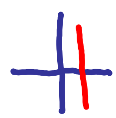
Complex Practice Exam 1

This practice exam contains ***sample*** *questions. The actual exam will have fewer questions, and may contain questions not listed here.*

1. Be prepared to explain the following concepts, definitions, or theorems:
   - A complex number, polar coordinates, rectangular coordinates
   - Add, Multiply, Sub, Div, Conjugate, abs Value, graphical interpretations of these
   - Complex roots
   - Mapping properties of complex functions
   - Arg(z) and arg(z)
   - The limit of a complex function f(z) as z approaches c is L
   - Continuity of a complex function f(z) at a point z = c
   - The complex derivative of a function f(z)
   - Analytic function and Entire function
   - CR equations
   - f(z) analytic & f'(z) = 0, f(z) analytic & f-conjugate analytic, f(z) analytic and |f(z)| constant
   - Harmonic function and harmonic conjugate of a function u (incl. how to find)
   - e^z, sin(z), cos(z), log(z), and Log(z)
   - Euler's Formula, De Moivre's Formula
   - Complex parametric functions z(t), their integrals and derivatives
   - Different paths (line segments and circles)
   - Contour Integrals

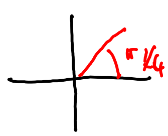
2. Describe the set of points z such that

   (a) $\mathrm{Re}(z) = 1$ $\quad$ $\mathrm{Re}(z) = x = 1 \Rightarrow$ vertical line

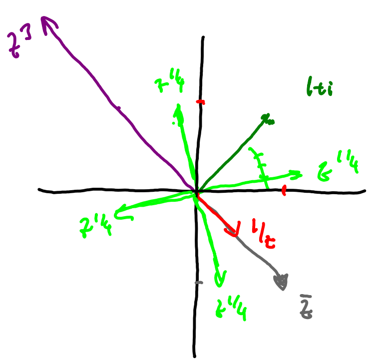
   (b) $|z-1| = 2$ $\quad$ circle, center at $z=1$ and radius $r=2$

   (c) $Arg(z) = \dfrac{\pi}{4}$

   ray

3. Let $z = 1+i$. Draw, in one coordinate system, $\bar{z}$, $\dfrac{1}{z}$, z^3, and $z^{\frac{1}{4}}$



4. Compute/simplify the following and find real and imag parts:

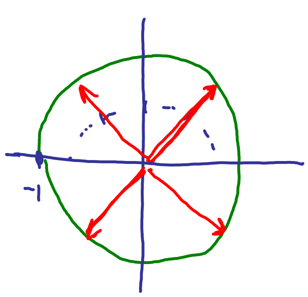
a) $\left|(\overline{1+i})(1-i)i\right| = |(1-i)(1-i)i| = |(1-2i-1)i| = 2$

(b) $\dfrac{i(1+i)^3}{(1-i)^2} = \dfrac{e^{\pi/2\, i}\left(\sqrt{2}\, e^{i\pi/4}\right)^3}{\left(\sqrt{2}\, e^{-i\pi/4}\right)^2} = \sqrt{2}\, e^{i(\pi/2 + 3\pi/4 + 2\pi/4)} = \sqrt{2}\, e^{i\, 7\pi/4} = 1-i$

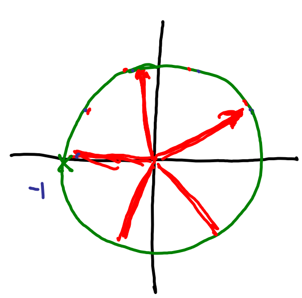
(c) $(1+i)^6 = \left(\sqrt{2}\, e^{i\pi/4}\right)^6 = 2^3 e^{i\frac{6\pi}{4}} = 8\, e^{\frac{3\pi}{2}} = -8i$

(d) $\dfrac{2+2i}{-\sqrt{3}+i} \cdot \dfrac{\sqrt{3}+i}{\sqrt{3}+i} = \dfrac{2\sqrt{3}-2+i(2+2\sqrt{3})}{3+1} = \dfrac{1}{2}\left(\sqrt{3}-1+i(\sqrt{3}+1)\right)$

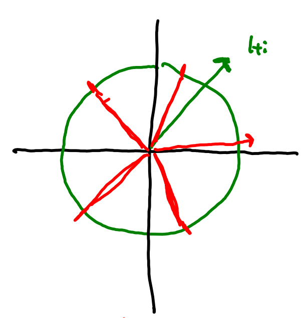
5. Find the fourth roots of -1, i.e. $\sqrt[4]{-1}$, and display them graphically. Do the same for the fifth roots of -1 and of (1+i).



$\sqrt[4]{-1}$



$\sqrt[5]{-1}$

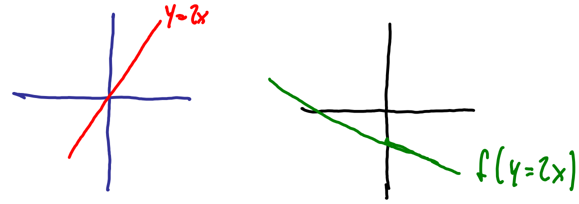


$\sqrt[5]{1+i}$

6. Find the image of the line $y = 2x$ under the map $f(z) = iz - 1$. i rotates by $\pi/2$, -1 shifts down

Set $x = t \Rightarrow y = 2x$ means
$z = t + i2t$

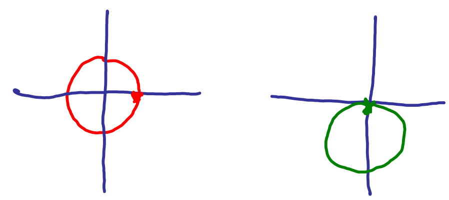
$\Rightarrow f(z) = i(t + i2t) - 1 =$
$= \underbrace{it}_{y} \underbrace{- 2t - 1}_{x} \Rightarrow x = -2t - 1 \Rightarrow x + 1 = -2t$
$y = t = -\frac{1}{2}x - \frac{1}{2}$ strange, should be -1



What is the image of the unit circle under the same map? Make sure to represent your answers algebraically as well as graphically.

circle $z = e^{it}$
$f(z) = ie^{it} - 1 = e^{i\pi/2} e^{it} - 1$
$= e^{i(t + \pi/2)} - 1$

circle shifted down



7. Consider the following questions, involving limits and continuity of complex functions. Remember that limits can be taken in different directions, and for complicated limits there is l'Hospital's rule.

a) If $f(z) = \dfrac{x - iy}{x + iy}$, then f is clearly undefined at $z = 0$. Can you define $f(0)$ in such a way that the new function is continuous at every point in the complex plane?

Let $x = 0$: $\lim_{y \to 0} \frac{0 - iy}{0 + iy} = -1$
let $y = 0$: $\lim_{x \to 0} \frac{x}{x} = 1$
} different, so $\lim_{z \to 0} f(z)$ does not exist, so can't make f cont. at $z = 0$

b) Say $f(z) = \dfrac{z^9 + z - 2i}{z^{15} + i}$ Can you define $f(i)$ in such a way that the new function is continuous at every point in the complex plane?

$f(i) = \dfrac{i^9 + i - 2i}{i^{15} + i} = \dfrac{0}{0}$ (?)

Define $f(z) = \begin{cases} \text{as before} & \text{if } z \neq i \\ -\frac{2}{3} & \text{if } z = i \end{cases}$

l'Hospital: $\lim_{z \to i} \dfrac{z^9 + z - 2i}{z^{15} + i} = \lim_{z \to i} \dfrac{9z^8 + 1}{15 z^{14}} = \dfrac{10}{-15} = -\dfrac{2}{3}$

c) Find $\lim_{z \to 1} \dfrac{1 + z^6}{1 + z^{10}}$ $= \frac{2}{2} = 1$

$\lim_{z \to i} \frac{1+z^6}{1-z^{10}}$ $= \frac{1+i^6}{1-i^{10}} \cdot \frac{0}{2} = 0$

$\lim_{z \to i} \frac{1+z^6}{1+z^{10}} = \frac{0}{0} = \lim_{z \to i} \frac{6z^5}{10z^9} = \frac{6i}{10i} = \frac{6}{10} = \frac{3}{5}$

l'Hospital

8. Consider the following questions about analytic functions.

   a) If $f(z) = \frac{1}{(z^2+1)^2}$ then determine where, if at all, the function is analytic.
   If it is analytic, find the complex derivative of *f*.

   f is analytic for $z \neq \pm i$

   $f'(z) = -2(z^2+1)^{-3} \cdot 2z$

   b) If $f(z) = x^3 - 3xy^2 + i(3x^2y - y^3)$ then determine where, if at all, the function is analytic. If it is analytic, find the complex derivative of *f*.

   $u(x,y) = x^3 - 3xy^2$ $\quad$ $v(x,y) = 3x^2y - y^3$

   $u_x = 3x^2 - 3y^2$ $\quad$ $v_y = 3x^2 - 3y^2$

   $u_y = -6xy$ $\quad$ $v_x = 6xy$

   $\Rightarrow$ CR hold for all z

   $f'(z) = u_x + iv_x =$

   $= \underline{3x^2 - 3y^2 + i6xy} = 3z^2$

9. Decide which of the following functions are analytic, and in which domain they are analytic. If a function is analytic, find its complex derivative:

   (a) $f(z) = \frac{e^z+1}{e^z-1}$ $\quad$ Not analytic if $e^z = 1 \Rightarrow z = 2k\pi i,\ k = 0, \pm 1, \pm 2, \ldots$

   $f'(z) = \frac{e^z(e^z-1) - e^z(e^z+1)}{(e^z-1)^2} = \frac{-2e^z}{(e^z-1)^2}$

(b) $f(z) = x^3 + 3ix^2y - 3xy^2 + x - iy^3 + iy$

$= \underbrace{x^3 - 3xy^2 + x}_{u} + i(\underbrace{3x^2y - y^3}_{v} + y)$

$u_x = 3x^2 - 3y^2 + 1$ $\quad$ $v_y = 3x^2 - 3y^2 + 1$ $\quad \Rightarrow$ analytic everywhere

$u_y = -6xy$ $\quad$ $v_x = 6xy$

$f'(z) = u_x + iv_x = 3x^2 - 3y^2 + 1 + i(6xy) = 3z^2 + 1$

10. Consider the function $u(x, y) = e^x \sin(y)$. Is it harmonic ? If so, find its harmonic conjugate. Do the same for

$u_x = e^x \sin(y)$, $u_{xx} = e^x \sin(y)$

$u_y = e^x \cos(y)$, $u_{yy} = -e^x \sin(y)$

$u_{xx} + u_{yy} = 0$

harmonic ✓

$u_x = e^x \sin(y) = v_y \Rightarrow v = -e^x \cos(y) + C(x)$

$v_x = -e^x \cos(y) + C'(x) = -u_y$ ✓ $\Rightarrow C'(x) = 0$

$\underline{f(z) = e^x \sin(y) - ie^x \cos(y) + C}$

(a) $u(x, y) = x^3 - 2xy + xy^3$

$u_x = 3x^2 - 2y + y^3$, $u_{xx} = 6x$

$u_y = -2x + 3xy^2$, $u_{yy} = 6xy$

$\Rightarrow u_{xx} + u_{yy} \neq 0 \Rightarrow$ not harmonic

(b) $u(x, y) = e^y \cos(x)$

$u_x = -e^y \sin(x)$, $u_{xx} = -e^y \cos(x)$

$u_y = e^y \cos(x)$, $u_{yy} = e^y \cos(x)$

$u_{xx} + u_{yy} = 0$

harmonic

$u_x = -e^y \sin(x) = v_y \Rightarrow v = -e^y \sin(x) + C(x)$

$v_x = -e^y \cos(x) + C'(x) = -u_y \Rightarrow C'(x) = 0$

$\Rightarrow f(z) = \underline{e^y \cos(x) - ie^y \sin(x) + C}$

11. Please find the following numerical answers:

(a) e^{2+2i} $= e^2 e^{2i} = \underline{e^2(\cos(2) + i\sin(2))}$

(b) $\cos(\pi+i)$

$\cos(z) = \frac{1}{2}\left(e^{iz}+e^{-iz}\right)$

$\cos(\pi+i) = \frac{1}{2}\left(e^{i(\pi+i)}+e^{-i(\pi+i)}\right) = \frac{1}{2}\left(e^{i\pi}e^{-1}+e^{-i\pi}e^{1}\right) =$

$= \frac{1}{2}\left(-e^{-1}-e^{1}\right) = -\frac{1}{2}\left(e^{1}+e^{-1}\right) = -\cosh(1)$

(c) $\sin\left(i-\frac{\pi}{2}\right)$

$\sin(z) = \frac{1}{2i}\left(e^{iz}-e^{-iz}\right)$

$\sin\left(i-\frac{\pi}{2}\right) = \frac{1}{2i}\left(e^{i(i-\pi/2)}-e^{-i(i-\pi/2)}\right) = \frac{1}{2i}\left(e^{-1}e^{-\frac{\pi}{2}i}-e^{1}e^{\frac{\pi}{2}i}\right) = \frac{1}{2i}\left(-ie^{-1}-ie^{1}\right) = -\frac{1}{2}\left(e^{1}+e^{-1}\right) = -\cosh(1)$

14 Solve the following equations for z.

(a) $z^4+1=0$,

$z = \sqrt[4]{-1} = \left(e^{\pi i}\right)^{1/4} = e^{i\left(\pi/4+\frac{2\pi k}{4}\right)} = e^{i\frac{2k+1}{4}\pi}$, $k=0,1,2,3$

$= e^{i\pi/4},\ e^{i\frac{3\pi}{4}},\ e^{i\frac{5\pi}{4}},\ e^{i\frac{7\pi}{4}}$

(b) $\left|e^{2z}\right|=3$ $\Rightarrow \left|e^{2x+2iy}\right| = 3 \Leftrightarrow e^{2x}=3 \Leftrightarrow x=\frac{1}{2}\ln(3)$, $\Rightarrow z = \frac{1}{2}\ln(3)+iy$, any y

(c) $\sin(z)=3i$ $\Leftrightarrow \frac{1}{2i}\left(e^{iz}-e^{-iz}\right) = 3i$ $\Rightarrow e^{iz}-e^{-iz} = -6$ $\mid e^{iz}$ redo later

$\frac{1}{2}\left(e^{iz}-e^{-iz}\right) = -3$ $\quad \left(e^{iz}\right)^2-1 = -6e^{iz}$ let $u=e^{iz}$

$\Rightarrow u^2+6u-1=0 \Rightarrow u_{1,2} = \frac{-6\pm\sqrt{36+4}}{2} = \frac{-6\pm\sqrt{40}}{2}$

(d) $e^{4z}=1$

$\Leftrightarrow e^{4z}=e^{0} \Rightarrow z = 0, \frac{\pi}{2}i, \pi i, \frac{3\pi}{2}i, \ldots \Rightarrow z = \frac{k\pi}{2}i$, $k=0,\pm1,\pm2,\ldots$

(e) $\cos(z)=i\sin(z)$ $\quad \frac{1}{2}\left(e^{iz}+e^{-iz}\right) = \frac{1}{2i}i\left(e^{iz}-e^{-iz}\right) \Leftrightarrow \cancel{e^{iz}}+e^{-iz} = \cancel{e^{iz}}-e^{-iz}$ redo later

$\Leftrightarrow 2e^{iz}=0 \Leftrightarrow e^{iz}=0$ never

$\Rightarrow$ no solution

15 Use the definition of derivative to show that the functions $f(z) = \mathbf{Re}(z)$ is nowhere differentiable.

$$f'(z) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{z\to z_0} \frac{x-x_0}{x-x_0+i(y-y_0)}$$

At $x=x_0$, $\lim_{y\to y_0} \frac{1}{i(y-y_0)}$ dne. so f is nowhere diff'ble

Use the CR equations to show that the function $f(z) = \bar{z}$ is nowhere differentiable.

$f(z) = \bar{z} = \underbrace{x}_{u} - i\underbrace{y}_{v}$ $\quad u_x = 1 \quad v_y = -1$ $\quad$ but $u_x \neq v_y$ so f is not diff'ble

$u_y = 0 \quad v_x = 0$

Show that if v is the harmonic conjugate of u, then the product $u\,v$ is harmonic.

$u+iv$ are analytic $\Rightarrow$ $u_x = v_y$, $u_y = -v_x$ and u, v are harmonic

$(uv)_x = u_x v + u v_x \Rightarrow (uv)_{xx} = u_{xx} v + u_x v_x + u_x v_x + u v_{xx}$

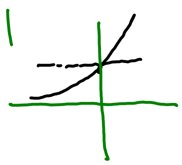
$(uv)_y = u_y v + u v_y \Rightarrow (uv)_{yy} = u_{yy} v + u_y v_y + u_y v_y + u v_{yy}$

$\Rightarrow (uv)_{xx} + (uv)_{yy} = (u_{xx}+u_{yy})v + 2u_x v_x + 2u_y v_y + u(v_{xx}+v_{yy}) =$

$= 0 \cdot v - 2u_x u_y + 2u_y u_x + u \cdot 0 = 0$ so harmonic

16 Show that $|e^z| \leq 1$ if $\mathbf{Re}(z) \leq 0$

$|e^{x+iy}| = |e^x e^{iy}| = |e^x||e^{iy}| = |e^x| = e^x < 1$



$e^x < 1 \iff x \leq 0$, or $\mathrm{Re}(z)$

17 State De Moivre's formula. Then use it to prove the trig identity $\sin(2x) = 2\sin(x)\cos(x)$

$(e^{it})^2 = e^{i2t} \iff (\cos(t) + i\sin(t))^2 = \cos(2t) + i\sin(2t)$

$\Rightarrow \cos^2(t) - \sin^2(t) + i\,2\cos(t)\sin(t) = \cos(2t) + i\sin(2t)$

$\Rightarrow \cos^2(t) - \sin^2(t) = \cos(2t)$ and $\underline{2\cos(t)\sin(t) = \sin(2t)}$ proved

18 Show that the function e^{iz} is periodic with period 2π

$e^{iz} = f(z) \iff f(z+2\pi) = e^{i(z+2\pi)} = e^{iz}\underbrace{e^{2\pi i}}_{1} = e^{iz} =$

19 Show that the function *sin(z)* is unbounded

let $z = it$: $|\sin(it)| = \left|\frac{1}{2i}\left(e^{i(it)} - e^{-i(it)}\right)\right|$

$= \frac{1}{2}\left|e^{-t} - e^{t}\right| \xrightarrow[\text{as } t\to\infty]{} \infty$ ✓

20 Show that the function $f(z) = z\bar{z} + z + \bar{z} + 2x$ can not be an analytic function.

$= x^2+y^2 + 2x + 2x = \underbrace{x^2+y^2+4x}_{u} + i\underbrace{0}_{v}$

$\Rightarrow u_x = 2x+4 = 0 \quad x=-4$

$u_y = 2y = 0 \quad \Rightarrow y=0$

$\Rightarrow$ not diff-ble in a nbhd. of $(-4,0)$ $\Rightarrow$ not analytic

21 Prove that $\sin^2(z) + \cos^2(z) = 1$ (Hint: take the derivative of $f(z) = \sin^2(z) + \cos^2(z)$)

$f(z) = \sin^2(z) + \cos^2(z)$

$f'(z) = 2\sin(z)\cos(z) - 2\cos(z)\sin(z) = 0 \quad \forall z$

$\Rightarrow f$ is constant. Since $f(0)=1$ that constant is 1

$\Rightarrow \sin^2(z) + \cos^2(z) = 1$

22 Prove the following theorem: If *f(z)* is an analytic function with values that are always imaginary, then the function must be constant.

did that

23 Prove the following theorem: if is a harmonic function in an open set U (i.e. h is twice continuously differentiable and $\frac{\partial^2}{\partial x^2}h + \frac{\partial^2}{\partial y^2}h = 0$ in the open set U), then the complex function $f(z) = \frac{\partial}{\partial x}h(x,y) - i\frac{\partial}{\partial y}h(x,y)$ is an analytic function in U.

$f(z) = \underbrace{h_x}_{u} - i\underbrace{h_y}_{v}$

$u_x = h_{xx}$ $\quad v_y = -h_{yy}$ $\quad$ since $h_{xx} + h_{yy} = 0$

$\Rightarrow u_x = v_y$

$u_y = h_{xy}$ $\quad v_x = -h_{yx}$ $\quad$ since h is twice cont. diffble $h_{xy} = h_{yx}$

$\Rightarrow u_y = -v_x$

So CR equations hold everywhere so f is analytic

24 Find complex parametric functions representing the following paths:

(a) a straight line from –i to i,

$$z(t) = -i + t(2i) \quad , t \in [0,1]$$

(b) the right half of a circle from –i to i,

$$z(t) = e^{it} \quad , t \in [-\pi/2, \pi/2]$$

(c) a straight line from -1 – 2i to 3 + 2i

$$z(t) = -1 - 2i + t(3 + 2i + 1 + 2i) =$$
$$= -1 - 2i + t(4 + 4i) \quad , t \in [0,1]$$

(d) a circle centered at 1+i of radius 2 – d

$$z(t) = 2e^{it} + 1 + i \quad , t \in [0, 2\pi)$$

25 Evaluate

a. z'(t) for $z(t) = \cos(2t) + i\sin(2t)$ $= e^{i2t}$

$z'(t) = -2\sin(2t) + 2i\cos(2t) = 2i e^{i2t}$

b. $\int_0^{\pi} z(t)dt$ for $z(t) = (5+4i)e^{3it}$

$\int_0^{\pi} (5+4i)e^{3it}dt = (5+4i)\int_0^{\pi} e^{3it}dt = (5+4i)\frac{1}{3i}e^{3it}\Big|_0^{\pi} =$

$= \frac{1}{3i}(5+4i)(e^{3\pi i}-1) = -\frac{2}{3i}(5+4i) =$

$= \underline{\underline{2/3\, i(5+4i)}}$

26 Evaluate

a. $\int_\gamma iz^2 + 3dz$ where γ is a line segment from -1-i to 1+i

$z(t) = -1-i+t(1+i+1+i) =$

$= -1-i+t(2+2i)$, $t \in [0,1]$

$\int_\gamma iz^2+3\,dz = \int_0^1 \left[i\left(-1-i+t(2+2i)\right)^2+3\right]\left[2+2i\right]dt =$

$= \ldots$ Rest left for reader

b. $\int_\gamma \frac{1}{\bar{z}}dz$ where γ is a circle radius 2 centered at the origin $z(t) = e^{it}$, $t \in [0, 2\pi]$

$\int_\gamma \frac{1}{\bar{z}}dz = \int_0^{2\pi} \frac{1}{e^{-it}} ie^{it}dt = i\int_0^{2\pi} e^{it}e^{it}dt = i\int_0^{2\pi} e^{2it}dt = i\frac{1}{2i}e^{2it}\Big|_0^{2\pi} = \frac{1}{2}(e^{4\pi i}-1) =$

$= \underline{\underline{0}}$