Chapter 2

Complex Hyperbolic Geometry

In complex hyperbolic geometry we consider an open set biholomorphic to an open ball in $\mathbb{C}^n$, and we equip it with a particular metric that makes it have constant negative holomorphic curvature. This is analogous to but different from the real hyperbolic space. In the complex case, the sectional curvature is constant on complex lines, but it changes when we consider real 2-planes which are not complex lines.

Complex hyperbolic geometry contains in it the real version: while every complex $\mathbb{C}^k$-*plane* in $\mathbb{H}^n_{\mathbb{C}}$ is biholomorphically isometric to $\mathbb{H}^k_{\mathbb{C}}$, every totally real k-plane in $\mathbb{H}^n_{\mathbb{C}}$ is isometric to the real hyperbolic space $\mathbb{H}^k_{\mathbb{R}}$. Moreover, every complex line in $\mathbb{H}^n_{\mathbb{C}}$ is biholomorphically isometric to $\mathbb{H}^2_{\mathbb{R}}$.

There are three classical models for complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$: the unit ball model in $\mathbb{C}^n$, the projective ball model in $\mathbb{P}^n_{\mathbb{C}}$ and the Siegel domain model. In this monograph we normally use the projective ball model, and the symbol $\mathbb{H}^n_{\mathbb{C}}$ will be reserved for that. These models for complex hyperbolic n-space are briefly discussed in Sections 2 and 3, where we also explain and define some basic notions which are used in the sequel, such as chains, bisectors and spinal spheres, horospherical coordinates and the Heisenberg geometry at infinity. We refer to [67], [71], [168], [200], [169] for rich accounts on complex hyperbolic geometry.

In Section 4 we discuss the geometry, dynamics and algebraic properties of the elements in the projective Lorentz group $\mathrm{PU}(n,1)$, which is the group of holomorphic isometries of $\mathbb{H}^n_{\mathbb{C}}$. We give in detail Goldman's proof of the classification according to trace, since this paves the ground for the discussion in Chapter 4.

In Section 5 we discuss methods and sources for constructing complex hyperbolic Kleinian groups, i.e., discrete subgroups of $\mathrm{PU}(n,1)$. For further reading on this we refer to the excellent survey articles [105], [168] and the bibliography in them. As explained in [168], the methods for constructing complex hyperbolic lattices can be roughly classified into four types: i) Arithmetic constructions; ii) reflection groups and construction of appropriate fundamental domains; iii) algebraic constructions, via the Yau-Miyaoka uniformisation theorem; and iv) as

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monodromy groups of certain hypergeometric functions. In fact, a fundamental problem in complex hyperbolic geometry is whether or not there are nonarithmetic lattices in dimensions ≥ 3 (see for instance [50]).

Finally, Section 6 is based on [45]. Here we discuss the definition of limit set for complex hyperbolic Kleinian groups and give some of its properties. This is analogous to the corresponding definition in real hyperbolic geometry. The similarities in both settings spring from the fact that in both situations one has the convergence property (see [105, §3.2]): every sequence of isometries either contains a convergent subsequence or contains a subsequence which converges to a constant map away from a point on the sphere at infinity. This property is not satisfied by discrete subgroups of $\mathrm{PSL}(n+1,\mathbb{C})$ in general, and this is why even the concept of limit set becomes intriguing in that general setting. This will be explored in the following chapters.

2.1 Some basic facts on Projective geometry

We recall that the complex projective space $\mathbb{P}^n_{\mathbb{C}}$ is defined as

$$\mathbb{P}^n_{\mathbb{C}} = (\mathbb{C}^{n+1} - \{0\})/\sim,$$

where "$\sim$" denotes the equivalence relation given by $x \sim y$ if and only if $x = \lambda y$ for some nonzero complex scalar λ. This is a compact connected complex n-dimensional manifold, diffeomorphic to the orbit space $\mathbb{S}^{2n+1}/\mathrm{U}(1)$, where $\mathrm{U}(1)$ is acting coordinate-wise on the unit sphere in $\mathbb{C}^{n+1}$.

We notice that the usual Riemannian metric on $\mathbb{S}^{2n+1}$ is invariant under the action of $\mathrm{U}(1)$ and therefore descends to a Riemannian metric on $\mathbb{P}^n_{\mathbb{C}}$, which is known as the *Fubini-Study* metric.

If $[\;]_n : \mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n_{\mathbb{C}}$ represents the quotient map, then a nonempty set $H \subset \mathbb{P}^n_{\mathbb{C}}$ is said to be a projective subspace of dimension k if there is a $\mathbb{C}$-linear subspace $\widetilde{H} \subset \mathbb{C}^{n+1}$ of dimension $k+1$ such that $[\widetilde{H}]_n = H$. If no confusion arises, we will denote the map $[\;]_n$ just by $[\;]$. Given a subset P in $\mathbb{P}^n_{\mathbb{C}}$, we define

$$\langle P \rangle = \bigcap \{ l \subset \mathbb{P}^n_{\mathbb{C}} \mid l \text{ is a projective subspace and } P \subset l \}.$$

Then $\langle P \rangle$ is a projective subspace of $\mathbb{P}^n_{\mathbb{C}}$, see for instance [125]. In particular, given $p, q \in \mathbb{P}^n_{\mathbb{C}}$ distinct points, $\langle \{p,q\} \rangle$ is the unique proper complex projective subspace passing through p and q. Such a subspace will be called a complex (projective) line and denoted by $\overleftrightarrow{p,q}$; this is the image under $[\;]_n$ of a two-dimensional linear subspace of $\mathbb{C}^{n+1}$. Observe that if ℓ_1, ℓ_2 are different complex lines in $\mathbb{P}^2_{\mathbb{C}}$, then $\ell_1 \cap \ell_2$ consists of exactly one point.

If $e_1, \ldots, e_{n+1}$ denotes the elements of the standard basis in $\mathbb{C}^{n+1}$, we will use the same symbols to denote their images under $[\;]_n$.

It is clear that every linear automorphism of $\mathbb{C}^{n+1}$ defines a holomorphic automorphism of $\mathbb{P}^n_{\mathbb{C}}$, and it is well known (see for instance [37]) that every automorphism of $\mathbb{P}^n_{\mathbb{C}}$ arises in this way. Thus one has

Theorem 2.1.1. *The group of projective automorphisms is:*

$$\mathrm{PSL}(n+1,\mathbb{C}) \;:=\; \mathrm{GL}\,(n+1,\mathbb{C})/(\mathbb{C}^*)^{n+1} \cong \mathrm{SL}(n+1,\mathbb{C})/\mathbb{Z}_{n+1}\,,$$

where $(\mathbb{C}^*)^{n+1}$ *is being regarded as the subgroup of diagonal matrices with a single nonzero eigenvalue, and we consider the action of* $\mathbb{Z}_{n+1}$ *(viewed as the roots of the unity) on* $\mathrm{SL}(n+1,\mathbb{C})$ *given by the usual scalar multiplication.*

This result is in fact a special case of a more general, well-known theorem, stating that every holomorphic endomorphism $f:\mathbb{P}^n_{\mathbb{C}}\to\mathbb{P}^n_{\mathbb{C}}$ is induced by a polynomial self-map $F=(F_0,\dots,F_n)$ of $\mathbb{C}^{n+1}$ such that $F^{-1}(0)=\{0\}$ and the components F_i are all homogeneous polynomials of the same degree. The case we envisage here is when these polynomials are actually linear.

We denote by $[[\;]]_{n+1}:\mathrm{SL}(n+1,\mathbb{C})\to\mathrm{PSL}(n+1,\mathbb{C})$ the quotient map. Given $\gamma\in\mathrm{PSL}(n+1,\mathbb{C})$ we say that $\tilde{\gamma}\in\mathrm{GL}\,(n+1,\mathbb{C})$ is a lift of γ if there is an scalar $r\in\mathbb{C}^*$ such that $r\tilde{\gamma}\in\mathrm{SL}(n,\mathbb{C})$ and $[[r\tilde{\gamma}]]_{n+1}=\gamma$.

Notice that $\mathrm{PSL}(n+1,\mathbb{C})$ acts transitively, effectively and by biholomorphisms on $\mathbb{P}^n_{\mathbb{C}}$, taking projective subspaces into projective subspaces.

There are two classical ways of decomposing the projective space that will play a significant role in the sequel; each provides a rich source of discrete subgroups of $\mathrm{PSL}(n+1,\mathbb{C})$. The first is by thinking of $\mathbb{P}^n_{\mathbb{C}}$ as being the union of $\mathbb{C}^n$ and the "hyperplane at infinity":

$$\mathbb{P}^n_{\mathbb{C}} \;=\; \mathbb{C}^n\cup\mathbb{P}^{n-1}_{\mathbb{C}}\,.$$

A way for doing so is by writing

$$\mathbb{C}^{n+1}=\mathbb{C}^n\times\mathbb{C}=\{(Z,z_{n+1})\,|\,Z=(z_1,\dots,z_n)\in\mathbb{C}^n\,\text{and}\,z_{n+1}\in\mathbb{C}\}\,.$$

Then every point in the hyperplane $\{(Z,1)\}$ determines a unique line through the origin in $\mathbb{C}^{n+1}$, i.e., a point in $\mathbb{P}^n_{\mathbb{C}}$; and every point in $\mathbb{P}^n_{\mathbb{C}}$ is obtained in this way except for those corresponding to lines (or "directions") in the hyperplane $\{(Z,0)\}$, which form the "hyperplane at infinity" $\mathbb{P}^{n-1}_{\mathbb{C}}$.

It is clear that every affine map of $\mathbb{C}^{n+1}$ leaves invariant the hyperplane at infinity $\mathbb{P}^{n-1}_{\mathbb{C}}$. Furthermore, every such map carries lines in $\mathbb{C}^{n+1}$ into lines in $\mathbb{C}^{n+1}$, so the map naturally extends to the hyperplane at infinity. This gives a natural inclusion of the affine group

$$\mathrm{Aff}\,(\mathbb{C}^n)\;\cong\mathrm{GL}\,(n,\mathbb{C})\ltimes\mathbb{C}^n\,,$$

in the projective group $\mathrm{PSL}(n+1,\mathbb{C})$. Hence every discrete subgroup of $\mathrm{Aff}\,(\mathbb{C}^n)$ is a discrete subgroup of $\mathrm{PSL}(n+1,\mathbb{C})$.

The second classical way of decomposing the projective space that plays a significant role in this monograph leads to complex hyperbolic geometry, which we study in the following section. For this we think of $\mathbb{C}^{n+1}$ as being a union

$V_- \cup V_0 \cup V_+$, where each of these sets consists of the points $(Z, z_{n+1}) \in \mathbb{C}^{n+1}$ satisfying that $\|Z\|^2$ is respectively smaller, equal to or larger than $|z_{n+1}|$.

We will see in the following section that the projectivisation of V_- is an open $(2n)$-ball $\mathbb{B}$ in $\mathbb{P}^n_{\mathbb{C}}$, bounded by $[V_0]$, which is a sphere. This ball $\mathbb{B}$ serves as model for complex hyperbolic geometry. Its full group of holomorphic isometries is $\mathrm{PU}(n,1)$, the subgroup of $\mathrm{PSL}(n+1,\mathbb{C})$ of projective automorphisms that preserve $\mathbb{B}$. This gives a second natural source of discrete subgroups of $\mathrm{PSL}(n+1,\mathbb{C})$, those coming from complex hyperbolic geometry.

We finish this section with some results about subgroups of $\mathrm{PSL}(n+1,\mathbb{C})$ that will be used later in the text.

Proposition 2.1.2. *Let $\Gamma \subset \mathrm{PSL}(n+1,\mathbb{C})$ be a discrete group. Then Γ is finite if and only if every element in Γ has finite order.*

This proposition follows from the theorem below (see [182, Theorem 8.29]) and its corollary; see [160], [41] for details.

Theorem 2.1.3 (Jordan)**.** *For any $n \in \mathbb{N}$ there is an integer $S(n)$ with the following property: If $G \subset \mathrm{GL}(n,\mathbb{C})$ is a finite subgroup, then G admits an abelian normal subgroup N such that* $\mathrm{card}(G) \leq S(n)\mathrm{card}(N)$.

Corollary 2.1.4. *Let G be a countable subgroup of* $\mathrm{GL}(3,\mathbb{C})$*, then there is an infinite commutative subgroup N of G.*

Proof of Proposition 2.1.2. If every element in G has finite order, then by Selberg's lemma (see for instance [186]) it follows that G has an infinite set of generators, say $\{\gamma_m\}_{m\in\mathbb{N}}$. Define

$$A_m = \langle \gamma_1, \ldots, \gamma_m \rangle,$$

then by Selberg's lemma A_m is finite and by Theorem 2.1.3 there is a normal commutative subgroup $N(A_m)$ of A_m such that

$$\mathrm{card}(A_m) \leq S(3)\mathrm{card}(N(A_m)).$$

Assume, without loss of generality, that $\mathrm{card}(A_m) = k_0\mathrm{card}(N(A_m))$ for some k_0 and every m. Set

$$n_m = \max\{o(g) : g \in N(A_m)\},$$

where $o(g)$ represents the order of g, and consider the following cases:

Case 1. The sequence $(n_m)_{m\in\mathbb{N}}$ is unbounded.

In this case we can assume that

$$k_0 n_j < n_{j+1} \text{ for all } j.$$

Now for each m, consider $\gamma_m \in N(A_m)$ such that $o(\gamma_m) = n_m$. Thus $\gamma_m^{k_0} \in \bigcap_{m\leq j} N(A_j)$ and $\gamma_m^{k_0} \neq \gamma_j^{k_0}$. Hence $\langle \gamma_m^{k_0} : m \in \mathbb{N}\rangle$ is an infinite commutative subgroup of G.

Case 2. The sequence $(n_m)_{m\in\mathbb{N}}$ is bounded.

We may assume that $n_m = c_0$ for every index m. Let us construct the following sequence:

Step 1. Assume that $\text{card}(N(A_1)) > k_0 c_o^3$. For each $m > 1$, consider the map

$$\phi_{1,m} : N(A_1) \longrightarrow A_m/N(A_m)$$

given by $l \mapsto N(A_m)l$. Since $\text{card}(N(A_1)) > \text{card}(A_m/N(A_m)) = k_0$, we deduce that $\phi_{1,m}$ is not injective. Then there is an element $w_1 \neq id$ and a subsequence $(B_n)_{n\in\mathbb{N}} \subset (A_n)_{n\in\mathbb{N}}$ such that $\text{card}(N(B_1)) > k_0 c_o^4$ and $w_1 \in \bigcap_{m\in\mathbb{N}} N(B_m)$.

Step 2. For every $m\in\mathbb{N}$ consider the map $\phi_{2,m} : N(B_1)/\langle w_1\rangle \longrightarrow B_m/N(B_m)$ given by $\langle w_1\rangle l \mapsto N(B_m)l$. As in step 1 we can deduce that there is an element w_2 and a subsequence $(C_n)_{n\in\mathbb{N}} \subset (B_n)_{n\in\mathbb{N}}$ such that $\text{card}(N(C_1)) > k_0 c_o^5$ and $w_2 \in \bigcap_{m\in\mathbb{N}} N(C_m) - \langle w_1\rangle$.

Step 3. For $m \in \mathbb{N}$ consider the map $\phi_{3,m} : N(C_1)/\langle w_1, w_2\rangle \longrightarrow C_m/N(C_m)$ given by $\langle w_1, w_2\rangle l \mapsto N(C_m)l$. As in step 2 we deduce that there is an element w_3 and a subsequence $(D_n)_{n\in\mathbb{N}} \subset (C_n)_{n\in\mathbb{N}}$ such that $\text{card}(N(A_1)) > k_0 c_o^6$ and $w_3 \in \bigcap_{m\in\mathbb{N}} N(D_m) - \langle w_1, w_2\rangle$.

Continuing this process ad infinitum we deduce that $\langle w_m : m \in \mathbb{N}\rangle$ is an infinite commutative subgroup. $\square$

2.2 Complex hyperbolic geometry. The ball model

There are three classical models for complex hyperbolic n-space, namely:

(i) the unit ball model in $\mathbb{C}^n$;

(ii) the projective ball model in $\mathbb{P}^n_{\mathbb{C}}$; and

(iii) the Siegel domain model in $\mathbb{C}^n$.

Let us discuss first the ball models for complex hyperbolic space. For this, let $\mathbb{C}^{n,1}$ denote the vector space $\mathbb{C}^{n+1}$ equipped with the Hermitian form $\langle\, , \rangle$ given by

$$\langle u, v\rangle = u_1\overline{v}_1 + \cdots + u_n\overline{v}_n - u_{n+1}\overline{v}_{n+1}\,,$$

where $u = (u_1, u_2, \ldots, u_{n+1})$ and $v = (v_1, v_2, \ldots, v_{n+1})$. This form corresponds to the Hermitian matrix

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

One obviously has $\langle u, v\rangle = uHv^*$, where v^* is the Hermitian adjoint of v, i.e., it is the column vector with entries $\bar{v}_1$, $i = 1, \ldots, n+1$. Notice H has n positive eigenvalues and a negative one, so it has signature $(n, 1)$.

As before, we think of $\mathbb{C}^{n+1} \approx \mathbb{C}^{n,1}$ as being the union $V_- \cup V_0 \cup V_+$ of negative, null and positive vectors z, depending respectively (in the obvious way) on the sign of $\langle z, z\rangle = |z_1|^2 + \cdots + |z_n|^2 - |z_{n+1}|^2$. It is clear that each of the three sets $V_- \cup V_0 \cup V_+$ is a union of complex lines; that is, if a vector v is in V_-, then every complex multiple of v is a negative vector, and similarly for V_0 and V_+. The set V_0 is often called *the cone of light*, or the space of null vectors for the quadratic form $Q(z) = \langle z, z\rangle$.

We now look at the intersection of V_0 and V_- with the hyperplane in $\mathbb{C}^{n,1}$ defined by $z_{n+1} = 1$. For V_0 we get the $(2n-1)$-sphere

$$\mathbb{S} := \{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} \,\big|\, |z_1|^2 + \cdots + |z_n|^2 = 1\}.$$

For V_- we get the ball $\mathbb{B}$ bounded by $\mathbb{S}$:

$$\mathbb{B} := \{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} \,\big|\, |z_1|^2 + \cdots + |z_n|^2 < 1\}.$$

This ball, equipped with the complex hyperbolic metric serves as a model for complex hyperbolic geometry, this is the unit ball model of complex hyperbolic space. We refer to [169] for details on this model and for beautiful explanations about the way in which it relates to the Siegel domain model that we explain in the following section.

To get the complex hyperbolic space we must endow $\mathbb{B}$ with the appropriate metric. We do so in a similar way to how we did it in Chapter 1 for the real hyperbolic space. Consider the group $\mathrm{U}(n,1)$ of elements in $\mathrm{GL}\,(n+1,\mathbb{C})$ that preserve the above Hermitian form. That is, we consider matrices A satisfying $A^*HA = H$, where A^* is the Hermitian transpose of A (that is, each column vector v with entries $v_0, v_1, \dots, v_n$, is replaced by its transpose v^*, the row vector $(\bar{v}_0, \bar{v}_1, \dots, \bar{v}_n)$). It is easy to see that $\mathrm{U}(n,1)$ acts transitively on $\mathbb{B}$ with isotropy $\mathrm{U}(n)$ (see [67, Lemma 3.1.3]). In fact $\mathrm{U}(n,1)$ acts transitively on the space of negative lines in $\mathbb{C}^{n,1}$.

Let $\underline{0} = (0, \dots, 0, 1)$ denote the centre of the ball $\mathbb{B}$, consider the space $T_{\underline{0}}\mathbb{B} \cong \mathbb{C}^n$ tangent to $\mathbb{B}$ at $\underline{0}$, and put on it the usual Hermitian metric on $\mathbb{C}^n$. Now we use the action of $\mathrm{U}(n,1)$ to spread the metric to all tangent spaces $T_x\mathbb{H}^n_{\mathbb{C}}$, using that the action is transitive and the isotropy is $\mathrm{U}(n)$, which preserves the usual metric on $\mathbb{C}^n$. We thus get a Hermitian metric on $\mathbb{B}$, which is clearly homogeneous. This is the complex hyperbolic metric, and the ball $\mathbb{B}$, equipped with this metric, serves as a model for complex hyperbolic n-space $\mathbb{H}^n_{\mathbb{C}}$. This is the *unit ball model for complex hyperbolic space*. The boundary $\partial\mathbb{H}^n_{\mathbb{C}}$ is called *the sphere at infinity* (it is called *the absolute* in [67]).

Notice that this way of constructing a model for the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$ is entirely analogous to the method used in Chapter 1 to construct the real hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$. Yet, there is one significant difference. In the real case the action of $\mathrm{Iso}(\mathbb{H}^n_{\mathbb{R}})$ on the unit ball has isotropy $O(n)$ and this group acts transitively on the spaces of lines and 2-planes through a given point. Thence $\mathbb{H}^n_{\mathbb{R}}$ has constant sectional curvature. In the complex case, the corresponding isotropy group is $\mathrm{U}(n)$, which acts transitively on the space of complex lines through a given point, but it does not act transitively on the space of real 2-planes: a totally real plane cannot be taken into a complex line by an element in $\mathrm{U}(n)$. Therefore the sectional curvature of $\mathbb{H}^n_{\mathbb{C}}$ is not constant, though it has constant holomorphic curvature.

Observe that for $n = 1$ one gets the complex hyperbolic line $\mathbb{H}^1_{\mathbb{C}}$. This corresponds to the unit ball

$$\{(z_1, z_2) \in \mathbb{C}^2 \big| \, |z_1| < 1 \,\text{and}\, z_2 = 1\,\}\,.$$

Notice that $\mathrm{U}(1)$ is isomorphic to $\mathrm{SO}(2)$, hence $\mathbb{H}^1_{\mathbb{C}}$ is biholomorphically isometric to the open ball model of the real hyperbolic space $\mathbb{H}^2_{\mathbb{R}}$. Moreover, since $\mathrm{U}(n,1)$ acts transitively on the space of negative lines in $\mathbb{C}^{n,1}$, every such line can be taken into the line spanned by the vector $\{(0,\dots,0,1)\} \subset \mathbb{C}^{n,1}$ and the above considerations essentially show that the induced metric on the unit ball in this complex line corresponds to the usual real hyperbolic metric on the ball model for $\mathbb{H}^2_{\mathbb{R}}$. That is: *every complex line that meets $\mathbb{H}^n_{\mathbb{C}}$ determines an embedded copy of $\mathbb{H}^1_{\mathbb{C}} \cong \mathbb{H}^2_{\mathbb{R}}$* (see [67, §1.4] or [169, §5.2] for clear accounts on $\mathbb{H}^1_{\mathbb{C}}$).

It is now easy to construct the projective ball model for complex hyperbolic space, which is the model we actually use in the sequel. For this we notice that if a complex line through the origin $0 \in \mathbb{C}^{n+1}$ is null, then it meets the above sphere $\mathbb{S}$ at exactly one point. Hence the projectivisation $(V_0\setminus\{0\})/\mathbb{C}^*$ of V_0 is diffeomorphic to the $(2n-1)$-sphere $\mathbb{S}$. Similar considerations apply for the negative lines, so the projectivisation $[V_-]$ is the open $2n$-ball $[\mathbb{B}]$ bounded by the sphere $[S] = [V_-]$.

The ball $[\mathbb{B}]$ in $\mathbb{P}^n_{\mathbb{C}}$ can be equipped with the metric coming from the complex hyperbolic metric in $\mathbb{B}$, and we get the *projective ball model for complex hyperbolic space*. From now on, unless it is stated otherwise, the symbol $\mathbb{H}^n_{\mathbb{C}}$ will denote this model for complex hyperbolic space. The corresponding Hermitian metric is the *Bergmann metric*, up to multiplication by a constant. It is clear from the above construction that the projective Lorentz group $\mathrm{PU}(n,1)$ acts on $\mathbb{H}^n_{\mathbb{C}}$ as its the group of holomorphic isometries.

In [67, 3.1.7] there is an algebraic expression for the distance function in complex hyperbolic space which is useful, among other things, for making computations. For this, recall first that in Euclidean space the distance function is determined by the usual inner product, and this is closely related with the angle between vectors x, y:

$$\cos(\angle(x,y)) \;=\; \frac{|x\cdot y|}{\|x\|\|y\|} \;=\; \sqrt{\hat\delta(x,y)}\,,$$

where $\hat\delta(x,y) \;=\; \frac{(x\cdot y)(y\cdot x)}{(x\cdot x)(y\cdot y)}$. Similarly, the Bergmann metric can be expressed (up to multiplication by a constant) as follows: given points $[x], [y] \in \mathbb{H}^n_{\mathbb{C}}$, their complex hyperbolic distance ρ is:

$$\rho([x],[y]) = 2\cosh^{-1}(\sqrt{\delta(x,y)});$$

$$\delta(x,y) = \frac{\langle x,y\rangle\langle y,x\rangle}{\langle x,x\rangle\langle y,y\rangle},$$

where $\langle\, ,\,\rangle$ is now the Hermitian product given by the quadratic form Q used to construct $\mathbb{H}^n_{\mathbb{C}}$.

2.2.1 Totally geodesic subspaces

Given a point $z \in \mathbb{H}^n_{\mathbb{C}}$ one wishes to know what the geodesics through z, and more generally the totally geodesic subspaces of $\mathbb{H}^n_{\mathbb{C}}$, look like. Here is a first answer. Consider a complex projective line $\mathcal{L}$ in $\mathbb{P}^n_{\mathbb{C}}$ passing by z. Then $\mathcal{L}\cap\mathbb{H}^n_{\mathbb{C}}$ is a holomorphic submanifold of $\mathbb{H}^n_{\mathbb{C}}$ and we already know (see the previous section or [67, Theorem 3.1.10]) that $\mathcal{L}\cap\mathbb{H}^n_{\mathbb{C}}$ is isometric to $\mathbb{H}^1_{\mathbb{C}}$; the latter can be regarded as $\mathbb{H}^2_{\mathbb{R}}$ equipped with Poincare's ball model for real hyperbolic geometry. This "2-plane" $\mathcal{L}\cap\mathbb{H}^n_{\mathbb{C}}$ is totally geodesic, i.e., every geodesic in $\mathbb{H}^n_{\mathbb{C}}$ joining two points in $\mathcal{L}\cap\mathbb{H}^n_{\mathbb{C}}$ is actually contained in $\mathcal{L}\cap\mathbb{H}^n_{\mathbb{C}} \cong \mathbb{H}^1_{\mathbb{C}}$. This type of surfaces in $\mathbb{H}^n_{\mathbb{C}}$ are called *complex geodesics* (they are also called *complex slices*). They have constant negative curvature for the Bergman metric, and we assume this metric has been scaled so that these slices have constant sectional curvature -1 (see [67]). The intersection of the projective line $\mathcal{L}$ with the boundary $\partial\mathbb{H}^n_{\mathbb{C}}$ is a circle $\mathbb{S}^1$. This kind of circles in the sphere at infinity, which bound a complex slice, are called *chains.*

Notice that two distinct points z_1, z_2 in $\mathbb{H}^n_{\mathbb{C}}$ determine a unique line in $\mathbb{P}^n_{\mathbb{C}}$, so there is a unique complex geodesic $\mathcal{L}$ passing through them. There is also a unique real geodesic in $\mathcal{L}\cong\mathbb{H}^2_{\mathbb{R}}$ passing through these points. Of course this statement can be easily adapted to include the case when either one, or both, of these points is in the boundary $\partial\mathbb{H}^n_{\mathbb{C}}$.

Each real geodesic in $\mathbb{H}^n_{\mathbb{C}}$ is determined by its end points in the sphere $\partial\mathbb{H}^n_{\mathbb{C}}$. For each point $q\in\partial\mathbb{H}^n_{\mathbb{C}}$, the real geodesics in $\mathbb{H}^n_{\mathbb{C}}$ which end at q are parametrised by the points in $\mathbb{R}^{2n-1}\approx\partial\mathbb{H}^n_{\mathbb{C}}\setminus q$ and they form a *parabolic pencil* (see [67, Sections 7.27, 7.28]). Each of these real geodesics σ is contained in a complex geodesic Σ asymptotic to q, and the set of all complex geodesics end at q has the natural structure of an $(n-1)$-dimensional complex affine plane.

In fact, since each complex geodesic Σ corresponds to the intersection of the ball $\mathbb{H}^n_{\mathbb{C}}$ with a complex projective line $\mathcal{L}$, one has that Σ is asymptotic to all points in $\mathcal{L}\cap\partial\mathbb{H}^n_{\mathbb{C}}$, which form a circle $\mathbb{S}^1$. If the complex geodesic Σ ends at $q\in\partial\mathbb{H}^n_{\mathbb{C}}$, it follows that the real geodesics in Σ asymptotic to q are parametrised by $\mathbb{R}\approx(\mathcal{L}\cap\partial\mathbb{H}^n_{\mathbb{C}})\setminus\{q\}$.

More generally, if $\mathcal{P}$ is a complex projective subspace of $\mathbb{P}^n_{\mathbb{C}}$ of dimension k that passes through the point $z\in\mathbb{H}^n_{\mathbb{C}}$, then $\mathcal{P}\cap\mathbb{H}^n_{\mathbb{C}}$ is obviously a complex holomorphic submanifold of $\mathbb{H}^n_{\mathbb{C}}$. Then Theorem 3.1.10 in [67] tells us that $\mathcal{P}\cap\mathbb{H}^n_{\mathbb{C}}$ is actually a totally geodesic subspace of $\mathbb{H}^n_{\mathbb{C}}$ which is biholomorphically isometric to $\mathbb{H}^k_{\mathbb{C}}$. Such a holomorphic submanifold of $\mathbb{H}^n_{\mathbb{C}}$ is called a $\mathbb{C}^k$-*plane*; so a $\mathbb{C}^k$-*plane* is a complex geodesic. The boundary of a $\mathbb{C}^k$-*plane* is a sphere of real dimension

$2k-1$ called a $\mathbb{C}^k$-*chain*. This is the set of points where the corresponding projective plane meets the sphere at infinity $\partial\mathbb{H}^n_{\mathbb{C}}$. A $\mathbb{C}^1$-*chain* is simply a chain.

Goldman shows in his book that behind $\mathbb{C}^k$-planes, there is only another type of totally geodesic subspaces in $\mathbb{H}^n_{\mathbb{C}}$: The totally real projective subspaces:

Definition 2.2.1. Let $\widetilde{\mathcal{R}}^{k+1}$ be a linear real subspace of $\mathbb{C}^{n,1}$ of real dimension $k+1$ which contains negative vectors. We say that $\widetilde{\mathcal{R}}^{k+1}$ is *totally real* with respect to the Hermitian form Q if $J(\widetilde{\mathcal{R}}^{k+1})$ is Q-orthogonal to $\widetilde{\mathcal{R}}^{k+1}$, where J denotes complex multiplication by i. A *totally real subspace of* $\mathbb{H}^n_{\mathbb{C}}$ means the intersection with $\mathbb{H}^n_{\mathbb{C}}$ of the projectivisation $\mathcal{R}^k := [\widetilde{\mathcal{R}}^{k+1}]$ of a totally real projective subspace $\widetilde{\mathcal{R}}^{k+1}$ of $\mathbb{C}^{n,1}$. Such a plane in $\mathbb{H}^n_{\mathbb{C}}$ is called an $\mathbb{R}^k$-*plane*. (Of course this can only happen if $k \leq n$.) An $\mathbb{R}^2$-plane is called *a real slice*.

It is easy to see that $\mathrm{PU}(n,1)$ acts transitively on the set of all $\mathbb{R}^k$-planes in $\mathbb{H}^n_{\mathbb{C}}$, for every k. One has (see [67, Section 3.1]):

Theorem 2.2.2. *Every totally geodesic submanifold of* $\mathbb{H}^n_{\mathbb{C}}$ *is either a* $\mathbb{C}^k$*-plane or an* $\mathbb{R}^k$*-plane. In particular* $\mathbb{H}^n_{\mathbb{C}}$ *has no totally geodesic real submanifolds of codimension* 1 *(for* $n > 1$*). Furthermore:*

(i) *Every* $\mathbb{C}^k$*-plane, with its induced metric, is biholomorphically isometric to* $\mathbb{H}^k_{\mathbb{C}}$*. Every complex line in* $\mathbb{H}^n_{\mathbb{C}}$ *is biholomorphically isometric to Poincaré's ball model of* $\mathbb{H}^2_{\mathbb{R}}$*.*

(ii) *Ever* $\mathbb{R}^k$*-plane, with its induced metric, is isometric to the real hyperbolic space* $\mathbb{H}^k_{\mathbb{R}}$ *equipped with the Beltrami-Klein model for hyperbolic geometry.*

(iii) *In particular,* $\mathbb{H}^n_{\mathbb{C}}$ *has two types of real* 2*-planes which are both totally geodesic: complex slices, and real slices. Furthermore, it is at these two types of* 2*-planes where* $\mathbb{H}^n_{\mathbb{C}}$ *attains its bounds regarding sectional curvature: the sectional curvature in* $\mathbb{H}^n_{\mathbb{C}}$ *varies in the interval* $[-1, -\frac{1}{4}]$ *with the upper bound corresponding to the curvature of real slices and the lower one being attained at complex slices.*

2.2.2 Bisectors and spines

Recall that in real hyperbolic geometry the group of isometries is generated by inversions on spheres of codimension 1 that meet orthogonally the sphere at infinity. This type of spheres are totally geodesic. These spheres also determine the sides of the Dirichlet fundamental domains. In complex hyperbolic space, there are no totally real submanifolds of codimension 1, and a reasonable substitute are the bisectors (or equidistant hypersurfaces), that we now define. These were introduced by Giraud (see [67]) and also used by Moser, who called them spinal surfaces, to construct the first examples of nonarithmetic lattices in $\mathrm{PU}(n,1)$.

Given points z_1, z_2 in $\mathbb{H}^n_{\mathbb{C}}$, we denote by $\rho(z_1, z_2)$ their complex hyperbolic distance.

Definition 2.2.3. Let z_1, z_2 be two distinct points in $\mathbb{H}^n_{\mathbb{C}}$.

(i) The *bisector of* z_1 and z_2 is the set

$$\mathfrak{E}\{z_1, z_2\} \;=\; \{z \in \mathbb{H}^n_{\mathbb{C}} \,|\, \rho(z, z_1) = \rho(z, z_2)\,\}\,.$$

(ii) The boundary of a bisector is called *a spinal sphere* in $\partial\mathbb{H}^n_{\mathbb{C}}$.

(iii) The complex geodesic $\Sigma = \Sigma(z_1, z_2)$ in $\mathbb{H}^n_{\mathbb{C}}$ spanned by z_1, z_2 is called the *complex spine* (or simply the $\mathbb{C}$-*spine*) of the bisector $\mathfrak{E}\{z_1, z_2\}$ with respect to z_1 and z_2.

(iv) The *spine* $\sigma = \sigma(z_1, z_2)$ of $\mathfrak{E}\{z_1, z_2\}$ with respect to z_1 and z_2 is the intersection of the bisector with the complex spine of z_1 and z_2:

$$\sigma := \mathfrak{E}\{z_1, z_2\} \cap \Sigma \;=\; \{z \in \Sigma \,|\, \rho(z, z_1) = \rho(z, z_2)\,\}\,.$$

Notice that by the above discussion, the complex spine Σ is the intersection with $\mathbb{H}^n_{\mathbb{C}}$ of projectivisation of a complex linear 2-space in $\mathbb{C}^{n,1}$. Thence one has a (holomorphic) orthogonal projection $\pi_\Sigma : \mathbb{H}^n_{\mathbb{C}} \to \Sigma$ induced by the orthogonal projection in $\mathbb{C}^{n,1}$ (with respect to the corresponding quadratic form). Then one has the following theorem, which is essentially the *Slice Decomposition Theorem* of Giraud and Mostow (see Theorem 5.1, its corollary 5.1.3 and lemma 5.1.4 in [67]). Recall that a bisector is a real hypersurface in the complex manifold $\mathbb{H}^n_{\mathbb{C}}$ and therefore comes equipped with a natural CR-structure and a Levi-form.

Theorem 2.2.4. *The bisector $\mathfrak{E}$ is a real analytic hypersurface in $\mathbb{H}^n_{\mathbb{C}}$ diffeomorphic to $\mathbb{R}^{2n-1}$, which fibres analytically over the spine σ with projection being the restriction to $\mathfrak{E}$ of the orthogonal projection $\pi_\Sigma : \mathbb{H}^n_{\mathbb{C}} \to \Sigma$:*

$$\mathfrak{E} = \pi_\Sigma^{-1}(\sigma) = \bigcup_{s\in\sigma} \pi_\Sigma^{-1}((s)\,.$$

Furthermore, each bisector $\mathfrak{E}$ is Levi-flat and the slices $\pi_\Sigma^{-1}((s) \subset \mathfrak{E}$ are its maximal holomorphic submanifolds. Hence $\mathfrak{E}$, the spine, the complex spine and the slices are independent of the choice of points z_1, z_2 used to define them.

Of course, spinal spheres are diffeomorphic to $\mathbb{S}^{2n-2}$. We remark too [67, Theorem 5.1.6] that the above association $\mathfrak{E} \rightsquigarrow \sigma$ defines a bijective correspondence between bisectors and geodesics in $\mathbb{H}^n_{\mathbb{C}}$: every real geodesic σ is contained in a unique complex geodesic Σ; then one has an orthogonal projection $\pi_\Sigma : \mathbb{H}^n_{\mathbb{C}} \to \Sigma$ and the bisector is $\mathfrak{E} = \pi_\Sigma^{-1}(\sigma)$.

Another nice property of bisectors is that just as they decompose naturally into complex hyperplanes, as described by the Slice Decomposition Theorem above, they also decompose naturally into totally real geodesic subspaces and one has the corresponding Meridianal Decomposition Theorem (see [67, Theorem 5.1.10]).

2.3 The Siegel domain model

There is another classical model for complex hyperbolic geometry, the Siegel domain (or paraboloid) model. While the ball models describe complex projective space regarded from within, the Siegel domain model describes this space as regarded from a point at infinity. Thence it is in some sense analogous to the upper-half plane model for real hyperbolic geometry. The basic references for this section are [67], [71] and [169].

Consider $\mathbb{C}^n$ as $\mathbb{C}^{n-1} \times \mathbb{C}$ with coordinates (w', w_n), $w' \in \mathbb{C}^{n-1}$, and let $\langle\langle\ ,\ \rangle\rangle$ be the usual Hermitian product in $\mathbb{C}^{n-1}$. So $\langle\langle w', w' \rangle\rangle = w'_1 \bar{w}'_1 + \cdots + w'_{n-1} \bar{w}'_{n-1}$. The *Siegel domain* $\mathfrak{S}^n$ consists of the points in $\mathbb{C}^n$ satisfying

$$2\mathrm{Re}(w_n) > \langle\langle w', w' \rangle\rangle \,.$$

Its boundary is a paraboloid in $\mathbb{C}^n$.

Now consider the embedding of $\mathfrak{S}^n$ in $\mathbb{P}^n_{\mathbb{C}}$ given by

$$(w', w_n) \overset{B}{\mapsto} [w', \frac{1}{2} - w_n, \frac{1}{2} + w_n] \,.$$

We claim that the image $B(\mathfrak{S}^n)$ is the ball $\mathbb{B}$ of negative points that serves as a model for complex hyperbolic $\mathbb{H}^n_{\mathbb{C}}$. To see this, notice that given points $w = (w', w_n)$, $z = (z', z_n) \in \mathbb{C}^n$, the Hermitian product of their image in $\mathbb{P}^n_{\mathbb{C}}$ (induced by the product $\langle\ ,\ \rangle$ in $\mathbb{C}^{n,1}$) takes the form

$$\langle B(w), B(z) \rangle = \langle\langle w', z' \rangle\rangle - w_n - \overline{z}_n \,.$$

In particular $\langle B(w), B(w) \rangle = \langle\langle w', w' \rangle\rangle - \mathrm{Re} w_n$. Hence $B(w)$ is a negative point in $\mathbb{P}^n_{\mathbb{C}}$ if and only if w is in $\mathfrak{S}^n$.

Now consider the null vector $\tilde{p}_\infty := (0', -1, 1) \in \mathbb{C}^{n,1}$ and its image p_∞ in $\mathbb{P}^n_{\mathbb{C}}$. Let H_∞ be the unique complex projective hyperplane in $\mathbb{P}^n_{\mathbb{C}}$ which is tangent to $\mathbb{H}^n_{\mathbb{C}}$ at p_∞. This hyperplane consists of all points $[z] \in \mathbb{P}^n_{\mathbb{C}}$ whose homogeneous coordinates $[z_1 : \ldots : z_{n+1}]$ satisfy $z_n = z_{n+1}$. Therefore the image of B does not meet H_∞ and B provides an affine coordinate chart for $\mathbb{P}^n_{\mathbb{C}} \setminus H_\infty$, carrying the Siegel domain $\mathfrak{S}^n$ into the ball $\mathbb{H}^n_{\mathbb{C}}$.

Notice that the boundary of $\mathfrak{S}^n$ is the paraboloid $\{2\mathrm{Re}(w_n) = \langle\langle w', w' \rangle\rangle\}$, and its image under B is the sphere $\partial \mathbb{H}^n_{\mathbb{C}}$ minus the null point $p_\infty := [0', -1, 1] \in \mathbb{P}^n_{\mathbb{C}}$, where $0' := (0, \ldots, 0) \in \mathbb{C}^{n-1}$. That is:

$$\partial \mathfrak{S}^n \cong \partial \mathbb{H}^n_{\mathbb{C}} \setminus \{q_\infty\} \,.$$

One thus has an induced complex hyperbolic metric on $\mathfrak{S}^n$, induced by the Bergman metric on $\mathbb{H}^n_{\mathbb{C}}$ (see [71, p. 520] for the explicit formula).

Definition 2.3.1. For each positive real number u, the *horosphere* in $\mathfrak{S}^n$ (centred at q_∞) of level u is the set

$$\mathfrak{H}_u := \{ w = (w', w_n) \in \mathfrak{S}^n \,\big|\, \mathrm{Re} w_n - \langle\langle w', w' \rangle\rangle \;=\; u \} \,.$$

We also set $\mathfrak{H}_0 := \partial \mathfrak{S}^n$.

Hence, horospheres in $\mathfrak{S}^n$ are paraboloids, the translates of the boundary. When regarded in $\mathbb{H}^n_{\mathbb{C}}$ they become spheres, tangent to $\partial\mathbb{H}^n_{\mathbb{C}}$ at $q_\infty = [0', -1, 1$. This definition can be easily adapted horospheres in $\mathbb{H}^n_{\mathbb{C}}$ centred at every point in $\partial\mathbb{H}^n_{\mathbb{C}}$.

2.3.1 Heisenberg geometry and horospherical coordinates

Recall that the classical Heisenberg group is the group of 3×3 triangular matrices of the form

$$H = \begin{pmatrix} 1 & a & t \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

where the coefficients are real numbers. This is a 3-dimensional nilpotent Lie group diffeomorphic to $\mathbb{R}^3$. Its group structure, coming from the multiplication of matrices, is a semi-direct product $\mathbb{C} \ltimes \mathbb{R}$ where $\mathbb{C}$ and $\mathbb{R}$ are being regarded as additive groups:

$$\big((a,b),t\big) \cdot \big((a',b'),t'\big) \mapsto \big((a+a', b+b'), t+t'+a\cdot b'\big)\,,$$
$$\big(v,t\big) \cdot \big(v',t'\big) \mapsto \big(v+v', t+t'+a\cdot b'\big)\,,$$

where $v = (a,b)$ and $v' = (a',b')$.

More generally, let V be a finite-dimensional real vector space, equipped with a symplectic form ω. This means that ω is a nondegenerate skew symmetric bilinear form on V. One has a Heisenberg group $H = H(V,\omega)$ associated with the pair (V,ω). This group is a semi-direct product $V \ltimes \mathbb{R}$, where the group structure is given by the following law:

$$(v_1,t_1)\cdot(v_2,t_2) \;=\; (v_1+v_2, t_1+t_2+2\omega(v_1,v_2))\,;$$

the factor 2 is included for conventional reasons. This group is a central extension of the additive group V and there is an exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow H(V,\omega) \longrightarrow V \longrightarrow 0\,.$$

A *Heisenberg space* is a principal $H(V,\omega)$-homogeneous space N, say a smooth manifold, for some Heisenberg group as above. In other words, H acts (say by the left) transitively on N with trivial isotropy. So N is actually parametrised by H and it can be equipped with a Lie group structure coming from that in H.

Consider now the isotropy subgroup G_∞ of the null point $p_\infty := [0', -1, 1] \in \partial\mathbb{H}^n_{\mathbb{C}}$ under the action of $\mathrm{PU}(n,1)$. Let $\mathfrak{N}$ be the set of unipotent elements in G_∞. It is proved in [67, §4.2] (see also [71]) that $\mathfrak{N}$ is isomorphic to a Heisenberg group as above. This group is a semidirect product $\mathbb{C}^{n-1} \ltimes \mathbb{R}$ and consists of the so-called *Heisenberg translations* $\{T_{\zeta,t}\}$. These are more easily defined in $\mathfrak{S}^n$. For each $\zeta \in \mathbb{C}^{n-1}$, $t \in \mathbb{R}$ and $(w', w_n) \in \mathfrak{S}^n \subset \mathbb{C}^n$ one has:

$$(w', w_n) \quad \xrightarrow{T_{\zeta,u}} \quad \big(w'+\zeta\;,\; w_n + \langle\langle w', \zeta\rangle\rangle + \frac{1}{2}\langle\langle \zeta,\zeta\rangle\rangle - \frac{1}{2}it\big)\,.$$

Notice that each orbit in $\mathfrak{S}^n$ is a horosphere $\mathfrak{H}_u$, and $\mathfrak{N}$ acts simply transitively on $\mathfrak{H}_0 := \partial\mathfrak{S}^n$.

One also has the one-parameter group $\mathfrak{D} = \{D_u\}$ of *real Heisenberg dilatations*: For each $u > 0$ define

$$(w', w) \quad \xrightarrow{D_t} \quad \left(\sqrt{u}\, w' \,,\, u\, w_n\right).$$

Each dilatation carries horospheres into horospheres.

These two groups of transformations were used in [71] to equip the complex hyperbolic space with horospherical coordinates, obtained by identifying $\mathfrak{S}^n$ with the orbit of a "marked point" under the action of the group $\mathfrak{N} \cdot \mathfrak{D}$ generated by Heisenberg translations and Heisenberg dilatations. The choice of "marked point" is the obvious one: $(0', \frac{1}{2}) \in \mathfrak{S}^n$, the inverse image under the map B of the centre $\underline{0} = [0', 0, 1] \in \mathbb{H}^n_{\mathbb{C}}$.

We thus get an identification $(\mathbb{C}^{n-1} \times \mathbb{R} \times \mathbb{R}_+) \xrightarrow{\cong} \mathbb{H}^n_{\mathbb{C}}$ given by

$$(\zeta, t, u) \quad \mapsto \quad \Big(\zeta \,,\, \frac{1}{2}\big(1 - \langle\langle\zeta,\zeta\rangle\rangle - u + it\big) \,,\, \frac{1}{2}\big(1 - \langle\langle\zeta,\zeta\rangle\rangle - u + it\big)\Big).$$

Following [71] we call $(\zeta, t, u) \in \mathbb{C}^{n-1} \times \mathbb{R} \times \mathbb{R}_+$ the *horospherical coordinates* of the corresponding point in $\mathbb{H}^n_{\mathbb{C}}$.

From the previous discussion we also get a specific identification of $\partial\mathbb{H}^n_{\mathbb{C}} \setminus q_\infty \cong \mathfrak{N}$. Thence the sphere at infinity can be thought of as being "the horosphere" of level 0, and if we remove from it the point q_∞, then it carries the structure of a Heisenberg space $\mathbb{C}^{n-1} \times \mathbb{R}$ where the group operation is

$$(\zeta, t) \cdot (\zeta', t') \;=\; \left(\zeta + \zeta' \,, t + t' + 2\Im(\langle\langle\zeta, \zeta'\rangle\rangle\right).$$

We now consider a point $q \in \partial\mathbb{H}^n_{\mathbb{C}}$, the family $\{\mathfrak{H}_u(q)\}$ of horospheres centred at q, and the pencil of all real geodesics in $\mathbb{H}^n_{\mathbb{C}}$ ending at q. Just as in real hyperbolic geometry, one has (see [67, §4.2] or [71, §1.3]) that every such geodesic is orthogonal to every horosphere $\mathfrak{H}_u(q)$. Thence, for every $u, u' \geq 0$, the geodesic from q to a point $x \in \mathfrak{H}_u(q)$ meets the horosphere $\mathfrak{H}_{u'}(q)$ at exactly one point. This gives a canonical identification $\Pi : \mathfrak{H}_u(q) \to \mathfrak{H}_{u'}(q)$ called *the geodesic perspective map*. In particular we get an identification between $\partial\mathbb{H}^n_{\mathbb{C}} = \mathfrak{H}_0(q)$ and every other horosphere.

2.3.2 The geometry at infinity

Just as there is a deep relation between real hyperbolic geometry and the conformal geometry on the sphere at infinity (as described in Chapter 1), so too there is a deep relation between the geometry of complex hyperbolic space and a geometry on its sphere at infinity. In this case the relevant geometry is the spherical CR (or Heisenberg) geometry. In both cases (real and complex hyperbolic geometry) this relation can be explained by means of the geodesic perspective introduced above.

In real hyperbolic geometry, geodesic perspective from a point q in the sphere at infinity identifies the various horospheres centred at q, and the corresponding maps between these spheres are conformal. This yields to the fact (explained in Chapter 1) that real hyperbolic geometry in the open ball degenerates into conformal geometry in the sphere at infinity. There is an analogous phenomenon in complex hyperbolic space, which we now briefly explain.

Recall that a CR-structure on a manifold means a codimension 1 sub-bundle of its tangent bundle which is complex and satisfies certain integrability conditions. In general, CR-manifolds arise as boundaries of complex manifolds, as for instance real hypersurfaces in complex manifolds, which carry a natural CR-structure.

In our setting, every horosphere, including $\partial\mathbb{H}^n_{\mathbb{C}} = \mathfrak{H}_0$, is a real hypersurface in $\mathbb{P}^n_{\mathbb{C}}$ and therefore carries a natural CR-structure. This is determined at each point by the unique complex $(n-1)$-dimensional subspace subspace of the bundle tangent to the corresponding horosphere. In fact, since real geodesics are orthogonal to all horospheres, the CR-structure can be regarded as corresponding to the Hermitian orthogonal complement of the line field tangent to the geodesics emanating from q, the centre of the horosphere in question. Therefore, it is clear that the geodesic perspective maps preserve the CR-structures on horospheres.

We are in fact interested in a refinement of this notion: spherical CR-structures:

Definition 2.3.2. A manifold M of dimension $2n-1$ has a *spherical CR-structure* if it has an atlas which is locally modeled on the sphere $\mathbb{S}^{2n-1}$ with coordinate changes lying in the group $\mathrm{PU}(n,1)$.

Amongst CR-manifolds, the spherical CR-manifolds are characterised as being those for which the Cartan connection on the CR-bundle has vanishing curvature.

Now consider the sphere $\mathbb{S}^{2n-1} \cong \partial\mathbb{H}^n_{\mathbb{C}}$. Then one has that $\mathrm{PU}(n,1)$ acts on it by automorphisms that preserve the spherical CR-structure, and one actually has that every CR-automorphism of the sphere is an element in $\mathrm{PU}(n,1)$. Thus one has that every CR-automorphism of the sphere at infinity corresponds to a holomorphic isometry of $\mathbb{H}^n_{\mathbb{C}}$.

In particular, the Heissenberg group $\mathfrak{N}$ acts by left multiplication on the sphere at infinity, and by holomorphic isometries on complex hyperbolic space.

2.4 Isometries of the complex hyperbolic space

This section is based on [67]. We consider the ball model for $\mathbb{H}^n_{\mathbb{C}} \subset \mathbb{P}^n_{\mathbb{C}}$, equipped with the Bergman metric, and we recall that $\mathrm{PU}(n,1)$ is the group of holomorphic isometries of the complex hyperbolic n-space $\mathbb{H}^n_{\mathbb{C}}$. When $n = 1$ the space $\mathbb{H}^1_{\mathbb{C}}$ coincides with $\mathbb{H}^2_{\mathbb{R}}$, the group $\mathrm{PU}(n,1)$ is $\mathrm{PSL}(2,\mathbb{R})$, and we know that its elements are classified into three types: elliptic, parabolic and hyperbolic. This classification is determined by their dynamics (the number and location of their fixed points), and also according to their trace. We will see that a similar classification holds for isometries of $\mathbb{H}^2_{\mathbb{C}}$, and to some extent also for those of $\mathbb{H}^n_{\mathbb{C}}$.

2.4.1 Complex reflections

Recall that in Euclidean geometry reflections play a fundamental role. Given a hyperplane $H \subset \mathbb{R}^n$, a way for defining the reflection on H is to look at its orthogonal complement $H^\perp$, and consider the orthogonal projections,

$$\pi_H : \mathbb{R}^n \to H \quad \text{and} \quad \pi_{H^\perp} : \mathbb{R}^n \to H^\perp .$$

Then the reflection on H is the map

$$x \mapsto \pi_H(x) - \pi_{H^\perp}(x) .$$

This notion extends naturally to complex geometry in the obvious way. Yet, in complex geometry this definition is too rigid and it is convenient to make it more flexible. For instance a hyperplane in $\mathbb{C}$ is just a point, say the origin 0, and the reflection above yields to the antipodal map, while we would like to get the full group $\mathrm{U}(1)$. Thus, more generally, given a complex hyperplane $H \subset \mathbb{C}^n$, to define a reflection on H we consider the orthogonal projections

$$\pi_H : \mathbb{R}^n \to H \quad \text{and} \quad \pi_{H^\perp} : \mathbb{R}^n \to H^\perp ,$$

as before, but now with respect to the usual Hermitian product in $\mathbb{C}^n$. Then a reflection on H is any map of the form

$$x \mapsto \pi_H(x) + \zeta \cdot \pi_{H^\perp}(x) ,$$

where ζ is a unit complex number. In this way we get, for instance, that the group of such reflections in $\mathbb{C}^2$ is $\mathrm{U}(2) \cong \mathrm{SU}(2) \times \mathrm{U}(1) \cong \mathbb{S}^3 \times \mathbb{S}^1$.

Sometimes the name "complex reflection" requires also that the complex number ζ be a root of unity, so that the corresponding automorphism has finite order. And it is also usual to extend this concept so that complex reflection means an automorphism of $\mathbb{C}^n$ that leaves a hyperplane fix-point invariant. This includes for instance, automorphisms constructed as above but considering different Hermitian products, and this brings us closer to the subject we envisage here.

Denote by $\langle\, ,\, \rangle$ the Hermitian product on $\mathbb{C}^{n,1}$ corresponding to the quadratic form Q of signature $(n, 1)$. Let F be a complex linear subspace of $\mathbb{C}^{n,1}$ such that the restriction of the product $\langle\, ,\, \rangle$ to F is nondegenerate. Then there is an orthogonal direct-sum decomposition

$$\mathbb{C}^{n,1} = F \oplus F^\perp ,$$

where $F^\perp$ is the Q-orthogonal complement of F:

$$F^\perp := \{ z \in \mathbb{C}^{n,1} \,|\, \langle z, f \rangle = 0 \quad \forall\, f \in F \} .$$

Let $\pi_F, \pi_{F^\perp}$ be the corresponding orthogonal projections of $\mathbb{C}^{n,1}$ into F and $F^\perp$ respectively, so one has $\pi_{F^\perp}(z) = z - \pi_F(z)$. Then following [67, p. 68] we have:

Definition 2.4.1. For each unit complex number ζ, define *the complex reflection in F with reflection factor* ζ to be the element in $\mathrm{U}(n,1)$ defined by

$$\varrho_F^{\zeta}(z) = \pi_F(x) + \zeta\, \pi_{F^\perp}(z)\,.$$

When $\zeta = -1$ the complex reflection is said to be an *inversion.*

For instance, if F is 1-dimensional and V is a nonzero vector in this linear space, then

$$\pi_F(z) = \frac{\langle z, V\rangle}{\langle V, V\rangle}\, V\,,$$

and therefore the complex reflection in F with reflection factor ζ is

$$\varrho_F^{\zeta}(z) = \zeta z + (1-\zeta)\frac{\langle z, V\rangle}{\langle V, V\rangle}\, V\,.$$

In particular, every complex reflection is conjugate to an inversion and in this case the formula is

$$I_F(z) = -z + 2\,\frac{\langle z, V\rangle}{\langle V, V\rangle}\, V\,. \tag{2.4.1}$$

Notice that classically reflections are taken with respect to a hyperplane, which is not required here: We now have reflections with respect to points, lines, etc., which is in fact a concept coming from classical geometry, where one speaks of symmetries with respect to points, lines, planes, etc.

Of course our interest is in looking at the projectivisations of these maps, that we call complex reflections as well.

Example 2.4.2. Consider in $\mathbb{C}^{2,1}$ the vector $v = (-1, 1, 0)$, which is positive for the Hermitian product

$$\langle z, w\rangle = z_1\bar{w}_1 + z_2\bar{w}_2 - z_3\bar{w}_3\,.$$

The inversion on the line F spanned by v is the map

$$I_v(z_1, z_2, z_3) = (-z_2, -z_1, -z_3).$$

Notice that $\mathbb{H}^2_{\mathbb{C}}$ is contained in the coordinate patch of $\mathbb{P}^2_{\mathbb{C}}$ with homogeneous coordinates

$[u_1 : u_2 : 1]$. In these coordinates the automorphism of $\mathbb{H}^2_{\mathbb{C}}$ determined by I_v is the map $[u_1 : u_2 : 1] \mapsto [u_2 : u_1 : 1]$. Notice also that in this example the points in $\mathbb{H}^2_{\mathbb{C}}$ with homogeneous coordinates $[u : u : 1]$ are obviously fixed points of the inversion I_v. Indeed these points form a complex geodesic in $\mathbb{H}^2_{\mathbb{C}}$, which is the projectivisation of the orthogonal complement of F:

$$F^\perp = \{(w_1, w_2, w_3) \in \mathbb{C}^{2,1} \mid w_1 = -w_2\}\,.$$

Now consider in $\mathbb{C}^2$ the inversion with respect to the negative vector $w = (0, 0, 1)$. The inversion is: $I_w(z) = (-1 + 2z_3)\, z$. Its fixed point set in $\mathbb{P}^2_{\mathbb{C}}$ consists of the point $[0:0:1]$, which is the centre of the ball $\mathbb{H}^2_{\mathbb{C}}$, together with the complex line in $\mathbb{P}^2_{\mathbb{C}}$ consisting of points with homogeneous coordinates $[u_1 : u_2 : 0]$. These points are the projectivisation of the orthogonal complement of the line spanned by w; they are all positive vectors (or the origin).

Thence in the first case, the complex reflection has a complex geodesic of fixed points in $\mathbb{H}^2_{\mathbb{C}}$ and the corresponding projective map has a circle of fixed points in $\mathbb{S}^3 = \partial\mathbb{H}^2_{\mathbb{C}}$, plus another fixed point far from $\mathbb{H}^2_{\mathbb{C}}$. In the second example the reflection has a single fixed point in $\mathbb{H}^2_{\mathbb{C}}$ and the remaining fixed points form a complex projective line in $\mathbb{P}^2_{\mathbb{C}} \setminus (\overline{\mathbb{H}}^2_{\mathbb{C}})$. This illustrates the two types of complex reflections one has in $\mathbb{H}^2_{\mathbb{C}}$. Of course similar considerations apply in higher dimensions, but in that case there is a larger set of possibilities.

2.4.2 Dynamical classification of the elements in $\mathrm{PU}(2,1)$

Every automorphism γ of $\mathbb{H}^n_{\mathbb{C}}$ lifts to a unitary transformation $\tilde{\gamma} \in \mathrm{SU}(n,1)$. Just as for classical Möbius transformations in $\mathrm{PSL}(2,\mathbb{C})$ their geometry and dynamics is studied by looking at their liftings to $\mathrm{SL}(2,\mathbb{C})$, here too we study the geometry and dynamics of γ by looking at their liftings $\tilde{\gamma} \in \mathrm{U}(n,1)$. The fixed points of γ correspond to eigenvectors of $\tilde{\gamma}$. By the Brouwer fixed point theorem, every automorphism of the compact ball $\overline{\mathbb{H}}^n_{\mathbb{C}} := \mathbb{H}^n_{\mathbb{C}} \cup \partial\mathbb{H}^n_{\mathbb{C}}$ has a fixed point.

The following definition generalises to complex hyperbolic spaces the corresponding notions from the classical theory of Möbius transformations.

Definition 2.4.3. An element $g \in \mathrm{PU}(n,1)$ is called *elliptic* if it has a fixed point in $\mathbb{H}^n_{\mathbb{C}}$; it is *parabolic* if it has a unique fixed point in $\partial\mathbb{H}^n_{\mathbb{C}}$, and *loxodromic* (or *hyperbolic*) if it fixes a unique pair of points in $\partial\mathbb{H}^n_{\mathbb{C}}$.

In fact this classification can be refined as follows. Recall that a square matrix is unipotent if all its eigenvalues are 1.

Definition 2.4.4. An elliptic transformation in $\mathrm{PU}(n,1)$ is: *regular* if it can be represented by an element in $\mathrm{SU}(n,1)$ whose eigenvalues are pairwise different, or a *complex reflection* otherwise (and this can be either with respect to a point or to a complex geodesic). There are two classes of parabolic transformations in $\mathrm{PU}(n,1)$: *unipotent* if it can be represented as a unipotent element of $\mathrm{PU}(n,1)$, and *ellipto-parabolic* otherwise. A loxodromic element is strictly hyperbolic if it has a lifting whose eigenvalues are all real.

One has (see [67, p. 201]) that if γ is ellipto-parabolic, then there exists a unique invariant complex geodesic in $\mathbb{H}^n_{\mathbb{C}}$ on which γ acts as a parabolic element of $\mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{Iso}\,\mathbb{H}^1_{\mathbb{C}} \cong \mathrm{Iso}\,\mathbb{H}^2_{\mathbb{R}}$. Furthermore, around this geodesic, γ acts as a nontrivial unitary automorphism of its normal bundle.

From now on in this section, we restrict to the case $n = 2$ and we think of $\mathbb{H}^2_{\mathbb{C}}$ as being the ball in $\mathbb{P}^2_{\mathbb{C}}$ consisting of points whose homogeneous coordinates satisfy $|z_1|^2 + |z_2|^2 < |z_3|^2$.

Let $g \in \mathrm{PU}(2,1)$ be an elliptic element. Since $\mathrm{PU}(2,1)$ acts transitively on $\mathbb{H}^2_{\mathbb{C}}$, we can assume that $[0:0:1]$ is fixed by g. If $\tilde{g}$ denotes a lift to $\mathrm{SU}(2,1)$ of g then $(0,0,1)$ is an eigenvector of $\tilde{g}$, so it is of the form

$$\tilde{g} = \begin{pmatrix} A & 0 \\ 0 & \lambda \end{pmatrix},$$

where $A \in \mathrm{U}(2)$ and $\lambda \in \mathbb{S}^1$. Then every eigenvector of $\tilde{g}$ has module 1 and g generates a cyclic group with compact closure.

Conversely, if $\tilde{g}$ is as above (an element in $\mathrm{U}(2) \times \mathbb{S}^1$), or a conjugate of such an element, then the transformation g induced by $\tilde{g}$ is elliptic.

When g is regular elliptic, then it has precisely three fixed points in $\mathbb{P}^2_{\mathbb{C}}$, which correspond to $(0,0,1)$ and two other distinct eigenvectors, both being positive vectors with respect to the Hermitian product $\langle \cdot, \cdot \rangle$. If g is elliptic but not regular, then there exist two cases: if g is a reflection with respect to a point x in $\mathbb{H}^2_{\mathbb{C}}$, then the set of fixed points of g is the polar to x which does not meet $\mathbb{H}^2_{\mathbb{C}} \cup \partial \mathbb{H}^2_{\mathbb{C}}$. If g is a reflection with respect to a complex geodesic, then g has a whole circle of fixed points contained in $\partial \mathbb{H}^2_{\mathbb{C}}$. Therefore the definitions of elliptic, loxodromic and parabolic elements are disjoint.

Now we assume $g \in \mathrm{PU}(2,1)$ is a loxodromic element. Let $\tilde{g} \in \mathrm{SU}(2,1)$ be a lift of g, we denote by x and y the fixed points of g and by $\tilde{x}$ and $\tilde{y}$ some respective lifts to $\mathbb{C}^{2,1}$. Let Σ be the complex geodesic determined by x and y, and let σ be the geodesic determined by x and y. We can assume that $\Sigma = \mathbb{H}^1_{\mathbb{C}} \times 0$; in other words, we can assume $x = [-1:0:1]$, $y = [1:0:1]$. To see this, notice that if $z \in \sigma$, then there exists $h \in \mathrm{PU}(2,1)$ such that $h(z) = (0,0)$, so $h(L)$ contains the origin of $B^2 = \mathbb{H}^2_{\mathbb{C}}$. The stabiliser of the origin is $\mathrm{U}(2)$ and it acts transitively on the set of complex lines through the origin, so there exists $h_1 \in \mathrm{PU}(2,1)$ such that $h_1(\Sigma) = \mathbb{H}^1_{\mathbb{C}} \times 0$ and $h_1(\sigma)$ contains the origin. Composing with a rotation in $\mathbb{H}^1_{\mathbb{C}}$, we prove the statement.

We see that any vector c polar to Σ is an eigenvector of $\tilde{g}$. In fact, if $v \in \mathbb{C}^{2,1}$ and $\mathbb{P}_{\mathbb{C}}(v) \in \Sigma$, then $\langle \tilde{g}(c), \tilde{g}(v) \rangle = \langle c, v \rangle = 0$, then $\tilde{g}(c)$ is polar to $g(\Sigma) = \Sigma$, but we know that the complex dimension of the orthogonal complement (respect to $\langle \cdot, \cdot \rangle$) of the vector subspace inducing Σ is 1.

Now, $c = (0,1,0)$ is a vector polar to Σ, then we can assume $[0:1:0]$ is a fixed point of g. Thus $\tilde{g}$ has the form

$$\begin{pmatrix} a & 0 & b \\ 0 & e^{-2i\theta} & 0 \\ b & 0 & a \end{pmatrix},$$

and the transformation of $\mathrm{PSL}(2,\mathbb{C})$ induced by the matrix

$$e^{-i\theta} \begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

is a hyperbolic transformation preserving the unitary disc of $\mathbb{C}$ and with fixed points 1, -1. With this information it is not hard to see that we can take $\tilde{g}$ of the form

$$\begin{pmatrix} e^{i\theta}\cosh u & 0 & e^{i\theta}\sinh u \\ 0 & e^{-2i\theta} & 0 \\ e^{i\theta}\sinh u & 0 & e^{i\theta}\cosh u \end{pmatrix}.$$

We notice that $\tilde{g}$ has an eigenvalue in the open unitary disc, one in $\mathbb{S}^1$ and another outside the closed unitary disc. Since they are parabolic, all of these have a unique fixed point in $\partial\mathbb{H}^2_{\mathbb{C}}$, but they have a different behaviour in $\mathbb{P}^2_{\mathbb{C}} \setminus \mathbb{H}^2_{\mathbb{C}}$.

Example 2.4.5. a) The transformation induced by the matrix

$$\tilde{g} = \begin{pmatrix} 1+i/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1-i/2 \end{pmatrix}$$

is a unipotent transformation with a whole line of fixed points in $\mathbb{P}^2_{\mathbb{C}}$, tangent to $\partial\mathbb{H}^2_{\mathbb{C}}$.

b) The transformation in $\mathrm{SU}(2,1)$ given by the matrix

$$\tilde{g} = \begin{pmatrix} \frac{1}{3} & \frac{2\sqrt{2}}{3} & 0 \\ -\frac{4\sqrt{2}}{3} & \frac{2}{3} & \sqrt{3} \\ -\frac{2\sqrt{6}}{3} & \frac{\sqrt{3}}{3} & 2 \end{pmatrix}$$

induces a unipotent automorphism of $\mathbb{H}^2_{\mathbb{C}}$ with one single fixed point in $\mathbb{P}^2_{\mathbb{C}}$.

c) The transformation in $\mathrm{SU}(2,1)$ given by

$$\begin{pmatrix} \frac{2+\epsilon i}{2}e^{i\theta} & 0 & \frac{\epsilon i}{2}e^{i\theta} \\ 0 & e^{-2i\theta} & 0 \\ -\frac{\epsilon i}{2}e^{i\theta} & 0 & \frac{2-\epsilon i}{2}e^{i\theta} \end{pmatrix},$$

where $\epsilon \neq 0$, induces an ellipto-parabolic automorphism of $\mathbb{H}^2_{\mathbb{C}}$, with fixed points $[-1:0:1] \in \partial\mathbb{H}^2_{\mathbb{C}}$ and $[0:1:0]$.

Remark 2.4.6. We recall that complex hyperbolic space has (nonconstant) negative sectional curvature. As such, the classification given above of its isometries, into elliptic, parabolic and loxodromic (or hyperbolic), actually fits into a similar classification given in the general setting of the isometries of spaces of nonpositive curvature, and even more generally, for CAT(0)-spaces. We refer to [13], [32] for thorough accounts of this subject. The concept of CAT(0)-spaces captures the essence of nonpositive curvature and allows one to reflect many of the basic properties of such spaces, as for instance $\mathbb{H}^n_{\mathbb{R}}$, and $\mathbb{H}^n_{\mathbb{C}}$, in a much wider setting. The origins of CAT(0)-spaces, and more generally CAT(κ)-spaces, are in the work of A. D. Alexandrov, where he gives several equivalent definitions of what it means for a metric space to have curvature bounded above by a real number κ. The terminology "CAT(κ)" was coined by M. Gromov in 1987 and the initials are in honour of E. Cartan, A. Alexandrov and V. Toponogov, each of whom considered similar conditions.

Yet, we notice that our main focus in this work concerns automorphisms of $\mathbb{P}^n_{\mathbb{C}}$, whose sectional curvature, when we equip it with the Fubini-Study metric, is strictly positive, ranging from $1/4$ to 1. Moreover, the action of $\mathrm{PSL}(n+1,\mathbb{C})$ is not by isometries with respect to this metric. Thence, these transformations do not fit in the general framework of isometries of CAT(0)-spaces. Even so, we will see in the following chapter that the elements of $\mathrm{PSL}(3,\mathbb{C})$ are also naturally classified into elliptic, parabolic and hyperbolic, both in terms of their geometry and dynamics, and also algebraically.

2.4.3 Traces and conjugacy classes in $\mathrm{SU}(2,1)$

We now describe Goldman's classification of the elements in $\mathrm{PU}(2,1)$ by means of the trace of their liftings to $\mathrm{SU}(2,1)$.

Let $\tau : \mathrm{SU}(2,1) \to \mathbb{C}$ be the function mapping an element in $\mathrm{SU}(2,1)$ to its trace. Notice that a holomorphic automorphism of $\mathbb{H}^2_{\mathbb{C}}$ has a trace which is well-defined up to multiplication by a cubic root of unity. The following result of linear algebra is well known:

Lemma 2.4.7. *Let E be a Hermitian vector space, and let A be a unitary automorphism of E. The set of eigenvalues of A is invariant under the inversion $\imath$ in the unitary circle of $\mathbb{C}$:*

$$\begin{aligned} \imath : \mathbb{C}^* &\to \mathbb{C}^* \\ z &\mapsto \frac{1}{\bar{z}}. \end{aligned}$$

Proof. We assume the Hermitian form on E is given by a Hermitian matrix M. The automorphism A is unitary if and only if

$$\bar{A}^t M A = M.$$

In other words,

$$A = M^{-1}(\bar{A}^t)^{-1} M.$$

Thus A has the same eigenvalues as $(\bar{A}^t)^{-1}$, which means that λ is an eigenvalue of A if and only if $(\overline{\lambda})^{-1}$ is an eigenvalue. □

Notice that when $\tilde{g} \in \mathrm{SU}(2,1)$, then $\tilde{g}$ has at least an eigenvalue of module 1. Moreover, the eigenvalues not lying in the unitary circle are given in an $\imath$-invariant pair. Particularly, if $\tilde{g}$ has two eigenvalues of the same module, then every eigenvalue has module 1.

Lemma 2.4.8. *The monic polynomial $\chi(t) = t^3 - xt^2 + yt - 1$ with complex coefficients has repeated roots if and only if*

$$\begin{aligned}\tilde{f}(x,y) &= -x^2y^2 + 4(x^3+y^3) - 18xy + 27\\ &= \begin{vmatrix} 1 & -x & y & -1 & 0\\ 0 & 1 & -x & y & -1\\ 3 & -2x & y & 0 & 0\\ 0 & 3 & -2x & y & 0\\ 0 & 0 & 3 & -2x & y \end{vmatrix}\\ &= 0.\end{aligned}$$

Proof. We assume χ has repeated roots, then χ and its derivative χ' have a common root. We suppose

$$\chi(t) = (t-a_1)(t-a_2)(t-a_3)$$

and

$$\chi'(t) = 3(t-a_1)(t-a_4).$$

Then

$$3(t-a_4)\chi(t) = (t-a_2)(t-a_3)\chi'(t),$$

or which is the same,

$$3t\chi(t) - 3a_4\chi(t) - t^2\chi'(t) + (a_2+a_3)t\chi'(t) - a_2a_3\chi'(t) \equiv 0.$$

This means that the vectors obtained from the polynomials $t\chi(t)$, $\chi(t)$, $t^2\chi'(t)$, $t\chi'(t)$, $\chi'(t)$, taking the coefficients of the terms with degree ≤ 4, are linearly independent. But such vectors are precisely those row vectors of the determinant defining $\tilde{f}(x,y)$. Thence $\tilde{f}(x,y) = 0$.

Conversely, if we assume that $\tilde{f}(x,y) = 0$, then there exist complex numbers $c_1, \ldots, c_5$, not all zero, such that

$$c_1t\chi(t) + c_2\chi(t) + c_3t^2\chi'(t) + c_4t\chi'(t) + c_5\chi'(t) \equiv 0\,.$$

Then $(c_1t + c_2)\chi(t) = -(c_3t^2 + c_4t + c_5)\chi'(t)$, which implies that $\chi(t)$ and $\chi'(t)$ have a common root because $deg(\chi) = 3$. Therefore $\chi(t)$ has a repeated root. □

We denote by $C_3 = \{\omega, \omega^2, 1\} \subset \mathbb{C}$ the set of cubic roots of unity, and $3C_3$ denotes the set $\{3\omega, 3\omega^2, 3\}$. We observe that there is a short exact sequence

$$1 \to C_3 \to \mathrm{SU}(2,1) \to \mathrm{PU}(2,1) \to 1.$$

Let $\tau : \mathrm{SU}(2,1) \to \mathbb{C}$ be the function which assigns to an element of $\mathrm{SU}(2,1)$ its trace. Goldman's classification theorem involves the real polynomial $f : \mathbb{C} \to \mathbb{R}$ defined by $f(z) = |z|^4 - 8\mathrm{Re}(z^3) + 18|z|^2 - 27$. In other words $f(z) = -\tilde{f}(z, \bar{z})$, where

$$\tilde{f}(x,y) = -x^2y^2 + 4(x^3+y^3) - 18xy + 27\,,$$

is the discriminant of the (characteristic) polynomial $\chi(t) = t^3 - xt^2 + yt - 1$.

Theorem 2.4.9. [67, Theorem 6.2.4] *The map $\tau : \mathrm{SU}(2,1) \to \mathbb{C}$ defined by the trace is surjective, and if $A_1, A_2 \in \mathrm{SU}(2,1)$ satisfy $\tau(A_1) = \tau(A_2) \in \mathbb{C} - f^{-1}(0)$, then they are conjugate. Furthermore, supposing $A \in \mathrm{SU}(2,1)$ one has:*

1) *A is regular elliptic if and only if $f(\tau(A)) < 0$.*
2) *A is loxodromic if and only if $f(\tau(A)) > 0$.*
3) *A is ellipto-parabolic if and only if A is not elliptic and $\tau(A) \in f^{-1}(0) - 3C_3$.*
4) *A is a complex reflection if and only if A is elliptic and $\tau(A) \in f^{-1}(0) - 3C_3$.*
5) *$\tau(A) \in 3C_3$ if and only if A represents a unipotent parabolic element.*

Proof. Let $\chi_A(t)$ be the characteristic polynomial of A:

$$\chi_A(t) = t^3 - xt^2 + yt - 1.$$

The eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of A are the roots of $\chi_A(t)$. We have

$$x = \tau(A) = \lambda_1 + \lambda_2 + \lambda_3,$$

and

$$\lambda_1\lambda_2\lambda_3 = \det(A) = 1\,. \tag{2.4.10}$$

The coefficient y of χ_A is equal to

$$y \;=\; \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 \;=\; \overline{\lambda_1} + \overline{\lambda_2} + \overline{\lambda_3} \;=\; \overline{\tau(A)}.$$

Thus,

$$\chi_A(t) = t^3 - \tau(A)t^2 + \overline{\tau(A)}\,t - 1.$$

If $A \in \mathrm{SU}(2,1)$, then its eigenvalues satisfy equation 2.4.10 and the set

$$\tilde{\lambda} = \{\lambda_1, \lambda_2, \lambda_3\},$$

of these eigenvalues satisfies:

$$\lambda \in \tilde{\lambda} \;\Rightarrow\; \imath(\lambda) = \bar{\lambda}^{-1} \in \tilde{\lambda}\,. \tag{2.4.11}$$

Let $\tilde{\Lambda}$ (respectively Λ) be the set of unordered triples of complex numbers satisfying (2.4.10) (respectively (2.4.11)). Then $\imath$ induces an involution in $\tilde{\Lambda}$ (denoted by the same) whose set of fixed points is Λ. The function

$$\begin{aligned} \chi : \tilde{\Lambda} &\to \mathbb{C}^2 \\ \tilde{\lambda} &\mapsto (\lambda_1 + \lambda_2 + \lambda_3, \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) \end{aligned}$$

is bijective, with inverse function

$$\begin{aligned} \mathbb{C}^2 &\to \tilde{\Lambda} \\ (x, y) &\mapsto \{t \in \mathbb{C} : t^3 - xt + yt - 1 = 0\}. \end{aligned}$$

The involution j on $\mathbb{C}^2$ defined as

$$j(x,y) = (\bar{y}, \bar{x})$$

satisfies

$$\chi \circ \imath = j \circ \chi$$

and χ restricted to the set of fixed points of $\imath$ is a bijection on the set of fixed points of j in $\mathbb{C}^2$, which is the image of

$$\begin{aligned} e : \mathbb{C} &\to \mathbb{C}^2 \\ z &\mapsto (z, \bar{z}). \end{aligned}$$

One has $e^{-1} \circ \chi|_\Lambda = \tau$. Let $\tilde{\Lambda}_{\mathrm{sing}} \subset \tilde{\Lambda}$ be the set of unordered triples $\{\lambda_1, \lambda_2, \lambda_3\}$, not all the λ_j being different. Lemma 2.4.8 implies that χ restricted to $\tilde{\Lambda}_{\mathrm{sing}}$ is a bijection on the set

$$\{(x,y) \in \mathbb{C}^2 : \tilde{f}(x,y) = 0\},$$

where

$$\begin{aligned} \tilde{f}(x,y) &= \begin{vmatrix} 1 & -x & y & -1 & 0 \\ 0 & 1 & -x & y & -1 \\ 3 & -2x & y & 0 & 0 \\ 0 & 3 & -2x & y & 0 \\ 0 & 0 & 3 & -2x & y \end{vmatrix} \\ &= -x^2y^2 + 4(x^3 + y^3) - 18xy + 27. \end{aligned}$$

We define $\Lambda_0 = \Lambda - \tilde{\Lambda}_{\mathrm{sing}}$ and notice that $\tau|_{\Lambda \cap \tilde{\Lambda}_{\mathrm{sing}}} : \Lambda \cap \tilde{\Lambda}_{\mathrm{sing}} \to f^{-1}(0)$ and $\tau|_{\Lambda_0} : \Lambda_0 \to \mathbb{C} - f^{-1}(0)$ are bijections. In fact, in order to prove that $\tau|_{\Lambda \cap \tilde{\Lambda}_{\mathrm{sing}}}$ is injective it is enough to see that $\tau = e^{-1} \circ \chi$ is the composition of injective functions. Now, if $z \in f^{-1}(0)$, then $0 = f(z) = -\tilde{f}(z, \bar{z}) = -\tilde{f}(e(z))$, but there exists $\lambda \in \Lambda$ such that $\chi(\lambda) = e(z)$, then $\tilde{f}(\chi(\lambda)) = 0$, which implies that $\lambda \in \Lambda \cap \tilde{\Lambda}_{\mathrm{sing}}$. Moreover $\tau(\lambda) = e^{-1} \circ \chi(\lambda) = z$. Therefore, $\tau|_{\Lambda \cap \tilde{\Lambda}_{\mathrm{sing}}}$ is onto. The proof of the fact that $\tau|_{\Lambda_0}$ is bijective is straightforward.

We have proved that $\tau : \mathrm{SU}(2,1) \to \mathbb{C}$ is onto. However we have implicitly assumed that Λ is the image of a correspondence defined in $\mathrm{SU}(2,1)$. Such correspondence is defined by

$$\begin{aligned} L : \mathrm{SU}(2,1) &\to \Lambda \\ A &\mapsto \{\lambda_1, \lambda_2, \lambda_3\}, \end{aligned}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of A. We must check that L is onto. We define $\lambda = \{\lambda_1, \lambda_2, \lambda_3\} \in \Lambda$. Notice there are only two possibilities:

(i) $|\lambda_i| = 1$ for $i = 1, 2, 3$; in this case

$$A = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} \in \mathrm{SU}(2,1),$$

and $L(A) = \lambda$.

(ii) $\lambda_3 = re^{i\theta}$, $0 < r < 1$ and we can take $\lambda_2 = e^{-2i\theta}$, $\lambda_1 = (1/r)e^{i\theta}$. We take $r = e^{-u}$ for some $u \in \mathbb{R}^+$, then

$$A = \begin{pmatrix} \cosh(u)e^{i\theta} & 0 & \sinh(u)e^{i\theta} \\ 0 & e^{-2i\theta} & 0 \\ \sinh(u)e^{i\theta} & 0 & \cosh(u)e^{i\theta} \end{pmatrix} \in \mathrm{SU}(2,1),$$

and $L(A) = \lambda$.

If $A_1, A_2 \in \mathrm{SU}(2,1)$ satisfy $\tau(A_1) = \tau(A_2) \in \mathbb{C} - f^{-1}(0)$, then the characteristic polynomials of A_1 and A_2 are equal, which implies they have the same eigenvalues (they are different amongst them). Then A_1 and A_2 are conjugate of the same diagonal matrix, so A_1 and A_2 are conjugate.

We define $\Lambda_0^l = \{\lambda \in \Lambda : \lambda \cap \mathbb{S}^1$ has one single element $\}$. We can suppose that

$$|\lambda_1| > 1, \quad |\lambda_2| < 1, \quad |\lambda_3| = 1\,. \tag{2.4.12}$$

In particular, $\Lambda_0^l \subset \Lambda_0$, and given that λ_1 satisfies 2.4.12 , we obtain unique λ_2, λ_3 by means of the relations

$$\lambda_2 = (\overline{\lambda_1})^{-1}, \qquad \lambda_3 = \frac{\overline{\lambda_1}}{\lambda_1}$$

which proves that Λ_0^l is homeomorphic to the exterior of the unitary disc of $\mathbb{C}$ and therefore it is connected. [The topology on $\tilde{\Lambda}$ is the topology induced by the bijective function $\chi : \tilde{\Lambda} \to \mathbb{C}^2$ and the topology on Λ is the subspace topology.]

We denote by $\Lambda_0^e = \Lambda_0 - \Lambda_0^l$. If $\lambda \in \mathbb{R} \setminus \{0, 1, -1\}$, then

$$\tau(\lambda, \lambda^{-1}, 1) = \lambda + \lambda^{-1} + 1$$

which shows that $\tau|_{\Re(\Lambda_0^l)} : \Re(\Lambda_0^l) \to \mathbb{R} \setminus [-1, 3]$ is a bijection, where $\Re(\Lambda_0^l)$ denotes the set of triples in Λ_0^l: We note that $\tau|_{\Re(\Lambda_0^l)}$ is injective because it is the composition of injective functions and it is not hard to check that it is onto. Notice that f is positive in $\mathbb{R} \setminus [-1, 3]$. In fact, if $x \in \mathbb{R}$, then

$$f(x) = x^4 - 8x^3 + 18x^2 - 27 = (x+1)(x-3)^3.$$

We claim that $\tau|_{\Lambda_0^l} : \Lambda_0^l \to f^{-1}(\mathbb{R}^+)$ is bijective. In fact, first notice that $\tau(\Lambda_0^l) \subset f^{-1}(\mathbb{R}^+)$, for otherwise, using that Λ_0^l is connected, we can find a triplet

$\lambda \in \Lambda_0^l$ such that $f(\tau(\lambda)) = 0$, which means $\lambda \in \Lambda_0 \cap \tilde{\Lambda}_{\text{sing}} = \varnothing$, a contradiction. That τ is injective follows from the fact that it is a composition of the injective functions e^{-1} and χ.

In order to prove $\tau(\Lambda_0^l) = f^{-1}(\mathbb{R}^+)$, we take $z \in f^{-1}(\mathbb{R}^+)$, we know that $z = \tau(\lambda)$, for some $\lambda \in \Lambda$. We assume that $\lambda \notin \Lambda_0^l$, then every element in λ has module 1, which implies that

$$|z| = |\tau(\lambda)| \leq 3,$$

and then

$$|\Re(z^3)| \leq |z^3| \leq 27,$$

therefore,

$$f(z) \leq (3)^4 - 8(-27) + 18(3)^2 - 27 = 0,$$

which is a contradiction.

We now claim that the function $\tau|_{\Lambda_0^e}$ is a bijection on $f^{-1}(\mathbb{R}^-)$. We prove first that $\tau(\Lambda_0^e) \subset f^{-1}(\mathbb{R}^-)$. In fact, if $\lambda \in \Lambda_0^e$, then the elements in λ are different and they have module 1. We know that $\tau|_{\Lambda_0^e}$ is injective, because it is the restriction of the injective function $\tau|_{\Lambda_0}$. Finally, $\tau(\Lambda_0^e) = f^{-1}(\mathbb{R}^-)$, because $\tau(\Lambda_0^l) = f^{-1}(\mathbb{R}^+)$.

Hence $f(\tau(A)) < 0$ if and only if the eigenvalues of A are distinct unitary complex numbers, which happens if and only if A is regular elliptic. One has $f(\tau(A)) > 0$ if and only if A has exactly one eigenvalue in $\mathbb{S}^1$, if and only if A is loxodromic.

Now we consider the case when $f(\tau(A)) = 0$. Clearly $A \in \mathrm{PU}(2,1)$ is unipotent if and only if it has a lift to $\mathrm{SU}(2,1)$ having equal eigenvalues, and therefore $\frac{1}{3}\tau(A) \in C_3$. Conversely, if $\frac{1}{3}\tau(A) = \omega \in C_3$, then

$$\chi_A(t) = t^3 - 3\omega t^2 + 3\omega^2 t - 1 = (t-\omega)^3,$$

thus A has three repeated eigenvalues and it is projectively equivalent to a unipotent matrix.

Finally we consider the case when $\tau(A) \in f^{-1}(0) - 3C_3$. Then A has an eigenvalue $\zeta \in \mathbb{S}^1$ of multiplicity 2 and the other eigenvalue is equal to ζ^{-2}. Given that $\tau(A) \notin 3C_3$, we have $\zeta \neq \zeta^{-2}$. There are two cases depending on the Jordan canonical form of A: if A is diagonalizable, then A is elliptic (a complex reflection). This case splits in two cases, depending on whether the ζ-eigenspace V_ζ is positive or indefinite: if V_ζ is indefinite, then A is a complex reflection with respect to the complex geodesic corresponding to V_ζ; if V_ζ is positive, then A is a complex reflection with respect to the point corresponding to the ζ^{-2}-eigenspace. If A is not diagonalizable, then A has a repeated eigenvalue of module 1 and it has Jordan canonical form

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix}.$$

In this case the eigenvector corresponding to λ is e_1 and A is ellipto-parabolic. $\square$

Corollary 2.4.13. *Let $\imath : \mathbb{C}^* \to \mathbb{C}^*$ be the inversion on the unitary circle; i.e., $\imath(z) = 1/\bar{z}$. If the set of eigenvalues of $A \in \mathrm{SL}(3, \mathbb{C})$ is invariant under $\imath$, then:*

(i) *The eigenvalues of A are unitary complex numbers, and pairwise different if and only if $f(\tau(A)) < 0$.*

(ii) *Precisely one eigenvalue is unitary if and only if $f(\tau(A)) > 0$.*

This corollary can be proved using the proof of Lemma 2.4.8. This is useful for proving Theorem 4.3.3, which is an extension to the elements in $\mathrm{PSL}(3, \mathbb{C})$ of the classification Theorem 2.4.9.

Remark 2.4.14. In [44] the authors look at the group of quaternionic Möbius transformations that preserve the unit ball in the quaternionic space $\mathcal{H}$, and they use this to classify the isometries of the hyperbolic space $\mathbb{H}^4_{\mathbb{R}}$ into six types, in terms of their fixed points and whether or not they are conjugate in $\mathrm{U}(1, 1; \mathcal{H})$ to an element of $\mathrm{U}(1, 1; \mathbb{C})$. In [72] the author uses also quaternionic transformations to refine the classification of the elements in $\mathrm{Conf}_+(\mathbb{S}^4)$ using algebraic invariants, and in [73] the authors characterize algebraically the isometries of quaternionic hyperbolic spaces (see also [43]). In Chapter 10 we follow [202] and use Ahlfor's characterization of the group $\mathrm{Conf}_+(\mathbb{S}^4) \cong \mathrm{Iso}_+(\mathbb{H}^5_{\mathbb{R}})$ as quaternionic Möbius transformations in order to describe the canonical embedding $\mathrm{Conf}_+(\mathbb{S}^4) \hookrightarrow \mathrm{PSL}(4, \mathbb{C})$ that appears in twistor theory, and use this to construct discrete subgroups of $\mathrm{PSL}(4, \mathbb{C})$.

2.5 Complex hyperbolic Kleinian groups

As before, let $\mathrm{U}(n, 1) \subset \mathrm{GL}\,(n + 1, \mathbb{C})$ be the group of linear transformations that preserve the quadratic form $|z_1|^2 + \cdots + |z_n|^2 - |z_{n+1}|^2$. We let V_- be the set of negative vectors in $\mathbb{C}^{n,1}$ for this quadratic form. We know already that the projectivization $[V_-]$ is a $2n$-ball that serves as a model for complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$ and $\mathrm{PU}(n, 1)$ is its group of holomorphic isometries (see Section 2.2).

Definition 2.5.1. A subgroup $\Gamma \subset \mathrm{PSL}(n + 1, \mathbb{C})$ is a *complex hyperbolic Kleinian group* if it is conjugate to a discrete subgroup of $\mathrm{PU}(n, 1)$.

A fundamental problem in complex hyperbolic geometry is the construction and understanding of complex hyperbolic Kleinian groups.

This problem goes back to the work of Picard and Giraud (see the appendix in [67]), and it has been subsequently addressed by many authors, as for instance Mostow, Deligne, Hirzebruch and many others, as a means of generalising the classical theory of automorphic forms and functions, and also as a means of constructing complex manifolds with a rich geometry. Complex hyperbolic Kleinian groups can also be regarded as being discrete faithful representations of a group Γ into $\mathrm{PU}(n, 1)$.

Two basic questions are the construction of lattices in $\mathrm{PU}(n, 1)$, and the existence of lattices which are not commensurable with arithmetic lattices. Let us explain briefly what this means.

A discrete subgroup Γ of $\mathrm{PU}(n,1)$ (and more generally of a locally compact group G), equipped with a Haar measure, is said to be a *lattice* if the quotient $\mathrm{PU}(n,1)/\Gamma$ has finite volume. The lattice is said to be *uniform (or cocompact)* if the quotient $\mathrm{PU}(n,1)/\Gamma$ is actually compact.

A subgroup Γ of $\mathrm{U}(n,1)$ is *arithmetic* if there is an embedding $\mathrm{U}(n,1) \overset{\iota}{\hookrightarrow} \mathrm{GL}\,(N,\mathbb{C})$, for some N, such that the image of Γ is *commensurable* with the intersection of $\iota(\mathrm{U}(n,1))$ with $\mathrm{GL}\,(N,\mathbb{Z})$. That is, $\iota(\Gamma) \cap \mathrm{GL}\,(N,\mathbb{Z})$ has finite index in $\iota(\Gamma)$ and in $\mathrm{GL}\,(N,\mathbb{Z})$.

Complex hyperbolic space is, like real hyperbolic space, a noncompact symmetric space of rank 1, and an important problem in the study of noncompact symmetric spaces is the relationship between arithmetic groups and lattices: while all arithmetic groups are lattices (by [29]), the question of whether or not all lattices are arithmetic is rather subtle. We know from Margulis' work that for symmetric spaces of rank ≥ 2, all irreducible lattices are arithmetic. The rank 1 symmetric spaces of noncompact type come in three infinite families, real, complex and quaternionic hyperbolic spaces: $\mathbb{H}^n_{\mathbb{R}}, \mathbb{H}^n_{\mathbb{C}}, \mathbb{H}^n_{\mathcal{H}}$, and one has also the Cayley (or octonionic) hyperbolic plane $\mathbb{H}^2_{\mathbb{O}}$. We know by [81] (and work by K. Corlette) that all lattices on $\mathbb{H}^n_{\mathcal{H}}$ and $\mathbb{H}^2_{\mathbb{O}}$ are arithmetic, and we also know by [80] that there are nonarithmetic lattices acting on real hyperbolic spaces of all dimensions. In complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$, we know that there are nonarithmetic lattices for $n = 2, 3$, by [50] respectively. The question of existence of nonarithmetic lattices on $\mathbb{H}^n_{\mathbb{C}}$ is open for $n \geq 4$ and this is one of the major open problems in complex hyperbolic geometry (see [50]).

There is a huge wealth of knowledge in literature about complex hyperbolic Kleinian groups published the last decades by various authors, as for instance P. Deligne, G. Mostow, W. Goldman, J. Parker, R. Schwartz, N. Gusevskii, E. Falbel, P. V. Koseleff, D. Toledo, M. Kapovich, E. Z. Xia, among others. Here we mention just a few words that we hope will give a taste of the richness of this branch of mathematics.

2.5.1 Constructions of complex hyperbolic lattices

We particularly encourage the reader to look at the beautiful articles [105] and [168] for wider and deeper surveys related to the topic of complex hyperbolic Kleinian groups. Kapovich's article is full of ideas and provides a very deep understanding of real and complex hyperbolic Kleinian groups. Parker's article explains the known ways and sources for producing complex hyperbolic lattices (see also [169]). Below we briefly mention some of Parker's explanations for lattices, and in the section below we give other interesting examples of constructions that produce complex hyperbolic groups which are not lattices.

Parker classifies the methods for constructing complex hyperbolic lattices in the following four main types, which of course overlap. All these are also present in the monograph [50], whose main goal is to investigate commensurability among lattices in $\mathrm{PU}(n,1)$.

i) Arithmetic lattices: The natural inclusion of the integers in the real numbers is the prototype of an arithmetic group. This yields naturally to the celebrated modular group $\mathrm{PSL}(2,\mathbb{Z})$ in $\mathrm{PSL}(2,\mathbb{R})$. That construction was generalised to higher-dimensional complex hyperbolic lattices by Picard in 1882, and then studied by a number of authors. For instance, let d be a positive square-free integer, let $\mathbb{Q}(i\sqrt{d})$ be the corresponding quadratic number field and $\mathcal{O}_d$ its ring of integers, a discrete subgroup of $\mathbb{C}$. Let H be a Hermitian matrix with signature $(2,1)$ and entries in $\mathcal{O}_d$. Let $\mathrm{SU}(H)$ be the group of unitary matrices that preserve H, and let $\mathrm{SU}(H;\mathcal{O}_d)$ be the subgroup of $\mathrm{SU}(H)$ consisting of matrices whose entries are in $\mathcal{O}_d$. Then $\mathrm{SU}(H;\mathcal{O}_d)$ is a lattice in $\mathrm{SU}(H)$. These type of arithmetic groups are known as Picard modular groups. We refer to Parker's article for a wide bibliography concerning these and other arithmetic constructions in the context of complex hyperbolic geometry.

ii) The second major technique mentioned by Parker for producing lattices in complex hyperbolic spaces is to consider objects that are parametrised by some $\mathbb{H}^n_{\mathbb{C}}$, with the property that the corresponding group of automorphisms is a complex hyperbolic lattice. For instance we know that the modular group $\mathrm{PSL}(2,\mathbb{Z})$ can be regarded as being the monodromy group of elliptic functions. Similar results were known to Poincaré, Schwartz and others for real hyperbolic lattices. In complex hyperbolic geometry, the first examples of this type of lattices were again given by Picard in 1885. He considered the moduli space of certain hypergeometric functions and showed that their monodromy groups were lattices in $\mathrm{PU}(2,1)$, though his proof of the discreteness of the groups was not complete. This was settled and extended in [50], where the authors study the monodromy groups of a certain type of integrals, generalising the classical work of Schwarz and Picard. Under a certain integrality condition that they call (INT), they prove that the monodromy group Γ is a lattice in $\mathrm{PU}(n,1)$; yet, for $d > 5$ this condition is never satisfied. They also give criteria for Γ to be arithmetic. Further research along similar lines was developed in [50], as well as by various other authors, as for instance Le Vavasseur, Terada, Thurston, Parker and others. Alternative approaches along this same general line of research have been followed by Allcock, Carlson, Toledo and others. Thurston's approach in [224] is particularly interesting and gives an alternative way of interpreting the (INT)-condition. We refer to Section 3 in [168] for an account on the construction of lattices arising as monodromy groups of hypergeometric functions.

iii) A third way for constructing discrete groups is by looking at lattices generated by complex reflections, or more generally by finding appropriate fundamental domains. This approach was introduced by Giraud (see Appendix A in [67]). Typically, a fundamental domain is a locally finite polyhedron P with some combinatorial structure that tells us how to identify its faces, called the sides, by maps in $\mathrm{PU}(n,1)$. Given this information, Poincare's polyhedron theorem gives conditions under which the group generated by the sides pairing maps is discrete, and it gives a presentation of the group. One way for doing so is to construct the Dirichlet fundamental domain $D_\Gamma(z_0)$ as in Chapter 1: Assume $z_0 \in \mathbb{H}^n_{\mathbb{C}}$ is not fixed by any nontrivial element in a given group Γ, then $D_\Gamma(z_0)$ is the set of points in $\mathbb{H}^n_{\mathbb{C}}$ that are closer to z_0 than to any other point in its orbit. Its sides are contained in bisectors. Mostow used this approach in [155] to give the first examples of nonarithmetic lattices in $\mathrm{PU}(n,1)$ $(n \leq 3)$. Alternative methods for constructing lattices in this way have been given by Deraux, Falbel, Paupert, Parker, Goldman and others. Again, we refer to [168] for more on this topic.

iv) A fourth way for constructing complex hyperbolic lattices is using algebraic geometry. In fact, the Yau-Miyaoka uniformization theorem ([151]) states that if M is a compact complex 2-manifold whose Chern classes satisfy $c_1^2 = 3c_2$, then M is either $\mathbb{P}^2_{\mathbb{C}}$ or a complex hyperbolic manifold, i.e., the quotient of $\mathbb{H}^2_{\mathbb{C}}$ by some cocompact lattice. Thence the fundamental groups of such surfaces with $c_1^2 = 3c_2$ are uniform lattices in $\mathrm{PU}(2,1)$. Yet, techniques for having a direct geometric construction of such surfaces were not available until the Ph. D. Thesis of R. A. Livné [Harvard, Cambridge, Mass., 1981]. A variant on Livné's technique, using abelian branched covers of surfaces, was subsequently used in [93] to construct an infinite sequence of noncompact surfaces satisfying $\overline{c}_1^2 = 3\overline{c}_2$. See also [94] and [195].

2.5.2 Other constructions of complex hyperbolic Kleinian groups

In the previous subsection we briefly explained methods for constructing complex hyperbolic lattices. In the case of $\mathbb{H}^1_{\mathbb{C}} \cong \mathbb{H}^2_{\mathbb{R}}$ these are the so-called Fuchsian groups of the first kind, i.e., discrete subgroups of $\mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{PU}(1,1)$ whose limit set is the whole sphere at infinity. And we know that Fuchsian groups of the second kind are indeed very interesting. It is thus natural to ask about discrete subgroups of $\mathrm{PU}(n,1)$ which are not lattices. This is in itself a whole area of research, that we will not discuss here, and we refer for this to the bibliography, particularly to [67], [105].

One of the classical ways of doing so is by taking discrete subgroups of $\mathrm{PSL}(2,\mathbb{R})$, considering representations of these in $\mathrm{PU}(2,1)$ and then looking at their deformations. More generally (see [67, Section 4.3.7]) one may consider $\Gamma \subset \mathrm{U}(n,1)$ a discrete subgroup and consider the natural inclusion $\mathrm{U}(n,1) \hookrightarrow \mathrm{U}(n+1,1)$. The composition

$$\Gamma \hookrightarrow \mathrm{U}(n,1) \hookrightarrow \mathrm{U}(n+1,1) \longrightarrow \mathrm{PU}(n+1,1)$$

defines a representation of Γ as a group of isometries of $\mathbb{H}^{n+1}_{\mathbb{C}}$. If Γ is a lattice, then there are strong local rigidity theorems, due to Goldman, Goldman-Millson, Toledo (for n=1) and Corlette (for n=2). If we now consider Γ to be discrete but not a lattice, then there is a rich deformation theory, and there are remarkable contributions by various authors.

We now give two interesting examples along this line of research, with references that can guide the interested reader into further reading.

Example 2.5.2 (Complex hyperbolic triangle groups)**.** In Chapter 1 we discussed the classical hyperbolic triangle groups. These are discrete subgroups of $\mathrm{PSL}(2,\mathbb{R})$ generated by inversions on the sides (edges) of a triangle in the hyperbolic space $\mathbb{H}^2_{\mathbb{R}}$, bounded by geodesics, with inner angles $\pi/p, \pi/q$ and π/r for some integers $p, q, r > 1$; in fact some of these integers can be ∞, corresponding to triangles having one or more vertices on the visual sphere. When all vertices of the triangle are at infinity, this is called an ideal triangle.

In their article [70], W. Goldman and J. Parker give a method of constructing and studying complex hyperbolic ideal triangle groups. These are representations in $\mathrm{PU}(2,1)$ of hyperbolic ideal triangle groups, such that each standard generator of the triangle group maps to a complex reflection, taking good care of the way in which products of pairs of generators are mapped. The fixed point set of a complex reflection is a complex slice (see Section 2.4.1).

Roughly speaking, the technique for constructing such groups begins with the embedding of a Fuchsian subgroup $\Gamma_0 \hookrightarrow \mathrm{PSL}(2,\mathbb{R})$, and deforms the representation inside $\mathrm{Hom}(\Gamma, \mathrm{PU}(2,1))$. Thus Γ_0 preserves the real hyperbolic plane $\mathbb{H}^2$, but in general the deformed groups will not preserve any totally geodesic 2-plane (which are either complex lines or totally real 2-planes, intersected with the ball).

More precisely, Goldman and Parker look at the space of representations for a given triangle group, and for this they consider a triple of points (u_1, u_2, u_3) in ∂B^2, the boundary of the complex ball B^2. Let C_1, C_2, C_3 be the corresponding complex geodesics they span. Let Σ be the free product of three groups of order 2, and let $\phi : \Sigma \to \mathrm{PU}(2,1)$ be the homomorphism taking the generators of Σ into the inversions (complex reflections) on C_1, C_2, C_3. Conjugacy classes of such homomorphisms correspond to $\mathrm{PU}(2,1)$-equivalence classes of triples (u_1,u_2,u_3), and such objects are parametrised by their Cartan angular invariant $\phi(C_1, C_2, C_3)$, $-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$. The problem they address is: When is the subgroup $\Gamma \subset \mathrm{PU}(2,1)$, obtained in this way, discrete? They prove that if the embedding of Γ is discrete, then $|\phi(x_0, x_1, x_2)| \leq \tan^{-1}\sqrt{125/3}$, and they conjectured that the condition $|\phi(x_0, x_1, x_2)| \leq \tan^{-1}\sqrt{125/3}$ was also sufficient to have a discrete embedding. This was referred to as the Goldman-Parker conjecture, and this was proved by R. Schwartz in [197] (see also [199]).

In this same vein one has R. Schwartz' article [198], where he discusses triangle subgroups of $\mathrm{PU}(2,1)$ obtained also by complex reflections. Recall there is a simple formula (2.4.1) for the general complex reflection: Let V_+ and V_- be as above and choose a vector $c \in V_+$. For every nondegenerate vector $u \in \mathbb{C}^{2,1}$ the inversion on u is

$$I_c(u) = -u + \frac{2\langle u, c\rangle}{\langle c, c\rangle}\, c\,, \tag{2.5.3}$$

and every complex reflection is conjugate to a map of this type. One may also consider the Hermitian cross product (see [67, p. 45])

$$u \boxtimes v = (u_3 v_2 - u_2 v_3 \, , \, u_1 v_3 - u_3 v_1 \, , \, u_1 v_2 - u_2 v_1) \tag{2.5.4}$$

which satisfies: $\langle u, u \boxtimes v \rangle = \langle v, u \boxtimes v \rangle = 0$.

The two equations 2.5.3 and 2.5.4 enable us to generate discrete groups defined by complex reflections as follows: Take three vectors $V_1, V_2, V_3 \in V_-$ and set $c_j = V_{j-1} \boxtimes V_{j+1}$ (indices are taken modulo 3). For simplicity set $I_j = I_{c_j}$. Then the complex reflection I_j leaves invariant the complex geodesic determined by the points $[V_{j-1}]$ and $[V_{j+1}]$. The group $\Gamma := \langle I_1, I_2, I_3 \rangle$ is a complex-reflection triangle group determined by the triangle with vertices $[V_1], [V_2], [V_3]$, which is discrete if the vertices are chosen appropriately, as mentioned above. These groups furnish some of the simplest examples of complex hyperbolic Kleinian groups having a rich deformation theory (see for instance [198], [197], [56]).

This construction gives rise to manifolds with infinite volume obtained as quotient spaces $M = \mathbb{H}^2_{\mathbb{C}}/\Gamma$, which are the interior of a compact manifold-with-boundary, whose boundary ∂M is a real hyperbolic 3-manifold, quotient of a domain $\Omega \subset \partial \mathrm{H}^2_{\mathbb{C}} \cong \mathbb{S}^3$ by Γ (c.f. [198]).

Example 2.5.5 (Complex hyperbolic Kleinian groups with limit set a wild knot)**.** As mentioned above, a way of producing interesting complex hyperbolic Kleinian groups is by taking a Fuchsian group $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ and considering a representation of it in $\mathrm{PU}(2, 1)$ (or more generally in $\mathrm{PU}(n, 1)$). A specially interesting case is when Γ is the fundamental group of a hyperbolic surface, say of finite area. There has been a lot of progress in the study and classification of complex hyperbolic Kleinian groups which are isomorphic to such a surface group, but this is yet a mysterious subject which is being explored by several authors. The most natural way for this is by considering the *Teichmüller space* $T(\Gamma)$ of discrete, faithful, type-preserving representations of Γ in $\mathrm{PU}(2, 1)$; type-preserving means that every element in Γ that can be represented by a loop enclosing a single puncture is carried into a parabolic element in $\mathrm{PU}(2, 1)$, i.e., an element having exactly one fixed point on the boundary sphere of the ball in $\mathbb{P}^2_{\mathbb{C}}$ that serves as a model for $\mathbb{H}^2_{\mathbb{C}}$.

In [53] the authors prove that if Γ is the fundamental group of a noncompact surface of finite area, then the Teichmüller space $T(\Gamma)$ is not connected. For this they construct a geometrically finite quasi-Fuchsian group Γ acting on $H^2_{\mathbb{C}}$ whose limit set is a wild knot, and they show that this group can be also embedded in $\mathrm{PU}(2, 1)$ in such a way that the two embeddings are in different components of $T(\Gamma)$. They also prove that in both cases the two representations have the same Toledo invariant, thence this invariant does not distinguish different connected components of $T(\Gamma)$ when the surface has punctures, unlike the case where the surface is compact.

Let us sketch the construction of Dutenhefner-Gusevskii (see their article for more details). Consider the 3-dimensional Heisenberg group $\mathfrak{N}$, which is diffeomorphic to $\mathbb{C} \times \mathbb{R}$. Notice that this group carries naturally the *Heisenberg norm*

$$\|(\zeta, t)\| = \left|\|\zeta\|^2 + it\right|^{\frac{1}{2}}.$$

The corresponding metric on $\mathfrak{N}$ is *the Cygan metric*

$$\rho_0\big((\zeta, t), (\zeta', t')\big) = \|(\zeta - \zeta', t - t' + 2\Im\langle\langle\zeta - \zeta'\rangle\rangle)\|\,.$$

We use horospherical coordinates $\{(\zeta, t)\}$ for $\mathbb{H}^2_{\mathbb{C}}$, which allow us to identify $\mathfrak{N}$ with the horospheres in $\mathbb{H}^2_{\mathbb{C}}$ as explained in Subsection 2.3.1; these are the sets of points in $\mathbb{H}^2_{\mathbb{C}}$ of a constant "height".

Now consider a knot K and a finite collection $S = \{S_k, S'_k\}, k = 1 \dots n$, of Heisenberg spheres (in the Cygan metric) placed along K, satisfying the following condition: there is an enumeration $T_1; \dots; T_{2n}$ of the spheres of this family such that each T_k lies outside all the others, except that T_k and T_{k+1} are tangent, for $k = 1, \dots, 2n - 1$, and T_{2n} and T_1 are tangent. Such a collection S of Heisenberg spheres is called a *Heisenberg string of beads*, see Figure 2.1. Let g_k be elements from $\mathrm{PU}(2, 1)$ such that:

(i) $g_k(S_k) = S'_k$,

(ii) $g_k(\mathrm{Ext}(S_k)) \subset \mathrm{Int}(S'_k)$,

(iii) g_k maps the points of tangency of S_k to the points of tangency of S'_k.

Let Γ be the group generated by g_k. Suppose now that Γ is Kleinian and the region D lying outside all the spheres of the family S is a fundamental domain for Γ. Then one can show that, under these conditions, the limit set of the group Γ is a wild knot.

The main difficulties in that construction are in finding a suitable knot, and appropriate spheres and pairing transformations g_k, so that the region D one gets is a *special Ford fundamental domain*, so that one can use Poincaré's polyhedron theorem (proved in [84]) for complex hyperbolic space, to ensure among other things that the group is Kleinian.

In order to construct a knot and a family of spheres as above, having the properties we need, they consider the granny knot K in $\mathcal{H}$; this is the connected sum of two right-handed trefoil knots (could also be left-handed). This has the property that it can be placed in $\mathcal{H}$ so as to be symmetric with respect to the reflection in the y-axis $\{v, (x + iy) \in \mathcal{H} \mid x = v = 0\}$, and also with respect to reflection in the vertical axis $\{x = y = 0\}$. These reflections are restrictions of elements in $\mathrm{PU}(2, 1)$. Then they further choose K to be represented by a polygonal knot L, with the same symmetry properties and so that the edges of L are either segments of "horizontal" lines or segments of "vertical chains". Using this knot, they can show that there is a family of spheres and transformations with the required properties.

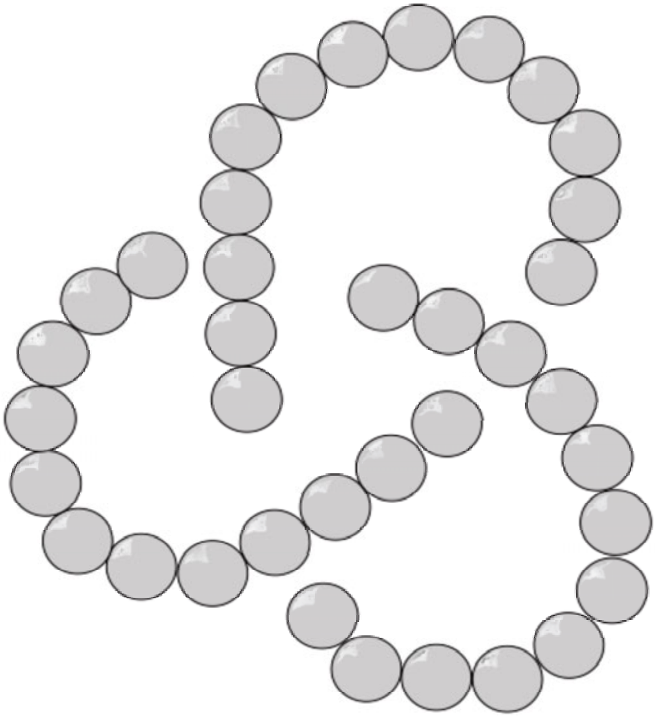

Figure 2.1: A Heisenberg string of beads

2.6 The Chen-Greenberg limit set

Consider now a discrete subgroup G of $\mathrm{PU}(n,1)$. As before, we take as a model for complex hyperbolic n-space $\mathbb{H}^n_{\mathbb{C}}$ the ball $\mathbb{B} \cong \mathbb{B}^{2n}$ in $\mathbb{P}^n_{\mathbb{C}}$ consisting of points with homogeneous coordinates satisfying

$$|z_1|^2 + \cdots + |z_n|^2 < |z_{n+1}|^2$$

whose boundary is a sphere $\mathbb{S} := \partial\mathbb{H}^n_{\mathbb{C}} \cong \mathbb{S}^{2n-1}$, and we equip $\mathbb{B}$ with the Bergman metric ρ to get $\mathbb{H}^n_{\mathbb{C}}$.

The following theorem can be found in [45]) and it is essentially a consequence of Arzelà-Ascoli's theorem, since G is acting on $\mathbb{H}^n_{\mathbb{C}}$ by isometries (see [185]).

Theorem 2.6.1. *Let G be a subgroup of* $\mathrm{PU}(n,1)$. *The following four conditions are equivalent:*

(i) *The subgroup $G \subset \mathrm{PU}(n,1)$ is discrete.*

(ii) *G acts properly discontinuously on $\mathbb{H}^n_{\mathbb{C}}$.*

(iii) *The region of discontinuity of G in $\mathbb{H}^n_{\mathbb{C}}$ is all of $\mathbb{H}^n_{\mathbb{C}}$.*

(iv) *The region of discontinuity of G in $\mathbb{H}^n_{\mathbb{C}}$ is nonempty.*

It follows that the orbit of every $x \in \mathbb{H}^n_{\mathbb{C}}$ must accumulate in $\partial\mathbb{H}^n_{\mathbb{C}}$.

Definition 2.6.2. If G is a subgroup of $\mathrm{PU}(n,1)$, the *Chen-Greenberg limit set* of G, denoted by $\Lambda_{CG}(G)$, is the set of accumulation points of the G-orbit of any point in $\mathbb{H}^n_{\mathbb{C}}$.

The following lemma is a slight generalisation of lemma 4.3.1 in [45]. This is essentially the *convergence property* for complex hyperbolic Kleinian groups (see page 15), and it implies that $\Lambda_{CG}(G)$ does not depend on the choice of the point in $\mathbb{H}^n_{\mathbb{C}}$.

Lemma 2.6.3. *Let p be a point in $\mathbb{H}^n_{\mathbb{C}}$ and let (g_n) be a sequence of elements in $\mathrm{PU}(n,1)$ such that $g_n(p) \underset{m\to\infty}{\longrightarrow} q \in \partial\mathbb{H}^n_{\mathbb{C}}$. Then for all $p' \in \mathbb{H}^n_{\mathbb{C}}$ we have that $g_n(p') \underset{m\to\infty}{\longrightarrow} q$. Moreover, if $K \subset \mathbb{H}^n_{\mathbb{C}}$ is a compact set, then the sequence of functions $g_n|_K$ converges uniformly to the constant function with value q.*

The expression $B_\rho(x, C)$ denotes the ball with centre at $x \in \mathbb{H}^n_{\mathbb{C}}$ and radius $C > 0$ with respect to the Bergman metric in $\mathbb{H}^n_{\mathbb{C}}$.

Proof. Assume the sequence $(g_n(p))$ converges to a point $q \in \partial\mathbb{H}^n_{\mathbb{C}}$, and we assume there exists $p' \in \mathbb{H}^n_{\mathbb{C}}$ such that the sequence $(g_n(p'))$ does not converge to q. Then there is a subsequence of $(g_n(p'))$ converging to a point $q' \in \overline{\mathbb{H}^n_{\mathbb{C}}}$, $q' \neq q$. So we may suppose that $g_n(p) \underset{m\to\infty}{\longrightarrow} q$ and $g_n(p') \underset{m\to\infty}{\longrightarrow} q'$. Let us denote by $[p, p']$ and $[q, q']$ the geodesic segments (with respect to the Bergman metric) joining p with p' and q with q', respectively. The distance from p to p' in the Bergman metric is equal to the length of $[p, p']$, and to $[g_n(p), g_n(p')]$, but this length goes to ∞ as $n \to \infty$, a contradiction. Therefore $g_n(p') \to q$ as $n \to \infty$.

Now we prove the convergence is uniform. If K is a compact subset of $\mathbb{H}^n_{\mathbb{C}}$, then there exists $C > 0$ such that $K \subset B_\rho(\mathbf{0}, C)$, where $\mathbf{0}$ denotes the origin in the ball model for $\mathbb{H}^n_{\mathbb{C}}$. The first part of this proof implies that $g_n(\mathbf{0}) \to q$. Given that g_n is an isometry of $\mathbb{H}^n_{\mathbb{C}}$, we have that $g_n(B_\rho(\mathbf{0}, C)) = B_\rho(g_n(\mathbf{0}), C)$ for each n, and the result follows from the lemma below. □

Lemma 2.6.4. *If (x_n) is a sequence of elements in $\mathbb{H}^n_{\mathbb{C}}$ such that $x_n \to q \in \partial\mathbb{H}^n_{\mathbb{C}}$ as $n \to \infty$, and $C > 0$ is a fixed positive number, then the Euclidean diameter of the ball $B_\rho(x_n, C)$ goes to zero as $n \to \infty$.*

Proof. We use the model of the ball $B^n \subset \mathbb{C}^n$ for $\mathbb{H}^n_{\mathbb{C}}$. Let $x \in B^n$. Every complex geodesic through x has the form

$$\Sigma_y = \{x + \zeta y \mid \zeta \in \mathbb{C},\ |x + \zeta y| < 1\}\ ,\ y \in \mathbb{C}^n,\ |y| = 1.$$

The Euclidean distance of Σ_y to the origin of B^n is equal to

$$r(y) := |x - \langle\langle x, y\rangle\rangle y|,$$

where the symbol $\langle\langle x, y\rangle\rangle$ means the classical Hermitian product of the vectors $x, y \in \mathbb{C}^n$. Also, Σ_y is a Euclidean disc of Euclidean radius

$$(1 - r(y)^2)^{1/2} = (1 - |x|^2 + |\langle\langle x, y\rangle\rangle|^2)^{1/2} =: R(y).$$

Moreover, the Bergman metric in Σ_y has the form

$$\frac{4R(y)^2 dz d\bar{z}}{(R(y)^2 - |z|^2)^2}.$$

The intersection $B_\rho(x, C) \cap \Sigma_y$ is a hyperbolic disk of hyperbolic radius equal to C in Σ_y, and its hyperbolic centre $x \in \Sigma_y$ has Euclidean distance $|\langle\langle x, y\rangle\rangle|$ from the Euclidean centre of Σ_y, which is the point $x - \langle\langle x, y\rangle\rangle y$. Then $B_\rho(x, C) \cap \Sigma_y$ is a Euclidean disc of Euclidean radius

$$C_e(y) = R(y) \tanh(C/2) \frac{R(y)^2 - |\langle\langle x, y\rangle\rangle|^2}{R(y)^2 - \tanh^2(C/2)|\langle\langle x, y\rangle\rangle|^2}.$$

When x is fixed, the radius $C_e(y)$ is a continuous function of $y \in \mathbb{S}^{2n-1}$. Let y_M be such that $C_e(y_M) \geq C_e(y)$ for all $y \in \mathbb{S}^{2n-1}$. Let X_1, $X_2 \in B_\rho(x, C)$, then there exist $y_1, y_2 \in \mathbb{S}^{2n-1}$ such that $X_k \in \Sigma_{y_k} \cap B_\rho(x, C)$, $k = 1, 2$. If σ_k denotes the Euclidean centre of the disc $\Sigma_{y_k} \cap B_\rho(x, C)$, one has that

$$|X_1 - X_2| \leq |X_1 - \sigma_1| + |X_2 - \sigma_2| \leq 2C_e(y_1) + 2C_e(y_2) \leq 4C_e(y_M).$$

Finally we see that, for each $y \in \mathbb{S}^{2n-1}$, $C_e(y) \to 0$ as $x \to \partial\mathbb{H}^n_{\mathbb{C}}$. □

It is clear from the definitions that the limit set $\Lambda_{CG}(G)$ is a closed, G-invariant set, and it is empty if and only if G is finite (since every sequence in a compact set contains convergent subsequences). Moreover the following proposition says that the action on $\Lambda_{CG}(G)$ is minimal (cf. [45]).

Proposition 2.6.5. *Let G be a nonelementary subgroup of* $\mathrm{PU}(n, 1)$*. If $X \subset \partial\mathbb{H}^n_{\mathbb{C}}$ is a G-invariant closed set containing more than one point, then $\Lambda_{CG}(G) \subset X$. Thence every orbit in $\Lambda_{CG}(G)$ is dense in $\Lambda_{CG}(G)$.*

Proof. Let q be a point in $\Lambda_{CG}(G)$. There exists a sequence (g_n) of elements of G such that $g_n(p) \to q$ for all $p \in \mathbb{H}^n_{\mathbb{C}}$. Let $x, y \in X$, $x \neq y$. We assume, taking subsequences if necessary, that $g_n(x) \to \hat{x}$ and $g_n(y) \to \hat{y}$, where $\hat{x} \neq q$, $\hat{y} \neq q$. Let $p' \in \mathbb{H}^n_{\mathbb{C}}$ be a point in the geodesic determined by x and y. Then $g_n(p')$ is in the geodesic determined by $g_n(x)$ and $g_n(y)$. Given that $q \neq \hat{x}$ and $q \neq \hat{y}$, we have that$q = \lim_{n\to\infty} g_n(p')$ belongs to the geodesic determined by $\lim_{n\to\infty} g_n(x) = \hat{x}$ and $\lim_{n\to\infty} g_n(y) = \hat{y}$, a contradiction. Thus, $g_n(x) \to q$ or $g_n(y) \to q$, so $q \in X$. Therefore $\Lambda_{CG}(G) \subset X$. □

Theorem 2.6.6. *Let G be a discrete group such that $\Lambda_{CG}(G)$ has more than two points, then it has infinitely many points.*

Proof. Assume that $\Lambda_{CG}(G)$ is finite with at least three points. Then

$$\widetilde{G} = \bigcap_{x \in \Lambda_{CG}(G)} \mathrm{Isot}(x, G)$$

is a normal subgroup of G with finite index. Then for each $\gamma \in \widetilde{G}$, it follows that $\gamma(x) = x$, for each $x \in \Lambda_{CG}(\Gamma)$. Then, the classification of elements in $\mathrm{PU}(n,1)$ yields that each element in $\widetilde{G}$ is elliptic. Hence $\widetilde{G}$ is finite, which is a contradiction. $\square$

Definition 2.6.7. (cf. Definition 3.3.9) The group G is *elementary* if $\Lambda_{CG}(G)$ has at most two points.

Notice that Proposition 2.6.5 implies that if G is nonelementary and $\Lambda_{CG}(G)$ is not all of $\partial\mathbb{H}^n_{\mathbb{C}}$, then $\Lambda_{CG}(G)$ is a nowhere dense perfect set. In other words, $\Lambda_{CG}(G)$ has empty interior and every orbit in $\Lambda_{CG}(G)$ is dense in $\Lambda_{CG}(G)$.

The following corollary is an immediate consequence of Proposition 2.6.5:

Corollary 2.6.8. *If $G \subset \mathrm{PU}(1,n)$ is a nonelementary discrete group, then $\Lambda(G)$ is the unique closed minimal G-invariant set, for the action of G in $\overline{\mathbb{H}^n}$.*

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