CLASS : XIII (VXYZ) **Dpp on Complex Number (After 1st Lecture)** **DPP. NO.- 1**

$Q.1 The sequence $S = i + 2i^2 + 3i^3 + ...$ upto 100 terms simplifies to where $i = \sqrt{-1}$:
(A*) $50(1 - i)$ (B) $25i$ (C) $25(1 + i)$ (D) $100(1 - i)$

$Q.2 If $z + z^3 = 0$ then which of the following must be true on the complex plane?
(A) $Re(z) < 0$ (B*) $Re(z) = 0$ (C) $Im(z) = 0$ (D) $z^4 = 1$
[Hint: $z(1 + z^2) = 0 \Rightarrow z = 0$ or $z^2 = i^2 \Rightarrow z = 0$ or $z = \pm i \Rightarrow Re(z) = 0$] **[13th (25-9-2005)]**

Q.3 Number of integral values of n for which the quantity $(n + i)^4$ where $i^2 = -1$, is an integer is
(A) 1 (B) 2 (C*) 3 (D) 4
[Sol. $(n + i)^4 = n^4 + 4n^3i + 6n^2i^2 + 4ni^3 + i^4$ **[12th, 06-01-2008]**
$= n^4 - 6n^2 + 1 + i(4n^3 - 4n)$
for this to be integer
$4n^3 - 4n = 4n(n^2 - 1)$ must be zero
$\Rightarrow$ $n = 0$ or $n = \pm 1$ $\Rightarrow$ 3 values $\Rightarrow$ (C)]

$Q.4 Let $i = \sqrt{-1}$. The product of the real part of the roots of $z^2 - z = 5 - 5i$ is
(A) -25 (B*) -6 (C) -5 (D) 25
[Hint: roots are $3 - i$ and $-2 + i$ $\Rightarrow$ -6]

Q.5 There is only one way to choose real numbers M and N such that when the polynomial $5x^4 + 4x^3 + 3x^2 + Mx + N$ is divided by the polynomial $x^2 + 1$, the remainder is 0. If M and N assume these unique values, then M – N is
(A) -6 (B) -2 (C*) 6 (D) 2
[Sol. Let $P(x) = 5x^4 + 4x^3 + 3x^2 + Mx + N$ **[12th & 13th 15-10-2006]**
let $Q(x) = x^2 + 1$
if the quotient is Q
then $P(x) = Q(x^2 + 1)$
if $x = i$ then $P(i) = 0$
if $x = -i$ then $P(-i) = 0$
hence $5 - 4i - 3 + Mi + N = 0$
hence $N + Mi = -2 + 4i$
$\therefore$ $N = -2$; $M = 4$
$\therefore$ $M - N = 6$ Ans.]

$Q.6 In the quadratic equation $x^2 + (p + iq)x + 3i = 0$, p & q are real. If the sum of the squares of the roots is 8 then
(A) $p = 3, q = -1$ (B) $p = -3, q = -1$ (C*) $p = \pm 3, q = \pm 1$ (D) $p = -3, q = 1$
[Hint: $\alpha + \beta = -(p + iq)$; $\alpha\beta = 3i$
Given: $\alpha^2 + \beta^2 = 8$
$(\alpha + \beta)^2 - 2\alpha\beta = 8$
$(p + iq)^2 - 6i = 8$
$p^2 - q^2 + i(2pq - 6) = 8 \Rightarrow p^2 - q^2 = 8$ and $pq = 3$
$\Rightarrow$ $p = 3$ & $q = 1$ or $p = -3$ and $q = 1$]

Q.7 The complex number z satisfying $z + |z| = 1 + 7i$ then the value of $|z|^2$ equals
(A*) 625 (B) 169 (C) 49 (D) 25

[Sol. $z = x + iy$ **[11th, 16-11-2008, P-2]**

$\therefore \quad x + iy + \sqrt{x^2 + y^2} = 1 + 7i$

$x + \sqrt{x^2 + y^2} = 1 \quad \text{... (1)}$

and $y = 7 \quad \text{... (2)}$

$\therefore \quad x + \sqrt{x^2 + 49} = 1$

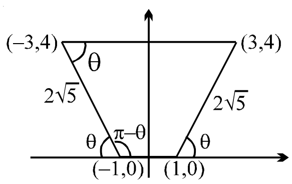
$x^2 + 49 = 1 + x^2 - 2x$

$2x = -48$

$x = -24$

$\therefore \quad |z|^2 = x^2 + y^2 = 625$ **Ans.**]

Q.8 The figure formed by four points $1 + 0i$; $-1 + 0i$; $3 + 4i$ & $\dfrac{25}{-3-4i}$ on the argand plane is :
(A) a parallelogram but not a rectangle (B) a trapezium which is not equilateral
(C*) a cyclic quadrilateral (D) none of these

[Hint: 

; Note that opposite angles are supplementary]

Q.9 If $z = (3 + 7i)(p + iq)$ where $p, q \in I - \{0\}$, is purely imaginary then minimum value of $|z|^2$ is
(A) 0 (B) 58 (C) $\dfrac{3364}{3}$ (D*) 3364

[Hint: $z = (3p - 7q) + i(3q + 7p)$
for purely imaginary $3p = 7q \Rightarrow p = 7$ or $q = 3$ (for least value)
$|z| = |3 + 7i|\,|p + iq| \Rightarrow |z|^2 = 58(p^2 + q^2) = 58[7^2 + 9] = 58^2 \Rightarrow$ (D)]

Q.10 Number of values of z (real or complex) simultaneously satisfying the system of equations
$1 + z + z^2 + z^3 + \dots\dots + z^{17} = 0$ and $1 + z + z^2 + z^3 + \dots\dots + z^{13} = 0$ is
(A*) 1 (B) 2 (C) 3 (D) 4

[Sol. $1 - z^{18} = 0$; $1 - z^{14} = 0 \Rightarrow z^{14} = 1$ or $z^{18} = 1$
since one is extraneous root $z = -1$ is the common root.]

$Q.11 If $\dfrac{x-3}{3+i} + \dfrac{y-3}{3-i} = i$ where $x, y \in R$ then
(A) $x = 2$ & $y = -8$ (B*) $x = -2$ & $y = 8$ (C) $x = -2$ & $y = -6$ (D) $x = 2$ & $y = 8$

Q.12 Number of complex numbers z satisfying $z^3 = \bar{z}$ is
(A) 1 (B) 2 (C) 4 (D*) 5

[Sol. $z = 0$; $z = \pm 1$; $z = \pm i$;

$z^3 = \bar{z} \Rightarrow |z|^3 = |\bar{z}| = |z|$

note that $z^n = |\bar{z}|$ has n + 2 solutions

hence $|z| = 0$ or $|z|^2 = 1$

again $z^4 = z\bar{z} = |z|^2 = 1 \Rightarrow z^4 = 1 \Rightarrow$ total number of roots are 5

Note that the equation $z^n = \bar{z}$ will have (n + 2) solutions.]

Q.13 If $x = 9^{1/3}\ 9^{1/9}\ 9^{1/27}$ ......ad inf
$y = 4^{1/3}\ 4^{-1/9}\ 4^{1/27}$ ...... ad inf and $z = \sum_{r=1}^{\infty} (1+i)^{-r}$
then , the argument of the complex number $w = x + yz$ is

(A) 0 (B) $\pi - \tan^{-1}\left(\frac{\sqrt{2}}{3}\right)$ (C*) $-\tan^{-1}\left(\frac{\sqrt{2}}{3}\right)$ (D) $-\tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$

[Sol. $x = 9^{\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+.....} = 9^{\frac{\frac{1}{3}}{1-\frac{1}{3}}} = 9^{\frac{1}{2}} = 3$

$y = 4^{\frac{1}{3}-\frac{1}{9}+\frac{1}{27}+.....} = 4^{\frac{\frac{1}{3}}{1+\frac{1}{3}}} = 4^{\frac{1}{4}} = \sqrt{2}$

$$z = \sum_{r=1}^{\infty}(1+i)^{-r} = \frac{1}{1+i} + \frac{1}{(1+i)^2} + \frac{1}{(1+i)^3} + ... = \frac{\frac{1}{1+i}}{1-\frac{1}{1+i}} = \frac{1}{i} = -i$$

Let $\omega = x + yz = 3 - \sqrt{2}\,i$ (4th quad.) $\Rightarrow$ Arg $\omega = -\tan^{-1}\left(\frac{\sqrt{2}}{3}\right)$ $\Rightarrow$ (C)]

$Q.14 Let $z = 9 + bi$ where b is non zero real and $i^2 = -1$. If the imaginary part of z^2 and z^3 are equal, then b^2 equals
(A) 261 (B*) 225 (C) 125 (D) 361

[Sol. $z^2 = 81 - b^2 + 18bi$ **[13th, 05-08-2007]** **[1st dpp of complex no.]**
$z^3 = 729 + 243bi - 27b^2 - b^3i$
hence $243b - b^3 = 18b$ and
$243 - b^2 = 18$
$b^2 = 225$ Ans.]

One or more than one is/are correct:

$Q.15 If the expression $(1 + ir)^3$ is of the form of $s(1 + i)$ for some real 's' where 'r' is also real and $i = \sqrt{-1}$, then the value of 'r' can be

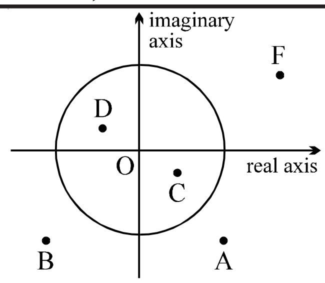
(A) $\cot\frac{\pi}{8}$ (B*) $\sec \pi$ (C*) $\tan\frac{\pi}{12}$ (D*) $\tan\frac{5\pi}{12}$

[Sol. We have $(1 + ri)^3 = s(1 + i)$ **[13th, 04-10-2009, P-1]**
$1 + 3ri + 3r^2i^2 + r^3i^3 = s(1 + i)$ **[12th, 22-06-2008]**
$1 - 3r^2 + i(3r - r^3) = s + si$ $\Rightarrow$ $1 - 3r^2 = s = 3r - r^3$
Hence $1 - 3r^2 = 3r - r^3$
$\Rightarrow$ $r^3 - 3r^2 - 3r + 1 = 0$ $\Rightarrow$ $(r^3 + 1) - 3r(r + 1) = 0$ $\Rightarrow$ $(r + 1)(r^2 + 1 - r - 3r) = 0$
$\therefore$ $r = -1$ or $r^2 - 4r + 1 = 0$

$\Rightarrow$ $r = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm 2\sqrt{3}}{2}$ $\Rightarrow$ $r = 2+\sqrt{3}$ or $2-\sqrt{3}$ $\Rightarrow$ **B, C, D**]

CLASS : XIII (VXYZ) **Dpp on Complex Number (After 2nd Lecture)** ***DPP. NO.-2***

$Q.1 The digram shows several numbers in the complex plane. The circle is the unit circle centered at the origin. One of these numbers is the reciprocal of F, which is



(A) A (B) B
(C*) C (D) D

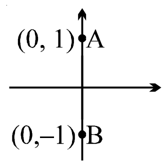
[Sol. Let F as a + bi, $a, b \in R$
where we see from the diagram that a, b > 0 and $a^2 + b^2 > 1$ (as F lies outside the unit circle)

Since $\alpha = \dfrac{1}{a+bi} = \dfrac{a-bi}{a^2+b^2} = \dfrac{a}{a^2+b^2} - \dfrac{b}{a^2+b^2}\,i$,
(real part + ve and imaginary part – ve and both less than unity)
we see that the reciprocal of F is in IV quadrant, since the real part is positive and the imaginary part is negative. Also, the magnitude of the reciprocal is

$$\frac{1}{a^2+b^2}\sqrt{a^2+(-b)^2} = \frac{1}{\sqrt{a^2+b^2}} < 1$$

Thus the only possibility is point C.] **[19-2-2006, 12th & 13th]**

$Q.2 If $z = x + iy$ & $\omega = \dfrac{1-iz}{z-i}$ then $|\omega| = 1$ implies that, in the complex plane
(A) z lies on the imaginary axis (B*) z lies on the real axis
(C) z lies on the unit circle (D) none



[Sol. $w = \dfrac{-i(z+i)}{z-i}$; $|w| = \left|\dfrac{z+i}{z-i}\right| = 1$ $\Rightarrow$ $|z+i| = |z-i|$
$\Rightarrow$ z lies on the perpendicular bisector of the segment joining (0, 1) and (0, – 1) which is x-axis
$\Rightarrow$ z lies on x-axis
$\Rightarrow$ Im (z) is real]

Q.3 On the complex plane locus of a point z satisfying the inequality
$2 \le |z - 1| < 3$ denotes
(A) region between the concentric circles of radii 3 and 1 centered at (1, 0)
(B) region between the concentric circles of radii 3 and 2 centered at (1, 0) excluding the inner and outer boundaries.
(C) region between the concentric circles of radii 3 and 2 centered at (1, 0) including the inner and outer boundaries.
(D*) region between the concentric circles of radii 3 and 2 centered at (1, 0) including the inner boundary and excluding the outer boundary. **[12th test (09-10-2005)]**

$Q.4 The complex number z satisfies $z + |z| = 2 + 8i$. The value of $|z|$ is
(A) 10 (B) 13 (C*) 17 (D) 23

[Sol. Let $z = a + bi$.
$|z|^2 = a^2 + b^2$.
So, $z + |z| = 2 + 8i$
$a + bi + \sqrt{a^2+b^2} = 2 + 8i$

$a + \sqrt{a^2 + b^2} = 2$, $b = 8$

$a + \sqrt{a^2 + 64} = 2$

$a^2 + 64 = (2 - a)^2 = a^2 - 4a + 4$,

$4a = -60$, $a = -15$. Thus, $a^2 + b^2 = 225 + 64 = 289$

$\therefore$ $|z| = \sqrt{a^2 + b^2} = \sqrt{289} = 17$ Ans.]

Q.5 Let $Z_1 = (8 + i)\sin\theta + (7 + 4i)\cos\theta$ and $Z_2 = (1 + 8i)\sin\theta + (4 + 7i)\cos\theta$ are two complex numbers. If $Z_1 \cdot Z_2 = a + ib$ where $a, b \in R$ then the largest value of $(a + b)$ $\forall$ $\theta \in R$, is

(A) 75 (B) 100 (C*) 125 (D) 130

[Sol. $Z_1 = (8\sin\theta + 7\cos\theta) + i(\sin\theta + 4\cos\theta)$ **[13th, 10-08-2008, P-1]**

$Z_2 = (\sin\theta + 4\cos\theta) + i(8\sin\theta + 4\cos\theta)$

hence $\left.\begin{matrix} Z_1 = x + iy \\ Z_2 = y + ix \end{matrix}\right]$ where $x = (8\sin\theta + 7\cos\theta)$ and $y = (\sin\theta + 4\cos\theta)$

$Z_1 \cdot Z_2 = (xy - xy) + i(x^2 + y^2) = 0 \Rightarrow a = 0;\ b = x^2 + y^2$

now, $x^2 + y^2 = (8\sin\theta + 7\cos\theta)^2 + (\sin\theta + 4\cos\theta)^2$

$= 65\sin^2\theta + 65\cos^2\theta + 120\sin\theta \cdot \cos\theta$

$= 65 + 60\sin 2\theta$

hence $Z_1 \cdot Z_2\big|_{max} = 125$ **Ans.**]

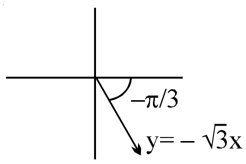
Q.6 The locus of z, for arg $z = -\pi/3$ is

(A) same as the locus of z for arg $z = 2\pi/3$

(B) same as the locus of z for arg $z = \pi/3$

(C*) the part of the straight line $\sqrt{3}\,x + y = 0$ with $(y < 0, x > 0)$

(D) the part of the straight line $\sqrt{3}\,x + y = 0$ with $(y > 0, x < 0)$

[Hint: 

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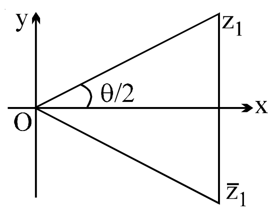
Q.7 If z_1 & $\bar{z}_1$ represent adjacent vertices of a regular polygon of n sides with centre at the origin & if $\dfrac{\text{Im}\, z_1}{\text{Re}\, z_1} = \sqrt{2} - 1$ then the value of n is equal to :

(A*) 8 (B) 12 (C) 16 (D) 24

[Hint: $\dfrac{y}{x} = \tan\dfrac{\theta}{2} = \sqrt{2} - 1 = \tan\dfrac{\pi}{8}$

$= \dfrac{\theta}{2} = \dfrac{\pi}{8} \Rightarrow \theta = 45° \Rightarrow n = \dfrac{360°}{45°} = 8$

if $\dfrac{y}{x} = 2 - \sqrt{3} \Rightarrow n = 12$]



$Q.8 If z_1, z_2 are two complex numbers & a, b are two real numbers then, $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 =$

(A) $(a+b)^2\left[|z_1|^2+|z_2|^2\right]$ (B) $(a+b)\left[|z_1|^2+|z_2|^2\right]$

(C) $\left(a^2-b^2\right)\left[|z_1|^2+|z_2|^2\right]$ (D*) $\left(a^2+b^2\right)\left[|z_1|^2+|z_2|^2\right]$

Q.9$_{15complex}$ The value of $e\left(\text{CiS}(-i)-\text{CiS}(i)\right)$ is equal to

(A) 0 (B) $1 - e$ (C) $e - \frac{1}{e}$ (D*) $e^2 - 1$

[Sol. using $\text{CiS}\ \theta = e^{i\theta}$ **[13th, 25-01-2009]**

$E = e\left[e^{-i^2} - e^{i^2}\right] = e[e - e^{-1}] = e^2 - 1$ **Ans.**]

&Q.10 All real numbers *x* which satisfy the inequality $\left|1+4i-2^{-x}\right| \le 5$ where $i = \sqrt{-1}$, $x \in R$ are

(A*) $[-2, \infty)$ (B) $(-\infty, 2]$ (C) $[0, \infty)$ (D) $[-2, 0]$

[12th test (29-10-2005)]

[Sol. $(1 - 2^{-x})^2 + 16 \le 25$; $(1 - 2^{-x}) - 3^2 \le 0$; $(4 - 2^{-x})(-2 - 2^{-x}) \le 0$
$(2^{-x} - 4)(2^{-x} + 2) \ge 0$]

Q.11 For $Z_1 = \sqrt[6]{\frac{1-i}{1+i\sqrt{3}}}$; $Z_2 = \sqrt[6]{\frac{1-i}{\sqrt{3}+i}}$; $Z_3 = \sqrt[6]{\frac{1+i}{\sqrt{3}-i}}$ which of the following holds good?

(A) $\sum |Z_1|^2 = \frac{3}{2}$ (B*) $|Z_1|^4 + |Z_2|^4 = |Z_3|^{-8}$

(C) $\sum |Z_1|^3 + |Z_2|^3 = |Z_3|^{-6}$ (D) $|Z_1|^4 + |Z_2|^4 = |Z_3|^8$

[Hint: $|z_1| = \left|\frac{1-i}{1+i\sqrt{3}}\right|^{\frac{1}{6}} = \left|\frac{\sqrt{2}}{2}\right|^{\frac{1}{6}} = 2^{-\frac{1}{12}}$

|||ly $|z_2| = 2^{-\frac{1}{12}}$; $|z_3| = 2^{-\frac{1}{12}}$ hence the result]

Q.12 Number of real or purely imaginary solution of the equation, $z^3 + iz - 1 = 0$ is :

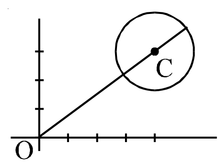
(A*) zero (B) one (C) two (D) three

[Hint: Let x be the real solution .
$\Rightarrow$ $x^3 - 1 + xi = 0$ $\Rightarrow$ $x^3 - 1 = 0$ & $x = 0$ which is not possible
note that the equation has no purely imaginary root as well.]

$Q.13 A point 'z' moves on the curve $|z - 4 - 3i| = 2$ in an argand plane. The maximum and minimum values of $|z|$ are

(A) 2, 1 (B) 6, 5 (C) 4, 3 (D*) 7, 3

[Sol. $|(x - 4) + i(y - 3)| = 2$
circle with centre (4, 3) and radius 2 ;
Hence OC = 5
$|z|_{max} = 5 + 2 = 7$
$|z|_{min} = 5 - 2 = 3$]

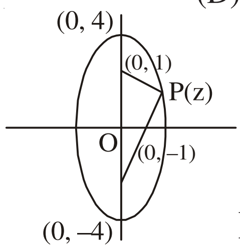


Q.14 If z is a complex number satisfying the equation $|z+i|+|z-i|=8$, on the complex plane then maximum value of $|z|$ is

(A) 2 (B*) 4 (C) 6 (D) 8

[Sol. If $|z+i|+|z-i|=8$,

[12th, 04-01-2008]



$PF_1 + PF_2 = 8$

$\therefore \quad |z|_{max} = 4 \quad \Rightarrow$ **(B)**]

Q.15 Let $z_r\ (1 \le r \le 4)$ be complex numbers such that $|z_r| = \sqrt{r+1}$
and $|30\, z_1 + 20\, z_2 + 15\, z_3 + 12\, z_4| = k\, |z_1 z_2 z_3 + z_2 z_3 z_4 + z_3 z_4 z_1 + z_4 z_1 z_2|$.
Then the value of k equals

(A) $|z_1 z_2 z_3|$ (B) $|z_2 z_3 z_4|$ (C) $|z_3 z_4 z_1|$ (D*) $|z_4 z_1 z_2|$

[Sol. We have $\left|\frac{z_1}{2}+\frac{z_2}{3}+\frac{z_3}{4}+\frac{z_4}{5}\right| = \frac{k}{60}|z_1 z_2 z_3 z_4|\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}+\frac{1}{z_4}\right|$ **[12th, 06-12-2009, P-2]**

Now, $z_1\,\bar{z}_1 = 2$, $z_2\,\bar{z}_2 = 3$, $z_3\,\bar{z}_3 = 4$ and $z_4\,\bar{z}_4 = 5$

So, $k = \frac{60}{|z_1 z_2 z_3 z_4|} = \frac{60}{\sqrt{2}\sqrt{3}\sqrt{4}\sqrt{5}} = \sqrt{30} = |z_4\, z_1\, z_2|$ **Ans.**

Note for objective take $z_1 = \sqrt{2}$; $z_2 = \sqrt{3}$; $z_3 = 2$; $z_4 = \sqrt{5}$]

CLASS : XIII (VXYZ) **Dpp on Complex Number (After 3rd Lecture)** *DPP. NO.-3*

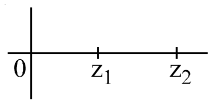
Q.1 If z_1 & z_2 are two non-zero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then Arg z_1 – Arg z_2 is equal to:

(A) $-\pi$ (B) $-\pi/2$ (C*) 0 (D) $\pi/2$

[Hint: $|z_1 + z_2| = |z| + |z_2|$

$\Rightarrow \sqrt{(r_1\cos\theta_1 + r_2\cos\theta_2) + i(r_1\sin\theta_1 + r_2\sin\theta_2)} = r_1 + r_2$

$\sqrt{r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)} = r_1 + r_2$



This is possible only if $\theta_1 = \theta_2$

$\Rightarrow$ 0, z_1 and z_2 are collinear with z_1 and z_2 on the same side of the origin

$\Rightarrow$ Arg z_1 = Arg z_2]

Q.2 Let Z be a complex number satisfying the equation

$(Z^3 + 3)^2 = -16$ then $|Z|$ has the value equal to

(A) $5^{1/2}$ (B*) $5^{1/3}$ (C) $5^{2/3}$ (D) 5

[Sol. $(Z^3 + 3)^2 = 16i^2$

$Z^3 + 3 = 4i$ or $-4i$

$Z^3 = -3 + 4i$ or $-3 - 4i$

$|Z|^3 = |-3 + 4i| = 5$

$|Z|^3 = 5 \Rightarrow |Z| = 5^{1/3}$]

Q.3 If z_1, z_2, z_3 are 3 distinct complex numbers such that $\frac{3}{|z_2 - z_3|} = \frac{4}{|z_3 - z_1|} = \frac{5}{|z_1 - z_2|}$,

then the value of $\frac{9}{z_2 - z_3} + \frac{16}{z_3 - z_1} + \frac{25}{z_1 - z_2}$ equals

(A*) 0 (B) 3 (C) 4 (D) 5

[Sol. We have $\frac{3}{|z_2 - z_3|} = \frac{4}{|z_3 - z_1|} = \frac{5}{|z_1 - z_2|} = k$ (let) **[12th, 20-12-2009, complex]**

$\Rightarrow \frac{9}{|z_2 - z_3|^2} = \frac{16}{|z_3 - z_1|^2} = \frac{25}{|z_1 - z_2|^2} = k^2$

Now $\frac{9}{|z_2 - z_3|^2} = k^2 \Rightarrow \frac{9}{z_2 - z_3} = k^2(\bar{z}_2 - \bar{z}_3)$ ....(1) [As $|z|^2 = z\bar{z}$]

Illly $\frac{16}{|z_3 - z_1|^2} = k^2 \Rightarrow \frac{16}{z_3 - z_1} = k^2(\bar{z}_3 - \bar{z}_1)$ ....(2)

Illly $\frac{25}{|z_1 - z_2|^2} = k^2 \Rightarrow \frac{25}{z_1 - z_2} = k^2(\bar{z}_1 - \bar{z}_2)$ ....(3)

$\therefore$ On adding (1), (2) and (3), we get

$\frac{9}{z_2 - z_3} + \frac{16}{z_3 - z_1} + \frac{25}{z_1 - z_2} = k^2(\bar{z}_2 - \bar{z}_3 + \bar{z}_3 - \bar{z}_1 + \bar{z}_1 - \bar{z}_2) = 0$ **Ans.**]

\$Q.4 The points representing the complex number z for which $|z+5|^2 - |z-5|^2 = 10$ lie on

(A*) a straight line (B) a circle

(C) a parabola (D) the bisector of the line joining $(5,0)$ & $(-5,0)$

[Hint: $(z+5)(\bar{z}+5) - (z-5)(\bar{z}-5) = 10$ or $5(z+\bar{z}) + 25 + 5(z+\bar{z}) - 25 = 10$

$2 \cdot 2x = 2 \Rightarrow x = \frac{1}{2} \Rightarrow$ (A)]

Q.5 If $x = \frac{1+\sqrt{3}\,i}{2}$ then the value of the expression, $y = x^4 - x^2 + 6x - 4$, equals

(A*) $-1 + 2\sqrt{3}\,i$ (B) $2 - 2\sqrt{3}\,i$ (C) $2 + 2\sqrt{3}\,i$ (D) none

[Sol. $x = \frac{1+\sqrt{3}\,i}{2} = -\omega^2$ **[12th & 13th 03-12-2006]**

$\therefore \quad y = \omega^8 - \omega^4 - 6\omega^2 - 4 = \omega^2 - \omega - 6\omega^2 - 4 = 5\omega^2 - \omega - 4$

$= \underbrace{-1-\omega-\omega^2}_{\text{zero}} - 4\omega^2 - 3 = +4\left(\frac{1+i\sqrt{3}}{2}\right) - 3 = 2(1 - i\sqrt{3}) - 3 = -1 + 2\sqrt{3}\,i$ Ans.]

\$Q.6 Consider two complex numbers α and β as

$\alpha = \left(\frac{a+bi}{a-bi}\right)^2 + \left(\frac{a-bi}{a+bi}\right)^2$, where $a, b \in R$ and $\beta = \frac{z-1}{z+1}$, where $|z| = 1$, then

(A) Both α and β are purely real (B) Both α and β are purely imaginary

(C*) α is purely real and β is purely imaginary (D) β is purely real and α is purely imaginary

[Hint: Note that $\alpha = \bar{\alpha} \Rightarrow \alpha$ is real **[12th test (29-10-2005)]**

and $\beta + \bar{\beta} = \frac{z-1}{z+1} + \frac{\bar{z}-1}{\bar{z}+1} = \frac{(z-1)(\bar{z}+1)+(z+1)(\bar{z}-1)}{(z+1)(\bar{z}+1)} = \frac{2z\bar{z}-2}{D^r} = 0$

as $z\bar{z} = |z|^2 = 1$ (given)]

Q.7 Let Z is complex satisfying the equation

$z^2 - (3+i)z + m + 2i = 0$, where $m \in R$. Suppose the equation has a real root.

The additive inverse of non real root, is

(A) $1 - i$ (B) $1 + i$ (C*) $-1 - i$ (D) -2

[Sol. Let α be the real root **[12th test (29-10-2005)]**

$\alpha^2 - (3+i)\alpha + m + 2i = 0$

$(\alpha^2 - 3\alpha + m) + i(2 - \alpha) = 0$

$\therefore \alpha = 2$ (real root)

$\therefore \quad 4 - 6 + m = 0 \Rightarrow m = 2$

Product of the roots $= 2(1+i)$ with one root as 2

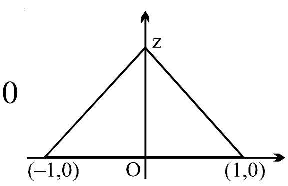
non real root $= 1 + i$, addivitve inverse is $-1 - i$ **Ans**]

Q.8 The minimum value of $|1+z| + |1-z|$ where z is a complex number is :

(A*) 2 (B) 3/2 (C) 1 (D) 0

[Hint : distance of z (1,0) & (–1, 0), will be minimum with z is at 'O'

$y \le |z| + 1 + |z| + 1 = 2 + 2|z| = 2$ where $z = 0$]



Q.9 If $i = \sqrt{-1}$, then $4 + 5\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{334} + 3\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{365}$ is equal to

(A) $1 - i\sqrt{3}$ (B) $-1 + i\sqrt{3}$ (C*) $i\sqrt{3}$ (D) $-i\sqrt{3}$

[JEE '99, 2 out of 200]

Q.10 Let $|z - 5 + 12i| \le 1$ and the least and greatest values of $|z|$ are m and n and if l be the least positive value of $\frac{x^2 + 24x + 1}{x}$ $(x > 0)$, then l is

(A) $\frac{m+n}{2}$ (B*) $m + n$ (C) m (D) n

[Hint: $|z|_{least} = 13 - 1 = 12 = m$; $|z|_{greatest} = 13 + 1 = 14 = n$

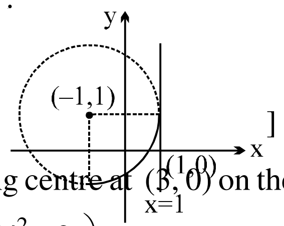
also $l = x + \frac{1}{x} + 24$; $\therefore$ $l = 26$; Hence $l = m + n$]

Q.11 The system of equations $\left.\begin{array}{ll} |z + 1 - i| & = 2 \\ \text{Re } z & \ge 1 \end{array}\right\}$ where z is a complex number has :

(A) no solution (B*) exactly one solution

(C) two distinct solutions (D) infinite solution

[Hint: $z = 1 + i$ only satisfies both]



Q.12 Let C_1 and C_2 are concentric circles of radius 1 and 8/3 respectively having centre at (3, 0) on the argand plane. If the complex number z satisfies the inequality, $\log_{1/3}\left(\frac{|z-3|^2 + 2}{11|z-3| - 2}\right) > 1$ then :

(A*) z lies outside C_1 but inside C_2 (B) z lies inside of both C_1 and C_2

(C) z lies outside both of C_1 and C_2 (D) none of these

[Hint: note that $11|z - 3| - 2 > 0$

$\frac{|z-3|^2 + 2}{11|z-3| - 2} < \frac{1}{3}$; put $|z - 3| = t$ $\Rightarrow$ $(3t - 8)(t - 1) < 0$ $\Rightarrow 1 < |z - 3| < 8/3$

$\Rightarrow$ z lies between the two concentric circles]

Q.13 Identify the incorrect statement.

(A) no non zero complex number z satisfies the equation, $\bar{z} = -4z$

(B) $\bar{z} = z$ implies that z is purely real

(C) $\bar{z} = -z$ implies that z is purely imaginary

(D*) if z_1, z_2 are the roots of the quadratic equation $az^2 + bz + c = 0$ such that $\text{Im}(z_1 z_2) \ne 0$ then a, b, c must be real numbers .

[Hint : (D) If $\text{Im}(z_1 z_2) \ne 0 \Rightarrow z_1$ and z_2 are not conjugates of each other. A quadratic equation having complex roots will have real co-efficients if and only if the roots are conjugates of each other $\Rightarrow$ False]

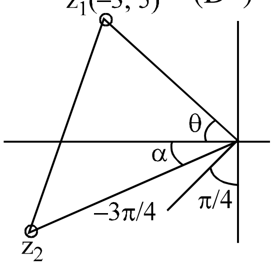
Q.14 The equation of the radical axis of the two circles represented by the equations, $|z - 2| = 3$ and $|z - 2 - 3i| = 4$ on the complex plane is :

(A) $3y + 1 = 0$ (B*) $3y - 1 = 0$ (C) $2y - 1 = 0$ (D) none

[Hint : square both the sides , use $z\bar{z} = |z|^2$ and subtract]

Q.15 If $z_1 = -3 + 5i$; $z_2 = {}^{-}5 - 3i$ and z is a complex number lying on the line segment joining z_1 & z_2 then arg z can be :

(A) $-\frac{3\pi}{4}$ (B) $-\frac{\pi}{4}$ (C) $\frac{\pi}{6}$ (D*) $\frac{5\pi}{6}$



[Hint: $\tan\theta = \frac{5}{3} \Rightarrow \theta > \frac{\pi}{4}$

$\tan\alpha = \frac{3}{5} \Rightarrow \alpha < \frac{\pi}{4}$

$\Rightarrow$ A/B/C cannot be the answer]

Q.16 Given $z = f(x) + i\,g(x)$ where $f, g : (0, 1) \to (0, 1)$ are real valued functions then, which of the following holds good?

(A) $z = \frac{1}{1-ix} + i\left(\frac{1}{1+ix}\right)$ (B*) $z = \frac{1}{1+ix} + i\left(\frac{1}{1-ix}\right)$

(C) $z = \frac{1}{1+ix} + i\left(\frac{1}{1+ix}\right)$ (D) $z = \frac{1}{1-ix} + i\left(\frac{1}{1-ix}\right)$

[Hint: Choice A on simplification gives, $z = \frac{1+x}{1+x^2} + i\frac{1+x}{1+x^2}$

for $x = 0.5$; $f(0.5) > 1$ which is out of range $\Rightarrow$ A is not correct

Choice B ; $z = \frac{1-x}{1+x^2} + i\frac{1-x}{1+x^2}$

$f(x)$ & $g(x) \in (0, 1)$ if $x \in (0, 1) \Rightarrow$ B is correct

Choice C; $z = \frac{1+x}{1+x^2} + \frac{1-x}{1+x^2}i \Rightarrow$ C is not correct;

Choice D; $z = \frac{1-x}{1+x^2} + \frac{1+x}{1+x^2}i \Rightarrow$ D is not correct]

$Q.17 $z_1 = \frac{a}{1-i}$; $z_2 = \frac{b}{2+i}$; $z_3 = a - bi$ for $a, b \in R$

if $z_1 - z_2 = 1$ then the centroid of the triangle formed by the points z_1, z_2, z_3 in the argand's plane is given by

(A*) $\frac{1}{9}(1 + 7i)$ (B) $\frac{1}{3}(1 + 7i)$ (C) $\frac{1}{3}(1 - 3i)$ (D) $\frac{1}{9}(1 - 3i)$

[Sol. $z_1 = \frac{a(1+i)}{2}$; $z_2 = \frac{b(2-i)}{5}$

$\frac{a(1+i)}{2} - \frac{b(2-i)}{5} = 1$

$5a(1 + i) - 2b(2 - i) = 10$

$(5a - 4b - 10) + i(5a + 2b) = 0$

$5a - 4b = 10$; $5a = -2b$

$-6b = 10 \Rightarrow b = \frac{-5}{3}$

$5a = -2\left(\frac{-5}{3}\right) = \frac{10}{3} \Rightarrow a = \frac{2}{3}$]

Q.18 Consider the equation $10z^2 - 3iz - k = 0$, where z is a complex variable and $i^2 = -1$. Which of the following statements is True?
(A) For all real positive numbers k, both roots are pure imaginary.
(B*) For negative real numbers k, both roots are pure imaginary.
(C) For all pure imaginary numbers k, both roots are real and irrational.
(D) For all complex numbers k, neither root is real.

[Sol. Use the quadratic formula to obtain $z = \frac{3i \pm \sqrt{-9+40k}}{20}$ **[19-2-2006, 12th & 13th]**
which has discriminant $D = -9 + 40k$. If $k = 1$, then $D = 31$, so (A) is false.
If k is a negative real number, then D is a negative real number, so (B) is true.
If $k = i$, then $D = -9 + 40i = 16 + 40i + 25i^2 = (4 + 5i)^2$, and the roots are $\frac{1}{5} + \frac{2}{5}i$ and $-\frac{1}{5} - \frac{1}{10}i$, so (C) is false.
If $k = 0$ (which is a complex number), then the roots are 0 and $\frac{3}{10}i$, so (D) is false.]

Q.19 Number of complex numbers z such that $|z| = 1$ and $\left|\frac{z}{\bar{z}} + \frac{\bar{z}}{z}\right| = 1$ is
(A) 4 (B) 6 (C*) 8 (D) more than 8

[Sol. Let $z = \cos x + i \sin x$, $x \in [0, 2\pi)$. Then

$$1 = \left|\frac{z}{\bar{z}} + \frac{\bar{z}}{z}\right| = \frac{|z^2 + \bar{z}^2|}{|z|^2} = |\cos 2x + i \sin 2x + \cos 2x - i \sin 2x| = 2|\cos 2x|$$

hence $\cos 2x = 1/2$ or $\cos 2x = -1/2$
If $\cos 2x = 1/2$, then

$$x_1 = \frac{\pi}{6},\ x_2 = \frac{5\pi}{6},\ x_3 = \frac{7\pi}{6},\ x_4 = \frac{11\pi}{6}$$

If $\cos 2x = -\frac{1}{2}$, then

$$x_5 = \frac{\pi}{3},\ x_6 = \frac{2\pi}{3},\ x_7 = \frac{4\pi}{3},\ x_8 = \frac{5\pi}{3}$$

Hence there are eight solutions
$z_k = \cos x_k + i \sin x_k$, $k = 1, 2, \ldots, 8$]

Alternatively :
$|z| = 1 \Rightarrow z = \frac{1}{\bar{z}}$
hence $\left|\frac{z}{\bar{z}} + \frac{\bar{z}}{z}\right| = 1$; $z = e^{i\theta}$
$\left|e^{i2\theta} + e^{-i2\theta}\right| = 1$
$|2\cos 2\theta| = 1$
$\cos 2\theta = \frac{1}{2}$ or $-\frac{1}{2}$

Q.20 Number of ordered pairs(s) (a, b) of real numbers such that $(a + ib)^{2008} = a - ib$ holds good, is
(A) 2008 (B) 2009 (C*) 2010 (D) 1

[Sol. Let $z = a + ib \Rightarrow \bar{z} = a - ib$ **[12th, 04-01-2009]**
hence we have $z^{2008} = \bar{z}$
$\therefore$ $|z|^{2008} = |\bar{z}| = |z|$
$|z|\left[|z|^{2007} - 1\right] = 0$
$|z| = 0$ or $|z| = 1$; if $|z| = 0 \Rightarrow z = 0 \Rightarrow (0, 0)$
if $|z| = 1$ $z^{2009} = z\bar{z} = |z|^2 = 1 \Rightarrow$ 2009 values of z $\Rightarrow$ Total = 2010 **Ans.**]

CLASS : XIII (VXYZ) **Dpp on Complex Number (After 4th Lecture)** **DPP. NO.- 4**

Q.1 Consider $az^2 + bz + c = 0$, where a, b, c $\in$ R and $4ac > b^2$.

(i) If z_1 and z_2 are the roots of the equation given above, then which one of the following complex numbers is purely real?

(A) $z_1\bar{z}_2$ (B) $\bar{z}_1 z_2$ (C) $z_1 - z_2$ (D*) $(z_1 - z_2)i$

(ii) In the argand's plane, if A is the point representing z_1, B is the point representing z_2 and $z = \frac{\overrightarrow{OA}}{\overrightarrow{OB}}$ then

(A) z is purely real (B) z is purely imaginary
(C*) $|z| = 1$ (D) ΔAOB is a scalene triangle.

[Sol.
(i) As a, b, c are real number and $b^2 - 4ac < 0$

$\therefore$ z_1 and z_2 are complex conjugates of each other

$\Rightarrow$ $z_1 - z_2 = 2\,\text{Im.}(z_1)i$ $\Rightarrow$ $(z_2 - z_1)i$ is purely real $\Rightarrow$ (D)

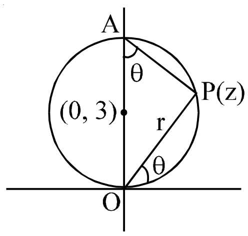
(ii) As z_1 and z_2 are the complex conjugate of each other $\Rightarrow$ $|z_1| = |z_2|$

$$\therefore \quad |z| = \left|\frac{\overrightarrow{OA}}{\overrightarrow{OB}}\right| = \frac{\left|\overrightarrow{OA}\right|}{\left|\overrightarrow{OB}\right|} = \frac{|z_1|}{|z_2|} = 1 \quad]$$

Q.2 Let z be a complex number having the argument θ, $0 < \theta < \pi/2$ and satisfying the equality $|z - 3i| = 3$. Then $\cot\theta - \frac{6}{z}$ is equal to :

(A) 1 (B) -1 (C*) i (D) $-i$

[Hint : $z = r(\cos\theta + i\sin\theta)$ now $r = OA\sin\theta = 6\sin\theta$



$$z = 6\sin\theta(\cos\theta + i\sin\theta) \qquad \frac{6}{z} = \frac{1}{\sin\theta(\cos\theta + i\sin\theta)}$$

$$= \frac{\cos\theta - i\sin\theta}{\sin\theta} = -i + \cot\theta \Rightarrow \cot\theta - \frac{6}{z} = i \Rightarrow C\,]$$

Q.3 If the complex number z satisfies the condition $|z| \geq 3$, then the least value of $\left|z + \frac{1}{z}\right|$ is equal to :

(A) 5/3 (B*) 8/3 (C) 11/3 (D) none of these

[Hint : $$\left|z + \frac{1}{z}\right| \geq |z| - \frac{1}{|z|}$$

$$\left|z + \frac{1}{z}\right|_{\text{least}} \geq 3 - \frac{1}{3} \geq \frac{8}{3} \quad]$$

Q.4 Given $z_p = \cos\left(\frac{\pi}{2^P}\right) + i\sin\left(\frac{\pi}{2^P}\right)$, then $\lim\limits_{n\to\infty}(z_1 z_2 z_3 \ldots z_n) =$

(A) 1 (B*) -1 (C) i (D) $-i$

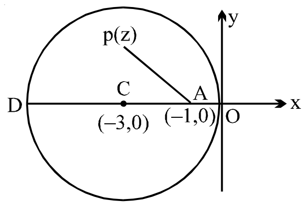
[Hint: $z_p = e^{\frac{i\pi}{2^p}}$; $z_1 = e^{\frac{i\pi}{2}}$; $z_2 = e^{\frac{i\pi}{2^2}}$ and so on .....

$$\lim_{n\to\infty} z_1 z_2 \ldots\ldots\ldots z_n = \lim_{n\to\infty} e^{i\pi\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots\ldots\ldots\ldots+\frac{1}{2^n}\right)}$$

$$= e^{i\pi}\left(\frac{1/2}{1-1/2}\right) = e^{i\pi} = \cos\pi + i\sin\pi = -1$$]

Q.5 The maximum & minimum values of $|z+1|$ when $|z+3| \le 3$ are :
(A*) (5 , 0) (B) (6 , 0) (C) (7 , 1) (D) (5 , 1)

[Hint: $|z+3| \le 3$ denotes set of points on or inside a circle with centre (– 3, 0) and radius 3 . $|z+1|$ denotes the distance of P from A of $|z+1|_{min} = 0$ & $|z+1|_{max} = AD$]

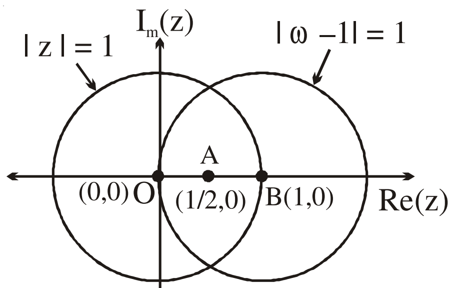


Q.6 If $z^3 + (3 + 2i)z + (-1 + ia) = 0$ has one real root, then the value of 'a' lies in the interval (a ∈ R)
(A) (– 2, – 1) (B*) (– 1, 0) (C) (0, 1) (D) (1, 2)

[Hint: Let $z = \alpha$ be a real root
$\alpha^3 + (3 + 2i)\alpha + (-1 + ia) = 0$
$(\alpha^3 + 3\alpha - 1) + i(a + 2\alpha) = 0$
∴ $\alpha^3 + 3\alpha - 1 = 0$ and $\alpha = -a/2$
∴ $-\frac{a^3}{8} - \frac{3a}{2} - 1 = 0$
$a^3 + 12a + 8 = 0$
Let $f(a) = a^3 + 12a + 8$
∴ $f(-1) < 0$ and $f(0) > 0$
hence $a \in (-1, 0)$]

Q.7 If $|z| = 1$ and $|\omega - 1| = 1$ where $z, \omega \in C$, then the largest set of values of $|2z - 1|^2 + |2\omega - 1|^2$ equals
(A) [1, 9] (B) [2, 6] (C) [2, 12] (D*) [2, 18]

[Sol. Least distance and greatest distance of any z and ω from the point $\left(\frac{1}{2}, 0\right)$ are $\frac{1}{2}$ and $\frac{3}{2}$ respectively.



$$\therefore \quad \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \le \left|z - \frac{1}{2}\right|^2 + \left|\omega - \frac{1}{2}\right|^2 \le \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2$$

Hence $2 \le |2z - 1|^2 + |2\omega - 1|^2 \le 18$

Alternatively: $(2z-1)(2\bar{z}-1) + (2\omega-1)(2\bar{\omega}-1)$ **[12th, 20-12-2009, complex]**

$4 + 1 - 2(z + \bar{z}) + 4 - 2(\omega + \bar{\omega}) + 1$
$10 - 2[2\,\text{Re}\,z + 2\,\text{Re}\,\omega]$
$10 - 4[\text{Re}\,z + \text{Re}\,\omega]$]

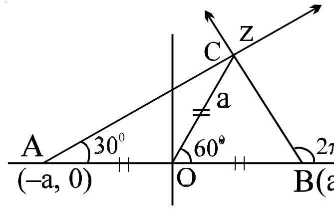
Q.8 If $\text{Arg}(z+a) = \frac{\pi}{6}$ and $\text{Arg}(z-a) = \frac{2\pi}{3}$; $a \in R^+$, then

(A) z is independent of a (B) $|a| = |z+a|$

(C) $z = a \text{ Cis } \frac{\pi}{6}$ (D*) $z = a \text{ Cis } \frac{\pi}{3}$

[Sol. Refer the figure z lies on the point of intersection of the rays from A and B. ΔACB is a right angle and OBC is an equilateral triangle

$\Rightarrow OC = a \Rightarrow z = a \text{ Cis } \frac{\pi}{3} \Rightarrow$ (D)



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Q.9 If z_1, z_2, z_3 are the vertices of the ΔABC on the complex plane which are also the roots of the equation, $z^3 - 3\alpha z^2 + 3\beta z + x = 0$, then the condition for the ΔABC to be equilateral triangle is

(A*) $\alpha^2 = \beta$ (B) $\alpha = \beta^2$ (C) $\alpha^2 = 3\beta$ (D) $\alpha = 3\beta^2$

[Hint: $z_1 + z_2 + z_3 = 3\alpha$; $\sum z_1 z_2 = 3\beta$

If ABC is equilateral $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

$(z_1 + z_2 + z_3)^2 = 3\sum z_1 z_2$

$9\alpha^2 = 3 \cdot 3\beta = 9\beta \Rightarrow \alpha^2 = \beta$]

Q.10 The locus represented by the equation, $|z-1| + |z+1| = 2$ is :

(A) an ellipse with focii (1, 0) ; (–1, 0)

(B) one of the family of circles passing through the points of intersection of the circles $|z-1| = 1$ and $|z+1| = 1$

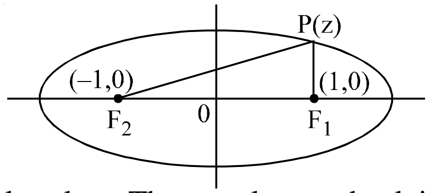
(C) the radical axis of the circles $|z-1| = 1$ and $|z+1| = 1$

(D*) the portion of the real axis between the points (1, 0) ; (–1, 0) including both.

[Hint: Note that $|z-1| + |z+1|$ denotes the sum of the distances of P from F_1 and F_2

since $|z_1 + 1| + |z_1 - 1| = 2$

hence locus will not be the ellipse]



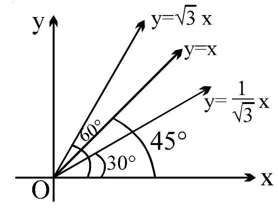
Q.11 The points $z_1 = 3 + \sqrt{3}\,i$ and $z_2 = 2\sqrt{3} + 6i$ are given on a complex plane. The complex number lying on the bisector of the angle formed by the vectors z_1 and z_2 is :

(A) $z = \frac{(3+2\sqrt{3})}{2} + \frac{\sqrt{3}+2}{2}i$ (B*) $z = 5 + 5i$

(C) $z = -1 - i$ (D) none

[Hint: Note that $z_1 = 3 + \sqrt{3}\,i$ lies on the line $y = \frac{1}{\sqrt{3}}x$ &

$z_2 = 2\sqrt{3} + 6i$ lies on the line $y = \sqrt{3}x$. Hence $z = 5 + 5i$ will only lie on the bisector of z_1 & z_2 i.e. $y = x$



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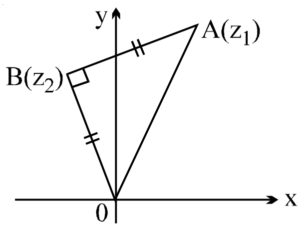
Q.12 Let z_1 & z_2 be non zero complex numbers satisfying the equation, $z_1^2 - 2z_1z_2 + 2z_2^2 = 0$. The geometrical nature of the triangle whose vertices are the origin and the points representing z_1 & z_2 is :
(A*) an isosceles right angled triangle
(B) a right angled triangle which is not isosceles
(C) an equilateral triangle
(D) an isosceles triangle which is not right angled .

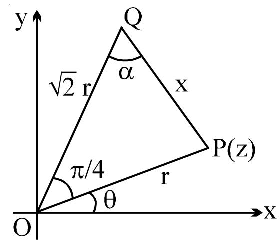


[Hint: $\frac{z_1}{z_2} = z \Rightarrow z^2 - 2z + 2 = 0 \Rightarrow z = 1 \pm i$

$\Rightarrow \frac{z_1}{z_2} = 1 \pm i \Rightarrow z_1 = z_2 \pm z_2 i \Rightarrow z_1 - z_2 = \pm z_2 i$

$\Rightarrow$ $z_1 - z_2$ is perpendicular to z_2 and $|z_1 - z_2| = |z_2|$]

Q.13 Let P denotes a complex number z on the Argand's plane, and Q denotes a complex number $\sqrt{2|z|^2}\ \text{CiS}\left(\frac{\pi}{4}+\theta\right)$ where $\theta = \text{amp } z$. If 'O' is the origin, then the Δ OPQ is :
(A) isosceles but not right angled (B) right angled but not isosceles
(C*) right isosceles (D) equilateral .

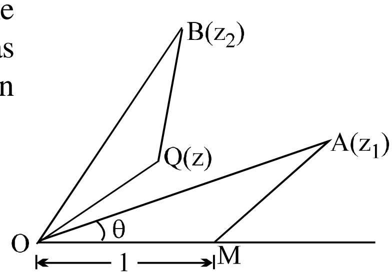


[Hint: $Z_P = r\,\text{Cis}\theta$; $Z_Q = \sqrt{2}\,|z|\,\text{Cis}\left(\theta + \frac{\pi}{4}\right) = \sqrt{2}\,r\left[\cos\left(\theta+\frac{\pi}{4}\right) + i\sin\left(\theta+\frac{\pi}{4}\right)\right]$

$$\cos\frac{\pi}{4} = \frac{2r^2 + r^2 - x^2}{2\,.\sqrt{2}\,r\,.\,r} = \frac{3r^2 - x^2}{2\sqrt{2}\,r^2}$$

$\therefore \quad 1 = \frac{3r^2 - x^2}{2r^2} \quad \Rightarrow \quad r^2 = x^2 \quad \Rightarrow \quad x = r$]

Q.14 On the Argand plane point 'A' denotes a complex number z_1. A triangle OBQ is made directily similiar to the triangle OAM, where OM = 1 as shown in the figure. If the point B denotes the complex number z_2, then the complex number corresponding to the point 'Q' is



(A) $z_1 z_2$ (B) $\frac{z_1}{z_2}$
(C*) $\frac{z_2}{z_1}$ (D) $\frac{z_1 + z_2}{z_2}$

[Sol. $\frac{z_2}{|z_2|} = \frac{z}{|z|}e^{i\theta}$ .....(1); $\frac{z_1}{|z_1|} = 1\,e^{i\theta}$ ....(2)

substitute the value of $e^{i\theta}$ from (2) in (1)

$\frac{z}{|z|} = \frac{z_2}{|z_2|}\cdot\frac{|z_1|}{z_1} \Rightarrow \frac{z}{|z|} = \frac{z_2/z_1}{|z_2/z_1|}$; $z = \frac{z_2}{z_1}$ **Ans.**]

Q.15 z_1 & z_2 are two distinct points in an argand plane. If $a|z_1| = b|z_2|$, (where a, b ∈ R) then the point

$\frac{a z_1}{b z_2} + \frac{b z_2}{a z_1}$ is a point on the :

(A*) line segment $[-2, 2]$ of the real axis (B) line segment $[-2, 2]$ of the imaginary axis
(C) unit circle $|z| = 1$ (D) the line with $\arg z = \tan^{-1} 2$.

[Hint: Assuming $\arg z_1 = \theta$ and $\arg z_2 = \theta + \alpha$.

$$\frac{a z_1}{b z_2} + \frac{b z_2}{a z_1} = \frac{a|z_1|e^{i\theta}}{b|z_2|e^{i(\theta+\alpha)}} + \frac{b|z_2|e^{i(\theta+\alpha)}}{a|z_1|e^{i\theta}} = e^{-i\alpha} + e^{i\alpha} = 2\cos\alpha$$

Alternatively: Let $\alpha = \frac{az_1}{bz_2}$; $\frac{1}{\alpha} = \frac{bz_2}{az_1}$; Also $|\alpha| = \frac{|az_1|}{|bz_2|} = \frac{a|z_1|}{b|z_2|} = 1 \Rightarrow \alpha = \frac{1}{\bar{\alpha}}$

$$\Rightarrow \alpha + \frac{1}{\alpha} = \alpha + \bar{\alpha} = 2\,\text{Re.}(\alpha) = 2\cos\alpha\ ]$$

Q.16 When the polynomial $5x^3 + Mx + N$ is divided by $x^2 + x + 1$ the remainder is 0. The value of $(M + N)$ is equal to

(A) – 3 (B) 5 (C*) – 5 (D) 15

[Sol. Let $f(x) = 5x^3 + Mx + N$, also $x^2 + x + 1 = (x - \omega)(x - \omega^2)$ **[19-2-2006, 12th & 13th]**

$f(\omega) = 5 + M\omega + N = 0$

$f(\omega^2) = 5 + M\omega^2 + N = 0$

$\Rightarrow$ $M = 0;\ N = -5$ $\Rightarrow$ $M + N = -5$ Ans.]

Q.17 If $z = \frac{\pi}{4}(1+i)^4\left(\frac{1-\sqrt{\pi}\,i}{\sqrt{\pi}+i} + \frac{\sqrt{\pi}-i}{1+\sqrt{\pi}\,i}\right)$ then $\left(\frac{|z|}{\text{amp}\,z}\right)$ equals

(A) 1 (B) π (C) 3π (D*) 4

[Hint: $$z = \frac{\pi}{2}\left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^4 = -\frac{\pi}{2}\left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^4 = -2\pi\left[\frac{2(\pi+1)}{\sqrt{\pi} + \pi i + i - \sqrt{\pi}}\right]$$

Alternatively: $$z = \frac{\pi}{4}(1+i)^4\left(\frac{(1-\pi)+(\pi+1)}{\sqrt{\pi}+\pi i + i - \sqrt{\pi}}\right); \qquad z = \frac{\pi}{2}(1+i)^4 \cdot \frac{1}{i}$$

$$\therefore \quad |z| = \frac{\pi}{2}\cdot 4 = 2\pi; \qquad \text{amp.}(z) = 0 + 4\cdot\frac{\pi}{4} - \frac{\pi}{2} \qquad \Rightarrow \qquad \frac{\phi z_1}{\text{amp.}(z)} = 4 \quad \textbf{Ans.}\ ]$$

One ore more than one is/are correct:

Q.18 Let z_1, z_2, z_3 be non-zero complex numbers satisfying the equation $z^4 = iz$.
Which of the following statement(s) is/are correct?

(A*) The complex number having least positive argument is $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

(B*) $\sum_{k=1}^{3} \text{Amp}(z_k) = \frac{\pi}{2}$

(C) Centroid of the triangle formed by z_1, z_2 and z_3 is $\left(\frac{1}{\sqrt{3}}, \frac{-1}{3}\right)$

(D) Area of triangle formed by z_1, z_2 and z_3 is $\frac{3\sqrt{3}}{2}$ **[12th, 20-12-2009, complex]**

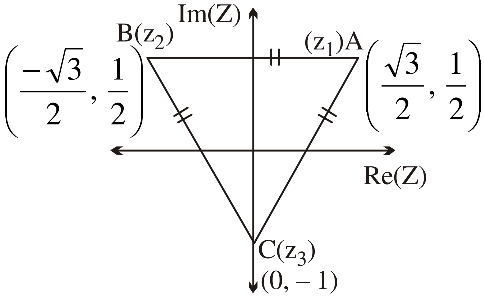
[Sol We have $z^4 = iz \Rightarrow z^3 = i$

$\Rightarrow \quad z = e^{i(4k+1)\frac{\pi}{6}}$ (Using D.M.T.)

Put $k = 0, 1, 2$, we get

$z_1 = e^{i\frac{\pi}{6}}$, $z_2 = e^{i\frac{5\pi}{6}}$ and $z_3 = e^{i\frac{3\pi}{2}}$

Clearly triangle formed by z_1, z_2 and z_3 is equilateral.



$\therefore$ centroid of ΔABC is (0, 0) and Area $(\Delta ABC) = \frac{3\sqrt{3}}{4}$]

Q.19 If $z \in C$, which of the following relation(s) represents a circle on an Argand diagram?

(A) $|z - 1| + |z + 1| = 3$ (B*) $(z - 3 + i)(\bar{z} - 3 - i) = 5$

(C*) $3|z - 2 + i| = 7$ (D*) $|z - 3| = 2$

[Sol. (A) is obviously ellipse **[11th, 27-01-2008]**

(B) $(z - \alpha)(\bar{z} - \bar{\alpha}) = 5$ where $\alpha = 3 - i$; $\bar{\alpha} = 3 + i$

$|z - \alpha|^2 = 5 \Rightarrow |z - \alpha| = \sqrt{5}$ circle with centre $(3, -1)$ and radius $= \sqrt{5} \Rightarrow$ **(B)** is correct

(C) $|z - (2 - i)| = \frac{7}{3} \Rightarrow$ circle with centre $(2, -1)$ and radius $= \frac{7}{3} \Rightarrow$ **(C)** is correct

(D) $|z - 3| = 2 \Rightarrow$ circle with centre (3, 0) and radius = 2 $\Rightarrow$ **(D)** is correct]

Q.20 Let z_1, z_2, z_3 be three complex number such that

$$|z_1| = |z_2| = |z_3| = 1 \text{ and } \frac{z_1^2}{z_2z_3} + \frac{z_2^2}{z_1z_3} + \frac{z_3^2}{z_1z_2} + 1 = 0$$

then $|z_1 + z_2 + z_3|$ can take the value equal to

(A*) 1 (B*) 2 (C) 3 (D) 4

[Sol. Given $|z_1| = |z_2| = |z_3| = 1 \Rightarrow z_1 = \frac{1}{\bar{z}_1}$ etc. **[12th, 07-12-2008, P-2]**

also $\frac{z_1^2}{z_2z_3} + \frac{z_2^2}{z_1z_3} + \frac{z_3^2}{z_1z_2} + 1 = 0 \Rightarrow (z_1)^3 + (z_2)^3 + (z_3)^3 + z_1z_2z_3 = 0$

$\Rightarrow (z_1)^3 + (z_2)^3 + (z_3)^3 - 3z_1z_2z_3 = -4z_1z_2z_3$

$(z_1 + z_2 + z_3)[(z_1)^2 + (z_2)^2 + (z_3)^2 - \sum z_1z_2] = -4z_1z_2z_3$

$\sum z_1\left[\left(\sum z_1\right)^2 - 3\sum z_1z_2\right] = -4z_1z_2z_3$

let $z_1 + z_2 + z_3 = z \Rightarrow \bar{z}_1 + \bar{z}_2 + \bar{z}_3 = \bar{z}$

$z\left[z^2 - 3\sum z_1z_2\right] = -4z_1z_2z_3$

$z^3 = 3z\sum z_1z_2 - 4z_1z_2z_3$

$$z^3 = z_1z_2z_3\left[3z\left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right) - 4\right] = z_1z_2z_3\left[3z\left(\bar{z}_1 + \bar{z}_2 + \bar{z}_3\right) - 4\right]$$

$z^3 = z_1 z_2 z_3 \left[3\,|\,z\,|^2 - 4\right]$

$\therefore \quad |\,z\,|^3 = \left|3|\,z\,|^2 - 4\right| \qquad ....(1)$

now $\quad$ if $|\,z\,| \geq \dfrac{2}{\sqrt{3}}$

then $\quad |\,z\,|^3 = 3|\,z\,|^2 - 4 \quad \Rightarrow \quad |\,z\,|^3 - 3|\,z\,|^2 + 4 = 0$

$\Rightarrow \quad |\,z\,|^2 (|\,z\,| - 2) - |\,z\,|(|\,z\,| - 2) - 2(|\,z\,| - 2) = 0 \quad \Rightarrow \quad (|\,z\,| - 2)(|\,z\,|^2 - |\,z\,| - 2) = 0$

$\Rightarrow \quad (|\,z\,| - 2)(|\,z\,| - 2)(|\,z\,| + 1) = 0 \quad \Rightarrow \quad |\,z\,| = 2$ $\quad$ or $\quad |\,z\,| = -1$ (rejected)

now $\quad$ if $0 < |\,z\,| < \dfrac{2}{\sqrt{3}}$ then equation (1) becomes

$|\,z\,|^3 = 4 - 3|\,z\,|^2 \quad \Rightarrow \quad |\,z\,|^3 + 3|\,z\,|^2 - 4 = 0$

$\Rightarrow \quad |\,z\,|^2 (|\,z\,| - 1) + 4\,|\,z\,|(|\,z\,| - 1) + 4(|\,z\,| - 1) = 0 \quad \Rightarrow \quad (|\,z\,| - 1)(|\,z\,|^2 + 4|\,z\,| + 4) = 0$

$\Rightarrow \quad (|\,z\,| - 1)(|\,z\,| + 2)^2 = 0 \quad \Rightarrow \quad |\,z\,| = +1$ $\quad$ or $\quad |\,z\,| = -2$ (rejected)

hence $\quad |\,z\,| = \{1, 2\}$ where $|\,z\,| = |\,z_1 + z_2 + z_3\,| \quad \Rightarrow$ **A, B**

NOTE: $\quad z_1 = 1;\ z_2 = i$ and $z_3 = -i$

and $\quad z_1 = 1;\ z_2 = -w$ and $z_3 = w^2$

also gives the result]

Q.1 A root of unity is a complex number that is a solution to the equation, $z^n = 1$ for some positive integer *n*. Number of roots of unity that are also the roots of the equation $z^2 + az + b = 0$, for some integer *a* and *b* is

(A) 6 (B*) 8 (C) 9 (D) 10

[Sol. Let α is a non real complex root of unity that is also a root of the equation $z^2 + az + b = 0$, then $\overline{\alpha}$ will also be its root. ($|\alpha| = 1$) **[13th, 09-03-2008]**

Hence $\alpha + \overline{\alpha} = -a$

$\therefore$ $|a| = |\alpha + \overline{\alpha}| \le |\alpha| + |\overline{\alpha}| = 2$

and $b = \alpha\overline{\alpha} = 1$

Hence we must check those equation for which $-2 \le a \le 2$ and $b = 1$

i.e. $z^2 + 2z + 1 = 0$; $z^2 + z + 1 = 0$; $z^2 + 1 = 0$

$z^2 - 2z + 1 = 0$; $z^2 - z + 1 = 0$

hence roots are $\pm 1, \pm i$; $\dfrac{-1 \pm \sqrt{-3}}{2}, \dfrac{1 \pm \sqrt{-3}}{2}$ i.e. 8 **Ans.**]

Q.2 z is a complex number such that $z + \dfrac{1}{z} = 2\cos 3°$, then the value of $z^{2000} + \dfrac{1}{z^{2000}} + 1$ is equal to

(A*) 0 (B) – 1 (C) $\sqrt{3} + 1$ (D) $1 - \sqrt{3}$

[Sol. Let $z = \cos\theta + i\sin\theta = = e^{i\theta}$; $\dfrac{1}{z} = \cos\theta - i\sin\theta = e^{-i\theta}$ **[13th test (14-8-2005)]**

so that $z + \dfrac{1}{z} = 2\cos\theta \quad (\theta = 3°)$

now $z^{2000} + \dfrac{1}{z^{2000}} + 1$

$e^{i\,2000\,\theta} + e^{-i\,2000\,\theta} + 1 = 2\cos(2000\,\theta) + 1 = 2\cos(6000°) + 1$ (as $\theta = 3°$)

$= 2\cos\left(\dfrac{100\pi}{3}\right) + 1 = 2\cos\left(\dfrac{4\pi}{3}\right) + 1 = -1 + 1 = 0$ Ans.]

Q.3 The complex number ω satisfying the equation $\omega^3 = 8i$ and lying in the second quadrant on the complex plane is

(A*) $-\sqrt{3} + i$ (B) $-\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i$ (C) $-2\sqrt{3} + i$ (D) $-\sqrt{3} + 2i$

[Hint: $\omega = 2 \cdot i^{1/3} = 2\left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right)^{1/3} = 2\left[\cos\dfrac{2n\pi + \frac{\pi}{2}}{3} + i\sin\dfrac{2n\pi + \frac{\pi}{2}}{3}\right]$

put n = 1 $= 2\left[\cos\dfrac{5\pi}{6} + i\sin\dfrac{5\pi}{6}\right] = -\sqrt{3} + i$ Ans.]

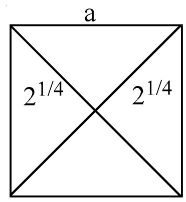
Q.4 If $z^4 + 1 = \sqrt{3}\,i$

(A) z^3 is purely real
(B) z represents the vertices of a square of side $2^{1/4}$
(C) z^9 is purely imaginary
(D*) z represents the vertices of a square of side $2^{3/4}$.

[Sol. $z^4 = -1 + \sqrt{3}\,i = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$z^4 = 2w^2 \quad \Rightarrow \quad$ A, C are not possible

root are $z_1 = 2^{1/4}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$; $\quad z_2 = 2^{1/4}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ etc.



$\therefore \quad a = \sqrt{2^{1/2} + 2^{1/2}} = (2^{3/2})^{1/2} = 2^{3/4} \Rightarrow$ (D)]

Q.5 The complex number z satisfies the condition $\left|z - \frac{25}{z}\right| = 24$. The maximum distance from the origin of co-ordinates to the point z is :

(A*) 25 (B) 30 (C) 32 (D) none of these

Q.6 If the expression $x^{2m} + x^m + 1$ is divisible by $x^2 + x + 1$, then :

(A) m is any odd integer
(B) m is divisible by 3
(C*) m is not divisible by 3
(D) none of these

[Sol. $x^{2m} + x^m + 1$ div. by $x^2 + x + 1$ i.e. $(x - \omega)(x - \omega^2)$

$\Rightarrow \quad \omega^{2m} + \omega^m + 1$ must be equal to zero

$\Rightarrow \quad 1^m + \omega^m + (\omega^2)^m = 0 \Rightarrow$ m is not divisible by 3]

Q.7 If $z_1 = 2 + 3i$, $z_2 = 3 - 2i$ and $z_3 = -1 - 2\sqrt{3}\,i$ then which of the following is true?

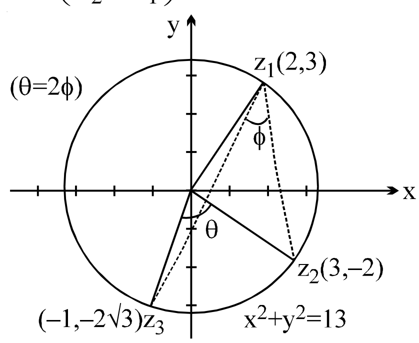
(A) $\arg\left(\frac{z_3}{z_2}\right) = \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$
(B) $\arg\left(\frac{z_3}{z_2}\right) = \arg\left(\frac{z_2}{z_1}\right)$

(C*) $\arg\left(\frac{z_3}{z_2}\right) = 2\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$
(D) $\arg\left(\frac{z_3}{z_2}\right) = \frac{1}{2}\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$

[Hint: Note that $|z_1| = |z_2| = |z_3| = \sqrt{13}$

Hence z_1, z_2, z_3 lies on a circle with centre (0, 0)

and $r = \sqrt{13}$ as shown



now $\quad \text{Arg}\,\frac{z_2}{z_3} = 2\text{Arg}\,\frac{z_2 - z_1}{z_3 - z_1}$

$\therefore \quad \text{Arg}\,\frac{z_3}{z_2} = 2\text{Arg}\,\frac{z_3 - z_1}{z_2 - z_1} \quad \Rightarrow \quad$ (C)]

Q.8 If ***m*** and ***n*** are the smallest positive integers satisfying the relation

$$\left(2\text{Cis}\frac{\pi}{6}\right)^m = \left(4\text{Cis}\frac{\pi}{4}\right)^n$$, then $(m + n)$ has the value equal to

(A) 120 (B) 96 (C*) 72 (D) 60

[Sol. $2^{m-2n} \cdot \left[\cos\frac{m\pi}{6} + i\sin\frac{m\pi}{6}\right] = \left[\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right]$

for equality m = 2n

$\frac{n\pi}{4} = \frac{m\pi}{6} + 2k\pi \qquad k \in I$ **[13th, 17-02-2008]**

put m = 2n

$\frac{n\pi}{4} = \frac{n\pi}{3} + 2k\pi\ ; \qquad -\left(\frac{n\pi}{12}\right) = 2k\pi$ (ignore (–)ve sign)

n = 24k; m = 48k; for m, n to be smallest m + n = 72 **Ans.**]

Q.9 If z is a complex number satisfying the equation

$Z^6 + Z^3 + 1 = 0.$

If this equation has a root $re^{i\theta}$ with $90° < \theta < 180°$ then the value of 'θ' is

(A) 100° (B) 110° (C*) 160° (D) 170°

[Sol. Let $Z^3 = t$ **[12[th] (27-11-2005)]**

hence equation becomes

$t^2 + t + 1 = 0 \quad \Rightarrow \quad t = \omega \quad \text{or} \quad \omega^2$

$Z^3 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = e^{\left(2m\pi + \frac{2\pi}{3}\right)i}$

$Z = e^{\frac{\left(2m\pi + \frac{2\pi}{3}\right)}{3}i}$

put m = 1 to get $\theta = \frac{8\pi}{9} \in (90°, 180°) = 160°$ Ans.]

Q.10 Least positive argument of the 4[th] root of the complex number $2 - i\sqrt{12}$ is

(A) π/6 (B*) 5π/12 (C) 7π/12 (D) 11π/12

[Sol. $z^4 = 2\left(1 - \sqrt{3}\,i\right) = 4\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 4\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]$

$z = \sqrt{2}\left[\cos\frac{2m\pi - (\pi/3)}{4} + i\sin\frac{2m\pi - (\pi/3)}{4}\right]$ **[13th, 25-01-2009]**

$m = 1,\ z = \sqrt{2}\left[\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)\right]$]

Q.11 P(z) is the point moving in the Argand's plane satisfying arg(z – 1) – arg(z + i) = π then, P is

(A) a real number, hence lies on the real axis.

(B) an imaginary number, hence lies on the imaginary axis.

(C*) a point on the hypotenuse of the right angled triangle OAB formed by O ≡ (0, 0); A ≡ (1, 0); B ≡ (0, – 1).

(D) a point on an arc of the circle passing through A ≡ (1, 0); B ≡ (0, – 1).

[Sol. $\text{amp.}\left(\frac{z-1}{z+i}\right) = \pi \Rightarrow \frac{z-1}{z+i}$ is real $\Rightarrow$ z moves on the lines joining (0, – 1) and (1, 0)]

Q.12 Number of ordered pair(s) (z, ω) of the complex numbers z and ω satisfying the system of equations, $z^3 + \overline{\omega}^7 = 0$ and $z^5 . \omega^{11} = 1$ is :

(A) 7 (B) 5 (C) 3 (D*) 2

[Hint: (i, i) and (–i, –i)

$(z^3) = (-(\overline{\omega})^7) \Rightarrow |z|^3 = |\overline{\omega}|^7 = |\omega|^7$ or $|z|^{15} = |\omega|^{35}$ — (1)

again $z^5 . \omega^{11} = 1 \Rightarrow |z|^5 . |\omega|^{11} = 1$ or $|z|^{15} |\omega|^{33} = 1$ — (2)

from (1) and (2) $\Rightarrow |z| = |\omega| = 1$

again $-(\overline{\omega})^{35} = \frac{1}{\omega^{33}} \Rightarrow -(\overline{\omega})^2 = 1 \Rightarrow (\overline{\omega})^2 = -1 = i^2$

$\Rightarrow \overline{\omega} = i$ or $-i$ $\Rightarrow \omega = -i$ or i]

Q.13 If $p = a + b\omega + c\omega^2$; $q = b + c\omega + a\omega^2$ and $r = c + a\omega + b\omega^2$ where a, b, c $\neq 0$ and ω is the complex cube root of unity, then :

(A) $p + q + r = a + b + c$ (B) $p^2 + q^2 + r^2 = a^2 + b^2 + c^2$

(C*) $p^2 + q^2 + r^2 = 2(pq + qr + rp)$ (D) none of these

[Hint: $p + q + r = a + b\omega + c\omega^2$
$b + c\omega + a\omega^2$
$c + a\omega + b\omega^2$

hence $p + q + r = (a + b + c)(1 + \omega + \omega^2) = 0$ ....(1)

$\Rightarrow (p + q + r)^2 = 0$

$$\Rightarrow p^2 + q^2 + r^2 = -2pqr\left[\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right]$$

$$= -2pqr\left[\frac{1}{a + b\omega + c\omega^2} + \frac{1}{b + c\omega + a\omega^2} + \frac{1}{c + a\omega + b\omega^2}\right]$$

$$= -2pqr\left[\frac{1}{\omega^2(a\omega + b\omega^2 + c)} + \frac{1}{\omega(b\omega^2 + c + a\omega)} + \frac{1}{c + a\omega + b\omega^2}\right]$$

$$= \frac{-2pqr}{a\omega + b\omega^2 + c}\left[\frac{1}{\omega^2} + \frac{1}{\omega} + \frac{1}{1}\right] = 0$$ ....(2) hence $p^2 + q^2 + r^2 = 2(pq + qr + rp)$]

Q.14 If A and B be two complex numbers satisfying $\frac{A}{B} + \frac{B}{A} = 1$. Then the two points represented by A and B and the origin form the vertices of

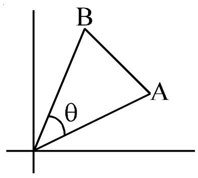
(A*) an equilateral triangle

(B) an isosceles triangle which is not equilateral

(C) an isosceles triangle which is not right angled

(D) a right angled triangle

[Hint: $A^2 - AB + B^2 = 0$

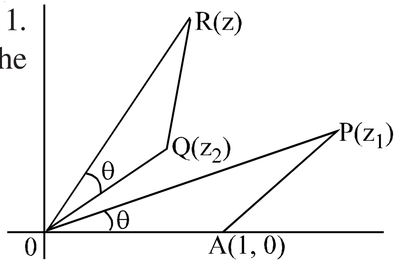


Let $\frac{A}{B} = z$; $\Rightarrow z + \frac{1}{z} = 1$ hence $z^2 - z + 1 = 0 \Rightarrow z = -\omega$ or $-\omega^2$

$\Rightarrow A = -B\omega$ or $A = -B\omega^2 \Rightarrow |A| = |B|$

and amp (A) – amp (B) = amp $(-\omega)$ = amp (– 1) + amp $(\omega) = \pi + \frac{2\pi}{3}$]

Q.15 On the complex plane triangles OAP & OQR are similiar and l(OA) = 1. If the points P and Q denotes the complex numbers z_1 & z_2 then the complex number 'z' denoted by the point R is given by :



(A*) $z_1 z_2$ (B) $\frac{z_1}{z_2}$

(C) $\frac{z_2}{z_1}$ (D) $\frac{z_1 + z_2}{z_2}$

[Hint: $\frac{OR}{OQ} = \frac{OP}{OA} \Rightarrow OR \cdot OA = OQ \cdot OP$

or $OR = |z_2|\,|z_1|$ (OA = 1)

Also $\angle ROA = \angle ROQ + \angle QOA = \theta + \phi$ (say arg of z_2)

$= \theta + \phi$ $=$ arg z_1 + arg z_2 = arg $(z_1 z_2)$

Hence complex number corresponding to the point R is $z_1 z_2$

Alternatively: $\frac{z}{|z|} = \frac{z_2}{|z_2|}e^{i\theta}$ ....(1)

$\frac{z_1}{|z_1|} = 1 \cdot e^{i\theta}$ ....(2)

$\frac{z}{|z|} \cdot e^{i\theta} = \frac{z_2}{|z_2|}e^{i\theta} \cdot \frac{z_1}{|z_1|}$

$\frac{z}{|z|} = \frac{z_1 z_2}{|z_1 z_2|} \Rightarrow z = z_1 z_2 \Rightarrow$ (A)]

Q.16 If $1, \alpha_1, \alpha_2, \ldots\ldots, \alpha_{2008}$ are $(2009)^{th}$ roots of unity, then the value of $\sum_{r=1}^{2008} r(\alpha_r + \alpha_{2009-r})$ equals

(A) 2009 (B) 2008 (C) 0 (D*) – 2009

[Sol. Let $S = 1(\alpha_1 + \alpha_{2008}) + 2(\alpha_2 + \alpha_{2007}) + 3(\alpha_3 + \alpha_{2006}) + \ldots\ldots\ldots + 2008(\alpha_{2008} + \alpha_1)$ ....(1)

Also $S = 2008(\alpha_{2008} + \alpha_1) + 2007(\alpha_2 + \alpha_{2007}) + \ldots\ldots\ldots + 2(\alpha_2 + \alpha_{2007}) + 1(\alpha_1 + \alpha_{2008})$ ....(2)

(writing in reverse order)

∴ On adding (1) and (2), we get **[12th, 20-12-2009, complex]**

$2S = 2009[2(\alpha_1 + \alpha_2 + \alpha_3 + \ldots\ldots\ldots\alpha_{2008})]$

$2S = 2009[2(\underbrace{1 + \alpha_1 + \alpha_2 + \alpha_3 + \ldots\ldots\ldots\alpha_{2008}}_{\text{zero}} - 1)]$

Hence $S = -2009$ **Ans.**

Note that $(\alpha_1$ and $\alpha_{2008})$, $(\alpha_2$ and $\alpha_{2007})$, $(\alpha_3$ and $\alpha_{2006})$, ..., $(\alpha_{1004}$ and $\alpha_{1005})$ are conjugate of each other.]

Paragraph for question nos. 17 to 19

For the complex number $w = \frac{4z - 5i}{2z + 1}$

Q.17 The locus of z, when w is a real number other than 2, is
(A) a point circle
(B) a straight line with slope $-\frac{5}{2}$ and y-intercept $\frac{5}{4}$
(C*) a straight line with slope $\frac{5}{2}$ and y-intercept $\frac{5}{4}$
(D) a straight line passing through the origin

Q.18 The locus of z, when w is a purely imaginary number is
(A) a circle with centre $\left(\frac{1}{2}, -\frac{5}{4}\right)$ passing through origin.
(B*) a circle with centre $\left(-\frac{1}{4}, \frac{5}{8}\right)$ passing through origin.
(C) a circle with centre $\left(\frac{1}{4}, -\frac{5}{8}\right)$ and radius $\frac{\sqrt{29}}{8}$
(D) any other circle

Q.19 The locus of z, when | w | = 1 is
(A) a circle with centre $\left(-\frac{5}{8}, \frac{1}{4}\right)$ and radius $\frac{1}{2}$
(B) a circle with centre $\left(\frac{1}{4}, -\frac{5}{8}\right)$ and radius $\frac{1}{2}$
(C) a circle with centre $\left(\frac{5}{8}, -\frac{1}{4}\right)$ and radius $\frac{1}{2}$
(D*) any other circle

[Hint:

(i) $w = \frac{4z - 5i}{2z + 1}$ $(w \neq z)$ **[13th, 23-11-2008]**

if w is real then $w = \overline{w}$

$$\overline{w} = \underbrace{\frac{4\overline{z} + 5i}{2\overline{z} + 1}}_{\overline{w}} = \underbrace{\frac{4z - 5i}{2z + 1}}_{w}$$

$$(4\overline{z} + 5i)(2z + 1) = (4z - 5i)(2\overline{z} + 1)$$

$$8z\overline{z} + 4\overline{z} + 10zi + 5i = 8z\overline{z} + 4z - 10\overline{z}i - 5i$$

$$4(z - \overline{z}) - 10i(z + \overline{z}) - 10i = 0$$

$$8iy - 20ix - 10i = 0$$

$$4y - 10x - 5 = 0 \quad \Rightarrow \quad 10x - 4y + 5 = 0$$

(ii) If w is purely imaginary then

$$w + \overline{w} = 0$$

$$\frac{4z-5i}{2z+1} + \frac{4\overline{z}+5i}{2\overline{z}+1} = 0$$

$$(4\overline{z}+5i)(2z+1) + (4z-5i)(2\overline{z}+1) = 0$$

simplifying $\quad 16z\overline{z} + 4(z+\overline{z}) + 10i(z-\overline{z}) = 0$

$$16(x^2+y^2) + 8x - 20y = 0$$

$$x^2 + y^2 + \frac{x}{2} - \frac{5}{4}y = 0 \quad]$$

Paragraph for question nos. 20 to 22

Let A, B, C be three sets of complex numbers as defined below.

$A = \{z : |z+1| \le 2 + \text{Re}(z)\}$, $\quad B = \{z : |z-1| \ge 1\}$ and $\quad C = \left\{z : \left|\frac{z-1}{z+1}\right| \ge 1\right\}$

Q.20 The number of point(s) having integral coordinates in the region $A \cap B \cap C$ is

(A) 4 (B*) 5 (C) 6 (D) 10

Q.21 The area of region bounded by $A \cap B \cap C$ is

(A*) $2\sqrt{3}$ (B) $\sqrt{3}$ (C) $4\sqrt{3}$ (D) 2

Q.22 The real part of the complex number in the region $A \cap B \cap C$ and having maximum amplitude is

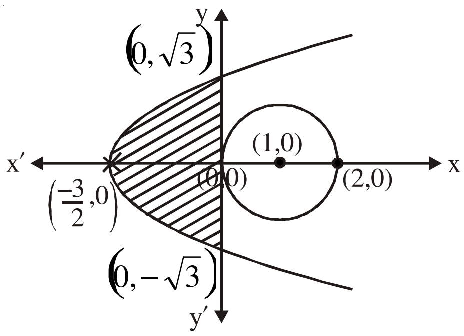
(A) –1 (B*) $\frac{-3}{2}$ (C) $\frac{1}{2}$ (D) – 2

[Sol. For A, $|z+1| \le 2 + \text{Re}(z)$ **[12th, 20-12-2009, complex]**

$\Rightarrow \quad (x+1)^2 + y^2 \le 4 + 4x + x^2$

$\Rightarrow \quad y^2 \le 3 + 2x$

$\Rightarrow \quad y^2 \le 2\left(x + \frac{3}{2}\right) \qquad .....(1)$



For B, $|z-1| \ge 1$

$\Rightarrow \quad (x-1)^2 + y^2 \ge 1 \qquad ......(2)$

For C, $|z-1|^2 \ge |z+1|^2$

$\Rightarrow \quad (z-1)(\overline{z}-1) \ge (z+1)(\overline{z}+1)$

$\Rightarrow \quad (z\overline{z} - \overline{z} - z + 1) \ge (z\overline{z} + \overline{z} + z + 1)$

$\Rightarrow \quad z + \overline{z} \le 0$

i.e. $\quad x \le 0 \qquad .....(3)$

(i) (–1,0), (–1, 1), (–1,–1), (0,0), (0,1), (0,–1) but z = – 1 is not in the domain in set C

$\therefore$ Total number of point(s) having integral coordinates in the region $A \cap B \cap C$ is 6.

(ii) Required area $= 2\int_{-3/2}^{0} \sqrt{2\left(x+\frac{3}{2}\right)}\, dx = 2\sqrt{3}$ (square units)

(iii) Clearly $z = \frac{-3}{2} + i0$ is the complex number in the region $A \cap B \cap C$ and having maximum amplitude.

$\therefore \quad \text{Re}(z) = \frac{-3}{2}$]

CLASS : XIII (VXYZ)   Dpp on Complex Number (After 6th Lecture)   DPP. NO.- 6

Q.1 If the six solutions of $x^6 = -64$ are written in the form $a + bi$, where a and b are real, then the product of those solutions with $a > 0$, is
(A*) 4 (B) 8 (C) 16 (D) 64

[Hint: Use De Moivre's theorem get the product of roots with + ve real part

$$z = 2(-1)^{1/6} = 2\left[\cos\frac{(2m+1)\pi}{6} + i\sin\frac{(2m+1)\pi}{6}\right]$$

put $m = 0$ or $m = 5$ for positive real part to get
$z_1 z_2 = 4e^{i\pi/6} \cdot e^{(11\pi/6)i} = 4e^{2\pi i} = 4$]

Q.2 Number of imaginary complex numbers satisfying the equation, $z^2 = \bar{z}\, 2^{1-|z|}$ is
(A) 0 (B) 1 (C*) 2 (D) 3

[Sol. $z^2 = \bar{z} \cdot 2^{1-|z|}$ **[12th, 06-01-2008]**

$z^3 = |z|^2\, 2^{1-|z|}$ ....(1) $\Rightarrow$ $|z| = 2^{1-|z|}$

hence z^3 is purely +ve real (as $z \neq 0$) $\Rightarrow$ z is + ve real

hence $z = r e^{i\frac{2k\pi}{3}}$ $k = 0, 1, 2$

we therefore need to solve

$r = 2^{1-r}$ $\Rightarrow$ $2^r = \dfrac{2}{r}$ $\Rightarrow$ $r = 1$

$\therefore$ $z = e^{i\frac{2k\pi}{3}}$

hence $z = 1, \omega, \omega^2$
but 1 is not imaginary
hence $z = w$ or w^2 $\Rightarrow$ (C)]

Q.3 If z_1 & z_2 are two complex numbers & if $\arg\dfrac{z_1 + z_2}{z_1 - z_2} = \dfrac{\pi}{2}$ but $|z_1 + z_2| \neq |z_1 - z_2|$ then the figure formed by the points represented by $0, z_1, z_2$ & $z_1 + z_2$ is :
(A) a parallelogram but not a rectangle or a rhombous
(B) a rectangle but not a square
(C*) a rhombous but not a square
(D) a square

Q.4 If $z_n = \cos\dfrac{\pi}{(2n+1)(2n+3)} + i\sin\dfrac{\pi}{(2n+1)(2n+3)}$, then $\underset{n\to\infty}{\text{Limit}}\,(z_1 . z_2 . z_3 . ... z_n) =$

(A) $\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}$ (B*) $\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}$ (C) $\cos\dfrac{5\pi}{6} + i\sin\dfrac{5\pi}{6}$ (D) $\cos\dfrac{3\pi}{2} + i\sin\dfrac{3\pi}{2}$

[Hint: $z_n = e^{\frac{i\pi}{2}\left(\frac{1}{2n+1}-\frac{1}{2n+3}\right)}$

$$\prod_{n=1}^{\infty} z_n = e^{\frac{i\pi}{2}\left[\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+........+\left(\frac{1}{2n+1}-\frac{1}{2n+3}\right)\right]} = e^{\frac{i\pi}{2}\left[\left(\frac{1}{3}-\frac{1}{2n+3}\right)\right]} = e^{\frac{i\pi}{6}} \text{ as } n \to \infty \Rightarrow \text{(B)]}$$

Q.5 The straight line $(1+2i)z+(2i-1)\bar{z} = 10i$ on the complex plane, has intercept on the imaginary axis equal to

(A*) 5 (B) $\frac{5}{2}$ (C) $-\frac{5}{2}$ (D) – 5

[Hint: put $z = iy$ $(1 + 2i)\, iy - (2i - 1)\, i\, y = 10\, i$

$2y + 0\, y = 10 \Rightarrow y = 5$

Note: For x-intercept put $z = x + 0i \Rightarrow x = 5/2$

Alternatively: put $z+\bar{z} = 0 \Rightarrow \bar{z} = -z \Rightarrow (1 + 2i)z - z(2i - 1) = 10i$

$2z = 10i \Rightarrow z = 5i$; $y = 5$] **[13th Test (24-03-2005)]**

Q.6 If $\cos\theta + i\sin\theta$ is a root of the equation $x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n = 0$ then the value of $\sum_{r=1}^{n} a_r \cos r\theta$ equals (where all coefficient are real)

(A) 0 (B) 1 (C*) –1 (D) none

[Hint : Divide the equation by x^n and put $x = \cos\theta + i\sin\theta$.

Equate real and imaginary part]

Q.7 Let A(z_1) and B(z_2) represent two complex numbers on the complex plane. Suppose the complex slope of the line joining A and B is defined as $\frac{z_1-z_2}{\bar{z}_1-\bar{z}_2}$. Then the lines l_1 with complex slope ω_1 and l_2 with complex slope ω_2 on the complex plane will be perpendicular to each other if

(A*) $\omega_1 + \omega_2 = 0$ (B) $\omega_1 - \omega_2 = 0$ (C) $\omega_1\omega_2 = -1$ (D) $\omega_1\omega_2 = 1$

[Hint: l_1 is perpendicular to l_2 **[12th test (29-10-2005)]**

$\Rightarrow \frac{z_1-z_2}{z_3-z_4}$ is purely imaginary

$$\frac{z_1-z_2}{z_3-z_4} + \frac{\bar{z}_1-\bar{z}_2}{\bar{z}_3-\bar{z}_4} = 0$$

$$\frac{z_1-z_2}{\bar{z}_1-\bar{z}_2} + \frac{z_3-z_4}{\bar{z}_3-\bar{z}_4} = 0 \Rightarrow \omega_1 + \omega_2 = 0$$

Note: If l_1 parallel to l_2 then

$$\frac{z_1-z_2}{z_3-z_4} = \frac{\bar{z}_1-\bar{z}_2}{\bar{z}_3-\bar{z}_4} \Rightarrow \omega_1 = \omega_2 \text{]}$$

Q.8 If the equation, $z^4 + a_1z^3 + a_2z^2 + a_3z + a_4 = 0$, where a_1, a_2, a_3, a_4 are real coefficients different from zero has a pure imaginary root then the expression $\frac{a_3}{a_1a_2} + \frac{a_1a_4}{a_2a_3}$ has the value equal to:

(A) 0 (B*) 1 (C) – 2 (D) 2

[Hint : Let xi be the root where $x \neq 0$ and $x \in R$ (as if $x = 0$ satisfies then $a_4 = 0$ which contradicts)

$x^4 - a_1 x^3 i - a_2 x^2 + a_3 x i + a_4 = 0$

$x^4 - a_2 x^2 + a_4 = 0$ ....(1) and

$a_1 x^3 - a_3 x = 0$ ....(2)

From equation (2) : $a_1 x^2 - a_3 = 0 \Rightarrow x^2 = a_3/a_1$.(as $x \neq 0$)

Putting the value of x^2 in equation .....(1)

$\frac{a_3^2}{a_1^2} - \frac{a_2 a_3}{a_1} + a_4 = 0$ or $a_3^2 + a_4 a_1^2 = a_1 a_2 a_3$ or $\frac{a_3}{a_1 a_2} + \frac{a_1 a_4}{a_2 a_3} = 1$ (dividing by $a_1 a_2 a_3$)]

Q.9 Suppose A is a complex number & $n \in N$, such that $A^n = (A + 1)^n = 1$, then the least value of *n* is

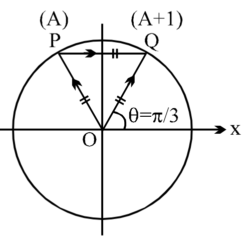
(A) 3 (B*) 6 (C) 9 (D) 12

[Hint : Let $A = x + iy$; $|A| = 1 \Rightarrow x^2 + y^2 = 1$ and

$|A + 1| = 1 \Rightarrow (x + 1)^2 + y^2 = 1 \Rightarrow x = -\frac{1}{2}$ and $y = \pm \frac{\sqrt{3}}{2} \Rightarrow (A) = \omega$ or ω^2

$\Rightarrow (\omega)^n = (1 + \omega)^n = (-\omega^2)^n \Rightarrow$ n must be even and divisible by 3

Alternatively :



$A = \frac{-1+i\sqrt{3}}{2}$; $A + 1 = \frac{1+i\sqrt{3}}{2}$

and $n \arg A = n \arg(A+1) = \arg 1 = 2n\pi$

$\Rightarrow \arg A = \arg(A+1) = 2\pi$

now let $\vec{OQ} = A+1$ and $\vec{OP} = A$

$\Rightarrow \vec{OP}$ & $\vec{OQ}$ vectors must be turned a minimum number of times to coincide with positive x – axis $\Rightarrow$ 6]

Q.10 Intercept made by the circle $z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + r = 0$ on the real axis on complex plane, is

(A) $\sqrt{(\alpha+\bar{\alpha})-r}$ (B) $\sqrt{(\alpha+\bar{\alpha})^2-2r}$ (C) $\sqrt{(\alpha+\bar{\alpha})^2+r}$ (D*) $\sqrt{(\alpha+\bar{\alpha})^2-4r}$

[Sol. Points where the circle cuts the x-axis $z = \bar{z}$. **[12th (27-11-2005)]**

Hence substituting $z = \bar{z}$ in the equation of circle

$z^2 + \bar{\alpha}z + \alpha z + r = 0$

$z^2 + (\alpha + \bar{\alpha})z + r = 0$

$AB = |z_1 - z_2| = \sqrt{(z_1 + z_2)^2 - 4z_1 z_2} = \sqrt{(\alpha+\bar{\alpha})^2 - 4r} \Rightarrow$ (D)

<u>Alternatively</u>: put $z = x$ and $\bar{z} = x$ to get $x^2 + \bar{\alpha}x + \alpha x + r = 0$ which is the same equation]

Q.11 If Z_r ; r = 1, 2, 3,..., 50 are the roots of the equation $\sum_{r=0}^{50} (Z)^r = 0$, then the value of $\sum_{r=1}^{50} \frac{1}{Z_r - 1}$ is

(A) – 85 (B*) – 25 (C) 25 (D) 75

[Hint : $E = \frac{1}{z_1 - 1} + \frac{1}{z_2 - 1} + ... + \frac{1}{z_{50} - 1}$, where $z_1, z_2, ..., z_{50}$ are the roots of the equation $z^{51} - 1 = 0$ other than 1.

$= -25 + \left(\frac{1}{2} + \frac{1}{z_1 - 1}\right) + \left(\frac{1}{2} + \frac{1}{z_2 - 1}\right) + ... + \left(\frac{1}{2} + \frac{1}{z_{50} - 1}\right)$

Note that (1^{st} + last) and (2^{nd} + 2^{nd} last) will vanish using $z_r = z^r$ and $z^{51} = 1$

<u>Alternatively</u>: Let $1 + z + z^2 + ... + z^{50} = (z - z_1)(z - z_2)(z - z_{50})$

differentiate both sides w.r.t. z after taking logarithm on both the sides.

$$\frac{1+2z+3z^2+....+50z^{49}}{1+z+z^2+.....+z^{50}} = \frac{1}{z-z_1} + \frac{1}{z-z_2} + ... + \frac{1}{z-z_{50}}$$. Now put z = 1

we get, $\frac{50\cdot 51}{2\cdot 51} = -\left[\frac{1}{z_1-1} + \frac{1}{z_2-1} + ... + \frac{1}{z_{50}-1}\right]$

$\therefore \quad \sum \frac{1}{z_r - 1} = -25$ Ans.]

Q.12 All roots of the equation, $(1 + z)^6 + z^6 = 0$:
(A) lie on a unit circle with centre at the origin
(B) lie on a unit circle with centre at (- 1, 0)
(C) lie on the vertices of a regular polygon with centre at the origin
(D*) are collinear

[Hint: $z = -\frac{1}{2}\left(1 + i\cot\frac{2r+1}{12}\pi\right)$, r = 1, 2, 3,4, 5]

Q.13 If z & w are two complex numbers simultaneously satisfying the equations,

$z^3 + w^5 = 0$ and $z^2 . \overline{w}^4 = 1$, then :

(A*) z and w both are purely real
(B) z is purely real and w is purely imaginary
(C) w is purely real and z is purely imaginarly
(D) z and w both are imaginary .

[Hint: $z^3 = -\omega^5 \Rightarrow |z|^3 = |\omega|^5 \Rightarrow |z|^6 = |\omega|^{10}$ ....(1) [Ans. (1, –1) or (–1, 1)]

and $z^2 = \frac{1}{\overline{\omega}^4} \Rightarrow |z|^2 = \frac{1}{|\omega|^4} \Rightarrow |z|^6 = \frac{1}{|\omega|^{12}}$ ....(2)

From (1) & (2) $|\omega| = 1$ & $|z| = 1 \Rightarrow z\overline{z} = \omega\overline{\omega} = 1$

Again $z^6 = \omega^{10}$ — (3) and $z^6 . \overline{\omega}^{12} = 1$

$z^6 = \frac{1}{\overline{\omega}^{12}} = \omega^{10}$ (from 3) $\Rightarrow (\omega\,\overline{\omega})^{10}\,(\overline{\omega})^2 = 1 \Rightarrow (\overline{\omega})^2 = 1$

$\Rightarrow \overline{\omega} = 1$ or $-1 \Rightarrow \omega = 1$ or -1

if $\omega = 1$ then $z^3 + 1 = 0$ and $z^2 = 1 \Rightarrow z = -1$

if $\omega = -1$ then $z^3 - 1 = 0$ and $z^2 = 1 \Rightarrow z = 1$

Hence $z = 1$ & $\omega = -1$ or $z = -1$ & $\omega = 1$]

Q.14 A function f is defined by $f(z) = (4 + i)z^2 + \alpha z + \gamma$ for all complex numbers z, where α and γ are complex numbers. If f (1) and f (i) are both real then the smallest possible value of $|\alpha| + |\gamma|$ equals

(A) 1 (B*) $\sqrt{2}$ (C) 2 (D) $2\sqrt{2}$

[Sol. Let $\alpha = a + ib$ and $\gamma = c + id$ **[13th, 25-01-2009]**

where a, b, c, d $\in$ R. We have to minimise $\sqrt{a^2+b^2} + \sqrt{c^2+d^2}$

now $\quad f(z) = (4+i)z^2 + z(a+ib) + (c+id)$

$\quad f(1) = 4 + i + a + ib + c + id$ is real

or $\quad (4+a+c) + i(1+b+d)$ is real

hence $\quad b + d + 1 = 0 \quad .....(1)$

$\quad f(i) = -(4+i) + i(a+ib) + (c+id)$ is real

$\quad f(i) = -4 - b + c + i(a + d - 1)$ is real

$\quad a + d = 1 \quad .....(2)$

from (1) and (2) $\quad a - b = 2 \quad .....(3)$

hence there is no restriction on 'c'. Let $c = 0$

hence $\quad |\alpha| + |\gamma| = \sqrt{a^2+b^2} + \sqrt{d^2}$

$$= \sqrt{4+2ab} + |d| \geq \sqrt{4+2ab} \geq \sqrt{2}$$

with equality if $d = 0$; $a = 1$ and $b = -1 \quad \Rightarrow \quad$ **(B)**]

Q.15 Given f (z) = the real part of a complex number z. For example, $f(3-4i) = 3$. If $a \in N$, $n \in N$ then the value of $\sum_{n=1}^{6a} \log_2 \left| f\left(\left(1+i\sqrt{3}\right)^n \right) \right|$ has the value equal to

(A) $18a^2 + 9a$ (B) $18a^2 + 7a$ (C) $18a^2 - 3a$ (D*) $18a^2 - a$

[Sol. $\left(1+i\sqrt{3}\right)^n = \left[2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^n = 2^n\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right)$

$f\left(\left(1+i\sqrt{3}\right)^n\right)$ = real part of $z = 2^n \cos\frac{n\pi}{3}$ **[12th, 04-01-2009]**

$$\therefore \quad \sum_{n=1}^{6a} \log_2 \left| 2^n \cos\frac{n\pi}{3} \right| = \sum_{n=1}^{6a} \left(n + \log_2 \left| \cos\frac{n\pi}{3} \right| \right) = \frac{6a(6a+1)}{2} + \underbrace{(-1-1+0-1-1+0)}_{\text{a such term}}$$

$$= 3a(6a+1) - 4a = 18a^2 - a$$ **Ans.**]

Q.16 It is given that complex numbers z_1 and z_2 satisfy $|z_1| = 2$ and $|z_2| = 3$. If the included angle of their corresponding vectors is 60° then $\left|\frac{z_1+z_2}{z_1-z_2}\right|$ can be expressed as $\frac{\sqrt{N}}{7}$ where N is natural number then N equals

(A) 126 (B) 119 (C*) 133 (D) 19

[Sol. Using cosine rule **[12th, 04-01-2009, P-1]**

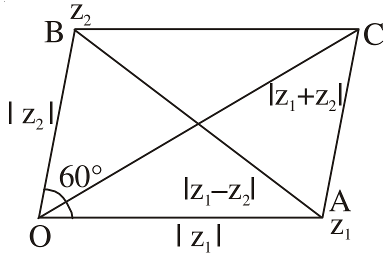
$$|z_1 + z_2| = \sqrt{|z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos 120°}$$

$$= \sqrt{4 + 9 + 2\cdot 3} = \sqrt{19}$$

and $\quad |z_1 - z_2| = \sqrt{|z_1|^2 + |z_2|^2 - 2|z_1||z_1|\cos 60°}$

$$= \sqrt{4+9-6} = \sqrt{7}$$

$$\therefore \quad \left|\frac{z_1+z_2}{z_1+z_2}\right| = \sqrt{\frac{19}{7}} = \frac{\sqrt{133}}{7} \quad \Rightarrow N = 133$$ **Ans.**]



Q.17 Let $f(x) = x^3 + ax^2 + bx + c$ be a cubic polynomial with real coefficients and all real roots. Also $| f(i) | = 1$ where $i = \sqrt{-1}$

Statement-1: All 3 roots of $f(x) = 0$ are zero

because

Statement-2: $a + b + c = 0$

(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.
(B*) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.
(C) Statement-1 is true, statement-2 is false.
(D) Statement-1 is false, statement-2 is true.

[Sol. Let $x_1, x_2, x_3 \in R$ be the roots of $f(x) = 0$ **[12th, 07-12-2008]**

$\therefore \quad f(x) = (x - x_1)(x - x_2)(x - x_3)$

$f(i) = (i - x_1)(i - x_2)(i - x_3)$

$| f(i) | = | x_1 - i | \; | x_2 - i | \; | x_3 - i | = 1$

$\therefore \quad \sqrt{x_1^2+1}\,\sqrt{x_2^2+1}\,\sqrt{x_3^2+1} = 1$

This is possible only if $x_1 = x_2 = x_3 = 0$

$\Rightarrow \quad f(x) = x^3 \quad \Rightarrow \quad a = 0 = b = c \quad \Rightarrow \quad a + b + c = 0$

(sum of coeffiients zero can not imply that all zero roots)]

Q.18 All complex numbers 'z' which satisfy the relation $\left|z - |z+1|\right| = \left|z + |z-1|\right|$ on the complex plane lie on the

(A) line $y = 0$ or an ellipse with foci $(-1, 0)$ and $(1, 0)$
(B) radical axis of the circles $| z - 1 | = 1$ and $| z + 1 | = 1$
(C) circle $x^2 + y^2 = 1$
(D*) line $x = 0$ or on a line segment joining $(-1, 0)$ to $(1, 0)$

[Sol. Given $| z - |z + 1| |^2 = | z + |z - 1| |^2$ **[13th, 01-02-2009]**

$\therefore \quad \left(z - |z+1|\right)\left(\bar{z} - |z+1|\right) = \left(z + |z-1|\right)\left(\bar{z} + |z-1|\right)$

$z\bar{z} - z|z+1| - \bar{z}|z+1| + |z+1|^2 = z\bar{z} + z|z-1| + \bar{z}|z-1| + |z-1|^2$

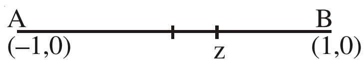
$|z+1|^2 - |z-1|^2 = (z+\bar{z})\left[|z-1| + |z+1|\right]$

$(z+1)(\bar{z}+1) - (z-1)(\bar{z}-1) = (z+\bar{z})\left[|z-1| + |z+1|\right]$

$\left(z\bar{z} + z + \bar{z} + 1\right) - \left(z\bar{z} - z - \bar{z} + 1\right) = (z+\bar{z})\left[|z-1| + |z+1|\right]$

$2(z+\bar{z}) = (z+\bar{z})\left[|z+1| + |z-1|\right]$

$(z+\bar{z})\left[|z+1| + |z-1| - 2\right] = 0$



$\Rightarrow$ either $z + \bar{z} = 0 \quad \Rightarrow \quad z$ is purely imaginary

$\Rightarrow$ z lies on y axis $\Rightarrow \quad x = 0$

or $|z + 1| + |z - 1| = 2$

$\Rightarrow$ z lie on the line segment joining $(-1, 0)$ and $(1, 0)$ $\Rightarrow$ **(D)**]

One ore more than one is/are correct:

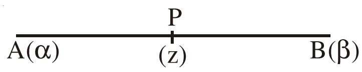
Q.19 Let A and B be two distinct points denoting the complex numbers α and β respectively. A complex number z lies between A and B where $z \neq \alpha$, $z \neq \beta$. Which of the following relation(s) hold good?

(A*) $|\alpha - z| + |z - \beta| = |\alpha - \beta|$

(B*) $\exists$ a positive real number 't' such that $z = (1 - t)\alpha + t\beta$

(C*) $\begin{vmatrix} z-\alpha & \bar{z}-\bar{\alpha} \\ \beta-\alpha & \bar{\beta}-\bar{\alpha} \end{vmatrix} = 0$ (D*) $\begin{vmatrix} z & \bar{z} & 1 \\ \alpha & \bar{\alpha} & 1 \\ \beta & \bar{\beta} & 1 \end{vmatrix} = 0$

[Sol. AP + PB = AB



$|z - \alpha| + |\beta - z| = |\beta - \alpha| \Rightarrow$ A is True **[Dpp-6, complex]** **[13th, 01-02-2009]**

Now $z = \alpha + t(\beta - \alpha)$

$= (1 - t)\alpha + t\beta$ where $t \in (0,1)$ $\Rightarrow$ B is True

again $\dfrac{z-\alpha}{\beta-\alpha}$ is real $\Rightarrow$ $\dfrac{z-\alpha}{\beta-\alpha} = \dfrac{\bar{z}-\bar{\alpha}}{\bar{\beta}-\bar{\alpha}}$

$\Rightarrow$ $\begin{vmatrix} z-\alpha & \bar{z}-\bar{\alpha} \\ \beta-\alpha & \bar{\beta}-\bar{\alpha} \end{vmatrix} = 0$ **Ans.**

also $\begin{vmatrix} z & \bar{z} & 1 \\ \alpha & \bar{\alpha} & 1 \\ \beta & \bar{\beta} & 1 \end{vmatrix} = 0$ if and only if $\begin{vmatrix} z-\alpha & \bar{z}-\bar{\alpha} & 0 \\ \alpha & \bar{\alpha} & 1 \\ \beta-\alpha & \bar{\beta}-\bar{\alpha} & 0 \end{vmatrix} = 0$

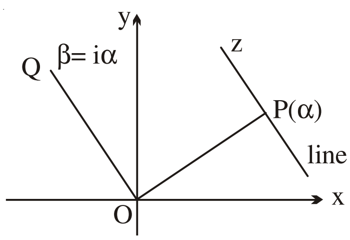
$\Rightarrow$ $\begin{vmatrix} (z-\alpha) & \bar{z}-\bar{\alpha} \\ \beta-\bar{\alpha} & \bar{\beta}-\bar{\alpha} \end{vmatrix} = 0$ **Ans.**]

Q.20 Equation of a straight line on the complex plane passing through a point P denoting the complex number α and perpendicular to the vector $\overrightarrow{OP}$ where 'O' is the origin can be written as

(A) $\text{Im}\left(\dfrac{z-\alpha}{\alpha}\right) = 0$ (B*) $\text{Re}\left(\dfrac{z-\alpha}{\alpha}\right) = 0$ (C) $\text{Re}(\bar{\alpha} z) = 0$ (D*) $\bar{\alpha} z + \alpha\bar{z} - 2|\alpha|^2 = 0$

[Sol. Required line is passing through $P(\alpha)$ and parallel to the vector $\overrightarrow{OQ}$ **[12th, 07-12-2008]**

hence $z = \alpha + i\lambda\alpha$, $\lambda \in R$



$\dfrac{z-\alpha}{\alpha}$ = purely imaginary

$\Rightarrow$ $\text{Re}\left(\dfrac{z-\alpha}{\alpha}\right) = 0$ $\Rightarrow$ **(B)** (multiply N^r and D^r by $\bar{\alpha}$)

$\Rightarrow$ $\text{Re}\left((z-\alpha)\bar{\alpha}\right) = 0$ $\Rightarrow$ $\text{Re}\left(z\bar{\alpha} - |\bar{\alpha}|\right) = 0$

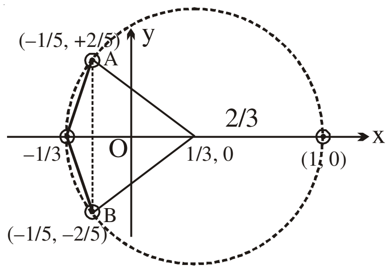
also $\dfrac{z-\alpha}{\alpha} + \dfrac{\bar{z}-\bar{\alpha}}{\bar{\alpha}} = 0$

$\bar{\alpha}(z-\alpha) + \alpha(\bar{z}-\bar{\alpha}) = 0$

$\bar{\alpha} z + \alpha\bar{z} - 2|\alpha|^2 = 0$ $\Rightarrow$ **(D)**]

Q.21 Which of the following represents a point on an argands' plane, equidistant from the roots of the equation $(z+1)^4 = 16z^4$?

(A) (0, 0) (B) $\left(-\frac{1}{3}, 0\right)$ (C*) $\left(\frac{1}{3}, 0\right)$ (D) $\left(0, \frac{2}{\sqrt{5}}\right)$



[Hint: $\left(\frac{z+1}{z}\right)^4 = 16 \Rightarrow \frac{z+1}{z} = 2$ or -2 or i or $-i$

Roots are $1\,;\, -\frac{1}{3}\,;\, \left(-\frac{1}{5}-\frac{2}{5}i\right)$ and $\left(-\frac{1}{5}+\frac{2}{5}i\right)$

Note that $\left(-\frac{1}{3}, 0\right)$ and (1, 0) are equidistant from $\left(\frac{1}{3}, 0\right)$ and since it lies on the perpendicular bisector of AB, it will be equidistant from A and B also.

Alternatively: $|z+1| = 2|z|$

$(z+1)(\bar{z}+1) = 4(z\bar{z})$ .....(1)

This is the equation of circle with centre (1/3, 0) which is equidistant from the root of the equation.]]

Q.22 If z is a complex number which simultaneously satisfies the equations

$3|z-12| = 5|z-8i|$ and $|z-4| = |z-8|$ then the Im(z) can be

(A) 15 (B) 16 (C*) 17 (D*) 8

[Sol. Let $z = x + iy$ **[12th, 06-01-2008]**

from 2[nd] equation $x = 6$ put in (1)

$3|(x-12)+yi| = 5|x+(y-8)i|$

$9[36+y^2] = 25[36+(y-8)^2]$ (substituting $x = 6$)

$9 \cdot 36 + 9y^2 = 25 \cdot 36 + 25[y^2 + 64 - 16y]$

$16y^2 - 25 \cdot 16y + 36 \cdot 16 + 25 \cdot 64 = 0$

$y^2 - 25y + 36 + 100 = 0$

$y^2 - 25y + 136 = 0$

$(y-17)(y-8) = 0$

then $y = 17$ or $y = 8 \Rightarrow$ (C), (D)]

Q.23 Let z_1, z_2, z_3 are the coordinates of the vertices of the triangle $A_1A_2A_3$. Which of the following statements are equivalent.

(A*) $A_1A_2A_3$ is an equilateral triangle.

(B*) $(z_1 + \omega z_2 + \omega^2 z_3)(z_1 + \omega^2 z_2 + \omega z_3) = 0$, where ω is the cube root of unity.

(C*) $\frac{z_2 - z_1}{z_3 - z_2} = \frac{z_3 - z_2}{z_1 - z_3}$ (D*) $\begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_2 & z_3 & z_1 \end{vmatrix} = 0$

Q.24 If $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_{n-1}$ are the imaginary n^{th} roots of unity then the product $\prod_{r=1}^{n-1}(i-\alpha_r)$ (where $i=\sqrt{-1}$) can take the value equal to

(A*) 0 (B*) 1 (C*) i (D*) (1 + i)

[Sol. $\frac{z^n-1}{z-1} = (z-\alpha_1)(z-\alpha_2)...........(z-\alpha_{n-1})$ **[13th, 08-03-2009, P-2]**

put $z = i$

$$\prod_{r=1}^{n-1}(i-\alpha_r) = \frac{i^n-1}{i-1} = \begin{cases} 0 & \text{if } n=4k \\ 1 & \text{if } n=4k+1 \\ 1+i & \text{if } n=4k+2 \\ i & \text{if } n=4k+3 \end{cases}]$$

[MATCH THE COLUMN]

&Q.25 Match the equation in z, in **Column-I** with the corresponding values of arg(z) in **Column-II**.

Column-I (equations in z)	**Column-II** (principal value of arg (z))
(A) $z^2 - z + 1 = 0$	(P) $-\ 2\pi/3$
(B) $z^2 + z + 1 = 0$	(Q) $-\ \pi/3$
(C) $2z^2 + 1 + i\sqrt{3} = 0$	(R) $\pi/3$
(D) $2z^2 + 1 - i\sqrt{3} = 0$	(S) $2\pi/3$

[Ans. (A) Q, R; (B) P, S; (C) Q, S; (D) P, R]

[Sol. (A) $z = \frac{1\pm\sqrt{3}i}{2} = \frac{1+i\sqrt{3}}{2}$ or $\frac{1-i\sqrt{3}}{2}$ **[12th, 07-12-2008, P-2][11th, 27-12-2009, P-2]**

$\text{amp } z = \frac{\pi}{3}$ or $\text{amp } z = -\frac{\pi}{3}$ $\Rightarrow$ **Q, R**

(B) $z = \frac{-1\pm\sqrt{3}i}{2} = \frac{-1+i\sqrt{3}}{2}$ or $\frac{-1-i\sqrt{3}}{2}$

$\text{amp } z = \frac{2\pi}{3}$ or $-\frac{2\pi}{3}$ $\Rightarrow$ **P, S**

(C) $2z^2 = -1-i\sqrt{3}$ $\Rightarrow$ $z^2 = \frac{-1-i\sqrt{3}}{2}$

$z^2 = w^2$

$\therefore$ $z = w$ or $z = -w$

$z = \cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)$ or $z = \cos\left(\frac{-\pi}{3}\right)+i\sin\left(\frac{-\pi}{3}\right)$

$\Rightarrow$ $\text{amp } z = -\frac{\pi}{3}$ or $\frac{2\pi}{3}$ $\Rightarrow$ **Q, S**

(D) $2z^2 + 1 - i\sqrt{3} = 0$ $\Rightarrow$ $z^2 = \frac{-1+i\sqrt{3}}{2}$

$z^2 = w = w^4$

$\therefore$ $z = w^2$ or $-w^2$

$z = \frac{-1-i\sqrt{3}}{2}$ $\qquad$ $z = \frac{1+i\sqrt{3}}{2}$

$\therefore \qquad \cos\left(\frac{-2\pi}{3}\right)+i\sin\left(\frac{-2\pi}{3}\right) \qquad \cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)$

amp (z) = $\frac{-2\pi}{3}$ or $\frac{\pi}{3}$ $\Rightarrow$ **P, R**]

Dpp-1

Q.9 Square root of $x^2 + \frac{1}{x^2} - \frac{4}{i}\left(x - \frac{1}{x}\right) - 6$ where $x \in R$ is equal to :

(A*) $\pm\left(x - \frac{1}{x} + 2i\right)$ (B) $\pm\left(x - \frac{1}{x} - 2i\right)$

(C) $\pm\left(x + \frac{1}{x} + 2i\right)$ (D) $\pm\left(x + \frac{1}{x} - 2i\right)$

[Hint: $E = \left(x - \frac{1}{x}\right)^2 + 4i\left(x - \frac{1}{x}\right) + 4i^2 = \left[\left(x - \frac{1}{x}\right) + 2i\right]^2 \Rightarrow$ A]

Dpp-2

Q.5 If S is the set of points in the complex plane such that $z(3 + 4i)$ is a real number then S denotes a
(A) circle (B) hyperbola (C*) line (D) parabola

[Hint: $\text{Im}(3 + 4i)(x + iy) = 0$
$3y + 4x = 0 \Rightarrow$ (B)]

Dpp-3

Q.3 Let $i = \sqrt{-1}$. Define a sequence of complex number by $z_1 = 0$, $z_{n+1} = z_n^2 + i$ for $n \geq 1$. In the complex plane, how far from the origin is z_{111}?
(A) 1 (B*) $\sqrt{2}$ (C) $\sqrt{3}$ (D) $\sqrt{110}$

[Hint: $z_3, z_7, z_{11}, z_{15}.......... z_{111}$ will have the same value $= -1 + i \Rightarrow$ result
i.e. periodicity with period 4]

Dpp-4

$Q.7 If $x = a + bi$ is a complex number such that $x^2 = 3 + 4i$ and $x^3 = 2 + 11i$ where $i = \sqrt{-1}$, then (a + b) equal to
(A) 2 (B*) 3 (C) 4 (D) 5

[Sol. $x = \frac{x^3}{x^2} = \frac{2+11i}{3+4i} \times \frac{3-4i}{3-4i} = \frac{6+(33-8)i-44i^2}{9-16i^2} = \frac{(6+44)+25i}{9+16} = \frac{50+25i}{25} = 2 + i$]

[19-2-2006, 12th & 13th]

Q.18 $\left(\sqrt[3]{3} + \left(3^{5/6}\right)i\right)^3$ is an integer where $i = \sqrt{-1}$. The value of the integer is equal to
(A) 24 (B*) – 24 (C) – 22 (D) – 21

[Sol. $[3^{1/3}(1 + \sqrt{3}\,i)]^3 = 3\left(1+\sqrt{3}\,i\right)^3 = 3 \cdot 8\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^3 = -24$ Ans.] **[13th 15-10-2006]**

Dpp-5

Paragraph of questions nos. 19 to 21

Consider the two complex numbers z and w such that $w = \frac{z-1}{z+2} = a + bi$, where $a, b \in R$.

Q.19 If $z = \text{CiS}\,\theta$ then, which of the following does hold good?

(A) $\cos\theta = \frac{1-5a}{1+4a}$ (B) $\sin\theta = \frac{9b}{1-4a}$

(C*) $(1 + 5a)^2 + (3b)^2 = (1 - 4a)^2$ (D) All of these

Q.20 Which of the following is the value of $-\frac{b}{a}$, whenever it exists?

(A) $3\tan\left(\frac{\theta}{2}\right)$ (B) $\frac{1}{3}\tan\left(\frac{\theta}{2}\right)$ (C) $-\frac{1}{3}\cot\theta$ (D*) $3\cot\left(\frac{\theta}{2}\right)$

Q.21 Which of the following equals $| z |$?

(A) $| w |$ (B*) $(a + 1)^2 + b^2$ (C) $a^2 + (b + 2)^2$ (D) $(a + 1)^2 + (b + 1)^2$

[Sol.

(19) Consider $z = \text{CiS}\,\theta$ and $a + ib = \frac{z-1}{z+2}$

$$\Rightarrow \quad a + ib = \frac{\text{Cis}\theta - 1}{\text{Cis}\theta + 2} = \frac{(\cos\theta - 1) + i\sin\theta}{(\cos\theta + 2) + i\sin\theta} = \frac{((\cos\theta - 1) + i\sin\theta)((\cos\theta + 2) - i\sin\theta)}{(\cos\theta + 2)^2 + \sin^2\theta}$$

$$= \frac{((\cos\theta - 1)(\cos\theta + 2) + \sin^2\theta) + i((\cos\theta + 2)\sin\theta - (\cos\theta - 1)\sin\theta)}{\cos^2\theta + 4\cos\theta + 4 + \sin^2\theta}$$

$$= \frac{(\cos^2\theta + \cos\theta + \sin^2\theta - 2) + i(3\sin\theta)}{4\cos\theta + 5} = \frac{\cos\theta - 1}{4\cos\theta + 5} + i\frac{3\sin\theta}{4\cos\theta + 5}$$

$$\Rightarrow \quad a = \frac{\cos\theta - 1}{4\cos\theta + 5}\,;\; b = \frac{3\sin\theta}{4\cos\theta + 5} \qquad ....(1)$$

$$\Rightarrow \quad 4a\cos\theta + 5a = \cos\theta - 1 \quad \Rightarrow \quad (4a - 1)\cos\theta = -(1 + 5a) \text{ or } \cos\theta = \frac{1+5a}{1-4a} \qquad ....(2)$$

Also, $4b\cos\theta + 5b = 3\sin\theta$.

i.e. $3\sin\theta = \frac{4b(1+5a)}{(1-4a)} + 5b = \frac{4b + 20ab + 5b - 20ab}{(1-4a)}$

$$\Rightarrow \quad 3\sin\theta = \frac{9b}{1-4a} \quad \text{or} \quad \sin\theta = \frac{3b}{1-4a} \qquad ....(3)$$

as, $\sin^2\theta + \cos^2\theta = 1$

$$\frac{9b^2}{(1-4a)^2} + \frac{(1+5a)^2}{(1-4a)^2} = 1 \quad \text{i.e.} \quad (1 + 5a)^2 + 9b^2 = (1 - 4a)^2 \quad ....(4)$$ **Ans.**

(20) from (1) $\frac{b}{a} = \frac{3\sin\theta}{\cos\theta - 1} \Rightarrow -\frac{b}{a} = \frac{6\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)} = 3\cot\left(\frac{\theta}{2}\right)$ Ans.

(21) from (4)

$\Rightarrow$ $25a^2 + 10a + 1 + 9b^2 = 16a^2 - 8a + 1$ $\Rightarrow$ $9a^2 + 18a + 9b^2 = 0$ or $a^2 + 2a + b^2 = 0$

i.e. $a^2 + 2a + 1 + b^2 = 1$ $\Rightarrow$ $(a + 1)^2 + b^2 = 1$

but $| z | \geq 0$ $\therefore$ $| z | = 1$

hence $| z | = 1$ $\Rightarrow$ $(a + 1)^2 + b^2 = | z |$ **Ans.**]

Dpp-1

Q.4 Given $i=\sqrt{-1}$, the value of the sum

$$\frac{1}{1+i}+\frac{1}{1-i}+\frac{1}{-1+i}+\frac{1}{-1-i}+\frac{2}{1+i}+\frac{2}{1-i}+\frac{2}{-1+i}+\frac{2}{-1-i}+$$

$$+\frac{3}{1+i}+\frac{3}{1-i}+\frac{3}{-1+i}+\frac{3}{-1-i}+\ldots\ldots..\frac{n}{1+i}+\frac{n}{1-i}+\frac{n}{-1+i}+\frac{n}{-1-i}, \text{ is}$$

(A) $2n^2+2n$ (B) $2\,i\,n^2+2\,i\,n$ (C) $(1+i)n^2$ (D*) none of these

[Sol. $\frac{1}{1+i}+\frac{1}{1-i}+\frac{1}{-1+i}+\frac{1}{-1-i}=\frac{(1-i)+(1+i)}{(1+i)(1-i)}+\frac{(-1-i)+(-1+i)}{(-1+i)(-1-i)}=\frac{2}{2}-\frac{2}{2}=0$

The next four terms of the sum will also give 0, since they are twice the first four terms, and so on, the entire sum is 0. The correct answer is (D)]

Q.7 If a point P denoting the complex number z moves on the complex plane such that, $|\text{Re } z| + |\text{Im } z| = 1$ then the locus of z is:

(A*) a square (B) a circle

(C) two intersecting lines (D) a line

[Hint: $|x| + |y| = 1$]

Dpp-2

Q.9 If $\frac{3+2i\sin x}{1-2i\sin x}$ is purely imaginary then x =

(A) $n\pi \pm \frac{\pi}{6}$ (B*) $n\pi \pm \frac{\pi}{3}$ (C) $2n\pi \pm \frac{\pi}{3}$ (D) $2n\pi \pm \frac{\pi}{6}$

Q.12 Let $z = 1-\sin\alpha + i\cos\alpha$ where $\alpha \in (0, \pi/2)$, then the modulus and the principal value of the argument of z are respectively :

(A*) $\sqrt{2(1-\sin\alpha)}, \left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$ (B) $\sqrt{2(1-\sin\alpha)}, \left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$

(C) $\sqrt{2(1+\sin\alpha)}, \left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$ (D) $\sqrt{2(1+\sin\alpha)}, \left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$

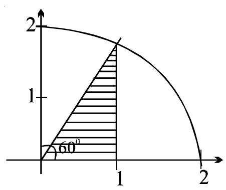
[Sol. $z = 1-\sin\alpha + i\cos\alpha$

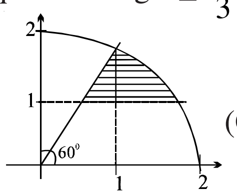
$|z| = \sqrt{(1-\sin\alpha)^2+\cos^2\alpha} = \sqrt{2-2\sin\alpha} = \sqrt{2(1-\sin\alpha)}$

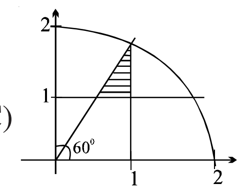
$\text{amp } z = \tan^{-1}\left(\frac{\cos\alpha}{1-\sin\alpha}\right) = \tan^{-1}\left(\frac{\cos\alpha/2+\sin\alpha/2}{\cos\alpha/2-\sin\alpha/2}\right) = \tan^{-1}\left(\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)\right) = \left(\frac{\pi}{4}+\frac{\alpha}{2}\right) \Rightarrow$ (A)]

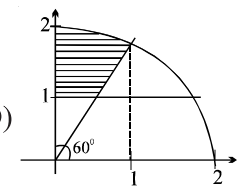
Dpp-3

Q.19 The region represented by inequalities $\text{Arg } Z \le \frac{\pi}{3}$; $|Z| \le 2$; $\text{Im}(z) \ge 1$ in the Argand diagram is given by

(A) 

(B*) 

(C) 

(D) 

Dpp-4

Q.5 Let A, B, C represent the complex numbers z_1, z_2, z_3 respectively on the complex plane. If the circumcentre of the triangle ABC lies at the origin, then the orthocentre is represented by the complex number :

(A) $z_1 + z_2 - z_3$ (B) $z_2 + z_3 - z_1$ (C) $z_3 + z_1 - z_2$ (D*) $z_1 + z_2 + z_3$

[Hint : Use O, G, C collinear]

Q.4 Given that z satisfies $z + \frac{1}{z} = 2\cos 13°$, find an angle B so that $0 < B < \frac{\pi}{2}$ and $z^2 + \frac{1}{z^2} = 2\cos B$.

(A) 23° (B) 24° (C) 25° (D*) 26°

Dpp-5

Q.1 If $\alpha = e^{i2\pi/n}$, then $(11 - \alpha)(11 - \alpha^2) \ldots\ldots (11 - \alpha^{n-1}) =$

(A) 11^{n-1} (B*) $\frac{11^n - 1}{10}$ (C) $\frac{11^{n-1} - 1}{10}$ (D) $\frac{11^{n-1} - 1}{11}$

[Sol. We have $x^n - 1 = (x - 1)(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_{n-1})$; note that α is the n^{th} root of unity

$\therefore \quad \frac{x^n - 1}{x - 1} = (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_{n-1})$

put $x = 11$ we get the result]

Q.11 If $D = \begin{vmatrix} a & \omega b & \omega^2 c \\ \omega^2 b & c & \omega a \\ \omega c & \omega^2 a & b \end{vmatrix}$; $D' = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

where ω is the non real cube root of unity then which of the following does not hold good?

(A) $D = 0$ if $(a + b + c) = 0$ and a ,b, c all distinct

(B) $D' = 0$ if $a = b = c$ and $(a + b + c) \neq 0$

(C*) $D = -D'$

(D) $D = D'$

[Hint : On expanding $D = D'$]

Q.3 If α, β be the roots of the equation $u^2 - 2u + 2 = 0$ & if $\cot\theta = x + 1$, then $\frac{(x+\alpha)^n - (x+\beta)^n}{\alpha - \beta}$ (where $n \in N$) is equal to

(A*) $\frac{\sin n\theta}{\sin^n \theta}$ (B) $\frac{\cos n\theta}{\cos^n \theta}$ (C) $\frac{\sin n\theta}{\cos^n \theta}$ (D) $\frac{\cos n\theta}{\sin^n \theta}$

[Hint: $u^2 - 2u + 2 = 0 \Rightarrow u = 1 \pm i \Rightarrow \alpha = 1 + i$ and $\beta = 1 - i$; also $x = \cot\theta - 1$

L.H.S. $\frac{[(\cot\theta - 1) + (1 + i)]^n - [(\cot\theta - 1) + (1 - i)]^n}{2i}$; using $x = \cot\theta - 1$

$= \frac{(\cos\theta + i\sin\theta)^n - (\cos\theta - i\sin\theta)^n}{\sin^n\theta \; 2i} = \frac{2i\sin n\theta}{\sin^n\theta \; 2i} = \frac{\sin n\theta}{\sin^n\theta}$]

Q.8 $\dfrac{(\cos\theta - i\sin\theta)^4}{(\sin\theta + i\cos\theta)^5} =$

(A) $\cos\theta - i\sin\theta$ (B) $\cos 9\theta - i\sin 9\theta$ (C) $\sin 9\theta - i\cos 9\theta$ (D*) $\sin\theta - i\cos\theta$

Q.9 If $p^2 - p + 1 = 0$ then the value of p^{3n} is $(n \in I)$:

(A*) 1, –1 (B) 1 (C) –1 (D) 0

[Hint: $p^2 - p + 1 = 0 \Rightarrow p = -\omega$ or $-\omega^2$, Hence $p^{3n} = -1$ or 1]

Q.13 Let z be the root of the equation $z^5 - 1 = 0$ such that $z \neq 1$. Then the value of $\sum_{r=15}^{50} z^r$ is equal to

(A*) 1 (B) i (C) – 1 (D) 0

[Sol. $z^5 - 1 = 0, \quad z \neq 1$ **[12th, 06-01-2008]**

now $S = z^{15} + z^{16} + z^{17} + \ldots\ldots + z^{50}$

$= z^{15}[1 + z + z^2 + \ldots\ldots.. + z^{35}]$

$= z^{15}\dfrac{[z^{36} - 1]}{z - 1}$

but z^{15} and z^{35} both are 1

$\therefore \quad S = \dfrac{z-1}{z-1} = 1$ Ans.]

Dpp-6

Q.1 If $z^2 - z + 1 = 0$ then the value of $\left(z + \frac{1}{z}\right)^2 + \left(z^2 + \frac{1}{z^2}\right)^2 + \left(z^3 + \frac{1}{z^3}\right)^2 + \ldots\ldots.. + \left(z^{24} + \frac{1}{z^{24}}\right)^2$

is equal to

(A) 24 (B) 32 (C*) 48 (D) None

[Hint: $z = -\omega$ or $-\omega^2$, also $\left(z^3 + \frac{1}{z^3}\right)^2 = 4$ etc (8 such pairs out of 24)

$\Rightarrow$ 32 + 16 = 48] **[12th Test (26-12-2004)]**

Q.4 If α & β are imaginary cube roots of unity then $\alpha^n + \beta^n$ is equal to :

(A*) $2\cos\frac{2n\pi}{3}$ (B) $\cos\frac{2n\pi}{3}$ (C) $2i\sin\frac{2n\pi}{3}$ (D) $i\sin\frac{2n\pi}{3}$

[Hint: $\alpha^n + \beta^n = 2\text{Re}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^n = 2\cos\frac{2n\pi}{3}$ Ans.]

Q.16 $P(z_1), Q(z_2), R(z_3)$ and $S(z_4)$ are four complex numbers representing the vertices of a rhombus which is not a square taken in order on the complex plane, then which one of the following hold(s) good?

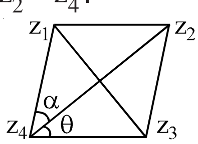
(A*) $\dfrac{z_1 - z_4}{z_2 - z_3}$ is purely real (B) amp $\dfrac{z_1 - z_4}{z_2 - z_4} \neq$ amp $\dfrac{z_2 - z_4}{z_3 - z_4}$

(C*) $\dfrac{z_1 - z_3}{z_2 - z_4}$ is purely imaginary (D*) $|z_1 - z_3| \neq |z_2 - z_4|$

[Hint: Obviously diagonals of rhombus are not equal

$\Rightarrow$ D is correct

$\therefore \quad \theta = \alpha \Rightarrow$ B is correct] **[12th (27-11-2005)]**



Q.17 If $1, z_1, z_2, z_3, ..., z_{n-1}$ be the n^{th} roots of unity and ω be a non real complex cube root of unity then the product $\prod_{r=1}^{n-1} (\omega - z_r)$ can be equal to

(A*) 0 (B*) 1 (C) – 1 (D*) $1 + \omega$

[Hint: $x^n - 1 = (x - 1)(x - z_1)(x - z_2) ...... (x - z_{n-1})$

$$\frac{x^n - 1}{x - 1} = (x - z_1)(x - z_2) ...... (x - z_{n-1}) \qquad \text{put } x = \omega$$

$$\prod_{r=1}^{n-1} (\omega - z_r) = \frac{\omega^n - 1}{\omega - 1} = \begin{bmatrix} 0 & \text{if } n \text{ is a multiple of } 3 \\ 1 & \text{if } n \text{ is of the form of } 3k+1,\ k \in I \\ 1+\omega & \text{if } n \text{ is of the form of } 3k+2,\ k \in I \end{bmatrix}$$]

Q.6 The number of solutions of the equation $z^2 + z = 0$ where z is a complex number, is :

(A) 4 (B) 3 (C*) 2 (D) 1

[Hint: $z(z + 1) = 0 \Rightarrow z = 0$ or $z = -1 \Rightarrow$ (C)]

Q.20 If q_1, q_2, q_3 are the roots of the equation, $x^3 + 64 = 0$, then the value of the determinant $\begin{vmatrix} q_1 & q_2 & q_3 \\ q_2 & q_3 & q_1 \\ q_3 & q_1 & q_2 \end{vmatrix}$ is

(A) 1 (B) 4 (C) 10 (D*) zero

[Hint: $q_1 = -4,\ q_2 = -4\omega,\ q_3 = -4\omega^2$]

Dpp-2

Q.10 The complex number $z = x + iy$ which satisfy the equation $\left|\frac{z - 5i}{z + 5i}\right| = 1$ lie on :

(A*) the x-axis (B) the straight line $y = 5$

(C) a circle passing through the origin (D) the y-axis

[Hint: perpendicular bisector of the line segment joining (0, 5) and (0, – 5) i.e. x-axis]

Dpp-3

Q.5 Let z_1, z_2, z_3 be three distinct complex numbers satisfying $|z_1 - 1| = |z_2 - 1| = |z_3 - 1|$.
If $z_1 + z_2 + z_3 = 3$ then z_1, z_2, z_3 must represent the vertices of :

(A*) an equilateral triangle (B) an isosceles triangle which is not equilateral

(C) a right triangle (D) nothing definite can be said .

[Hint: (1, 0) is equidistant from $z_1, z_2, z_3 \Rightarrow (1, 0)$ is circumcentre, also $\frac{z_1 + z_2 + z_3}{3} = 1$

$\Rightarrow$ (1, 0) is also the centroid $\Rightarrow$ A] **[to be changed]**

Q.18 If the equation, $z^3+(3+i)z^2-3z-(m+i)=0$ where $m \in R$ has at least one real root then m can have the value equal to

(A) 1 or 3 (B) 2 or 5 (C) 3 or 5 (D*) 1 or 5

[Sol. If α is a real root then

$\alpha^3+(3+i)\alpha^2-3\alpha-(m+i)=0$

$\therefore \quad \alpha^3+3\alpha^2-3\alpha-m=0$

and $\quad \alpha^2-1=0 \Rightarrow \quad \alpha=1$ or -1

if $\quad \alpha=1 \Rightarrow \quad m=1$

$\alpha=-1 \quad \Rightarrow \quad m=5 \Rightarrow \quad$ (D)]

Dpp-4

Q.19 $\sqrt{-1-\sqrt{-1-\sqrt{-1 \ldots\ldots \infty}}}$ is equal to :

(A*) ω or ω^2 (B) $-\omega$ or $-\omega^2$ (C) $1+i$ or $1-i$ (D) $-1+i$ or $-1-i$

where ω is the imaginary cube root of unity and $i=\sqrt{-1}$

[Hint: $z=\sqrt{-1-z} \Rightarrow z^2+z+1=0 \Rightarrow z=\omega$ or ω^2]

Q.1 If ω be a complex n^{th} root of unity, then $\sum_{r=1}^{n}(ar+b)\,\omega^{r-1}$ is equal to :

(A) $\frac{n\,(n+1)\,a}{2}$ (B) $\frac{nb}{1-n}$ (C*) $\frac{na}{\omega-1}$ (D) none

[Hint: $\sum_{r=1}^{n}(ar+b)\omega^{r-1} = (a+b)+(2a+b)\omega+(3a+b)\omega^2+\ldots\ldots\ldots\ldots+(na+b)\omega^{n-1}$

$= b\,(\underbrace{1+\omega+\omega^2+\ldots\ldots\ldots\ldots+\omega^{n-1}}_{\text{zero}}) + a\,(1+2\omega+3\omega^2+\ldots\ldots\ldots\, n\omega^{n-1})$

Now $\quad S = 1+2\omega+3\omega^2+\ldots\ldots\ldots\ldots\ldots n\omega^{n-1}$

$S\omega = +\omega+2\omega^2+\ldots\ldots\ldots\ldots\ldots\ldots+(n-1)\omega^{n-1}+n\omega^n$

$- \quad - \quad - \quad\quad\quad\quad -$

$S(1-\omega) = \underbrace{1+\omega+\omega^2+\ldots\ldots\ldots\ldots\ldots\ldots+\omega^{n-1}}_{\text{zero}} - n\omega^n = -n \quad\quad$ (as $\omega^n=1$)

$S=\frac{n}{\omega-1} \Rightarrow E=\frac{na}{\omega-1} \Rightarrow C$]

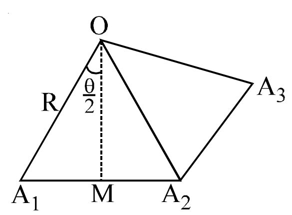
Dpp-5

Q.11 If A_r (r = 1, 2, 3, ... , n) are the vertices of a regular polygon inscribed in a circle of radius R, then $(A_1A_2)^2+(A_1A_3)^2+(A_1A_4)^2+\ldots\ldots+(A_1A_n)^2=$

(A) $\frac{nR^2}{2}$ (B*) $2nR^2$ (C) $4R^2\cot\frac{\pi}{2n}$ (D) $(2n-1)R^2$

[Hint: $\quad A_1A_2 = 2R\sin\frac{\theta}{2} \quad \left(\theta=\frac{2\pi}{n}\right)$

$(A_1A_2)^2 = 4R^2\sin^2\frac{\theta}{2} = 2R^2(1-\cos\theta)$



Hence

$$\text{L. H. S.} = 2R^2 \left[(1-\cos\theta) + (1-\cos 2\theta) + ... + (1-\cos(n-1)\theta) + (1-\cos n\theta)\right]$$

$$= 2R^2 \left[n - (\cos\theta + \cos 2\theta + ... + \cos n\theta\right]$$

$$= 2nR^2 \quad (\text{As } \cos\theta + \cos 2\theta + ... + \cos n\theta \text{ vanishes if } \theta = \frac{2\pi}{n}\,] \text{ Hence}$$

$$\text{L. H. S.} = 2R^2 \left[(1-\cos\theta) + (1-\cos 2\theta) + ... + (1-\cos(n-1)\theta) + (1-\cos n\theta)\right]$$

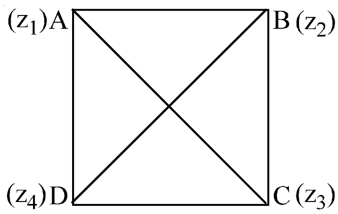
$$= 2R^2 \left[n - (\cos\theta + \cos 2\theta + ... + \cos n\theta\right]$$

$$= 2nR^2 \quad (\text{As } \cos\theta + \cos 2\theta + ... + \cos n\theta \text{ vanishes if } \theta = \frac{2\pi}{n}\,]$$

Q.14 If z_1, z_2, z_3, z_4 are the vertices of a square in that order, then which of the following do(es) not hold good?

(A) $\frac{z_1 - z_2}{z_3 - z_2}$ is purely imaginary (B) $\frac{z_1 - z_3}{z_2 - z_4}$ is purely imaginary

(C*) $\frac{z_1 - z_2}{z_3 - z_4}$ is purely imaginary (D) none of these



[Hint: AB is $\perp$ to BC $\Rightarrow \frac{z_1 - z_2}{z_3 - z_2}$ is pure imaginary

AC is $\perp$ to BD $\Rightarrow \frac{z_1 - z_3}{z_2 - z_4}$ is pure imaginary

AB is || to CD $\Rightarrow \frac{z_1 - z_2}{z_3 - z_4}$ is purely real $\Rightarrow$ C is incorrect]

Q.15 Given α, β respectively the fifth and the fourth non-real roots of unity, then the value of $(1 + \alpha)(1 + \beta)(1 + \alpha^2)(1 + \beta^2)(1 + \beta^3)(1 + \alpha^4)$ is

(A*) 0 (B) $(1 + \alpha + \alpha^2)(1 - \beta^2)$

(C) $(1 + \alpha)(1 + \beta + \beta^2)$ (D) 1

[Sol. As α is the fifth non-real root of unity

$\therefore \quad \alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1 = 0$

β is the fourth non real root of unity

$\therefore \quad \beta^3 + \beta^2 + \beta + 1 = 0$

Consider $(1 + \alpha)(1 + \alpha^2)(1 + \alpha^4)(1 + \beta)(1 + \beta^2)(1 + \beta^3)$

$= (1 + \alpha + \alpha^2 + \alpha^3)(1 + \alpha^4)(1 + \beta + \beta^2 + \beta^3)(1 + \beta^3) = 0$ Ans]

Dpp-4

[Hint:16 $\frac{OB}{OQ} = \frac{OA}{OM} = OA \quad (\therefore\ OM = 1)$

$OQ = \frac{OB}{OA}$ or $|z| = \frac{|z_2|}{|z_1|}$

Also amp $\frac{\vec{OB}}{\vec{OA}} = \text{amp } \vec{OB} - \text{amp } \vec{OA}$

or $\angle BOM - \angle AOM = \angle BOM - \angle BOQ \quad (\angle AOM = \angle BOQ = \theta)$

$\angle QOM = \text{amp } z$

Hence complex number corresponding to the point $Q = \frac{z_2}{z_1}$]

Q.19 If ω is an imaginary cube root of unity, then the value of ,

$(p+q)^3 + (p\omega + q\omega^2)^3 + (p\omega^2 + q\omega)^3$ is :

(A) $p^3 + q^3$ (B*) $3(p^3 + q^3)$

(C) $3(p^3 + q^3) - pq(p+q)$ (D) $3(p^3 + q^3) + pq(p+q)$

[Sol. $X = p + q$; $Y = p\omega + q\omega^2$ & $Z = p\omega^2 + q\omega \Rightarrow X + Y + Z = 0$

$\Rightarrow X^3 + Y^3 + Z^3 = 3XYZ$

$= 3(p+q)(p\omega + q\omega^2)(p\omega^2 + q\omega) = 3(p+q)(p^2 - pq + q^2) = 3(p^3 + q^3)$]

Dpp-6

Q.1 The expression $\left[\frac{1+\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}}{1+\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}}\right]^8 =$

(A) 1 (B*) -1 (C) i (D) $-i$

[Hint: Put $\sin\frac{\pi}{8} + i\cos\frac{\pi}{8} = z$ hence LHS $= \left(\frac{1+z}{1+\frac{1}{z}}\right)^8 = z^8 = \left(\sin\frac{\pi}{8} + i\cos\frac{\pi}{8}\right)^8$

$= \left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right)^8 = \cos 3\pi = -1$]

Q.4 If the equation of the perpendicular bisector of the line joining two complex numbers P(z_1) and Q(z_2) are the complex plane is $\bar{\alpha}z + \alpha\bar{z} + r = 0$ then α and r are respectively are

(A) $z_2 - z_1$ and $|z_1|^2 + |z_2|^2$ (B) $z_1 - z_2$ and $|z_1|^2 - |z_2|^2$

(C) $\bar{z}_2 - \bar{z}_1$ and $|z_1|^2 + |z_2|^2$ (D*) $z_2 - z_1$ and $|z_1|^2 - |z_2|^2$

[Sol. $(z - z_1)(\bar{z} - \bar{z}_1) = (z - z_2)(\bar{z} - \bar{z}_2)$ **[12th Test 16-1-2005]**

$-z\bar{z}_1 - z_1\bar{z} + z_1\bar{z}_1 = -z\bar{z}_2 - z_2\bar{z} + z_2\bar{z}_2$

$z(\bar{z}_2 - \bar{z}_1) + (z_2 - z_1)\bar{z} + |z_1|^2 - |z_2|^2 = 0$

or $\bar{\alpha}z + \alpha\bar{z} + r = 0 \Rightarrow \alpha = z_2 - z_1$ and $r = |z_1|^2 - |z_2|^2$]

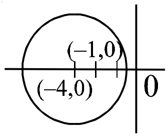
Q.6 If $|z+4| \le 3$, $z \in C$, then the greatest and least value of $|z+1|$ are :

(A) (7 , 1) (B) (6 , 1) (C*) (6 , 0) (D) none

[Hint: $(Z+1) = |(Z+4) + (-3)| \le |Z+4)| + 3$;

hence $|Z+1| \le |Z+4| + |-3| = 6$

$\therefore$ $|Z+1| \ge 0$]



[Alternative: note that $|Z+1|$ denotes the distance of Z from (–1,0)]