COMPLEX NUMBERS

EXAMPLES & SOLUTIONS

Department of Mathematics & Philosophy

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COMPLEX NUMBERS:EXAMPLES & SOLUTIONS

Question (01)

(i) Find the real values of x and y such that $\frac{(1-i)x+2i}{3-i}+\frac{(2+3i)y+i}{3+i}=-i$

(ii) Find the real values of x and y are the complex numbers $3-ix^2y$ and $-x^2-y-4i$ conjugate of each other.

(iii) Find the square roots of $4+4i$

(iv) Find the complex number Z satisfying the equation $\left|\frac{Z-12}{Z-8i}\right|=\frac{5}{3}$ and $\left|\frac{Z-4}{Z-8}\right|=1$

(v) Find real θ such that $\frac{3+2i\,sin\theta}{1-2i\sin\theta}$ is

(a) real

(b) imaginary

Solution

(i) $$\frac{(1-i)x+2i}{3-i}+\frac{(2+3i)y+i}{3+i}=-i$$

$$\frac{\{(1-i)x+2i\}(3+i)+\{(2+3i)y+i\}(3-i)}{(3-i)(3+i)}=-i$$

$$\frac{(3+i)(1-i)x+6i+2i^2+(2+3i)(3-i)y+3i-i^2}{9-i^2}=-i$$

$$\frac{(3+i-3i-i^2)x+6i+(6+9i-2i-3i^2)y+3i+i^2}{9+1}=-i$$

$$(4-2i)x+(9-7i)y+9i-1=-10i$$

$$[4x+9y-1]+i(19-2x-7y)=0$$

$$\therefore\ 4x+9y-1=0 \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$19-2x-7y=0 \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

By solving equations (1) and (2)

$$x=\frac{-82}{5} \qquad y=\frac{37}{5}$$

(ii) $3-ix^2y=\overline{-x^2-y-4i}$

$$3-ix^2y=-(x^2+y)+4i$$

$$\therefore -(x^2+y)=3 \qquad -x^2y=4$$

$$\therefore x^2=-\frac{4}{y}$$

$$\frac{4}{y}-y=3$$

$$y^2+3y-4=0$$

$$(y+4)(y-1)=0$$

$y=-4$ or $y=1$

when $y=-4$ $\quad x^2=\frac{-4}{-4}=1 \quad \therefore x=\pm 1$

$y=1 \qquad x^2=\frac{-4}{1}=-4$ (Not real)

$\therefore x=\pm 1$ and $y=-4$

(iii) Let $z^2=4+4i$ and $z=(x+iy)$

$$\therefore z=\sqrt{4+4i} \qquad z^2=(x+iy)(x+iy)$$

$$=x^2+2ixy+i^2y^2$$

$$z^2=(x^2-y^2)+i2xy$$

$$(x^2 - y^2) + i2xy = 4 + 4i$$

$$\therefore x^2 - y^2 = 4 \qquad 2xy = 4$$

$$y = 2/x$$

$$x^2 - \left\{2/x\right\}^2 = 4$$

$$x^4 - 4x^2 + 4 = 0$$

$$(x^2 - 2)^2 = 0$$

$$\therefore x^2 = 2$$

$$x = \pm\sqrt{2}$$

$$\therefore y = 2/\pm\sqrt{2} = \pm\sqrt{2}$$

$z = \sqrt{2} + \sqrt{2}\,i$ or $z = -\sqrt{2} - \sqrt{2}\,i$

(iv) $\left|\dfrac{z-12}{z-8i}\right| = \dfrac{5}{3}$ ....(1)

$\left|\dfrac{z-4}{z-8}\right| = 1$ ....(2)

Let $z = x + iy$

$z - 12 = (x-12) + iy$ $\qquad |z-12| = \sqrt{(x-12)^2 + y^2}$

$z - 8i = x + i(y-8)$ $\qquad |z-8i| = \sqrt{x^2 + (y-8)^2}$

$z - 4 = (x-4) + iy$ $\qquad |z-4| = \sqrt{(x-4)^2 + y^2}$

$z - 8 = (x-8) + iy$ $\qquad |z-8| = \sqrt{(x-8)^2 + y^2}$

From (1); $\dfrac{|z-12|}{|z-8i|} = \dfrac{5}{3} \Leftrightarrow 3\sqrt{(x-12)^2 + y^2} = 5\sqrt{x^2 + (y-8)^2}$

$9\left\{(x-12)^2 + y^2\right\} = 25\left[x^2 + (y-8)^2\right]$ ....(1)

From (2) $\frac{|z-4|}{|z-8|}=1$

$\sqrt{(x-4)^2+y^2} = \sqrt{(x-8)^2+y^2}$

$(x-4)^2+y^2=(x-8)^2+y^2$

$x^2-8x+16+y^2 = x^2-16x+64+y^2$

$8x=48$

$\therefore x=6$

$\therefore$ form (1); $9\left[(6-12)^2+y^2\right]=25\left[6^2+(y-8)^2\right]$

$9\left[(36+y^2)\right]=25\left[36+(y-8)^2\right]$

$16y^2-400y+(2500-324)=0$

$16y^2-400y+2176=0$

$y^2-25y+136=0$

$(y-17)\quad(y-8)=0$

$y=17$ or $y=8$

$z=6+17i$ or $z=6+8i$

(v) $z=\frac{3+2i\sin\theta}{1-2i\sin\theta}=\frac{(3+2i\sin\theta)(1+2i\sin\theta)}{(1-2i\sin\theta)(1+2i\sin\theta)}$

$=\frac{(3-4\sin^2\theta)}{1+4\sin^2\theta}+\frac{8i\sin\theta}{1+4\sin^2\theta}$

If z is real $\therefore \frac{8\sin\theta}{1+4\sin^2\theta}=0 \qquad \theta=\pi/2$

$\theta=n\pi+(-1)^n\pi/2 \qquad n\in\mathbb{Z}$

If z is imaginary $\frac{3-4\sin^2\theta}{1+4\sin^2\theta}=0$

$$\sin^2\theta = \tfrac{3}{4}$$

$$\sin\theta = \pm\frac{\sqrt{3}}{2}$$

$$\theta = \pm\frac{\pi}{3}$$

$$\theta = n\pi \pm \frac{\pi}{3} \qquad n \in \mathbb{Z}$$

Question (02)

(i) Express the following complex numbers in the polar form

(a) $\left(\dfrac{3-i}{2+i}\right)^2$

(b) $\dfrac{(1-i)}{(1+i)(1+\sqrt{3}\,i)}$

(c) $\dfrac{1+7i}{(2-i)^2}$

(d) $0+0i$

(ii) Find the modulus and the principal value of the argument of the following

complex numbers

(a) $\dfrac{1-3i}{1-2i}$

(b) $\dfrac{1-i}{1+i} - \dfrac{1+i}{1-i}$

(c) $1+\cos\alpha + i\sin\alpha$

(d) $\dfrac{1-(1-i)^2}{1+2i}$

(iii)Find the modulus and the principal value of the argument of each of the following complex numbers

(a) $(1-\sqrt{3}i)(1-i)$

(b) $\dfrac{(-3+\sqrt{3}i)}{(4-4i)}$

(c) $(-3+\sqrt{3}i)(4-4i)$

(iii) Find the square roots of

(a) $5+12i$

(b) $15+8i$

(c) $24+10i$

Solution

(2)(a) $z=\left(\dfrac{3-i}{2+i}\right)^2=\left\{\dfrac{(3-i)(2-i)}{(2+i)(2-i)}\right\}^2=\left(\dfrac{6+i^2-5i}{5}\right)^2=(1-i)^2$

$$=\left[\sqrt{2}\left\{\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right\}\right]^2$$

$$=\left\{\sqrt{2}(\cos(-\pi/4))+i\sin(-\pi/4))\right\}^2$$

$$=(\sqrt{2})^2\left\{\cos(-\pi/4)+i\sin(-\pi/4)\right\}\left\{\cos(-\pi/4)+i\sin(-\pi/4)\right\}$$

$$=2\left[\cos(-\pi/4-\pi/4)+i\sin(-\pi/4-\pi/4)\right]$$

$$=2\left[\cos(-\pi/2)+i\sin(-\pi/2)\right]$$

(b) $z=\dfrac{1-i}{(1+i)(1+\sqrt{3}i)}=\dfrac{(1-i)(1-i)(1-\sqrt{3}i)}{(1+i)(1-i)(1+\sqrt{3}i)(1-\sqrt{3}i)}$

$$=\frac{\left[1+i^2-2i\right](1-\sqrt{3}\,i)}{(1-i^2)(1-3i^2)}=\frac{-2i(1-\sqrt{3}i)}{2.4}$$

$$=\frac{1}{4}(-\sqrt{3}-i)$$

$$=\frac{2}{4}\left[-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right]$$

$$=\frac{1}{2}\{\cos(-5\pi/6)+i\sin(-5\pi/6)\}$$

(c). $z=\frac{1+7i}{(2-i)^2}=\frac{1+7i}{2^2+i^2-4i}=\frac{1+7i}{(3-4i)}=\frac{(1+7i)(3+4i)}{(3-4i)(3+4i)}$

$$=\frac{3+28i^2+25i}{25}=\frac{-25+25i}{25}=(-1+i)$$

$$=\sqrt{2}\left[-1/\sqrt{2}+1/\sqrt{2}\,i\right]$$

$$=\sqrt{2}\left[\cos 3\pi/4+i\sin 3\pi/4\right]$$

(d) $z=0+0i$ is not possible to write in the form $z=r\{\cos\theta+i\sin\theta\}$

Where $r>0 \quad -\pi\le\theta\le\pi$

Since there is no θ value such that $\cos\theta=0$ and $\sin\theta=0$

$\therefore$ There is no argument for Zero Complex number.

(iii)(a) $z=\frac{1-3i}{1+2i}=\frac{(1-3i)}{(1+2i)}\frac{(1-2i)}{(1-2i)}=\frac{1+6i^2-5i}{1-4i^2}=\frac{-5-5i}{5}$

$$=-1-i=\sqrt{2}\left\{-1/\sqrt{2}+(-1/\sqrt{2})i\right\}$$

$$=\sqrt{2}\left[\cos(-3\pi/4)+i\sin(-3\pi/4)\right]$$

$\therefore |z|=\sqrt{2}$ and $\arg(z)=-3\pi/4$

(b) $z=\frac{1-i}{1+i}-\frac{(1+i)}{(1-i)}=\frac{(1-i)^2-(1+i)^2}{(1+i)(1-i)}=\frac{[1-i+1+i][1-i-1-i]}{1-i^2}$

$$=\frac{2(-2i)}{2}=-2i=2\left[\cos(-\pi/2)+i\sin(-\pi/2)\right]$$

$|z| = 2 \ \ \arg(z) = -\pi/2$

(c) $z = 1 + \cos\alpha + i\sin\alpha$

$= 2\cos^2(\alpha/2) + 2i\sin(\alpha/2)\cos(\alpha/2)$

$z = 2\cos(\alpha/2)\{\cos(\alpha/2) + i\sin(\alpha/2)\}$

$|z| = 2\cos\alpha/2 \qquad \arg(z) = \alpha/2$

(d). $z = \dfrac{1-(1-i)^2}{1+2i} = \dfrac{1-(1-2i+i^2)}{1+2i} = \dfrac{1+2i}{1+2i} = 1$

$= 1\{\cos 0 + i\sin 0\}$

$|z| = 1 \ \ \arg(z) = 0$

(iii)(a) $z = (1-\sqrt{3}i)(1-i)$

$= 2\left[\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i\right].\sqrt{2}\left[\dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}}i\right]$

$= 2\left[\cos(-\dfrac{\pi}{3}) + i\sin(-\dfrac{\pi}{3})\right].\sqrt{2}\left[\cos(-\dfrac{\pi}{4}) + i\sin(-\dfrac{\pi}{4})\right]$

$= 2\sqrt{2}\left[\cos(-(\pi/3 + \pi/4)) + \sin(-(\pi/3 + \pi/4))\right]$

$= 2\sqrt{2}\left[\cos(-7\pi/12) + i\sin(-7\pi/12)\right]$

$Arg(z) = -7\pi/12 \qquad |z| = 2\sqrt{2}$

Question (03)

(a)Given that the complex number Z and its conjugate $\overline{Z}$ satisfy the equation $Z\overline{Z}+2iZ=12+6i$ find the possible values of Z .

(b)If $Z=x+iy$ and $Z^2=a+ib$ where x,y,a,b are real,prove that $2x^2=\sqrt{a^2+b^2}+a$

By solving the equation $Z^4+6Z^2+25=0$ for Z^2 ,or otherwise express each of the four roots of the equation in the form $x+iy$.

Solution

(a). $z\overline{z}+2iz=12+6i$ Let $z=x+iy \quad x,y\in\mathbb{R}$

$\therefore \overline{z}=x-iy$

$z\overline{z}=(x^2+y^2)$

$x^2+y^2+2i(x+iy)=12+6i$

$\therefore (x^2+y^2-2y)+2xi=12+6i$

$\therefore x^2+y^2-2y=12$ and $2x=6$

$\therefore x=3$

$3^2+y^2-2y=12$

$y^2-2y-3=0$

$(y-3)(y+1)=0$

$\therefore y=3$ or $y=-1$

$z=3+3i$ or $z=3-i$

(b) $z=x+iy$

$z^2=(x+iy)^2=(x^2-y^2)+i2xy$

$z^2=a+ib$

$\therefore a = x^2 - y^2$ and $b = 2xy$

$$y = \frac{b}{2x}$$

$$a = x^2 - \left\{\frac{b}{2x}\right\}^2$$

$$a = x^2 - \frac{b^2}{4x^2}$$

$$\therefore\ 4x^4 - 4ax^2 - b^2 = 0$$

$$x^2 = \frac{4a \pm \sqrt{16a^2 + 16b^2}}{8}$$

$$x^2 = \frac{a \pm \sqrt{a^2 + b^2}}{2}$$

Since $x \in \mathbb{R}$ $\therefore\ x^2 \geq 0$

$$\therefore\ 2x^2 = a + \sqrt{a^2 + b^2}$$

$$z^4 + 6z^2 + 25 = 0$$

$$(z^2 + 3)^2 + 16 = 0$$

$$\left(z^2 + 3\right)^2 = (4i)^2$$

$\therefore$ $z^2 = -3 + 4i$ or $z^2 = -3 - 4i$

Let $z = x + iy$ and $z^2 = a + ib = -3 + 4i$

$$\therefore 2x^2 = \sqrt{(-3)^2 + 4^2} - 3 = 2$$

$x^2 = 1$ $\therefore x = \pm 1$ and $y = \dfrac{b}{2x} = \dfrac{-4}{\pm 2} = \pm 2$

$z = 1 - 2i$ or $z = (-1 + 2i)$

$\therefore$ The roots of the equation $z^4 + 6z^2 + 25 = 0$ are $1 + 2i$, $-1 + 2i$, $-1 - 2i$, $1 - 2i$

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