Concurrency and directed algebraic topology

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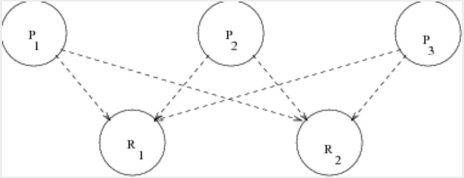
- In computer science, concurrency is a property of systems in which several computations are executing simultaneously, and potentially interacting with each other.
- The computations may be executing on multiple cores in the same chip, preemptively time-shared threads on the same processor, or executed on physically separated processors.
- A number of mathematical models have been developed for general concurrent computation including Petri nets, process calculi, the Parallel Random Access Machine model, the Actor model and the Reo Coordination Language.
- Specific applications to static program analysis – design of automated tools to verify correctness etc. of a concurrent program regardless of specific timed execution.

Mutual exclusion

occurs, when n processes P_i compete for m resources R_j.



Only k processes can be served at any given time.

Semaphores

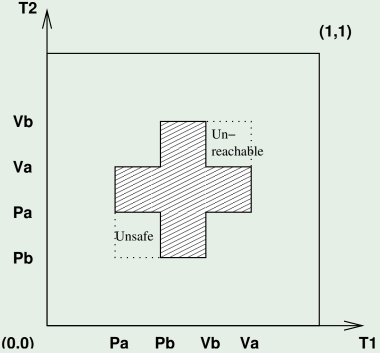
Semantics: A processor has to lock a resource and to relinquish the lock later on!

Description/abstraction: $P_i : \ldots PR_j \ldots VR_j \ldots$ (E.W. Dijkstra)

P: probeer; V: verhoog

A geometric model: Schedules in "progress graphs"

Semaphores: The Swiss flag example



PV-diagram from
$P_1 : P_aP_bV_bV_a$
$P_2 : P_bP_aV_aV_b$

Executions are directed paths – since time flow is irreversible – avoiding a forbidden region (shaded). D-paths that are dihomotopic (through a 1-parameter deformation consisting of dipaths) correspond to equivalent executions. Deadlocks, unsafe and unreachable regions may occur.

The state space – a d-space

A linear PV-program is modeled as the complement of a forbidden region F consisting of a number of holes in an n-cube:

- Hole = isothetic hyperrectangle (box) $R^i =]a_1^i, b_1^i[\times \cdots \times]a_n^i, b_n^i[\subset I^n, 1 \leq i \leq l$: with minimal vertex $\mathbf{a}^i$ and maximal vertex $\mathbf{b}^i$.
- State space $X = \vec{I}^n \setminus F,\ F = \bigcup_{i=1}^l R^i$
  X inherits a partial order from $\vec{I}^n$.
  d-paths are order preserving – form $\vec{P}(X)$.
  $(X, \vec{P}(X))$ a d-space.

More general concurrent programs $\rightsquigarrow$ HDA

Higher Dimensional Automata (HDA, V. Pratt; 1990):

- Cubical complexes: like simplicial complexes, with (partially ordered) hypercubes instead of simplices as building blocks.
- d-paths are order preserving
- Directed loops part of the model (important!)

Definition

- X a d-space, $a, b \in X$.
  $p : \vec{I} \to X$ a d-path in X (continuous and "order-preserving") from a to b.
- $\vec{P}(X)(a, b) = \{p : \vec{I} \to X | \, p(0) = a, p(b) = 1, p \text{ a d-path}\}$.
  Trace space $\vec{T}(X)(a, b) = \vec{P}(X)(a, b)$ modulo increasing reparametrizations.
  In our case: $\vec{P}(X)(a, b) \simeq \vec{T}(X)(a, b)$.
- A dihomotopy in $\vec{P}(X)(a, b)$ is a map $H : \vec{I} \times I \to X$ such that $H_t \in \vec{P}(X)(a, b), \; t \in I$; ie a path in $\vec{P}(X)(a, b)$.

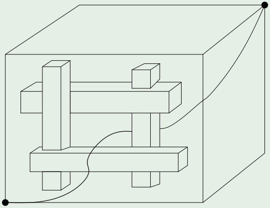
Aim : A model for calculation of invariants

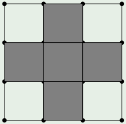
Description of $\vec{P}(X)(a, b)$ (up to homotopy equivalence) as explicit finite dimensional (prod-)simplicial complex.
In particular: its path components, ie the dihomotopy classes of d-paths (executions).

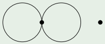
Example: State space, directed paths and trace space

Problem: How are they related?

State space and trace space for a semaphore HDA







(d-)State space:
a 3D cube $\vec{I}^3 \setminus F$
minus 4 box obstructions
pairwise connected

Path space model contained in torus $(\partial\Delta^2)^2$ – homotopy equivalent to a wedge of two circles and a point: $(S^1 \vee S^1) \sqcup *$

Homotopy of d-paths $\not\Rightarrow$ dihomotopy

Tool: Subspaces of X and of $\vec{P}(X)(\mathbf{0},\mathbf{1})$

$X = \vec{I}^n \setminus F$, $F = \bigcup_{i=1}^{l} R^i$; $R^i = [\mathbf{a}^i, \mathbf{b}^i]$; $\mathbf{0}, \mathbf{1}$ the two corners in I^n.

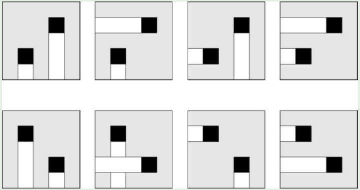
Definition

1. $X_{ij} = \{x \in X \mid x \leq \mathbf{b}^i \Rightarrow x_j \leq a^i_j\}$ –
   direction j restricted at hole i
2. M a binary $l \times n$-matrix: $X_M = \bigcap_{m_{ij}=1} X_{ij}$ –
   Which directions are restricted at which hole?

Examples: two holes in 2D – one hole in 3D (dark)

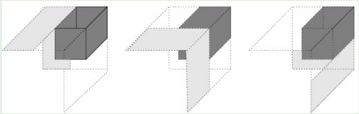
$M =$

$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$



$M =$

[100] [010] [001]



Covers by contractible (or empty) subspaces

Bookkeeping with binary matrices

Binary matrices

$M_{l,n}$ poset ($\leq$) of binary $l \times n$-matrices
$M_{l,n}^{R,*}$ no row vector is the zero vector –
every hole obstructed in at least one direction

A cover by contractible subspaces

Theorem

1. $$\vec{P}(X)(\mathbf{0},\mathbf{1}) = \bigcup_{M \in M_{l,n}^{R,*}} \vec{P}(X_M)(\mathbf{0},\mathbf{1}).$$
2. *Every path space $\vec{P}(X_M)(\mathbf{0},\mathbf{1})$, $M \in M_{l,n}^{R,*}$, is empty or contractible.* *Which is which?*

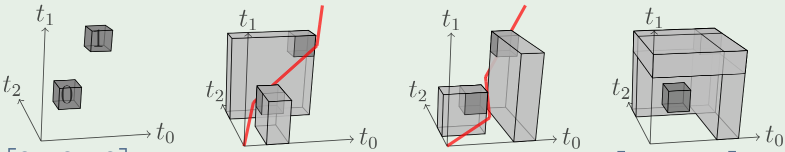
Proof.

Subspaces X_M, $M \in M_{l,n}^{R,*}$ are closed under $\vee$ = l.u.b. □

Example: A 3D-cube with two sub-cubes deleted

Category of binary matrices describes contractible or empty subspaces

$P_a.V_a.P_b.V_b \mid P_a.V_a.P_b.V_b \mid P_a.V_a.P_b.V_b$



$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$
state space	alive	alive	dead

Poset category and realization

Alive matrices: $\{M \in M_{2,3}(\mathbf{Z}/2) \mid$ no row $= [0,0,0]$ or $[1,1,1]\}$.
Associated (prod-)simplicial complex: $S^1 \times S^1$.

Nerve lemma

Given an open covering $\mathcal{U}$ of a space X such that every non-empty intersection of sets in $\mathcal{U}$ is contractible, then $X \simeq \mathcal{N}(\mathcal{U})$ – the nerve of the covering:
A simplicial complex with one n-simplex for every non-empty intersection of $n+1$ sets in $\mathcal{U}$.

General idea: HDA without d-loops

- Find decomposition of state space into subspaces so that d-path spaces in each piece – and intersections of such – are either contractible or empty.
- Describe the poset category corresponding to non-empty intersections using binary matrices.

HDA with d-loops

- L_1-length yields a homomorphism $l : \pi_1(X) \to \mathbf{Z}$.
- The associated length covering $\tilde{X}$ has only trivial d-loops.
- $\vec{P}(X)(x_0, x_1) \simeq \coprod_n \vec{P}(X)(\tilde{x}_0, \tilde{x}_1^n)$

A combinatorial model and its geometric realization

Combinatorics
poset category
$\mathcal{C}(X)(\mathbf{0},\mathbf{1}) \subseteq M_{l,n}^{R,*} \subseteq M_{l,n}$
$M \in \mathcal{C}(X)(\mathbf{0},\mathbf{1})$ "alive"

Topology:
prodsimplicial complex
$\mathbf{T}(X)(\mathbf{0},\mathbf{1}) \subseteq (\Delta^{n-1})^l$
$\Delta_M = \Delta_{m_1} \times \cdots \times \Delta_{m_l} \subseteq \mathbf{T}(X)(\mathbf{0},\mathbf{1})$ – one simplex Δ_{m_i} for every hole

$\Leftrightarrow \vec{P}(X_M)(\mathbf{0},\mathbf{1}) \neq \emptyset$.

Theorem (A variant of the nerve lemma)

$\vec{P}(X)(\mathbf{0},\mathbf{1}) \simeq \mathbf{T}(X)(\mathbf{0},\mathbf{1}) \simeq \Delta\mathcal{C}(X)(\mathbf{0},\mathbf{1})$.

From $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$ to properties of path space

Questions answered by homology calculations using $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$

Questions

- Is $\vec{P}(X)(\mathbf{0},\mathbf{1})$ path-connected, i.e., are all (execution) d-paths dihomotopic (lead to the same result)?
- Determination of path-components?
- Are components simply connected?
  Other topological properties?

Strategies – Attempts

- Implementation of $\mathcal{C}(X)(\mathbf{0},\mathbf{1}), \mathbf{T}(X)(\mathbf{0},\mathbf{1})$ in ALCOOL at CEA/LIX-lab. (France): Goubault, Haucourt, Mimram.
- The poset category $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$ leads to an associated chain complex of vector spaces over a field – departure for calculation of homology of $\vec{T}(X)(\mathbf{0},\mathbf{1})$.
- Use fast algorithms (eg Marian Mrozek's CrHom etc) to calculate the homology groups of these chain complexes even for very big complexes: M. Juda (Krakow).

Use and users

Verification, static analysis of concurrent programs: Need only check correctness for one representative in each dihomotopy class – all in the same class yield the same result.

Model checkers Competes well with respect to time and memory consumption.

Industrial customers Électricité de France, Airbus (on experimental basis)

Relations, challenges

- Explore relations to use of topological methods in distributed computing (Herlihy etal.)
- Complexity issues: Exponential growth. Exploit inductive calculations. Not easy: involves (homotopy) colimits.
- Incorporation of loops need further concern, both conceptual and implementation.

Other topics

Connections between concurrency and directed algebraic topology

- Detection of deadlocks and unsafe areas (instrumental in algorithms distinguishing live and dead matrices)
- Determination of fundamental category and other topological invariants of a state space (NB: d-paths non-reversible in general
- To which extent does variation of end points result in more/fewer d-homotopy classes? $\rightsquigarrow$ division of state space into components. Persistence?
- Directed coverings and (bi-)simulations

Want to know more?

References

- Kozlov, Combinatorial Algebraic Topology, Springer, 2008.
- Grandis, Directed Algebraic Topology, Cambridge UP, 2009.
- Fajstrup, Goubault, Raussen, Algebraic Topology and Concurrency, Theor. Comput. Sci. **357** (2006), 241 – 278.
- R, Simplicial models for trace spaces, AGT **10** (2010), 1683 – 1714.
- R, Execution spaces for simple HDA, Appl. Alg. Eng. Comm. Comp. **23** (2012), 59 – 84.
- R, Simplicial models for trace spaces II: General Higher Dimensional Automata, AGT **12**, 2012; in print.
- Fajstrup etal., Trace Spaces: an efficient new technique for State-Space Reduction, Proceedings ESOP, Lect. Notes Comput. Sci. **7211** (2012), 274 – 294.
- Jardine, Path categories and resolutions, Homology, Homotopy Appl. **12** (2010), 231 – 244.

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