.6816	0.6816	0.0000
	5	5	20	31	36	39	03
	1.	1.	0.6728	0.8613	0.9489	0.9489	0.0000
	0	0	13	1	82	85	03
1.0	0.	0.	-	-	-	0.0000	0.0000
	5	5	0.2082	0.0543	0.0000	00	19
	1.	1.	46	57	019	0.4794	0.0000
	0	0	0.1417	0.3857	0.4794	26	25
	0.	0.	97	25	01	0.4794	0.0000
	5	5	0.1417	0.3857	0.4794	26	25
	1.	1.	97	25	01	0.8414	0
	0	0	0.4571	0.7313	0.8414	48	
		23	69	48			

The approximate solution of equation (5.2) obtained by the present methods is close at hand to the exact solution. It is to

be noted that only the fourth-order term of the HDM and HPSTM were used to evaluate the approximate solutions for Figures 1. It is evident that the efficiency of the present method can be noticeably improved by computing additional terms of $u(x, t)$ when the HDM is used.

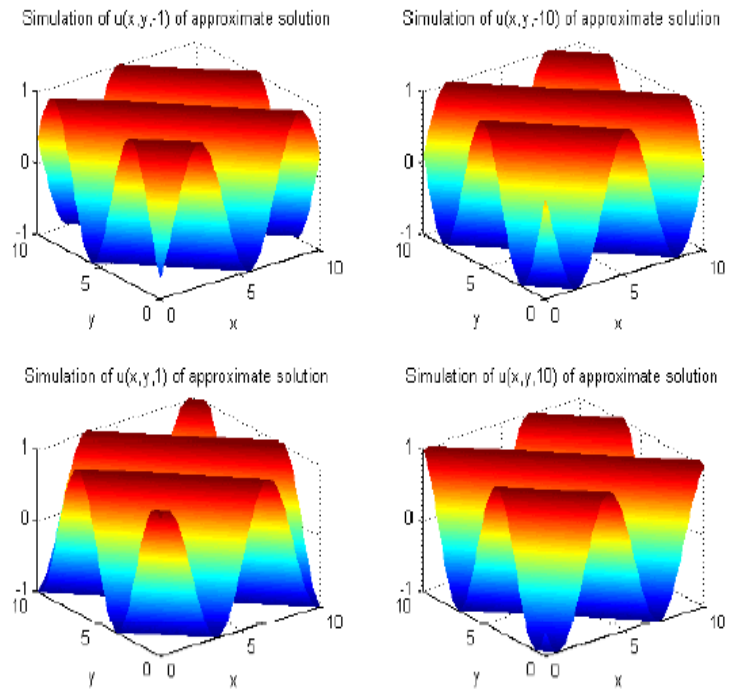


Figure 1: Numerical simulation of the approximated solution of equation (5.2)

VII. CONCLUSION

We studied two integral transform methods for solving fractional nonlinear partial differential equation. The first method namely homotopy perturbation Sumudu transform method is the coupling of the Sumudu transform and the HPM using He's polynomials. The second method namely Homotopy decomposition method is the combination of Adomian decomposition method and HPM using He's polynomials. These two methods are very powerful and efficient techniques for solving different kinds of linear and nonlinear fractional differential equations arising in different fields of science and engineering. However, the HDM has an advantage over the HPSTM which is that it solves the nonlinear problems using only the inverse operator which is simple the fractional integral. Also we do not need to use any order inverse transform to find the components of the series solutions as in the case of HPSTM. In addition the calculations involved in HDM are very simple and straightforward. In conclusion the HDM is a friendlier method.

REFERENCES

- [1] K. B. Oldham and J. Spanier, "The Fractional Calculus", Academic Press, New York, NY, USA, (1974).
- [2] I. Podlubny, "Fractional Differential Equations", Academic Press, New York, NY, USA, (1999).
- [3] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, "Theory and Applications of Fractional Differential Equations", Elsevier, Amsterdam, The Netherlands, (2006).
- [4] I. Podlubny, "Fractional Differential Equations", Academic Press, San Diego, Calif, USA, (1999).
- [5] M. Caputo, "Linear models of dissipation whose Q is almost frequency independent, part II," *Geophysical Journal International*, vol. 13, no. 5, pp. 529–539, (1967).
- [6] A. A. Kilbas, H. H. Srivastava, and J. J. Trujillo, "Theory and Applications of Fractional Differential Equations", Elsevier, Amsterdam, The Netherlands, (2006).
- [7] K. S. Miller and B. Ross, "An Introduction to the Fractional Calculus and Fractional Differential Equations", Wiley, New York, NY, USA, (1993).
- [8] S. G. Samko, A. A. Kilbas, and O. I. Marichev, "Fractional Integrals and Derivatives: Theory and Applications", Gordon and Breach, Yverdon, Switzerland, (1993).
- [9] G. M. Zaslavsky, "Hamiltonian Chaos and Fractional Dynamics", Oxford University Press, (2005).
- [10] A. Yildirim, "An algorithm for solving the fractional nonlinear Schrödinger equation by means of the homotopy perturbation method," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 10, no. 4, (2009). pp. 445–450,
- [11] J. H. He, "Asymptotic methods for solitary solutions and compactons," *Abstract and Applied Analysis*, vol. 2012, (2012). Article ID 916793, 130 pages,
- [12] J.-H. He, "Homotopy perturbation technique," *Computer Methods in Applied Mechanics and Engineering*, vol. 178, no. 3-4, (1999), pp.257–262.
- [13] J.-H. He, "Homotopy perturbation method: a new nonlinear analytical technique," *Applied Mathematics and Computation*, vol. 135, no. 1, (2003). pp. 73–79.
- [14] J.-H. He, "New interpretation of homotopy perturbation method. Addendum: 'some asymptotic methods for strongly nonlinear equations'," *International Journal of Modern Physics B*, vol. 20, no. 18, (2006). pp. 2561–2568.
- [15] D. D. Ganji, "The application of He's homotopy perturbation method to nonlinear equations arising in heat transfer," *Physics Letters A*, vol. 355, no. 4-5, (2006). pp. 337–341.
- [16] A. Yildirim, "An algorithm for solving the fractional nonlinear Schrödinger equation by means of the homotopy perturbation method," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 10, no. 4, (2009), pp. 445–450.
- [17] D. D. Ganji and M. Rafei, "Solitary wave solutions for a generalized Hirota-Satsuma coupled KdV equation by homotopy perturbation method," *Physics Letters A*, vol. 356, no. 2, (2006)., pp. 131– 137.
- [18] M. M. Rashidi, D. D. Ganji, and S. Dinarvand, "Explicit analytical solutions of the generalized Burger and Burger-Fisher equations by homotopy perturbation method," *Numerical Methods for Partial Differential Equations*, vol. 25, no. 2, (2009). pp. 409–417
- [19] H. Aminikhah and M. Hemmatnezhad, "An efficient method for quadratic Riccati differential equation," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 4, (2010). pp. 835–839.
- [20] S. H. Kachapi and D. D. Ganji, *Nonlinear Equations: Analytical Methods and Applications*, Springer, (2012).
- [21] Atangana Abdon. "New Class of Boundary Value Problems", *Inf. Sci. Lett.* Vol 1 no. 2, (2012) pp 67-76
- [22] Y. M. Qin and D.Q. Zeng, "Homotopy perturbation method for the q-diffusion equation with a source term," *Communications in Fractional Calculus*, vol. 3, no. 1, (2012). pp. 34–37,
- [23] M. Javidi and M. A. Raji, "Combination of Laplace transform and homotopy perturbation method to solve the parabolic partial differential equations," *Communications in Fractional Calculus*, vol. 3, no. 1, , (2012).pp. 10–19.
- [24] J. S. Duan, R. Rach, D. Buleanu, and A. M. Wazwaz, "A review of the Adomian decomposition method and its applications to fractional differential equations," *Communications in Fractional Calculus*, vol. 3, no. 2, (2012). pp. 73–99.
- [25] D.D. Ganji, "Asemi-Analytical technique for non-linear settling particle equation of motion," *Journal of Hydro-Environment Research*, vol. 6, no. 4, (2012). pp. 323–327,
- [26] A. Atangana and Aydin Secer. "Time-fractional Coupled-the Korteweg-de Vries Equations" *Abstract Applied Analysis*, In press (2013)
- [27] Atangana A., Botha J.F. "Analytical solution of groundwater flow equation via Homotopy Decomposition Method, *J Earth Sci Climate Change*, vol 3, (2012) pp 115. doi:10.4172/2157-7617.1000115
- [28] J. Singh, D. Kumar, and Sushila, "Homotopy perturbation Sumudu transform method for nonlinear equations,"

Advances in Applied Mathematics and Mechanics, vol. 4, (2011), pp. pp 165–175.

[29] Watugala, G. K., “Sumudu transform: a new integral transform to solve differential equations and control engineering problems.” *International Journal of Mathematical Education in Science and Technology* vol 24, (1993), pp 35–43.

[30] Weerakoon, S., “Application of Sumudu transform to partial differential equations” *International Journal of Mathematical Education in Science and Technology*, vol 25 (1994), pp 277–283.

31-Asiru, M.A. “Classroom note: Application of the Sumudu transform to discrete dynamic systems” *International Journal of Mathematical Education in Science and Technology*, vol 34 no 6, (2003) , pages. 944-949

[32] Airu, M.A. “Further properties of the Sumudu transform and its applications” *International Journal of Mathematical Education in Science and Technology*, vol 33 no 3, (2002), pp. 441-449

[33] Jagdev Singh, Devendra Kumar, and A. Kılıçman “Homotopy Perturbation Method for Fractional Gas Dynamics Equation Using Sumudu Transform”. *Abstract and Applied Analysis* Volume 2013 (2013), pp 8