COMPLEX NUMBERS PRACTICE (part 2)

ROOTS OF COMPLEX NUMBERS

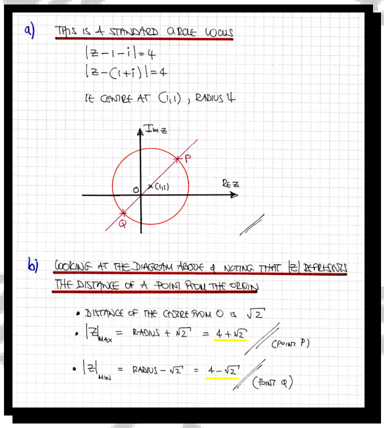
Question 1

$$z^4 = -16, \; z \in \mathbb{C}.$$

a) Solve the above equation, giving the answers in the form $a + bi$, where a and b are real numbers.

b) Plot the roots of the equation as points in an Argand diagram.

$$\boxed{z = \sqrt{2}(\pm 1 \pm i)}$$



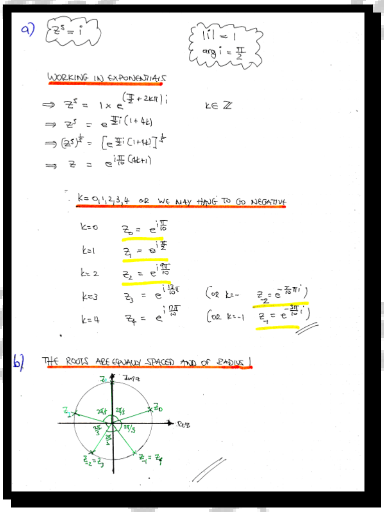
Question 2

$$z^5 = \mathrm{i}, \quad z \in \mathbb{C}.$$

a) Solve the equation, giving the roots in the form $r\mathrm{e}^{\mathrm{i}\theta}$, $r > 0$, $-\pi < \theta \le \pi$.

b) Plot the roots of the equation as points in an Argand diagram.

$$\boxed{z = \mathrm{e}^{\mathrm{i}\frac{\pi}{10}}, \quad z = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}, \quad z = \mathrm{e}^{\mathrm{i}\frac{9\pi}{10}}, \quad z = \mathrm{e}^{-\mathrm{i}\frac{3\pi}{10}}, \quad z = \mathrm{e}^{-\mathrm{i}\frac{7\pi}{10}}}$$



Question 3

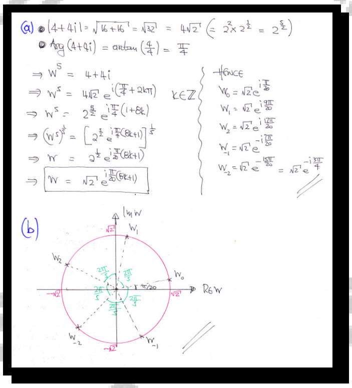
$$z = 4 + 4\mathrm{i}\,.$$

a) Find the fifth roots of z.

Give the answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, $r > 0$, $-\pi < \theta \le \pi$.

b) Plot the roots as points in an Argand diagram.

$$\boxed{\sqrt{2}\,\mathrm{e}^{\mathrm{i}\frac{\pi}{20}}, \sqrt{2}\,\mathrm{e}^{\mathrm{i}\frac{9\pi}{20}}, \sqrt{2}\,\mathrm{e}^{\mathrm{i}\frac{17\pi}{20}}, \sqrt{2}\,\mathrm{e}^{-\mathrm{i}\frac{7\pi}{20}}, \sqrt{2}\,\mathrm{e}^{-\mathrm{i}\frac{3\pi}{4}}}$$



Question 4

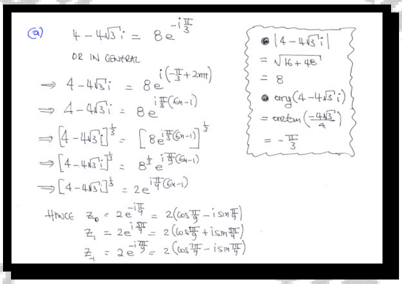
$$z = 4 - 4\sqrt{3}\mathrm{i}.$$

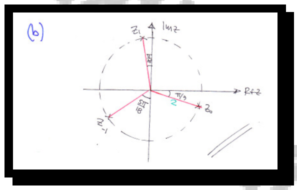
a) Find the cube roots of z.

Give the answers in polar form $r(\cos\theta + \mathrm{i}\sin\theta)$, $r > 0$, $-\pi < \theta \le \pi$.

b) Plot the roots as points in an Argand diagram.

$$\boxed{z = 2\left(\cos\frac{\pi}{9} - \mathrm{i}\sin\frac{\pi}{9}\right),\ z = 2\left(\cos\frac{5\pi}{9} + \mathrm{i}\sin\frac{5\pi}{9}\right),\ z = 2\left(\cos\frac{7\pi}{9} - \mathrm{i}\sin\frac{7\pi}{9}\right)}$$





Question 5

The following complex number relationships are given

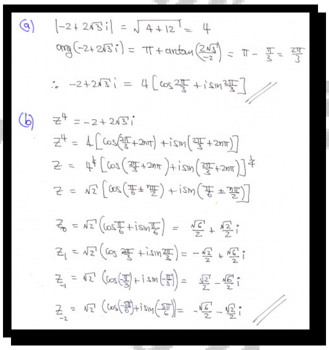
$$w = -2 + 2\sqrt{3}\mathrm{i}, \quad z^4 = w.$$

a) Express w in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $-\pi < \theta \le \pi$.

b) Find the possible values of z, giving the answers in the form $x + \mathrm{i}y$, where x and y are real numbers.

$$\boxed{w = 2\left[\cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2\pi}{3}\right)\right]},$$

$$\boxed{z = \tfrac{1}{2}\left(\sqrt{6} + \mathrm{i}\sqrt{2}\right),\ z = \tfrac{1}{2}\left(-\sqrt{2} + \mathrm{i}\sqrt{6}\right),\ z = \tfrac{1}{2}\left(\sqrt{2} - \mathrm{i}\sqrt{6}\right),\ z = \tfrac{1}{2}\left(-\sqrt{6} - \mathrm{i}\sqrt{2}\right)}$$


(a) $|-2+2\sqrt{3}i| = \sqrt{4+12} = 4$

$\arg(-2+2\sqrt{3}i) = \pi + \arctan\left(\frac{2\sqrt{3}}{-2}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$\therefore -2+2\sqrt{3}i = 4\left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right]$

(b) $z^4 = -2+2\sqrt{3}i$

$z^4 = 4\left[\cos\left(\frac{2\pi}{3}+2n\pi\right) + i\sin\left(\frac{2\pi}{3}+2n\pi\right)\right]$

$z = 4^{\frac{1}{4}}\left[\cos\left(\frac{2\pi}{3}+2n\pi\right) + i\sin\left(\frac{2\pi}{3}+2n\pi\right)\right]^{\frac{1}{4}}$

$z = \sqrt{2}\left[\cos\left(\frac{\pi}{6} \pm \frac{n\pi}{2}\right) + i\sin\left(\frac{\pi}{6} \pm \frac{n\pi}{2}\right)\right]$

$z_0 = \sqrt{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i$

$z_1 = \sqrt{2}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}i$

$z_{-1} = \sqrt{2}\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right) = \frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{2}i$

$z_{-2} = \sqrt{2}\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right) = -\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i$

Question 6

Find the cube roots of the imaginary unit i, giving the answers in the form $a + b\mathrm{i}$, where a and b are real numbers.

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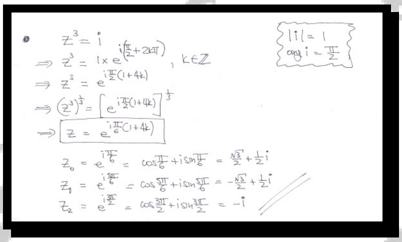
$$\boxed{z_1 = \frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i},\ z_2 = -\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i},\ z_3 = -\mathrm{i}}$$


$z^3 = i$

$\Rightarrow z^3 = 1 \times e^{i\left(\frac{\pi}{2}+2k\pi\right)}, \quad k \in \mathbb{Z}$

$\Rightarrow z^3 = e^{i\frac{\pi}{2}(1+4k)}$

$\Rightarrow (z^3)^{\frac{1}{3}} = \left[e^{i\frac{\pi}{2}(1+4k)}\right]^{\frac{1}{3}}$

$\Rightarrow \boxed{z = e^{i\frac{\pi}{6}(1+4k)}}$

$z_0 = e^{i\frac{\pi}{6}} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$

$z_1 = e^{i\frac{5\pi}{6}} = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$

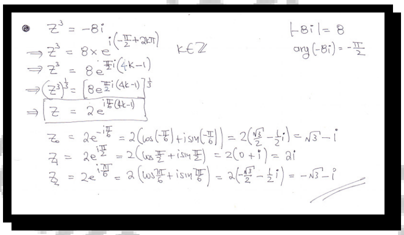
$z_2 = e^{i\frac{3\pi}{2}} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i$

$|i| = 1$, $\arg i = \frac{\pi}{2}$

Question 7

Find the cube roots of the complex number $-8i$, giving the answers in the form $a+bi$, where a and b are real numbers.

$$\boxed{z_1 = \sqrt{3} - i,\ \ z_2 = -\sqrt{3} - i,\ \ z_3 = 2i}$$



$z^3 = -8i$ $\qquad |-8i| = 8$, $\arg(-8i) = -\frac{\pi}{2}$

$\Rightarrow z^3 = 8 \times e^{i\left(-\frac{\pi}{2} + 2k\pi\right)}$ $\quad k \in \mathbb{Z}$

$\Rightarrow z^3 = 8e^{\frac{\pi}{2}i(4k-1)}$

$\Rightarrow (z^3)^{\frac{1}{3}} = \left[8e^{\frac{\pi}{2}i(4k-1)}\right]^{\frac{1}{3}}$

$\Rightarrow \boxed{z = 2e^{i\frac{\pi}{6}(4k-1)}}$

$z_0 = 2e^{-i\frac{\pi}{6}} = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right) = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = \sqrt{3} - i$

$z_1 = 2e^{i\frac{\pi}{2}} = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 2(0 + i) = 2i$

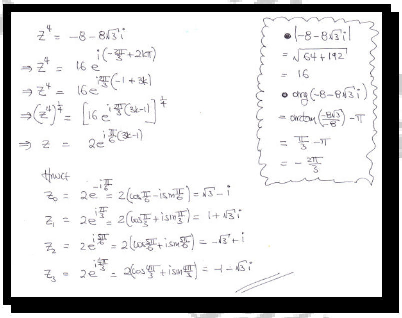
$z_2 = 2e^{i\frac{7\pi}{6}} = 2\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = -\sqrt{3} - i$

Question 8

$$z^4 = -8 - 8\sqrt{3}i,\ z \in \mathbb{C}.$$

Solve the above equation, giving the answers in the form $a+bi$, where a and b are real numbers.

$$\boxed{z = \sqrt{3} - i},\ \boxed{z = 1 + \sqrt{3}i},\ \boxed{z = -\sqrt{3} + i},\ \boxed{z = -1 - \sqrt{3}i}$$



$z^4 = -8 - 8\sqrt{3}i$

$\Rightarrow z^4 = 16e^{i\left(-\frac{2\pi}{3} + 2k\pi\right)}$

$\Rightarrow z^4 = 16e^{i\frac{2\pi}{3}(-1+3k)}$

$\Rightarrow (z^4)^{\frac{1}{4}} = \left[16e^{i\frac{2\pi}{3}(3k-1)}\right]^{\frac{1}{4}}$

$\Rightarrow z = 2e^{i\frac{\pi}{6}(3k-1)}$

$|-8-8\sqrt{3}i| = \sqrt{64+192} = 16$

$\arg(-8-8\sqrt{3}i) = \arctan\left(\frac{-8\sqrt{3}}{-8}\right) - \pi = \frac{\pi}{3} - \pi = -\frac{2\pi}{3}$

Hence

$z_0 = 2e^{-i\frac{\pi}{6}} = 2\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right) = \sqrt{3} - i$

$z_1 = 2e^{i\frac{\pi}{3}} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 1 + \sqrt{3}i$

$z_2 = 2e^{i\frac{5\pi}{6}} = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = -\sqrt{3} + i$

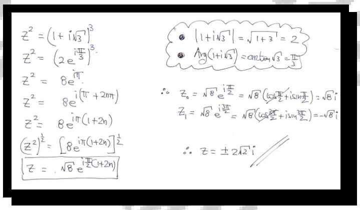
$z_3 = 2e^{i\frac{4\pi}{3}} = 2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = -1 - \sqrt{3}i$

Question 9

$$z^2 = \left(1 + \mathrm{i}\sqrt{3}\right)^3, \quad z \in \mathbb{C}.$$

Solve the above equation, giving the answers in the form $a + b\mathrm{i}$, where a and b are real numbers.

$$\boxed{z = \pm \mathrm{i}2\sqrt{2}}$$



Question 10

$$z^3 = 32 + 32\sqrt{3}\,\mathrm{i}, \quad z \in \mathbb{C}.$$

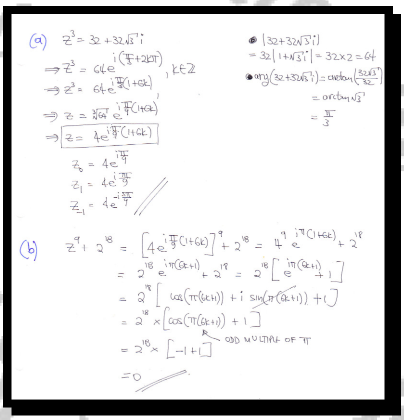
a) Solve the above equation.

Give the answers in exponential form $z = r\mathrm{e}^{\mathrm{i}\theta}$, $r > 0$, $-\pi < \theta \le \pi$.

b) Show that these roots satisfy the equation

$$w^9 + 2^{18} = 0.$$

$$\boxed{z = 4\mathrm{e}^{\mathrm{i}\frac{\pi}{9}},\ 4\mathrm{e}^{\mathrm{i}\frac{7\pi}{9}},\ 4\mathrm{e}^{-\mathrm{i}\frac{5\pi}{9}}}$$



Question 11

$$z^7 - 1 = 0, \ z \in \mathbb{C}.$$

One of the roots of the above equation is denoted by ω, where $0 < \arg \omega < \frac{\pi}{3}$.

a) Find ω in the form $\omega = r e^{i\theta}$, $\quad r > 0, \quad 0 < \theta \le \frac{\pi}{3}$.

b) Show clearly that

$$1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = 0.$$

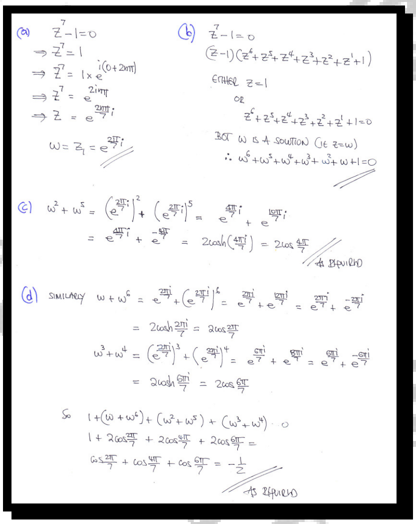
c) Show further that

$$\omega^2 + \omega^5 = 2\cos\left(\frac{4\pi}{7}\right).$$

d) Hence, using the results from the previous parts deduce that

$$\cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right) = -\frac{1}{2}.$$

$$\boxed{\omega = e^{i\frac{2\pi}{7}}}$$

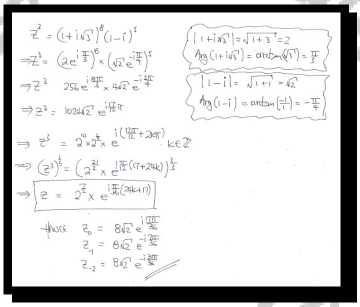


Question 12

$$z^3 = \left(1 + \mathrm{i}\sqrt{3}\right)^8 (1 - \mathrm{i})^5, \ z \in \mathbb{C}.$$

Find the three roots of the above equation, giving the answers in the form $k\sqrt{2}\,\mathrm{e}^{\mathrm{i}\theta}$, where $-\pi < \theta \le \pi$, $k \in \mathbb{Z}$.

$$\boxed{z = 8\sqrt{2}\,\mathrm{e}^{\mathrm{i}\theta}, \quad \theta = -\frac{31\pi}{36}, \ -\frac{7\pi}{36}, \ \frac{17\pi}{36}}$$



TRIGONOMETRIC IDENTITIES QUESTIONS

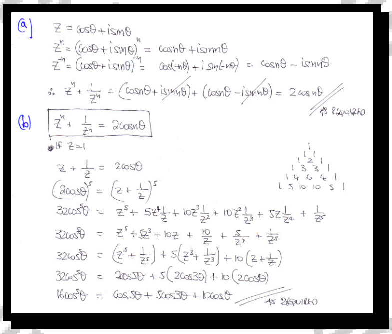
Question 1

If $z = \cos\theta + \mathrm{i}\sin\theta$, show clearly that …

a) … $z^n + \dfrac{1}{z^n} \equiv 2\cos n\theta$.

b) … $16\cos^5\theta \equiv \cos 5\theta + 5\cos 3\theta + 10\cos\theta$.

proof



(a)

$z = \cos\theta + i\sin\theta$

$z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$z^{-n} = (\cos\theta + i\sin\theta)^{-n} = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta$

$\therefore\ z^n + \frac{1}{z^n} = (\cos n\theta + i\sin n\theta) + (\cos n\theta - i\sin n\theta) = 2\cos n\theta$ AS REQUIRED

(b)

$z^n + \frac{1}{z^n} = 2\cos n\theta$

• If $z=1$

$z + \frac{1}{z} = 2\cos\theta$

$(2\cos\theta)^5 = \left(z + \frac{1}{z}\right)^5$

$32\cos^5\theta = z^5 + 5z^4\frac{1}{z} + 10z^3\frac{1}{z^2} + 10z^2\frac{1}{z^3} + 5z\frac{1}{z^4} + \frac{1}{z^5}$

$32\cos^5\theta = z^5 + 5z^3 + 10z + \frac{10}{z} + \frac{5}{z^3} + \frac{1}{z^5}$

$32\cos^5\theta = \left(z^5 + \frac{1}{z^5}\right) + 5\left(z^3 + \frac{1}{z^3}\right) + 10\left(z + \frac{1}{z}\right)$

$32\cos^5\theta = 2\cos 5\theta + 5(2\cos 3\theta) + 10(2\cos\theta)$

$16\cos^5\theta = \cos 5\theta + 5\cos 3\theta + 10\cos\theta$ AS REQUIRED

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Question 2

It is given that

$$\sin 5\theta \equiv 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta \,.$$

a) Use de Moivre's theorem to prove the validity of the above trigonometric identity.

It is further given that

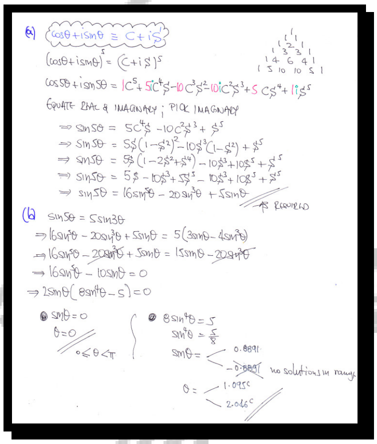
$$\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta \,.$$

b) Solve the equation

$$\sin 5\theta = 5\sin 3\theta \quad \text{for} \quad 0 \le \theta < \pi \,,$$

giving the solutions correct to 3 decimal places.

$\boxed{\theta = 0,\ 1.095^c,\ 2.046^c}$



Question 3

The complex number z is given by

$$z = e^{i\theta}, \; -\pi < \theta \le \pi .$$

a) Show clearly that

$$z^n + \frac{1}{z^n} \equiv 2\cos n\theta .$$

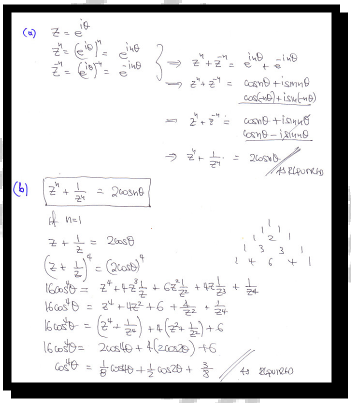
b) Hence show further that

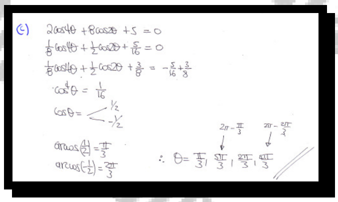
$$\cos^4 \theta \equiv \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8} .$$

c) Solve the equation

$$2\cos 4\theta + 8\cos 2\theta + 5 = 0, \; 0 \le \theta < 2\pi .$$

$$\boxed{\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}}$$





Question 4

The complex number z is given by

$$z = e^{i\theta}, \ -\pi < \theta \le \pi .$$

a) Show clearly that

$$z^n + \frac{1}{z^n} \equiv 2\cos n\theta .$$

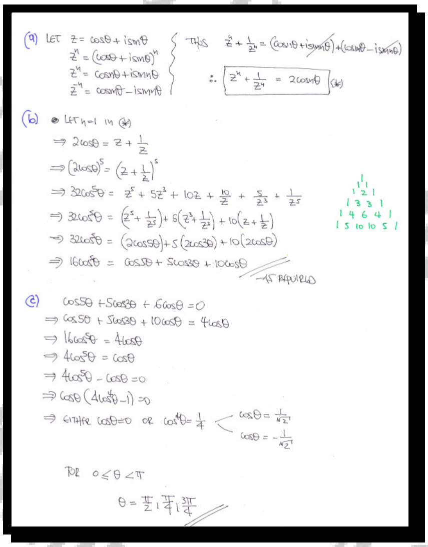
b) Hence show further that

$$16\cos^5\theta \equiv \cos 5\theta + 5\cos 3\theta + 10\cos\theta .$$

c) Use the results of part (a) and (b) to solve the equation

$$\cos 5\theta + 5\cos 3\theta + 6\cos\theta = 0, \ 0 \le \theta < \pi .$$

$$\boxed{\theta = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}}$$



(a) LET $z = \cos\theta + i\sin\theta$

$z^n = (\cos\theta + i\sin\theta)^n$

$z^n = \cos n\theta + i\sin n\theta$

$z^{-n} = \cos n\theta - i\sin n\theta$

THUS $z^n + \frac{1}{z^n} = (\cos n\theta + i\sin n\theta) + (\cos n\theta - i\sin n\theta)$

$\therefore \boxed{z^n + \frac{1}{z^n} = 2\cos n\theta}$ (*)

(b) • LET $n=1$ IN (*)

$\Rightarrow 2\cos\theta = z + \frac{1}{z}$

$\Rightarrow (2\cos\theta)^5 = \left(z + \frac{1}{z}\right)^5$

$\Rightarrow 32\cos^5\theta = z^5 + 5z^3 + 10z + \frac{10}{z} + \frac{5}{z^3} + \frac{1}{z^5}$

$\Rightarrow 32\cos^5\theta = \left(z^5 + \frac{1}{z^5}\right) + 5\left(z^3 + \frac{1}{z^3}\right) + 10\left(z + \frac{1}{z}\right)$

$\Rightarrow 32\cos^5\theta = (2\cos 5\theta) + 5(2\cos 3\theta) + 10(2\cos\theta)$

$\Rightarrow 16\cos^5\theta = \cos 5\theta + 5\cos 3\theta + 10\cos\theta$

AS REQUIRED

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

(c) $\cos 5\theta + 5\cos 3\theta + 6\cos\theta = 0$

$\Rightarrow \cos 5\theta + 5\cos 3\theta + 10\cos\theta = 4\cos\theta$

$\Rightarrow 16\cos^5\theta = 4\cos\theta$

$\Rightarrow 4\cos^5\theta = \cos\theta$

$\Rightarrow 4\cos^5\theta - \cos\theta = 0$

$\Rightarrow \cos\theta(4\cos^4\theta - 1) = 0$

$\Rightarrow$ EITHER $\cos\theta = 0$ OR $\cos^4\theta = \frac{1}{4}$ — $\cos\theta = \frac{1}{\sqrt{2}}$, $\cos\theta = -\frac{1}{\sqrt{2}}$

FOR $0 \le \theta < \pi$

$\theta = \frac{\pi}{2}, \frac{\pi}{4}, \frac{3\pi}{4}$

Question 5

De Moivre's theorem asserts that

$$\left(\cos\theta + \mathrm{i}\sin\theta\right)^n \equiv \cos n\theta + \mathrm{i}\sin n\theta,\ \theta \in \mathbb{R},\ n \in \mathbb{Q}.$$

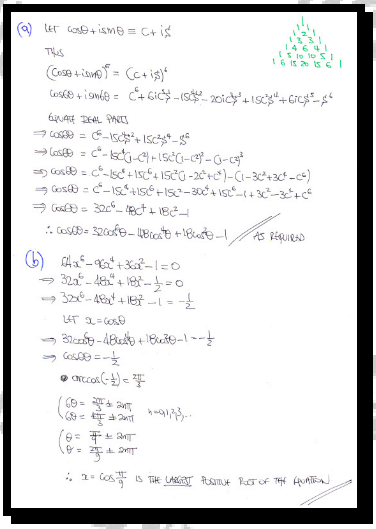
a) Use the theorem to prove the validity of the following trigonometric identity.

$$\cos 6\theta \equiv 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1.$$

b) Use the result of part (a) to find, in exact form, the largest positive root of the equation

$$64x^6 - 96x^4 + 36x^2 - 1 = 0.$$

$$\boxed{x = \cos\left(\frac{\pi}{9}\right)}$$



Question 6

Euler's identity states

$$e^{i\theta} \equiv \cos\theta + i\sin\theta\,,\ \theta \in \mathbb{R}\,.$$

a) Use the identity to show that

$$e^{in\theta} + e^{-in\theta} \equiv 2\cos n\theta\,.$$

b) Hence show further that

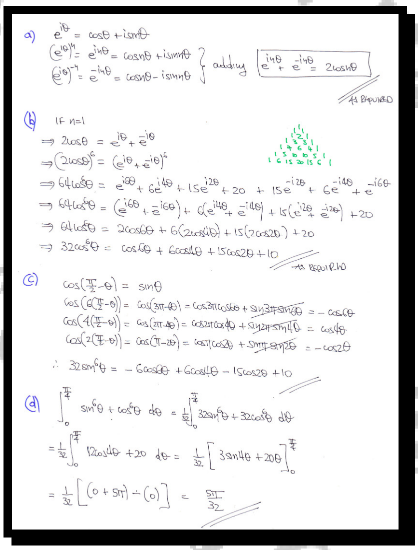
$$32\cos^6\theta \equiv \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10\,.$$

c) Use the fact that $\cos\left(\frac{\pi}{2} - \theta\right) \equiv \sin\theta$ to find a similar expression for $32\sin^6\theta$.

d) Determine the exact value of

$$\int_0^{\frac{\pi}{4}} \sin^6\theta + \cos^6\theta \ d\theta\,.$$

$\boxed{32\sin^6\theta = -\cos 6\theta + 6\cos 4\theta - 15\cos 2\theta + 10}$, $\boxed{\frac{5\pi}{32}}$



a) $e^{i\theta} = \cos\theta + i\sin\theta$

$(e^{i\theta})^n = e^{in\theta} = \cos n\theta + i\sin n\theta$

$(e^{i\theta})^{-n} = e^{-in\theta} = \cos n\theta - i\sin n\theta$

adding $\boxed{e^{in\theta} + e^{-in\theta} = 2\cos n\theta}$ AS REQUIRED

(b) IF $n=1$

$\Rightarrow 2\cos\theta = e^{i\theta} + e^{-i\theta}$

$\Rightarrow (2\cos\theta)^6 = (e^{i\theta} + e^{-i\theta})^6$

$\Rightarrow 64\cos^6\theta = e^{i6\theta} + 6e^{i4\theta} + 15e^{i2\theta} + 20 + 15e^{-i2\theta} + 6e^{-i4\theta} + e^{-i6\theta}$

$\Rightarrow 64\cos^6\theta = (e^{i6\theta} + e^{-i6\theta}) + 6(e^{i4\theta} + e^{-i4\theta}) + 15(e^{i2\theta} + e^{-i2\theta}) + 20$

$\Rightarrow 64\cos^6\theta = 2\cos 6\theta + 6(2\cos 4\theta) + 15(2\cos 2\theta) + 20$

$\Rightarrow 32\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$ AS REQUIRED

(c) $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

$\cos\left(6\left(\frac{\pi}{2} - \theta\right)\right) = \cos(3\pi - 6\theta) = \cos 3\pi\cos 6\theta + \sin 3\pi \sin 6\theta = -\cos 6\theta$

$\cos\left(4\left(\frac{\pi}{2} - \theta\right)\right) = \cos(2\pi - 4\theta) = \cos 2\pi\cos 4\theta + \sin 2\pi\sin 4\theta = \cos 4\theta$

$\cos\left(2\left(\frac{\pi}{2} - \theta\right)\right) = \cos(\pi - 2\theta) = \cos\pi\cos 2\theta + \sin\pi\sin 2\theta = -\cos 2\theta$

$\therefore 32\sin^6\theta = -6\cos 6\theta + 6\cos 4\theta - 15\cos 2\theta + 10$

(d) $\int_0^{\frac{\pi}{4}} \sin^6\theta + \cos^6\theta \ d\theta = \frac{1}{32}\int_0^{\frac{\pi}{4}} 32\sin^6\theta + 32\cos^6\theta \ d\theta$

$= \frac{1}{32}\int_0^{\frac{\pi}{4}} 12\cos 4\theta + 20 \ d\theta = \frac{1}{32}\left[3\sin 4\theta + 20\theta\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{32}\left[(0 + 5\pi) - (0)\right] = \frac{5\pi}{32}$

Question 7

De Moivre's theorem asserts that

$$(\cos\theta + i\sin\theta)^n \equiv \cos n\theta + i\sin n\theta, \ \theta \in \mathbb{R}, \ n \in \mathbb{Q}.$$

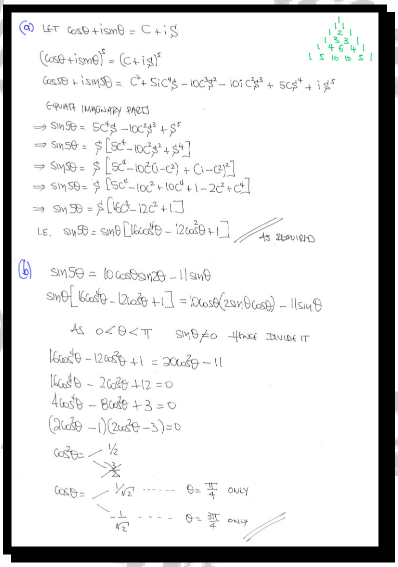
a) Use the theorem to prove validity of the following trigonometric identity

$$\sin 5\theta = \sin\theta\left(16\cos^4\theta - 12\cos^2\theta + 1\right).$$

b) Hence, or otherwise, solve the equation

$$\sin 5\theta = 10\cos\theta\sin 2\theta - 11\sin\theta, \ 0 < \theta < \pi.$$

$$\boxed{\theta = \frac{\pi}{4}, \frac{3\pi}{4}}$$


(a) LET $\cos\theta + i\sin\theta = C + iS$

$(\cos\theta + i\sin\theta)^5 = (C + iS)^5$

$\cos 5\theta + i\sin 5\theta = C^5 + 5iC^4S - 10C^3S^2 - 10iC^2S^3 + 5CS^4 + iS^5$

EQUATE IMAGINARY PARTS

$\Rightarrow \sin 5\theta = 5C^4S - 10C^2S^3 + S^5$

$\Rightarrow \sin 5\theta = S[5C^4 - 10C^2S^2 + S^4]$

$\Rightarrow \sin 5\theta = S[5C^4 - 10C^2(1-C^2) + (1-C^2)^2]$

$\Rightarrow \sin 5\theta = S[5C^4 - 10C^2 + 10C^4 + 1 - 2C^2 + C^4]$

$\Rightarrow \sin 5\theta = S[16C^4 - 12C^2 + 1]$

I.E. $\sin 5\theta = \sin\theta[16\cos^4\theta - 12\cos^2\theta + 1]$ AS REQUIRED

(b) $\sin 5\theta = 10\cos\theta\sin 2\theta - 11\sin\theta$

$\sin\theta[16\cos^4\theta - 12\cos^2\theta + 1] = 10\cos\theta(2\sin\theta\cos\theta) - 11\sin\theta$

AS $0 < \theta < \pi$ $\sin\theta \neq 0$ HENCE DIVIDE IT

$16\cos^4\theta - 12\cos^2\theta + 1 = 20\cos^2\theta - 11$

$16\cos^4\theta - 32\cos^2\theta + 12 = 0$

$4\cos^4\theta - 8\cos^2\theta + 3 = 0$

$(2\cos^2\theta - 1)(2\cos^2\theta - 3) = 0$

$\cos^2\theta = \frac{1}{2}$ ~~$\frac{3}{2}$~~

$\cos\theta = \frac{1}{\sqrt{2}}$ .... $\theta = \frac{\pi}{4}$ ONLY

$\cos\theta = -\frac{1}{\sqrt{2}}$ .... $\theta = \frac{3\pi}{4}$ ONLY

Question 8

It is given that

$$\sin 5\theta \equiv \sin\theta\left(16\cos^4\theta - 12\cos^2\theta + 1\right).$$

a) Use de Moivre's theorem to prove the validity of the above trigonometric identity.

Consider the general solution of the trigonometric equation

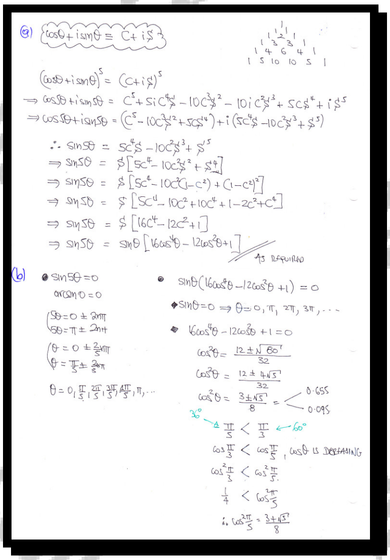
$$\sin 5\theta = 0.$$

b) Find exact simplified expressions for

$$\cos^2\left(\frac{\pi}{5}\right) \text{ and } \cos^2\left(\frac{2\pi}{5}\right),$$

fully justifying each step in the workings.

$\boxed{\cos^2\left(\frac{\pi}{5}\right) = \frac{3+\sqrt{5}}{8}}$, $\boxed{\cos^2\left(\frac{2\pi}{5}\right) = \frac{3-\sqrt{5}}{8}}$


(a) $\cos\theta + i\sin\theta \equiv C + iS$

$(\cos\theta + i\sin\theta)^5 = (C+iS)^5$

$\Rightarrow \cos 5\theta + i\sin 5\theta = C^5 + 5iC^4S - 10C^3S^2 - 10iC^2S^3 + 5CS^4 + iS^5$

$\Rightarrow \cos 5\theta + i\sin 5\theta = (C^5 - 10C^3S^2 + 5CS^4) + i(5C^4S - 10C^2S^3 + S^5)$

$\therefore \sin 5\theta = 5C^4S - 10C^2S^3 + S^5$

$\Rightarrow \sin 5\theta = S[5C^4 - 10C^2S^2 + S^4]$

$\Rightarrow \sin 5\theta = S[5C^4 - 10C^2(1-C^2) + (1-C^2)^2]$

$\Rightarrow \sin 5\theta = S[5C^4 - 10C^2 + 10C^4 + 1 - 2C^2 + C^4]$

$\Rightarrow \sin 5\theta = S[16C^4 - 12C^2 + 1]$

$\Rightarrow \sin 5\theta = \sin\theta[16\cos^4\theta - 12\cos^2\theta + 1]$ AS REQUIRED

(b) • $\sin 5\theta = 0$

$\arcsin 0 = 0$

$5\theta = 0 \pm 2n\pi$, $5\theta = \pi \pm 2n\pi$

$\theta = 0 \pm \frac{2}{5}n\pi$, $\theta = \frac{\pi}{5} \pm \frac{2}{5}n\pi$

$\theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}, \pi, \ldots$

• $\sin\theta(16\cos^4\theta - 12\cos^2\theta + 1) = 0$

• $\sin\theta = 0 \Rightarrow \theta = 0, \pi, 2\pi, 3\pi, \ldots$

• $16\cos^4\theta - 12\cos^2\theta + 1 = 0$

$\cos^2\theta = \frac{12 \pm \sqrt{80}}{32}$

$\cos^2\theta = \frac{12 \pm 4\sqrt{5}}{32}$

$\cos^2\theta = \frac{3 \pm \sqrt{5}}{8} = 0.655, 0.095$

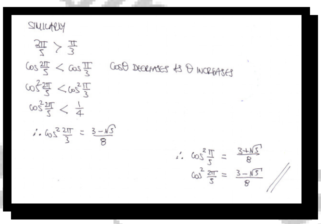
$36°$ $\frac{\pi}{5} < \frac{\pi}{3}$ $60°$

$\cos\frac{\pi}{3} < \cos\frac{\pi}{5}$, $\cos\theta$ IS DECREASING

$\cos^2\frac{\pi}{3} < \cos^2\frac{\pi}{5}$

$\frac{1}{4} < \cos^2\frac{\pi}{5}$

$\therefore \cos^2\frac{2\pi}{5} = \frac{3+\sqrt{5}}{8}$


SIMILARLY

$\frac{2\pi}{5} > \frac{\pi}{3}$

$\cos\frac{2\pi}{5} < \cos\frac{\pi}{3}$ $\cos\theta$ DECREASES AS θ INCREASES

$\cos^2\frac{2\pi}{5} < \cos^2\frac{\pi}{3}$

$\cos^2\frac{2\pi}{5} < \frac{1}{4}$

$\therefore \cos^2\frac{2\pi}{3} = \frac{3-\sqrt{5}}{8}$

$\therefore \cos^2\frac{\pi}{5} = \frac{3+\sqrt{5}}{8}$

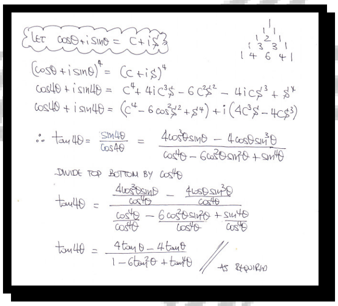
$\cos^2\frac{2\pi}{5} = \frac{3-\sqrt{5}}{8}$

Question 9

By considering the binomial expansion of $(\cos\theta + \mathrm{i}\sin\theta)^4$ show that

$$\tan 4\theta \equiv \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}.$$

proof



Question 10

By using de Moivre's theorem followed by a suitable trigonometric identity, show clearly that …

a) … $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$.

b) … $\cos 6\theta \equiv \left(2\cos^2\theta - 1\right)\left(16\cos^4\theta - 16\cos^2\theta + 1\right)$

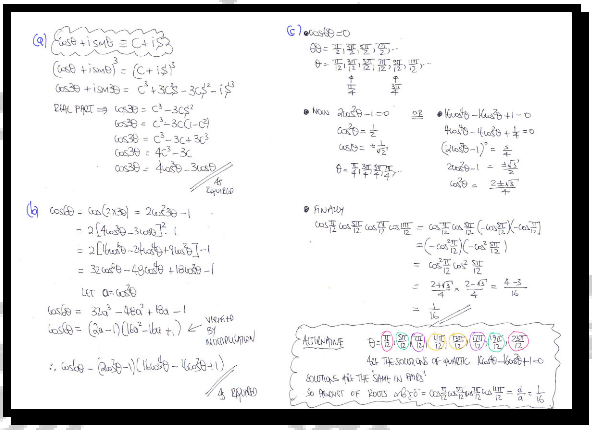
Consider the solutions of the equation.

$$\cos 6\theta = 0\,,\ 0 \le \theta \le \pi\ .$$

c) By fully justifying each step in the workings, find the exact value of

$$\cos\frac{\pi}{12}\cos\frac{5\pi}{12}\cos\frac{7\pi}{12}\cos\frac{11\pi}{12}\ .$$

$\boxed{\frac{1}{16}}$



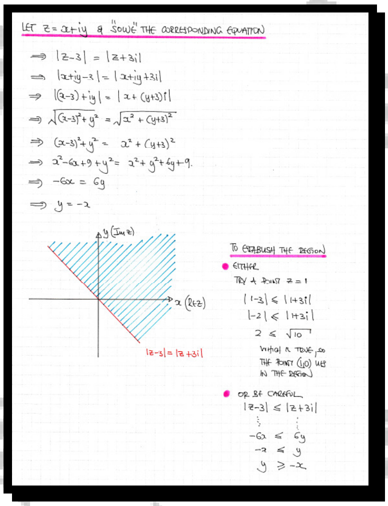
COMPLEX LOCI

Question 1

By finding a suitable Cartesian locus in the complex z plane, shade the region R that satisfies the inequality

$$|z-3| \le |z+3\mathrm{i}|.$$

$$\boxed{x+y \ge 0}$$



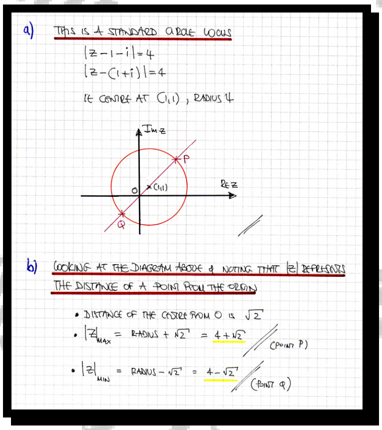
Question 2

$$|z-1-i|=4 \; , \; z \in \mathbb{C} .$$

a) Sketch the locus of the points that satisfy the above equation in a standard Argand diagram.

b) Find the minimum and maximum values of $|z|$ for points that lie on this locus.

$\boxed{z_{\min} = 4-\sqrt{2}}$, $\boxed{z_{\min} = 4+\sqrt{2}}$



Question 3

The complex number z represents the point $P(x, y)$ in the Argand diagram.

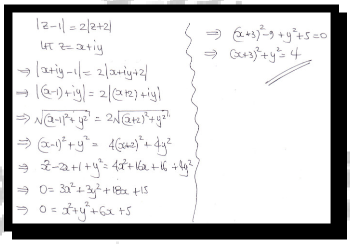
Given that

$$|z-1| = 2|z+2|,$$

show that the locus of P is given by

$$(x+3)^2 + y^2 = 4.$$

proof



Question 4

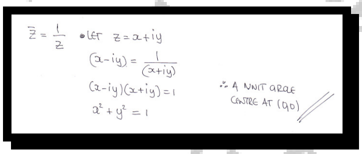
The complex number $z = x + \mathrm{i}y$ represents the point P in the complex plane.

Given that

$$\overline{z} = \frac{1}{z}, \ z \neq 0$$

determine a Cartesian equation for the locus of P.

$x^2 + y^2 = 1$



Question 5

Sketch, on the same Argand diagram, the locus of the points satisfying each of the following equations.

a) $|z-3+\mathrm{i}|=3$.

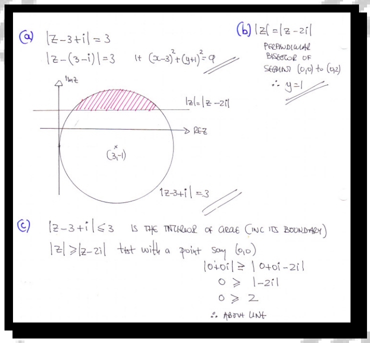
b) $|z|=|z-2\mathrm{i}|$.

Give in each case a Cartesian equation for the locus.

c) Shade in the sketch the region that is satisfied by both these inequalities

$$|z-3+\mathrm{i}|\le 3$$
$$|z|\ge|z-2\mathrm{i}|$$

$\boxed{(x-3)^2+(y+1)^2=9}$, $\boxed{y=1}$



Question 6

a) Sketch on the same Argand diagram the locus of the points satisfying each of the following equations.

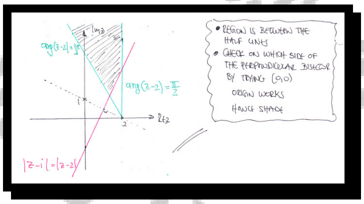
i. $|z-\mathrm{i}| = |z-2|$.

ii. $\arg(z-2) = \dfrac{\pi}{2}$.

b) Shade in the sketch the region that is satisfied by both these inequalities

$$|z-\mathrm{i}| \le |z-2| \quad \text{and} \quad \frac{\pi}{2} \le \arg(z-2) \le \frac{2\pi}{3}.$$

sketch



Question 7

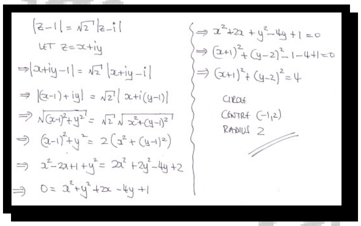
The complex number z represents the point $P(x, y)$ in the Argand diagram.

Given that

$$|z-1| = \sqrt{2}|z-i|,$$

show that the locus of P is a circle, stating its centre and radius.

$\boxed{(x+1)^2 + (y-2)^2 = 4}$, $\boxed{(-1,2), r=2}$



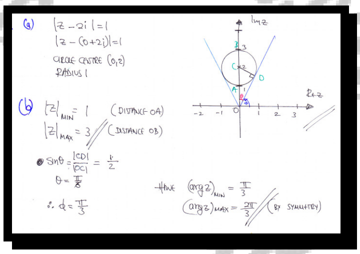
Question 8

$$|z-2i| = 1, \ z \in \mathbb{C}.$$

a) In the Argand diagram, sketch the locus of the points that satisfy the above equation.

b) Find the minimum value and the maximum value of $|z|$, and the minimum value and the maximum of $\arg z$, for points that lie on this locus.

$\boxed{|z|_{\text{min}} = 1}$, $\boxed{|z|_{\text{max}} = 3}$, $\boxed{\arg z_{\text{min}} = \frac{\pi}{3}}$, $\boxed{\arg z_{\text{max}} = \frac{2\pi}{3}}$



Question 9

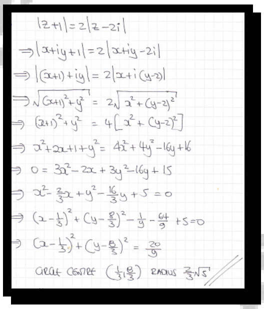
The complex number z represents the point $P(x, y)$ in the Argand diagram.

Given that

$$|z+1| = 2|z-2\text{i}|,$$

show that the locus of P is a circle and state its radius and the coordinates of its centre.

$$\boxed{\left(\frac{1}{3},\frac{8}{3}\right), \quad r=\frac{2}{3}\sqrt{5}}$$



Question 10

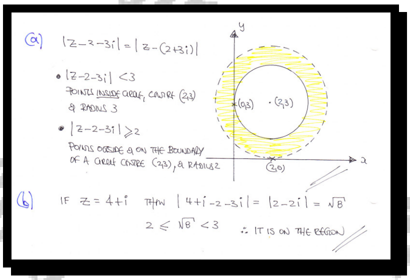
The complex number $z = x + \mathrm{i}y$ satisfies the relationship

$$2 \le |z - 2 - 3\mathrm{i}| < 3 .$$

a) Shade **accurately** in an Argand diagram the region represented by the above relationship.

b) Determine algebraically whether the point that represents the number $4+\mathrm{i}$ lies inside or outside this region.

inside the region



Question 11

Two sets of loci in the Argand diagram are given by the following equations

$$|z| = |z+2| \quad \text{and} \quad |z| = 2, \quad z \in \mathbb{C}.$$

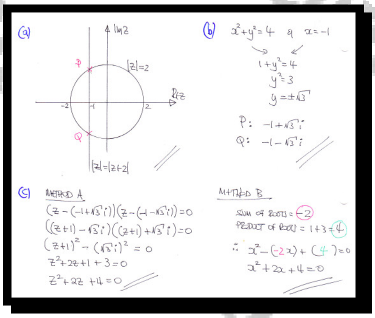
a) Sketch both these loci in the same Argand diagram.

The points P and Q in the Argand diagram satisfy both loci equations.

b) Write the complex numbers represented by P and Q, in the form $a + ib$, where a and b are real numbers.

c) Find a quadratic equation with real coefficients, whose solutions are the complex numbers represented by the points P and Q.

$\boxed{z = -1 \pm \sqrt{3}}$, $\boxed{z^2 + 2z + 4 = 0}$



Question 12

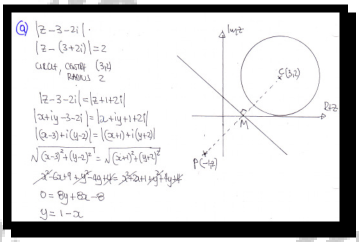
a) Sketch in the same Argand diagram the locus of the points satisfying each of the following equations

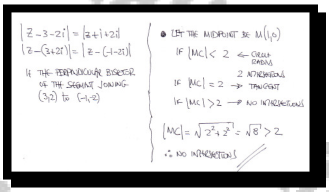
i. $|z-3-2\mathrm{i}|=2$.

ii. $|z-3-2\mathrm{i}|=|z+1+2\mathrm{i}|$.

b) Show by a **geometric** calculation that no points lie on both loci.

proof





Question 13

The point A represents the complex number on the z plane such that

$$|z - 6\mathrm{i}| = 2|z - 3|,$$

and the point B represents the complex number on the z plane such that

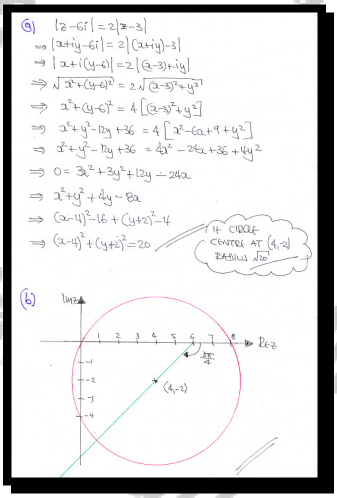
$$\arg(z - 6) = -\frac{3\pi}{4}.$$

a) Show that the locus of A as z varies is a circle, stating its radius and the coordinates of its centre.

b) Sketch, on the same z plane, the locus of A and B as z varies.

c) Find the complex number z, so that the point A coincides with the point B.

$\boxed{C(4,-2),\ r = \sqrt{20}}$, $\boxed{z = \left(4 - \sqrt{10}\right) + \mathrm{i}\left(-2 - \sqrt{10}\right)}$



(a) $|z-6i| = 2|z-3|$

$\Rightarrow |x+iy-6i| = 2|(x+iy)-3|$

$\Rightarrow |x+i(y-6)| = 2|(x-3)+iy|$

$\Rightarrow \sqrt{x^2+(y-6)^2} = 2\sqrt{(x-3)^2+y^2}$

$\Rightarrow x^2+(y-6)^2 = 4\left[(x-3)^2+y^2\right]$

$\Rightarrow x^2+y^2-12y+36 = 4\left[x^2-6x+9+y^2\right]$

$\Rightarrow x^2+y^2-12y+36 = 4x^2-24x+36+4y^2$

$\Rightarrow 0 = 3x^2+3y^2+12y-24x$

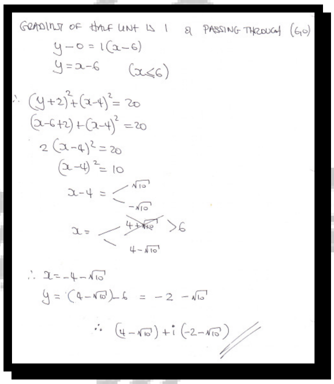
$\Rightarrow x^2+y^2+4y-8x$

$\Rightarrow (x-4)^2-16+(y+2)^2-4$

$\Rightarrow (x-4)^2+(y+2)^2 = 20$

IT CIRCLE CENTRE AT $(4,-2)$ RADIUS $\sqrt{20}$

(b)



GRADIENT OF HALF LINE IS 1 & PASSING THROUGH (6,0)

$y - 0 = 1(x-6)$

$y = x - 6 \quad (x \leqslant 6)$

$\therefore (y+2)^2+(x-4)^2 = 20$

$(x-6+2)+(x-4)^2 = 20$

$2(x-4)^2 = 20$

$(x-4)^2 = 10$

$x - 4 = \sqrt{10}$ or $-\sqrt{10}$

$x = $ ~~$4+\sqrt{10}$~~ >6 or $4-\sqrt{10}$

$\therefore x = -4-\sqrt{10}$

$y = (4-\sqrt{10})-6 = -2-\sqrt{10}$

$\therefore (4-\sqrt{10})+i(-2-\sqrt{10})$

Question 14

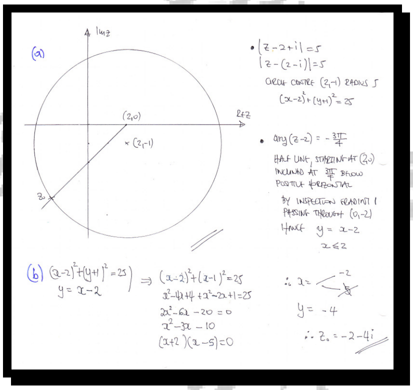
$$|z-2+\mathrm{i}|=5.$$

$$\arg(z-2)=-\frac{3\pi}{4}.$$

a) Sketch each of the above complex loci in the same Argand diagram.

b) Determine, in the form $x+\mathrm{i}y$, the complex number z_0 represented by the intersection of the two loci of part (a).

$$\boxed{z_0=-2-4\mathrm{i}}$$



Question 15

The locus of the point z in the Argand diagram, satisfy the equation

$$|z-2+\mathrm{i}|=\sqrt{3}.$$

a) Sketch the locus represented by the above equation.

The half line L with equation

$$y=mx-1, \quad x \geq 0, \quad m>0,$$

touches the locus described in part (a) at the point P.

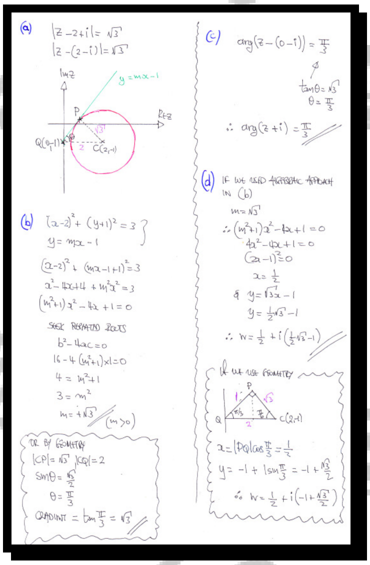
b) Find the value of m.

c) Write the equation of L, in the form

$$\arg(z-z_0)=\theta, \quad z_0 \in \mathbb{C}, \quad -\pi<\theta \leq \pi.$$

d) Find the complex number w, represented by the point P.

$$\boxed{m=\sqrt{3}}, \quad \boxed{\arg(z+\mathrm{i})=\frac{\pi}{3}}, \quad \boxed{w=\frac{1}{2}+\mathrm{i}\left(\frac{\sqrt{3}}{2}-1\right)}$$



Question 16

The complex numbers z_1 and z_2 are given by

$$z_1 = 1 + i\sqrt{3} \qquad \text{and} \qquad z_2 = iz_1 .$$

a) Label accurately the points representing z_1 and z_2, in an Argand diagram.

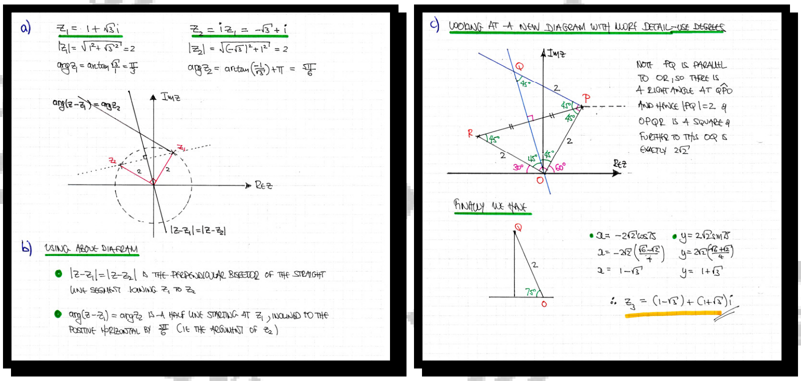
b) On the same Argand diagram, sketch the locus of the points z satisfying …

i. … $|z - z_1| = |z - z_2|$.

ii. … $\arg(z - z_1) = \arg z_2$.

c) Determine, in the form $x + iy$, the complex number z_3 represented by the intersection of the two loci of part (**b**).

$$\boxed{z_3 = \left(1 - \sqrt{3}\right) + i\left(1 + \sqrt{3}\right)}$$



Question 17

The complex number z lies in the region R of an Argand diagram, defined by the inequalities

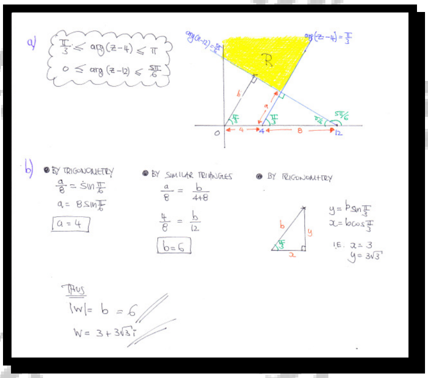
$$\frac{\pi}{3} \le \arg(z-4) \le \pi \quad \text{and} \quad 0 \le \arg(z-12) \le \frac{5\pi}{6}.$$

a) Sketch the region R, indicating clearly all the relevant details.

The complex number w lies in R, so that $|w|$ is minimum.

b) Find $|w|$, further giving w in the form $u+\mathrm{i}v$, where u and v are real numbers.

$\boxed{|w|=3}$, $\boxed{w=3+3\sqrt{3}\mathrm{i}}$



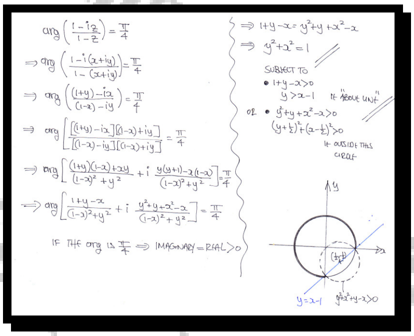
Question 18

The point P represents the number $z = x + \mathrm{i}y$ in an Argand diagram and further satisfies the equation

$$\arg\left(\frac{1-\mathrm{i}z}{1-z}\right) = \frac{\pi}{4}\ ,\ z \neq -\mathrm{i}\ .$$

Use an algebraic method to find an equation of the locus of P and sketch this locus accurately in an Argand diagram.

$x^2 + y^2 = 1$, such that $y > x - 1$



Question 19

The complex number $x + \mathrm{i}y$ in the z plane of an Argand diagram satisfies the inequality

$$x^2 + y^2 + x > 0.$$

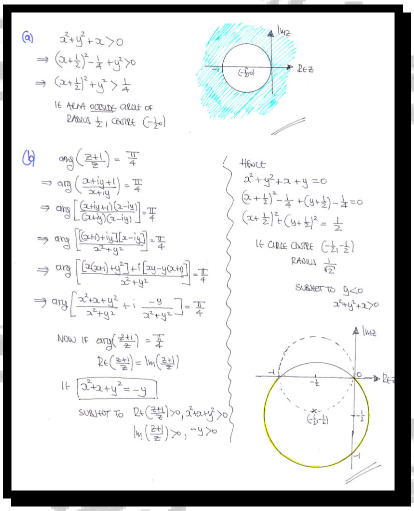
a) Sketch the region represented by this inequality.

A locus in the z plane of an Argand diagram is given by the equation

$$\arg\left(\frac{z+1}{z}\right) = \frac{\pi}{4}.$$

b) Sketch the locus represented by this equation.

sketch



Question 20

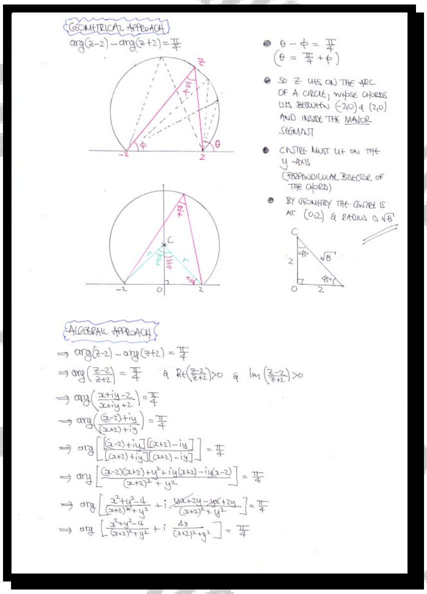
The complex number z satisfies the relationship

$$\arg(z-2)-\arg(z+2)=\frac{\pi}{4}.$$

Show that the locus of z is a circular arc, stating …

- … the coordinates of its endpoints.
- … the coordinates of its centre.
- … the length of its radius.

$\boxed{(-2,0),(2,0)}$, $\boxed{(0,2)}$, $\boxed{r=2\sqrt{2}}$



GEOMETRICAL APPROACH

$\arg(z-2)-\arg(z+2)=\frac{\pi}{4}$

- $\theta-\phi=\frac{\pi}{4}$ $\left(\theta=\frac{\pi}{4}+\phi\right)$
- SO z LIES ON THE ARC OF A CIRCLE, WHOSE CHORDS LIES BETWEEN $(-2,0)$ & $(2,0)$ AND INSIDE THE MAJOR SEGMENT
- CENTRE MUST LIE ON THE y-AXIS (PERPENDICULAR BISECTOR OF THE CHORD)
- BY GEOMETRY THE CENTRE IS AT $(0,2)$ & RADIUS IS $\sqrt{8}$

ALGEBRAIC APPROACH

$\Rightarrow \arg(z-2)-\arg(z+2)=\frac{\pi}{4}$

$\Rightarrow \arg\left(\frac{z-2}{z+2}\right)=\frac{\pi}{4}$ & $\text{Re}\left(\frac{z-2}{z+2}\right)>0$ & $\text{Im}\left(\frac{z-2}{z+2}\right)>0$

$\Rightarrow \arg\left(\frac{x+iy-2}{x+iy+2}\right)=\frac{\pi}{4}$

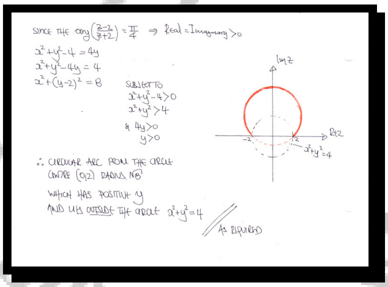
$\Rightarrow \arg\left(\frac{(x-2)+iy}{(x+2)+iy}\right)=\frac{\pi}{4}$

$\Rightarrow \arg\left[\frac{[(x-2)+iy][(x+2)-iy]}{[(x+2)+iy][(x+2)-iy]}\right]=\frac{\pi}{4}$

$\Rightarrow \arg\left[\frac{(x-2)(x+2)+y^2+iy(x+2)-iy(x-2)}{(x+2)^2+y^2}\right]=\frac{\pi}{4}$

$\Rightarrow \arg\left[\frac{x^2+y^2-4}{(x+2)^2+y^2}+i\frac{yx+2y-yx+2y}{(x+2)^2+y^2}\right]=\frac{\pi}{4}$

$\Rightarrow \arg\left[\frac{x^2+y^2-4}{(x+2)^2+y^2}+i\frac{4y}{(x+2)^2+y^2}\right]=\frac{\pi}{4}$



SINCE THE $\arg\left(\frac{z-2}{z+2}\right)=\frac{\pi}{4}$ $\Rightarrow$ Real = Imaginary >0

$x^2+y^2-4=4y$

$x^2+y^2-4y=4$

$x^2+(y-2)^2=8$

SUBJECT TO

$x^2+y^2-4>0$

$x^2+y^2>4$

& $4y>0$

$y>0$

$\therefore$ CIRCULAR ARC FROM THE CIRCLE CENTRE $(0,2)$ RADIUS $\sqrt{8}$ WHICH HAS POSITIVE y AND LIES OUTSIDE THE CIRCLE $x^2+y^2=4$

AS REQUIRED

COMPLEX FUNCTIONS

Question 1

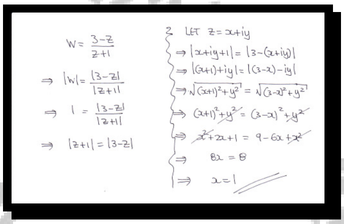
A transformation from the z plane to the w plane is defined by the complex function

$$w = \frac{3 - z}{z + 1}, \quad z \neq -1.$$

The locus of the points represented by the complex number $z = x + \mathrm{i}y$ is transformed to the circle with equation $|w| = 1$ in the w plane.

Find, in Cartesian form, an equation of the locus of the points represented by the complex number z.

$$\boxed{x = 1}$$



Question 2

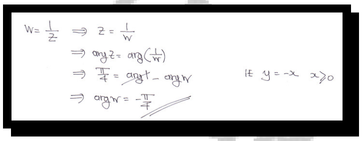
Find an equation of the locus of the points which lie on the half line with equation

$$\arg z = \frac{\pi}{4}, \quad z \neq 0$$

after it has been transformed by the complex function

$$w = \frac{1}{z}.$$

$$\boxed{\arg w = -\frac{\pi}{4}}$$



Question 3

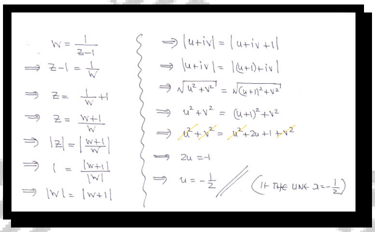
The complex function

$$w = \frac{1}{z-1}, \quad z \neq 1, z \in \mathbb{C}, \quad z \neq 1$$

transforms the point represented by $z = x + \mathrm{i}y$ in the z plane into the point represented by $w = u + \mathrm{i}v$ in the w plane.

Given that z satisfies the equation $|z| = 1$, find a Cartesian locus for w.

$$\boxed{u = -\frac{1}{2}}$$



Question 4

The complex function $w = f(z)$ is given by

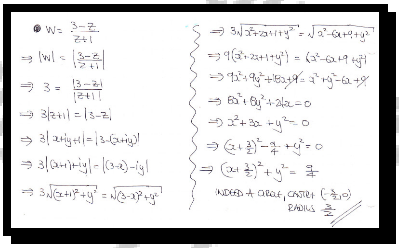
$$w = \frac{3-z}{z+1} \quad \text{where } z \in \mathbb{C}, \quad z \neq -1.$$

A point P in the z plane gets mapped onto a point Q in the w plane.

The point Q traces the circle with equation $|w| = 3$.

Show that the locus of P in the z plane is also a circle, stating its centre and its radius.

centre $\left(-\frac{3}{2}, 0\right)$, radius $= \frac{3}{2}$


$w = \dfrac{3-z}{z+1}$

$\Rightarrow |w| = \left|\dfrac{3-z}{z+1}\right|$

$\Rightarrow 3 = \dfrac{|3-z|}{|z+1|}$

$\Rightarrow 3|z+1| = |3-z|$

$\Rightarrow 3|x+iy+1| = |3-(x+iy)|$

$\Rightarrow 3|(x+1)+iy| = |(3-x)-iy|$

$\Rightarrow 3\sqrt{(x+1)^2+y^2} = \sqrt{(3-x)^2+y^2}$

$\Rightarrow 3\sqrt{x^2+2x+1+y^2} = \sqrt{x^2-6x+9+y^2}$

$\Rightarrow 9(x^2+2x+1+y^2) = (x^2-6x+9+y^2)$

$\Rightarrow 9x^2+9y^2+18x+9 = x^2+y^2-6x+9$

$\Rightarrow 8x^2+8y^2+24x = 0$

$\Rightarrow x^2+3x+y^2 = 0$

$\Rightarrow \left(x+\frac{3}{2}\right)^2 - \frac{9}{4} + y^2 = 0$

$\Rightarrow \left(x+\frac{3}{2}\right)^2 + y^2 = \frac{9}{4}$

INDEED A CIRCLE, CENTRE $\left(-\frac{3}{2}, 0\right)$
RADIUS $\frac{3}{2}$

Question 5

The general point $P(x, y)$ which is represent by the complex number $z = x + iy$ in the z plane, lies on the locus of

$$|z| = 1.$$

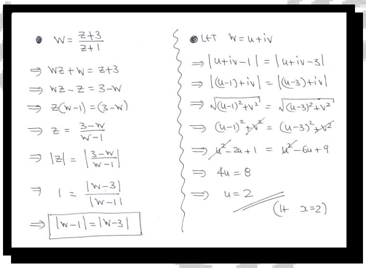
A transformation from the z plane to the w plane is defined by

$$w = \frac{z+3}{z+1}, \quad z \neq -1,$$

and maps the point $P(x, y)$ onto the point $Q(u, v)$.

Find, in Cartesian form, the equation of the locus of the point Q in the w plane.

$\boxed{u = 2}$



Question 6

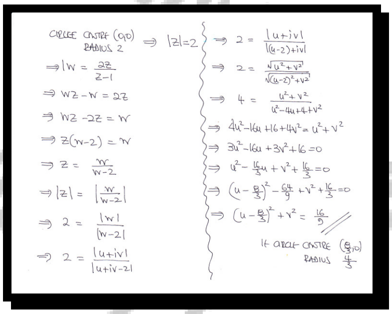
The point P represented by $z = x + \mathrm{i}y$ in the z plane is transformed into the point Q represented by $w = u + \mathrm{i}v$ in the w plane, by the complex transformation

$$w = \frac{2z}{z-1}, \ z \neq 1.$$

The point P traces a circle of radius 2, centred at the origin O.

Find a Cartesian equation of the locus of the point Q.

$$\boxed{\left(u - \frac{8}{3}\right)^2 + v^2 = \frac{16}{9}}$$



Question 7

The complex numbers $z = x + iy$ and $w = u + iv$ are represented by the points P and Q, respectively, in separate Argand diagrams.

The two numbers are related by the equation

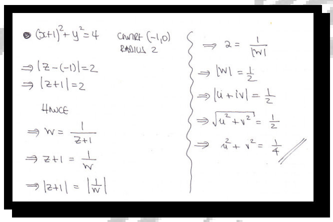
$$w = \frac{1}{z+1}, \quad z \neq -1.$$

If P is moving along the circle with equation

$$(x+1)^2 + y^2 = 4,$$

find in Cartesian form an equation of the locus of the point Q.

$$\boxed{u^2 + v^2 = \frac{1}{4}}$$



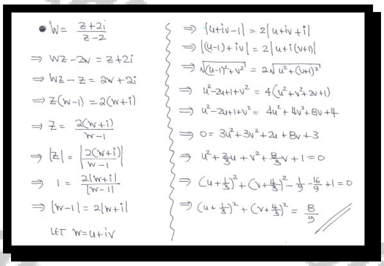
Question 8

A transformation from the z plane to the w plane is defined by the equation

$$w = \frac{z+2\text{i}}{z-2}, \quad z \neq 2.$$

Find in the w plane, in Cartesian form, the equation of the image of the circle with equation $|z| = 1$, $z \in \mathbb{C}$.

$$\boxed{\left(u+\frac{1}{3}\right)^2 + \left(v+\frac{4}{3}\right)^2 = \frac{8}{9}}$$



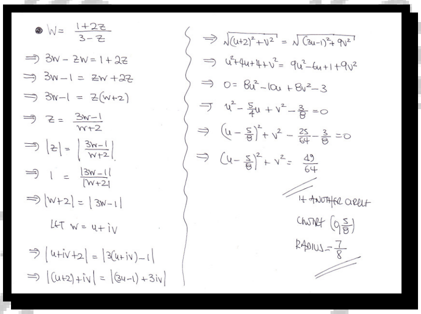
Question 9

A transformation from the z plane to the w plane is given by the equation

$$w = \frac{1+2z}{3-z}, \quad z \neq 3.$$

Show that the in the w plane, the image of the circle with equation $|z| = 1$, $z \in \mathbb{C}$, is another circle, stating its centre and its radius .

$$\boxed{\left(u-\frac{5}{8}\right)^2 + v^2 = \frac{49}{64}}, \quad \boxed{\text{centre}\left(\frac{5}{8}, 0\right)}, \quad \boxed{r = \frac{7}{8}}$$



Question 10

The complex numbers $z = x + iy$ and $w = u + iv$ are represented by the points P and Q, respectively, in separate Argand diagrams.

The two numbers are related by the equation

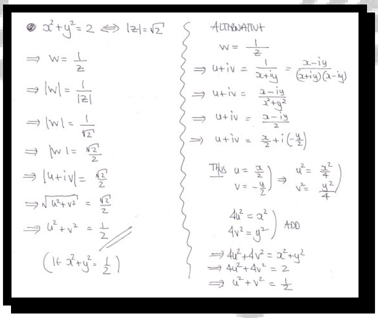
$$w = \frac{1}{z}, \quad z \neq 0 .$$

If P is moving along the circle with equation

$$x^2 + y^2 = 2 ,$$

find in Cartesian form an equation for the locus of the point Q.

$$\boxed{u^2 + v^2 = \frac{1}{2}}$$



Question 11

The complex numbers $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$ are represented by the points P and Q on separate Argand diagrams.

In the z plane, the point P is tracing the line with equation $y = x$.

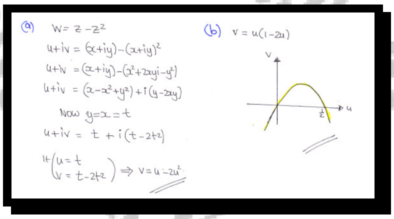
The complex numbers z and w are related by

$$w = z - z^2.$$

a) Find, in Cartesian form, the equation of the locus of Q in the w plane.

b) Sketch the locus traced by Q.

$$\boxed{v = u - 2u^2 \text{ or } y = x - 2x^2}$$



Question 12

The complex numbers $z = x + iy$ and $w = u + iv$ are represented by the points P and Q on separate Argand diagrams.

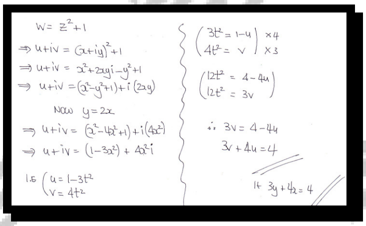
In the z plane, the point P is tracing the line with equation $y = 2x$.

Given that he complex numbers z and w are related by

$$w = z^2 + 1$$

find, in Cartesian form, the locus of Q in the w plane.

$\boxed{4u + 3v = 4 \quad \text{or} \quad 4x + 3y = 4}$



Question 13

A transformation of the z plane to the w plane is given by

$$w=\frac{1+3z}{1-z},\ z\in\mathbb{C},\ z\neq 1,$$

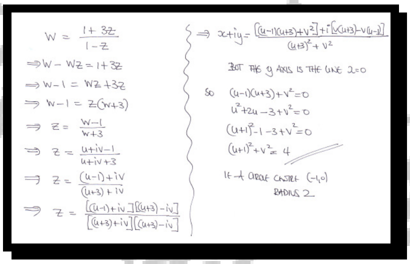
where $z=x+\text{i}y$ and $w=u+\text{i}v$.

The set of points that lie on the y axis of the z plane, are mapped in the w plane onto a curve C.

Show that a Cartesian equation of C is

$$(u+1)^2+v^2=4.$$

proof



$W = \frac{1+3Z}{1-Z}$

$\Rightarrow W - WZ = 1+3Z$

$\Rightarrow W-1 = WZ+3Z$

$\Rightarrow W-1 = Z(W+3)$

$\Rightarrow Z = \frac{W-1}{W+3}$

$\Rightarrow Z = \frac{u+iv-1}{u+iv+3}$

$\Rightarrow Z = \frac{(u-1)+iv}{(u+3)+iv}$

$\Rightarrow Z = \frac{[(u-1)+iv][(u+3)-iv]}{[(u+3)+iv][(u+3)-iv]}$

$\Rightarrow x+iy = \frac{[(u-1)(u+3)+v^2]+i[v(u+3)-v(u-1)]}{(u+3)^2+v^2}$

BUT THE y AXIS IS THE LINE $x=0$

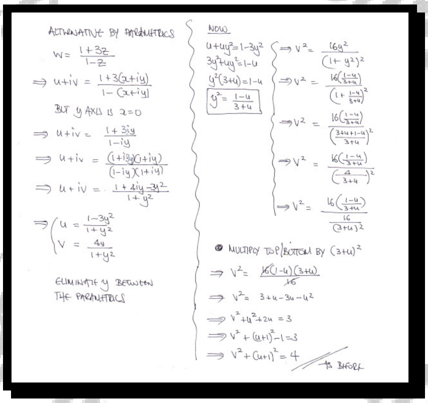
SO $(u-1)(u+3)+v^2=0$

$u^2+2u-3+v^2=0$

$(u+1)^2-1-3+v^2=0$

$(u+1)^2+v^2=4$

IE A CIRCLE CENTRE $(-1,0)$ RADIUS 2



ALTERNATIVE BY PARAMETRICS

$W = \frac{1+3Z}{1-Z}$

$\Rightarrow u+iv = \frac{1+3(x+iy)}{1-(x+iy)}$

BUT y AXIS IS $x=0$

$\Rightarrow u+iv = \frac{1+3iy}{1-iy}$

$\Rightarrow u+iv = \frac{(1+3iy)(1+iy)}{(1-iy)(1+iy)}$

$\Rightarrow u+iv = \frac{1+4iy-3y^2}{1+y^2}$

$\Rightarrow \begin{cases} u = \frac{1-3y^2}{1+y^2} \\ v = \frac{4y}{1+y^2} \end{cases}$

ELIMINATE y BETWEEN THE PARAMETRICS

NOW

$u+uy^2=1-3y^2$

$3y^2+uy^2=1-u$

$y^2(3+u)=1-u$

$y^2=\frac{1-u}{3+u}$

$\Rightarrow v^2 = \frac{16y^2}{(1+y^2)^2}$

$\Rightarrow v^2 = \frac{16\left(\frac{1-u}{3+u}\right)}{\left(1+\frac{1-u}{3+u}\right)^2}$

$\Rightarrow v^2 = \frac{16\left(\frac{1-u}{3+u}\right)}{\left(\frac{3+u+1-u}{3+u}\right)^2}$

$\Rightarrow v^2 = \frac{16\left(\frac{1-u}{3+u}\right)}{\left(\frac{4}{3+u}\right)^2}$

$\Rightarrow v^2 = \frac{16\left(\frac{1-u}{3+u}\right)}{\frac{16}{(3+u)^2}}$

MULTIPLY TOP/BOTTOM BY $(3+u)^2$

$\Rightarrow v^2 = \frac{16(1-u)(3+u)}{16}$

$\Rightarrow v^2 = 3+u-3u-u^2$

$\Rightarrow v^2+u^2+2u=3$

$\Rightarrow v^2+(u+1)^2-1=3$

$\Rightarrow v^2+(u+1)^2=4$

AS BEFORE

Question 14

The complex function $w = f(z)$ is given by

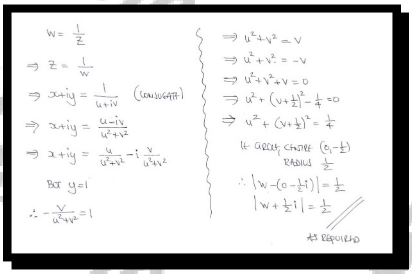
$$w = \frac{1}{z}, \ z \in \mathbb{C}, \ z \neq 0 .$$

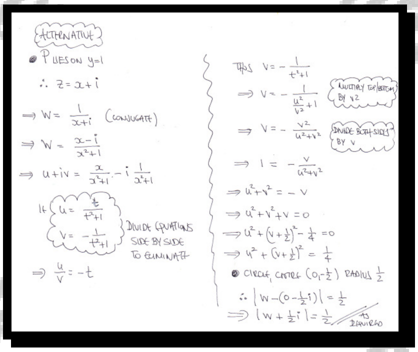
This function maps a general point $P(x, y)$ in the z plane onto the point $Q(u, v)$ in the w plane.

Given that P lies on the line with Cartesian equation $y = 1$, show that the locus of Q is given by

$$\left| w + \frac{1}{2}\mathrm{i} \right| = \frac{1}{2} .$$

proof





Question 15

A transformation of the z plane onto the w plane is given by

$$w = \frac{az+b}{z+c}, \ z \in \mathbb{C}, z \neq -c$$

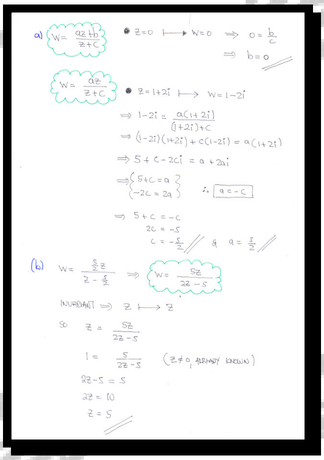
where a, b and c are real constants.

Under this transformation the point represented by the number $1+2i$ gets mapped to its complex conjugate and the origin remains invariant.

a) Find the value of a, the value of b and the value of c.

b) Find the number, other than the number represented by the origin, which remains invariant under this transformation.

$\boxed{a = \frac{5}{2}}$, $\boxed{b = 0}$, $\boxed{c = -\frac{5}{2}}$, $\boxed{z = 5}$



Question 16

A transformation of the z plane to the w plane is given by

$$w = \frac{1}{z-2}, \ z \in \mathbb{C}, \ z \neq 2$$

where $z = x + iy$ and $w = u + iv$.

The line with equation

$$2x + y = 3$$

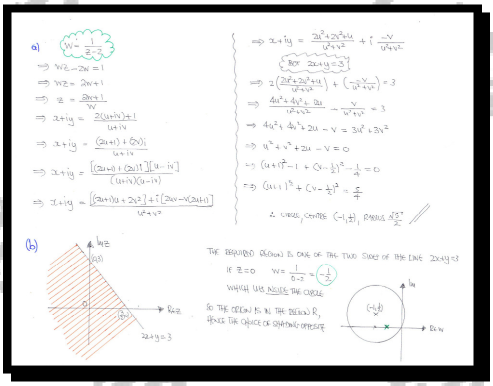
is mapped in the w plane onto a curve C.

a) Show that C represents a circle and determine the coordinates of its centre and the size of its radius.

The points of a region R in the z plane are mapped onto the points which lie inside C in the w plane.

b) Sketch and shade R in a suitable labelled Argand diagram, fully justifying the choice of region.

centre at $\left(-1, \frac{1}{2}\right)$, radius $= \frac{\sqrt{5}}{2}$



Question 17

A transformation of the z plane to the w plane is given by

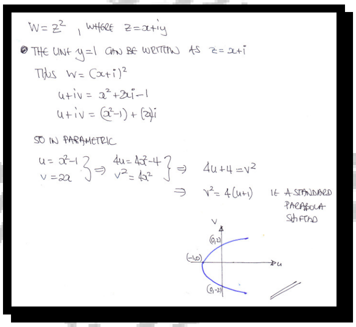
$$w = z^2, \ z \in \mathbb{C},$$

where $z = x + iy$ and $w = u + iv$.

The line with equation $y = 1$ is mapped in the w plane onto a curve C.

Sketch the graph of C, marking clearly the coordinates of all points where the graph of C meets the coordinate axes.

sketch



Question 18

A transformation of points from the z plane onto points in the w plane is given by the complex relationship

$$w = z^2, \quad z \in \mathbb{C},$$

where $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$.

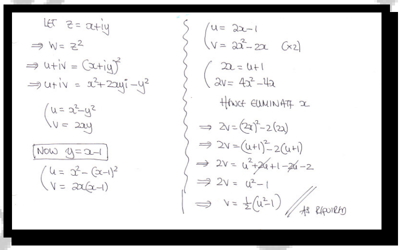
Show that if the point P in the z plane lies on the line with equation

$$y = x - 1,$$

the locus of this point in the w plane satisfies the equation

$$v = \frac{1}{2}\left(u^2 - 1\right).$$

proof



Question 19

A complex transformation from the z plane to the w plane is defined by

$$w = \frac{z+\mathrm{i}}{3+\mathrm{i}z}, \ z \in \mathbb{C}, \ z \neq 3\mathrm{i}.$$

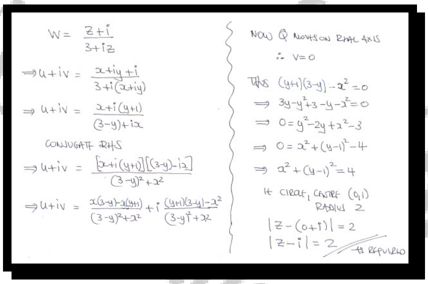
The point $P(x, y)$ is mapped by this transformation into the point $Q(u, v)$.

It is further given that Q lies on the real axis for all the possible positions of P.

Show that the P traces the curve with equation

$$|z-\mathrm{i}| = 2.$$

proof



$W = \frac{z+i}{3+iz}$

$\Rightarrow u+iv = \frac{x+iy+i}{3+i(x+iy)}$

$\Rightarrow u+iv = \frac{x+i(y+1)}{(3-y)+ix}$

CONJUGATE RHS

$\Rightarrow u+iv = \frac{[x+i(y+1)][(3-y)-ix]}{(3-y)^2+x^2}$

$\Rightarrow u+iv = \frac{x(3-y)-x(y+1)}{(3-y)^2+x^2} + i\,\frac{(y+1)(3-y)-x^2}{(3-y)^2+x^2}$

NOW Q MOVES ON REAL AXIS

$\therefore v=0$

THUS $(y+1)(3-y)-x^2=0$

$\Rightarrow 3y-y^2+3-y-x^2=0$

$\Rightarrow 0=y^2-2y+x^2-3$

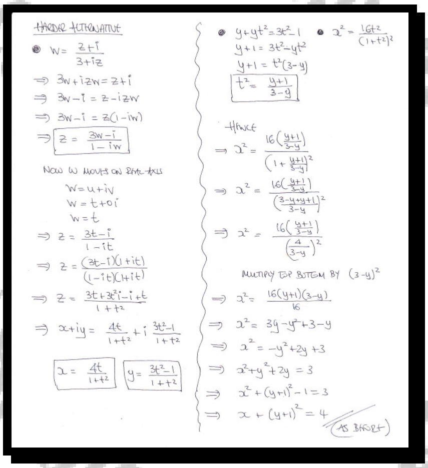
$\Rightarrow 0=x^2+(y-1)^2-4$

$\Rightarrow x^2+(y-1)^2=4$

IE CIRCLE, CENTRE $(0,1)$ RADIUS 2

$|z-(0+i)|=2$

$|z-i|=2$ AS REQUIRED



HARDER ALTERNATIVE

• $W = \frac{z+i}{3+iz}$

$\Rightarrow 3w+izw = z+i$

$\Rightarrow 3w-i = z-izw$

$\Rightarrow 3w-i = z(1-iw)$

$\Rightarrow z = \frac{3w-i}{1-iw}$

NOW W MOVES ON REAL AXIS

$w=u+iv$

$w=t+0i$

$w=t$

$\Rightarrow z = \frac{3t-i}{1-it}$

$\Rightarrow z = \frac{(3t-i)(1+it)}{(1-it)(1+it)}$

$\Rightarrow z = \frac{3t+3t^2i-i+t}{1+t^2}$

$\Rightarrow x+iy = \frac{4t}{1+t^2} + i\,\frac{3t^2-1}{1+t^2}$

$x = \frac{4t}{1+t^2}$ $\quad$ $y = \frac{3t^2-1}{1+t^2}$

• $y+yt^2 = 3t^2-1$

$y+1 = 3t^2-yt^2$

$y+1 = t^2(3-y)$

$t^2 = \frac{y+1}{3-y}$

• $x^2 = \frac{16t^2}{(1+t^2)^2}$

HENCE

$\Rightarrow x^2 = \frac{16\left(\frac{y+1}{3-y}\right)}{\left(1+\frac{y+1}{3-y}\right)^2}$

$\Rightarrow x^2 = \frac{16\left(\frac{y+1}{3-y}\right)}{\left(\frac{3-y+y+1}{3-y}\right)^2}$

$\Rightarrow x^2 = \frac{16\left(\frac{y+1}{3-y}\right)}{\left(\frac{4}{3-y}\right)^2}$

MULTIPLY TOP BOTTOM BY $(3-y)^2$

$\Rightarrow x^2 = \frac{16(y+1)(3-y)}{16}$

$\Rightarrow x^2 = 3y-y^2+3-y$

$\Rightarrow x^2 = -y^2+2y+3$

$\Rightarrow x^2+y^2+2y = 3$

$\Rightarrow x^2+(y+1)^2-1 = 3$

$\Rightarrow x+(y+1)^2 = 4$ (AS BEFORE)

Question 20

A transformation of the z plane to the w plane is given by

$$w = \frac{2z+1}{z}, \quad z \in \mathbb{C}, \quad z \neq 0$$

where $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$.

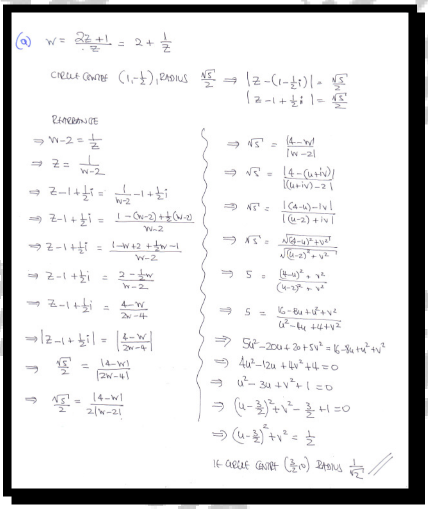
The circle C_1 with centre at $\left(1, -\frac{1}{2}\right)$ and radius $\frac{\sqrt{5}}{2}$ in the z plane is mapped in the w plane onto another curve C_2.

a) Show that C_2 is also a circle and determine the coordinates of its centre and the size of its radius.

The points inside C_1 in the z plane are mapped onto points of a region R in the w plane.

b) Sketch and shade R in a suitably labelled Argand diagram, fully justifying the choice of the region.

centre at $\left(\frac{3}{2}, 0\right)$, radius $= \frac{1}{\sqrt{2}}$



(a) $w = \frac{2z+1}{z} = 2 + \frac{1}{z}$

CIRCLE CENTRE $(1, -\frac{1}{2})$, RADIUS $\frac{\sqrt{5}}{2} \Rightarrow |z - (1 - \frac{1}{2}i)| = \frac{\sqrt{5}}{2}$

$|z - 1 + \frac{1}{2}i| = \frac{\sqrt{5}}{2}$

REARRANGE

$\Rightarrow w - 2 = \frac{1}{z}$

$\Rightarrow z = \frac{1}{w-2}$

$\Rightarrow z - 1 + \frac{1}{2}i = \frac{1}{w-2} - 1 + \frac{1}{2}i$

$\Rightarrow z - 1 + \frac{1}{2}i = \frac{1 - (w-2) + \frac{1}{2}(w-2)}{w-2}$

$\Rightarrow z - 1 + \frac{1}{2}i = \frac{1 - w + 2 + \frac{1}{2}w - 1}{w-2}$

$\Rightarrow z - 1 + \frac{1}{2}i = \frac{2 - \frac{1}{2}w}{w-2}$

$\Rightarrow z - 1 + \frac{1}{2}i = \frac{4-w}{2w-4}$

$\Rightarrow |z - 1 + \frac{1}{2}i| = \left|\frac{4-w}{2w-4}\right|$

$\Rightarrow \frac{\sqrt{5}}{2} = \frac{|4-w|}{|2w-4|}$

$\Rightarrow \frac{\sqrt{5}}{2} = \frac{|4-w|}{2|w-2|}$

$\Rightarrow \sqrt{5} = \frac{|4-w|}{|w-2|}$

$\Rightarrow \sqrt{5} = \frac{|4-(u+iv)|}{|(u+iv)-2|}$

$\Rightarrow \sqrt{5} = \frac{|(4-u)-iv|}{|(u-2)+iv|}$

$\Rightarrow \sqrt{5} = \frac{\sqrt{(4-u)^2+v^2}}{\sqrt{(u-2)^2+v^2}}$

$\Rightarrow 5 = \frac{(4-u)^2+v^2}{(u-2)^2+v^2}$

$\Rightarrow 5 = \frac{16-8u+u^2+v^2}{u^2-4u+4+v^2}$

$\Rightarrow 5u^2 - 20u + 20 + 5v^2 = 16 - 8u + u^2 + v^2$

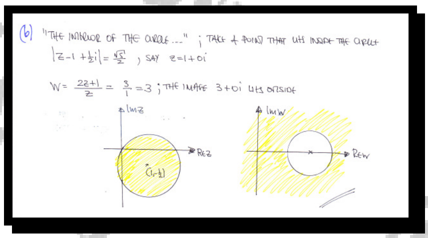
$\Rightarrow 4u^2 - 12u + 4v^2 + 4 = 0$

$\Rightarrow u^2 - 3u + v^2 + 1 = 0$

$\Rightarrow (u - \frac{3}{2})^2 + v^2 - \frac{9}{4} + 1 = 0$

$\Rightarrow (u - \frac{3}{2})^2 + v^2 = \frac{1}{2}$

IE CIRCLE CENTRE $(\frac{3}{2}, 0)$ RADIUS $\frac{1}{\sqrt{2}}$



(b) "THE INTERIOR OF THE CIRCLE ..." ; TAKE A POINT THAT LIES INSIDE THE CIRCLE $|z - 1 + \frac{1}{2}i| = \frac{\sqrt{5}}{2}$, SAY $z = 1 + 0i$

$w = \frac{2z+1}{z} = \frac{3}{1} = 3$; THE IMAGE $3 + 0i$ LIES OUTSIDE

Question 21

A transformation of the z plane to the w plane is given by

$$w = z + \frac{1}{z}, \ z \in \mathbb{C}, \ z \neq 0,$$

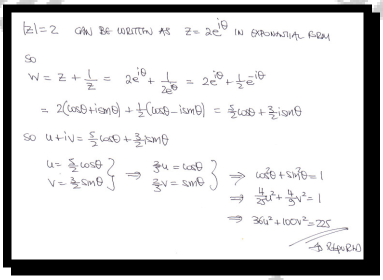
where $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$.

The locus of the points in the z plane that satisfy the equation $|z| = 2$ are mapped in the w plane onto a curve C.

By considering the equation of the locus $|z| = 2$ in exponential form, or otherwise, show that a Cartesian equation of C is

$$36u^2 + 100v^2 = 225.$$

proof



Question 22

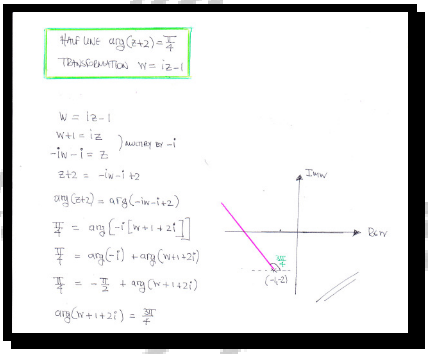
A transformation from the z plane to the w plane is defined by the equation

$$w = \mathrm{i}\,z - 1, \ z \in \mathbb{C}.$$

Sketch in the w plane, in Cartesian form, the equation of the image of the half line with equation

$$\arg(z+2) = \frac{\pi}{4}, \ z \in \mathbb{C}.$$

graph



Question 23

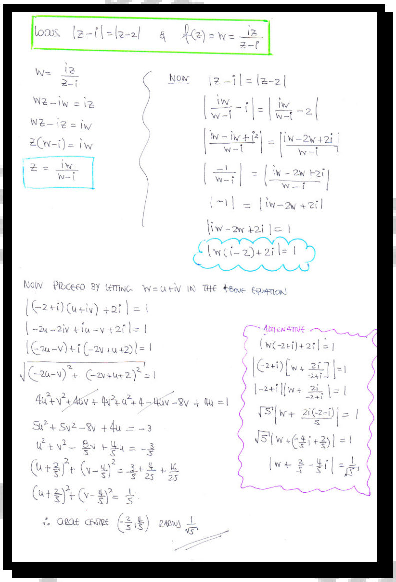
A transformation from the z plane to the w plane is defined by the equation

$$f(z) = \frac{\mathrm{i}\,z}{z-\mathrm{i}}, \quad z \in \mathbb{C}.$$

Find, in Cartesian form, the equation of the image of straight line with equation

$$|z-\mathrm{i}| = |z-2|, \quad z \in \mathbb{C}.$$

$$\boxed{\left(u+\tfrac{2}{5}\right)^2 + \left(v-\tfrac{4}{5}\right)^2 = \tfrac{1}{5}}$$



LOCUS $|z-i| = |z-2|$ & $f(z) = w = \frac{iz}{z-i}$

$w = \frac{iz}{z-i}$

$wz - iw = iz$

$wz - iz = iw$

$z(w-i) = iw$

$z = \frac{iw}{w-i}$

NOW $|z-i| = |z-2|$

$\left|\frac{iw}{w-i} - i\right| = \left|\frac{iw}{w-i} - 2\right|$

$\left|\frac{iw - iw + i^2}{w-i}\right| = \left|\frac{iw - 2w + 2i}{w-i}\right|$

$\left|\frac{-1}{w-i}\right| = \left|\frac{iw - 2w + 2i}{w-i}\right|$

$|-1| = |iw - 2w + 2i|$

$|iw - 2w + 2i| = 1$

$|w(i-2) + 2i| = 1$

NOW PROCEED BY LETTING $w = u + iv$ IN THE ABOVE EQUATION

$|(-2+i)(u+iv) + 2i| = 1$

$|-2u - 2iv + iu - v + 2i| = 1$

$|(-2u-v) + i(-2v+u+2)| = 1$

$\sqrt{(-2u-v)^2 + (-2v+u+2)^2} = 1$

$4u^2 + v^2 + 4uv + 4v^2 + u^2 + 4 - 4uv - 8v + 4u = 1$

$5u^2 + 5v^2 - 8v + 4u = -3$

$u^2 + v^2 - \frac{8}{5}v + \frac{4}{5}u = -\frac{3}{5}$

$\left(u+\frac{2}{5}\right)^2 + \left(v-\frac{4}{5}\right)^2 = -\frac{3}{5} + \frac{4}{25} + \frac{16}{25}$

$\left(u+\frac{2}{5}\right)^2 + \left(v-\frac{4}{5}\right)^2 = \frac{1}{5}$

$\therefore$ CIRCLE CENTRE $\left(-\frac{2}{5}, \frac{4}{5}\right)$ RADIUS $\frac{1}{\sqrt{5}}$

ALTERNATIVE

$|w(-2+i) + 2i| = 1$

$\left|(-2+i)\left[w + \frac{2i}{-2+i}\right]\right| = 1$

$|-2+i|\left|w + \frac{2i}{-2+i}\right| = 1$

$\sqrt{5}\left|w + \frac{2i(-2-i)}{5}\right| = 1$

$\sqrt{5}\left|w + \left(-\frac{4}{5}i + \frac{2}{5}\right)\right| = 1$

$\left|w + \frac{2}{5} - \frac{4}{5}i\right| = \frac{1}{\sqrt{5}}$

Question 24

The complex function $w = f(z)$ is given by

$$w = \frac{1}{1-z}, \quad z \neq 1.$$

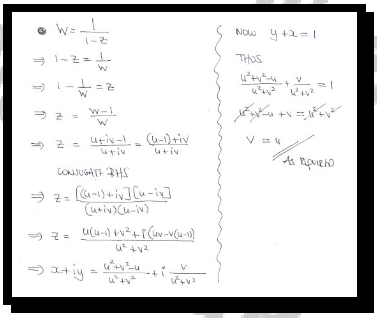
The point $P(x, y)$ in the z plane traces the line with Cartesian equation

$$y + x = 1.$$

Show that the locus of the **image** of P in the w plane traces the line with equation

$$v = u.$$

proof



Question 25

The complex function $w = f(z)$ satisfies

$$w = \frac{1}{z}, \ z \in \mathbb{C}, \ z \neq 0.$$

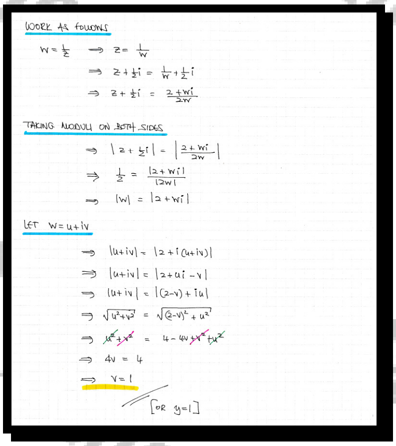
This function maps the point $P(x, y)$ in the z plane onto the point $Q(u, v)$ in the w plane.

It is further given that P traces the curve with equation

$$\left| z + \frac{1}{2}\mathrm{i} \right| = \frac{1}{2}.$$

Find, in Cartesian form, the equation of the locus of Q.

$\boxed{v = 1}$


WORK AS FOLLOWS

$w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$

$\Rightarrow z + \frac{1}{2}i = \frac{1}{w} + \frac{1}{2}i$

$\Rightarrow z + \frac{1}{2}i = \frac{2+wi}{2w}$

TAKING MODULI ON BOTH SIDES

$\Rightarrow |z + \frac{1}{2}i| = \left|\frac{2+wi}{2w}\right|$

$\Rightarrow \frac{1}{2} = \frac{|2+wi|}{|2w|}$

$\Rightarrow |w| = |2+wi|$

LET $w = u + iv$

$\Rightarrow |u+iv| = |2+i(u+iv)|$

$\Rightarrow |u+iv| = |2+ui-v|$

$\Rightarrow |u+iv| = |(2-v)+iu|$

$\Rightarrow \sqrt{u^2+v^2} = \sqrt{(2-v)^2+u^2}$

$\Rightarrow u^2+v^2 = 4-4v+v^2+u^2$

$\Rightarrow 4v = 4$

$\Rightarrow v = 1$

[OR $y=1$]

Question 26

$$z = \cos\theta + \mathrm{i}\sin\theta\,,\ -\pi < \theta \le \pi\,.$$

a) Show clearly that

$$\frac{2}{1+z} = 1 - \mathrm{i}\tan\frac{\theta}{2}\,.$$

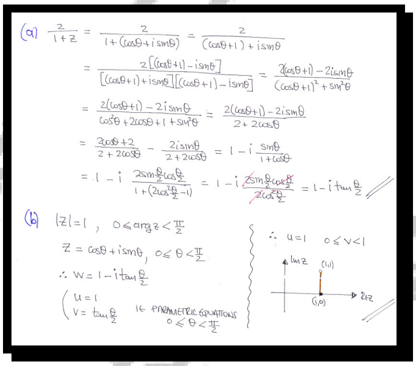
The complex function $w = f(z)$ is defined by

$$w = \frac{2}{1+z}\,,\ z \in \mathbb{C}\,,\ z \neq -1\,.$$

The circular arc $|z| = 1$, for which $0 \le \arg z < \frac{\pi}{2}$, is transformed by this function.

b) Sketch the image of this circular arc in a suitably labelled Argand diagram.

proof/sketch



Question 27

The complex function with equation

$$f(z)=\frac{1}{z^2},\ z\in\mathbb{C},\ z\neq 0$$

maps the complex number $x+\mathrm{i}y$ from the z plane onto the complex number $u+\mathrm{i}v$ in the w plane.

The line with equation

$$y=mx,\ x\neq 0,$$

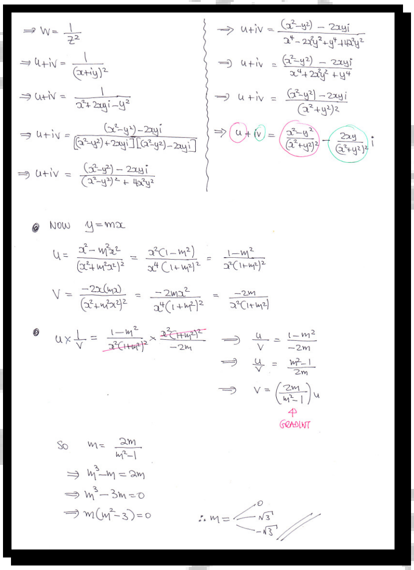
is mapped onto the line with equation

$$v=Mu,$$

where m and M are the respective gradients of the two lines.

Given that $m=M$, determine the three possible values of m.

$$\boxed{m=0,\pm\sqrt{3}}$$



Question 28

A complex transformation of points from the z plane onto points in the w plane is defined by the equation

$$w = z^2, \ z \in \mathbb{C}.$$

The point represented by $z = x + iy$ is mapped onto the point represented by $w = u + iv$.

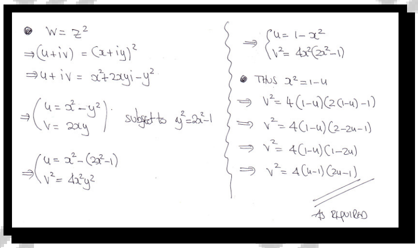
Show that if z traces the curve with Cartesian equation

$$y^2 = 2x^2 - 1,$$

the locus of w satisfies the equation

$$v^2 = 4(u-1)(2u-1).$$

proof



Question 29

The complex function $w = f(z)$ is defined by

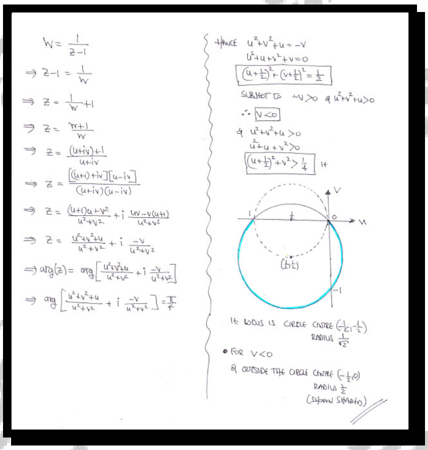
$$w = \frac{1}{z-1}, \; z \in \mathbb{C}, \; z \neq 1.$$

The half line with equation $\arg z = \frac{\pi}{4}$ is transformed by this function.

a) Find a Cartesian equation of the locus of the **image** of the half line.

b) Sketch the **image** of the locus in an Argand diagram.

$$\left(u+\frac{1}{2}\right)^2 + \left(v+\frac{1}{2}\right)^2 = \frac{1}{2}, \; v<0, \; u^2+v^2+u>0$$



Question 30

The complex function $w = f(z)$ is defined by

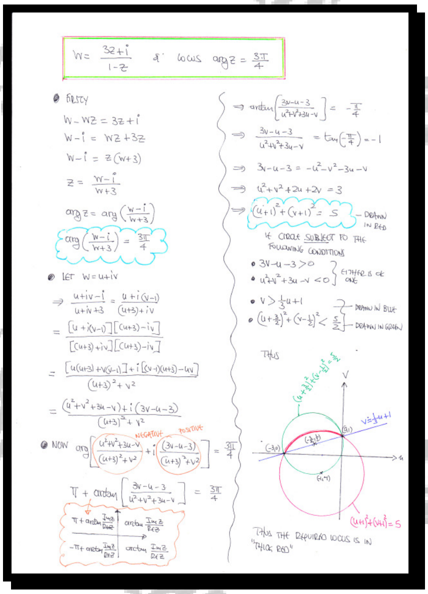
$$w = \frac{3z + \mathrm{i}}{1 - z}, \; z \in \mathbb{C}, \; z \neq 1.$$

The half line with equation $\arg z = \frac{3\pi}{4}$ is transformed by this function.

a) Find a Cartesian equation of the locus of the **image** of the half line.

b) Sketch the **image** of the locus in an Argand diagram.

$$\boxed{(u+1)^2 + (v+1)^2 = 5, \quad v > \tfrac{1}{3}u + 1}$$



COMPLEX SERIES

Question 1

The following convergent series C and S are given by

$$C = 1 + \frac{1}{2}\cos\theta + \frac{1}{4}\cos 2\theta + \frac{1}{8}\cos 3\theta \ldots$$
$$S = \frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta \ldots$$

a) Show clearly that

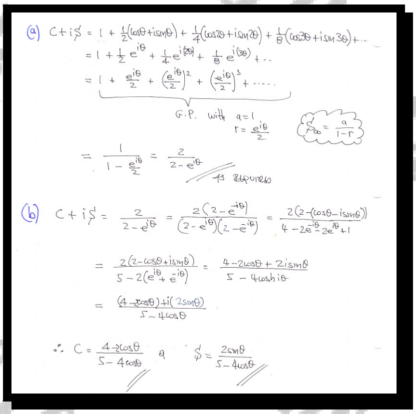
$$C + \mathrm{i}S = \frac{2}{2 - \mathrm{e}^{\mathrm{i}\theta}}.$$

b) Hence show further that

$$C = \frac{4 - 2\cos\theta}{5 - 4\cos\theta},$$

and find a similar expression for S.

$$\boxed{S = \frac{2\sin\theta}{5 - 4\cos\theta}}$$


(a) $C + iS = 1 + \frac{1}{2}(\cos\theta + i\sin\theta) + \frac{1}{4}(\cos 2\theta + i\sin 2\theta) + \frac{1}{8}(\cos 3\theta + i\sin 3\theta) + \ldots$

$= 1 + \frac{1}{2}e^{i\theta} + \frac{1}{4}e^{i(2\theta)} + \frac{1}{8}e^{i(3\theta)} + \ldots$

$= 1 + \frac{e^{i\theta}}{2} + \left(\frac{e^{i\theta}}{2}\right)^2 + \left(\frac{e^{i\theta}}{2}\right)^3 + \ldots$

G.P. with $a = 1$, $r = \frac{e^{i\theta}}{2}$ $\quad S_\infty = \frac{a}{1-r}$

$= \frac{1}{1 - \frac{e^{i\theta}}{2}} = \frac{2}{2 - e^{i\theta}}$ as required

(b) $C + iS = \frac{2}{2 - e^{i\theta}} = \frac{2(2 - e^{-i\theta})}{(2 - e^{i\theta})(2 - e^{-i\theta})} = \frac{2(2 - (\cos\theta - i\sin\theta))}{4 - 2e^{-i\theta} - 2e^{i\theta} + 1}$

$= \frac{2(2 - \cos\theta + i\sin\theta)}{5 - 2(e^{i\theta} + e^{-i\theta})} = \frac{4 - 2\cos\theta + 2i\sin\theta}{5 - 4\cosh i\theta}$

$= \frac{(4 - 2\cos\theta) + i(2\sin\theta)}{5 - 4\cos\theta}$

$\therefore C = \frac{4 - 2\cos\theta}{5 - 4\cos\theta}$ & $S = \frac{2\sin\theta}{5 - 4\cos\theta}$

Question 2

The following finite sums, C and S, are given by

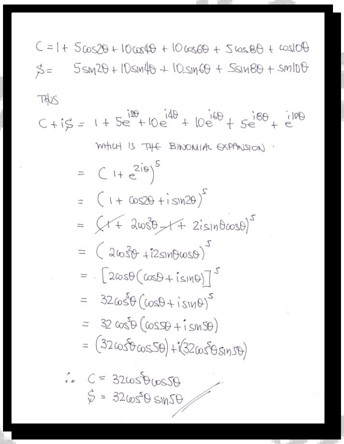
$$C = 1 + 5\cos 2\theta + 10\cos 4\theta + 10\cos 6\theta + 5\cos 8\theta + \cos 10\theta$$
$$S = 5\sin 2\theta + 10\sin 4\theta + 10\sin 6\theta + 5\sin 8\theta + \sin 10\theta$$

By considering the binomial expansion of $(1+A)^5$, show clearly that

$$C = 32\cos^5\theta\cos 5\theta,$$

and find a similar expression for S

$\boxed{S = 32\cos^5\theta\sin 5\theta}$



$C = 1 + 5\cos 2\theta + 10\cos 4\theta + 10\cos 6\theta + 5\cos 8\theta + \cos 10\theta$

$S = 5\sin 2\theta + 10\sin 4\theta + 10\sin 6\theta + 5\sin 8\theta + \sin 10\theta$

THUS

$C + iS = 1 + 5e^{i2\theta} + 10e^{i4\theta} + 10e^{i6\theta} + 5e^{i8\theta} + e^{i10\theta}$

WHICH IS THE BINOMIAL EXPANSION

$= (1 + e^{2i\theta})^5$

$= (1 + \cos 2\theta + i\sin 2\theta)^5$

$= (1 + 2\cos^2\theta - 1 + 2i\sin\theta\cos\theta)^5$

$= (2\cos^2\theta + i2\sin\theta\cos\theta)^5$

$= [2\cos\theta(\cos\theta + i\sin\theta)]^5$

$= 32\cos^5\theta(\cos\theta + i\sin\theta)^5$

$= 32\cos^5\theta(\cos 5\theta + i\sin 5\theta)$

$= (32\cos^5\theta\cos 5\theta) + i(32\cos^5\theta\sin 5\theta)$

$\therefore\ C = 32\cos^5\theta\cos 5\theta$

$S = 32\cos^5\theta\sin 5\theta$

Question 3

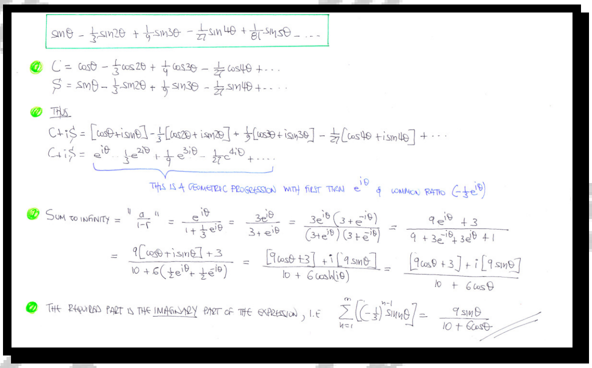
The following convergent series S is given below

$$S = \sin\theta - \frac{1}{3}\sin 2\theta + \frac{1}{9}\sin 3\theta - \frac{1}{27}\sin 4\theta \ ...$$

By considering the sum to infinity of a suitable geometric series involving the complex exponential function, show that

$$S = \frac{9\sin\theta}{10 + 6\cos\theta}.$$

proof



$\sin\theta - \frac{1}{3}\sin 2\theta + \frac{1}{9}\sin 3\theta - \frac{1}{27}\sin 4\theta + \frac{1}{81}\sin 5\theta - \dots$

$C = \cos\theta - \frac{1}{3}\cos 2\theta + \frac{1}{9}\cos 3\theta - \frac{1}{27}\cos 4\theta + \dots$

$S = \sin\theta - \frac{1}{3}\sin 2\theta + \frac{1}{9}\sin 3\theta - \frac{1}{27}\sin 4\theta + \dots$

THUS

$C + iS = [\cos\theta + i\sin\theta] - \frac{1}{3}[\cos 2\theta + i\sin 2\theta] + \frac{1}{9}[\cos 3\theta + i\sin 3\theta] - \frac{1}{27}[\cos 4\theta + i\sin 4\theta] + \dots$

$C + iS = e^{i\theta} - \frac{1}{3}e^{2i\theta} + \frac{1}{9}e^{3i\theta} - \frac{1}{27}e^{4i\theta} + \dots$

THIS IS A GEOMETRIC PROGRESSION WITH FIRST TERM $e^{i\theta}$ & COMMON RATIO $\left(-\frac{1}{3}e^{i\theta}\right)$

SUM TO INFINITY $= \text{"}\frac{a}{1-r}\text{"} = \frac{e^{i\theta}}{1 + \frac{1}{3}e^{i\theta}} = \frac{3e^{i\theta}}{3 + e^{i\theta}} = \frac{3e^{i\theta}(3 + e^{-i\theta})}{(3 + e^{i\theta})(3 + e^{-i\theta})} = \frac{9e^{i\theta} + 3}{9 + 3e^{-i\theta} + 3e^{i\theta} + 1}$

$= \frac{9[\cos\theta + i\sin\theta] + 3}{10 + 6\left(\frac{1}{2}e^{i\theta} + \frac{1}{2}e^{-i\theta}\right)} = \frac{[9\cos\theta + 3] + i[9\sin\theta]}{10 + 6\cosh(i\theta)} = \frac{[9\cos\theta + 3] + i[9\sin\theta]}{10 + 6\cos\theta}$

THE REQUIRED PART IS THE IMAGINARY PART OF THE EXPRESSION, I.E $\sum_{n=1}^{\infty}\left[\left(-\frac{1}{3}\right)^{n-1}\sin n\theta\right] = \frac{9\sin\theta}{10 + 6\cos\theta}$

Question 4

The sum C is given below

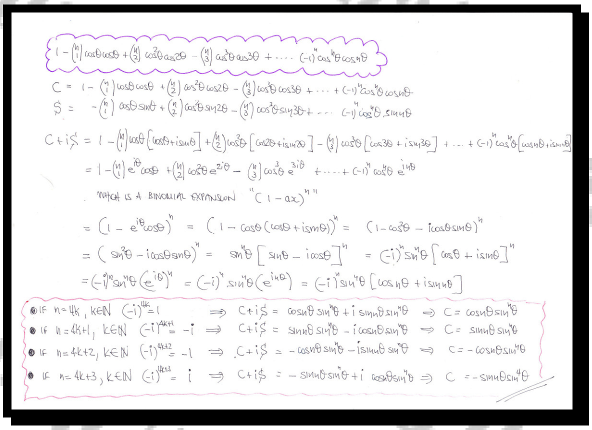
$$C = 1 - \binom{n}{1}\cos\theta\cos\theta + \binom{n}{2}\cos^2\theta\cos 2\theta - \binom{n}{1}\cos^3\theta\cos 3\theta + \ldots + (-1)^n\cos^n\theta\cos n\theta$$

Given that $n \in \mathbb{N}$ determine the 4 possible expressions for C.

Give the answers in exact simplified form.

$\boxed{n = 4k,\ k \in \mathbb{N} : C = \cos n\theta \sin^n\theta}$, $\boxed{n = 4k+1,\ k \in \mathbb{N} : C = \sin n\theta \sin^n\theta}$,

$\boxed{n = 4k+2,\ k \in \mathbb{N} : C = -\cos n\theta \sin^n\theta}$, $\boxed{n = 4k+3,\ k \in \mathbb{N} : C = -\sin n\theta \sin^n\theta}$



Question 5

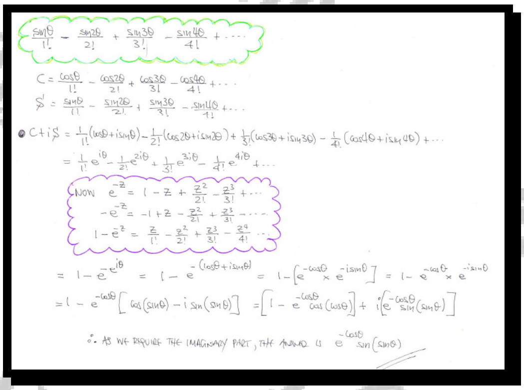
The following convergent series S is given below

$$S = \frac{\sin\theta}{1!} - \frac{\sin 2\theta}{2!} + \frac{\sin 3\theta}{3!} - \frac{\sin 4\theta}{4!} + \ldots$$

By considering the sum to infinity of a suitable series involving the complex exponential function, show that

$$S = e^{-\cos\theta}\sin(\sin\theta).$$

proof



$$\frac{\sin\theta}{1!} - \frac{\sin 2\theta}{2!} + \frac{\sin 3\theta}{3!} - \frac{\sin 4\theta}{4!} + \ldots$$

$$C = \frac{\cos\theta}{1!} - \frac{\cos 2\theta}{2!} + \frac{\cos 3\theta}{3!} - \frac{\cos 4\theta}{4!} + \ldots$$

$$S = \frac{\sin\theta}{1!} - \frac{\sin 2\theta}{2!} + \frac{\sin 3\theta}{3!} - \frac{\sin 4\theta}{4!} + \ldots$$

$$C + iS = \frac{1}{1!}(\cos\theta + i\sin\theta) - \frac{1}{2!}(\cos 2\theta + i\sin 2\theta) + \frac{1}{3!}(\cos 3\theta + i\sin 3\theta) - \frac{1}{4!}(\cos 4\theta + i\sin 4\theta) + \ldots$$

$$= \frac{1}{1!}e^{i\theta} - \frac{1}{2!}e^{2i\theta} + \frac{1}{3!}e^{3i\theta} - \frac{1}{4!}e^{4i\theta} + \ldots$$

NOW $e^{-z} = 1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \ldots$

$-e^{-z} = -1 + z - \frac{z^2}{2!} + \frac{z^3}{3!} - \ldots$

$1 - e^{-z} = \frac{z}{1!} - \frac{z^2}{2!} + \frac{z^3}{3!} - \frac{z^4}{4!} \ldots$

$$= 1 - e^{-e^{i\theta}} = 1 - e^{-(\cos\theta + i\sin\theta)} = 1 - \left[e^{-\cos\theta} \times e^{-i\sin\theta}\right] = 1 - e^{-\cos\theta} \times e^{-i\sin\theta}$$

$$= 1 - e^{-\cos\theta}\left[\cos(\sin\theta) - i\sin(\sin\theta)\right] = \left[1 - e^{-\cos\theta}\cos(\cos\theta)\right] + i\left[e^{-\cos\theta}\sin(\sin\theta)\right]$$

∴ AS WE REQUIRE THE IMAGINARY PART, THE ANSWER IS $e^{-\cos\theta}\sin(\sin\theta)$