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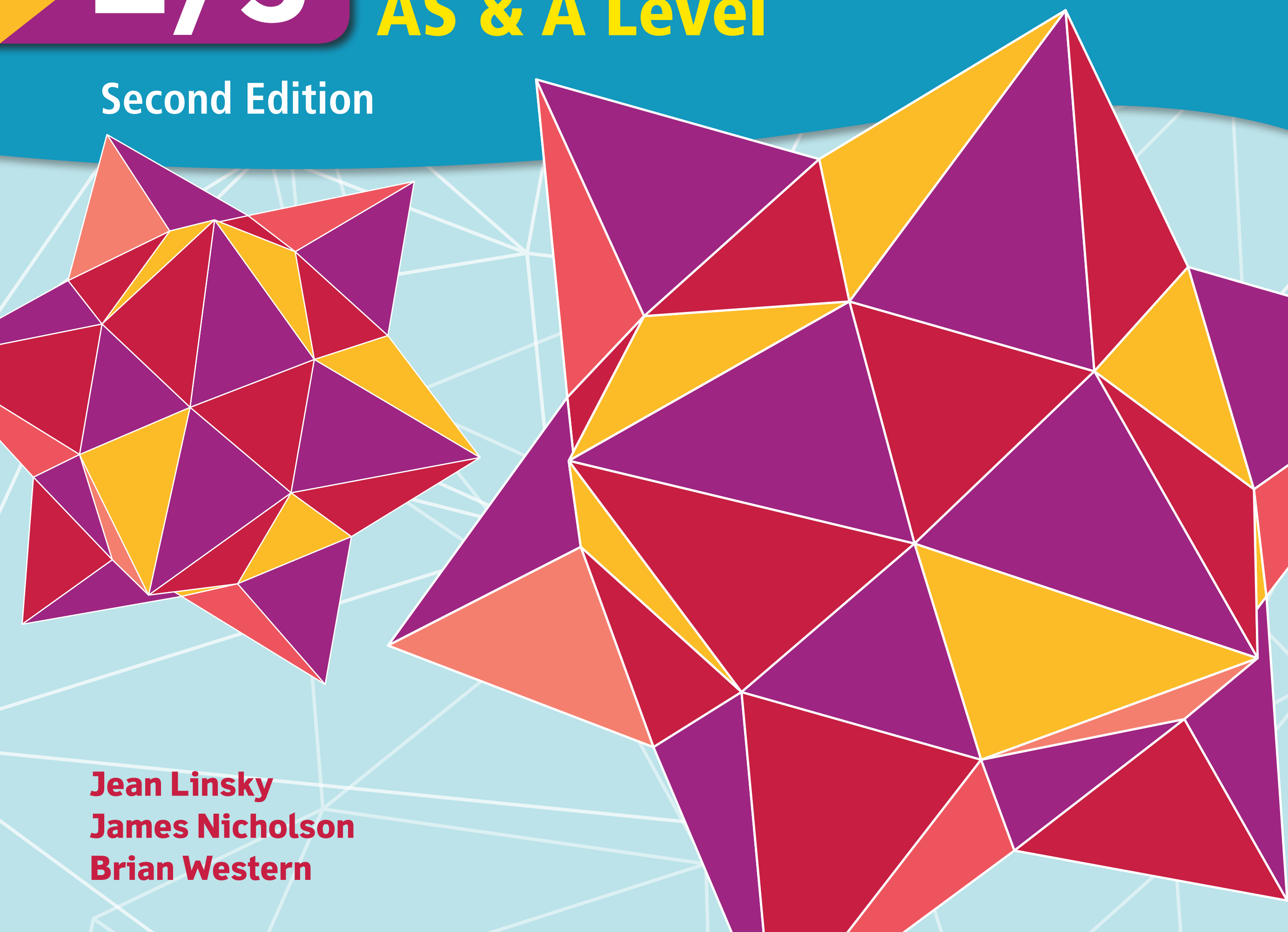
Complete Pure Mathematics 2/3 for Cambridge International AS & A Level

Second Edition



Jean Linsky
James Nicholson
Brian Western

Oxford excellence for Cambridge AS & A Level

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Contents

Introduction

About this book

This book has been written to cover the **Cambridge AS & A level International Mathematics (9709)** course, and is fully aligned to the syllabus. The first six chapters of the book cover material applicable to both Pure 2 and Pure 3, and the final five chapters cover Pure 3 material only.

In addition to the main curriculum content, you will find:

- 'Maths in real-life', showing how principles learned in this course are used in the real world.
- Chapter openers, which outline how each topic in the Cambridge 9709 syllabus is used in real-life.
- 'Did you know?' boxes (as shown below), which give interesting side-notes beyond the scope of the syllabus.

The book contains the following features:

Note

Did you know?

Advice on calculator use

Pure 2

Pure 3

Throughout the book, you will encounter worked examples and a host of rigorous exercises. The examples show you the important techniques required to tackle questions. The exercises are carefully graded, starting from a basic level and going up to exam standard, allowing you plenty of opportunities to practise your skills. Together, the examples and exercises put maths in a real-world context, with a truly international focus.

At the start of each chapter, you will see a list of objectives that are covered in the chapter. These objectives are drawn from the Cambridge AS and A level syllabus. Each chapter begins with a *Before you start* section and finishes with a *Summary exercise* and *Chapter summary*, ensuring that you fully understand each topic.

Each chapter contains key mathematical terms to improve understanding, highlighted in colour, with full definitions provided in the Glossary of terms at the end of the book.

The answers given at the back of the book are concise. However, when answering exam-style questions, you should show as many steps in your working as possible. All exam-style questions, as well as *Exam-style papers 2A, 2B, 3A* and *3B*, have been written by the authors.

About the authors

Brian Western has over 40 years of experience in teaching mathematics up to A Level and beyond, and is also a highly experienced examiner. He taught mathematics and further mathematics, and was an Assistant Headteacher in a large state school. Brian has written and consulted on a number of mathematics textbooks.

James Nicholson is an experienced teacher of mathematics at secondary level, having taught for 12 years at Harrow School and spent 13 years as Head of Mathematics in a large Belfast grammar school. He is the author of several A Level texts, and editor of the *Concise Oxford Dictionary of Mathematics*. He has also contributed to a number of other sets of curriculum and assessment materials, is an experienced examiner and has acted as a consultant for UK government agencies on accreditation of new specifications.

Jean Linsky has been a mathematics teacher for over 30 years, as well as Head of Mathematics in Watford, Herts, and is also an experienced examiner. Jean has authored and consulted on numerous mathematics textbooks.

A note from the authors

The aim of this book is to help students prepare for the Pure 2 and Pure 3 units of the Cambridge International AS and A Level mathematics syllabus, although it may also be useful in providing support material for other AS and A Level courses. The book contains a large number of practice questions, many of which are exam-style.

In writing the book we have drawn on our experiences of teaching mathematics and Further mathematics to A Level over many years as well as on our experiences as examiners, and our discussions with mathematics educators from many countries at international conferences.

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Student book & Cambridge syllabus matching grid

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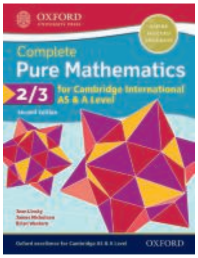


Student Book: *Complete Pure Mathematics 2 & 3 for Cambridge International AS & A Level*

Syllabus: Cambridge International AS & A Level Mathematics: Pure Mathematics 2 & 3 (9709)

PURE MATHEMATICS 2 & 3	Student Book
Syllabus overview	
Unit P2: Pure Mathematics 2 (Paper 2)	
Knowledge of the content of unit P1 is assumed and candidates may be required to demonstrate such knowledge in answering questions.	
1. Algebra	
• understand the meaning of $\lvert x\rvert$, sketch the graph of $y = \lvert ax + b\rvert$ and use relations such as $\lvert a\rvert = \lvert b\rvert \Leftrightarrow a^2 = b^2$ and $\lvert x - a\rvert < b \Leftrightarrow a - b < x < a + b$ when solving equations and inequalities;	Pages 2–17
• divide a polynomial, of degree not exceeding 4, by a linear or quadratic polynomial, and identify the quotient and remainder (which may be zero);	Pages 2–17
• use the factor theorem and the remainder theorem, e.g. to find factors, solve polynomial equations or evaluate unknown coefficients.	Pages 2–17
2. Logarithmic and exponential functions	
• understand the relationship between logarithms and indices, and use the laws of logarithms (excluding change of base);	Pages 18–39
• understand the definition and properties of e^x and $\ln x$, including their relationship as inverse functions and their graphs;	Pages 18–39
• use logarithms to solve equations and inequalities in which the unknown appears in indices;	Pages 18–39
• use logarithms to transform a given relationship to linear form, and hence determine unknown constants by considering the gradient and/or intercept.	Pages 18–39

3. Trigonometry	
• understand the relationship of the secant, cosecant and cotangent functions to cosine, sine and tangent, and use properties and graphs of all six trigonometric functions for angles of any magnitude;	Pages 40–65
• use trigonometrical identities for the simplification and exact evaluation of expressions and in the course of solving equations, and select an identity or identities appropriate to the context, showing familiarity in particular with the use of: – $\sec^2\theta \equiv 1 + \tan^2\theta$ and $\mathrm{cosec}^2\theta \equiv 1 + \cot^2\theta$, – the expansions of $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$, – the formulae for $\sin 2A$, $\cos 2A$ and $\tan 2A$, – the expressions of $a \sin\theta + b \cos\theta$ in the forms $R\sin(\theta \pm a)$ and $R\cos(\theta \pm a)$.	Pages 40–65
4. Differentiation	
• use the derivatives of e^x, $\ln x$, $\sin x$, $\cos x$, $\tan x$, together with constant multiples, sums, differences and composites;	Pages 68–90
• differentiate products and quotients;	Pages 68–90
• find and use the first derivative of a function which is defined parametrically or implicitly.	Pages 68–90
5. Integration	
• extend the idea of ‘reverse differentiation’ to include the integration of e^{ax+b}, $\frac{1}{ax+b}$, $\sin(ax+b)$, $\cos(ax+b)$ and $\sec^2(ax+b)$ (knowledge of the general method of integration by substitution is not required);	Pages 91–116
• use trigonometrical relationships (such as double-angle formulae) to facilitate the integration of functions such as $\cos^2 x$;	Pages 91–116
• use the trapezium rule to estimate the value of a definite integral, and use sketch graphs in simple cases to determine whether the trapezium rule gives an over-estimate or an under-estimate.	Pages 91–116
6. Numerical solution of equations	
• locate approximately a root of an equation, by means of graphical considerations and/or searching for a sign change;	Pages 117–133
• understand the idea of, and use the notation for, a sequence of approximations which converges to a root of an equation;	Pages 123–140
• understand how a given simple iterative formula of the form $x_{n+1} = F(x_n)$ relates to the equation being solved, and use a given iteration, or an iteration based on a given rearrangement of an equation, to determine a root to a prescribed degree of accuracy (knowledge of the condition for convergence is not included, but candidates should understand that an iteration may fail to converge).	Pages 123–140

Unit P3: Pure Mathematics (Paper 3)	
Knowledge of the content of unit P1 is assumed and candidates may be required to demonstrate such knowledge in answering questions.	
1. Algebra	
• understand the meaning of $\lvert x\rvert$, sketch the graph of $y = \lvert ax + b\rvert$ and use relations such as $\lvert a\rvert = \lvert b\rvert \Leftrightarrow a^2 = b^2$ and $\lvert x - a\rvert < b \Leftrightarrow a - b < x < a + b$ when solving equations and inequalities;	Pages 2–17
• divide a polynomial, of degree not exceeding 4, by a linear or quadratic polynomial, and identify the quotient and remainder (which may be zero);	Pages 2–17
• use the factor theorem and the remainder theorem, e.g. to find factors, solve polynomial equations or evaluate unknown coefficients;	Pages 2–17
• recall an appropriate form for expressing rational functions in partial fractions, and carry out the decomposition, in cases where the denominator is no more complicated than: - $(ax + b)(cx + d)(ex + f)$, - $(ax + b)(cx + d)^2$, - $(ax + b)(x^2 + c^2)$, and where the degree of the numerator does not exceed that of the denominator;	Pages 136–153
• use the expansion of $(1 + x)^n$, where n is a rational number and $\lvert x\rvert < 1$ (finding a general term is not included, but adapting the standard series to expand e.g. $(2 - \frac{1}{2}x)^{-1}$ is included).	Pages 152–169
2. Logarithmic and exponential functions	
• understand the relationship between logarithms and indices, and use the laws of logarithms (excluding change of base);	Pages 18–39
• understand the definition and properties of e^x and $\ln x$, including their relationship as inverse functions and their graphs;	Pages 18–39
• use logarithms to solve equations of the form $a^x = b$, and similar inequalities;	Pages 18–39
• use logarithms to transform a given relationship to linear form, and hence determine unknown constants by considering the gradient and/or intercept.	Pages 18–39

3. Trigonometry	
• understand the relationship of the secant, cosecant and cotangent functions to cosine, sine and tangent, and use properties and graphs of all six trigonometric functions for angles of any magnitude;	Pages 40–65
• use trigonometrical identities for the simplification and exact evaluation of expressions and in the course of solving equations, and select an identity or identities appropriate to the context, showing familiarity in particular with the use of: - $\sec^2\theta \equiv 1 + \tan^2\theta$ and $\operatorname{cosec}^2\theta \equiv 1 + \cot^2\theta$, - the expansions of $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$, - the formulae for $\sin 2A$, $\cos 2A$ and $\tan 2A$, - the expressions of $a \sin\theta + b\cos\theta$ in the forms $R\sin(\theta \pm a)$ and $R\cos(\theta \pm a)$.	Pages 40–65

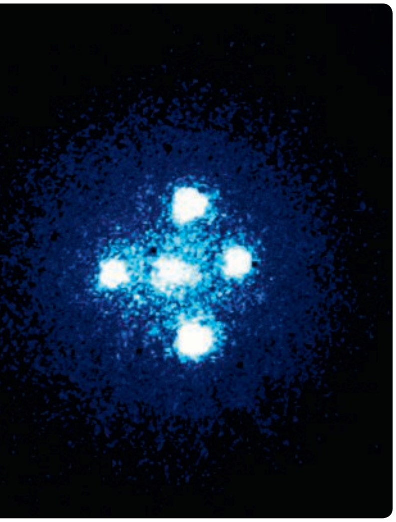
4. Differentiation	
• use the derivatives of e^x, $\ln x$, $\sin x$, $\cos x$, $\tan x$, $\tan^{-1} x$, together with constant multiples, sums, differences and composites;	Pages 68–90
• differentiate products and quotients;	Pages 68–90
• find and use the first derivative of a function which is defined parametrically or implicitly.	Pages 68–90

5. Integration	
• extend the idea of 'reverse differentiation' to include the integration of e^{ax+b}, $\frac{1}{ax+b}$, $\sin(ax+b)$, $\cos(ax+b)$, $\sec^2(ax+b)$ and $\frac{1}{x^2+a^2}$;	Pages 91–116 and Pages 170–197
• use trigonometrical relationships (such as double-angle formulae) to facilitate the integration of functions such as $\cos^2 x$;	Pages 97–122
• integrate rational functions by means of decomposition into partial fractions (restricted to the types of partial fractions specified in paragraph 1 above);	Pages 154–181
• recognise an integrand of the form $\frac{kf'(x)}{f'(x)}$, and integrate, for example, $\frac{x}{x^2+1}$ or $\tan x$;	Pages 170–197
• recognise when an integrand can usefully be regarded as a product, and use integration by parts to integrate, for example, $x\sin 2x$, x^2e^x or $\ln x$;	Pages 170–197
• use a given substitution to simplify and evaluate either a definite or an indefinite integral;	Pages 170–197
• use the trapezium rule to estimate the value of a definite integral, and use sketch graphs in simple cases to determine whether the trapezium rule gives an over-estimate or an under-estimate.	Pages 97–122

6. Numerical solution of equations	
• locate approximately a root of an equation, by means of graphical considerations and/or searching for a sign change;	Pages 117–133
• understand the idea of, and use the notation for, a sequence of approximations which converges to a root of an equation;	Pages 117–133
• understand how a given simple iterative formula of the form $x_{n+1} = \mathrm{F}(x_n)$ relates to the equation being solved, and use a given iteration, or an iteration based on a given rearrangement of an equation, to determine a root to a prescribed degree of accuracy (knowledge of the condition for convergence is not included, but candidates should understand that an iteration may fail to converge).	Pages 117–133
7. Vectors	
• use standard notations for vectors, i.e. $\begin{pmatrix} x \\ y \end{pmatrix}$, $x\mathbf{i}+y\mathbf{j}$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, $\overrightarrow{AB}$, $\mathbf{a}$;	Pages 182-214
• carry out addition and subtraction of vectors, and multiplication of a vector by a scalar, and interpret these operations in geometrical terms;	Pages 182-214
• calculate the magnitude of a vector, and use unit vectors, displacement vectors and position vectors;	Pages 182-214
• understand the significance of all the symbols used when the equation of a straight line is expressed in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, and find the equation of a line, given sufficient information;	Pages 182-214
• determine whether two lines are parallel, intersect or are skew, and find the point of intersection of two lines when it exists;	Pages 182-214
• use formulae to calculate the scalar product of two vectors, and use scalar products in problems involving lines and points.	Pages 182-214
8. Differential equations	
• formulate a simple statement involving a rate of change as a differential equation, including the introduction if necessary of a constant of proportionality;	Pages 215–240
• find by integration a general form of solution for a first order differential equation in which the variables are separable;	Pages 215–240
• use an initial condition to find a particular solution;	Pages 215–240
• interpret the solution of a differential equation in the context of a problem being modelled by the equation.	Pages 215–240

9. Complex numbers	
• understand the idea of a complex number, recall the meaning of the terms real part, imaginary part, modulus, argument, conjugate, and use the fact that two complex numbers are equal if and only if both real and imaginary parts are equal;	Pages 241–275
• carry out operations of addition, subtraction, multiplication and division of two complex numbers expressed in cartesian form $x + \mathrm{i}y$;	Pages 241–275
• use the result that, for a polynomial equation with real coefficients, any non-real roots occur in conjugate pairs;	Pages 241–275
• represent complex numbers geometrically by means of an Argand diagram;	Pages 241–275
• carry out operations of multiplication and division of two complex numbers expressed in polar form $r(\cos\theta + \mathrm{i}\sin\theta) \equiv r\,\mathrm{e}^{\mathrm{i}\theta}$;	Pages 241–275
• find the two square roots of a complex number;	Pages 241–275
• understand in simple terms the geometrical effects of conjugating a complex number and of adding, subtracting, multiplying and dividing two complex numbers;	Pages 241–275
• illustrate simple equations and inequalities involving complex numbers by means of loci in an Argand diagram, e.g. $\vert z - a \vert < k$, $\vert z - a \vert = \vert z - b \vert$, $\arg(z - a) = \alpha$.	Pages 241–275

1 Algebra



Algebra is used extensively in mathematics, chemistry, physics, economics and social sciences. For example, the study of polynomials in astrophysics has led to our understanding of gravitational lensing.

Gravitational lensing occurs when light from a distant source bends around a massive object (such as a galaxy) between a source and an observer. Multiple images of the same object may be seen. Here, the 'Einstein Cross', four images of a very distant supernova, is seen in a photograph taken by the Hubble telescope. The supernova is at a distance of approximately 8 billion light years, and is 20 times further away than the galaxy, which is at a distance of 400 million light years. The light from the supernova is bent in its path by the gravitational field of the galaxy. This bending produces the four bright outer images. The bright central region of the galaxy is seen as the central object. This phenomena was predicted by Einstein's general theory of relativity published in 1915, but was not observed until 1979.

Objectives

- Understand the meaning of $|x|$, sketch the graph of $y = |ax + b|$ and use relations such as $|a| = |b| \Leftrightarrow a^2 = b^2$ and $|x - a| < b \Leftrightarrow a - b < x < a + b$ when solving equations and inequalities.
- Divide a polynomial, of degree not exceeding 4, by a linear or quadratic polynomial, and identify the quotient and remainder (which may be zero).
- Use the factor theorem and the remainder theorem, e.g. to find factors, solve polynomial equations or evaluate unknown coefficients.

Before you start

You should know how to:

1. Do long division,

e.g. 357 ÷ 21

$$\begin{array}{r} 17 \\ 21\overline{)357} \\ \underline{21} \\ 147 \\ \underline{147} \\ 0 \end{array}$$

Therefore $\frac{357}{21} = 17$

2. Find the remainder,

e.g. 461 ÷ 37

$$\begin{array}{r} 12 \\ 37\overline{)461} \\ \underline{37} \\ 91 \\ \underline{74} \\ 17 \end{array}$$

Remainder = 17

Skills check:

1. Find the following using long division.

a) 608 ÷ 19

b) 2774 ÷ 38

c) 1081 ÷ 23

d) 1392 ÷ 24

2. Find the remainder of the following after doing long division.

a) 923 ÷ 21

b) 742 ÷ 32

c) 1527 ÷ 43

d) 4258 ÷ 26

1.1 The modulus function

The **modulus** of a real number is the magnitude of that number.

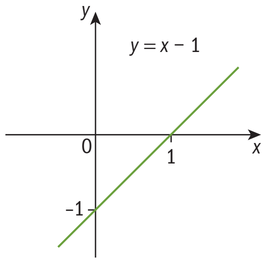
If we have a real number x, then the modulus of x is written as $|x|$. We say this as 'mod x'.

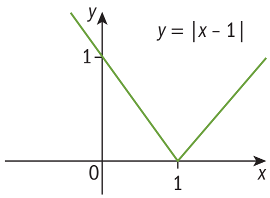
Thus $|2| = 2$ and $|-2| = 2$, and if we write $|x| < 2$ this means that $-2 < x < 2$.

The **modulus function** $f(x) = |x|$ is defined as

$$|x| = x \quad \text{for} \quad x \geq 0$$
$$|x| = -x \quad \text{for} \quad x < 0$$

Consider the impact that the modulus function has by looking at the graphs of $y = x - 1$ and $y = |x - 1|$.





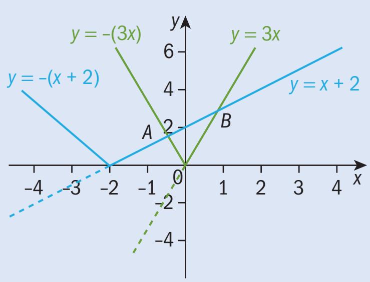
Note: For $f(x) < 0$, $|f(x)| = -f(x)$.

When graphing $y = |f(x)|$, we reflect the graph of $y = f(x)$ in the x-axis whenever $f(x) < 0$.

Example 1

Solve the equation $|x + 2| = |3x|$.

Method 1: Using a graph



Sketch the graphs and find where they intersect.

The lines cannot be drawn below the x-axis.

For $x < 0$, $|3x| = -(3x)$

For $x < -2$, $|x + 2| = -(x + 2)$

Graphs intersect at A and B.

At A, $x + 2 = -3x$

$4x = -2 \Rightarrow x = -\frac{1}{2}$

At A, the line $y = x + 2$ intersects the line $y = -(3x)$.

At B, $x + 2 = 3x$

$2x = 2 \Rightarrow x = 1$

At B, the line $y = x + 2$ intersects the line $y = 3x$.

$x = -\frac{1}{2}$ or $x = 1$

Continued on the next page

Method 2: Squaring both sides of the equation

$$(x+2)^2 = (3x)^2$$
$$x^2 + 4x + 4 = 9x^2$$
$$8x^2 - 4x - 4 = 0$$
$$2x^2 - x - 1 = 0$$
$$(2x+1)(x-1) = 0$$
$$x = -\frac{1}{2} \text{ or } x = 1$$

We do this to ensure both sides of the equation are positive.

Note: You can only use this method if the variable (e.g. x) is inside the modulus expression.

Method 3: Removing modulus signs by equating the left-hand side with both 'plus and minus' the right-hand side

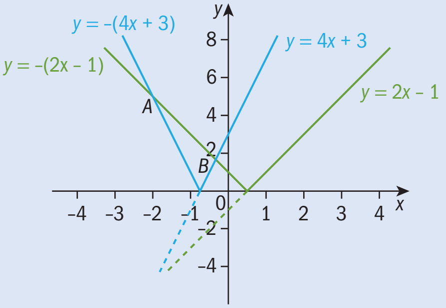
$$x + 2 = 3x \quad \text{or} \quad x + 2 = -3x$$
$$2x = 2 \quad \text{or} \quad 4x = -2$$
$$x = 1 \quad \text{or} \quad x = -\frac{1}{2}$$

We get the same result if we say $3x = \pm(x + 2)$.

Example 2

Solve the inequality $|4x + 3| > |2x - 1|$.

Method 1: Using a graph



Sketch the graphs and find where they intersect.

The lines cannot be drawn below the x-axis.

For $x < -\frac{3}{4}$, $|4x + 3| = -(4x + 3)$

For $x < \frac{1}{2}$, $|2x - 1| = -(2x - 1)$

Graphs meet at A and B.

At A, $-(4x + 3) = -(2x - 1)$

$$2x = -4, \quad x = -2$$

At A, the line $y = -(4x + 3)$ intersects the line $y = -(2x - 1)$.

At B, $4x + 3 = -(2x - 1)$

$$6x = -2, \quad x = -\frac{1}{3}$$

At B, the line $y = 4x + 3$ intersects the line $y = -(2x - 1)$.

We want the region where $|4x + 3| > |2x - 1|$.

This is where the 'blue' lines are above the 'green' lines.

$x < -2$ and $x > -\frac{1}{3}$

Continued on the next page

Method 2: Squaring both sides of the inequality

$$(4x+3)^2 > (2x-1)^2$$

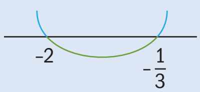
We can do this because we know both $|4x+3|$ and $|2x-1|$ are positive so squaring them will not change the inequality.

$$16x^2 + 24x + 9 > 4x^2 - 4x + 1$$

$$12x^2 + 28x + 8 > 0$$

$$3x^2 + 7x + 2 > 0$$

$$(3x+1)(x+2) > 0$$



Draw a rough sketch to see where $y > 0$. There are two separate blue regions that satisfy $y > 0$.

$x < -2$ or $x > -\frac{1}{3}$

There are two separate regions that satisfy $y > 0$.

Method 3: Removing modulus signs by equating the left-hand side with both 'plus and minus' the right-hand side

First find where the graphs of $y = |4x+3|$ and $y = |2x-1|$ intersect (the values of x at these points are called the critical values).

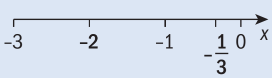
$4x + 3 = 2x - 1$ or $4x + 3 = -(2x - 1)$

$2x = -4$ or $6x = -2$

$x = -2$ or $x = -\frac{1}{3}$

This gives us the two critical values.

To find the correct inequalities you need to take values of x on either side of the critical values.



Take values on either side of $x = -2$ and $x = -\frac{1}{3}$.

e.g.

$x = -3$, $|4x+3| = 9$ and $|2x-1| = 7$

Thus $|4x+3| > |2x-1|$, so $x < -2$

One of the solution regions is less than -2.

is one region where $|4x+3| > |2x-1|$

When $x = -1$, $|4x+3| = 1$ and $|2x-1| = 3$

$|4x+3| > |2x-1|$ is false so there is no solution between -2 and $-\frac{1}{3}$, so we do **not** require this region.

When $x = 0$, $|4x+3| = 3$ and $|2x-1| = 1$

Thus $|4x+3| > |2x-1|$ so $x > -\frac{1}{3}$

One of the solution regions is greater than $-\frac{1}{3}$.

$x < -2$ or $x > -\frac{1}{3}$

Exercise 1.1

1. Solve each of these equations algebraically.
   a) $|1 - 2x| = 3$ b) $|x - 3| = |x + 1|$
   c) $|5x - 2| = |2x|$ d) $|5 - 4x| = 4$
   e) $|2x - 1| = |x + 2|$ f) $|x| = |4 - 2x|$
   g) $|3x + 1| = |4 - 2x|$ h) $|2x - 6| = |3x + 1|$
   i) $|x + 4| = |3x + 1|$ j) $|1 - 3x| = |5x - 3|$
   k) $3|x - 4| = |x + 2|$ l) $5|2x - 3| = 4|x - 5|$
2. Solve each of these inequalities algebraically.
   a) $|2x - 3| < |x|$ b) $|x - 1| \geq 4$
   c) $|x + 3| \geq |2x + 2|$ d) $|2x + 3| > x + 6$
3. Solve each of these inequalities graphically.
   a) $|x + 6| \leq 3|x - 2|$ b) $|3x - 2| < |x + 4|$
   c) $|2x| < |1 - x|$ d) $5 \leq |2x - 1|$
   e) $2|x - 1| < |x + 3|$ f) $|2x + 1| \geq |1 - 4x|$
   g) $|x + 2| < 2|x + 1|$ h) $|3x - 1| \leq |x + 3|$

1.2 Sketching linear graphs of the form $y = a|x| + b$

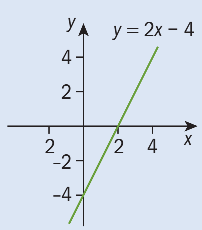
We have sketched graphs of the form $y = |ax + b|$ in section 1.1. Using the information given in P1 Chapter 3 on transformations, we can sketch graphs of the form $y = a|x| + b$.

Example 3

Sketch the following graphs.

a) $y = |2x - 4|$ b) $y = 2|x| - 4$

a)



First sketch the graph without the modulus.

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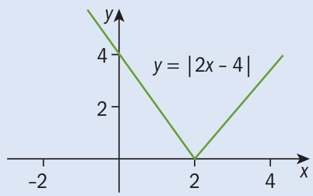
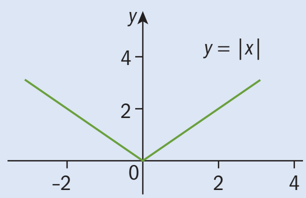
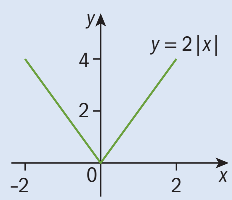


Then reflect the negative y values in the x-axis.

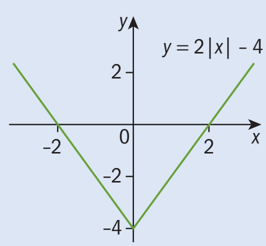
b)



First sketch the graph of $|x|$.



Then stretch the graph by a stretch factor of 2 in the direction of the y-axis.



Then translate the graph by the vector $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$.

When $y = 0$, $x = 2$ or -2.

Exercise 1.2

1. Sketch the following.

a) **i)** $y = |x+1|$ **ii)** $y = |x| + 1$

b) **i)** $y = |3x+2|$ **ii)** $y = 3|x| + 2$

c) **i)** $y = |2x-2|$ **ii)** $y = 2|x-2|$

d) **i)** $y = \left|\frac{1}{2}x + 3\right|$ **ii)** $y = \frac{1}{2}|x| + 3$

e) **i)** $y = |-x|$ **ii)** $y = -|x|$

f) **i)** $y = |3-x|$ **ii)** $y = 3 - |x|$

1.3 Division of polynomials

We can use long division to divide a polynomial by another polynomial.

Example 4

Divide $x^3 - 5x^2 + x + 10$ by $(x - 2)$.

$$\begin{array}{rl}
 & x^2 - 3x - 5 \\
x - 2 & \overline{)\, x^3 - 5x^2 + x + 10} \\
 & \underline{x^3 - 2x^2} \\
 & \quad -3x^2 + x + 10 \\
 & \quad \underline{-3x^2 + 6x} \\
 & \qquad -5x + 10 \\
 & \qquad \underline{-5x + 10}
\end{array}$$

$x^3 \div x = x^2$, so multiply $(x - 2)$ by x^2.

Subtract $(x^3 - 2x^2)$ from $(x^3 - 5x^2)$ and bring down $(+ x + 10)$.

$-3x^2 \div x = 3x$, so multiply $(x - 2)$ by $-3x$.

Subtract $(3x^2 + 6x)$ from $(3x^2 + x)$ and bring down $+ 10$.

We cannot continue the process because $(-5x + 10) - (-5x + 10) = 0$.

Thus $(x^3 - 5x^2 + x + 10) \div (x - 2) = x^2 - 3x - 5$

The expression $(x^3 - 5x^2 + x + 10)$ is called the **dividend**, $(x - 2)$ is called the **divisor**, and $(x^2 - 3x - 5)$ is called the **quotient**. When we subtract $(-5x + 10)$ from $(-5x + 10)$ we are left with nothing, so we say there is no remainder. Because there is no remainder, we can say that $(x - 2)$ is a **factor** of $x^3 - 5x^2 + x + 10$.

Example 5

Find the remainder when $4x^3 - 7x - 1$ is divided by $(2x + 1)$.

$$\begin{array}{rl}
 & 2x^2 - x - 3 \\
2x + 1 & \overline{)\, 4x^3 + 0x^2 - 7x - 1} \\
 & \underline{4x^3 + 2x^2} \\
 & \quad -2x^2 - 7x - 1 \\
 & \quad \underline{-2x^2 - x} \\
 & \qquad -6x - 1 \\
 & \qquad \underline{-6x - 3} \\
 & \qquad\qquad + 2
\end{array}$$

As there is no term in x^2 in the dividend, it is useful to write $0x^2$ as part of the dividend.

$(4x^3 \div 2x) = 2x^2$, so multiply $(2x + 1)$ by $2x^2$.

Subtract $(4x^3 + 2x^2)$ from $(4x^3 + 0x^2)$ and bring down $-7x - 1$.

$(-2x^2 \div 2x) = -x$, so multiply $(2x + 1)$ by $-x$.

Subtract $(-2x^2 - x)$ from $(-2x^2 - 7x)$ and bring down -1.

Subtract $(-6x - 3)$ from $(-6x - 1)$ to get a remainder of $+2$.

We cannot continue the process as 2 cannot be divided by $(2x + 1)$.

Thus, when $(4x^3 - 7x - 1)$ is divided by $(2x + 1)$, the remainder is 2.

Looking at Example 5 we can write
$(4x^3 - 7x - 1) = (2x^2 - x - 3)(2x + 1) + 2$

In general:

$$f(x) = \text{quotient} \times \text{divisor} + \text{remainder}$$

Exercise 1.3

1. Divide

a) $x^3 + 3x^2 + 3x + 2$ by $(x + 2)$

b) $x^3 - 2x^2 + 6x + 9$ by $(x + 1)$

c) $x^3 - 3x^2 + 6x - 8$ by $(x - 2)$

d) $x^3 + x^2 - 3x - 2$ by $(x + 2)$

e) $2x^3 - 6x^2 + 7x - 21$ by $(x - 3)$

f) $3x^3 - 20x^2 + 10x + 12$ by $(x - 6)$

g) $6x^4 + 5x^3 + 5x^2 + 10x + 7$ by $(3x^2 - 2x + 4)$.

Hint: In part (**g**), use the same method as when dividing by a linear expression. State any remainder.

2. Find the remainder when

a) $6x^3 + 28x^2 - 7x + 10$ is divided by $(x + 5)$

b) $2x^3 + x^2 + 5x - 4$ is divided by $(x - 1)$

c) $x^3 + 2x^2 - 17x - 2$ is divided by $(x - 3)$

d) $2x^3 + 3x^2 - 4x + 5$ is divided by $(x^2 + 2)$

e) $4x^3 - 5x + 4$ is divided by $(2x - 1)$

f) $3x^3 - x^2 + 1$ is divided by $(x + 2)$.

3. **Show that** $(2x + 1)$ is a factor of $2x^3 - 3x^2 + 2x + 2$.

4. **a)** Show that $(x - 1)$ is a factor of $x^3 - 6x^2 + 11x - 6$.

b) Hence factorise $x^3 - 6x^2 + 11x - 6$.

5. Show that when $4x^3 - 6x^2 + 5$ is divided by $(2x - 1)$ the remainder is 4.

6. Divide $x^3 + 1$ by $(x + 1)$.

7. Find the quotient and the remainder when $x^4 + 2x^3 + 3x^2 + 7$ is divided by $(x^2 + x + 1)$.

8. Find the quotient and the remainder when $2x^3 + 3x^2 - 4x + 5$ is divided by $(x + 2)$.

9. **a)** Show that $(2x - 1)$ is a factor of $12x^3 + 16x^2 - 5x - 3$.
   b) Hence factorise $12x^3 + 16x^2 - 5x - 3$.

10. The expression $2x^3 - 5x^2 - 16x + k$ has a remainder of -6 when divided by $(x - 4)$.
    Find the value of k.

11. Find the quotient and the remainder when $2x^4 - 8x^3 - 3x^2 + 7x - 7$ is divided by $(x^2 - 3x - 5)$.

12. The polynomial $x^4 + x^3 - 5x^2 + ax - 4$ is denoted by $p(x)$. It is given that $p(x)$ is divisible by $(x^2 + 2x - 4)$.
    Find the value of a.

1.4 The remainder theorem

You can find the remainder when a polynomial is divided by $(ax - b)$ by using the **remainder theorem**.

We know that if $f(x)$ is divided by $(x - a)$ then $f(x) = \text{quotient} \times (x - a) + \text{remainder}$.

When $x = a$, $f(a) = \text{quotient} \times (a - a) + \text{remainder} = \text{remainder}$.

Thus $f(a) = \text{remainder}$.

When a polynomial $f(x)$ is divided by $(x - a)$, the remainder is $f(a)$.

When a polynomial $f(x)$ is divided by $(ax - b)$, the remainder is $f\left(\frac{b}{a}\right)$.

Example 6

Find the remainder when $4x^3 + x^2 - 3x + 7$ is divided by $(x + 2)$.

$f(x) = 4x^3 + x^2 - 3x + 7$

Write the polynomial as a function.

$f(-2) = 4(-2)^3 + (-2)^2 - 3(-2) + 7$

$= -32 + 4 + 6 + 7 = -15$

$(x + 2) = 0 \Rightarrow x = -2$, so calculate $f(-2)$.

The remainder is -15.

Example 7

When $16x^4 - ax^3 + 8x^2 - 4x - 1$ is divided by $(2x - 1)$, the remainder is 3.

Find the value of a.

$\mathrm{f}(x) = 16x^4 - ax^3 + 8x^2 - 4x - 1$ ← Write the polynomial as a function.

$\mathrm{f}\left(\frac{1}{2}\right) = 16\left(\frac{1}{2}\right)^4 - a\left(\frac{1}{2}\right)^3 + 8\left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right) - 1$ ← $(2x - 1) = 0 \Rightarrow x = \frac{1}{2}$, so calculate $\mathrm{f}\left(\frac{1}{2}\right)$.

$= 1 - \frac{1}{8}a + 2 - 2 - 1 = 3$ ← Equate this to 3, since the remainder is 3.

$-\frac{1}{8}a = 3 \Rightarrow a = -24$

Exercise 1.4

1. Find the remainder when

a) $2x^3 + 8x^2 - x + 4$ is divided by $(x - 3)$

b) $5x^4 - 3x^3 - 2x^2 + x - 1$ is divided by $(x + 1)$

c) $x^3 + 4x^2 + 8x - 3$ is divided by $(2x + 1)$

d) $3x^3 - 2x^2 - 5x - 7$ is divided by $(2 - x)$

e) $9x^3 - 8x + 3$ is divided by $(1 - x)$

f) $243x^4 - 27x^3 + 6x + 4$ is divided by $(3x - 2)$.

2. When $ax^3 + 16x^2 - 5x - 5$ is divided by $(2x - 1)$ the remainder is -2. Find the value of a.

3. The polynomial $4x^3 - 4x^2 + ax + 1$, where a is a constant, is denoted by $\mathrm{p}(x)$. When $\mathrm{p}(x)$ is divided by $(2x - 3)$ the remainder is 13. Find the value of a.

4. The polynomial $x^3 + ax^2 + bx + 1$, where a and b are constants, is denoted by $\mathrm{p}(x)$. When $\mathrm{p}(x)$ is divided by $(x - 2)$ the remainder is 9 and when $\mathrm{p}(x)$ is divided by $(x + 3)$ the remainder is 19. Find the value of a and the value of b.

5. When $5x^3 + ax + b$ is divided by $(x - 2)$, the remainder is equal to the remainder obtained when the same expression is divided by $(x + 2)$.

a) **Explain** why b can take any value.

b) Find the value of a.

6. The polynomial $2x^4 + 3x^2 - x + 2$ is denoted by $p(x)$. Show that the remainder when $p(x)$ is divided by $(x + 2)$ is 8 times the remainder when $p(x)$ is divided by $(x - 1)$.

7. The polynomial $x^3 + ax + b$, where a and b are constants, is denoted by $p(x)$. When $p(x)$ is divided by $(x - 1)$ the remainder is 14 and when $p(x)$ is divided by $(x - 4)$ the remainder is 56. Find the values of a and b.

8. The polynomial $x^3 + ax^2 + 2$, where a is a constant, is denoted by $p(x)$. When $p(x)$ is divided by $(x + 1)$ the remainder is one more than when $p(x)$ is divided by $(x + 2)$. Find the value of a.

9. When $6x^2 + x + 7$ is divided by $(x - a)$, the remainder is equal to the remainder obtained when the same expression is divided by $(x + 2a)$, where $a \neq 0$. Find the value of a.

1.5 The factor theorem

We can deduce the **factor theorem** directly from the remainder theorem (section 1.4).

For any polynomial $f(x)$, if $f(a) = 0$ then the remainder when $f(x)$ is divided by $(x - a)$ is zero. Thus $(x - a)$ is a factor of $f(x)$.

For any polynomial $f(x)$, if $f\left(\frac{b}{a}\right) = 0$, then $(ax - b)$ is a factor of $f(x)$.

Example 8

The polynomial $x^3 - ax^2 + 2x + 8$, where a is a constant, is denoted by $p(x)$.

It is given that $(x - 2)$ is a factor of $p(x)$.

a) **Evaluate** a.

b) When a has this value, factorise $p(x)$ completely.

a) $p(2) = 8 - 4a + 4 + 8 = 0$

$4a = 20 \Rightarrow a = 5$

b) We can factorise $x^3 - 5x^2 + 2x + 8$ using either (**i**) long division or (**ii**) testing other factors using the factor theorem.

Continued on the next page

i)

$$
\begin{array}{r}
x^2 - 3x - 4 \\
x - 2\ \overline{)\,x^3 - 5x^2 + 2x + 8} \\
\underline{x^3 - 2x^2} \\
-3x^2 + 2x + 8 \\
\underline{-3x^2 + 6x} \\
-4x + 8 \\
\underline{-4x + 8}
\end{array}
$$

Put $a = 5$.

You would expect there to be no remainder since $x - 2$ is a factor.

and $x^2 - 3x - 4 = (x - 4)(x + 1)$

Factorise the quotient.

So $p(x) = (x - 2)(x - 4)(x + 1)$

ii) $f(+1) = 1 - 5 + 2 + 8 \neq 0$ $(x - 1)$ is not a factor

Try a value of x.

$f(-1) = -1 - 5 - 2 + 8 = 0$ $(x + 1)$ is a factor

$f(4) = 64 - 80 + 8 + 8 = 0$ $(x - 4)$ is a factor

So $p(x) = (x - 2)(x - 4)(x + 1)$

Note: Instead of performing this last trial we could work out that the final factor is $(x - 4)$ as we know $x \times x \times x = x^3$ and $(-2) \times (+1) \times (-4) = +8$.

Example 9

Solve $x^3 - 3x^2 - 4x + 12 = 0$.

Let $f(x) = x^3 - 3x^2 - 4x + 12$.

To solve, we must first factorise $x^3 - 3x^2 - 4x + 12$.

$f(1) = 1 - 3 - 4 + 12 \neq 0$

so $(x - 1)$ is not a factor.

Trial any value of x that is a factor of 12.

$f(2) = 8 - 12 - 8 + 12 = 0$

so $(x - 2)$ is a factor.

Alternatively, at this stage you could also do a long division to find the other two factors since you already know one factor.

$f(-2) = -8 - 12 + 8 + 12 = 0$

so $(x + 2)$ is a factor.

We can deduce that the third factor is $(x - 3)$.

$12 \div 2 \div -2 = -3$ (and the coefficient of x^3 is 1).

$f(3) = 27 - 27 - 12 + 12$

so $(x - 3)$ is a factor.

Thus $(x - 2)(x + 2)(x - 3) = 0$

$x = 2$ or $x = -2$ or $x = 3$

Example 10

The polynomial $ax^3 + x^2 + bx + 6$, where a and b are constants, is denoted by $p(x)$. It is given that $(2x - 1)$ is a factor of $p(x)$ and that when $p(x)$ is divided by $(x - 1)$ the remainder is -4. Find the values of a and b.

$p(x) = ax^3 + x^2 + bx + 6$

$p\left(\frac{1}{2}\right) = \frac{a}{8} + \frac{1}{4} + \frac{b}{2} + 6 = 0$ — $p\left(\frac{1}{2}\right) = 0$ as $(2x - 1)$ is a factor.

$a + 2 + 4b + 48 = 0$ — Multiply each term by 8.

$a + 4b = -50 \qquad (1)$

$p(1) = a + 1 + b + 6 = -4$ — $p(1) = -4$ as the remainder is -4.

$a + b = -11 \qquad (2)$

$(1) - (2) \Rightarrow 3b = -39$ — Solve the two equations simultaneously.

$b = -13$

$(2) \Rightarrow \quad a - 13 = -11$

$a = 2$

$a = 2$ and $b = -13$

Exercise 1.5

1. Factorise the following as a product of three linear factors.
In each case, one of the factors has been given.

a) $2x^3 - 5x^2 - 4x + 3$ One factor is $(x - 3)$.

b) $x^3 - 6x^2 + 11x - 6$ One factor is $(x - 2)$.

c) $5x^3 + 14x^2 + 7x - 2$ One factor is $(5x - 1)$.

d) $2x^3 + 3x^2 - 18x + 8$ One factor is $(x + 4)$.

e) $x^3 + x^2 - 4x - 4$ One factor is $(x + 2)$.

f) $6x^3 + 13x^2 - 4$ One factor is $(3x + 2)$.

2. Solve the following equations.

a) $2x^3 + 7x^2 - 7x - 12 = 0$

b) $2x^3 - 5x^2 - 14x + 8 = 0$

c) $x^3 - 6x^2 + 3x + 10 = 0$

d) $x^3 + 3x^2 - 6x - 8 = 0$

e) $2x^3 - 15x^2 + 13x + 60 = 0$

f) $3x^3 - 2x^2 - 7x - 2 = 0$

3. Show that $(x - 3)$ is a factor of $x^5 - 3x^4 + x^3 - 4x - 15$.

4. Factorise $x^4 + x^3 - 7x^2 - x + 6$ as a product of four linear factors.

5. $(x-2)$ is a factor of $x^3 - 3x^2 + ax - 10$. Evaluate the coefficient a.

6. a) Show that $(2x-5)$ is a factor of $4x^3 - 20x^2 + 19x + 15$.
   b) Hence factorise $4x^3 - 20x^2 + 19x + 15$ as a product of three linear factors.

7. The polynomial $ax^3 - 3x^2 - 5ax - 9$ is denoted by $p(x)$ where a is a real number. It is given that $(x-a)$ is a factor of $p(x)$. Find the possible values of a.

8. The polynomial $3x^3 + 2x^2 - bx + a$, where a and b are constants, is denoted by $p(x)$. It is given that $(x-1)$ is a factor of $p(x)$ and that when $p(x)$ is divided by $(x+1)$ the remainder is 10. Find the values of a and b.

9. The polynomial $ax^3 + bx^2 - 5x + 3$, where a and b are constants, is denoted by $p(x)$. It is given that $(2x-1)$ is a factor of $p(x)$ and that when $p(x)$ is divided by $(x-1)$ the remainder is -3. Find the remainder when $p(x)$ is divided by $(x+3)$.

10. Factorise $2x^4 + 5x^3 - 5x - 2$ as a product of four linear factors.

Summary exercise 1

1. Solve algebraically the equation $|5 - 2x| = 7$.

2. Solve algebraically the equation $|3x - 4| = |5 - 2x|$.

3. Sketch the following graphs:
   a) $y = 2|x| + 5$
   b) $y = 2 - |x|$.

4. Solve graphically the inequality $2|x - 2| < |x|$.

5. Solve graphically the inequality $|2x - 1| < |3x - 4|$.

EXAM-STYLE QUESTION

6. Solve the inequality $|x + 3| \geq 2|x - 3|$.

7. Solve the inequality $|x - 2| \leq 3|x + 1|$.

EXAM-STYLE QUESTION

8. Solve the inequality $2|x - a| > |2x + a|$ where a is a constant and $a > 0$.

9. Divide $2x^4 - 9x^3 + 13x^2 - 15x + 9$ by $(x-3)$.

10. Find the quotient and the remainder when $x^3 - 3x^2 + 6x + 1$ is divided by $(x-2)$.

EXAM-STYLE QUESTIONS

11. a) Show that $(x-4)$ is a factor of $x^3 - 3x^2 - 10x + 24$.
    b) Hence factorise $x^3 - 3x^2 - 10x + 24$.

12. The expression $x^3 + 3x^2 + 6x + k$ has a remainder of -3 when divided by $(x+1)$. Find the value of k.

13. The polynomial $ax^4 + bx^3 - 8x^2 + 6$ is denoted by $p(x)$. When $p(x)$ is divided by $(x^2 - 1)$ the remainder is $2x + 1$. Find the value of a and the value of b.

14. The polynomial $x^4 + ax^3 + bx^2 - 16x - 12$ is denoted by $p(x)$.
    $(x+1)$ and $(x-2)$ are factors of $p(x)$.
    a) Evaluate the coefficients a and b.
    b) Hence factorise $p(x)$ fully.

15. The polynomial $x^4 + x^3 - 22x^2 - 16x + 96$ is denoted by $p(x)$.

   a) Find the quotient when $p(x)$ is divided by $x^2 + x - 6$.

   b) Hence solve the equation $p(x) = 0$.

16. The polynomial $6x^3 - 23x^2 + ax + b$ is denoted by $p(x)$. When $p(x)$ is divided by $(x + 1)$ the remainder is -21. When $p(x)$ is divided by $(x - 3)$ the remainder is 11.

   a) Find the value of a and the value of b.

   b) Hence factorise $p(x)$ fully.

17. A polynomial is defined by $p(x) = x^3 + Ax^2 + 49x - 36$, where A is a constant. $(x - 9)$ is a factor of $p(x)$.

   a) Find the value of A.

   b) i) Find all the roots of the equation $p(x) = 0$.

      ii) Find all the roots of the equation $p(x^2) = 0$.

18. The polynomial $x^3 - 15x^2 + Ax + B$, where A and B are constants, is denoted by $p(x)$. $(x - 16)$ is a factor of $p(x)$. When $p(x)$ is divided by $(x - 2)$ the remainder is -56.

   a) Find the value of A and the value of B.

   b) i) Find all 3 roots of the equation $p(x) = 0$.

      ii) Find the 4 real roots of the equation $p(x^4) = 0$.

19. i) Find the quotient and remainder when
   $x^4 + 2x^3 + x^2 + 20x - 25$
   is divided by $(x^2 + 2x - 5)$.

   ii) It is given that, when
   $x^4 + 2x^3 + x^2 + px + q$
   is divided by $(x^2 + 2x - 5)$, there is no remainder. Find the values of the constants p and q.

   iii) When p and q have these values, show that there are exactly two real values of x satisfying the equation
   $x^4 + 2x^3 + x^2 + px + q = 0$
   and state what these values are. Give your answer in the form $a \pm \sqrt{b}$.

Chapter summary

The modulus function

- The modulus of a real number is the magnitude of that number.
- The modulus function $f(x) = |x|$ is defined as

  $|x| = x \quad \text{for} \quad x \geq 0$

  $|x| = -x \quad \text{for} \quad x < 0$

Sketching graphs of the modulus function

- When sketching the graph of $y = |f(x)|$ we reflect the section of the graph where $y < 0$ in the x-axis.
- When sketching the graph of $y = f(|x|)$ we sketch the section of the graph where $x > 0$ and then reflect this in the y-axis.

Division of polynomials

- When dividing algebraic expressions, for example $(4x^3 - 7x - 3) \div (2x + 1) = (2x^2 - x - 3)$, you need to know the following terms:
  - $(4x^3 - 7x - 3)$ is called the dividend.
  - $(2x + 1)$ is called the divisor.
  - $(2x^2 - x - 3)$ is called the quotient, and there is no remainder.
  - $(2x + 1)$ is a factor of $(4x^3 - 7x - 3)$.
- $f(x) = \text{quotient} \times \text{divisor} + \text{remainder}$

The remainder theorem

- When a polynomial $f(x)$ is divided by $(x - a)$, the remainder is $f(a)$.
- When a polynomial $f(x)$ is divided by $(ax - b)$, the remainder is $f\left(\frac{b}{a}\right)$.

The factor theorem

- For any polynomial $f(x)$, if $f(a) = 0$ then the remainder when $f(x)$ is divided by $(x - a)$ is zero. Thus $(x - a)$ is a factor of $f(x)$.
- For any polynomial $f(x)$, if $f\left(\frac{b}{a}\right) = 0$, then $(ax - b)$ is a factor of $f(x)$.

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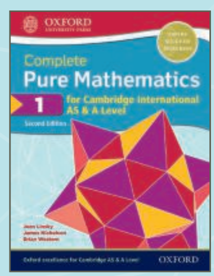

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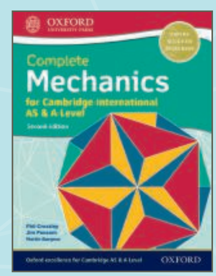

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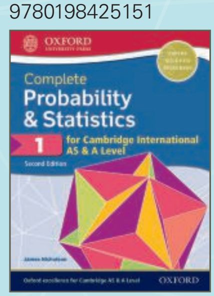

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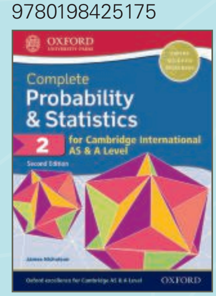

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