Transactions on Modelling and Simulation vol 35, © 2003 WIT Press, www.witpress.com, ISSN 1743-355X

Comparison of the performance of two error indicators for 2-D elastostatic problems

K. H. Muci-Küchler[1] & J. C. Miranda-Valenzuela[2]
[1]*Mechanical Engineering Department, South Dakota School of Mines and Technology, USA*
[2]*Departamento de Ingeniería Mecánica, ITESM – Campus Toluca, Mexico*

Abstract

The application of adaptive meshing techniques for the design of mechanical components is gaining increased importance in industry. Nowadays many companies are following structured product development methodologies that involve the evaluation of several alternative product concepts in a short amount of time. Under these circumstances, product design engineers are faced with the task of performing the stress analysis of the proposed concepts using numerical methods such as the FEM or the BEM. The key difficulty is that they do not necessarily have all the knowledge and expertise that is required to use those methods in the most adequate and cost-effective way. In order to overcome this difficulty, different adaptive meshing techniques have been proposed. The goal of these techniques is to provide, in an automatic fashion, accurate values for the stresses using as a starting point a model with a coarse mesh that only has enough elements to represent in an appropriate fashion the geometry and the boundary conditions of the problem. A key aspect in any adaptive meshing process is how to estimate the error in the numerical solution for the stresses in each element. In this regard, in recent years two alternatives that involve the use of the Tangent Derivative BIEs have been proposed. One makes direct use of the results provided by Hermite elements to estimate the error in the numerical solution for the stresses. The other uses a global reanalysis technique to improve the accuracy of the stresses provided by conventional elements and, at the same time, estimate the error in the solution. In this paper, the stress analysis of a bracket under plane stress conditions is used as a test case to compare the performance of those error indicators in the context of a practical application.

Transactions on Modelling and Simulation vol 35, © 2003 WIT Press, www.witpress.com, ISSN 1743-355X

1 Introduction

Nowadays it is widely recognized that to develop successful products it is extremely important to follow a structured product development process (PDP). The different phases that are typically followed in the development of products of moderate complexity can be summarized using the model proposed by Ulrich and Eppinger [1]. According to this model, the activities that take place during the PDP can be grouped into five phases: Concept development, system-level design, detail design, testing and refinement, and production ramp-up. The concept development phase is of particular interest since during this phase alternative product concepts are proposed and one is selected for further development. In general, the methods commonly used for concept selection rely on the qualitative evaluation of the performance of each concept with respect to the selection criteria. Although this approach is useful to narrow down a large number of concepts to the most promising ones, it may be prone to error when a finer resolution is needed to compare, in an objective fashion, several concepts that have a good performance with respect to most of the selection criteria.

In the development of mechanical components, it is very common to have among the important selection criteria items related to the stresses experienced by the part. In most cases, the complexity of the geometry, the types of loads and constraints, and/or other factors, do not allow an engineer to obtain an analytical solution. Thus, the only way to determine the value of the stresses without using physical prototypes is by using numerical methods like the Finite Element Method (FEM) or the Boundary Element Method (BEM). The difficulty is that, in general, the core product development team does not include engineers with the expertise required to obtain accurate results with those methods. Also, requesting help from experts outside the core team may be difficult since several designs have to be considered and the available resources may be limited.

Based on the preceding discussion, it is evident that there is a need for software tools that can perform in an automated fashion the stress analysis of a mechanical component using methods like the FEM or the BEM. The idea behind those tools is to provide accurate results for the stresses with minimum input and intervention from the user. Starting with a basic model, the software should perform, in an automatic fashion, all the steps that are needed to obtain the results for the stresses within a prescribed tolerance. In order to create in a reliable an efficient fashion the sequence of meshes required to achieve this goal, robust adaptive meshing techniques must be used.

Active research in the field of adaptive meshing with boundary elements started in the early 1980's and, since then, many papers dealing with this subject have been published. Kita and Kamiya [2] and Miranda-Valenzuela and Muci-Küchler [3] have presented a comprehensive summary of references related to this topic. For a variety of problems, different alternatives to estimate the error in the numerical solution corresponding to a given mesh have been explored. Also, several strategies on how to improve the discretization used in a given model have been proposed and tested. A key aspect in any adaptive meshing process is how to estimate the error in the numerical solution inside each element. In this

Transactions on Modelling and Simulation vol 35, © 2003 WIT Press, www.witpress.com, ISSN 1743-355X

regard, in recent years two alternatives that involve the use of the Tangent Derivative Boundary Integral Equations (TBIEs) have been proposed for the two-dimensional elastostatic problem. One makes direct use of the results provided by Hermite elements to estimate the error in the numerical solution for the stresses [4]. The other uses a global reanalysis technique to improve the accuracy of the stresses and, at the same time, estimate the error in the solution [5]. In this paper, an overview of those methods is presented and the stress analysis of a mechanical component is used as test case to compare their capability to lead adaptive meshing processes in the context of a practical application.

2 Boundary element formulation

In the case of the boundary elements that are commonly used for the solution of elastostatic problems, the displacements and the tractions are approximated inside each element using the nodal values of those quantities together with Lagrange shape functions. If we let $\vec{w}$ represent either the displacement or the traction vector, the components of $\vec{w}$ are approximated inside each element as:

$$w_i(\eta) = \sum_{n=1}^{N} L_n(\eta)\, w_i^{(n)} \tag{1}$$

where N is the number of functional nodes in the element, L_n are the shape functions, $w_i^{(n)}$ are the nodal values of w_i, and η is a local coordinate on the element. Since, this type of approximation gives rise to two unknowns per functional node, the Conventional Boundary Integral Equations (CBIEs) are collocated at those locations to generate the system of equations required to obtain a solution. For the two-dimensional elastostatic problem without body forces, the CBIEs can be written as [6]:

$$\int_S T_{ij}(\vec{x},\vec{\xi})[u_j(\vec{x}) - u_j(\vec{\xi})]\, dS(\vec{x}) = \int_S U_{ij}(\vec{x},\vec{\xi})\, t_j(\vec{x})\, dS(\vec{x}) \tag{2}$$

where the range of indices goes from 1 to 2 and the summation convention is in force. Here, $\vec{x}$ represents the field point, $\vec{\xi}$ is the source point, and u_i and t_i are the displacement and traction components. U_{ij} and T_{ij} are the standard displacement and tractions kernels and their definition can be found in several references including [7].

Hermite elements include the nodal values of the tangential derivative of the field variables as additional degrees of freedom in the functional representation for those quantities. Thus, the field variables are approximated inside each element as:

$$w_i(\eta)=\sum_{n=1}^{N}\left[H_n^{(1)}(\eta)\,w_i^{(n)}+H_n^{(2)}(\eta)\frac{\partial w_i^{(n)}}{\partial s}\right] \tag{3}$$

where $H_n^{(m)}$ are the shape functions, and $\partial w_i^{(n)}/\partial s$ are the nodal values of the tangential derivative of w_i. Since in this case there are four unknowns associated with each functional node, the CBIEs and the TDBIEs are simultaneously collocated at those locations in order to generate the system of equations required to find a solution. For the two-dimensional elastostatic problem, the TDBIEs, in their completely regularized form, are given by [6]:

$$\begin{gathered}\int_S V_{ij}(\vec{x},\vec{\xi})\left[u_j(\vec{x})-u_j(\vec{\xi})-\frac{\partial u_j}{\partial\zeta}r_m\zeta_m\right]dS(\vec{x})+\\ \int_S Y_{ij}(\vec{x},\vec{\xi})\,dS(\vec{x})\frac{\partial u_j}{\partial\zeta}(\vec{\xi})=\int_S[W_{ij}^o(\vec{x},\vec{\xi})t_j(\vec{x})-W_{ij}(\vec{x},\vec{\xi})t_j(\vec{\xi})]\,dS(\vec{x})\end{gathered} \tag{4}$$

where ζ_i are the components of the unit vector tangent to S at $\vec{\xi}$ and, for plane strain, the kernels V_{ij}, W_{ij}^o, W_{ij}, and Y_{ij} are given by [6]:

$$V_{ij}(\vec{x},\vec{\xi})=\zeta_k\frac{\partial T_{ij}}{\partial\xi_k}\ ;\ W_{ij}^o(\vec{x},\vec{\xi})=\zeta_k\frac{\partial U_{ij}}{\partial\xi_k}\ ;\ W_{ij}(\vec{x},\vec{\xi})=s_k\frac{\partial U_{ij}}{\partial\xi_k} \tag{5}$$

$$\begin{aligned}Y_{ij}(\vec{x},\vec{\xi})=\mu\Bigg\{&\frac{2\nu}{1-2\nu}\zeta_j(\zeta_k n_l-s_k\nu_l)\frac{\partial U_{il}}{\partial\xi_k}+\zeta_k n_l\zeta_l\frac{\partial U_{ij}}{\partial\xi_k}+\\ &\zeta_l(\zeta_k n_j-s_k\nu_j)\frac{\partial U_{il}}{\partial\xi_k}\Bigg\}\end{aligned} \tag{6}$$

In the above expressions $\vec{r}=\vec{x}-\vec{\xi}$, μ is the shear modulus, ν is the Poisson's ratio, s_i and n_i are the components of the normal and tangent unit vectors to S at $\vec{x}$, and ν_i are the components of the unit outward normal vector at $\vec{\xi}$.

Once all the degrees of freedom that are associated with the functional representation for the displacements and the tractions are known, it is possible to find the components of the stress tensor at any point on the boundary in terms of a local coordinate system. For plane strain, the in-plane stresses are given by:

$$\sigma_{(n)(n)}=t_i n_i\ ;\ \sigma_{(n)(s)}=t_i s_i\ ;\ \sigma_{(s)(s)}=\frac{1}{1-\nu}[2\mu\varepsilon_{(s)(s)}+\nu\sigma_{(n)(n)}] \tag{7}$$

Although $\sigma_{(n)(n)}$ and $\sigma_{(n)(s)}$ are found in a straightforward fashion, the computation of $\sigma_{(s)(s)}$ requires the prior calculation of the normal strain in the tangential direction $\varepsilon_{(s)(s)}=(\partial u_i/\partial s)s_i$. In general, the values of $\partial u_i/\partial s$ are

Transactions on Modelling and Simulation vol 35, © 2003 WIT Press, www.witpress.com, ISSN 1743-355X

obtained through the differentiation of the functional representation for the displacements. The only exception is when Hermite elements are used and the point under consideration is one of the functional nodes of the mesh.

For the Hermite elements an error indicator can be obtained as follows. Starting from the solution obtained using Hermite elements, a second, less accurate, "reduced" solution is generated without running another analysis by treating each Hermite element as if it was a conventional one with the same number and distribution of nodes. For that purpose, the nodal values of the displacements and the tractions that were obtained with the Hermite elements are used together with Lagrange shape functions to approximate the displacements and the tractions inside each element using eqn (1). Finally, the stresses corresponding to the Hermite and the "reduced" solution are computed and used to estimate the error inside each element as:

$$\lambda^{(e)} = \left(\int_{S^{(e)}} (\sigma_{\mathrm{VM1}}^{(e)} - \sigma_{\mathrm{VM2}}^{(e)})^2 dS \right)^{1/2} \tag{8}$$

where $\sigma_{\mathrm{VM1}}^{(e)}$ and $\sigma_{\mathrm{VM2}}^{(e)}$ are the von Mises stresses obtained from the Hermite element and the "reduced" solution, respectively.

The global reanalysis technique is based on performing two consecutive analyses using the same mesh for the discretization of the geometry of the boundary. In the first analysis, the CBIEs and Lagrangian elements are used to obtain the values of displacements and tractions at the functional nodes. In the second one (which constitutes the global reanalysis), the way in which the displacements are approximated inside the elements is changed from Lagrangian to Hermite, introducing the nodal values of the tangential derivatives of the displacements as additional degrees of freedom. If it is assumed that the nodal values of the displacements and the tractions remain practically the same as the ones obtained in the first analysis, then the only new unknowns are the nodal values of $\partial u_i / \partial s$. The TDBIEs are collocated at each functional node to generate the additional set of equations required to determine those new degrees of freedom. Once both solutions are available, the error inside each element can be estimated using eqn (8), where $\sigma_{\mathrm{VM1}}^{(e)}$ and $\sigma_{\mathrm{VM2}}^{(e)}$ are the von Mises stresses obtained from the global reanalysis and the conventional analysis, respectively.

3 Test case and numerical results

In order to compare the performance of the two error indicators in the context of a practical application, they were used to lead adaptive meshing processes for the stress analysis of a steel bracket under plane stress conditions. The dimensions of the bracket, the material properties, and the boundary conditions are shown in Fig. 1. The left end of the bracket is completely fixed and a constant pressure of 800 psi is applied over one quarter of the circular hole located at the bottom of the part. Fig. 1 also shows the results for the von Mises stress contours that were

obtained using the FEM software Algor™ and a very fine mesh. The finite element solution is included here as a reference since an analytical solution for the test case under consideration is not available.

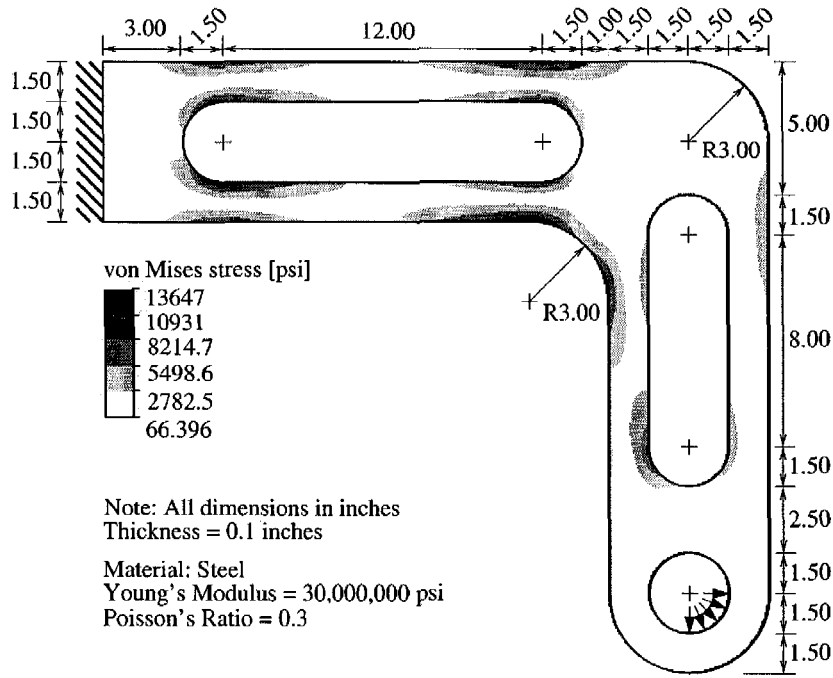


Figure 1: Geometry, boundary conditions, and stress contours for the test case.

For the adaptive meshing process, Hermite elements with three nodes were considered first. As shown in Fig. 2, the initial mesh consisted of a total of 46 elements, enough to describe accurately the geometry and boundary conditions for the problem. Geometrically, the elements were linear in all the straight parts of the boundary and quadratic elsewhere. In order to avoid the collocation of the TDBIEs at corners, partially discontinuous elements were used in and near the fixed end. The adaptive process for this model was run using an error indicator based on the von Mises stress as described by eqn (8). In this regard, the integral over the length of the element was computed using a numerical integration scheme with 9 Gaussian quadrature points. During the adaptive process, all the elements with a reported error above 0.65 times the largest error in the mesh were refined into two elements of equal size. In order to avoid problems related to numerical integration, the so called "compatibility condition for integration" suggested by Guiggiani [8] was applied. The adaptive process was stopped when two consecutive analyses reported a maximum value for the von Mises stress whose difference was less than 1%. The adaptive meshing process took four steps to complete and the final mesh together with the corresponding results for the von Mises stresses are shown in Fig. 3.

Transactions on Modelling and Simulation vol 35, © 2003 WIT Press, www.witpress.com, ISSN 1743-355X

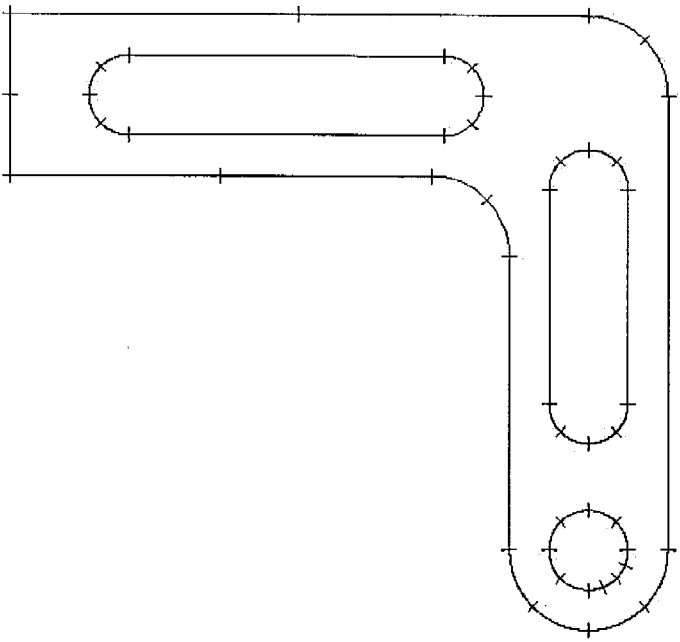


Figure 2: Initial mesh for the adaptive meshing process using Hermite elements.

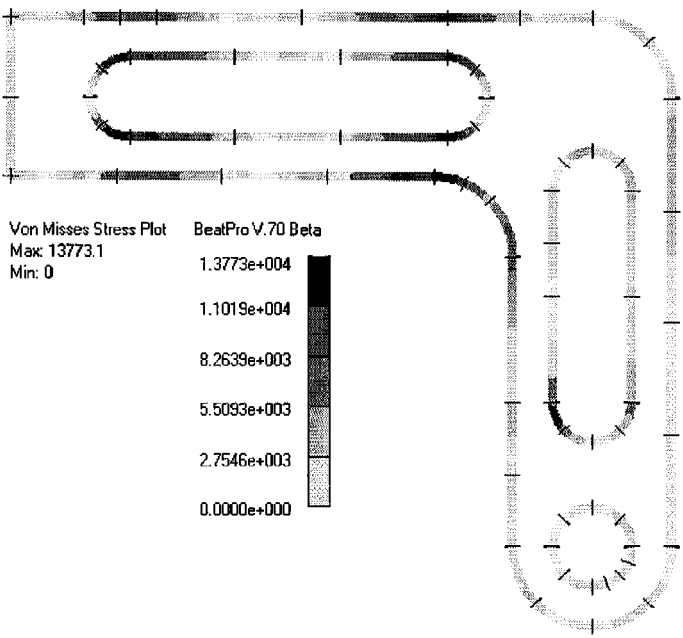


Figure 3: Final mesh for the adaptive meshing process using Hermite elements.

Transactions on Modelling and Simulation vol 35, © 2003 WIT Press, www.witpress.com, ISSN 1743-355X

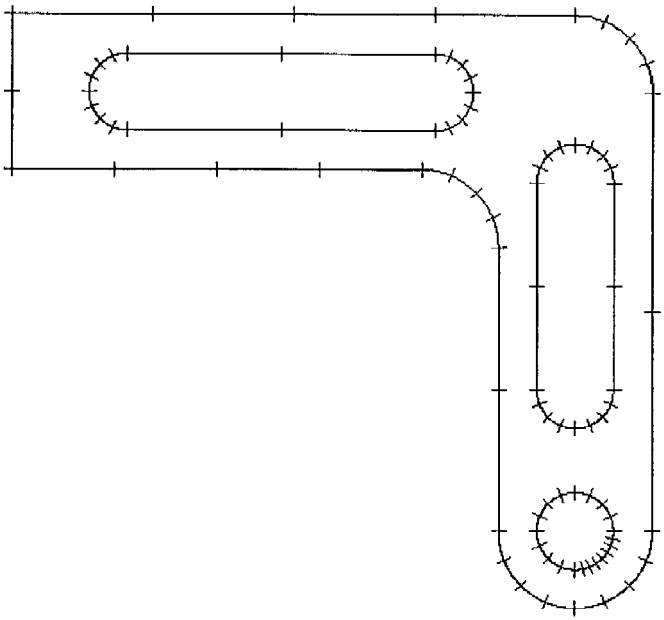


Figure 4: Initial mesh for the adaptive meshing process using global reanalysis.

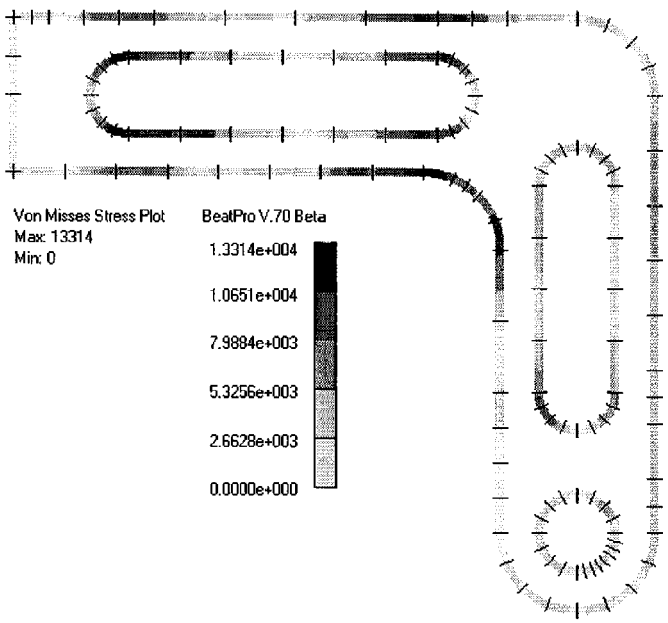
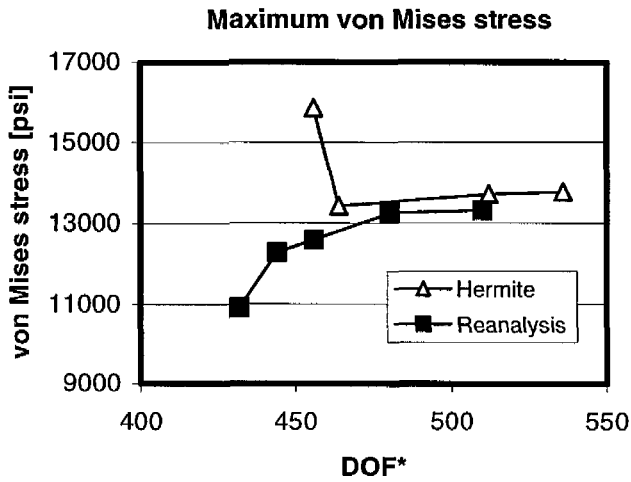


Figure 5: Final mesh for the adaptive meshing process using global reanalysis.

Transactions on Modelling and Simulation vol 35, © 2003 WIT Press, www.witpress.com, ISSN 1743-355X



* For global reanalysis, two runs with the indicated number of degrees of freedom were required

Figure 6: Convergence to the solution for the two adaptive meshing processes.

For the second adaptive meshing process, quadratic Lagrange elements were enriched through the application of the global reanalysis technique in order to compute an error indicator as described in eqn (8). In this case, the initial mesh consisted of 90 elements distributed as shown in Fig. 4. Partially discontinuous elements were also used as needed to avoid the collocation of TDBIEs at corners. In all other aspects, the adaptive meshing process was run with the same parameters as the ones that were employed for the Hermite elements. The adaptive meshing process took five steps to complete. The final mesh together with the corresponding results for the von Mises stresses are shown in Fig. 5.

As can be seen in Fig. 6, for both adaptive meshing processes the maximum value of the von Mises stress converges to a numeric value that is very close to the one that was obtained using a very fine finite element mesh. It should be clarified that the high value for the von Mises stress reported by the Hermite elements in the first point of the plot is a consequence of the use of discontinuous elements in the fixed boundary.

From the results corresponding to the final meshes generated by the adaptive processes, it can be seen that both capture the behavior of the solution over the entire boundary. From the numerical experiments carried out, it is safe to say that the adaptive processes are robust provided that good initial meshes are used and the boundary conditions are accurately represented. Although this requirement may seem difficult to achieve by novice users, easy modeling rules are of help. For the problem described here, two of these rules were proposed and tested. The first was to use at least two Hermite elements to model each 90° circular arc where homogeneous boundary conditions were applied and at least four where non-homogeneous boundary conditions were specified. The second was to avoid the use of partially discontinuous elements of considerable length.

Transactions on Modelling and Simulation vol 35, © 2003 WIT Press, www.witpress.com, ISSN 1743-355X

4 Conclusions

In this work, the application of adaptive boundary elements for the automated analysis of a mechanical component was presented. Based on the stress error indicators developed by the authors, the adaptive meshing schemes used appear to be robust enough to be employed during the PDP to evaluate de performance of different product concepts. From the analyses carried out, Hermite elements seem to provide the best balance between reliability and computational cost.

References

[1] Ulrich, K.T. & Eppinger, S.D. *Product Design and Development*, second edition. Irwin McGraw-Hill, 2000.

[2] Kita, E. & Kamiya, N. Error estimation and adaptive mesh refinement in the Boundary Element Method – An overview. *Engineering Analysis with Boundary Elements*, **25**, 479-498, 2001.

[3] Miranda-Valenzuela, J.C. & Muci-Küchler, K.H. *Adaptive Meshing with Boundary Elements*. WIT Press, Southampton, UK, 2002.

[4] Muci-Küchler, K.H. & Miranda-Valenzuela, J.C. A new error indicator based on stresses for adaptive meshing with Hermite boundary elements. *Engineering Analysis with Boundary Elements*, **23**, 657-670, 1999.

[5] Muci-Küchler, K.H., Miranda-Valenzuela, J.C. & Soriano-Soriano, S. Use of the tangent derivative boundary integral equations for the efficient computation of stresses and error indicators. *International Journal for Numerical Methods in Engineering*, **53**, 797-824, 2002.

[6] Muci-Küchler, K.H. & Rudolphi, T.J. Coincident collocation of Displacement and Tangent Derivative Boundary Integral Equations in elasticity. *International Journal for Numerical Methods in Engineering*, **36**, 2837-2849, 1993.

[7] Becker, A.A. *The Boundary Element Method in Engineering - A Complete Course*. McGraw-Hill International, 1992.

[8] Guiggiani, M. Error indicators for adaptive mesh refinement in the Boundary Element Method – A new approach. *International Journal for Numerical Methods in Engineering*, **29**, 1247-1269, 1990.