FLUID DYNAMICS

Master Degree Programme in Physics - UNITS
Physics of the Earth and of the Environment

Compressible Flow

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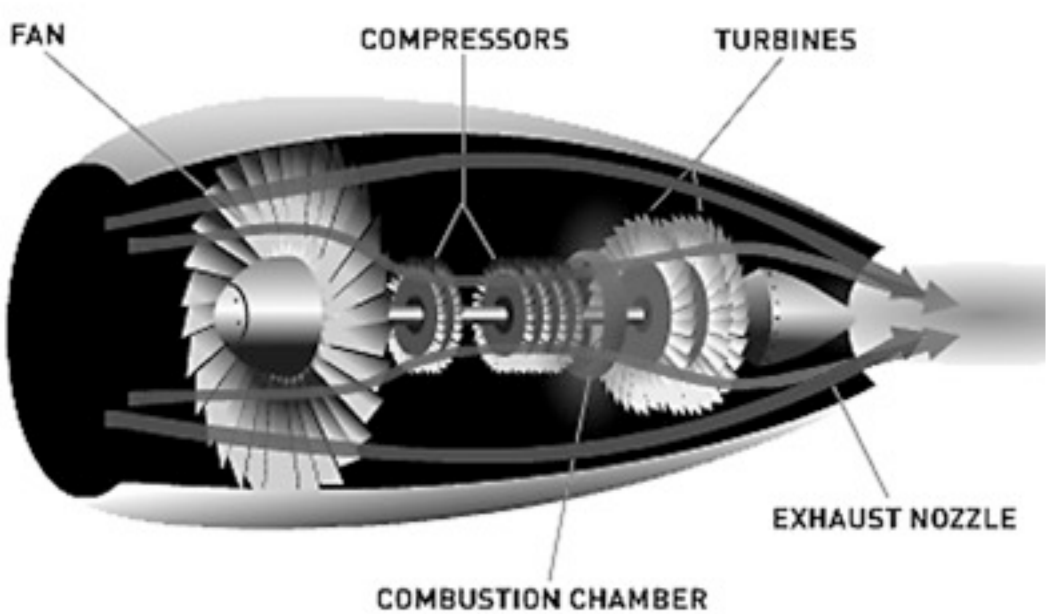
Overview

- We consider flows that involve significant changes in density. Such flows are called compressible flows, and they are frequently encountered in devices that involve the flow of gases at very high speeds.
- Compressible flow combines fluid dynamics and thermodynamics in that both are absolutely necessary to the development of the required theoretical background.
- We develop the general relations associated with compressible flows for an ideal gas with constant specific heats.

Stagnation Properties



- Definition of enthalpy

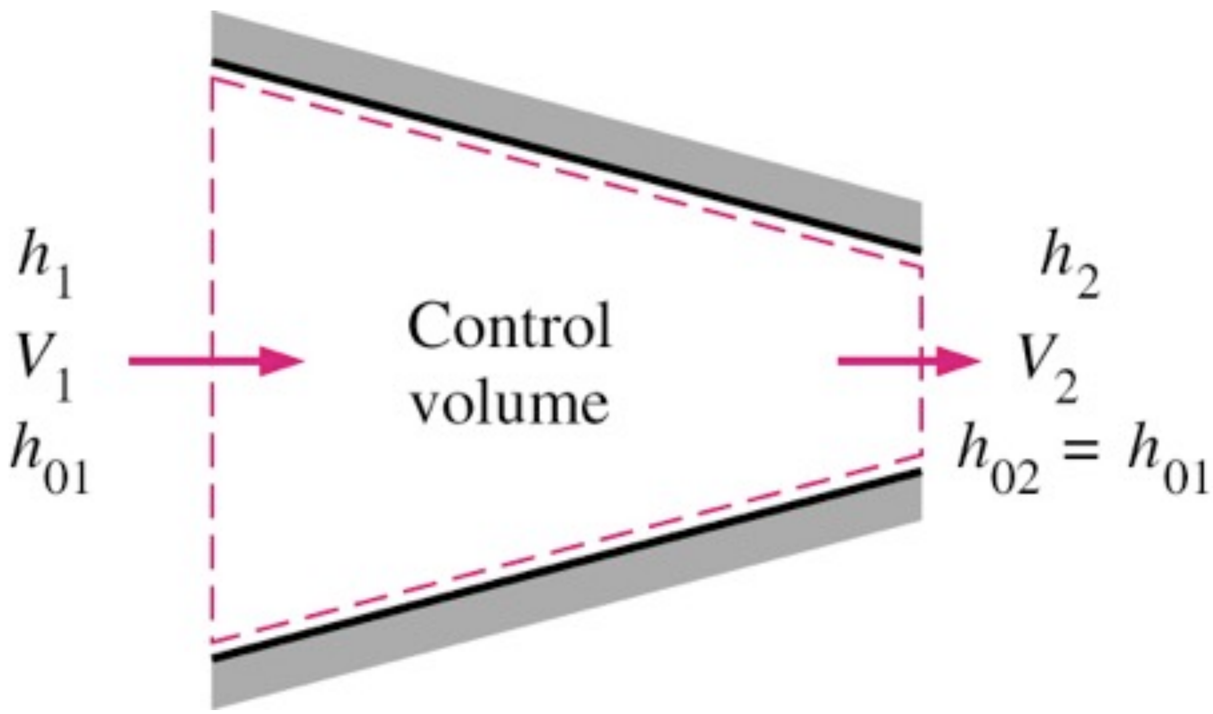
$$h = u + P/\rho$$

which is the sum of internal energy u and flow energy P/ρ

- For high-speed flows, enthalpy and kinetic energy are combined into stagnation enthalpy h_0

$$h_0 = h + \frac{V^2}{2}$$

Stagnation Properties



- Steady adiabatic flow through duct with no shaft/electrical work and no change in elevation and potential energy

$$\dot{E}_{in} = \dot{E}_{out}$$

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2}$$

$$h_{0,1} = h_{0,2}$$

- Therefore, stagnation enthalpy remains constant during steady-flow process

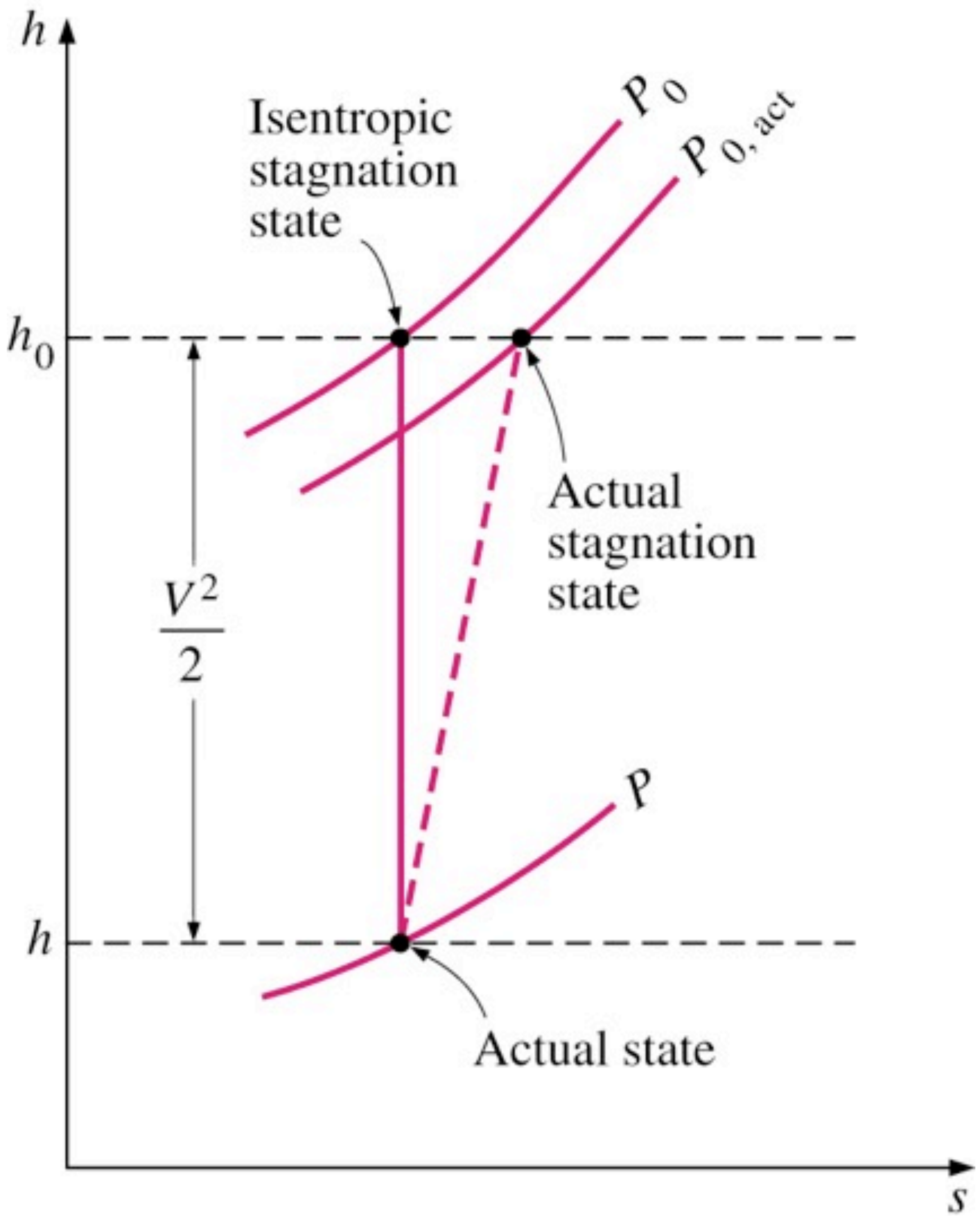
Stagnation Properties

- If a fluid were brought to a complete stop ($V_2 = 0$)

$$h_1 + \frac{V_1^2}{2} = h_2 = h_{0,2}$$

- Therefore, h_0 represents the enthalpy of a fluid when it is brought to rest adiabatically.
- During a stagnation process, kinetic energy is converted to enthalpy.
- Properties at this point are called stagnation properties (which are identified by subscript 0)

Stagnation Properties



- If the process is also reversible, the stagnation state is called the isentropic stagnation state.
  - Stagnation enthalpy is the same for isentropic and actual stagnation states
  - Actual stagnation pressure $P_{0,act}$ is lower than P_0 due to increase in entropy s as a result of fluid friction.
  - Nonetheless, stagnation processes are often approximated to be isentropic, and isentropic properties are referred to as stagnation properties

Stagnation Properties

- For an ideal gas, $h = C_p T$, which allows the h_0 to be rewritten

$$c_p T_0 = c_p T + \frac{V^2}{2} \Longrightarrow T_0 = T + \frac{V^2}{2c_p}$$

  - T_0 is the stagnation temperature. It represents *the temperature an ideal gas attains when it is brought to rest adiabatically.*
  - $(V^2/2c_p)$ corresponds to the temperature rise, and is called the dynamic temperature

- For ideal gas with constant specific heats, stagnation pressure and density can be expressed as

$$\frac{P_0}{P} = \left(\frac{T_0}{T}\right)^{k/(k-1)} \qquad \frac{\rho_0}{\rho} = \left(\frac{T_0}{T}\right)^{1/(k-1)}$$

Stagnation Properties

- When using stagnation enthalpies, there is no need to explicitly use kinetic energy in the energy balance.

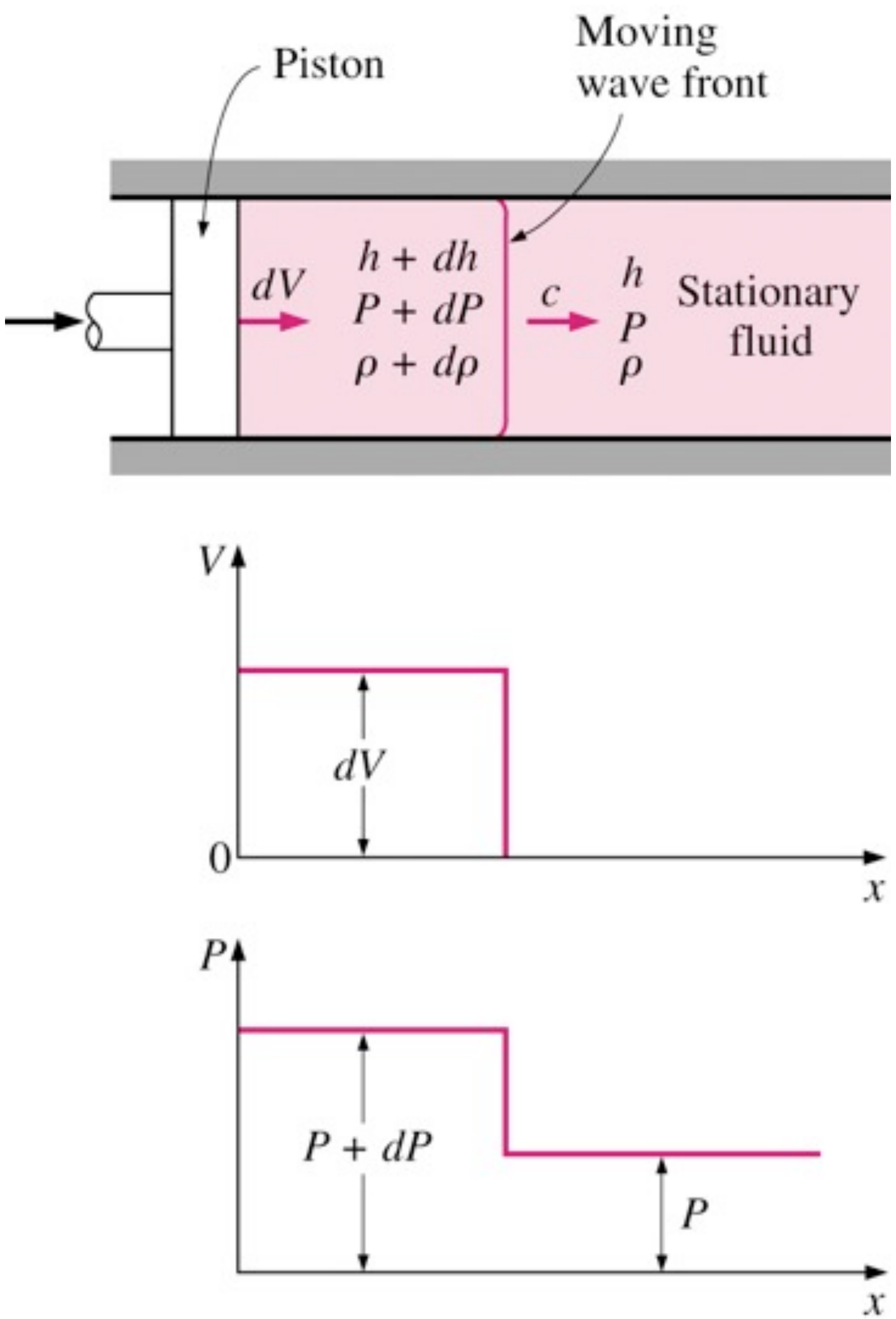
$$\dot{E}_{in} = \dot{E}_{out}$$

$$q_{in} + w_{in} + (h_{01} + gz_1) = q_{out} + w_{out} + (h_{02} + gz_2)$$

- Where h_{01} and h_{02} are stagnation enthalpies at states 1 and 2.
- If the fluid is an ideal gas with constant specific heats

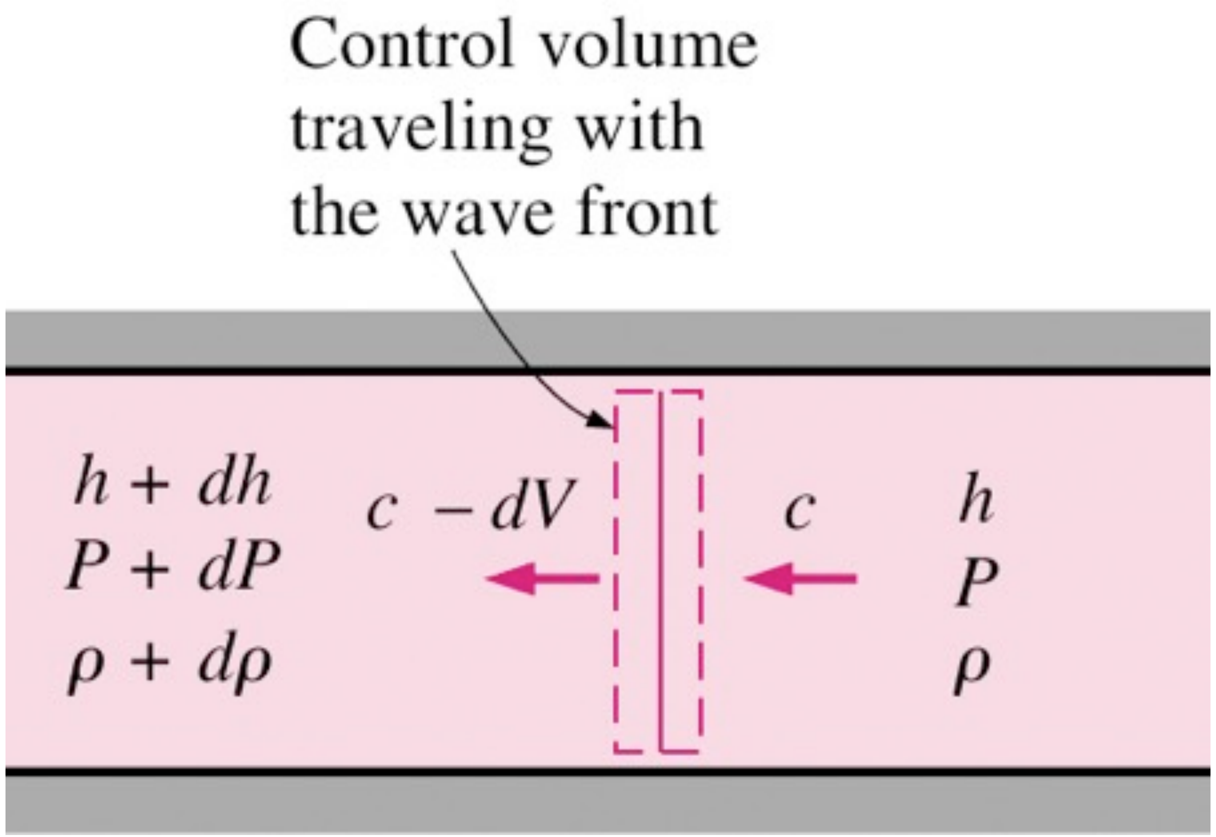
$$(q_{in} - q_{out}) + (w_{in} - w_{out}) = c_p (T_{02} - T_{01}) + g (z_2 - z_1)$$

Speed of Sound and Mach Number



- Important parameter in compressible flow is the speed of sound.
  - Speed at which infinitesimally small pressure wave travels
- Consider a duct with a moving piston
  - Creates a sonic wave moving to the right
  - Fluid to left of wave front experiences incremental change in properties
  - Fluid to right of wave front maintains original properties

Speed of Sound and Mach Number



- Construct CV that encloses wave front and moves with it
- Mass balance

$$\dot{m}_{right} = \dot{m}_{left}$$

$$\rho A c = (\rho + d\rho)\, A\, (c - dV)$$

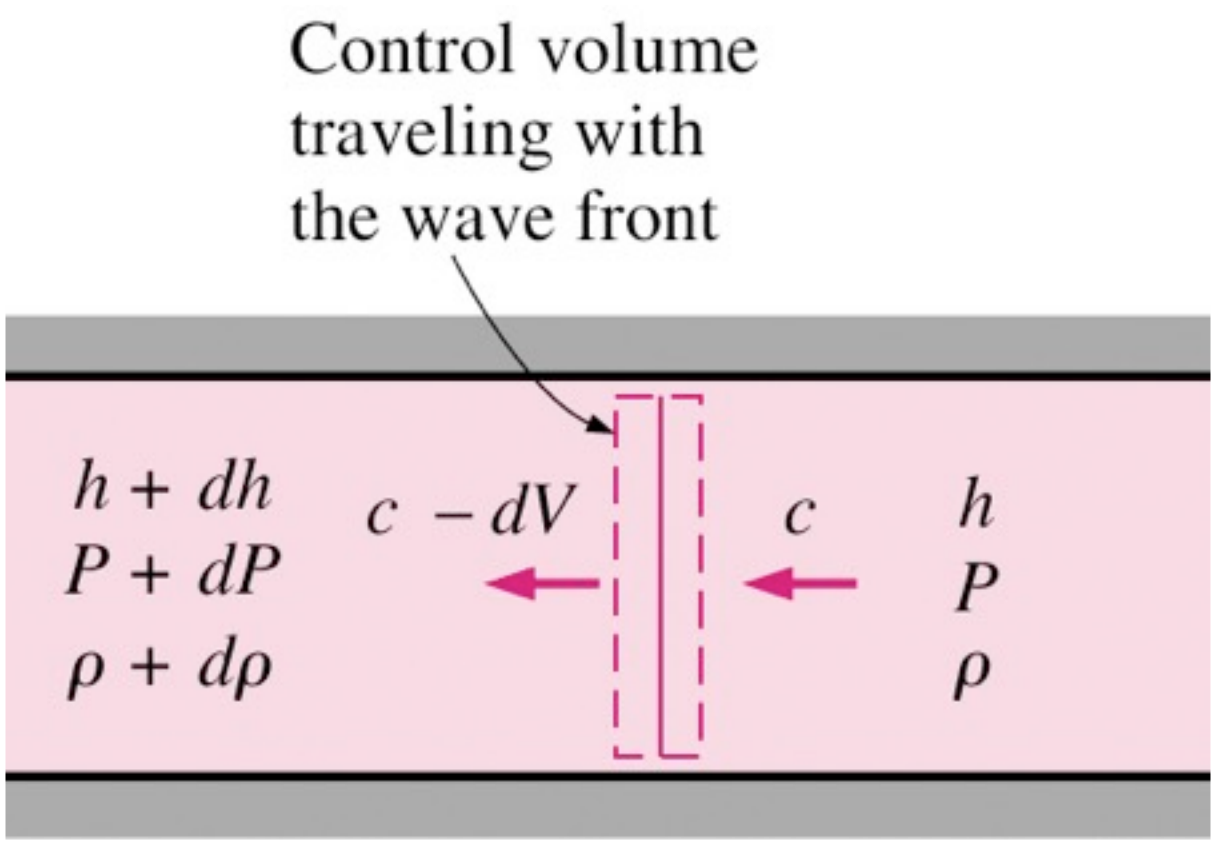
$$\rho A c = A\,(\rho c - \rho\, dV + c\, d\rho - d\rho\, dV)$$

cancel

Neglect H.O.T.

$$c\, d\rho - \rho\, dV = 0$$

Speed of Sound and Mach Number



- Energy balance $e_{in} = e_{out}$

$$h + \frac{c^2}{2} = h + dh + \frac{(c - dV)^2}{2}$$

$$h + \frac{c^2}{2} = h + dh + \frac{c^2 - 2c\,dV + dV^2}{2}$$

cancel

cancel

Neglect H.O.T.

$$dh - c\,dV = 0$$

Speed of Sound and Mach Number

- Using the thermodynamic relation

$$T\,\cancel{ds} = dh - dP/rho \Longrightarrow dh = dP/rho$$

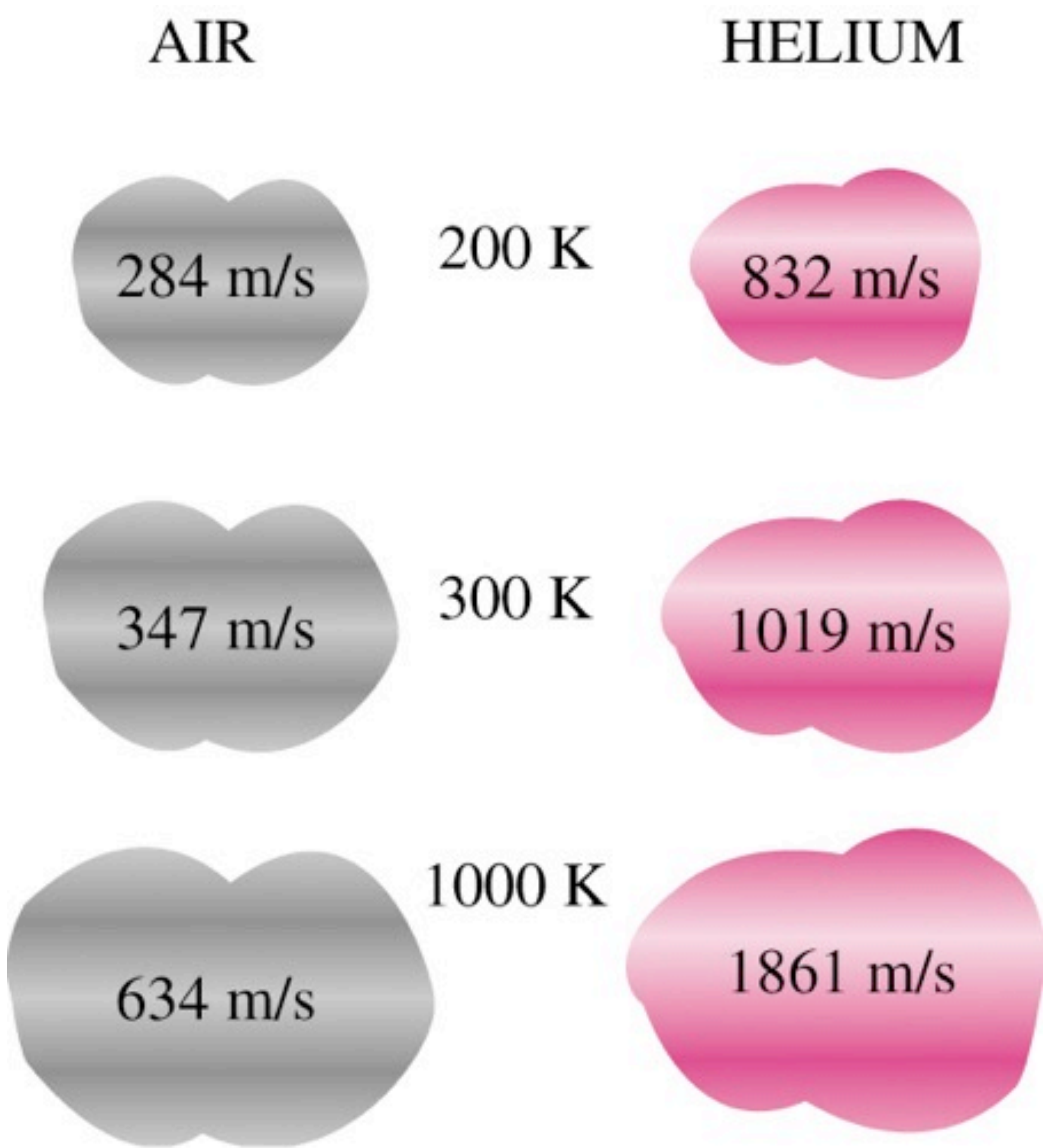
- Combining this with mass and energy conservation gives

$$c^2 = k\left(\frac{\partial P}{\partial \rho}\right)_T$$

- For an ideal gas $P = \rho RT$

$$c^2 = k\left(\frac{\partial P}{\partial \rho}\right)_T = k\left[\frac{\partial(\rho RT)}{\partial \rho}\right]_T = kRT \Longrightarrow \boxed{c = \sqrt{kRT}}$$

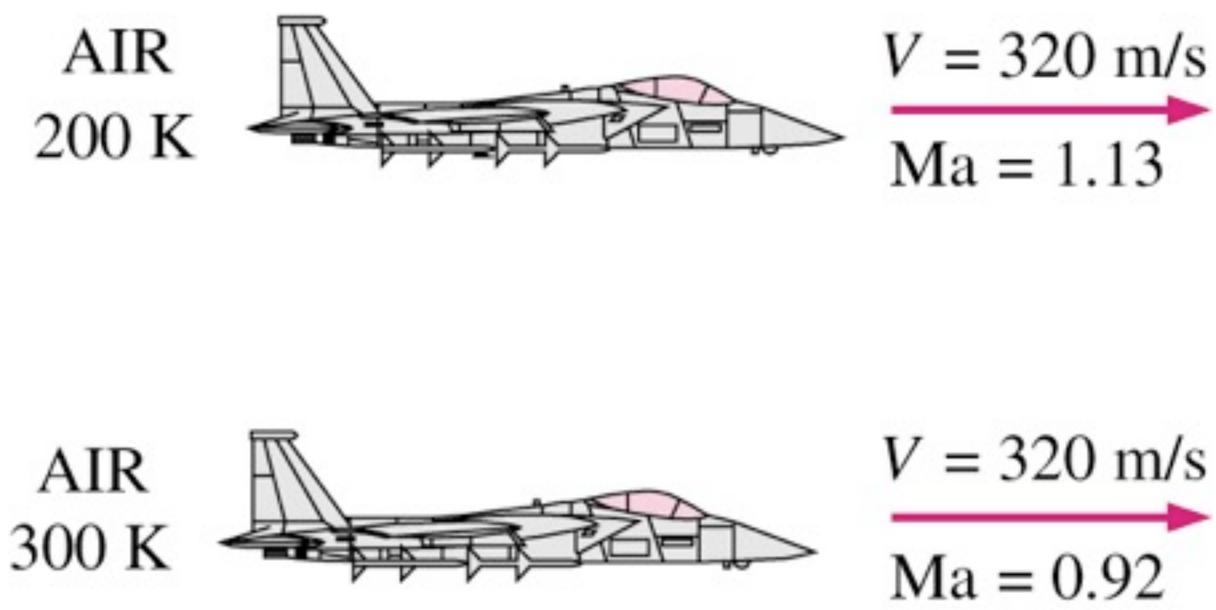
Speed of Sound and Mach Number



$$c = \sqrt{kRT}$$

- Since
  - R is constant
  - k is only a function of T
  - Speed of sound is only a function of temperature

Speed of Sound and Mach Number



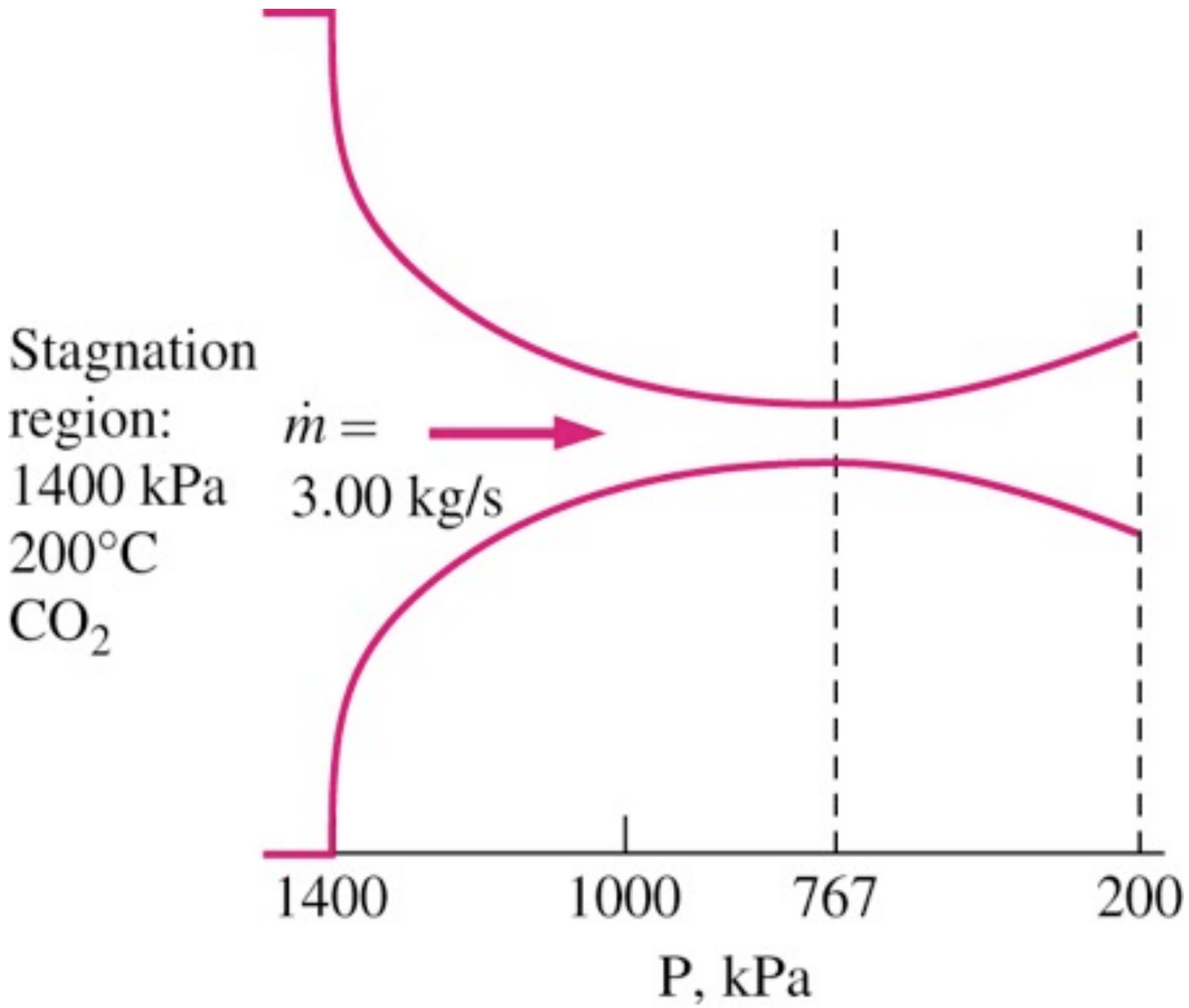
- Second important parameter is the Mach number *Ma*
- Ratio of fluid velocity to the speed of sound

$$Ma = \frac{V}{c}$$

- Flow regimes classified in terms of *Ma*

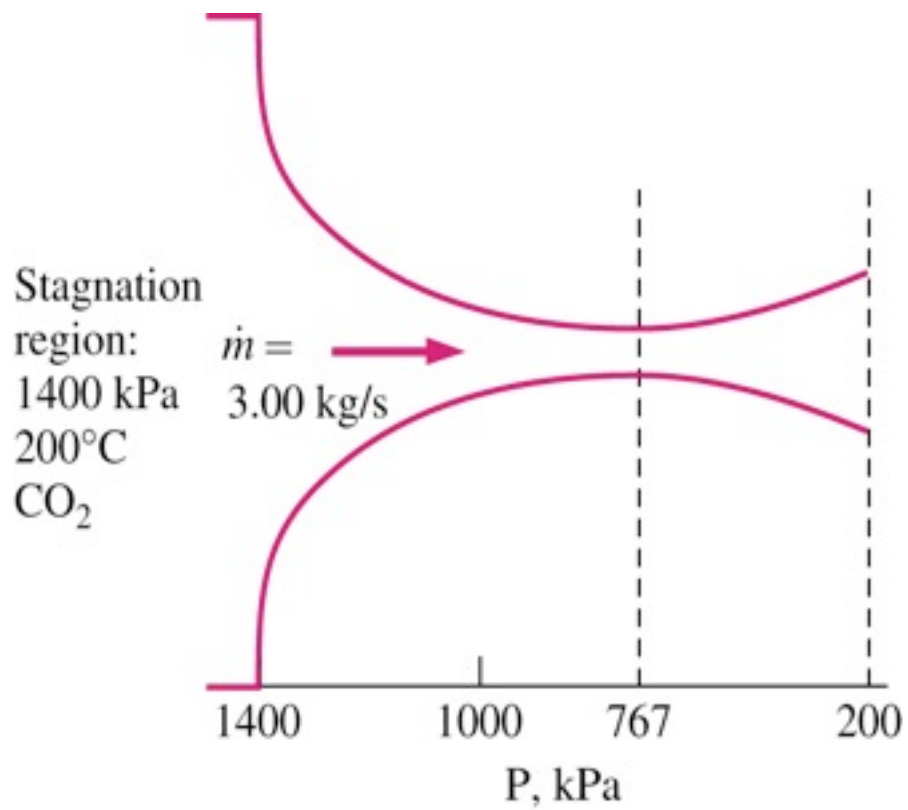
Ma < 1	: Subsonic
Ma = 1	: Sonic
Ma > 1	: Supersonic
Ma >> 1	: Hypersonic
Ma ≈ 1	: Transonic

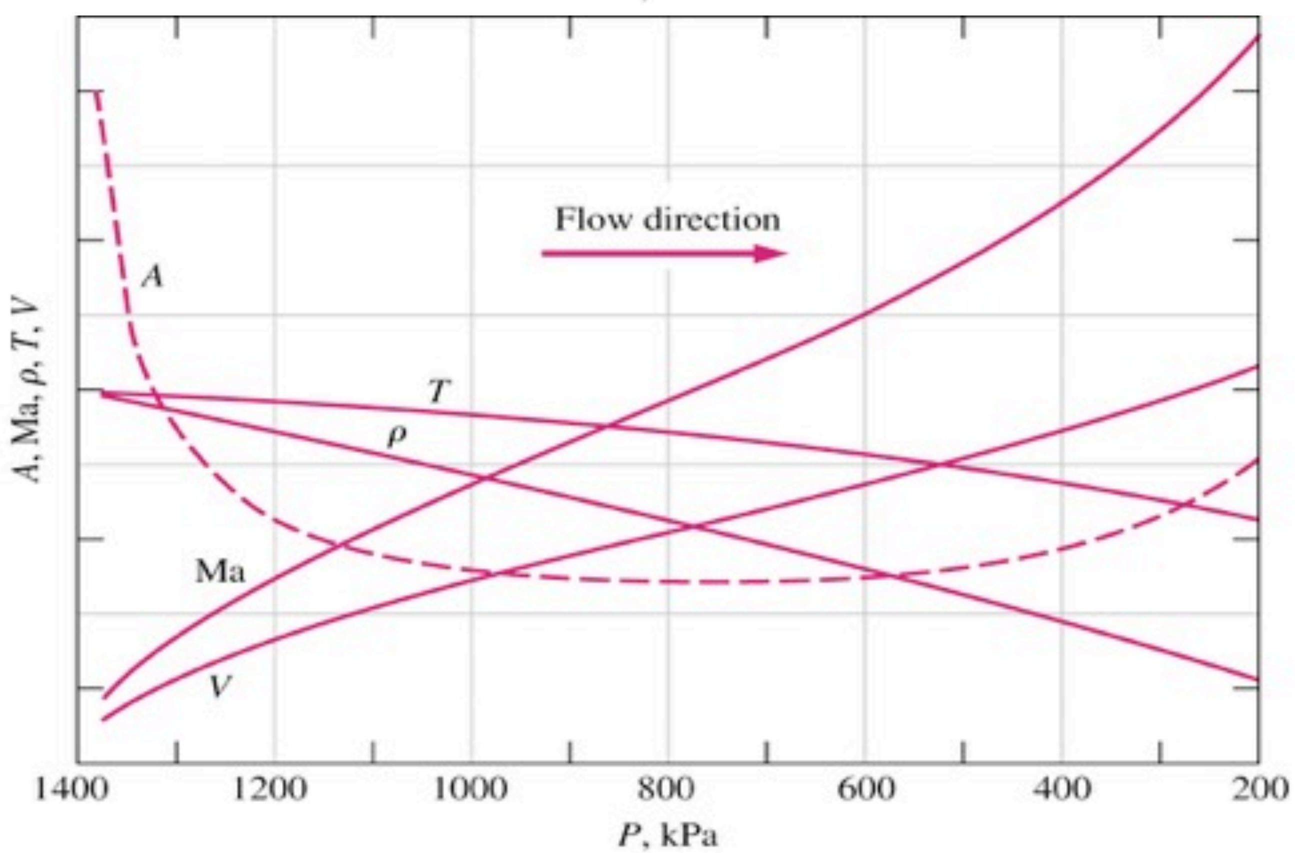
One-Dimensional Isentropic Flow



- For flow through nozzles, diffusers, and turbine blade passages, flow quantities vary primarily in the flow direction
  - Can be approximated as 1D isentropic flow
- Consider example of Converging-Diverging Duct

One-Dimensional Isentropic Flow





- Example illustrates:
  - *Ma* = 1 at the location of the smallest flow area, called the throat
  - Velocity continues to increase past the throat, and is due to decrease in density
  - Area decreases, and then increases. Known as a converging - diverging nozzle. Used to accelerate gases to supersonic speeds.

One-Dimensional Isentropic Flow

Variation of Fluid Velocity with Flow Area

- Relationship between V, ρ, and A are complex
- Derive relationship using continuity, energy, speed of sound equations

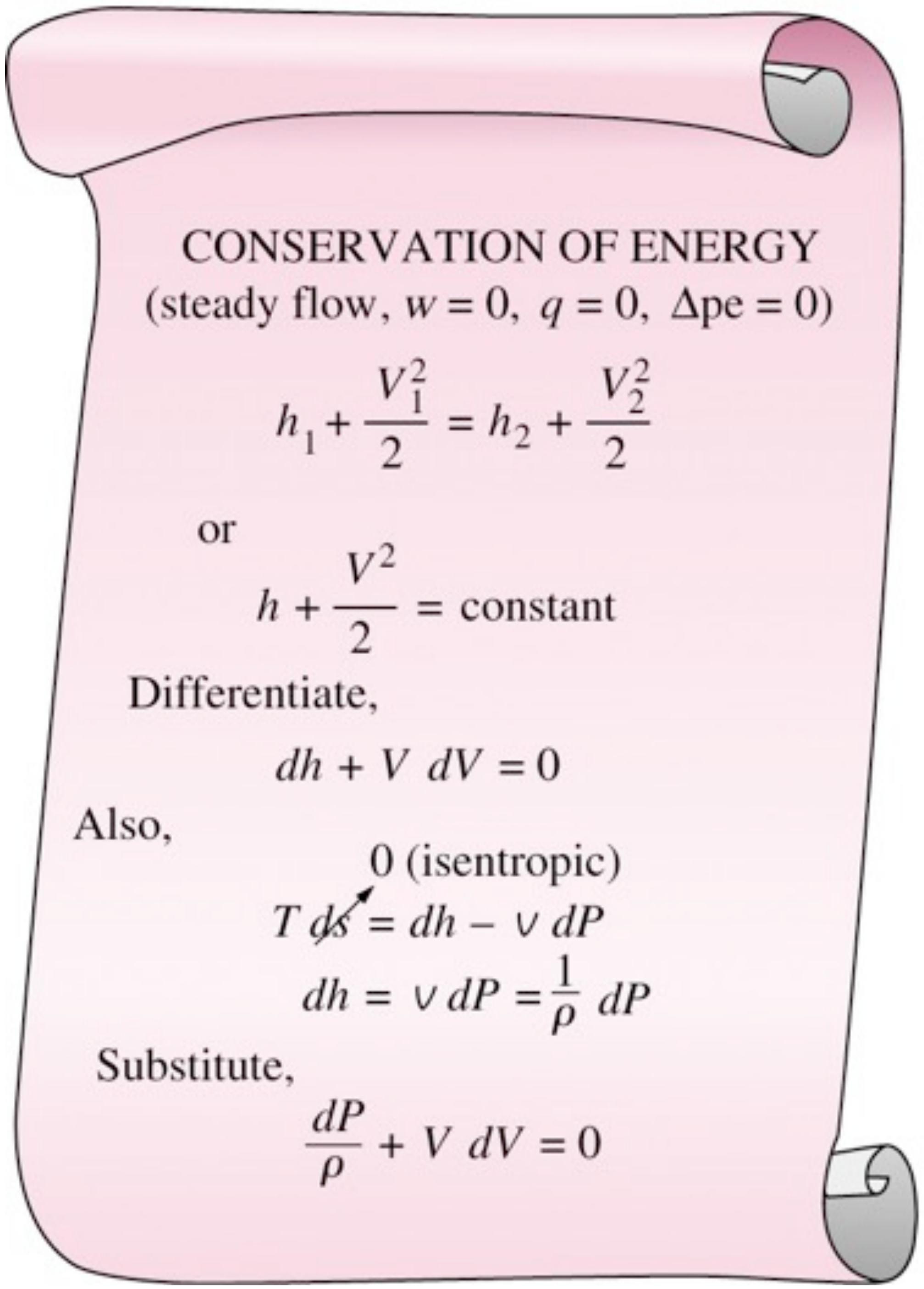
- Continuity $\dot{m} = \rho A V = \text{constant}$

  - Differentiate and divide by mass flow rate (ρAV)

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0$$

One-Dimensional Isentropic Flow

Variation of Fluid Velocity with Flow Area



- Derived relation (on image at left) is the differential form of Bernoulli's equation.
- Combining this with result from continuity gives

$$\frac{dA}{A} = \frac{dP}{\rho}\left(\frac{1}{V^2} - \frac{d\rho}{dP}\right)$$

- Using thermodynamic relations and rearranging

$$\boxed{\frac{dA}{A} = \frac{dP}{\rho V^2}\left(1 - Ma^2\right)}$$

One-Dimensional Isentropic Flow

Variation of Fluid Velocity with Flow Area

$$\frac{dA}{A} = \frac{dP}{\rho V^2}\left(1 - Ma^2\right)$$

- This is an important relationship
  - For $Ma < 1$, $(1 - Ma^2)$ is positive $\Rightarrow$ dA and dP have the same sign.
    - Pressure of fluid must increase as the flow area of the duct increases, and must decrease as the flow area decreases
  - For $Ma > 1$, $(1 - Ma^2)$ is negative $\Rightarrow$ dA and dP have opposite signs.
    - Pressure must increase as the flow area decreases, and must decrease as the area increases

One-Dimensional Isentropic Flow

Variation of Fluid Velocity with Flow Area

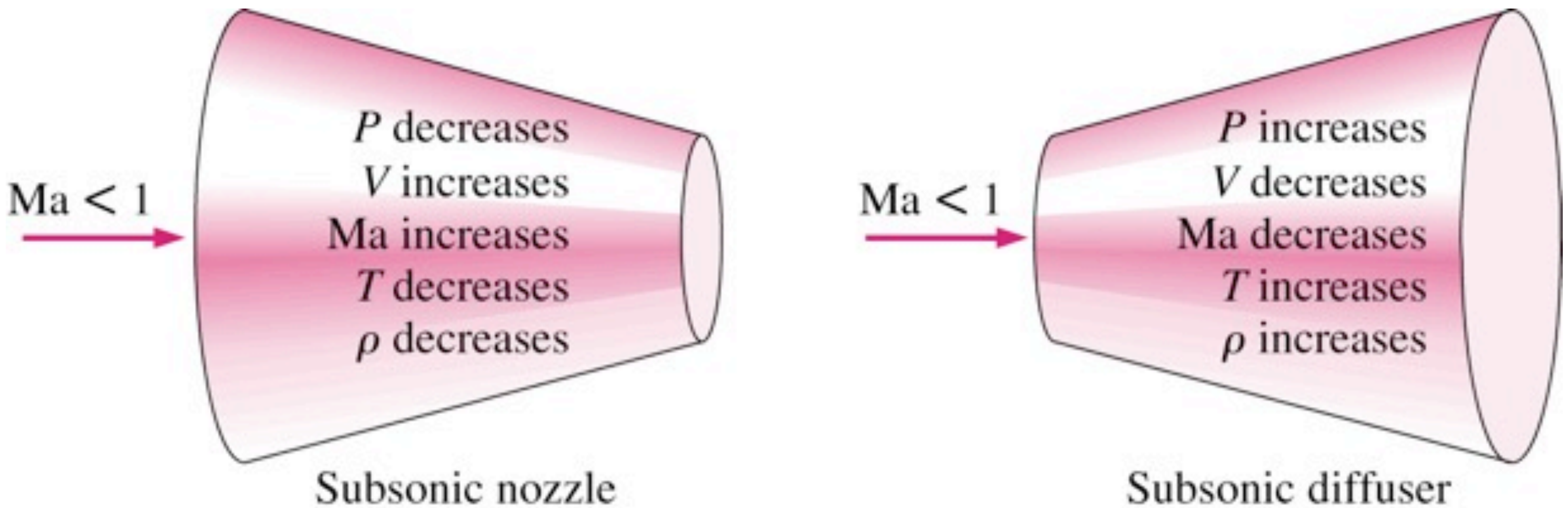
- A relationship between *dA* and *dV* can be derived by substituting $\rho V = -dP/dV$ (from the differential Bernoulli equation)

$$\frac{dA}{A} = -\frac{dV}{V}\left(1 - Ma^2\right)$$

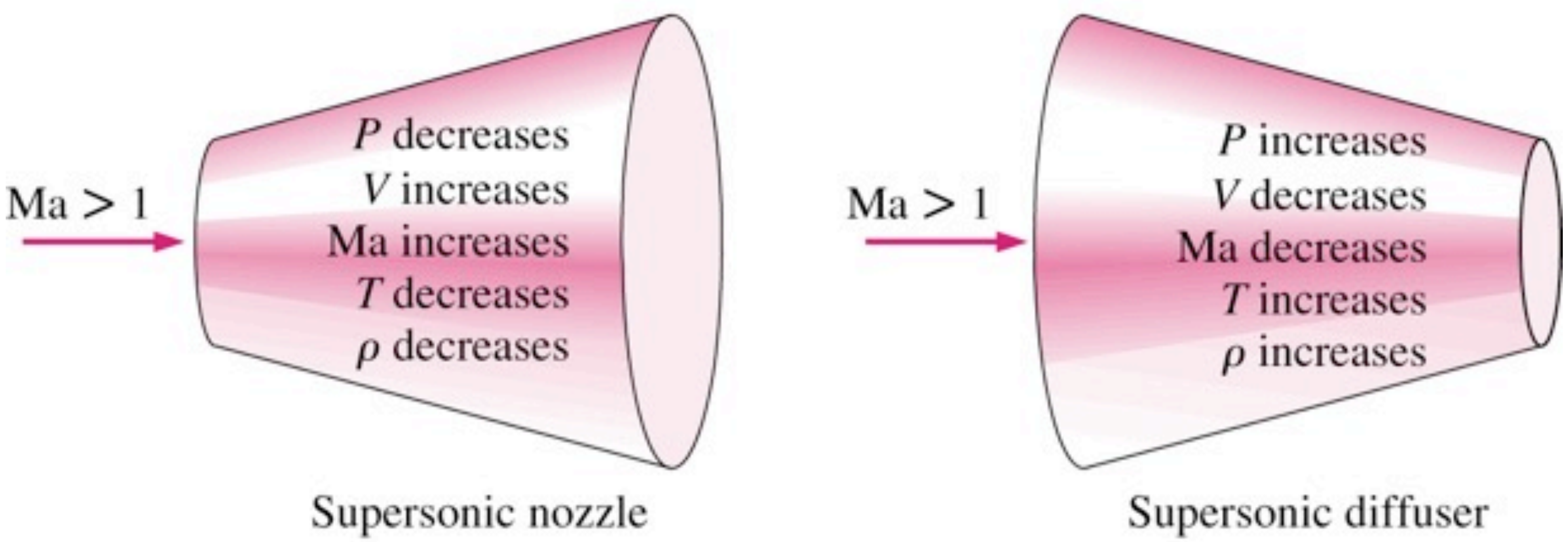
- Since *A* and *V* are positive
  - For subsonic flow (Ma < 1) $dA/dV < 0$
  - For supersonic flow (Ma > 1) $dA/dV > 0$
  - For sonic flow (Ma = 1) $dA/dV = 0$

One-Dimensional Isentropic Flow
Variation of Fluid Velocity with Flow Area

Comparison of flow properties in subsonic and supersonic nozzles and diffusers



(*a*) Subsonic flow



(*b*) Supersonic flow

One-Dimensional Isentropic Flow

Property Relations for Isentropic Flow of Ideal Gases

- Relations between static properties and stagnation properties in terms of *Ma* are useful.
- Earlier, it was shown that stagnation temperature for an ideal gas was

$$T_0 = T + \frac{V^2}{2c_p} \quad \Longrightarrow \quad \frac{T_0}{T} = 1 + \frac{V^2}{2c_p T}$$

- Using definitions, the dynamic temperature term can be expressed in terms of *Ma*

$$c_p = kR/(k-1), \quad c^2 = kRT, \quad Ma = V/c$$

$$\frac{V^2}{2c_p T} = \frac{V^2}{2[kR/(k-1)]T} = \left(\frac{k-1}{2}\right)\frac{V^2}{c^2} = \left(\frac{k-1}{2}\right)Ma^2$$

$$\boxed{\frac{T_0}{T} = 1 + \left(\frac{k-1}{2}\right)Ma^2}$$

One-Dimensional Isentropic Flow

Property Relations for Isentropic Flow of Ideal Gases

- Substituting T_0/T ratio into P_0/P and ρ_0/ρ relations:

$$\frac{P_0}{P} = \left[1 + \left(\frac{k-1}{2}\right) Ma^2\right]^{k/(k-1)}$$

$$\frac{\rho_0}{\rho} = \left[1 + \left(\frac{k-1}{2}\right) Ma^2\right]^{1/(k-1)}$$

- Numerical values of T_0/T, P_0/P and ρ_0/ρ can be compiled in Tables (e.g. for k=1.4)

- For *Ma* = 1, these ratios are called critical ratios

One-Dimensional Isentropic Flow

Property Relations for Isentropic Flow of Ideal Gases

TABLE 12–2

The critical-pressure, critical-temperature, and critical-density ratios for isentropic flow of some ideal gases

	Superheated steam, $k = 1.3$	Hot products of combustion, $k = 1.33$	Air, $k = 1.4$	Monatomic gases, $k = 1.667$
$\frac{P^*}{P_0}$	0.5457	0.5404	0.5283	0.4871
$\frac{T^*}{T_0}$	0.8696	0.8584	0.8333	0.7499
$\frac{\rho^*}{\rho_0}$	0.6276	0.6295	0.6340	0.6495

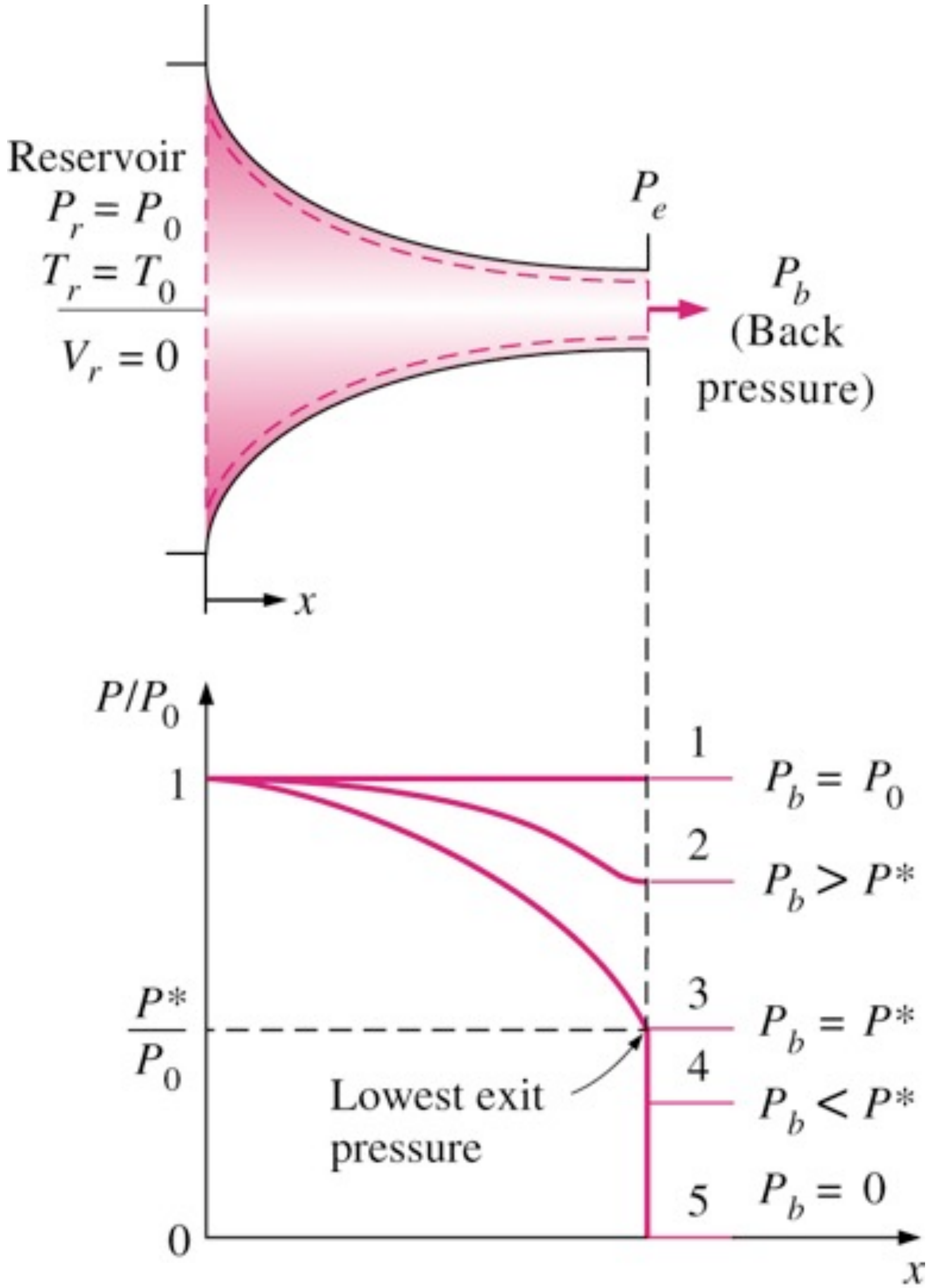
Isentropic Flow Through Nozzles

- Converging or converging-diverging nozzles are found in many engineering applications
  - Steam and gas turbines, aircraft and spacecraft propulsion, industrial blast nozzles, torch nozzles

- Here, we will study the effects of back pressure (pressure at discharge) on the exit velocity, mass flow rate, and pressure distribution along the nozzle

Isentropic Flow Through Nozzles

Converging Nozzles



- State 1: $P_b = P_0$, there is no flow, and pressure is constant.
- State 2: $P_b < P_0$, pressure along nozzle decreases.
- State 3: $P_b = P^*$, flow at exit is sonic, creating maximum flow rate called choked flow.
- State 4: $P_b < P^*$, there is no change in flow or pressure distribution in comparison to state 3
- State 5: $P_b = 0$, same as state 4.

Isentropic Flow Through Nozzles

Converging Nozzles

- Under steady flow conditions, mass flow rate is constant

$$\dot{m} = \rho A V = \left(\frac{P}{RT}\right) A \left(Ma\sqrt{kRT}\right) = P A Ma \sqrt{\frac{k}{RT}}$$

- Substituting T and P from the expressions on previous slides gives

$$\dot{m} = \frac{A\, Ma P_0 \sqrt{k/(RT_0)}}{[1 + (k-1)Ma^2/2]^{(k+1)/[2(k-1)]}}$$

  - Mass flow rate is a function of stagnation properties, flow area, and *Ma*

Isentropic Flow Through Nozzles

Converging Nozzles

- The maximum mass flow rate through a nozzle with a given throat area A^* is fixed by the P_0 and T_0 and occurs at $Ma = 1$

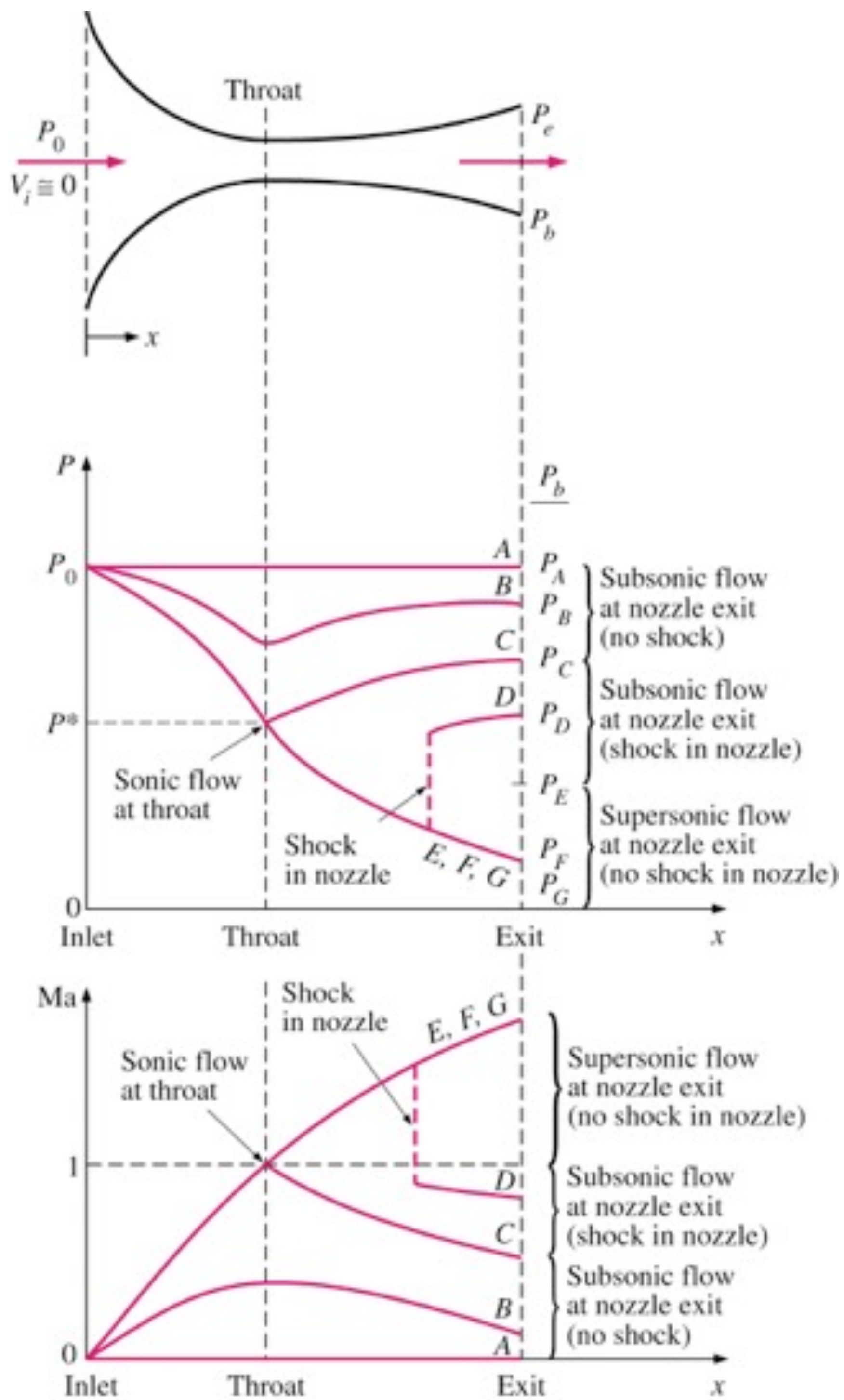
$$\dot{m} = A^* P_0 \sqrt{\frac{k}{RT_0}} \left(\frac{2}{k+1} \right)^{(k+1)/[2(k-1)]}$$

- This principle is important for chemical processes, medical devices, flow meters, and anywhere the mass flux of a gas must be known and controlled.

Isentropic Flow Through Nozzles
Converging-Diverging Nozzles

- The highest velocity in a converging nozzle is limited to the sonic velocity (*Ma* = 1), which occurs at the exit plane (throat) of the nozzle
- Accelerating a fluid to supersonic velocities (*Ma* > 1) requires a diverging flow section
  - Converging-diverging (C-D) nozzle
  - Standard equipment in supersonic aircraft and rocket propulsion
- Forcing fluid through a C-D nozzle does not guarantee supersonic velocity
  - Requires proper back pressure Pb

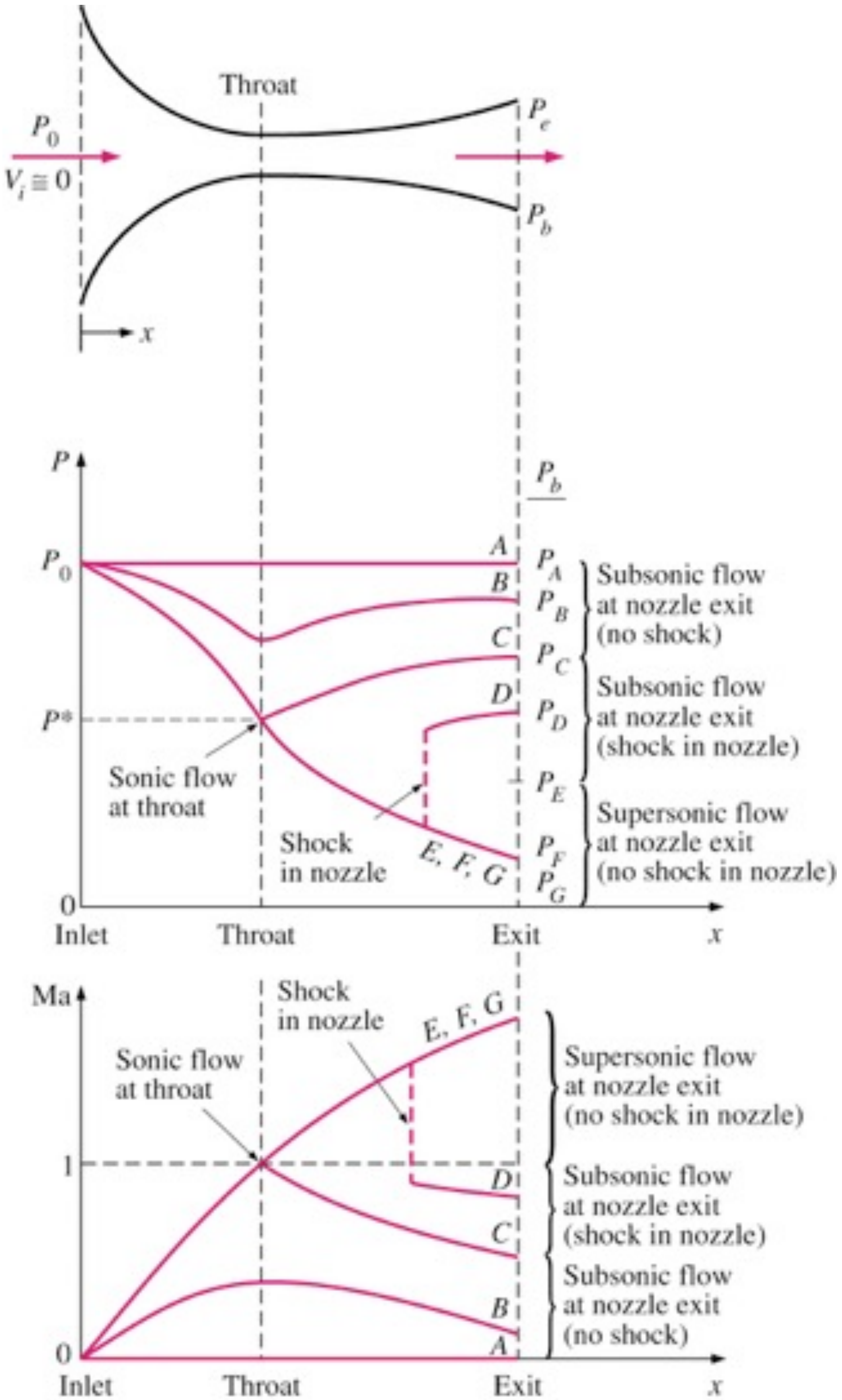
Isentropic Flow Through Nozzles
Converging-Diverging Nozzles



1. $P_0 > P_b > P_c$
   - Flow remains subsonic, and mass flow is less than for choked flow. Diverging section acts as diffuser
2. $P_b = P_C$
   - Sonic flow achieved at throat. Diverging section acts as diffuser. Subsonic flow at exit. Further decrease in Pb has no effect on flow in converging portion of nozzle

Isentropic Flow Through Nozzles

Converging-Diverging Nozzles



3. $P_C > P_b > P_E$

Fluid is accelerated to supersonic velocities in the diverging section as the pressure decreases. However, acceleration stops at location of normal shock. Fluid decelerates and is subsonic at outlet. As P_b is decreased, shock approaches nozzle exit.

4. $P_E > P_b > 0$

Flow in diverging section is supersonic with no shock forming in the nozzle. Without shock, flow in nozzle can be treated as isentropic.

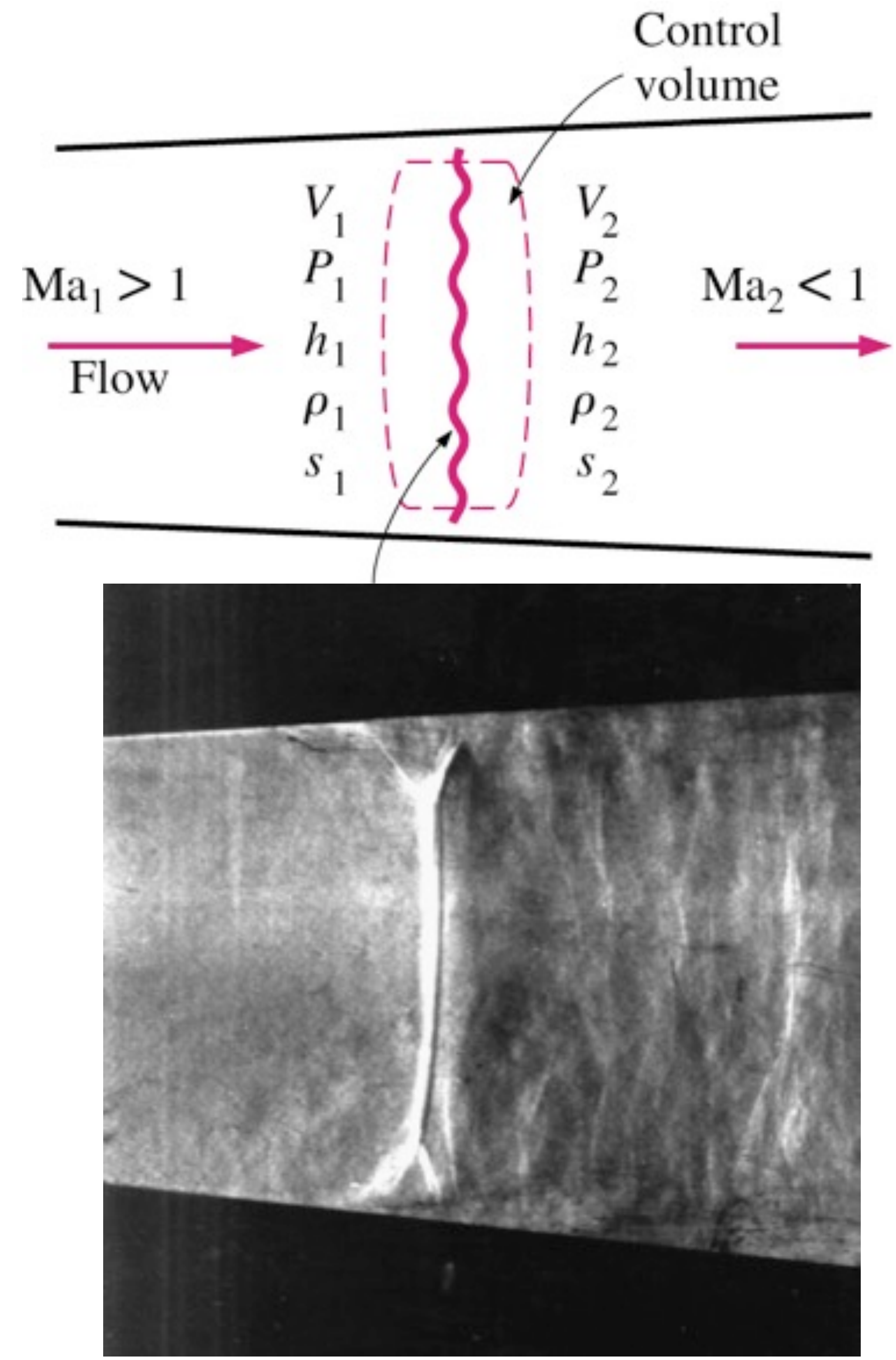
Shock Waves and Expansion Waves

- Review
  - Sound waves are created by small pressure disturbances and travel at the speed of sound
  - For some back pressures, abrupt changes in fluid properties occur in C-D nozzles, creating a shock wave
- Here, we will study the conditions under which shock waves develop and how they affect the flow.

Shock Waves and Expansion Waves

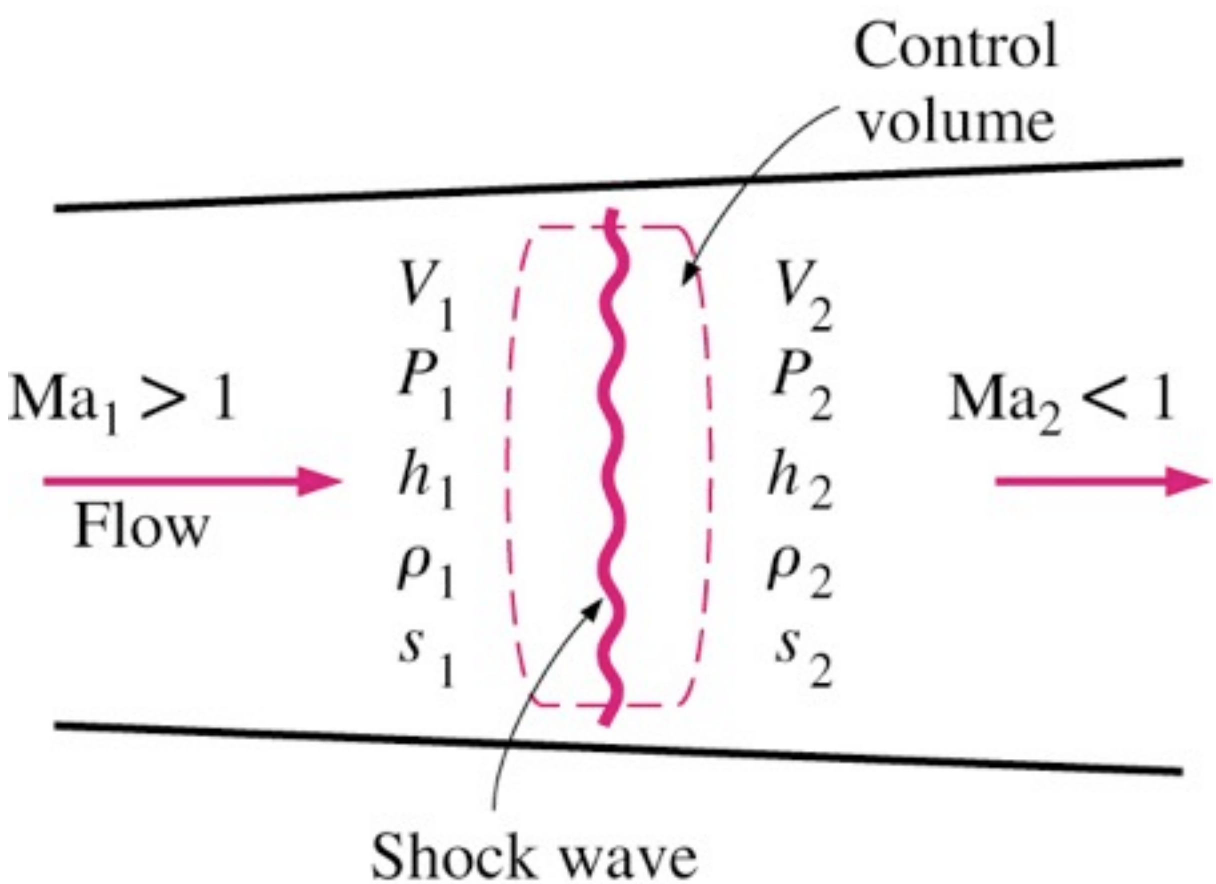
Normal Shocks

- Shocks which occur in a plane normal to the direction of flow are called normal shock waves
- Flow process through the shock wave is highly irreversible and cannot be approximated as being isentropic
- Develop relationships for flow properties before and after the shock using conservation of mass, momentum, and energy



Shock Waves and Expansion Waves

Normal Shocks



<u>Conservation of mass</u>

$$\rho_1 A V_1 = \rho_2 A V_2 \longrightarrow \rho_1 V_1 = \rho_2 V_2$$

<u>Conservation of energy</u>

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2} \longrightarrow h_{01} = h_{02}$$

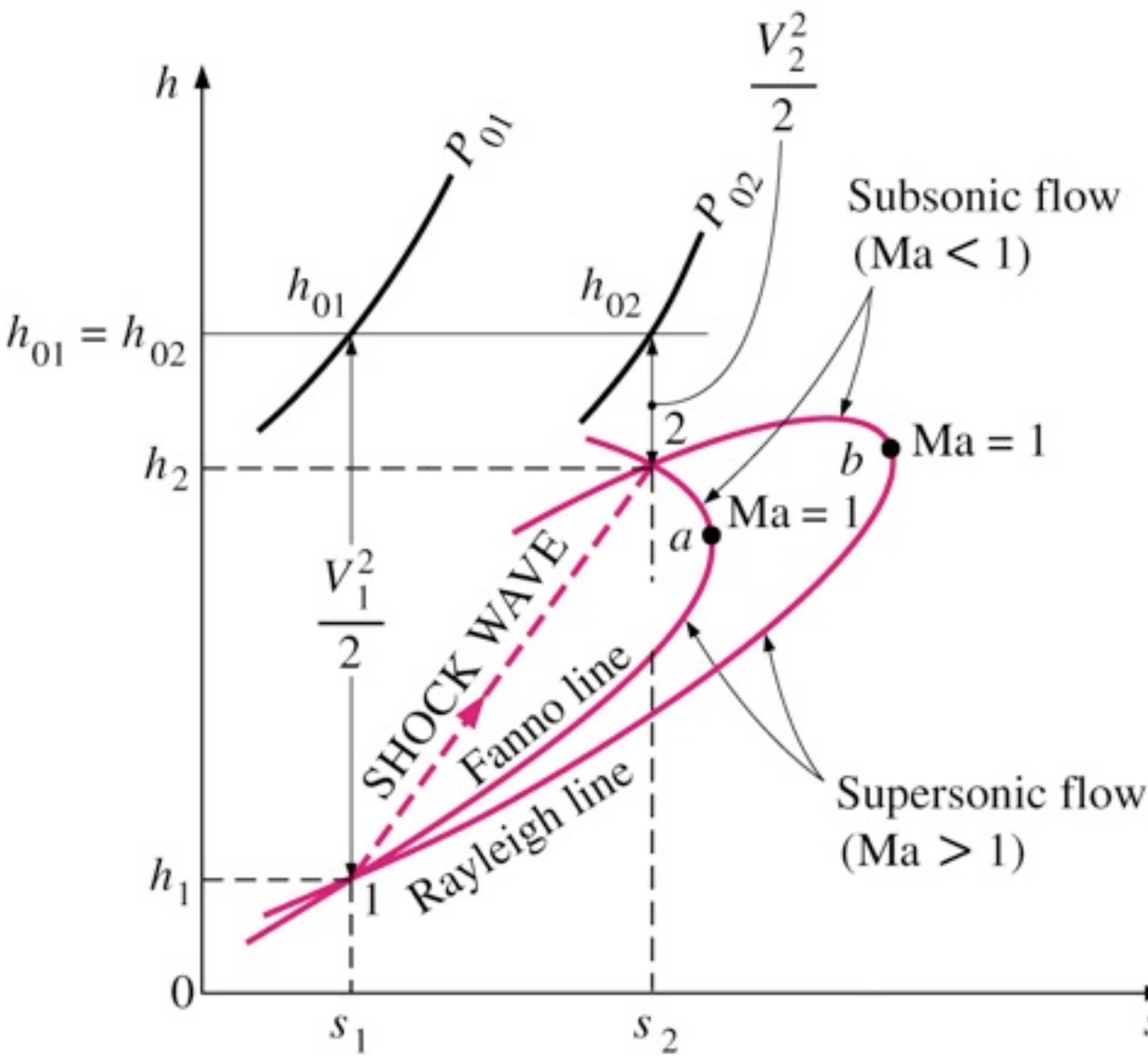
<u>Conservation of momentum</u>

$$A(P_1 - P_2) = \dot{m}(V_2 - V_1)$$

<u>Increase in entropy</u>

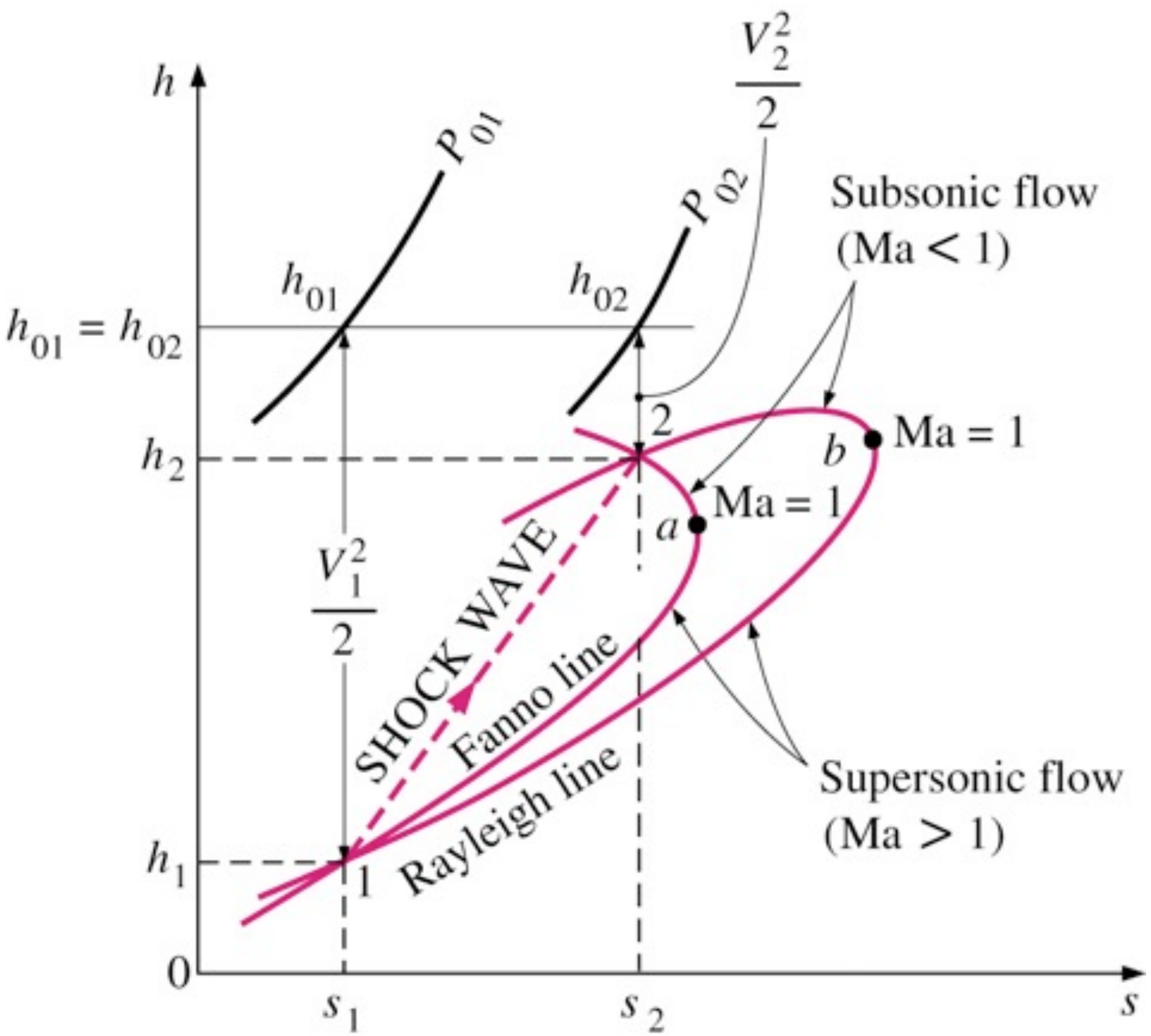
$$s_2 - s_1 \geq 0$$

Shock Waves and Expansion Waves
Normal Shocks



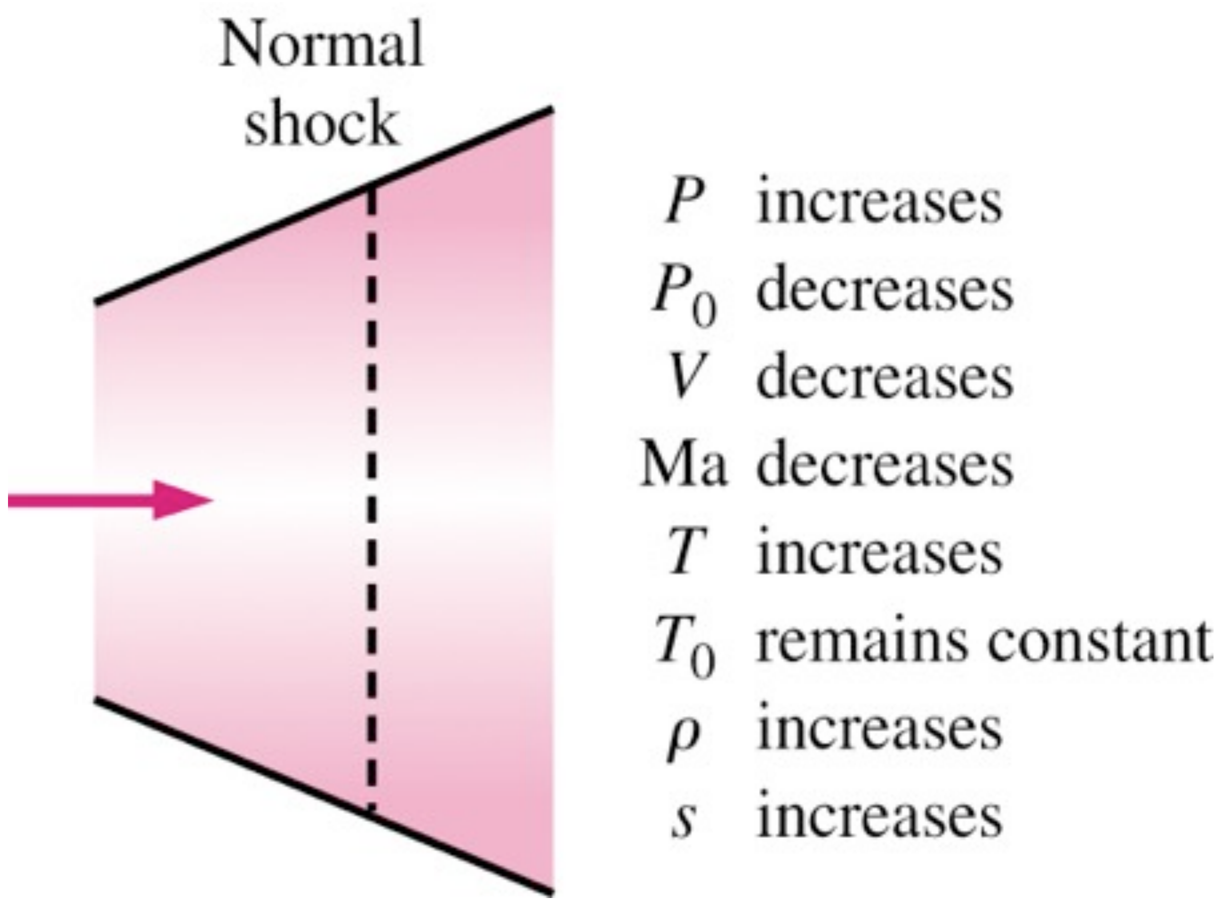
- Combine conservation of mass and energy into a single equation and plot on *h*-*s* diagram
  - Fanno Line: locus of states that have the same value of h_0 and mass flux
- Combine conservation of mass and momentum into a single equation and plot on *h*-*s* diagram
  - Rayleigh line
- Points of maximum entropy correspond to *Ma* = 1.
  - Above / below this point is subsonic / supersonic

Shock Waves and Expansion Waves
Normal Shocks



- There are 2 points where the Fanno and Rayleigh lines intersect : points where all 3 conservation equations are satisfied
  - Point 1: before the shock (supersonic)
  - Point 2: after the shock (subsonic)
- The larger *Ma* is before the shock, the stronger the shock will be.
- Entropy increases from point 1 to point 2 : expected since flow through the shock is adiabatic but irreversible

Shock Waves and Expansion Waves
Normal Shocks



- Equation for the Fanno line for an ideal gas with constant specific heats can be derived

$$\frac{P_2}{P_1} = \frac{Ma_1\sqrt{1+Ma_1^2(k-1)/2}}{Ma_2\sqrt{1+Ma_2^2(k-1)/2}}$$

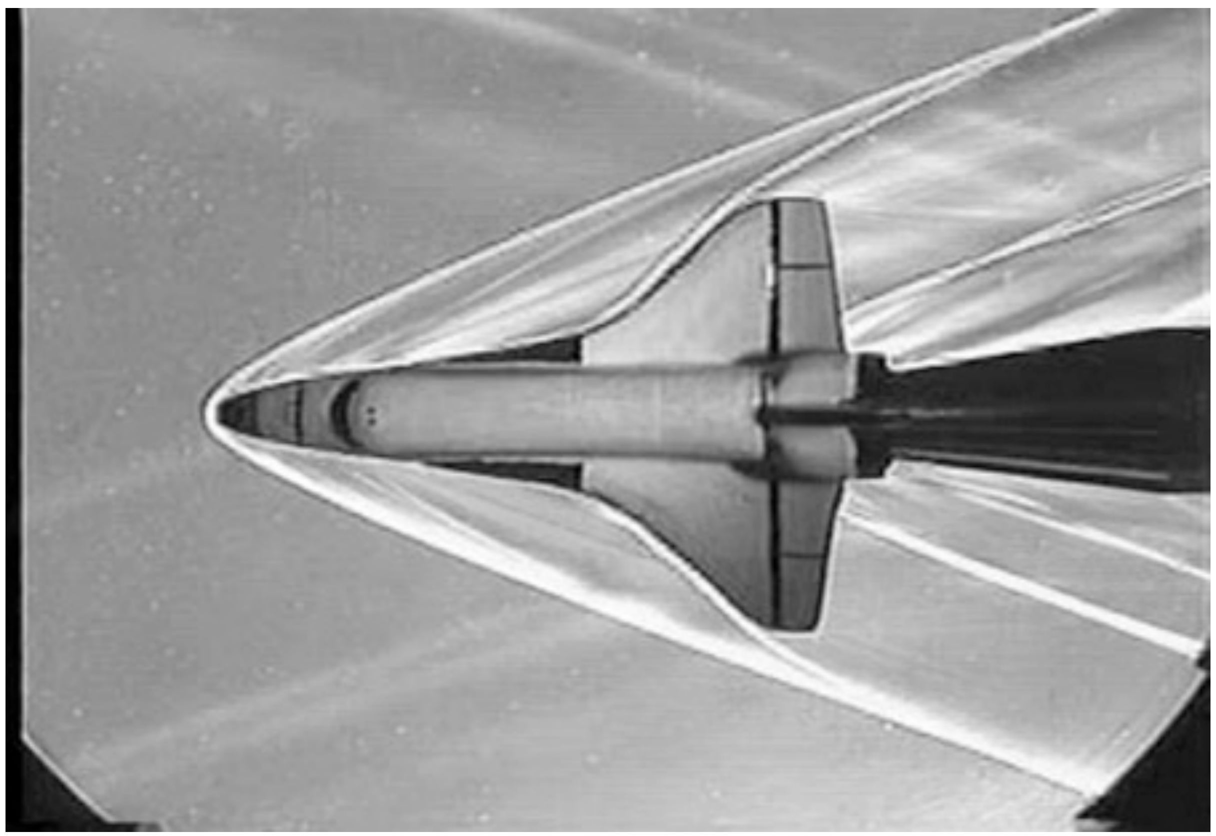
- Similar relation for Rayleigh line is

$$\frac{P_2}{P_1} = \frac{1+kMa_1^2}{1+kMa_2^2}$$

- Combining this gives the intersection points

$$Ma_2^2 = \frac{Ma_1^2 + 2/(k-1)}{2Ma_1^2 k/(k-1) - 1}$$

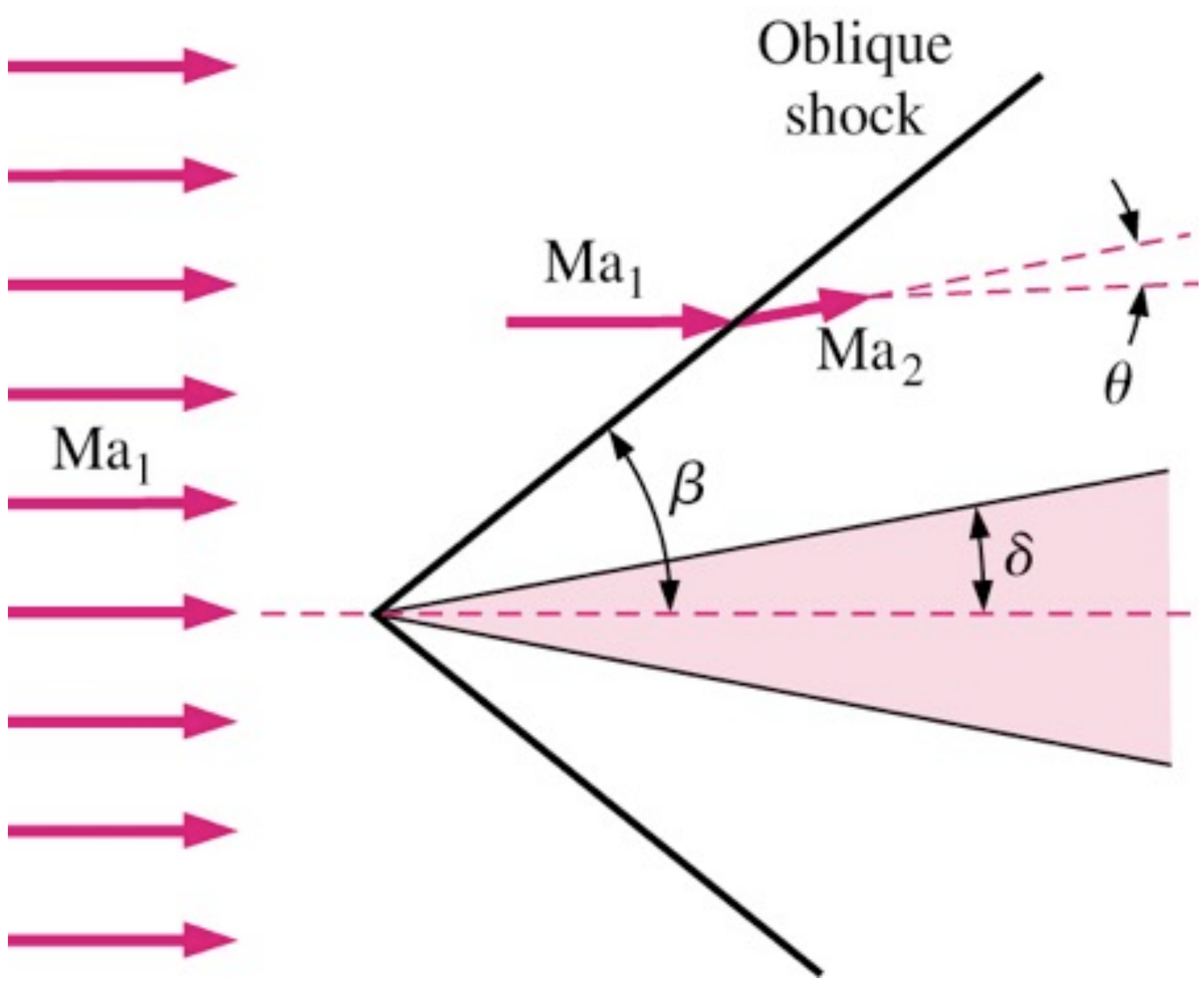
Shock Waves and Expansion Waves
Oblique Shocks



- Not all shocks are normal to flow direction.
- Some are inclined to the flow direction, and are called oblique shocks

Shock Waves and Expansion Waves

Oblique Shocks



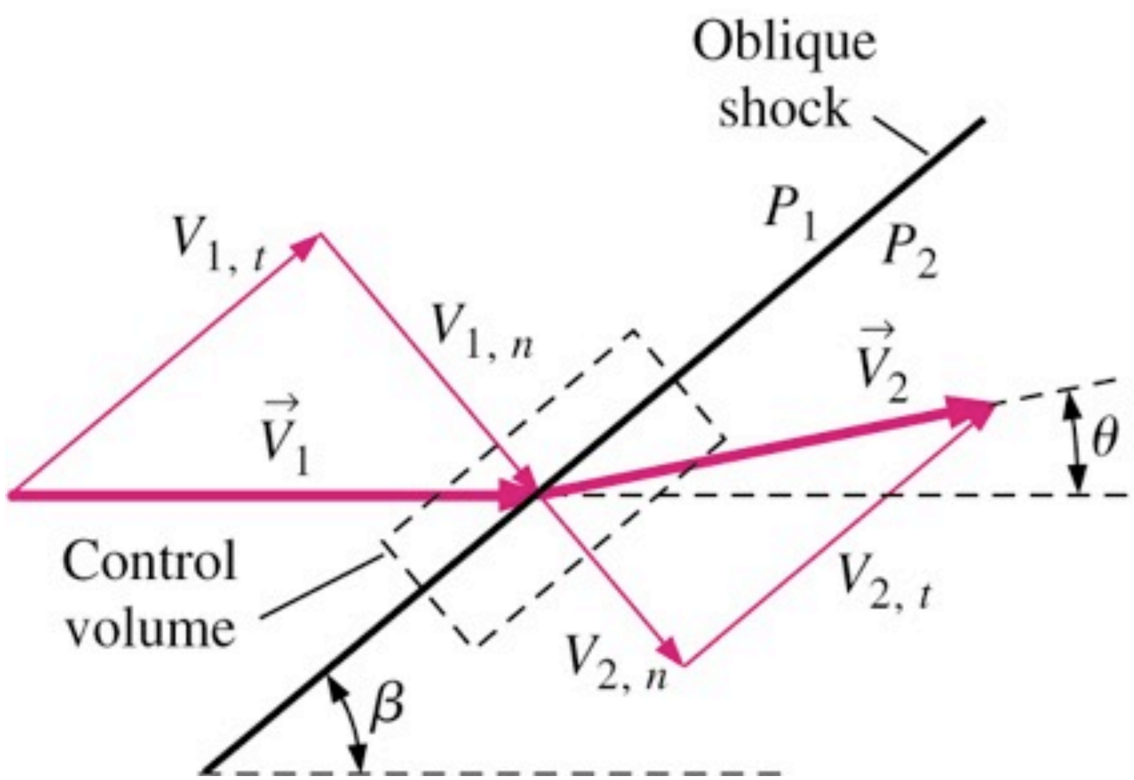
- At leading edge, flow is deflected through an angle θ called the turning angle
- Result is a straight oblique shock wave aligned at shock angle β relative to the flow direction
- Due to the displacement thickness, θ is slightly greater than the wedge half-angle δ.

Shock Waves and Expansion Waves

Oblique Shocks

- Like normal shocks, *Ma* decreases across the oblique shock, and are only possible if upstream flow is supersonic
- However, unlike normal shocks in which the downstream *Ma* is always subsonic, Ma_2 of an oblique shock can be subsonic, sonic, or supersonic depending upon Ma_1 and θ.

Shock Waves and Expansion Waves
Oblique Shocks



θ–β-Ma relationship

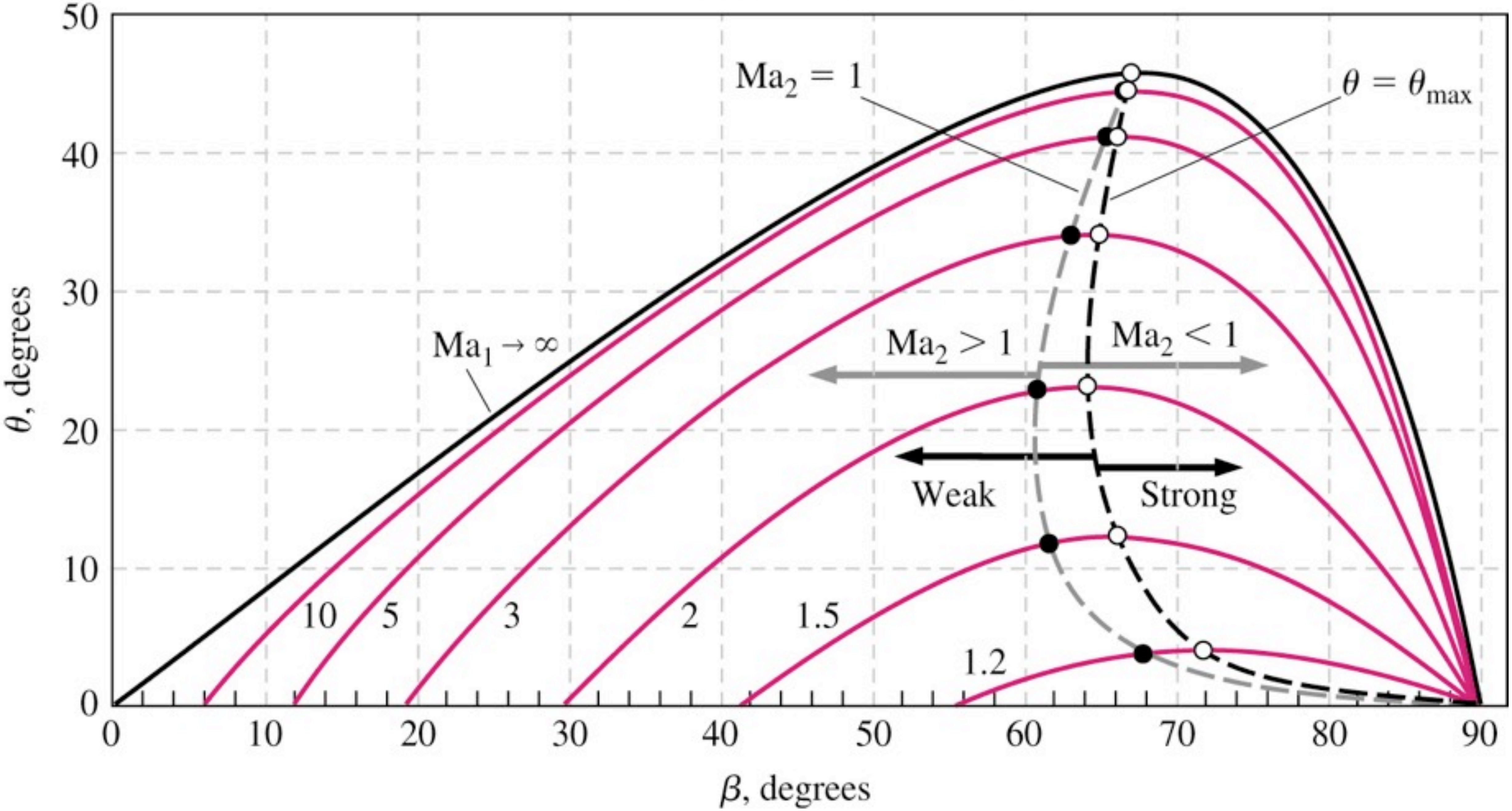
$$\tan\theta = \frac{2\cot\beta(Ma_1^2\sin^2\beta - 1)}{Ma_1^2(k + \cos 2\beta) + 2}$$

- All equations and shock tables for normal shocks apply to oblique shocks as well, provided that we use only the normal components of the Mach number
  - $Ma_{1,n} = V_{1,n}/c_1$
  - $Ma_{2,n} = V_{2,n}/c_2$

Shock Waves and Expansion Waves

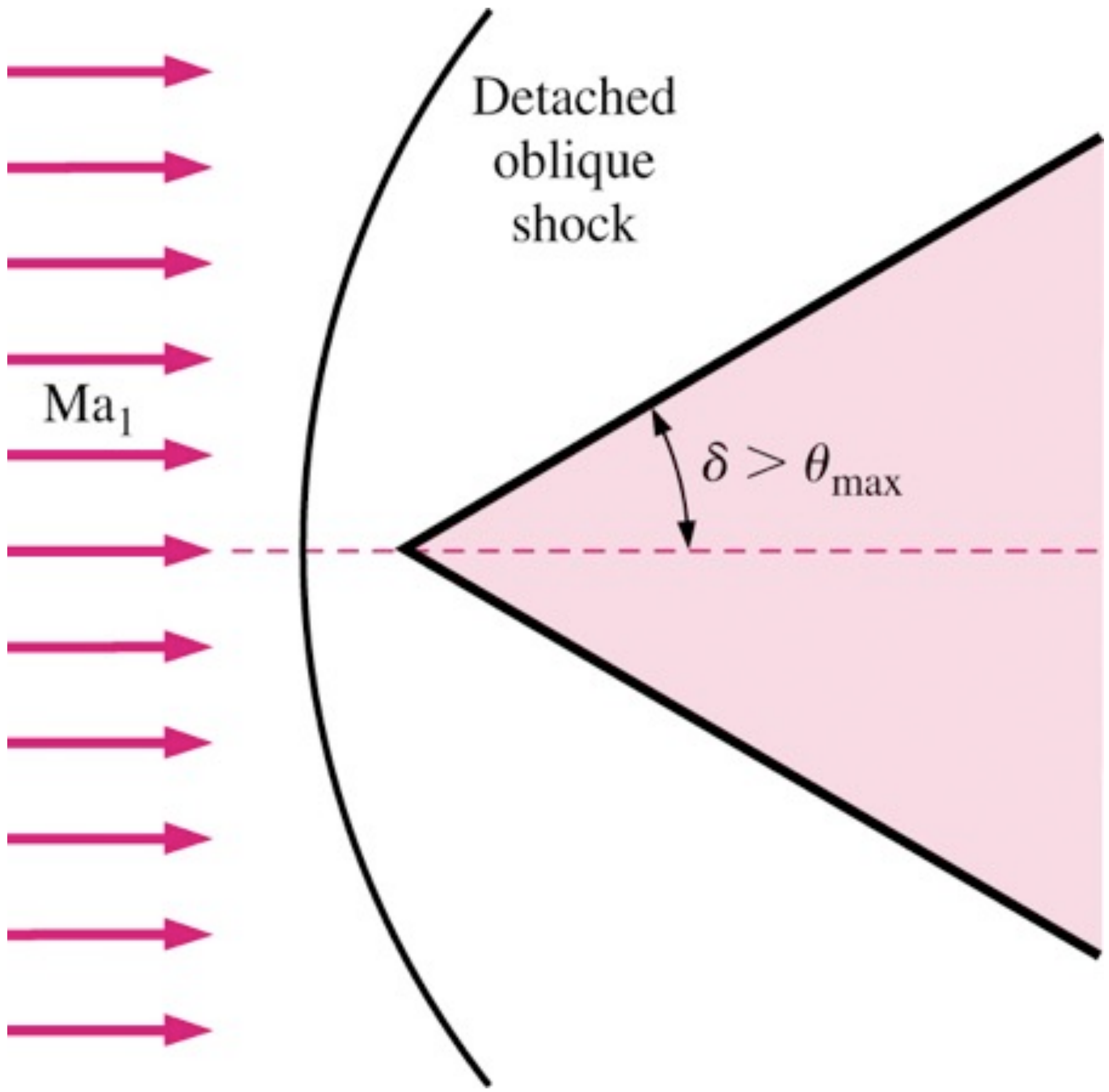
Oblique Shocks

$$\tan\theta = \frac{2\cot\beta(Ma_1^2\sin^2\beta - 1)}{Ma_1^2(k+\cos 2\beta)+2}$$



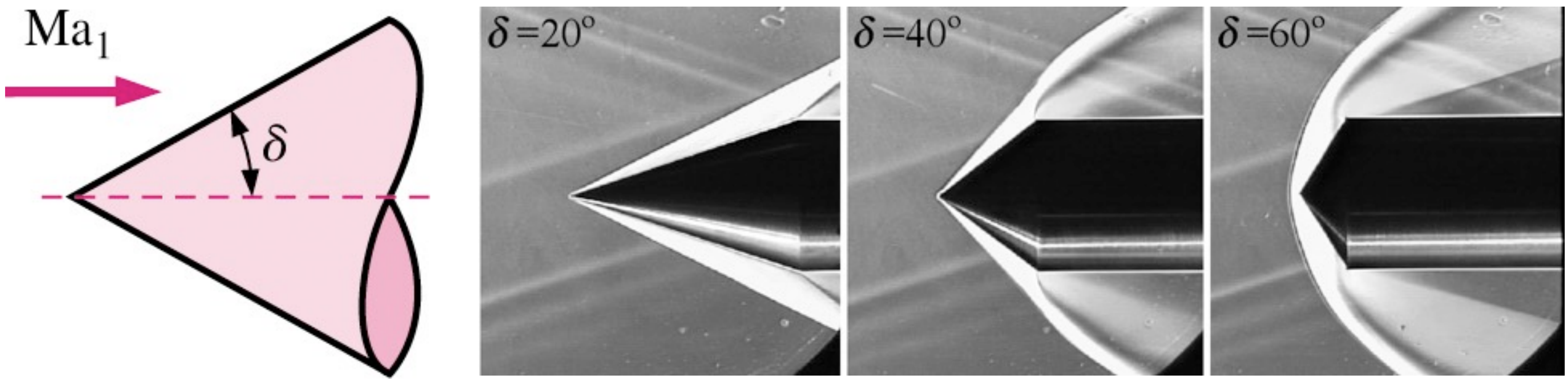
Shock Waves and Expansion Waves

Oblique Shocks



- If wedge half angle $\delta > \theta_{max}$, a detached oblique shock or bow wave is formed
- Much more complicated that straight oblique shocks.
- Requires CFD for analysis.

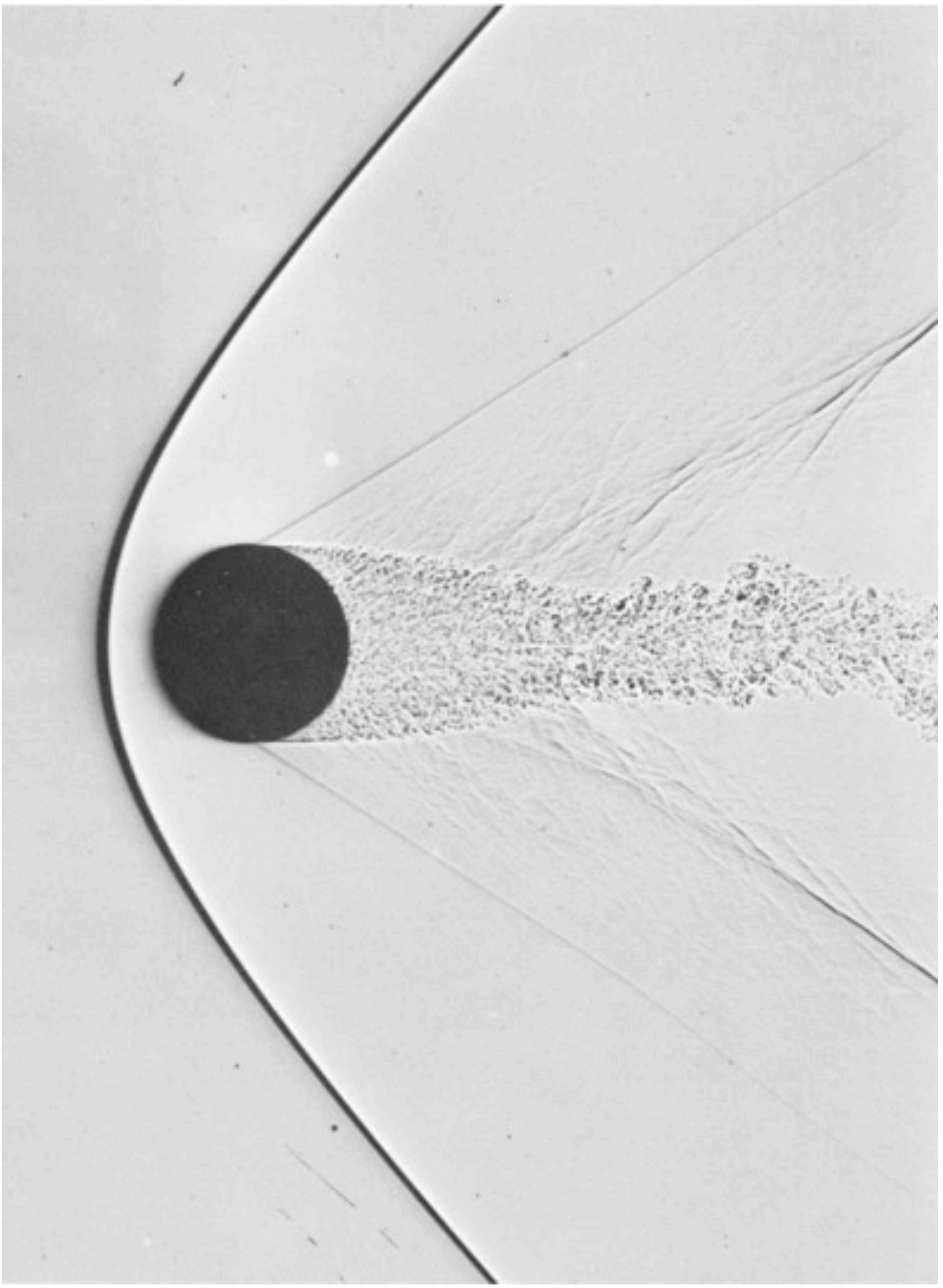
Shock Waves and Expansion Waves
Oblique Shocks



- Similar shock waves see for axisymmetric bodies, however, θ–β-*Ma* relationship and resulting diagram is different than for 2D bodies

Shock Waves and Expansion Waves

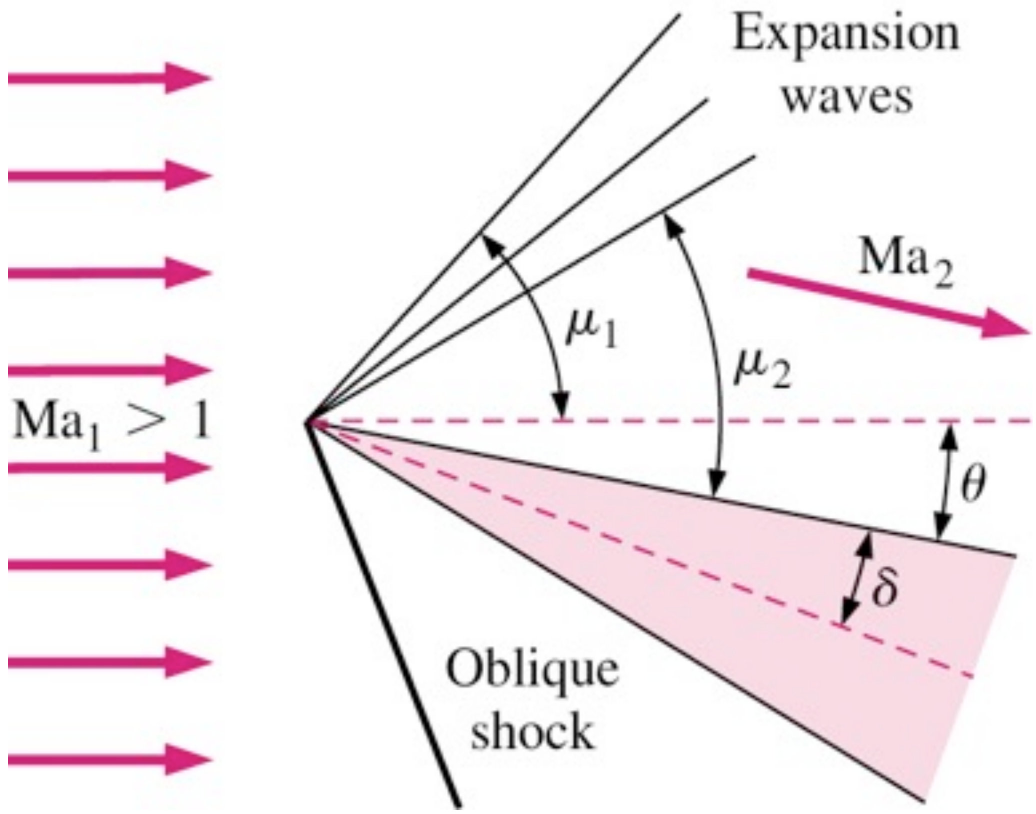
Oblique Shocks



- For blunt bodies, without a sharply pointed nose, $\delta = 90°$, and an attached oblique shock *cannot* exist regardless of *Ma*.

Shock Waves and Expansion Waves

Prandtl-Meyer Expansion Waves



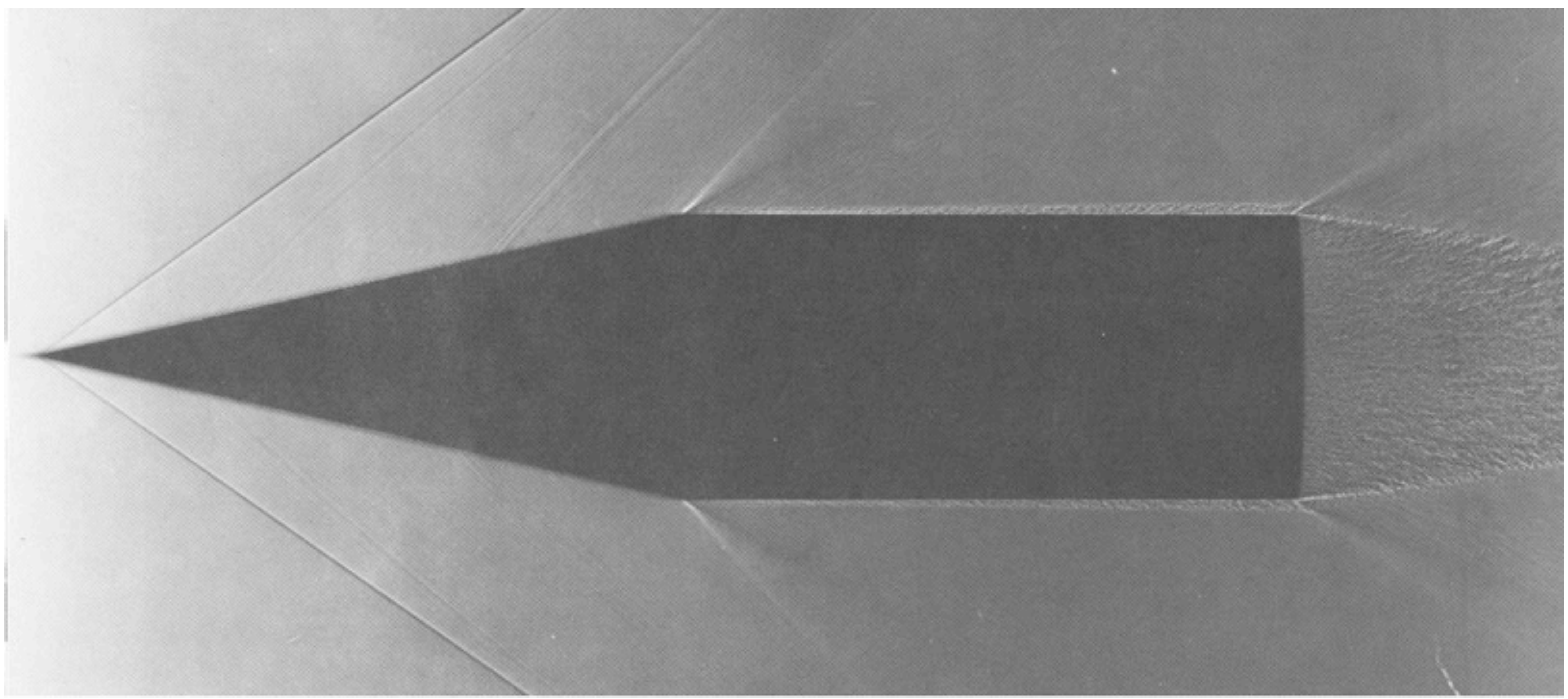
Flow turns *gradually* as each successful Mach wave turns the flow ay an infinitesimal amount

- In some cases, flow is turned in the opposite direction across the shock
- Example: wedge at angle of attack θ greater than wedge half angle δ
- This type of flow is called an expanding flow, in contrast to the oblique shock which creates a compressing flow
- Instead of a shock, a expansion fan appears, which is comprised of infinite number of Mach waves called Prandtl-Meyer expansion waves
- Each individual expansion wave is isentropic : flow across entire expansion fan is isentropic
- $Ma_2 > Ma_1$
- P, ρ, T *decrease* across the fan

Shock Waves and Expansion Waves

Prandtl-Meyer Expansion Waves

- Prandtl-Meyer expansion fans also occur in axisymmetric flows, as in the corners and trailing edges of the cone cylinder.



Shock Waves and Expansion Waves

Prandtl-Meyer Expansion Waves

Interaction between shock waves and expansions waves in "over expanded" supersonic jet