6

Complex Integration

6.1 Complex Integrals

In Chapter 3 we saw how the derivative of a complex function is defined. We now turn our attention to the problem of integrating complex functions. We will find that integrals of analytic functions are well behaved and that many properties from calculus carry over to the complex case. To introduce the integral of a complex function, we start by defining what is meant by the integral of a complex-valued function of a real variable. Let

$$f(t) = u(t) + iv(t) \quad \text{for } a \le t \le b,$$

where $u(t)$ and $v(t)$ are real-valued functions of the real variable t. If u and v are continuous functions on the interval, then from calculus we know that u and v are integrable functions of t. Therefore we make the following definition for the definite integral of f:

$$(1) \quad \int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.$$

Integrals of this type can be evaluated by finding the antiderivatives of u and v and evaluating the definite integrals on the right side of equation (1). That is, if $U'(t) = u(t)$ and $V'(t) = v(t)$, then we write

$$(2) \quad \int_a^b f(t)\,dt = U(b) - U(a) + i[V(b) - V(a)].$$

EXAMPLE 6.1 Let us show that

$$(3) \quad \int_0^1 (t - i)^3\,dt = \frac{-5}{4}.$$

Since the complex integral is defined in terms of real integrals, we write the integrand in equation (3) in terms of its real and imaginary parts: $f(t) = (t - i)^3 = t^3 - 3t + i(-3t^2 + 1)$. Here we see that u and v are given by $u(t) = t^3 - 3t$ and $v(t) = -3t^2 + 1$. The integrals of u and v are easy to compute, and we find that

$$\int_0^1 (t^3 - 3t)\,dt = \frac{-5}{4} \quad \text{and} \quad \int_0^1 (-3t^2 + 1)\,dt = 0.$$

Hence definition (1) can be used to conclude that

$$\int_0^1 (t - i)^3\, dt = \int_0^1 u(t)\, dt + i \int_0^1 v(t)\, dt = \frac{-5}{4}.$$

Our knowledge about the elementary functions can be used to find their integrals.

EXAMPLE 6.2 Let us show that

$$\int_0^{\pi/2} \exp(t + it)\, dt = \frac{1}{2}(e^{\pi/2} - 1) + \frac{i}{2}(e^{\pi/2} + 1).$$

Using the method suggested by equations (1) and (2), we obtain

$$\int_0^{\pi/2} \exp(t + it)\, dt = \int_0^{\pi/2} e^t \cos t\, dt + i \int_0^{\pi/2} e^t \sin t\, dt.$$

The integrals can be evaluated via integration by parts, and we have

$$\int_0^{\pi/2} \exp(t + it)\, dt = \frac{1}{2} e^t(\cos t + \sin t) + \frac{i}{2} e^t(\sin t - \cos t) \Big|_{t=0}^{t=\pi/2}$$
$$= \frac{1}{2}(e^{\pi/2} - 1) + \frac{i}{2}(e^{\pi/2} + 1).$$

Complex integrals have properties that are similar to those of real integrals. Let $f(t) = u(t) + iv(t)$ and $g(t) = p(t) + iq(t)$ be continuous on $a \le t \le b$. Then the integral of their sum is the sum of their integrals; so we can write

$$\text{(4)} \quad \int_a^b [f(t) + g(t)]\, dt = \int_a^b f(t)\, dt + \int_a^b g(t)\, dt.$$

It is convenient to divide the interval $a \le t \le b$ into $a \le t \le c$ and $c \le t \le b$ and integrate $f(t)$ over these subintervals. Hence we obtain the formula

$$\text{(5)} \quad \int_a^b f(t)\, dt = \int_a^c f(t)\, dt + \int_c^b f(t)\, dt.$$

Constant multiples are dealt with in the same manner as in calculus. If $c + id$ denotes a complex constant, then

$$\text{(6)} \quad \int_a^b (c + id) f(t)\, dt = (c + id) \int_a^b f(t)\, dt.$$

If the limits of integration are reversed, then

$$\text{(7)} \quad \int_b^a f(t)\, dt = -\int_a^b f(t)\, dt.$$

Let us emphasize that we are dealing with complex integrals. We write the integral of the product as follows:

$$(8)\quad \int_a^b f(t)g(t)\,dt = \int_a^b [u(t)p(t) - v(t)q(t)]\,dt + i\int_a^b [u(t)q(t) + v(t)p(t)]\,dt.$$

EXAMPLE 6.3 Let us prove equation (6). We start by writing

$$(c + id)f(t) = cu(t) - dv(t) + i[cv(t) + du(t)].$$

Using definition (1), the left side of equation (6) can be written as

$$(9)\quad c\int_a^b u(t)\,dt - d\int_a^b v(t)\,dt + ic\int_a^b v(t)\,dt + id\int_a^b u(t)\,dt,$$

which is easily seen to be equivalent to the product

$$(10)\quad (c + id)\left[\int_a^b u(t)\,dt + i\int_a^b v(t)\,dt\right].$$

It is worthwhile to point out the similarity between equation (2) and its counterpart in calculus. Suppose that U and V are differentiable on $a < t < b$ and $F(t) = U(t) + iV(t)$, then $F'(t)$ is defined to be

$$F'(t) = U'(t) + iV'(t),$$

and equation (2) takes on the familiar form

$$(11)\quad \int_a^b f(t)\,dt = F(b) - F(a), \quad \text{where } F'(t) = f(t).$$

This can be viewed as an extension of the fundamental theorem of calculus. In Section 6.5 we will see how the extension is made to the case of analytic functions of a complex variable. For now, note that we have the following important case of equation (11):

$$(12)\quad \int_a^b f'(t)\,dt = f(b) - f(a).$$

EXAMPLE 6.4 Let us use equation (11) to show that $\int_0^\pi \exp(it)\,dt = 2i$.

Solution If we let $F(t) = -i\exp(it) = \sin t - i\cos t$ and $f(t) = \exp(it) = \cos t + i\sin t$, then $F'(t) = f(t)$, and from equation (11) we obtain

$$\int_0^\pi \exp(it)dt = \int_0^\pi f(t)\,dt = F(\pi) - F(0) = -ie^{i\pi} + ie^0 = 2i.$$

EXERCISES FOR SECTION 6.1

For Exercises 1–4, use equations (1) and (2) to find the following definite integrals.

1. $\int_0^1 (3t - i)^2\, dt$ **2.** $\int_0^1 (t + 2i)^3\, dt$ **3.** $\int_0^{\pi/2} \cosh(it)\, dt$ **4.** $\int_0^2 \frac{t}{t + i}\, dt$

5. Find $\int_0^{\pi/4} t \exp(it)\, dt$.

6. Let m and n be integers. Show that

$$\int_0^{2\pi} e^{imt}e^{-int}\, dt = \begin{cases} 0 & \text{when } m \neq n, \\ 2\pi & \text{when } m = n. \end{cases}$$

7. Show that $\int_0^{\infty} e^{-zt}\, dt = 1/z$ provided that $\text{Re}(z) > 0$.

8. Let $f(t) = u(t) + iv(t)$ where u and v are differentiable. Show that $\int_a^b f(t)f'(t)\, dt = \frac{1}{2}[f(b)]^2 - \frac{1}{2}[f(a)]^2$.

9. Establish identity (4).

10. Establish identity (5).

11. Establish identity (7).

12. Establish identity (8).

6.2 Contours and Contour Integrals

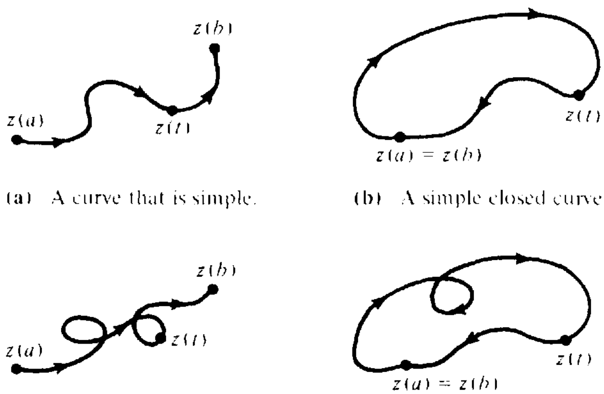
In Section 6.1 we learned how to evaluate integrals of the form $\int_a^b f(t)\, dt$, where f was complex-valued and $[a, b]$ was an interval on the real axis (so that t was real, with $t \in [a, b]$). In this section we shall define and evaluate integrals of the form $\int_C f(z)\, dz$, where f is complex-valued and C is a contour in the plane (so that z is complex, with $z \in C$). Our main result is Theorem 6.1, which will show how to transform the latter type of integral into the kind we investigated in Section 6.1.

We will use concepts first introduced in Section 1.6, where we defined the concept of a curve in the plane. Recall that to represent a curve C we used the parametric notation

(1) $C: z(t) = x(t) + iy(t) \quad \text{for } a \leq t \leq b,$

where $x(t)$ and $y(t)$ are continuous functions. We now want to place a few more restrictions on the type of curve that we will be studying. The following discussion will lead to the concept of a contour, which is a type of curve that is adequate for the study of integration.

Recall that C is said to be *simple* if it does not cross itself, which is expressed by requiring that $z(t_1) \neq z(t_2)$ whenever $t_1 \neq t_2$. A curve C with the property that $z(b) = z(a)$ is said to be a *closed curve.* If $z(b) = z(a)$ is the only point of intersection, then we say that C is a *simple closed curve.* As the parameter t increases from the value a to the value b, the point $z(t)$ starts at the *initial point* $z(a)$, moves along the curve C, and ends up at the *terminal points* $z(b)$. If C is simple, then $z(t)$ moves continuously from $z(a)$ to $z(b)$ as t increases, and the curve is given an *orientation*, which we indicate by drawing arrows along the curve. Figure 6.1 illustrates how the terms "simple" and "closed" can be used to describe a curve.



(a) A curve that is simple. (b) A simple closed curve.

(c) A curve that is *not* simple and *not* closed. (d) A closed curve that is *not* simple.

FIGURE 6.1 The terms "simple" and "closed" used to describe curves.

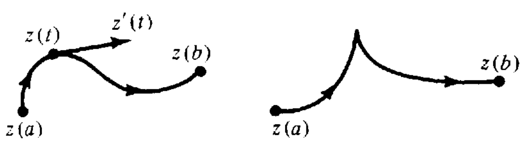
The complex-valued function $z(t)$ in equation (1) is said to be *differentiable* if both $x(t)$ and $y(t)$ are differentiable for $a \leq t \leq b$. Here the one-sided derivatives* of $x(t)$ and $y(t)$ are required to exist at the endpoints of the interval. The derivative $z'(t)$ with respect to t is defined by the equation

(2) $z'(t) = x'(t) + iy'(t)$ for $a \leq t \leq b$.

The curve C defined by equation (1) is said to be *smooth* if $z'(t)$, given by equation (2), is continuous and nonzero on the interval. If C is a smooth curve, then C has a nonzero tangent vector at each point $z(t)$, which is given by the vector $z'(t)$. If $x'(t_0) = 0$, then the tangent vector $z'(t_0) = iy'(t_0)$ is vertical. If $x'(t_0) \neq 0$, then the slope dy/dx of the tangent line to C at the point $z(t_0)$ is given by $y'(t_0)/x'(t_0)$. Hence the angle of inclination $\theta(t)$ of the tangent vector $z'(t)$ is defined for all values of t and is the continuous function given by

$$\theta(t) = \arg[z'(t)] = \arg[(x'(t) + iy'(t)].$$

Therefore a smooth curve has a continuously turning tangent vector. A smooth curve has no corners or cusps. Figure 6.2 illustrates this concept.



(a) A smooth curve. (b) A curve that is *not* smooth.

FIGURE 6.2 The term "smooth" used to describe curves.

---

*The derivative on the right $x'(a^+)$ and on the left $x'(b^-)$ are defined by the following limits:

$$x'(a^+) = \lim_{t \to a^+} \frac{x(t) - x(a)}{t - a} \quad \text{and} \quad x'(b^-) = \lim_{t \to b^-} \frac{x(t) - x(b)}{t - b}.$$

If C is a smooth curve, then ds, the differential of arc length, is given by

(3) $$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt = |z'(t)|\, dt.$$

Since $x'(t)$ and $y'(t)$ are continuous functions, then so is the function $\sqrt{[x'(t)]^2 + [y'(t)]^2}$, and the length L of the curve C is given by the definite integral

(4) $$L = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt = \int_a^b |z'(t)|\, dt.$$

Now consider C to be a curve with parameterization

$$C: z_1(t) = x(t) + iy(t) \quad \text{for } a \le t \le b.$$

The *opposite curve* $-C$ traces out the same set of points in the plane but in the reverse order, and it has the parameterization

$$-C: z_2(t) = x(-t) + iy(-t) \quad \text{for } -b \le t \le -a.$$

Since $z_2(t) = z_1(-t)$, it is easy to see that $-C$ is merely C traversed in the opposite sense. This is illustrated in Figure 6.3.

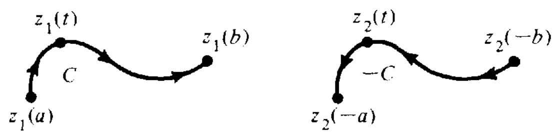


FIGURE 6.3 The curve C and its opposite curve $-C$.

A curve C that is constructed by joining finitely many smooth curves end to end is called a *contour*. Let $C_1, C_2, \ldots, C_n$ denote n smooth curves such that the terminal point of C_k coincides with the initial point of C_{k+1} for $k = 1, 2, \ldots, n - 1$. Then the contour C is expressed by the equation

(5) $$C = C_1 + C_2 + \cdots + C_n.$$

A synonym for contour is *path*.

EXAMPLE 6.5 Let us find a parameterization of the polygonal path C from $-1 + i$ to $3 - i$, which is shown in Figure 6.4.

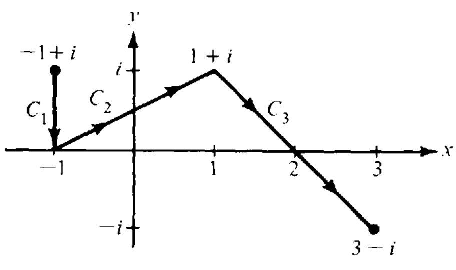


FIGURE 6.4 The polygonal path $C = C_1 + C_2 + C_3$ from $-1 + i$ to $3 - i$.

Solution The contour is conveniently expressed as three smooth curves $C = C_1 + C_2 + C_3$. A formula for the straight line segment joining two points was given by equation (2) in Section 1.6. If we set $z_0 = -1 + i$ and $z_1 = -1$, then the segment C_1 joining z_0 to z_1 is given by

$$C_1: z_1(t) = z_0 + t(z_1 - z_0) = (-1 + i) + t[-1 - (-1 + i)],$$

which can be simplified to obtain

$$C_1: z_1(t) = -1 + i(1 - t) \quad \text{for } 0 \le t \le 1.$$

In a similar fashion the segments C_2 and C_3 are given by

$$C_2: z_2(t) = (-1 + 2t) + it \qquad \text{for } 0 \le t \le 1 \quad \text{and}$$
$$C_3: z_3(t) = (1 + 2t) + i(1 - 2t) \qquad \text{for } 0 \le t \le 1.$$

We are now ready to define the integral of a complex function along a contour C in the plane with initial point A and terminal point B. Our approach is to mimic what is done in calculus. We create a partition $P_n = \{z_0 = A, z_1, z_2, \ldots, z_n = B\}$ of points that proceed along C from A to B and form the differences $\Delta z_k = z_k - z_{k-1}$ for $k = 1, 2, \ldots, n$. Between each pair of partition points z_{k-1} and z_k we select a point c_k on C, where the function $f(c_k)$ is evaluated (see Figure 6.5). These values are used to make a Riemann sum for the partition:

$$(6) \quad S(P_n) = \sum_{k=1}^{n} f(c_k)(z_k - z_{k-1}) = \sum_{k=1}^{n} f(c_k)\Delta z_k.$$

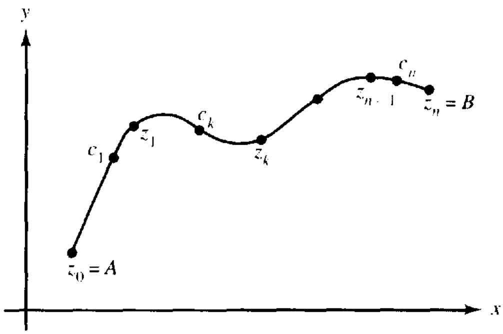


FIGURE 6.5 Partition points $\{z_k\}$, and function evaluation points $\{c_k\}$ for a Riemann sum along the contour C from $z = A$ to $z = B$.

Assume now that there exists a unique complex number L that is the limit of every sequence $\{S(P_n)\}$ of Riemann sums given in equation (6), where the maximum of $|\Delta z_k|$ tends toward 0, for the sequence of partitions. We define the number L as the value of the integral of $f(z)$ taken along the contour C. We thus have the following.

Definition 6.1 *Let C be a contour, then* $\int_C f(z)\,dz = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta z_k$, *provided the limit exists in the sense previously discussed.*

You will notice that in this definition, the value of the integral depends on the contour. In Section 6.3 the Cauchy-Goursat Theorem will establish the remarkable property that *if* $f(z)$ *is analytic*, then $\int_C f(z)\,dz$ is *independent* of the contour.

EXAMPLE 6.6 Use a Riemann sum to construct an approximation for the contour integral $\int_C \exp z\,dz$, where C is the line segment joining the point $A = 0$ to $B = 2 + i\frac{\pi}{4}$.

Solution Set $n = 8$ in equation (6) and form the partition P_8: $z_k = \frac{k}{4} + i\frac{\pi k}{32}$ for $k = 0, 1, 2, \ldots, 8$. For this situation, we have a uniform increment $\Delta z_k = \frac{1}{4} + i\frac{\pi}{32}$. For convenience we select $c_k = \frac{z_{k-1} + z_k}{2} = \frac{2k-1}{8} + i\frac{\pi(2k-1)}{64}$ for $k = 1, 2, \ldots, 8$. The points $\{z_k\}$ and $\{c_k\}$ are shown in Figure 6.6.

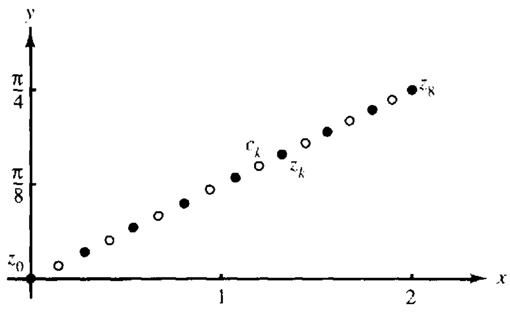


FIGURE 6.6 Partition and evaluation points for the Riemann sum $S(P_8)$.

One possible Riemann sum, then, is

$$S(P_8) = \sum_{k=1}^{8} f(c_k)\,\Delta z_k = \sum_{k=1}^{8} \exp\left[\frac{2k-1}{8} + i\frac{\pi(2k-1)}{64}\right]\left(\frac{1}{4} + i\frac{\pi}{32}\right).$$

By rounding the terms in this Riemann sum to two decimal digits, we obtain an approximation for the integral:

$$S(P_8) \approx (0.28 + 0.13i) + (0.33 + 0.19i) + (0.41 + 0.29i) + (0.49 + 0.42i)$$
$$+ (0.57 + 0.6i) + (0.65 + 0.84i) + (0.72 + 1.16i) + (0.78 + 1.57i),$$
$$S(P_8) \approx 4.23 + 5.20i.$$

This compares favorably with the precise value of the integral, which you will soon see equals $\exp\left(2 + i\frac{\pi}{4}\right) - 1 = -1 + e^2\frac{\sqrt{2}}{2} + ie^2\frac{\sqrt{2}}{2} \approx 4.22485 + 5.22485i$.

In general, obtaining an exact value for an integral given by Definition 6.1 is a daunting task. Fortunately, there is a beautiful theory that allows for an easy computation of many contour integrals. Suppose we have a parameterization of the contour C given by the function $z(t)$ for $a \le t \le b$. That is, C is the range of the function $z(t)$ over the interval $[a, b]$, as Figure 6.7 shows.

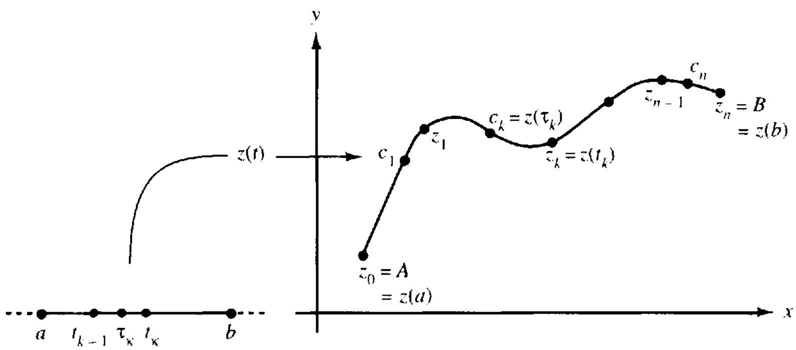


FIGURE 6.7 A parameterization of the contour C by $z(t)$ for $a \le t \le b$.

It follows that

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta z_k &= \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)(z_k - z_{k-1}) \\ &= \lim_{n\to\infty}\sum_{k=1}^{n} f(z(\tau_k))[z(t_k) - z(t_{k-1})],\end{aligned}$$

where the τ_k and t_k are those points contained in the interval $[a, b]$ with the property that $c_k = z(\tau_k)$ and $z_k = z(t_k)$, as is also shown in Figure 6.7. If for all k we multiply the kth term in the last sum by $\dfrac{t_k - t_{k-1}}{t_k - t_{k-1}}$, we get

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(z(\tau_k))\left[\frac{z(t_k) - z(t_{k-1})}{t_k - t_{k-1}}\right](t_k - t_{k-1}) = \lim_{n\to\infty}\sum_{k=1}^{n} f(z(\tau_k))\left[\frac{z(t_k) - z(t_{k-1})}{t_k - t_{k-1}}\right]\Delta t_k.$$

The quotient inside the last summation looks suspiciously like a derivative, and the entire quantity looks like a Riemann sum. Assuming no difficulties, this last expression should equal

$\int_a^b f(z(t))z'(t)\,dt$, as defined in Section 6.1.

It would be especially nice if we were to get the same limit *regardless of how we parameterize the contour C.* As the following theorem states, this is indeed the case.

Theorem 6.1 *Suppose $f(z)$ is a continuous complex-valued function defined on a set containing the contour C. Let $z(t)$ be any parameterization of C for $a \le t \le b$. Then*

$$\int_C f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt. \tag{7}$$

Proof We omit the proof of this theorem since it involves ideas (such as the theory of the Riemann-Stieltjes integral) that are beyond the scope of this book. A more rigorous development of the contour integral based on Riemann sums is found in advanced texts such as L. V. Ahlfors, *Complex Analysis*, 3rd ed. (New York: McGraw-Hill, 1979).

There are two important facets of Theorem 6.1 that are worth mentioning. First, the theorem makes the problem of evaluating complex-valued functions along contours easy since it reduces our task to one that requires the evaluation complex-valued functions over real intervals—a procedure we studied in Section 6.1. Second, according to the theorem this transformation yields the same answer regardless of the parameterization we choose for C, a truly remarkable fact.

EXAMPLE 6.7 Let us give an exact calculation of the integral in Example 6.6. That is, we want $\int_C \exp z\,dz$, where C is the line segment joining $A = 0$ to $B = 2 + i\frac{\pi}{4}$. According to equation (2) of Section 1.6, we can parameterize C by $z(t) = \left(2 + i\frac{\pi}{4}\right)t$, for $0 \le t \le 1$. Since $z'(t) = \left(2 + i\frac{\pi}{4}\right)$, according to Theorem 6.1 we have that

$$\begin{aligned}\int_C \exp z\,dz &= \int_0^1 \exp\left[\left(2 + i\frac{\pi}{4}\right)t\right]\left(2 + i\frac{\pi}{4}\right)dt\\ &= \left(2 + i\frac{\pi}{4}\right)\int_0^1 e^{2t}e^{i\pi t/4}\,dt\\ &= \left(2 + i\frac{\pi}{4}\right)\int_0^1 e^{2t}[\cos(\pi t/4) + i\sin(\pi t/4)]\,dt\\ &= \left(2 + i\frac{\pi}{4}\right)\left[\int_0^1 e^{2t}\cos(\pi t/4)\,dt + i\int_0^1 e^{2t}\sin(\pi t/4)\,dt\right].\end{aligned}$$

Each integral in the last expression can be done using integration by parts. We leave as an exercise that the final answer simplifies to $\exp\left(2 + i\frac{\pi}{4}\right) - 1$, as claimed in Example 6.6.

EXAMPLE 6.8 Evaluate $\int_C \frac{1}{z-2}\,dz$, where C is the upper semicircle with radius 1 centered at $x = 2$ oriented in a position (i.e., counterclockwise) direction.

Solution The function $z(t) = 2 + e^{it}$, for $0 \le t \le \pi$ is a parameterization for C. We apply Theorem 6.1 with $f(z) = \frac{1}{z-2}$. (Note: $f(z(t)) = \frac{1}{z(t)-2}$, and $z'(t) = ie^{it}$.) Hence,

$$\int_C \frac{1}{z-2}\,dz = \int_0^\pi \frac{1}{(2+e^{it})-2}\,ie^{it}\,dt = \int_0^\pi i\,dt = i\pi.$$

To help convince yourself that the value of the integral is independent of the parameterization chosen for the given contour, try working through this example with $z(t) = 2 + e^{i\pi t}$, for $0 \le t \le 1$.

There is a convenient bookkeeping device that helps us remember how to apply Theorem 6.1. Since $\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$, we can symbolically equate z with $z(t)$ and dz with $z'(t)\,dt$. This should be easy to remember because z is supposed to be a point on the contour C parameterized by $z(t)$, and $\frac{dz}{dt} = z'(t)$ according to the Leibniz notation for the derivative.

If $z(t) = x(t) + iy(t)$, then by the preceding paragraph we have

(8) $\quad dz = z'(t)\,dt = [x'(t) + iy'(t)]\,dt = dx + i\,dy,$

where dx and dy are the differentials for $x(t)$ and $y(t)$, respectively. (That is, dx is equated with $x'(t)\,dt$ and dy with $y'(t)\,dt$.) The expression dz is often called the *complex differential* of z. Just as dx and dy are intuitively considered to be small segments along the x and y axes in real variables, we can think of dz as representing a very tiny piece of the contour C. Moreover, if we write

(9) $\quad |dz| = |[x'(t) + iy'(t)]\,dt| = |[x'(t) + iy'(t)]|\,dt = \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt,$

then we know from calculus that the length of the curve C, $L(C)$, is given by

(10) $\quad L(C) = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt = \int_C |dz|,$

so we can think of $|dz|$ as representing the length of dz.

Suppose $f(z) = u(z) + iv(z)$, and $z(t) = x(t) + iy(t)$ is a parameterization for the contour C. Then

$$(11)\quad \int_C f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt$$
$$= \int_a^b [u(z(t)) + iv(z(t))][x'(t) + iy'(t)]\,dt$$
$$= \int_a^b [u(z(t))x'(t) - v(z(t))y'(t)]\,dt$$
$$+ i\int_a^b [v(z(t))x'(t) + u(z(t))y'(t)]\,dt$$
$$= \int_a^b (ux' - vy')\,dt + i\int_a^b (vx' + uy')\,dt,$$

where we are equating u with $u(z(t))$, x' with $x'(t)$, etc.

If we use the differentials given in equation (8), then equation (11) can be written in terms of line integrals of the real-valued functions u and v, giving

$$(12)\quad \int_C f(z)\,dz = \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy,$$

which is easy to remember if we recall that symbolically

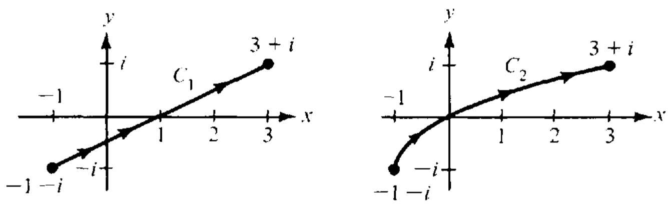
$$f(z)\,dz = (u + iv)(dx + i\,dy).$$

We emphasize that equation (12) is merely a notational device for applying equation (7) in Theorem 6.1. We recommend you carefully apply the theorem as illustrated in Examples 6.7 and 6.8 before using any shortcuts suggested by equation (12).

EXAMPLE 6.9 Let us show that

$$\int_{C_1} z\,dz = \int_{C_2} z\,dz = 4 + 2i,$$

where C_1 is the line segment from $-1 - i$ to $3 + i$ and C_2 is the portion of the parabola $x = y^2 + 2y$ joining $-1 - i$ to $3 + i$, as indicated in Figure 6.8.



(a) The line segment. (b) The portion of the parabola.

FIGURE 6.8 The two contours C_1 and C_2 joining $-1 - i$ to $3 + i$.

The line segment joining $(-1, -1)$ to $(3, 1)$ is given by the slope intercept formula $y = \frac{1}{2}x - \frac{1}{2}$, which can be written as $x = 2y + 1$. It is convenient to choose the parameterization $y = t$ and $x = 2t + 1$. Then the segment C_1 can be given by

$$C_1: z(t) = 2t + 1 + it \quad \text{and} \quad dz = (2 + i)\,dt \quad \text{for } -1 \le t \le 1.$$

Along C_1, we have $f(z(t)) = 2t + 1 + it$. Computing the value of the integral in equation (7), we obtain

$$\int_{C_1} z\,dz = \int_{-1}^{1} (2t + 1 + it)(2 + i)\,dt,$$

which can be evaluated by using straightforward techniques to obtain

$$\int_{C_1} z\,dz = \int_{-1}^{1} (3t + 2)\,dt + i\int_{-1}^{1} (4t + 1)\,dt = 4 + 2i.$$

Similarly, for the portion of the parabola $x = y^2 + 2y$ joining $(-1, -1)$ to $(3, 1)$, it is convenient to choose the parameterization $y = t$ and $x = t^2 + 2t$. Then C_2 can be given by

$$C_2: z(t) = t^2 + 2t + it \quad \text{and} \quad dz = (2t + 2 + i)\,dt \quad \text{for } -1 \le t \le 1.$$

Along C_2 we have $f(z(t)) = t^2 + 2t + it$. Computing the value of the integral in equation (7), we obtain

$$\int_{C_2} z\,dz = \int_{-1}^{1} (t^2 + 2t + it)(2t + 2 + i)\,dt$$

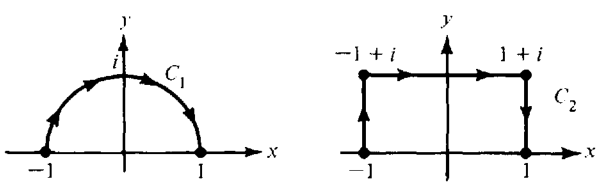
$$= \int_{-1}^{1} (2t^3 + 6t^2 + 3t)\,dt + i\int_{-1}^{1} (3t^2 + 4t)\,dt = 4 + 2i.$$

In this example, the value of the two integrals is the same. This does not hold in general, as is shown in Example 6.10.

EXAMPLE 6.10 Let us show that

$$\int_{C_1} \bar{z}\,dz = -\pi i, \quad \text{but} \quad \int_{C_2} \bar{z}\,dz = -4i,$$

where C_1 is the semicircular path from -1 to 1 and C_2 is the polygonal path from -1 to 1, respectively, that are shown in Figure 6.9.



(a) The semicircular path. (b) The polygonal path.

FIGURE 6.9 The two contours C_1 and C_2 joining -1 to 1.

Solution The semicircle C_1 can be parameterized by

$$C_1: z(t) = -\cos t + i \sin t \quad \text{and} \quad dz = (\sin t + i \cos t)\, dt \quad \text{for } 0 \leq t \leq \pi.$$

Along C_1 we have $f(z(t)) = -\cos t - i \sin t$. Computing the value of the integral equation in (7), we obtain

$$\int_{C_1} \bar{z}\, dz = \int_0^{\pi} (-\cos t - i \sin t)(\sin t + i \cos t)\, dt$$
$$= -i \int_0^{\pi} (\cos^2 t + \sin^2 t)\, dt = -\pi i.$$

The polygonal path C_2 must be parameterized in three parts, one for each line segment:

$$z_1(t) = -1 + it, \quad dz_1 = i\, dt, \quad f(z_1(t)) = -1 - it,$$
$$z_2(t) = -1 + 2t + i, \quad dz_2 = 2\, dt, \quad f(z_2(t)) = -1 + 2t - i,$$
$$z_3(t) = 1 + i(1 - t), \quad dz_3 = -i\, dt, \quad f(z_3(t)) = 1 - i(1 - t),$$

where all of the parameters t are to be taken on the interval $0 \leq t \leq 1$. The value of the integral in equation (7) is obtained by adding the three integrals along the above three segments, and the result is

$$\int_0^1 (-1 - it)i\, dt + \int_0^1 (-1 + 2t - i)2\, dt + \int_0^1 [1 - i(1 - t)](-i)\, dt.$$

A straightforward calculation now shows that

$$\int_{C_2} \bar{z}\, dz = \int_0^1 (6t - 3)\, dt + i \int_0^1 (-4)\, dt = -4i.$$

We remark that the value of the contour integral along C_1 is *not* the same as the value of the contour integral along C_2, although both integrals have the same initial and terminal points.

Contour integrals have properties that are similar to those of integrals of a complex function of a real variable, which were studied in Section 6.1. If C is given by equation (1), then the contour integral for the opposite contour $-C$ is given by

$$(13) \quad \int_{-C} f(z)\, dz = \int_{-b}^{-a} f(z(-\tau))[-z'(-\tau)]\, d\tau.$$

Using the change of variable $t = -\tau$ in equation (13) and identity (7) of Section 6.1, we obtain

$$(14) \quad \int_{-C} f(z)\, dz = -\int_C f(z)\, dz.$$

If two functions f and g can be integrated over the same path of integration C, then their sum can be integrated over C, and we have the familiar result

(15) $$\int_C [f(z) + g(z)]\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz.$$

Constant multiples are dealt with in the same manner as in identity (6) in Section 6.1:

(16) $$\int_C (c + id)f(z)\,dz = (c + id)\int_C f(z)\,dz.$$

If two contours C_1 and C_2 are placed end to end so that the terminal point of C_1 coincides with the initial point of C_2, then the contour $C = C_1 + C_2$ is a *continuation* of C_1, and we have the property

(17) $$\int_{C_1+C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz.$$

If the contour C has two parameterizations

$$C\colon z_1(t) = x_1(t) + iy_1(t) \quad \text{for } a \le t \le b \quad \text{and}$$
$$C\colon z_2(\tau) = x_2(\tau) + iy_2(\tau) \quad \text{for } \alpha \le \tau \le \beta,$$

and there exists a differentiable function $\tau = \phi(t)$ such that

(18) $$\alpha = \phi(a), \quad \beta = \phi(b), \quad \text{and} \quad \phi'(t) > 0 \quad \text{for } a < t < b,$$

then we say that $z_2(\tau)$ is a *reparameterization* of the contour C. If f is continuous on C, then we have

(19) $$\int_a^b f(z_1(t))z_1'(t)\,dt = \int_\alpha^\beta f(z_2(\tau))z_2'(\tau)\,d\tau.$$

Identity (19) shows that the value of a contour integral is invariant under a change in the parametric representation of its contour if the reparameterization satisfies equations (18).

There are a few important inequalities relating to complex integrals, which we now state.

Lemma 6.1 (Integral Triangle Inequality) *If $f(t) = u(t) + iv(t)$ is a continuous function of the real parameter t, then*

(20) $$\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt.$$

Proof Write the value of the integral in polar form:

(21) $$r_0 e^{i\theta_0} = \int_a^b f(t)\,dt \quad \text{and} \quad r_0 = \int_a^b e^{-i\theta_0} f(t)\,dt.$$

Taking the real part of the second integral in equations (21), we write

$$r_0 = \int_a^b \operatorname{Re}[e^{-i\theta_0} f(t)]\, dt.$$

Using equation (2) of Section 1.3, we obtain the relation

$$\operatorname{Re}[e^{-i\theta_0} f(t)] \leq |e^{-i\theta_0} f(t)| \leq |f(t)|.$$

The left and right sides can be used as integrands, and then familiar results from calculus can be used to obtain

$$r_0 = \int_a^b \operatorname{Re}[e^{-i\theta_0} f(t)]\, dt \leq \int_a^b |f(t)|\, dt.$$

Since

$$r_0 = \left|\int_a^b f(t)\, dt\right|,$$

we have established inequality (20).

Lemma 6.2 (ML Inequality) *If $f(z) = u(x, y) + iv(x, y)$ is continuous on the contour C, then*

(22) $$\left|\int_C f(z)\, dz\right| \leq ML,$$

where L is the length of the contour C and M is an upper bound for the modulus $|f(z)|$ on C.

Proof When inequality (20) is used with Theorem 6.1, we get

(23) $$\left|\int_C f(z)\, dz\right| = \left|\int_a^b f(z(t))z'(t)\, dt\right| \leq \int_a^b |f(z(t))z'(t)|\, dt.$$

Let M be the positive real constant such that

$$|f(z)| \leq M \quad \text{for all } z \text{ on } C.$$

Then equation (9) and inequality (23) imply that

$$\left|\int_C f(z)\, dz\right| \leq \int_a^b M|z'(t)|\, dt = ML.$$

Therefore inequality (22) is proved.

EXAMPLE 6.11 Let us use inequality (22) to show that

$$\left|\int_C \frac{1}{z^2+1}\,dz\right| \le \frac{1}{2\sqrt{5}},$$

where C is the straight line segment from 2 to $2+i$. Here $|z^2+1| = |z-i| \times |z+i|$, and the terms $|z-i|$ and $|z+i|$ represent the distance from the point z to the points i and $-i$, respectively. We refer to Figure 6.10 and use a geometric argument to see that

$$|z-i| \ge 2 \quad \text{and} \quad |z+i| \ge \sqrt{5} \quad \text{for } z \text{ on } C.$$

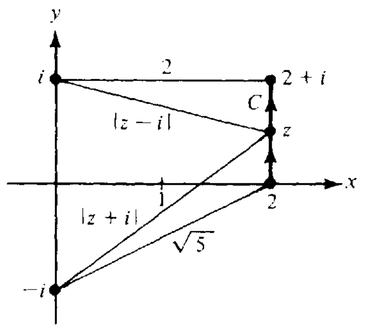


FIGURE 6.10 The distances $|z-i|$ and $|z+i|$ for z on C.

Here we have

$$|f(z)| = \frac{1}{|z-i|\,|z+i|} \le \frac{1}{2\sqrt{5}} = M,$$

and $L = 1$, so inequality (22) implies that

$$\left|\int_C \frac{1}{z^2+1}\,dz\right| \le ML = \frac{1}{2\sqrt{5}}.$$

EXERCISES FOR SECTION 6.2

1. Sketch the following curves.
   (a) $z(t) = t^2 - 1 + i(t+4)$ for $1 \le t \le 3$
   (b) $z(t) = \sin t + i\cos 2t$ for $-\pi/2 \le t \le \pi/2$
   (c) $z(t) = 5\cos t - i3\sin t$ for $\pi/2 \le t \le 2\pi$
2. Give a parameterization of the contour $C = C_1 + C_2$ indicated in Figure 6.11.
3. Give a parameterization of the contour $C = C_1 + C_2 + C_3$ indicated in Figure 6.12.

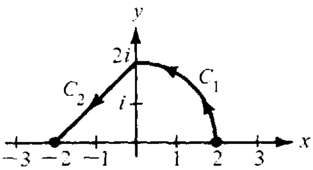


FIGURE 6.11 Accompanies Exercise 2.

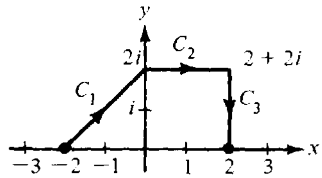


FIGURE 6.12 Accompanies Exercise 3.

4. Consider the integral $\int_C z^2 dz$, where C is the positively oriented upper semi-circle of radius 1, centered at 0.
   (a) Given a Riemann sum approximation for the above integral by selecting $n = 4$, and the following points: $z_k = e^{ik\pi/4}$; $c_k = e^{i(2k-1)\pi/8}$ for appropriate values of k.
   (b) Compute the integral exactly by selecting a parameterization for C and applying Theorem 6.1.

5. Show that the integral in Example 6.7 simplifies to $\exp\left(2 + i\frac{\pi}{4}\right) - 1$.

6. Evaluate $\int_C y\, dz$ for $-i$ to i along the following contours as shown in Figures 6.13(a) and 6.13(b).
   (a) The polygonal path C with vertices $-i$, $-1 - i$, -1, and i.
   (b) The contour C that is the left half of the circle $|z| = 1$.

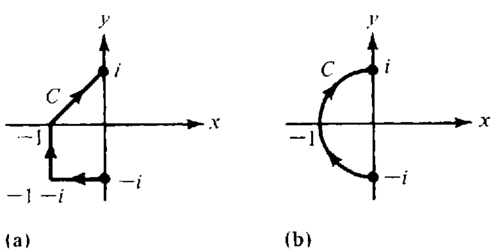


FIGURE 6.13 Accompanies Exercise 6.

7. Evaluate $\int_C x\, dz$ from -4 to 4 along the following contours as shown in Figures 6.14(a) and 6.14(b).
   (a) The polygonal path C with vertices -4, $-4 + 4i$, $4 + 4i$, and 4.
   (b) The contour C that is the upper half of the circle $|z| = 4$.

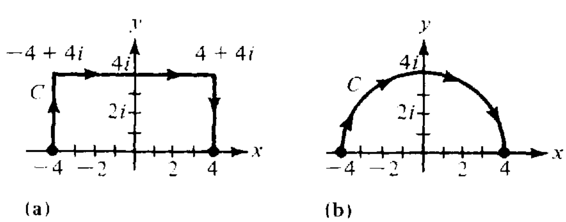


FIGURE 6.14 Accompanies Exercise 7.

8. Evaluate $\int_C z\,dz$, where C is the circle $|z| = 4$ taken with the counterclockwise orientation. *Hint*: Let C: $z(t) = 4\cos t + i4\sin t$ for $0 \le t \le 2\pi$.

9. Evaluate $\int_C \bar{z}\,dz$, where C is the circle $|z| = 4$ taken with the counterclockwise orientation.

10. Evaluate $\int_C (z + 1)\,dz$, where C given by C: $z(t) = \cos t + i\sin t$ for $0 \le t \le \pi/2$.

11. Evaluate $\int_C z\,dz$, where C is the line segment from i to 1 and $z(t) = t + (1 - t)i$ for $0 \le t \le 1$.

12. Evaluate $\int_C z^2\,dz$, where C is the line segment from 1 to $1 + i$ and $z(t) = 1 + it$ for $0 \le t \le 1$.

13. Evaluate $\int_C (x^2 - iy^2)\,dz$, where C is the upper semicircle C: $z(t) = \cos t + i\sin t$ for $0 \le t \le \pi$.

14. Evaluate $\int_C |z^2|\,dz$, where C given by C: $z(t) = t + it^2$ for $0 \le t \le 1$.

15. Evaluate $\int_C |z - 1|^2\,dz$, where C is the upper half of the circle $|z| = 1$ taken with the counterclockwise orientation.

16. Evaluate $\int_C (1/z)\,dz$, where C is the circle $|z| = 2$ taken with the clockwise orientation. *Hint*: C: $z(t) = 2\cos t - i2\sin t$ for $0 \le t \le 2\pi$.

17. Evaluate $\int_C (1/\bar{z})\,dz$, where C is the circle $|z| = 2$ taken with the clockwise orientation.

18. Evaluate $\int_C \exp z\,dz$, where C is the straight line segment joining 1 to $1 + i\pi$.

19. Show that $\int_C \cos z\,dz = \sin(1 + i)$, where C is the polygonal path from 0 to $1 + i$ that consists of the line segments from 0 to 1 and 1 to $1 + i$.

20. Show that $\int_C \exp z\,dz = \exp(1 + i) - 1$, where C is the straight line segment joining 0 to $1 + i$.

21. Evaluate $\int_C \bar{z}\exp z\,dz$, where C is the square with vertices 0, 1, $1 + i$, and i taken with the counterclockwise orientation.

22. Let $z(t) = x(t) + iy(t)$ for $a \le t \le b$ be a smooth curve. Give a meaning for each of the following expressions.

(a) $z'(t)$ (b) $|z'(t)|\,dt$ (c) $\int_a^b z'(t)\,dt$ (d) $\int_a^b |z'(t)|\,dt$

23. Let f be a continuous function on the circle $|z - z_0| = R$. Let the circle C have the parameterization C: $z(\theta) = z_0 + Re^{i\theta}$ for $0 \le \theta \le 2\pi$. Show that

$$\int_C f(z)\,dz = iR\int_0^{2\pi} f(z_0 + Re^{i\theta})e^{i\theta}\,d\theta.$$

24. Use the results of Exercise 23 to show that

(a) $\displaystyle\int_C \frac{1}{z - z_0}\,dz = 2\pi i$ and

(b) $\displaystyle\int_C \frac{1}{(z - z_0)^n}\,dz = 0$, where $n \ne 1$ is an integer,

where the contour C is the circle $|z - z_0| = R$ taken with the counterclockwise orientation.

25. Explain how contour integrals studied in complex analysis and line integrals studied in calculus are different. How are they similar?

26. Write a report on contour integrals. Include some of the more complicated techniques in your discussion. Resources include bibliographical items 5, 16, 81, 82, and 157.

6.3 The Cauchy-Goursat Theorem

The Cauchy-Goursat theorem states that within certain domains the integral of an analytic function over a simple closed contour is zero. An extension of this theorem will allow us to replace integrals over certain complicated contours with integrals over contours that are easy to evaluate. We will show how to use the technique of partial fractions together with the Cauchy-Goursat theorem to evaluate certain integrals. In Section 6.4 we will see that the Cauchy-Goursat theorem implies that an analytic function has an antiderivative. To start with, we need to introduce a few new concepts.

We saw in Section 1.6 that with each simple closed contour C there are associated two disjoint domains, each of which has C as its boundary. The contour C divides the plane into two domains. One domain is bounded and is called the *interior* of C, and the other domain is unbounded and is called the *exterior* of C. Figure 6.15 illustrates this concept. This result is known as the Jordan Curve Theorem.

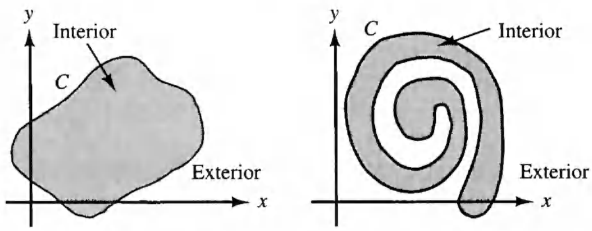
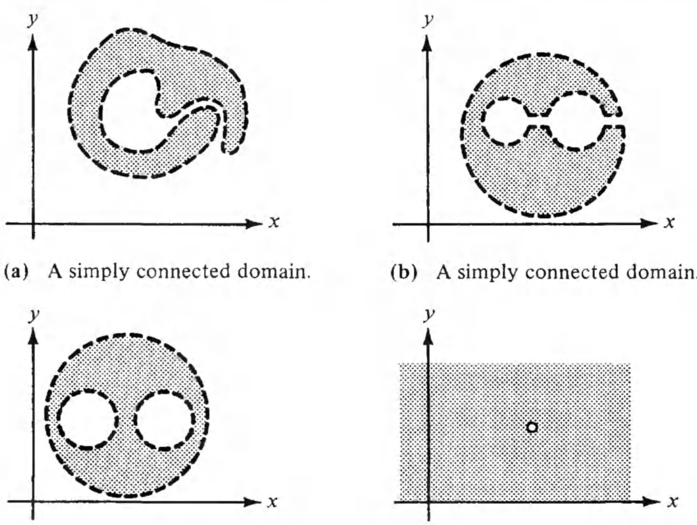


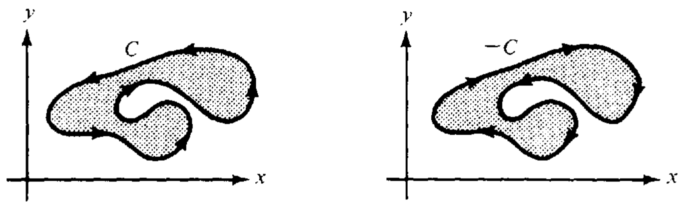
FIGURE 6.15 The interior and exterior of simple closed contours.

In Section 1.6 we saw that a domain D is an open connected set. In particular, if z_1 and z_2 are any pair of points in D, then they can be joined by a curve that lies entirely in D. A domain D is said to be *simply connected* if it has the property that any simple closed contour C contained in D has its interior contained in D. In other words, there are no "holes" in a simply connected domain. A domain that is not simply connected is said to be a *multiply connected domain.* Figure 6.16 illustrates the use of the terms "simply connected" and "multiply connected."



(a) A simply connected domain. (b) A simply connected domain.

Let the simple closed contour C have the parameterization C: $z(t) = x(t) + iy(t)$ for $a \le t \le b$. If C is parameterized so that the interior of C is kept on the left as $z(t)$ moves around C, then we say that C is oriented in the *positive* (counterclockwise) sense; otherwise, C is oriented *negatively*. If C is positively oriented, then $-C$ is negatively oriented. Figure 6.17 illustrates the concept of positive and negative orientation.



(a) A positively oriented contour. (b) A negatively oriented contour.

FIGURE 6.17 Simple closed contours that are positively and negatively oriented.

An important result from the calculus of real variables is known as Green's theorem and is concerned with the line integral of real-valued functions.

Theorem 6.2 (Green's Theorem) *Let C be a simple closed contour with positive orientation, and let R be the domain that forms the interior of C. If P and Q are continuous and have continuous partial derivatives P_x, P_y, Q_x, and Q_y at all points on C and R, then*

(1) $$\int_C P(x, y)\,dx + Q(x, y)\,dy = \iint_R [Q_x(x, y) - P_y(x, y)]\,dx\,dy.$$

Proof for a Standard Region* If R is a standard region, then there exist functions $y = g_1(x)$ and $y = g_2(x)$ for $a \le x \le b$ whose graphs form the lower and upper portions of C, respectively, as indicated in Figure 6.18. Since C is to be given the positive (counterclockwise) orientation, these functions can be used to express C as the sum of two contours C_1 and C_2 where

$$C_1: z_1(t) = t + ig_1(t) \qquad \text{for } a \le t \le b \quad \text{and}$$
$$C_2: z_2(t) = -t + ig_2(-t) \qquad \text{for } -b \le t \le -a.$$

We now use the functions $g_1(x)$ and $g_2(x)$ to express the double integral of $-P_y(x, y)$ over R as an iterated integral, first with respect to y and second with respect to x, as follows:

(2) $$-\iint_R P_y(x, y)\,dx\,dy = -\int_a^b \left[\int_{g_1(x)}^{g_2(x)} P_y(x, y)\,dy\right] dx.$$

*A standard region is bounded by a contour C, which can be expressed in the two forms $C = C_1 + C_2$ and $C = C_3 + C_4$ that are used in the proof.

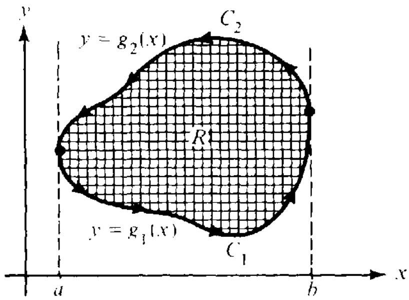


FIGURE 6.18 Integration over a standard region where $C = C_1 + C_2$.

Computing the first iterated integral on the right side of equation (2), we obtain

(3) $$-\iint_R P_y(x, y)\,dx\,dy = \int_a^b P(x, g_1(x))\,dx - \int_a^b P(x, g_2(x))\,dx.$$

In the second integral on the right side of equation (3) we can use the change of variable $x = -t$ and manipulate the integral to obtain

(4) $$-\iint_R P_y(x, y)\,dx\,dy = \int_a^b P(x, g_1(x))\,dx + \int_{-b}^{-a} P(-t, g_2(-t))(-1)\,dt.$$

When the two integrals on the right side of equation (4) are interpreted as contour integrals along C_1 and C_2, respectively, we see that

(5) $$-\iint_R P_y(x, y)\,dx\,dy = \int_{C_1} P(x, y)\,dx + \int_{C_2} P(x, y)\,dx = \int_C P(x, y)\,dx.$$

To complete the proof, we rely on the fact that for a standard region, there exist functions $x = h_1(y)$ and $x = h_2(y)$ for $c \le y \le d$ whose graphs form the left and right portions of C, respectively, as indicated in Figure 6.19. Since C has the positive orientation, it can be expressed as the sum of two contours C_3 and C_4, where

C_3: $z_3(t) = h_1(-t) - it$ for $-d \le t \le -c$ and
C_4: $z_4(t) = h_2(t) + it$ for $c \le t \le d$.

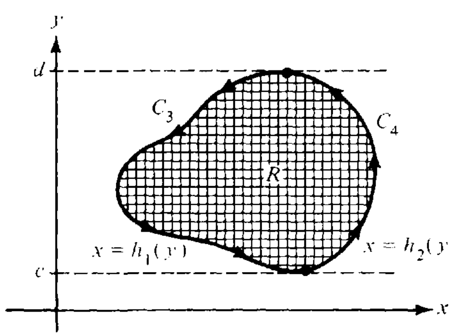


FIGURE 6.19 Integration over a standard region where $C = C_3 + C_4$.

Using the functions $h_1(y)$ and $h_2(y)$, we express the double integral of $Q_x(x, y)$ over R as an iterated integral:

$$(6)\quad \iint_R Q_x(x, y)\, dx\, dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} Q_x(x, y)\, dx \right] dy.$$

A similar derivation will show that equation (6) is equivalent to

$$(7)\quad \iint_R Q_x(x, y)\, dx\, dy = \int_C Q(x, y)\, dy.$$

When equations (5) and (7) are added, the result is equation (1), and the proof is complete.

We are now ready to state our main result in this section.

Theorem 6.3 (Cauchy-Goursat Theorem) *Let f be analytic in a simply connected domain D. If C is a simple closed contour that lies in D, then*

$$(8)\quad \int_C f(z)\, dz = 0.$$

Proof If we add the additional hypothesis that the derivative $f'(z)$ is also continuous, the proof is more intuitive. It was Augustin Cauchy who first proved this theorem under the hypothesis that $f'(z)$ is continuous. His proof, which we will now state, used Green's theorem.

Proof Using Green's Theorem We assume that C is oriented in the positive sense and use equation (12) in Section 6.2 to write

$$(9)\quad \int_C f(z)\, dz = \int_C u\, dx - v\, dy + i \int_C v\, dx + u\, dy.$$

If we use Green's theorem on the real part of the right side of equation (9) with $P = u$ and $Q = -v$, then we obtain

$$(10)\quad \int_C u\, dx - v\, dy = \iint_R (-v_x - u_y)\, dx\, dy,$$

where R is the region that is the interior of C. If we use Green's theorem on the imaginary part, the result will be

$$(11)\quad \int_C v\, dx + u\, dy = \iint_R (u_x - v_y)\, dx\, dy.$$

The Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ can be used in equations (10) and (11) to see that the value of equation (9) is given by

$$\int_C f(z)\,dz = \iint_R 0\,dx\,dy + i\iint_R 0\,dx\,dy = 0,$$

and the proof is complete.

A proof that does not require the continuity of $f'(z)$ was devised by Edward Goursat (1858–1936) in 1883.

Goursat's Proof of Theorem 6.3 We first establish the result for a triangular contour C with positive orientation. Construct four positively oriented contours C^1, C^2, C^3, and C^4 that are the triangles obtained by joining the midpoints of the sides of C as shown in Figure 6.20.

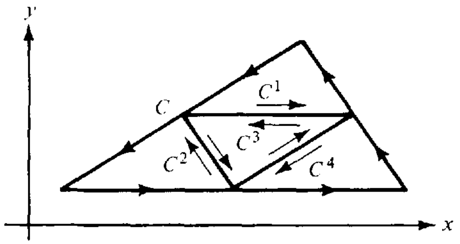


FIGURE 6.20 The triangular contours C and C^1, C^2, C^3, C^4.

Since each contour is positively oriented, if we sum the integrals along the four triangular contours, then the integrals along the segments interior to C cancel out in pairs. The result is

$$(12)\qquad \int_C f(z)\,dz = \sum_{k=1}^{4}\int_{C^k} f(z)\,dz.$$

Let C_1 be selected from C^1, C^2, C^3, and C^4 so that the following relation holds true:

$$(13)\qquad \left|\int_C f(z)\,dz\right| \le \sum_{k=1}^{4}\left|\int_{C^k} f(z)\,dz\right| \le 4\left|\int_{C_1} f(z)\,dz\right|.$$

We can proceed inductively and carry out a similar subdivision process to obtain a sequence of triangular contours $\{C_n\}$, where the interior of C_{n-1} lies in the interior of C_n and the following inequality holds:

$$(14)\qquad \left|\int_{C_n} f(z)\,dz\right| \le 4\left|\int_{C_{n+1}} f(z)\,dz\right| \quad \text{for } n = 1, 2, \ldots .$$

Let T_n denote the closed region that consists of C_n and its interior. Since the length of the sides of C_n go to zero as $n \to \infty$, there exists a unique point z_0 that belongs to all of the closed triangular regions T_n. Since f is analytic at the point z_0, there exists a function $\eta(z)$ with

(15) $$f(z) = f(z_0) + f'(z_0)(z - z_0) + \eta(z)(z - z_0).$$

Using equation (15) and integrating f along C_n, we find that

(16) $$\begin{aligned}\int_{C_n} f(z)\,dz &= \int_{C_n} f(z_0)\,dz + \int_{C_n} f'(z_0)(z - z_0)\,dz \\ &\quad + \int_{C_n} \eta(z)(z - z_0)\,dz \\ &= [f(z_0) - f'(z_0)z_0] \int_{C_n} 1\,dz \quad + \quad f'(z_0) \int_{C_n} z\,dz \\ &\quad + \int_{C_n} \eta(z)(z - z_0)\,dz \\ &= \int_{C_n} \eta(z)(z - z_0)\,dz.\end{aligned}$$

If $\varepsilon > 0$ is given, then a $\delta > 0$ can be found such that

(17) $$|z - z_0| < \delta \quad \text{implies that } |\eta(z)| < \frac{\varepsilon}{L^2},$$

where L is the length of the original contour C. An integer n can now be chosen so that C_n lies in the neighborhood $|z - z_0| < \delta$, as shown in Figure 6.21.

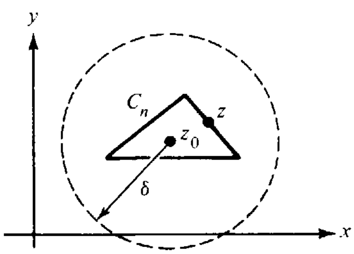


FIGURE 6.21 The contour C_n that lies in the neighborhood $|z - z_0| < \delta$.

Since the distance between a point z on a triangle and a point z_0 interior to the triangle is no greater than half the perimeter of the triangle, it follows that

(18) $$|z - z_0| < \tfrac{1}{2} L_n \quad \text{for all } z \text{ on } C_n,$$

where L_n is the length of the triangle C_n. From the preceding construction process, it follows that

(19) $$L_n = (\tfrac{1}{2})^n L \quad \text{and} \quad |z - z_0| < (\tfrac{1}{2})^{n+1} L \quad \text{for } z \text{ on } C_n.$$

We can use equations (14), (17), and (19) of this section and equation (23) of Section 6.2 to obtain the following estimate:

$$\begin{aligned}\left|\int_C f(z)\,dz\right| &\le 4^n \int_{C_n} |\eta(z)(z - z_0)|\,|dz| \\ &\le 4^n \int_{C_n} \frac{\varepsilon}{L^2}\left(\frac{1}{2}\right)^{n+1} L\,|dz| \\ &= \frac{2^{n-1}\varepsilon}{L}\int_{C_n} |dz| \\ &= \frac{2^{n-1}\varepsilon}{L}\left(\frac{1}{2}\right)^n L = \frac{\varepsilon}{2}.\end{aligned}$$

Since ε was arbitrary, it follows that equation (12) holds true for the triangular contour C. If C is a polygonal contour, then interior edges can be added until the interior is subdivided into a finite number of triangles. The integral around each triangle is zero, and the sum of all these integrals is equal to the integral around the polygonal contour C. Therefore equation (12) holds true for polygonal contours. The proof for an arbitrary simple closed contour is established by approximating the contour ''sufficiently close'' with a polygonal contour.

EXAMPLE 6.12 Let us recall that $\exp z$, $\cos z$, and z^n, where n is a positive integer are all entire functions and have continuous derivatives. The Cauchy-Goursat theorem implies that for any simple closed contour we have

$$\int_C \exp z\,dz = 0, \quad \int_C \cos z\,dz = 0, \quad \int_C z^n\,dz = 0.$$

EXAMPLE 6.13 If C is a simple closed contour such that the origin does not lie interior to C, then there is a simply connected domain D that contains C in which $f(z) = 1/z^n$ is analytic, as is indicated in Figure 6.22. The Cauchy-Goursat theorem implies that

$$\int_C \frac{1}{z^n}\,dz = 0 \quad \text{provided that the origin does not lie interior to } C.$$

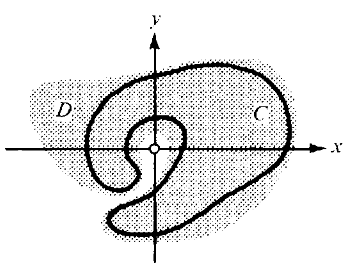


FIGURE 6.22 A simple connected domain D containing the simple closed contour C that does not contain the origin.

It is desirable to be able to replace integrals over certain complicated contours with integrals that are easy to evaluate. If C_1 is a simple closed contour that can be continuously deformed into another simple closed contour C_2 without passing through a point where f is not analytic, then the value of the contour integral of f over C_1 is the same as the value of the integral of f over C_2. To be precise, we state the following result.

Theorem 6.4 (Deformation of Contour) *Let C_1 and C_2 be two simple closed positively oriented contours such that C_1 lies interior to C_2. If f is analytic in a domain D that contains both C_1 and C_2 and the region between them, as shown in Figure 6.23, then*

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

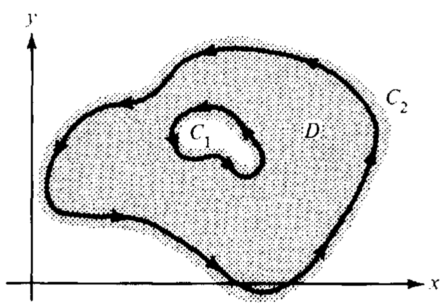
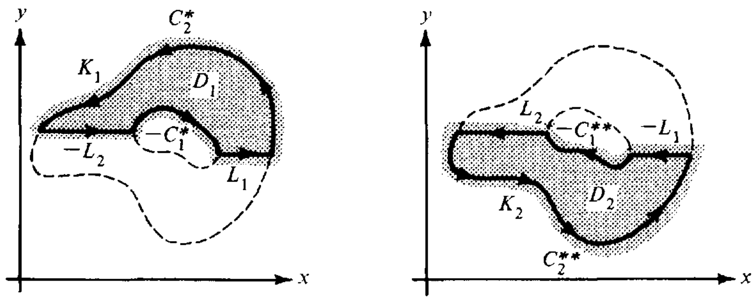


FIGURE 6.23 The domain D that contains the simple closed contours C_1 and C_2 and the region between them.

Proof Assume that both C_1 and C_2 have positive (counterclockwise) orientation. We construct two disjoint contours or *cuts* L_1 and L_2 that join C_1 to C_2. Hence the contour C_1 will be cut into two contours C_1^* and C_1^{**}, and the contour C_2 will be cut into C_2^* and C_2^{**}. We now form two new contours:

$$K_1 = -C_1^* + L_1 + C_2^* - L_2 \quad \text{and} \quad K_2 = -C_1^{**} + L_2 + C_2^{**} - L_1,$$

which are shown in Figure 6.24. The function f will be analytic on a simply connected domain D_1 that contains K_1, and f will be analytic on the simply connected domain D_2 that contains K_2, as is illustrated in Figure 6.24.



(a) The contour K_1 and domain D_1. (b) The contour K_2 and domain D_2.

FIGURE 6.24 The cuts L_1 and L_2 and the contours K_1 and K_2 used to prove the Deformation Theorem.

The Cauchy-Goursat theorem can be applied to the contours K_1 and K_2, and the result is

$$(20) \quad \int_{K_1} f(z)\, dz = 0 \quad \text{and} \quad \int_{K_2} f(z)\, dz = 0.$$

Adding contours, we observe that

$$(21) \quad K_1 + K_2 = -C_1^* + L_1 + C_2^* - L_2 - C_1^{**} + L_2 + C_2^{**} - L_1$$
$$= C_2^* + C_2^{**} - C_1^* - C_1^{**} = C_2 - C_1.$$

We can use identities (14) and (17) of Section 6.2 and equations (20) and (21) given in this section to conclude that

$$\int_{C_2} f(z)\, dz - \int_{C_1} f(z)\, dz = \int_{K_1} f(z)\, dz + \int_{K_2} f(z)\, dz = 0,$$

which completes the proof of Theorem 6.4.

We now state an important result that is proven by the deformation theorem. This result will occur several times in the theory to be developed and is an important tool for computations.

EXAMPLE 6.14 Let z_0 denote a fixed complex value. If C is a simple closed contour with positive orientation such that z_0 lies interior to C, then

$$(22) \quad \int_C \frac{dz}{z - z_0} = 2\pi i \quad \text{and}$$
$$\int_C \frac{dz}{(z - z_0)^n} = 0 \quad \text{where } n \neq 1 \text{ is an integer.}$$

Solution Since z_0 lies interior to C, we can choose R so that the circle C_R with center z_0 and radius R lies interior to C. Hence $f(z) = 1/(z - z_0)^n$ is analytic in a domain D that contains both C and C_R and the region between them, as shown in Figure 6.25. Let C_R have the parameterization

$$C_R: z(\theta) = z_0 + Re^{i\theta} \quad \text{and} \quad dz = i\,Re^{i\theta}\,d\theta \quad \text{for } 0 \le \theta \le 2\pi.$$

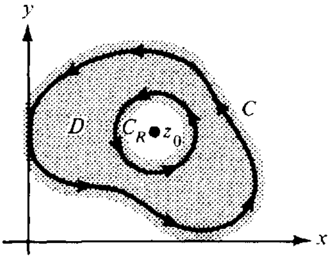


FIGURE 6.25 The domain D that contains both C and C_R.

The deformation theorem implies that the integral of f over C_R has the same value as the integral of f over C, and we obtain

$$\int_C \frac{dz}{z - z_0} = \int_{C_R} \frac{dz}{z - z_0} = \int_0^{2\pi} \frac{i\,Re^{i\theta}}{Re^{i\theta}}\,d\theta = i\int_0^{2\pi} d\theta = 2\pi i$$

and

$$\int_C \frac{dz}{(z - z_0)^n} = \int_{C_R} \frac{dz}{(z - z_0)^n} = \int_0^{2\pi} \frac{i\,Re^{i\theta}}{R^n e^{in\theta}}\,d\theta = iR^{1-n}\int_0^{2\pi} e^{i(1-n)\theta}\,d\theta$$

$$= \frac{R^{1-n}}{1-n}e^{i(1-n)\theta}\Big|_{\theta=0}^{\theta=2\pi} = \frac{R^{1-n}}{1-n} - \frac{R^{1-n}}{1-n} = 0.$$

The deformation theorem is an extension of the Cauchy-Goursat theorem to a doubly connected domain in the following sense. Let D be a domain that contains C_1 and C_2 and the region between them, as shown in Figure 6.25. Then the contour $C = C_2 - C_1$ is a parameterization of the boundary of the region R that lies between C_1 and C_2 so that the points of R lie to the left of C as a point $z(t)$ moves around C. Hence C is a positive orientation of the boundary of R, and Theorem 6.4 implies that

$$\int_C f(z)\,dz = 0.$$

We can extend Theorem 6.4 to multiply connected domains with more than one "hole." The proof, which is left for the reader, involves the introduction of several cuts and is similar to the proof of Theorem 6.4.

Theorem 6.5 (Extended Cauchy-Goursat Theorem)
Let $C, C_1, C_2, \ldots, C_n$ be simple closed positively oriented contours with the property that C_k lies interior to C for $k = 1, 2, \ldots, n$, and the set interior to C_k has no points in common with the set interior to C_j if $k \neq j$. Let f be analytic on a domain D that contains all the contours and the region between C and $C_1 + C_2 + \cdots + C_n$, which is shown in Figure 6.26. Then

$$(23)\quad \int_C f(z)\,dz = \sum_{k=1}^{n} \int_{C_k} f(z)\,dz.$$

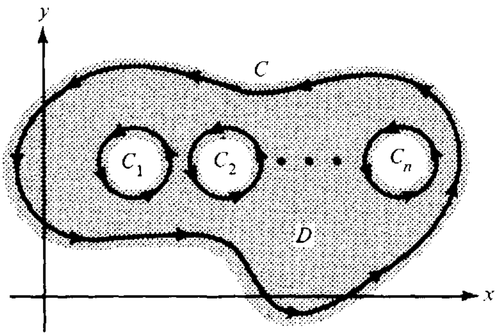


FIGURE 6.26 The multiply connected domain D and the contours C and $C_1, C_2, \ldots, C_n$ in the statement of the Extended Cauchy-Goursat Theorem.

EXAMPLE 6.15 If C is the circle $|z| = 2$ taken with positive orientation, then

$$(24)\quad \int_C \frac{2z\,dz}{z^2 + 2} = 4\pi i.$$

Solution Using partial fractions, the integral in equation (24) can be written as

$$(25)\quad \int_C \frac{2z\,dz}{z^2 + 2} = \int_C \frac{dz}{z + i\sqrt{2}} + \int_C \frac{dz}{z - i\sqrt{2}}.$$

Since the points $z = \pm i\sqrt{2}$ lie interior to C, Example 6.14 implies that

$$(26)\quad \int_C \frac{dz}{z \pm i\sqrt{2}} = 2\pi i.$$

The results in (26) can be used in (25) to conclude that

$$\int_C \frac{2z\,dz}{z^2 + 2} = 2\pi i + 2\pi i = 4\pi i.$$

EXAMPLE 6.16 If C is the circle $|z - i| = 1$ taken with positive orientation, then

$$(27) \quad \int_C \frac{2z\,dz}{z^2 + 2} = 2\pi i.$$

Solution Using partial fractions, the integral in equation (27) can be written as

$$(28) \quad \int_C \frac{2z\,dz}{z^2 + 2} = \int_C \frac{dz}{z + i\sqrt{2}} + \int_C \frac{dz}{z - i\sqrt{2}}.$$

In this case, only the point $z = i\sqrt{2}$ lies interior to C, so the second integral on the right side of equation (28) has the value $2\pi i$. The function $f(z) = 1/(z + i\sqrt{2})$ is analytic on a simply connected domain that contains C. Hence by the Cauchy-Goursat theorem the first integral on the right side of equation (28) is zero (see Figure 6.27). Therefore

$$\int_C \frac{2z\,dz}{z^2 + 2} = 0 + 2\pi i = 2\pi i.$$

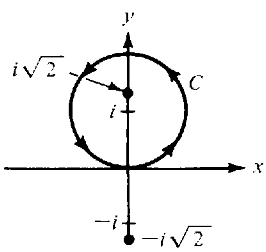
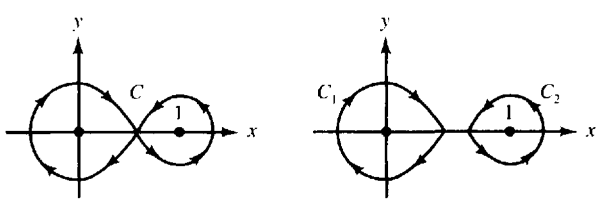


FIGURE 6.27 The circle $|z - i| = 1$ and the points $z = \pm i\sqrt{2}$.

EXAMPLE 6.17 Show that

$$\int_C \frac{z - 2}{z^2 - z}\,dz = -6\pi i$$

where C is the "figure eight" contour shown in Figure 6.28(a).



(a) The figure eight contour C. (b) The contours C_1 and C_2.

FIGURE 6.28 The contour $C = C_1 + C_2$.

Solution Partial fractions can be used to express the integral as

$$(29)\quad \int_C \frac{z-2}{z^2-z}\,dz = 2\int_C \frac{1}{z}\,dz - \int_C \frac{1}{z-1}\,dz.$$

Using the Cauchy-Goursat theorem and property (14) of Section 6.2 together with Example 6.13, we compute the value of the first integral on the right side of equation (29):

$$(30)\quad 2\int_C \frac{1}{z}\,dz = 2\int_{C_1} \frac{1}{z}\,dz + 2\int_{C_2} \frac{1}{z}\,dz$$
$$= -2\int_{-C_1} \frac{1}{z}\,dz + 0 = -2(2\pi i) = -4\pi i.$$

In a similar fashion we find that

$$(31)\quad -\int_C \frac{dz}{z-1} = -\int_{C_1} \frac{dz}{z-1} - \int_{C_2} \frac{dz}{z-1} = 0 - 2\pi i = -2\pi i.$$

The results of equations (30) and (31) can be used in equation (29) to conclude that

$$\int_C \frac{z-2}{z^2-z}\,dz = -4\pi i - 2\pi i = -6\pi i.$$

EXERCISES FOR SECTION 6.3

1. Determine the domain of analyticity for the following functions, and conclude that $\int_C f(z)\,dz = 0$, where C is the circle $|z| = 1$ with positive orientation.
   (a) $f(z) = \dfrac{z}{z^2+2}$ (b) $f(z) = \dfrac{1}{z^2+2z+2}$
   (c) $f(z) = \tan z$ (d) $f(z) = \operatorname{Log}(z+5)$
2. Show that $\int_C z^{-1}\,dz = 2\pi i$, where C is the square with vertices $1 \pm i$, $-1 \pm i$ with positive orientation.
3. Show that $\int_C (4z^2 - 4z + 5)^{-1}\,dz = 0$, where C is the unit circle $|z| = 1$ with positive orientation.
4. Find $\int_C (z^2 - z)^{-1}\,dz$ for the following contours.
   (a) The circle $|z - 1| = 2$ with positive orientation.
   (b) The circle $|z - 1| = \frac{1}{2}$ with positive orientation.
5. Find $\int_C (2z - 1)(z^2 - z)^{-1}\,dz$ for the following contours.
   (a) The circle $|z| = 2$ with positive orientation.
   (b) The circle $|z| = \frac{1}{2}$ with positive orientation.
6. Evaluate $\int_C (z^2 - z)^{-1}\,dz$, where C is the figure eight contour shown in Figure 6.28(a).
7. Evaluate $\int_C (2z - 1)(z^2 - z)^{-1}\,dz$, where C is the figure eight contour shown in Figure 6.28(a).
8. Evaluate $\int_C (4z^2 + 4z - 3)^{-1}\,dz = \int_C (2z - 1)^{-1}(2z + 3)^{-1}\,dz$ for the following contours.
   (a) The circle $|z| = 1$ with positive orientation.
   (b) The circle $|z + \frac{3}{2}| = 1$ with positive orientation.
   (c) The circle $|z| = 3$ with positive orientation.
9. Evaluate $\int_C (z^2 - 1)^{-1}\,dz$ for the contours given in Figure 6.29.

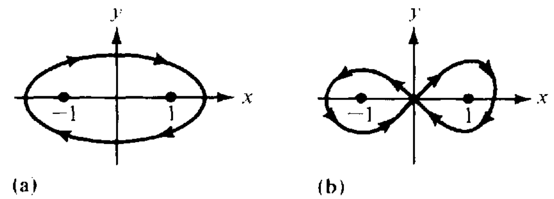


FIGURE 6.29 Accompanies Exercise 9.

10. Let C be the triangle with vertice 0, 1, and i with positive orientation. Parameterize C and show that

$$\int_C 1\,dz = 0 \quad \text{and} \quad \int_C z\,dz = 0.$$

11. Let the circle $|z| = 1$ be given the parameterization

$$C\colon z(t) = \cos t + i \sin t \quad \text{for } -\pi \le t \le \pi.$$

Use the principal branch of the square root function:

$$z^{1/2} = r^{1/2}\cos\frac{\theta}{2} + ir^{1/2}\sin\frac{\theta}{2} \quad \text{for } -\pi < \theta \le \pi$$

and find $\int_C z^{1/2}\,dz$.

12. Evaluate $\int_C |z|^2 \exp z\,dz$, where C is the unit circle $|z| = 1$ with positive orientation.

13. Let $f(z) = u(r, \theta) + iv(r, \theta)$ be analytic for all values of $z = re^{i\theta}$. Show that

$$\int_0^{2\pi} [u(r, \theta)\cos\theta - v(r, \theta)\sin\theta]\,d\theta = 0.$$

Hint: Integrate f around the circle $|z| = 1$.

14. Show by using Green's theorem that the area enclosed by a simple closed contour C is $\frac{1}{2}\int_C x\,dy - y\,dx$.

15. Compare the various methods for evaluating contour integrals. What are the limitations of each method?

6.4 The Fundamental Theorems of Integration

Let f be analytic in the simply connected domain D. The theorems in this section show that an antiderivative F can be constructed by contour integration. A consequence will be the fact that in a simply connected domain, the integral of an analytic function f along any contour joining z_1 to z_2 is the same, and its value is given by $F(z_2) - F(z_1)$. Hence we will be able to use the antiderivative formulas from calculus to compute the value of definite integrals.

Theorem 6.6 (Indefinite Integrals or Antiderivatives) *Let f be analytic in the simply connected domain D. If z_0 is a fixed value in D and if C is any contour in D with initial point z_0 and terminal point z, then the function given by*

(1) $F(z) = \int_C f(\xi)\, d\xi = \int_{z_0}^{z} f(\xi)\, d\xi$

is analytic in D and

(2) $F'(z) = f(z)$.

Proof We first establish that the integral is independent of the path of integration. Hence we will need to keep track only of the endpoints, and we can use the notation

$$\int_C f(\xi)\, d\xi = \int_{z_0}^{z} f(\xi)\, d(\xi).$$

Let C_1 and C_2 be two contours in D, both with the initial point z_0 and the terminal point z, as shown in Figure 6.30. Then $C = C_1 - C_2$ is a simple closed contour, and the Cauchy-Goursat theorem implies that

$$\int_{C_1} f(\xi)\, d\xi - \int_{C_2} f(\xi)\, d\xi = \int_{C_1 - C_2} f(\xi)\, d\xi = 0.$$

Therefore the contour integral in equation (1) is independent of path. Here we have taken the liberty of drawing contours that intersect only at the endpoints. A slight modification of the foregoing proof will show that a finite number of other points of intersection are permitted.

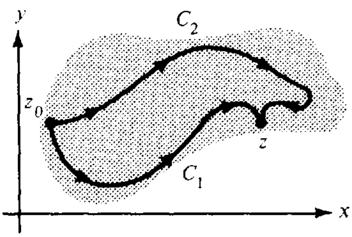


FIGURE 6.30 The contours C_1 and C_2 joining z_0 to z.

We now show that $F'(z) = f(z)$. Let z be held fixed, and let Δz be chosen small enough so that the point $z + \Delta z$ also lies in the domain D. Since z is held fixed, $f(z) = K$ where K is a constant, and equation (12) of Section 6.1 implies that

(3) $\int_{z}^{z+\Delta z} f(z)\, d\xi = \int_{z}^{z+\Delta z} K\, d\xi = K\,\Delta z = f(z)\,\Delta z.$

Using the additive property of contours and the definition of F given in equation (1), it follows that

(4) $$F(z + \Delta z) - F(z) = \int_{z_0}^{z+\Delta z} f(\xi)\, d\xi - \int_{z_0}^{z} f(\xi)\, d\xi$$
$$= \int_{C_2} f(\xi)\, d\xi - \int_{C_1} f(\xi)\, d\xi = \int_C f(\xi)\, d\xi,$$

where the contour C is the straight line segment joining z to $z + \Delta z$ and C_1 and C_2 join z_0 to z and z_0 to $z + \Delta z$, respectively, as shown in Figure 6.31.

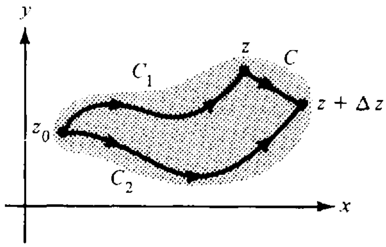


FIGURE 6.31 The contours C_1 and C_2 and the line segment $C = -C_1 + C_2$.

Since f is continuous at z, then if $\varepsilon > 0$, there is a $\delta > 0$ so that

(5) $\quad |f(\xi) - f(z)| < \varepsilon \quad$ whenever $|\xi - z| < \delta$.

If we require that $|\Delta z| < \delta$, then using equations (3) and (4), inequality (5), and inequality (22) of Section 6.2, we obtain the following estimate:

$$(6) \quad \left|\frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z)\right| = \frac{1}{|\Delta z|}\left|\int_C f(\xi)\,d\xi - \int_C f(z)\,d\xi\right|$$
$$\leq \frac{1}{|\Delta z|}\int_C |f(\xi) - f(z)|\,|d\xi|$$
$$< \frac{1}{|\Delta z|}\varepsilon|\Delta z| = \varepsilon.$$

Consequently, the left side of equation (6) tends to 0 as $\Delta z \to 0$; that is, $F'(z) = f(z)$, and the theorem is proven.

It is important to notice that the line integral of an analytic function is independent of path. An easy calculation shows

$$\int_{C_1} z\,dz = \int_{C_2} z\,dz = 4 + 2i,$$

where C_1 and C_2 were contours joining $-1 - i$ to $3 + i$. Since the integrand $f(z) = z$ is an analytic function, Theorem 6.6 implies that the value of the two integrals is the same; hence one calculation would suffice.

If we set $z = z_1$ in Theorem 6.6, then we obtain the following familiar result for evaluating a definite integral of an analytic function.

Theorem 6.7 (Definite Integrals) *Let f be analytic in a simply connected domain D. If z_0 and z_1 are two points in D, then*

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0) \tag{7}$$

where F is any antiderivative of f.

Proof If F is chosen to be the function in equation (1), then equation (7) holds true. If G is any other antiderivative of f, then $H(z) = G(z) - F(z)$ is analytic, and $H'(z) = 0$ for all points z in D. Hence $H(z) = K$ where K is a constant, and $G(z) = F(z) + K$. Therefore $G(z_1) - G(z_0) = F(z_1) - F(z_0)$, and Theorem 6.7 is proven.

Theorem 6.7 is an important method for evaluating definite integrals when the integrand is an analytic function. In essence, it permits us to use all the rules of integration that were introduced in calculus. For analytic integrands, application of Theorem 6.7 is easier to use than the method of parameterization of a contour.

EXAMPLE 6.18 Show that $\int_1^i \cos z\,dz = -\sin 1 + i \sinh 1$.

Solution An antiderivative of $f(z) = \cos z$ is $F(z) = \sin z$. Hence

$$\int_1^i \cos z\,dz = \sin i - \sin 1 = -\sin 1 + i \sinh 1.$$

EXAMPLE 6.19 Evaluate $\left(2 + \frac{i\pi}{4}\right) \int_0^1 e^{2t} e^{i\pi t/4}\,dt$.

Solution In Example 6.7, we broke the integrand up into its real and imaginary parts, which then required integration by parts. Using Theorem 6.7, however, we see that

$$\begin{aligned}\left(2 + \frac{i\pi}{4}\right) \int_0^1 e^{2t} e^{i\pi t/4} &= \left(2 + \frac{i\pi}{4}\right) \int_0^1 e^{t(2+i\pi/4)}\,dt \\ &= \left(2 + \frac{i\pi}{4}\right)\left(\frac{1}{2 + \frac{i\pi}{4}}\right) e^{t(2+i\pi/4)}\Big|_0^1 \\ &= e^{(2+i\pi/4)} - e^0 \\ &= e^{(2+i\pi/4)} - 1.\end{aligned}$$

EXAMPLE 6.20 Show that

$$\int_4^{8+6i} \frac{dz}{2z^{1/2}} = 1 + i,$$

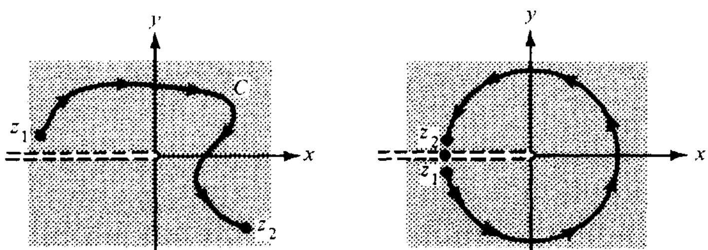
where $z^{1/2}$ is the principal branch of the square root function and the integral is to be taken along the line segment joining 4 to $8 + 6i$.

Solution Example 3.8 showed that if $F(z) = z^{1/2}$, then $F'(z) = 1/(2z^{1/2})$, where the principal branch of the square root function is used in both the formulas for F and F'. Hence

$$\int_4^{8+6i} \frac{dz}{2z^{1/2}} = (8 + 6i)^{1/2} - 4^{1/2} = 3 + i - 2 = 1 + i.$$

EXAMPLE 6.21 Let $D = \{z = re^{i\theta}: r > 0 \text{ and } -\pi < \theta < \pi\}$ be the simply connected domain shown in Figure 6.32. Then $F(z) = \text{Log } z$ is analytic in D, and its derivative is $F'(z) = 1/z$. If C is a contour in D that joins the point z_1 to the point z_2, then Theorem 6.7 implies that

$$\int_{z_1}^{z_2} \frac{dz}{z} = \int_C \frac{dz}{z} = \text{Log } z_2 - \text{Log } z_1.$$



(a) The path C joining z_1 and z_2.

(b) The path that is a portion of the unit circle $|z| = 1$.

FIGURE 6.32 The simply connected domain D in Examples 6.21 and 6.22.

EXAMPLE 6.22 As a consequence of Example 6.21, let us show that

$$\int_C \frac{dz}{z} = 2\pi i, \quad \text{where } C \text{ is the unit circle } |z| = 1,$$

taken with positive orientation.

Solution If we let z_2 approach -1 through the upper half plane and z_1 approaches -1 through the lower half plane, then we can integrate around the portion of the circle shown in Figure 6.32(b) and take limits to obtain

$$\int_C \frac{dz}{z} = \lim_{\substack{z_1 \to -1 \\ z_2 \to -1}} \int_{z_1}^{z_2} \frac{dz}{z} = \lim_{\substack{z_2 \to -1 \\ Im(z_2) > 0}} \text{Log } z_2 - \lim_{\substack{z_1 \to -1 \\ Im(z_1) < 0}} \text{Log } z_1$$
$$= i\pi - (-i\pi) = 2\pi i.$$

EXERCISES FOR SECTION 6.4

For Exercises 1–14, use antiderivatives to find the value of the definite integral.

1. $\int_{1+i}^{2+i} z^2 \, dz$
2. $\int_1^i \frac{1+z}{z} \, dz$ (use Log z)
3. $\int_2^{i\pi/2} \exp z \, dz$
4. $\int_i^{1+i} (z^2 + z^{-2}) \, dz$
5. $\int_{-i}^{1+i} \cos z \, dz$
6. $\int_0^{\pi - 2i} \sin \frac{z}{2} \, dz$
7. $\int_{-1-i\pi/2}^{2+\pi i} z \exp z \, dz$
8. $\int_{1-2i}^{1+2i} z \exp(z^2) \, dz$
9. $\int_0^i z \cos z \, dz$
10. $\int_0^i \sin^2 z \, dz$
11. $\int_1^{1+i} \text{Log } z \, dz$
12. $\int_2^{2+i} \frac{dz}{z^2 - z}$
13. $\int_2^{2+i} \frac{2z-1}{z^2 - z} \, dz$
14. $\int_2^{2+i} \frac{z-2}{z^2 - z} \, dz$
15. Show that $\int_{z_1}^{z_2} 1 \, dz = z_2 - z_1$ by parameterizing the line segment from z_1 to z_2.
16. Let z_1 and z_2 be points in the right half plane. Show that
$$\int_{z_1}^{z_2} \frac{dz}{z^2} = \frac{1}{z_1} - \frac{1}{z_2}.$$
17. Find
$$\int_9^{3+4i} \frac{dz}{2z^{1/2}},$$
where $z^{1/2}$ is the principal branch of the square root function and the integral is to be taken along the line segment from 9 to $3 + 4i$.
18. Find $\int_{-2i}^{2i} z^{1/2} \, dz$, where $z^{1/2}$ is the principal branch of the square root function and the integral is to be taken along the right half of the circle $|z| = 2$.
19. Using the equation
$$\frac{1}{z^2+1} = \frac{i}{2} \frac{1}{z+i} - \frac{i}{2} \frac{1}{z-i},$$
show that if z lies in the right half plane, then
$$\int_0^z \frac{d\xi}{\xi^2 + 1} = \arctan z = \frac{i}{2} \text{Log}(z+i) - \frac{i}{2} \text{Log}(z-i).$$
20. Let f' and g' be analytic for all z. Show that
$$\int_{z_1}^{z_2} f(z) g'(z) \, dz = f(z_2)g(z_2) - f(z_1)g(z_1) - \int_{z_1}^{z_2} f'(z)g(z) \, dz.$$
21. Compare the various methods for evaluating contour integrals. What are the limitations of each method?
22. Explain how the fundamental theorem of calculus studied in complex analysis and the fundamental theorem of calculus studied in calculus are different. How are they similar?

6.5 Integral Representations for Analytic Functions

We now present some major results in the theory of functions of a complex variable. The first result is known as Cauchy's integral formula and shows that the value of an analytic function f can be represented by a certain contour integral. The nth derivative, $f^{(n)}(z)$, will have a similar representation. In Chapter 7 we will show how the Cauchy integral formulae are used to prove Taylor's theorem, and we will establish the power series representation for analytic functions. The Cauchy integral formulae will also be a convenient tool for evaluating certain contour integrals.

Theorem 6.8 (Cauchy's Integral Formula) *Let f be analytic in the simply connected domain D, and let C be a simple closed positively oriented contour that lies in D. If z_0 is a point that lies interior to C, then*

$$(1) \quad f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz.$$

Proof Since f is continuous at z_0, if $\varepsilon > 0$ is given, there is a $\delta > 0$ such that

$$(2) \quad |f(z) - f(z_0)| < \varepsilon \quad \text{whenever } |z - z_0| < \delta.$$

Also the circle C_0: $|z - z_0| = \frac{1}{2}\delta$ lies interior to C as shown in Figure 6.33.

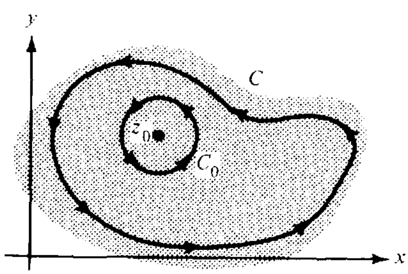


FIGURE 6.33 The contours C and C_0 in the proof of Cauchy's integral formula.

Since $f(z_0)$ is a fixed value, we can use the result of Exercise 24 of Section 6.2 to conclude that

$$(3) \quad f(z_0) = \frac{f(z_0)}{2\pi i} \int_{C_0} \frac{dz}{z - z_0} = \frac{1}{2\pi i} \int_{C_0} \frac{f(z_0)}{z - z_0}\, dz.$$

Using the deformation theorem we see that

$$(4) \quad \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz = \frac{1}{2\pi i} \int_{C_0} \frac{f(z)}{z - z_0}\, dz.$$

Using inequality (2), equations (3) and (4), and inequality (22) of Section 6.2, we obtain the following estimate:

$$(5)\quad \left|\frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-z_0} - f(z_0)\right| = \left|\frac{1}{2\pi i}\int_{C_0} \frac{f(z)\,dz}{z-z_0} - \frac{1}{2\pi i}\int_{C_0} \frac{f(z_0)\,dz}{z-z_0}\right|$$
$$\leq \frac{1}{2\pi}\int_{C_0} \frac{|f(z)-f(z_0)|}{|z-z_0|}\,|dz|$$
$$\leq \frac{1}{2\pi}\frac{\varepsilon}{(1/2)\delta}\pi\delta = \varepsilon.$$

Since ε can be made arbitrarily small, the theorem is proven.

EXAMPLE 6.23 Show that

$$\int_C \frac{\exp z}{z-1}\,dz = i2\pi e,$$

where C is the circle $|z| = 2$ with positive orientation.

Solution Here we have $f(z) = \exp z$ and $f(1) = e$. The point $z_0 = 1$ lies interior to C, so Cauchy's integral formula implies that

$$e = f(1) = \frac{1}{2\pi i}\int_C \frac{\exp z}{z-1}\,dz,$$

and multiplication by $2\pi i$ will establish the desired result.

EXAMPLE 6.24 Show that

$$\int_C \frac{\sin z}{4z+\pi}\,dz = \frac{-\sqrt{2}\pi i}{4},$$

where C is the circle $|z| = 1$ with positive orientation.

Solution Here we have $f(z) = \sin z$. We can manipulate the integral and use Cauchy's integral formula to obtain

$$\int_C \frac{\sin z}{4z+\pi}\,dz = \frac{1}{4}\int_C \frac{\sin z}{z+(\pi/4)}\,dz = \frac{1}{4}\int_C \frac{f(z)}{z-(-\pi/4)}\,dz$$
$$= \frac{1}{4}(2\pi i)f\left(\frac{-\pi}{4}\right) = \frac{\pi i}{2}\sin\left(\frac{-\pi}{4}\right) = \frac{-\sqrt{2}\pi i}{4}.$$

We now state a general result that shows how differentiation under the integral sign can be accomplished. The proof can be found in some advanced texts. See, for instance, Rolf Nevanlinna and V. Paatero, *Introduction to Complex Analysis* (Reading, Massachusetts: Addison-Wesley Publishing Company, 1969), Section 9.7.

Theorem 6.9 (Leibniz's Rule) *Let D be a simply connected domain, and let I: $a \le t \le b$ be an interval of real numbers. Let $f(z, t)$ and its partial derivative $f_z(z, t)$ with respect to z be continuous functions for all z in D and all t in I. Then*

$$(6) \quad F(z) = \int_a^b f(z, t)\, dt$$

is analytic for z in D, and

$$F'(z) = \int_a^b f_z(z, t)\, dt.$$

We now show how Theorem 6.8 can be generalized to give an integral representation for the nth derivative, $f^{(n)}(z)$. Leibniz's rule will be used in the proof, and we shall see that this method of proof will be a mnemonic device for remembering how the denominator is written.

Theorem 6.10 (Cauchy's Integral Formulae for Derivatives) *Let f be analytic in the simply connected domain D, and let C be a simple closed positively oriented contour that lies in D. If z is a point that lies interior to C, then*

$$(7) \quad f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z)^{n+1}}\, d\xi.$$

Proof We will establish the theorem for the case $n = 1$. We start by using the parameterization

$$C: \xi = \xi(t) \quad \text{and} \quad d\xi = \xi'(t)\, dt \quad \text{for } a \le t \le b.$$

We use Theorem 6.8 and write

$$(8) \quad f(z) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\xi - z}\, d\xi = \frac{1}{2\pi i} \int_a^b \frac{f(\xi(t))\xi'(t)\, dt}{\xi(t) - z}.$$

The integrand on the right side of equation (8) can be considered as a function $f(z, t)$ of the two variables z and t, where

$$(9) \quad f(z, t) = \frac{f(\xi(t))\xi'(t)}{\xi(t) - z} \quad \text{and} \quad f_z(z, t) = \frac{f(\xi(t))\xi'(t)}{(\xi(t) - z)^2}.$$

Using equations (9) and Leibniz's rule, we see that $f'(z)$ is given by

$$f'(z) = \frac{1}{2\pi i} \int_a^b \frac{f(\xi(t))\xi'(t)\, dt}{(\xi(t) - z)^2} = \frac{1}{2\pi i} \int_C \frac{f(\xi)\, d\xi}{(\xi - z)^2},$$

and the proof for the case $n = 1$ is complete. We can apply the same argument to the analytic function f' and show that its derivative f'' has representation (7) with $n = 2$. The principle of mathematical induction will establish the theorem for any value of n.

EXAMPLE 6.25 Let z_0 denote a fixed complex value. If C is a simple closed positively oriented contour such that z_0 lies interior to C, then

$$(10)\quad \int_C \frac{dz}{z - z_0} = 2\pi i \quad \text{and} \quad \int_C \frac{dz}{(z - z_0)^{n+1}} = 0,$$

where $n \geq 1$ is a positive integer.

Solution Here we have $f(z) = 1$ and the nth derivative is $f^{(n)}(z) = 0$. Theorem 6.8 implies that the value of the first integral in equations (10) is given by

$$\int_C \frac{dz}{z - z_0} = 2\pi i f(z_0) = 2\pi i,$$

and Theorem 6.10 can be used to conclude that

$$\int_C \frac{dz}{(z - z_0)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(z_0) = 0.$$

We remark that this is the same result that was proven earlier in Example 6.14. It should be obvious that the technique of using Theorems 6.8 and 6.10 is easier.

EXAMPLE 6.26 Show that

$$\int_C \frac{\exp z^2}{(z - i)^4}\, dz = \frac{-4\pi}{3e},$$

where C is the circle $|z| = 2$ with positive orientation.

Solution Here we have $f(z) = \exp z^2$, and a straightforward calculation shows that $f^{(3)}(z) = (12z + 8z^3) \exp z^2$. Using Cauchy's integral formulae with $n = 3$, we conclude that

$$\int_C \frac{\exp z^2}{(z - i)^4}\, dz = \frac{2\pi i}{3!} f^{(3)}(i) = \frac{2\pi i}{6} \frac{4i}{e} = \frac{-4\pi}{3e}.$$

EXAMPLE 6.27 Show that

$$\int_C \frac{\exp(i\pi z)\, dz}{2z^2 - 5z + 2} = \frac{2\pi}{3},$$

where C is the circle $|z| = 1$ with positive orientation.

Solution By factoring the denominator we obtain $2z^2 - 5z + 2 = (2z - 1)(z - 2)$. Only the root $z_0 = \frac{1}{2}$ lies interior to C. Now we set $f(z) = [\exp(i\pi z)]/(z - 2)$ and use Theorem 6.8 to conclude that

$$\int_C \frac{\exp(i\pi z)\,dz}{2z^2 - 5z + 2} = \frac{1}{2}\int_C \frac{f(z)\,dz}{z - (1/2)} = \frac{1}{2}(2\pi i)f(\tfrac{1}{2}) = \pi i\,\frac{\exp(i\pi/2)}{(1/2) - 2}$$
$$= \frac{2\pi}{3}.$$

We now state two important corollaries to Theorem 6.10.

Corollary 6.1 *If f is analytic in the domain D, then all derivatives f', f'', $\ldots, f^{(n)}, \ldots$ exist and are analytic in D.*

Proof For each point z_0 in D, there exists a closed disk $|z - z_0| \le R$ that is contained in D. The circle C: $|z - z_0| = R$ can be used in Theorem 6.10 to show that $f^{(n)}(z_0)$ exists for all n.

This result is interesting, since the definition of analytic function means that the derivative f' exists at all points in D. Here we find something more, that the derivatives of all orders exist!

Corollary 6.2 *If u is a harmonic function at each point (x, y) in the domain D, then all partial derivatives u_x, u_y, u_{xx}, u_{xy}, and u_{yy} exist and are harmonic functions.*

Proof For each point (x_0, y_0) in D there exists a closed disk $|z - z_0| \le R$ that is contained in D. A conjugate harmonic function v exists in this disk, so the function $f(z) = u + iv$ is an analytic function. We use the Cauchy-Riemann equations and see that $f'(z) = u_x + iv_x = v_y - iu_y$. Since f' is analytic, the functions u_x and u_y are harmonic. Again, we can use the Cauchy-Riemann equations to see that

$$f''(z) = u_{xx} + iv_{xx} = v_{yx} - iu_{yx} = -u_{yy} - iv_{yy}.$$

Since f'' is analytic, the functions u_{xx}, u_{xy}, and u_{yy} are harmonic.

EXERCISES FOR SECTION 6.5

For Exercises 1–15, assume that the contour C has positive orientation.

1. Find $\int_C (\exp z + \cos z)z^{-1}\,dz$, where C is the circle $|z| = 1$.
2. Find $\int_C (z + 1)^{-1}(z - 1)^{-1}\,dz$, where C is the circle $|z - 1| = 1$.
3. Find $\int_C (z + 1)^{-1}(z - 1)^{-2}\,dz$, where C is the circle $|z - 1| = 1$.
4. Find $\int_C (z^3 - 1)^{-1}\,dz$, where C is the circle $|z - 1| = 1$.

5. Find $\int_C (z \cos z)^{-1}\, dz$, where C is the circle $|z| = 1$.
6. Find $\int_C z^{-4} \sin z\, dz$, where C is the circle $|z| = 1$.
7. Find $\int_C z^{-3} \sinh(z^2)\, dz$, where C is the circle $|z| = 1$.
8. Find $\int_C z^{-2} \sin z\, dz$ along the following contours:
   (a) The circle $|z - (\pi/2)| = 1$. (b) The circle $|z - (\pi/4)| = 1$.
9. Find $\int_C z^{-n} \exp z\, dz$, where C is the circle $|z| = 1$ and n is a positive integer.
10. Find $\int_C z^{-2}(z^2 - 16)^{-1} \exp z\, dz$ along the following contours:
    (a) The circle $|z| = 1$. (b) The circle $|z - 4| = 1$.
11. Find $\int_C (z^4 + 4)^{-1}\, dz$, where C is the circle $|z - 1 - i| = 1$.
12. Find $\int_C (z^2 + 1)^{-1}\, dz$ along the following contours:
    (a) The circle $|z - i| = 1$. (b) The circle $|z + i| = 1$.
13. Find $\int_C (z^2 + 1)^{-1} \sin z\, dz$ along the following contours:
    (a) The circle $|z - i| = 1$. (b) The circle $|z + i| = 1$.
14. Find $\int_C (z^2 + 1)^{-2}\, dz$, where C is the circle $|z - i| = 1$.
15. Find $\int_C z^{-1}(z - 1)^{-1} \exp z\, dz$ along the following contours:
    (a) The circle $|z| = 1/2$. (b) The circle $|z| = 2$.

For Exercises 16–19, assume that the contour C has positive orientation.

16. Let $P(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3$ be a cubic polynomial. Find $\int_C P(z) z^{-n}\, dz$, where C is the circle $|z| = 1$ and n is a positive integer.
17. Let f be analytic in the simply connected domain D, and let C be a simple closed contour in D. Suppose that z_0 lies exterior to C. Find $\int_C f(z)(z - z_0)^{-1}\, dz$.
18. Let z_1 and z_2 be two complex numbers that lie interior to the simple closed contour C. Show that $\int_C (z - z_1)^{-1}(z - z_2)^{-1}\, dz = 0$.
19. Let f be analytic in the simply connected domain D, and let z_1 and z_2 be two complex numbers that lie interior to the simple closed contour C that lies in D. Show that

$$\frac{f(z_2) - f(z_1)}{z_2 - z_1} = \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{(z - z_1)(z - z_2)}.$$

State what happens when $z_2 \to z_1$.

20. The *Legendre polynomial* $P_n(z)$ is defined by

$$P_n(z) = \frac{1}{2^n n!} \frac{d^n}{dz^n} [(z^2 - 1)^n].$$

Use Cauchy's integral formula to show that

$$P_n(z) = \frac{1}{2\pi i} \int_C \frac{(\xi^2 - 1)^n\, d\xi}{2^n(\xi - z)^{n+1}},$$

where z lies inside C.

21. Discuss the importance of being able to define an analytic function $f(z)$ with the contour integral in formula (1). How does this differ from other definitions of a function that you have learned?
22. Write a report on Cauchy integral formula. Include examples of complicated examples discussed in the literature. Resources include bibliographical items 13, 59, 107, 110, 118, 119, and 187.

6.6 The Theorems of Morera and Liouville and Some Applications

In this section we investigate some of the qualitative properties of analytic and harmonic functions. Our first result shows that the existence of an antiderivative for a continuous function is equivalent to the statement that the integral of f is independent of the path of integration. This result is stated in a form that will serve as a converse to the Cauchy-Goursat theorem.

Theorem 6.11 (Morera's Theorem) *Let f be a continuous function in a simply connected domain D. If*

$$\int_C f(z)\,dz = 0$$

for every closed contour in D, then f is analytic in D.

Proof Select a point z_0 in D and define $F(z)$ by the following integral:

$$F(z) = \int_{z_0}^{z} f(\xi)\,d\xi.$$

The function $F(z)$ is uniquely defined because if C_1 and C_2 are two contours in D, both with initial point z_0 and terminal point z, then $C = C_1 - C_2$ is a closed contour in D, and

$$0 = \int_C f(\xi)\,d\xi = \int_{C_1} f(\xi)\,d\xi - \int_{C_2} f(\xi)\,d\xi.$$

Since $f(z)$ is continuous, then if $\varepsilon > 0$, there exists a $\delta > 0$ such that $|\xi - z| < \delta$ implies that $|f(\xi) - f(z)| < \varepsilon$. Now we can use the identical steps to those in the proof of Theorem 6.6 to show that $F'(z) = f(z)$. Hence $F(z)$ is analytic on D, and Corollary 6.1 implies that $F'(z)$ and $F''(z)$ are also analytic. Therefore $f'(z) = F''(z)$ exists for all z in D, and we have proven that $f(z)$ is analytic on D.

Cauchy's integral formula shows how the value $f(z_0)$ can be represented by a certain contour integral. If we choose the contour of integration C to be a circle with center z_0, then we can show that the value $f(z_0)$ is the integral average of the values of $f(z)$ at points z on the circle C.

Theorem 6.12 (Gauss's Mean Value Theorem) *If f is analytic in a simply connected domain D that contains the circle C: $|z - z_0| = R$, then*

$$(1)\quad f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{i\theta})\,d\theta.$$

Proof The circle C can be given the parameterization

(2) $C: z(\theta) = z_0 + Re^{i\theta}$ and $dz = i\,Re^{i\theta}\,d\theta$ for $0 \le \theta \le 2\pi$.

We can use the parameterization (2) and Cauchy's integral formula to obtain

$$f(z_0) = \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(z_0 + Re^{i\theta})i\,Re^{i\theta}\,d\theta}{Re^{i\theta}} = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{i\theta})\,d\theta,$$

and Theorem 6.12 is proven.

We now prove an important result concerning the modulus of an analytic function.

Theorem 6.13 (Maximum Modulus Principle) *Let f be analytic and nonconstant in the domain D. Then $|f(z)|$ does* not *attain a maximum value at any point z_0 in D.*

Proof by Contradiction Assume the contrary, and suppose that there exists a point z_0 in D such that

(3) $|f(z)| \le |f(z_0)|$ holds for all z in D.

If C_0: $|z - z_0| = R$ is any circle contained in D, then we can use identity (1) and property (22) of Section 6.2 to obtain

(4) $$|f(z_0)| = \left|\frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta\right| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{i\theta})|\,d\theta \quad \text{for } 0 \le r \le R.$$

But in view of inequality (3), we can treat $|f(z)| = |f(z_0 + re^{i\theta})|$ as a real-valued function of the real variable θ and obtain

(5) $$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{i\theta})|\,d\theta \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|\,d\theta = |f(z_0)| \quad \text{for } 0 \le r \le R.$$

If we combine inequalities (4) and (5), the result is the equation

$$|f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{i\theta})|\,d\theta,$$

which can be written as

(6) $$\int_0^{2\pi} (|f(z_0)| - |f(z_0 + re^{i\theta})|)\,d\theta = 0 \quad \text{for } 0 \le r \le R.$$

A theorem from the calculus of real-valued functions states that if the integral of a nonnegative continuous function taken over an interval is zero, then that function must be identically zero. Since the integrand in equation (6) is a nonnegative real-valued function, we conclude that it is identically zero; that is,

(7) $|f(z_0)| = |f(z_0 + re^{i\theta})|$ for $0 \le r \le R$ and $0 \le \theta \le 2\pi$.

If the modulus of an analytic function is constant, then the results of Example 3.13 show that the function is constant. Therefore identity (7) implies that

(8) $f(z) = f(z_0)$ for all z in the disk D_0: $|z - z_0| \le R$.

Now let Z denote an arbitrary point in D, and let C be a contour in D that joins z_0 to Z. Let $2d$ denote the minimum distance from C to the boundary of D. Then we can find consecutive points $z_0, z_1, z_2, \ldots, z_n = Z$ along C with $|z_{k+1} - z_k| \le d$, such that the disks D_k: $|z - z_k| \le d$ for $k = 0, 1, \ldots, n$ are contained in D and cover C, as shown in Figure 6.34.

Since each disk D_k contains the center z_{k+1} of the next disk D_{k+1}, it follows that z_1 lies in D_0, and from equation (8) we see that $f(z_1) = f(z_0)$. Hence $|f(z)|$ also assumes its maximum value at z_1. An argument identical to the one given above will show that

(9) $f(z) = f(z_1) = f(z_0)$ for all z in the disk D_1.

We can proceed inductively and show that

(10) $f(z) = f(z_{k+1}) = f(z_k)$ for all z in the disk D_{k+1}.

By using equations (8), (9), and (10) it follows that $f(Z) = f(z_0)$. Therefore f is constant in D. With this contradiction the proof of the theorem is complete.

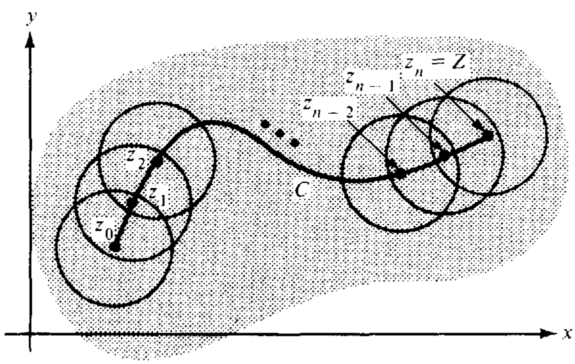


FIGURE 6.34 The "chain of disks" $D_0, D_1, \ldots, D_n$ that cover C.

The maximum modulus principle is sometimes stated in the following weaker form.

Theorem 6.13* (Maximum Modulus Principle) *Let f be analytic and nonconstant in the bounded domain D. If f is continuous on the closed region R that consists of D and all of its boundary points B, then $|f(z)|$ assumes its maximum value, and does so only at point(s) z_0 on the boundary B.*

EXAMPLE 6.28 Let $f(z) = az + b$, where the domain is the disk $D = \{z: |z| < 1\}$. Then f is continuous on the closed region $R = \{z: |z| \le 1\}$. Prove that

$$\max_{|z|\le 1} |f(z)| = |a| + |b|$$

and that this value is assumed by f at a point $z_0 = e^{i\theta_0}$ on the boundary of D.

Solution From the triangle inequality and the fact that $|z| \le 1$ it follows that

$$|f(z)| = |az + b| \le |az| + |b| \le |a| + |b|.$$

If we choose $z_0 = e^{i\theta_0}$, where $\theta_0 = \arg b - \arg a$, then

$$\arg az_0 = \arg a + (\arg b - \arg a) = \arg b,$$

so the vectors az_0 and b lie on the same ray through the origin. Hence $|az_0 + b| = |az_0| + |b| = |a| + |b|$, and the result is established.

Theorem 6.14 (Cauchy's Inequalities) *Let f be analytic in the simply connected domain D that contains the circle C: $|z - z_0| = R$. If $|f(z)| \le M$ holds for all points z on C, then*

$$(11) \quad |f^{(n)}(z_0)| \le \frac{n!M}{R^n} \quad \textit{for } n = 1, 2, \ldots .$$

Proof Let C have the parameterization

$$C: z(\theta) = z_0 + Re^{i\theta} \quad \text{and} \quad dz = i\,Re^{i\theta}\,d\theta \quad \text{for } 0 \le \theta \le 2\pi.$$

We can use Cauchy's integral formulae and write

$$(12) \quad f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z - z_0)^{n+1}} = \frac{n!}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + Re^{i\theta})i\,Re^{i\theta}\,d\theta}{R^{n+1}e^{i(n+1)\theta}}.$$

Using equation (12) and property (22) of Section 6.2, we obtain

$$|f^{(n)}(z_0)| \le \frac{n!}{2\pi R^n}\int_0^{2\pi} |f(z_0 + Re^{i\theta})|\,d\theta$$
$$\le \frac{n!}{2\pi R^n}\int_0^{2\pi} M\,d\theta = \frac{n!}{2\pi R^n} M2\pi = \frac{n!M}{R^n},$$

and Theorem 6.14 is established.

The next result shows that a nonconstant entire function cannot be a bounded function.

Theorem 6.15 (Liouville's Theorem) *If f is an entire function and is bounded for all values of z in the complex plane, then f is constant.*

Proof Suppose that $|f(z)| \le M$ holds for all values of z. Let z_0 denote an arbitrary point. Then we can use the circle C: $|z - z_0| = R$, and Cauchy's inequality with $n = 1$ implies that

$$(13) \quad |f'(z_0)| \le \frac{M}{R}.$$

If we let $R \to \infty$ in inequality (13), then we see that $f'(z_0) = 0$. Hence $f'(z) = 0$ for all z. If the derivative of an analytic function is zero for all z, then it must be a constant function. Therefore f is constant, and the theorem is proven.

EXAMPLE 6.29 The function $\sin z$ is *not* a bounded function.

Solution One way to see this is to observe that $\sin z$ is a nonconstant entire function, and therefore Liouville's theorem implies that $\sin z$ cannot be bounded. Another way is to investigate the behavior of real and imaginary parts of $\sin z$. If we fix $x = \pi/2$ and let $y \to \infty$, then we see that

$$\lim_{y\to+\infty} \sin\left(\frac{\pi}{2} + iy\right) = \lim_{y\to+\infty} \sin\frac{\pi}{2}\cosh y + i\cos\frac{\pi}{2}\sinh y$$
$$= \lim_{y\to+\infty} \cosh y = +\infty.$$

Liouville's theorem can be used to establish an important theorem of elementary algebra.

Theorem 6.16 (The Fundamental Theorem of Algebra) *If $P(z)$ is a polynomial of degree n, then P has at least one zero.*

Proof by Contradiction Assume the contrary and suppose that $P(z) \ne 0$ for all z. Then the function $f(z) = 1/P(z)$ is an entire function. We show that f is bounded as follows. First we write $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ and consider the equation

$$(14) \quad |f(z)| = \frac{1}{|P(z)|} = \frac{1}{|z|^n}\,\frac{1}{\left|a_n + \dfrac{a_{n-1}}{z} + \dfrac{a_{n-2}}{z^2} + \cdots + \dfrac{a_1}{z^{n-1}} + \dfrac{a_0}{z^n}\right|}.$$

Since $|a_k|/|z^{n-k}| = |a_k|/r^{n-k} \to 0$ as $|z| = r \to \infty$, it follows that

$$(15) \quad a_n + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \cdots + \frac{a_0}{z^n} \to a_n \quad \text{as } |z| \to \infty.$$

If we use statement (15) in equation (14), then the result is

$$|f(z)| \to 0 \quad \text{as } |z| \to \infty.$$

In particular, we can find a value of R such that

$$(16) \quad |f(z)| \le 1 \quad \text{for all } |z| \ge R.$$

Consider

$$|f(z)| = ([u(x, y)]^2 + [v(x, y)]^2)^{1/2},$$

which is a continuous function of the two real variables x and y. A result from calculus regarding real functions says that a continuous function on a closed and bounded set is bounded. Hence $|f(z)|$ is a bounded function on the closed disk

$$x^2 + y^2 \le R^2;$$

that is, there exists a positive real number K such that

$$(17) \quad |f(z)| \le K \quad \text{for all } |z| \le R.$$

Combining inequalities (16) and (17), it follows that $|f(z)| \le M = \max\{K, 1\}$ holds for all z. Liouville's theorem can now be used to conclude that f is constant. With this contradiction the proof of the theorem is complete.

Corollary 6.3 *Let P be a polynomial of degree n. Then P can be expressed as the product of linear factors. That is,*

$$P(z) = A(z - z_1)(z - z_2) \cdots (z - z_n)$$

where $z_1, z_2, \ldots, z_n$ are the zeros of P counted according to multiplicity and A is a constant.

EXERCISES FOR SECTION 6.6

For Exercises 1–4, express the given polynomial as a product of linear factors.

1. Factor $P(z) = z^4 + 4$.

2. Factor $P(z) = z^2 + (1 + i)z + 5i$.

3. Factor $P(z) = z^4 - 4z^3 + 6z^2 - 4z + 5$.

4. Factor $P(z) = z^3 - (3 + 3i)z^2 + (-1 + 6i)z + 3 - i$. *Hint*: Show that $P(i) = 0$.

5. Let $f(z) = az^n + b$, where the region is the disk $R = \{z: |z| \le 1\}$. Show that

$$\max_{|z| \le 1} |f(z)| = |a| + |b|.$$

6. Show that $\cos z$ is *not* a bounded function.

7. Let $f(z) = z^2$, where the region is the rectangle $R = \{z = x + iy: 2 \le x \le 3$ and $1 \le y \le 3\}$.
Find the following:
(a) $\max_R |f(z)|$ (b) $\min_R |f(z)|$
(c) $\max_R \text{Re}[f(z)]$ (d) $\min_R \text{Im}[f(z)]$
Hint for (*a*) and (*b*): $|z|$ is the distance from 0 to z.

8. Let $F(z) = \sin z$, where the region is the rectangle

$$R = \left\{z = x + iy: 0 \le x \le \frac{\pi}{2} \text{ and } 0 \le y \le 2\right\}.$$

Find $\max_R |f(z)|$. *Hint*: $|\sin z|^2 = \sin^2 x + \sinh^2 y$.

9. Let f be analytic in the disk $|z| < 5$, and suppose that $|f(\xi)| \le 10$ for values of ξ on the circle $|\xi - 1| = 3$. Find a bound for $|f^{(3)}(1)|$.

10. Let f be analytic in the disk $|z| < 5$, and suppose that $|f(\xi)| \le 10$ for values of ξ on the circle $|\xi - 1| = 3$. Find a bound for $|f^{(3)}(0)|$.

11. Let f be an entire function such that $|f(z)| \le M|z|$ holds for all z.
(a) Show that $f''(z) = 0$ for all z, and (b) conclude that $f(z) = az + b$.

12. Establish the following *minimum modulus principle*. Let f be analytic and nonconstant in the domain D. If $|f(z)| \ge m$, where $m > 0$ holds for all z in D, then $|f(z)|$ does *not* attain a minimum value at any point z_0 in D.

13. Let $u(x, y)$ be harmonic for all (x, y). Show that

$$u(x_0, y_0) = \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + R\cos\theta, y_0 + R\sin\theta)\, d\theta, \quad \text{where } R > 0.$$

Hint: Consider $f(z) = u(x, y) + iv(x, y)$.

14. Establish the following *maximum principle for harmonic functions*. Let $u(x, y)$ be harmonic and nonconstant in the simply connected domain D. Then u does not take on a maximum value at any point (x_0, y_0) in D. *Hint*: Let $f(z) = u(x, y) + iv(x, y)$ be analytic in D, and consider $F(z) = \exp[f(z)]$ where $|F(z)| = e^{u(x,y)}$.

15. Let f be an entire function that has the property $|f(z)| \ge 1$ for all z. Show that f is constant.

16. Let f be a nonconstant analytic function in the closed disk $R = \{z: |z| \le 1\}$. Suppose that $|f(z)| = K$ for all z on the circle $|z| = 1$. Show that f has a zero in D. *Hint*: Use both the maximum and minimum modulus principles.

17. Why is it important to study the fundamental theorem of algebra in a complex analysis course?

18. Look up the article on Morera's theorem and discuss what you found. Use bibliographical item 163.

19. Look up the article on Liouville's theorem and discuss what you found. Use bibliographical item 117.

20. Write a report on the fundamental theorem of algebra. Discuss ideas that you found in the literature that are not mentioned in the text. Resources include bibliographical items 6, 18, 29, 38, 60, 66, 150, 151, 170, and 184.

21. Write a report on zeros of polynomials and/or complex functions. Resources include bibliographical items 50, 65, 67, 102, 109, 120, 121, 122, 140, 152, 162, 171, 174, and 178.