Complex Numbers Revision Sheet – Question 4 of Paper 1

Introduction

Complex numbers are numbers that have a real part and an imaginary part.

The real part will be a number such as 3.
The imaginary part is represented by the letter i. *3 + i*

Examples –

$-4+3i$	*Real part – 4, imaginary part 3i*
$3+2i$	*Real part + 3, imaginary part 2i*
$-2-2i$	*Real part – 2, imaginary part – 2i*

In the exam you will normally see a letter (usually **z** or **w**) representing the complex number. The first thing we do when answering a question is replace these letters with the associated complex numbers. This will be your first step in answering questions.

Examples –

z = $-4+3i$	*This just means that wherever you see* ***z*** *in the question put in the complex number* $-4+3i$
w = $-2-2i$	*This just means that wherever you see* ***w*** *in the question put in the complex number* $-2-2i$

IT IS VERY IMPORTANT THAT THE FIRST THING YOU DO IN EACH QUESTION IS WRITE OUT THE QUESTION AGAIN REPLACING ANY LETTERS SUCH AS Z OR W WITH COMPLEX NUMBERS. THIS WILL HELP YOU FIGURE OUT WHAT YOU ARE BEING ASKED.

We use the letter **i** to represent the imaginary number $\sqrt{-1}$

$i=\sqrt{-1}$	$\sqrt{-1}$ *does not exist, no number squared gives – 1 that is why we say that it is imaginary*
$i^2=-1$	$i^2=\sqrt{-1}\sqrt{-1}=-1$

We never have an i^2 in an answer. We always replace it with –1.

Basic Operations – Simplify

Adding and Subtracting complex numbers –
We add or subtract the real numbers to the real numbers and the imaginary numbers to the imaginary numbers. We CANNOT add or subtract a real number and an imaginary number.

Example - **Simplify** **4 + 3i + 6 + 2i**

4 + 6 + 3i + 2i	*Real numbers together, i's together*
10 + 5i	*Add real to real (6 + 4), i's to i's (3i + 2i)*

Example - **Simplify** **6 – 4i + 5 + 2i**

6 + 5 –4i + 2i	*Real numbers together, i's together*
11 – 2i	*Add real to real (6 + 5), i's to i's (-4i + 2i)*

Example - $z_1 = 5 + 2i$ $z_2 = 3 + 6i$ **find** $z_1 + z_2$ **and** $z_1 - z_2$

$z_1 + z_2$ = 5 + 2i + 3 + 6i	*Replace* z_1 *and* z_2 *with the complex numbers*
= 5 + 3 + 2i + 6i	*Real numbers together, i's together*
= 8 + 8i	*Add real to real (5 + 3), i's to i's (2i + 6i)*

$z_1 - z_2$ = (5 + 2i) – (3 + 6i)	*Replace* z_1 *and* z_2 *with the complex numbers*
= 5 + 2i – 3 – 6i	*Remove brackets, careful with signs*
= 5 – 3 + 2i – 6i	*Real numbers together, i's together*
= 2 – 4i	*Add real to real (5 - 3), i's to i's (2i - 6i)*

Removing brackets –

We can multiply a number outside our complex numbers by removing brackets and multiplying.

Example - $z_1 = 5 + 2i$ $z_2 = 3 + 6i$

$2z_1 = 2(5 + 2i)$	*Multiply 2 by* z_1 *and simplify*
$= 10 + 4i$	

$3z_2 = 3(3 + 6i)$	*Multiply 3 by* z_2 *and simplify*
$= 9 + 18i$	

$4z_1 - 2z_2 = 4(5 + 2i) - 2(3 + 6i)$	*Write out the question replacing* z_1
$= 20 + 8i - 6 - 12i$	*and* z_2 *with the complex numbers*
$= 20 - 6 + 8i - 12i$	
$= 14 - 4i$	*Simplify*

Multiplying complex numbers –

Multiplying with complex numbers is very similar to multiplying in algebra by splitting the first bracket.

One important thing to remember is that $i^2 = -1$

Example - $w_1 = 5 + 2i$ $w_2 = 3 - 5i$ **Find** $w_1 w_2$

Find iw_1

$w_1 w_2 = (5+2i)(3-5i)$	*Replace* w_1 *and* w_2 *with the associated complex numbers*
$= 5(3-5i) + 2i(3-5i)$	*Open the first bracket as in algebra*
$= 15 - 25i + 6i - 10i^2$	*Remove brackets by multiplying*
$= 15 - 19i - 10(-1)$	*-25i + 6i = -19i,* $-10i^2 = -10(-1)$
$= 15 - 19i + 10$	*Simplify*
$= 5 - 19i$	*Simplify*

$iw_1 = i(5+2i)$	*Here they ask us to multiply i by* w_1 *so replace* w_1 *with complex number 5 + 2i*
$= 5i + 2i^2$	*Remove brackets by multiplying*
$= 5i + 2(-1)$	$2i^2 = 2(-1)$
$= 5i - 2$	*Simplify*
$= -2 + 5i$	*Rearrange putting real number first*

Division of Complex Numbers – The Conjugate

Before we can divide complex numbers we need to know what the **conjugate** of a complex is.
To find the **conjugate** of a complex number we just change the sign of the i part.
The conjugate of z is written $\bar{z}$.

Examples -

$z = 4 + 2i$	*then*	$\bar{z} = 4 - 2i$	*change sign of i part*
$w = -3 + 2i$	*then*	$\bar{w} = -3 - 2i$	*change sign of i part*
$w = 5 - 6i$	*then*	$\bar{w} = 5 + 6i$	*change sign of i part*

Division

To divide by a complex number we multiply above and below by the CONJUGATE of the bottom number (the number you are dividing by). This gets rid of the i value from the bottom. We should never have an i value on the bottom of an answer.
Remember anytime you see DIVISION in a question you must perform this operation.

Example - z = 4 – 3i and w = 3 + 2i Express $\frac{z}{w}$ **in the from** $a + bi$

$\frac{z}{w} = \frac{4-3i}{3+2i}$ *Replace z and w with complex numbers*

$\frac{4-3i}{3+2i} \times \frac{3-2i}{3-2i}$ *Multiply above and below* by **3 – 2i**

$= \frac{(4-3i)(3-2i)}{(3+2i)(3-2i)}$ *Top by the top, bottom by the bottom*

$= \frac{4(3-2i) - 3i(3-2i)}{3(3-2i) + 2i(3-2i)}$ *Open up the first brackets as in algebra*

$= \frac{12 - 8i - 9i + 6i^2}{9 - 6i + 6i - 4i}$ *Remove brackets by multiplying*

$= \frac{12 - 17i + 6(-1)}{9 - 4(-1)}$ *Simplify:* *Remember* $i^2 = -1$

$= \frac{12 - 17i - 6}{9 + 4}$ *Simplify*

$= \frac{6 - 17i}{13}$ *Simplify*

$= \frac{6}{13} - \frac{17}{13}i$ *Split into real and imaginary parts*

$= a + bi$ $\frac{z}{w}$ *is now in the form* $a + bi$ *as required*

Real with Real, i's with i's – Equality of Complex Numbers

If two complex numbers are equal then the real parts on the left of the '=' will be equal to the real parts on the right of the '=' and the imaginary parts will be equal to the imaginary parts. Remember a real part is any number **OR** letter that isn't attached to an i. Quite often you will need to use a lot of the skills you learned previously to simplify an equation with which you can let real = real and i's = i's.

Example - **3x + iy = 6 + 8i** **find x and y**

3x = 6 and iy = 8i *real = real **and** imaginary = imaginary*

$x = \frac{6}{3} = 2$ and y = 8 *solve equations*

Example - $(x+3)+i(y-1)=6+2i$ **find x and y**

(x + 3) = 6 and i(y – 1) = 2i *real = real **and** imaginary = imaginary*
x + 3 = 6 and y – 1 = 2
x = 6 – 3 and y = 2 + 1 *solve equations*
x = 3 and x = 3

Sometimes after we let the real = real and imaginary = imaginary we are left with a simultaneous equation before we can solve.

Example - **(4a – 2) + (a – 4)i = (4 – 2b) + 2bi** **find a and b**

(4a – 2) = (4 – 2b) and (a – 4)i = 2bi *real = real **and** i's = i's*
4a – 2 = 4 – 2b and a – 4 = 2b
4a + 2b = 4 + 2 and **a – 2b = 4** a's *and b's to the left, numbers to the right*
4a + 2b = 6 ***Equation 2*** *we are left with 2 simultaneous equations*
Equation 1

4a + 2b = 6 ***Equation 1***
a – 2b = 4 ***Equation 2***
5a = 10 *cancel the 2b's, 4a + a = 5a, 6 + 4 = 10*

$a = \frac{10}{5} = 2$ *Divide across by 5 to get a = 2*

We must put in a = 2 into either equation to get the b value

4a + 2b = 6 ***Equation 1***
4(2) + 2b = 6 *Replace the a value for a = 2*
8 + 2b = 6 *Simplify*
2b = 6 – 8 *b's to one side, numbers to the other*
2b = -2 *divide across by 2*

$b = \frac{-2}{2} = -1$ ***therefore a =2 and b = -1***

Modulus –

The modulus of a complex number is its **length** and is found through a formula we **MUST** learn. The modulus of z is written |z|

FORMULA – **if z = a + bi then**

|z| = |a + bi| = $\sqrt{a^2 + b^2}$

Example - **z = 8 + 6i** **find |z|**

|z| = | 8 + 6i | = $\sqrt{8^2 + 6^2} = \sqrt{64 + 36} = \sqrt{100} = 10$ *sub in values and solve*

Example - **z = -5 - 3i** **find |z|**

|z| = | -5 - 3i | = $\sqrt{-5^2 + -3^2} = \sqrt{25 + 9} = \sqrt{36} = 6$ *sub in values and solve*

It is important to note that $|z_1 + z_2|$ **is not the same as** $|z_1| + |z_2|$

Example - z_1 **= 3 + 4i and** z_2 **= 12 – 5i**

Investigate if $|z_1 + z_2| = |z_1| + |z_2|$

$|z_1 + z_2|$ = | 3 + 4i + 12 – 5i |
= | 3 + 12 + 4i – 5i |
= | 15 – i |
= $\sqrt{15^2 + (-1)^2}$
= $\sqrt{225 + 1}$
= $\sqrt{226}$ **= 15.03**

$|z_1| + |z_2|$ **=** | 3 + 4i | + | 12 – 5i |
= $\sqrt{3^2 + 4^2} + \sqrt{12^2 + (-5)^2}$
= $\sqrt{9 + 16} + \sqrt{144 + 25}$
= $\sqrt{25} + \sqrt{169}$
= 5 + 13 = **18**

15.03 $\neq$ **18** **therefore** $|z_1 + z_2| \neq |z_1| + |z_2|$

Roots –

If we get a quadratic equation in the complex numbers section it may involve the use of the formula to solve:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Normally we can never have a minus number under the root part but with complex numbers we will be able to change the root of a minus number into an i.

Example - **Solve** $z^2 - 8z + 25$ *Notice that they use z's instead of x's*

$z^2 - 8z + 25$	*Cannot be factorised so use FORMULA*
a = 1 b = - 8 c = 25	*Values of a, b and c for formula.*
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	*Formula*
$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(25)}}{2(1)}$	*Fill in values for a, b and c.*
$x = \frac{8 \pm \sqrt{64 - 100}}{2}$	*Simplify*
$x = \frac{8 \pm \sqrt{-36}}{2}$	*Simplify*
$x = \frac{8 \pm \sqrt{36}\sqrt{-1}}{2}$	*Split* $\sqrt{-36}$ *into* $\sqrt{36}\sqrt{-1}$
$x = \frac{8 \pm 6\sqrt{-1}}{2}$	$\sqrt{36} = 6$
$x = \frac{8 \pm 6i}{2}$	*Turn* $\sqrt{-1}$ *into i*
$x = 4 \pm 3i$	*Divide across by 2*
$z = 4 + 3i$ ***and*** $z = 4 - 3i$	*The roots of the equation*

Notice that roots occur in CONJUGATE pairs.
$4 + 3i$ *is the conjugate of* $4 - 3i$

However not all **ROOT** questions involve using the formula.
Sometimes you will be given one of the roots and asked to find the other root or asked to find the value of a certain letter.
Remember that if you know one of the roots then the other root is its CONJUGATE.
If something is a root of an equation then it satisfies that equation. That means you put it in for z and the equation will be true.
Turn over the page for examples of other types of ROOT questions.

Given a root find the letter……

Example - 5 – 3i is a root of the equation $z^2 - 10z + k = 0$ **find the value of k**

$z^2 - 10z + k = 0$	*Write out the equation*
$(5-3i)^2 - 10(5-3i) + k = 0$	*Fill in for z (the root is z = 5 – 3i)*
$5(5-3i) - 3i(5-3i) - 10(5-3i) + k = 0$	*Open up* $(5-3i)^2$
$25 - 15i - 15i + 9i^2 - 50 + 30i + k = 0$	*Remove brackets by multiplying*
$-25 + 9i^2 + k = 0$	*Simplify*
- 25 + 9(-1) + k = 0	*Turn* $\sqrt{-1}$ *into i*
-25 – 9 + k = 0	*Simplify*
-34 + k = 0	*Simplify*
k = 34	*k's to one side, numbers to the other*

If we had put **5 + 3i** in as z we would get the same answer as it is the other root. (Conjugate)

Given a root, what is the other root and find the values of two letters……..

Example - *2 – i* is a root of the equation $z^2 + pz + q = 0$ **Find p and q**

If 2 – i is a root then so is 2 + i *Conjugate*

Remember in ALGEBRA that if x = 3 was a root then x – 3 was a factor

The **roots** are z = 2 – i and z = 2 + i
Therefore the **factors** are z – 2 + i and z – 2 – i

Remember also in algebra that $(x-3)(x+4) = x^2 + x - 12$
Multiplying factors gives you a quadratic.

Multiplying the factors we get

(z – 2 + i)(z – 2 – i)	= z(z – 2 – i) – 2(z – 2 – i) + i(z – 2 – i)	*Split first bracket*
	$= z^2 - 2z - zi - 2z + 4 + 2i + iz - 2i - i^2$	*Multiply into brackets*
	$= z^2 - 4z + 4 - i^2$	*all the i's cancel*
	$= z^2 - 4z + 4 - (-1)$	$i^2 = -1$
	$= z^2 - 4z + 4 + 1$	
	$= z^2 - 4z + 5$	*Quadratic Equation*

Compare this to original quadratic equation.

$$z^2 - 4z + 5 = z^2 + pz + q$$

Therefore $p = -4$ and $q = 5$ *Values for p and q*

Argand Diagrams –

We can represent complex numbers on a graph using a real **Re** and imaginary **Im** axis similar to the x and y axis.

Complex numbers can be plotted quite simply.

$2 + 3i$	is represented by the point	(2, 3i)
$-4+2i$	is represented by the point	(-4, 2i)
$1 - 5i$	is represented by the point	(1, -5i)
$-3-i$	is represented by the point	(–3, -1i)
$4i$	is represented by the point	(0, 4i)
3	is represented by the point	(3, 0i)

With this topic you can be asked to do some basic operations on complex numbers BEFORE you can plot the points.

Example - $z_1 = 2 - i$ $z_2 = 1 + 3i$

Plot the points $z_1 z_2$ iz_1 $z_1 - 2\overline{z_2}$

1

$z_1 z_2 = (2-i)(1+3i)$	*Replace $z_1 z_2$ with complex numbers*
$= 2(1+3i) - i(1+3i)$	*Open first bracket*
$= 2 + 6i - i - 3i^2$	*Remove brackets by multiplying*
$= 2 + 5i - 3(-1)$	*Turn $\sqrt{-1}$ into i*
$= 2 + 5i + 3$	*Simplify*
$= 5 + 5i$	*Simplify*
(5, 5i)	**Plot this Point on diagram**

2

$iz_1 = i(2-i)$	*replace z_1 with complex number*
$= 2i - i^2$	*Remove brackets by multiplying*
$= 2 - (-1)$	*Turn $\sqrt{-1}$ into i*
$= 2 + 1$	*Simplify*
$= 3$	*Simplify* *Notice no i part*
(3, 0i)	**Plot this Point on diagram**

3

$z_1 - 2\overline{z_2} = (2-i) - 2(1-3i)$	*Replace z_1 and $\overline{z_2}$ (the conjugate of z_2)*
	if $z_2 = 1 + 3i$ **then** $\overline{z_2} = 1 - 3i$
$= 2 - i - 2 + 6i$	*Remove brackets by multiplying*
$= 5i$	*Simplify* *Notice no real part*
(0, 5i)	**Plot this Point on diagram**