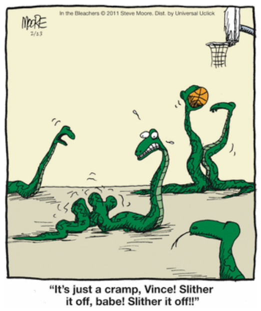
Civil Engineering Hydraulics
Mechanics of Fluids

Conservation of Angular Momentum



February 23rd, 1870 – In the United States, post-Civil War military control of Mississippi ends and it is readmitted to the Union.

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Review

- If we look at the angular momentum of a body, we consider the moment or torque that produces the rotation
- The expression is slightly different than for linear momentum

$$\vec{M} = I\vec{\alpha}$$

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Review

- Both expressions are similar in form but they do differ in the units
- Both expressions do involve vector quantities

$$\vec{M} = I\vec{\alpha} = I\frac{d\vec{\varpi}}{dt}$$

$$\vec{F} = m\vec{a} = m\frac{d\vec{V}}{dt}$$

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Review

- Now we will shift our attention to utilizing the conservation of angular momentum rather than the conservation of linear momentum
- Linear momentum will still be conserved, we are just looking at the system with a slightly different viewpoint

$$\vec{M} = I\vec{\alpha} = I\frac{d\vec{\varpi}}{dt}$$

Angular Momentum

- In the same way that in statics when we used the sum of the moments equal to 0 we did not violate the sum of the forces being equal to 0
- Now we can consider conserving both linear and angular momentum

$$\vec{M} = I\vec{\alpha} = I\frac{d\vec{\varpi}}{dt}$$

Angular Momentum

- A moment required a force to generate the moment and a moment arm
- A change in angular momentum requires a change in angular velocity and the moment of inertia about the axis of rotation
- If either of these quantities change, then there must be a corresponding change in the moments or torques acting on the system

$$\vec{M} = I\vec{\alpha} = I\frac{d\vec{\varpi}}{dt}$$

Angular Momentum

- A flow with a mass flow rate $\dot{m}$ and a tangential velocity of *v* (with respect to a circle and a center of rotation), has a linear momentum along the tangent which is equal to the product of the mass flow rate and the tangential velocity

$$\dot{m}\vec{v}$$

Angular Momentum

- If this represents a change in momentum, then a force is generated.
- The force in turn generates a moment about any point not on the line of action of the force.
- The moment generated by this force about the center of rotation is given by

$$\vec{F} = \dot{m}\vec{v}$$

$$\vec{M} = \vec{r} \otimes \vec{F} = \vec{r} \otimes \dot{m}\vec{v}$$

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Angular Momentum

- Since we started with a tangential velocity the radius vector is normal to the velocity vector and the magnitude of the moment can be calculated as

$$\vec{F} = \dot{m}\vec{v}$$
$$\vec{M} = \vec{r} \otimes \vec{F} = \vec{r} \otimes \dot{m}\vec{v}$$
$$M = r\dot{m}v$$

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Angular Momentum

- Tangential velocity can also be defined in terms of angular velocity

$$\vec{F} = \dot{m}\vec{v}$$
$$\vec{M} = \vec{r} \otimes \vec{F} = \vec{r} \otimes \dot{m}\vec{v}$$
$$M = r\dot{m}v$$
$$v = r\omega$$

Angular Momentum

- So the moment can be developed in terms of the angular velocity

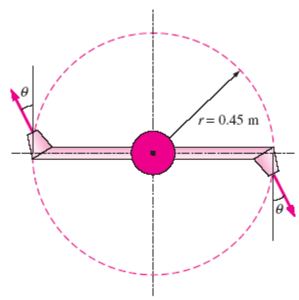
$$\vec{F} = \dot{m}\vec{v}$$
$$\vec{M} = \vec{r} \otimes \vec{F} = \vec{r} \otimes \dot{m}\vec{v}$$
$$M = r\dot{m}v$$
$$v = r\omega$$
$$M = r\dot{m}\left(r\omega\right) = r^2\dot{m}\omega$$

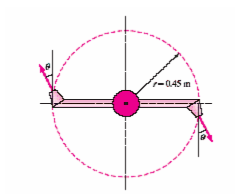
Angular Momentum

- Remember that the M is in response to a change in angular momentum

$$\vec{F} = \dot{m}\vec{v}$$
$$\vec{M} = \vec{r} \otimes \vec{F} = \vec{r} \otimes \dot{m}\vec{v}$$
$$M = r\dot{m}v$$
$$v = r\omega$$
$$M = r\dot{m}\left(r\omega\right) = r^2\dot{m}\omega$$

6–73 Water enters a two-armed lawn sprinkler along the vertical axis at a rate of 60 L/s, and leaves the sprinkler nozzles as 2-cm diameter jets at an angle of θ from the tangential direction, as shown in Fig. P6–73. The length of each sprinkler arm is 0.45 m. Disregarding any frictional effects, determine the rate of rotation *n.* of the sprinkler in rev/min for (*a*) θ=0°, (*b*) θ=30°, and (*c*) θ=60°.

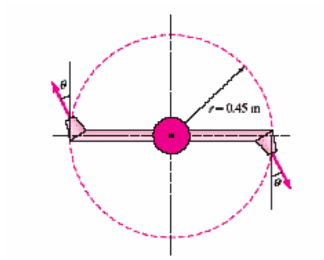




In order to determine the rate of rotation, n, we will need to find the tangential velocity of any point at a distance, r, from the center of rotation. The expression relating the tangential velocity and the rate of rotation is given as

$$\dot{n} = \frac{\omega}{2\pi}$$ where n is in radians per minute (rpm).

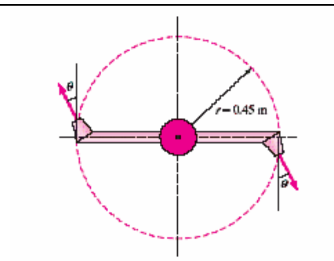
So to solve our problem(s), we will need to be able to solve for the angular velocity.



There is an expression relating angluar velocity to tangential velocity. The expression is

$$\omega = \frac{V_{tangential}}{r}$$

Since we know r, it is given in the problem, we need the tangential component of the velocity to solve the problem. We need to solve for the tangential velocity.



If we consider the circle developed by the radius of the sprinkler the control volume, we can write out momentum expression for that control volume.

$$\sum \vec{M} = \left(\sum \vec{r} \times \dot{m}\vec{V}\right)_{out} - \left(\sum \vec{r} \times \dot{m}\vec{V}\right)_{in}$$

Setting an x-y axis through the center of rotation of the sprinkler we can write the velocity as a vector at each jet.

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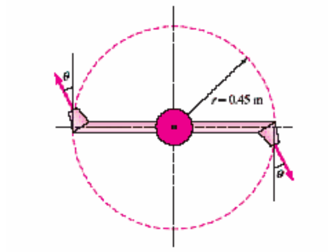

Setting an x-y axis through the center of rotation of the sprinkler we can write the velocity as a vector at each jet.

The velocity of the jet on each side of the system can be written as shown below. Since the sprinkler arms are opposed to each other we can set the x-axis along the axis arm.

$$\vec{V}_{left} = V\left(-\sin\theta\vec{i} + \cos\theta\vec{j}\right) \qquad \vec{V}_{right} = V\left(\sin\theta\vec{i} - \cos\theta\vec{j}\right)$$

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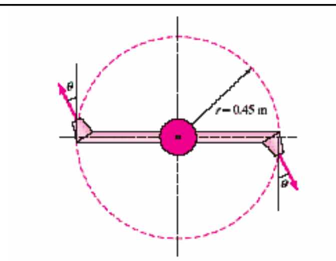


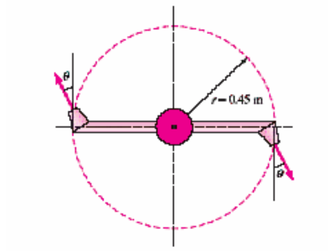
The change in angular momentum on the left side would be equal to the cross produce of the radius of the sprinkler times the velocity on the left side times the mass flow rate.

$$\vec{r}_{left} = r\left(-\vec{i}\right)$$

$$\frac{d\vec{H}_{left}}{dt} = \dot{m}_{left}\left(\vec{r}_{left} \otimes \vec{V}_{left}\right)$$

$$\frac{d\vec{H}_{left}}{dt} = \dot{m}_{left}\left(r\left(-\vec{i}\right) \otimes V\left(-\sin\theta\vec{i} + \cos\theta\vec{j}\right)\right)$$

$$\frac{d\vec{H}_{left}}{dt} = -\dot{m}_{left} rV\cos\theta\vec{k}$$



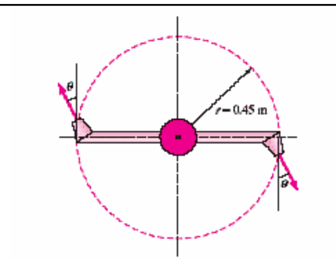
The change in angular momentum on the right side would be equal to the cross produce of the radius of the sprinkler times the velocity on the right side times the mass flow rate.

$$\vec{r}_{right} = r\left(\vec{i}\right)$$

$$\frac{d\vec{H}_{right}}{dt} = \dot{m}_{right}\left(\vec{r}_{right} \otimes \vec{V}_{right}\right)$$

$$\frac{d\vec{H}_{right}}{dt} = \dot{m}_{left}\left(r\left(\vec{i}\right) \otimes V\left(\sin\theta\vec{i} - \cos\theta\vec{j}\right)\right)$$

$$\frac{d\vec{H}_{right}}{dt} = -\dot{m}_{right} rV\cos\theta\vec{k}$$



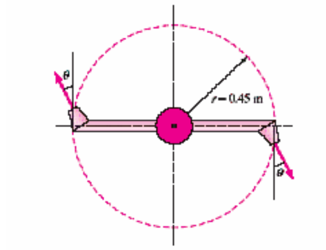
Since the mass flow rate is the same from both sides of the sprinkles the rate of change in momentum change across the system is equal to

$$\frac{d\vec{H}}{dt} = \frac{d\vec{H}_{right}}{dt} + \frac{d\vec{H}_{left}}{dt}$$

$$\frac{d\vec{H}}{dt} = -2\dot{m}rV\cos\theta\vec{k}$$

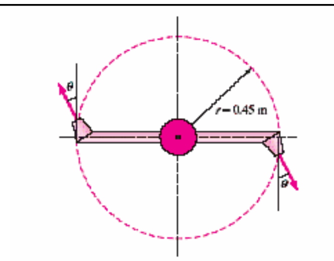
and in scalar terms

$$\frac{dH}{dt} = -2\dot{m}rV\cos\theta$$



We actually have two velocities acting here. The velocity of the nozzle and the velocity of the jet of water as it exits the nozzle. This isn't an easy thing to see but it is critical to the problem. If some torque was accelerating the nozzle, we would need to look at the relative velocity of the point with respect to center of the system.

In this case, we know that there are no applied moments on the system so we can say that the change in angular momentum is equal to 0.



We know that the mass flow rate is not equal to 0 and that the radius of the sprinkler is not equal to 0 so the relative velocity of the sprinkler to the water must be 0. What this means is the the only velocity component acting on the system is the velocity of the water exiting the sprinkler.

So we can state that the velocity of the sprinkler nozzle is the velocity of the jet. This also means that the tangential component of the jet's velocity if the tangential velocity of the sprinkler nozzle.

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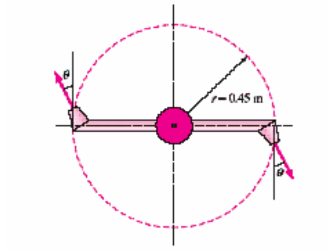

The velocity of the jet can be calculated using the volumetric flow rate and the area of the jet.

$Q := 60 \frac{L}{sec}$ $\quad$ $dia := 2cm$

$Q = 0.06 \frac{m^3}{s}$ $\quad$ $dia = 0.02\,m$

$A := \frac{\pi \cdot dia^2}{4}$ $\quad$ $A = 0.0003\,m^2$

$V := \frac{Q}{2 \cdot A}$ $\quad$ $V = 95.493 \frac{m}{s}$

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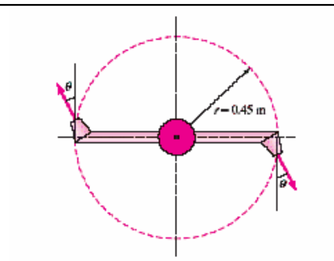
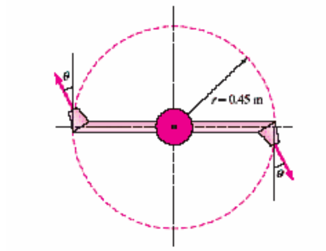


Depending on the orientation of the nozzle, θ, the tangential component of the velocity will change.

$$V_{tangential} = V \cos\theta$$

The tangential velocity can be related to the angular velocity using

$$V_{tangential} = V \cos\theta$$
$$V_{tangential} = r\omega$$
$$V \cos\theta = r\omega$$



The rotational speed can be related to the angular velocity using

$$\dot{n} = \frac{\omega}{2\pi}$$

$$V\cos\theta = r\omega$$

$$\dot{n} = \frac{V\cos\theta}{2\pi r}$$

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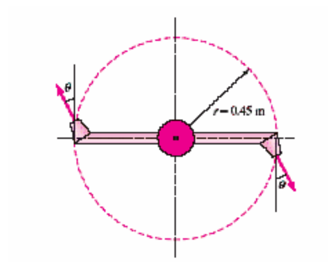


$r := 0.45\text{m}$

$n_{dot}(\theta) := \frac{V \cdot \cos(\theta)}{2 \cdot \pi \cdot r}$

$\text{rpm} \equiv \frac{1}{\text{min}}$

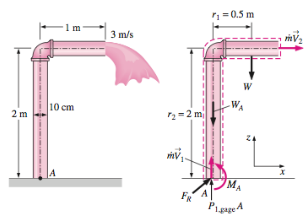
$n_{dot}(0) = 2026.42\,\text{rpm}$

$n_{dot}(30\text{deg}) = 1754.93\,\text{rpm}$

$n_{dot}(60\text{deg}) = 1013.21\,\text{rpm}$

Example

Underground water is pumped to a sufficient height through a 10-cm-diameter pipe that consists of a 2-m-long vertical and 1-m-long horizontal section, as shown in Fig. 6–37. Water discharges to atmospheric air at an average velocity of 3 m/s, and the mass of the horizontal pipe section when filled with water is 12 kg per meter length. The pipe is anchored on the ground by a concrete base. Determine the bending moment acting at the base of the pipe (point A) and the required length of the horizontal section that would make the moment at point A zero.



Homework 14-1

6–47 Water is flowing through a 12-cm-diameter pipe that consists of a 3-m-long vertical and 2-m-long horizontal section with a 90° elbow at the exit to force the water to be discharged downward, as shown in Fig. P6–47, in the vertical direction. Water discharges to atmospheric air at a velocity of 4 m/s, and the mass of the pipe section when filled with water is 15 kg per meter length. Determine the moment acting at the intersection of the vertical and horizontal sections of the pipe (point *A*). What would your answer be if the flow were discharged upward instead of downward?

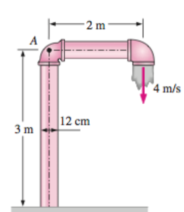


FIGURE P6–47

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Homework 14-2

6–48 A large lawn sprinkler with two identical arms is used to generate electric power by attaching a generator to its rotating head. Water enters the sprinkler from the base along the axis of rotation at a rate of 8 gal/s and leaves the nozzles in the tangential direction. The sprinkler rotates at a rate of 250 rpm in a horizontal plane. The diameter of each jet is 0.5 in, and the normal distance between the axis of rotation and the center of each nozzle is 2 ft. If the rotating head is somehow stuck, determine the moment acting on the head.