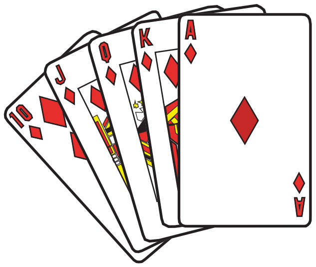
CompSci 230
Discrete Math for Computer Science
Counting II



November 7, 2013

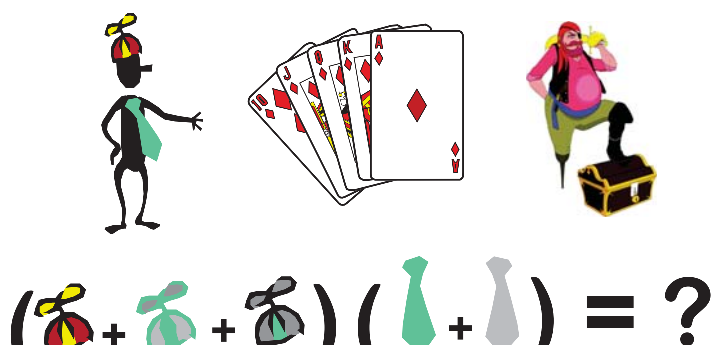
Prof. Rodger

Lecture adapted from Bruce Maggs/Lecture developed at Carnegie Mellon, primarily by Prof. Steven Rudich.

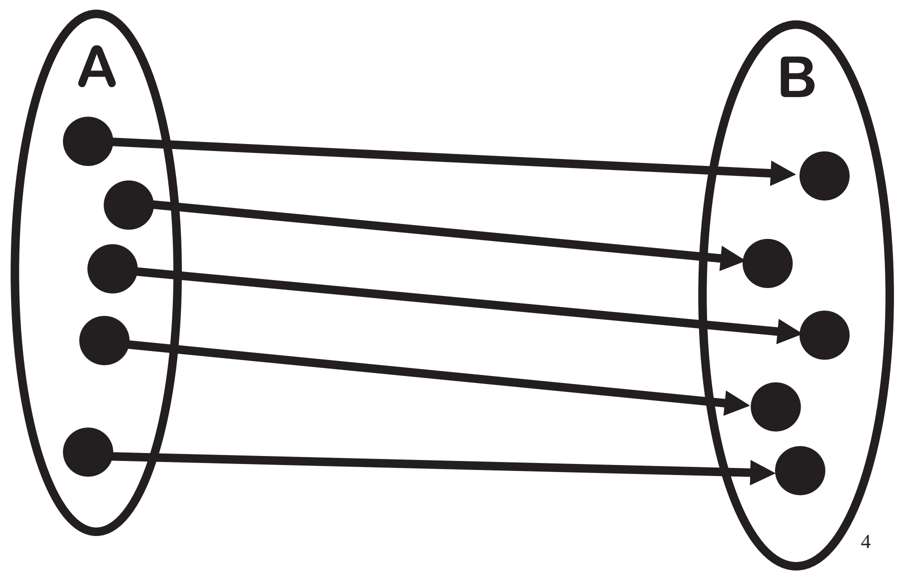
Announcements

- Read for next time Chap. 6.5-6.6
- Homework 6 due Tuesday
- Recitation this week – bring laptop

Counting II: Recurring Problems and Correspondences



1-1 onto Correspondence (just "correspondence" for short)



Correspondence Principle

If two finite sets can be placed into 1-1 onto correspondence, then they have the same size

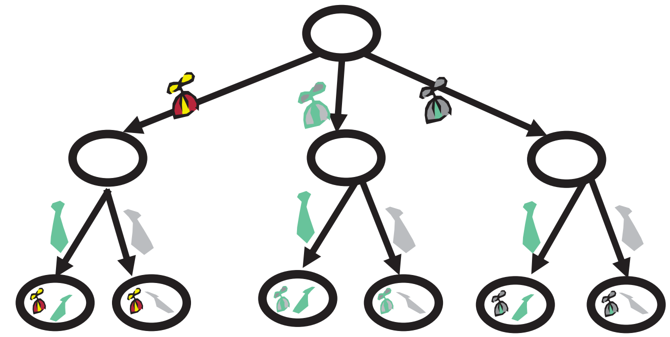
If a finite set A has a k-to-1 correspondence to finite set B, then |B| = |A|/k

The number of subsets of an n-element set is 2^n.

Sometimes it is easiest to count the number of objects with property Q, by counting the number of objects that do not have property Q.

The number of subsets of size r that can be formed from an n-element set is:

$$\frac{n!}{r!(n-r)!} = \binom{n}{r}$$



A choice tree provides a "choice tree representation" of a set S, if

1. Each leaf label is in S, and each element of S is some leaf label
2. No two leaf labels are the same

Product Rule (Rephrased)

Suppose every object of a set S can be constructed by a sequence of choices with P_1 possibilities for the first choice, P_2 for the second, and so on.

IF 1. Each sequence of choices constructs an object of type S

AND

2. No two different sequences create the same object

THEN

There are $P_1 P_2 P_3 \ldots P_n$ objects of type S

How Many Different Orderings of Deck With 52 Cards?

What object are we making? Ordering of a deck

Construct an ordering of a deck by a sequence of 52 choices:

52 possible choices for the first card;

51 possible choices for the second card;

: :

1 possible choice for the 52^{nd} card.

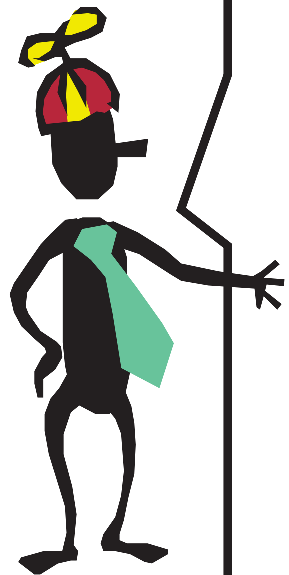
By product rule: $52 \times 51 \times 50 \times \ldots \times 2 \times 1 = 52!$

The Sleuth's Criterion

There should be a unique way to create an object in S.

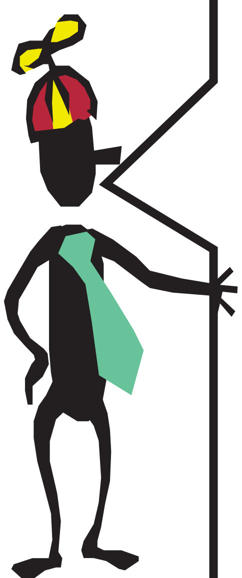
In other words:

For any object in S, it should be possible to reconstruct the (unique) sequence of choices which lead to it.



The three big mistakes people make in associating a choice tree with a set S are:

1. Creating objects not in S
2. Leaving out some objects from the set S
3. Creating the same object two different ways



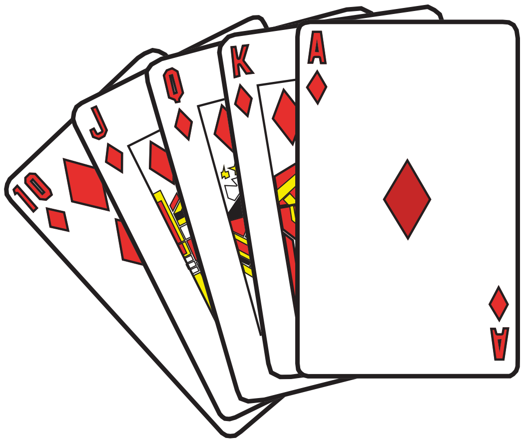
DEFENSIVE THINKING ask yourself:

Am I creating objects of the right type?

Can I reverse engineer my choice sequence from any given object?

Let's use our principles to extend our reasoning to different types of objects

Counting Poker Hands



52 Card Deck, 5 card hands

4 possible suits:
♥ ♦ ♣ ♠

13 possible ranks:
2,3,4,5,6,7,8,9,10,J,Q,K,A

Pair: set of two cards of the same rank
Straight: 5 cards of consecutive rank
Flush: set of 5 cards with the same suit

Ranked Poker Hands

Straight Flush: a straight and a flush

4 of a kind: 4 cards of the same rank

Full House: 3 of one rank and 2 of another

Flush: a flush, but not a straight

Straight: a straight, but not a flush

3 of a kind: 3 of the same rank, but not a full house or 4 of a kind

2 Pair: 2 pairs, but not 4 of a kind or a full house

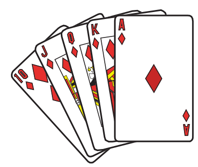
A Pair

Straight Flush
Choices for rank? Possible suits?

4 of a Kind
Choices of rank? Other choices?

Flush
Choices of suit? Choices of cards?

Straight
Choices of lowest card? Suits?



Storing Poker Hands:
How many bits per hand?

I want to store a 5-card poker hand using the smallest number of bits (space efficient)

Order the 2,598,560 Poker Hands Lexicographically (or in any fixed way)

To store a hand all I need is to store its index, which requires $\lceil \log_2(2{,}598{,}560) \rceil = 22$ bits

Hand 0000000000000000000000
Hand 0000000000000000000001
Hand 0000000000000000000010
.
.
.

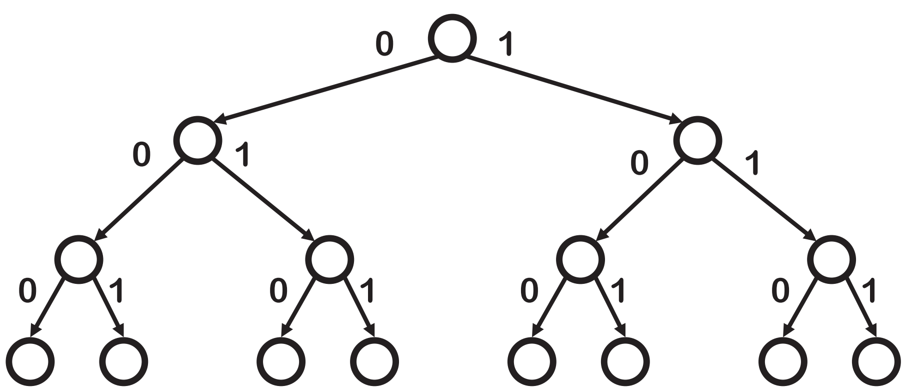
22 Bits is OPTIMAL

$2^{21} = 2{,}097{,}152 < 2{,}598{,}560$

Thus there are more poker hands than there are 21-bit strings

Hence, you can't have a 21-bit string for each hand

Binary (Boolean) Choice Tree



A binary (Boolean) choice tree is a choice tree where each internal node has degree 2

Usually the choices are labeled 0 and 1

22 Bits is OPTIMAL

$2^{21} = 2{,}097{,}152 < 2{,}598{,}560$

A binary choice tree of depth 21 can have at most 2^{21} leaves.

Hence, there are not enough leaves for all 5-card hands.

An n-element set can be stored so that each element uses $\lceil \log_2(n) \rceil$ bits

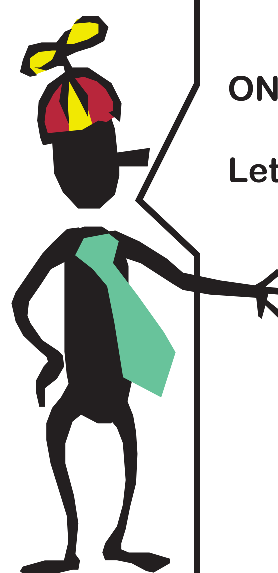
Furthermore, any representation of the set will have some string of at least that length

Information Counting Principle:

If each element of a set can be represented using k bits, the size of the set is bounded by 2^k

Information Counting Principle:

Let S be a set represented by a depth-k binary choice tree, the size of the set is bounded by 2^k

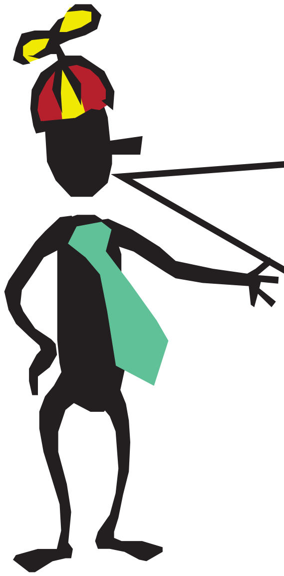


ONGOING MEDITATION:

Let S be any set and T be a binary choice tree representation of S

Think of each element of S being encoded by binary sequences of choices that lead to its leaf

We can also start with a binary encoding of a set and make a corresponding binary choice tree



Now, for something completely different…

How many ways to rearrange the letters in the word "SYSTEMS"?

SYSTEMS

7 places to put the Y,
6 places to put the T,
5 places to put the E,
4 places to put the M,
and the S's are forced

SYSTEMS

Let's pretend that the S's are distinct:

$S_1 Y S_2 TEMS_3$

Arrange n symbols: r_1 of type 1, r_2 of type 2, …, r_k of type k

$$\binom{n}{r_1}\binom{n-r_1}{r_2}\cdots\binom{n - r_1 - r_2 - \dots - r_{k-1}}{r_k}$$

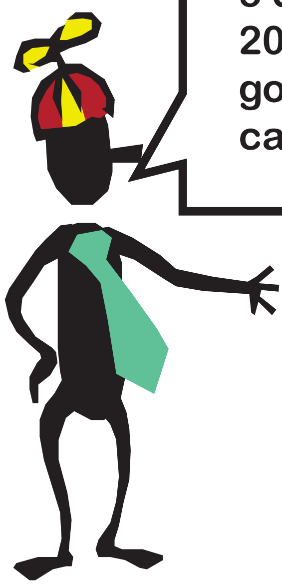
$$= \frac{n!}{(n-r_1)!r_1!}\ \frac{(n-r_1)!}{(n-r_1-r_2)!r_2!}\ \cdots$$

$$= \frac{n!}{r_1!r_2!\ \dots\ r_k!}$$

DUKEBLUEDEVILS

Remember:
The number of ways to arrange n symbols with r_1 of type 1, r_2 of type 2, …, r_k of type k is:

$$\frac{n!}{r_1!r_2! \ldots r_k!}$$



5 distinct pirates want to divide 20 identical, indivisible bars of gold. How many different ways can they divide up the loot?

Sequences with 20 G's and 4 /'s

GG/G//GGGGGGGGGGGGGGGGG/

represents the following division among the pirates: 2, 1, 0, 17, 0

In general, the ith pirate gets the number of G's after the i-1st / and before the ith /

This gives a correspondence between divisions of the gold and sequences with 20 G's and 4 /'s

How many different ways to divide up the loot?

Sequences with 20 G's and 4 /'s

How many different ways can n distinct pirates divide k identical, indivisible bars of gold?

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

How many integer solutions to the following equations?

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Think of x_k are being the number of gold bars that are allotted to pirate k

$$\binom{24}{4}$$

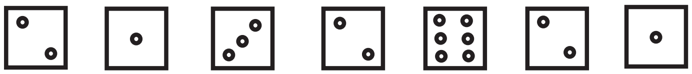
How many integer solutions to the following equations?

$$x_1 + x_2 + x_3 + \ldots + x_n = k$$

$$x_1, x_2, x_3, \ldots, x_n \geq 0$$

Identical/Distinct Dice

Suppose that we roll seven dice



How many different outcomes are there, if order matters?

What if order doesn't matter?
(E.g., Yahtzee!)

How did we get that last one, when order doesn't matter?

Multisets

A multiset is a set of elements, each of which has a multiplicity

The size of the multiset is the sum of the multiplicities of all the elements

Example:

Counting Multisets

The number of ways to choose a multiset of size k from n types of elements is:

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

Back to the Pirates

How many ways are there of choosing 20 pirates from a set of 5 pirates, with repetitions allowed?

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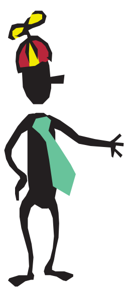
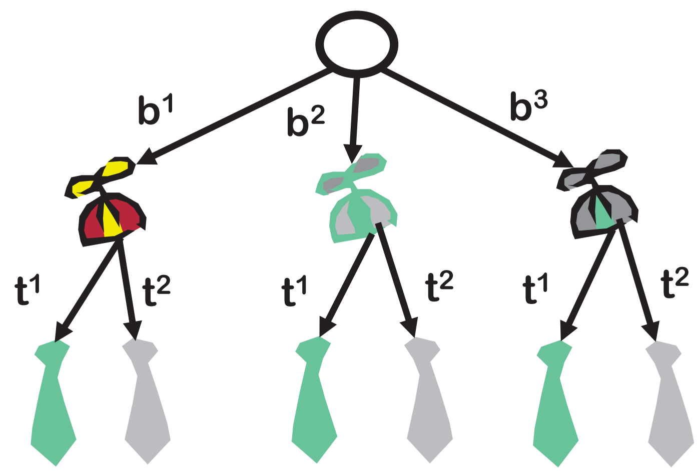
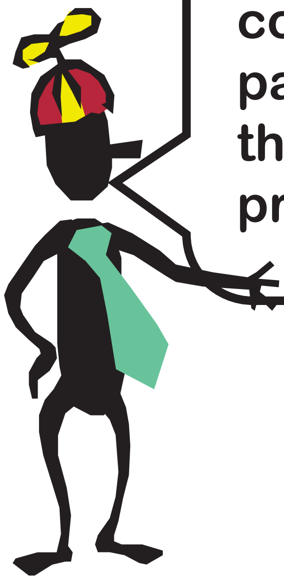


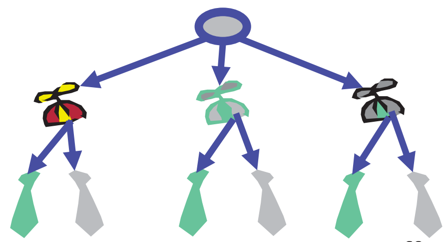
Polynomials Express Choices and Outcomes

Products of Sum = Sums of Products

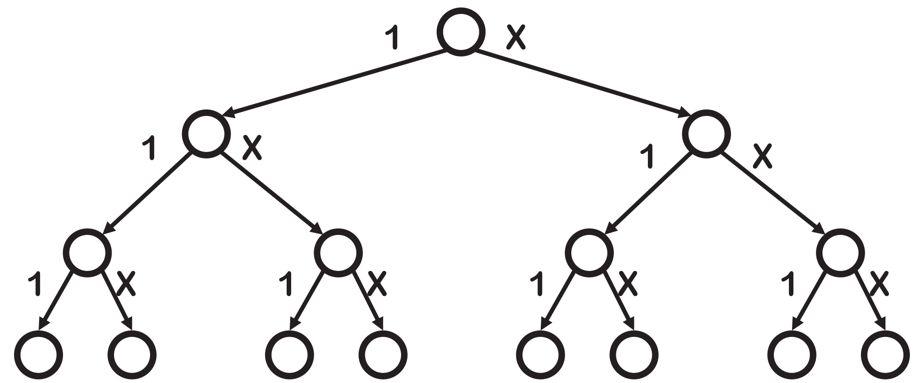




There is a correspondence between paths in a choice tree and the cross terms of the product of polynomials!



Choice Tree for Terms of $(1+X)^3$



Combine like terms to get:

What is a Closed Form Expression For c_k?

$$(1+X)^n = c_0 + c_1X + c_2X^2 + \ldots + c_nX^n$$

$$(1+X)(1+X)(1+X)(1+X)\ldots(1+X)$$

After multiplying things out, but before combining like terms, we get 2^n cross terms, each corresponding to a path in the choice tree

c_k, the coefficient of X^k, is the number of paths with exactly k X's

The Binomial Formula

$$(1+X)^n = \binom{n}{0}X^0 + \binom{n}{1}X^1 + \ldots + \binom{n}{n}X^n$$

Binomial Coefficients

binomial expression

The Binomial Formula

$(1+X)^0 = 1$

$(1+X)^1 = 1 + 1X$

$(1+X)^2 = 1 + 2X + 1X^2$

$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$

Coefficients?

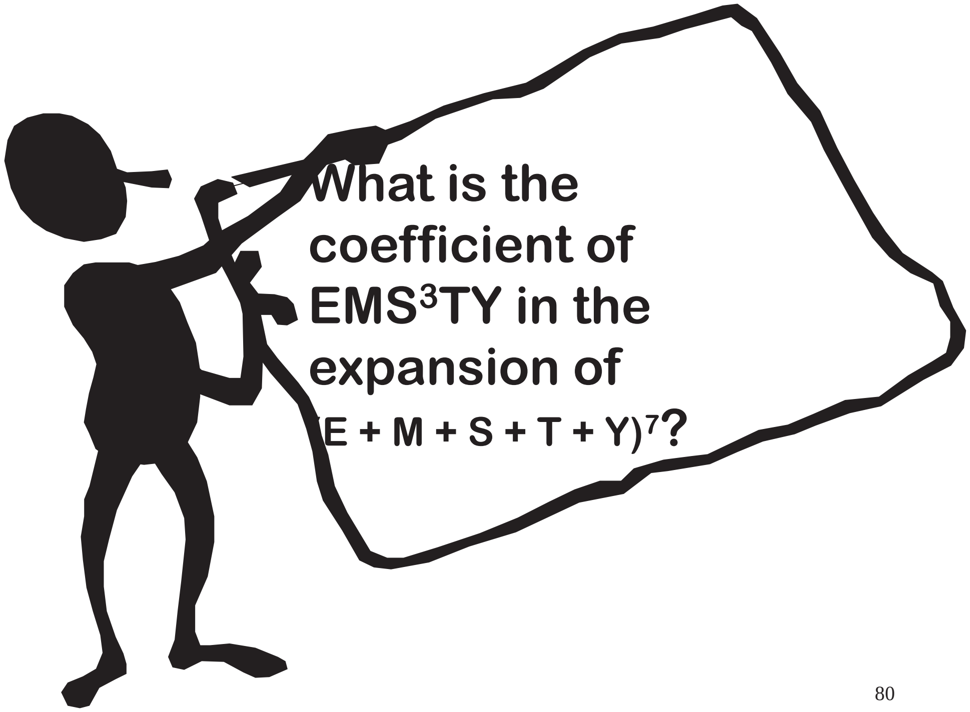
The Binomial Formula

$$(X+Y)^n = \binom{n}{0} X^n Y^0 + \binom{n}{1} X^{n-1} Y^1 + \ldots + \binom{n}{k} X^{n-k} Y^k + \ldots + \binom{n}{n} X^0 Y^n$$

The Binomial Formula

$$(X+Y)^n = \sum_{k=0}^{n} \binom{n}{k} X^{n-k} Y^k$$

What is the coefficient of EMSTY in the expansion of $(E + M + S + T + Y)^5$?


What is the coefficient of EMS^3TY in the expansion of $(E + M + S + T + Y)^7$?

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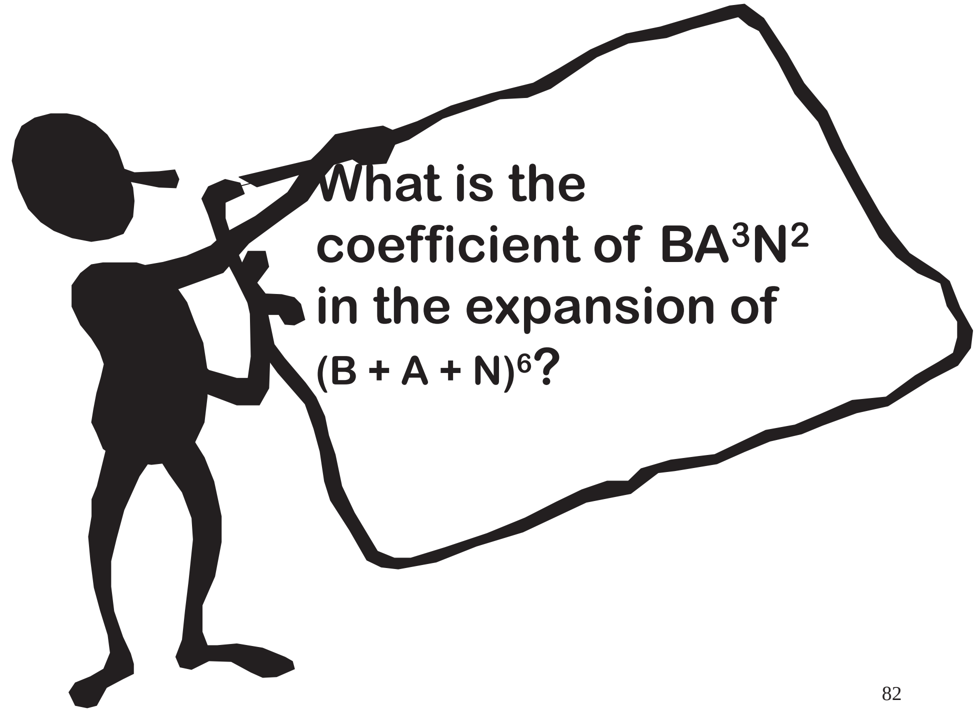
What is the coefficient of BA^3N^2 in the expansion of $(B + A + N)^6$?

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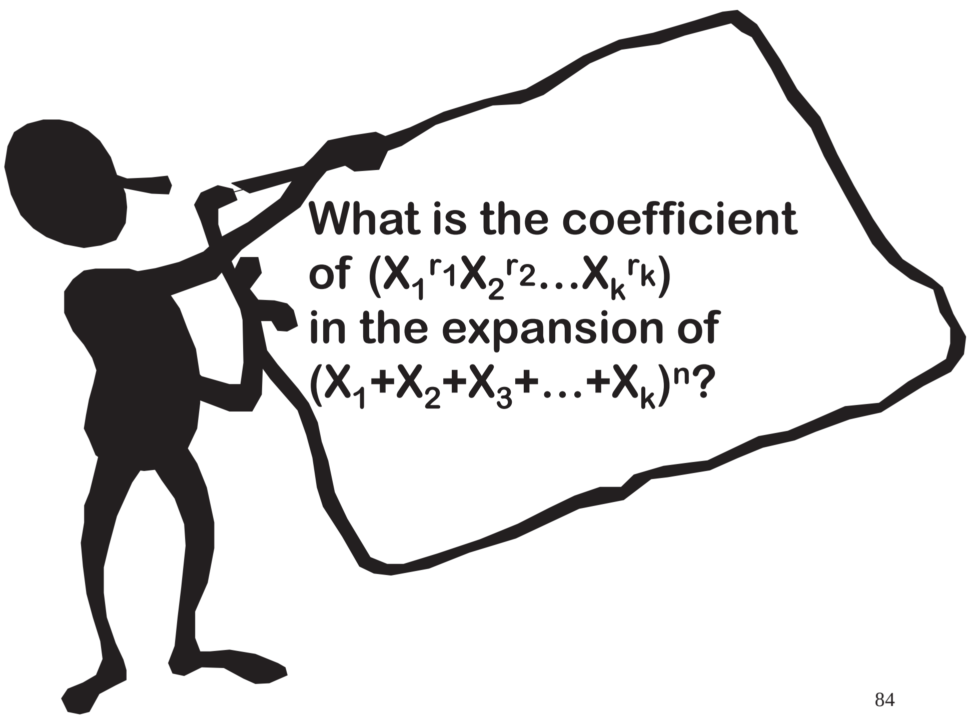
What is the coefficient of $(X_1^{r_1}X_2^{r_2}\ldots X_k^{r_k})$ in the expansion of $(X_1+X_2+X_3+\ldots+X_k)^n$?

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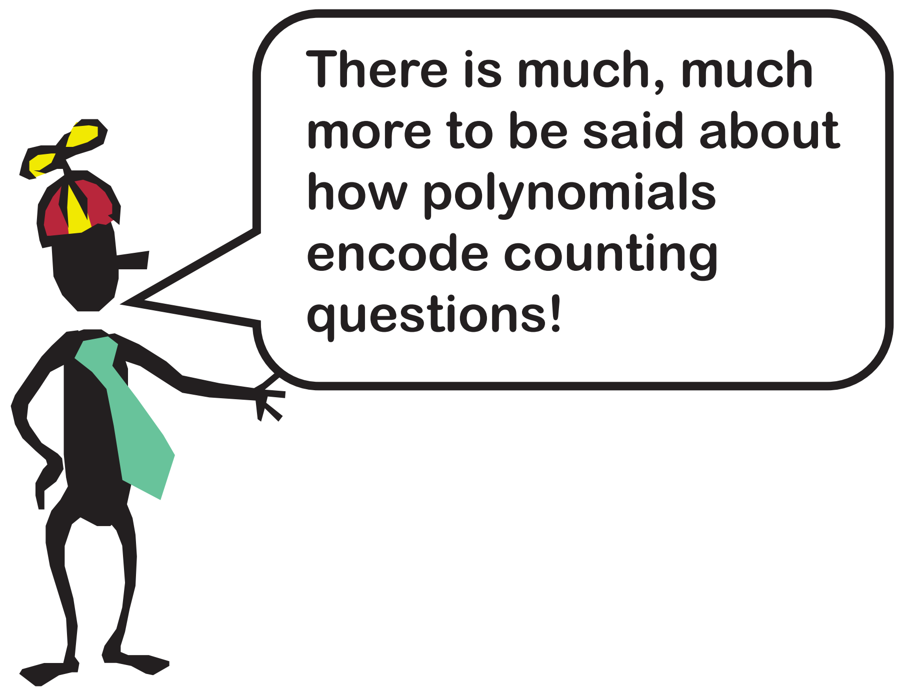
There is much, much more to be said about how polynomials encode counting questions!

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