Consistency in Tomographic Reconstruction by Iterative Methods

by

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Abstract

Tomographic image reconstruction using limited projection data requires incorporation of all available a priori information in the reconstruction process. This necessitates the use of transformations between projection and image spaces. The consistency of these transformations is important in elevating the confidence in the solution. It is also observed that better reconstructions can be obtained through consistent transformations.

Introduction

In several applications of the computerized tomography (CT), the available projection data is not enough for an acceptable reconstruction. This may be because of various reasons including: time constraints as in the case of imaging moving parts; physical constraints such as imaging of very large objects; and cost constraints limiting the number of detectors used. To obtain acceptable reconstructions using small numbers of projections is also important in reducing the overall X-ray exposure in medical tomography and reducing the scan time in nondestructive testing of materials in a high speed production environment. Clearly, in order to have an acceptable reconstruction the missing information in the limited projection data must be supplied from some other source. This necessitates the use of a reconstruction algorithm which incorporates all available a priori information about the reconstructed image and the projection process.

Recently, in many papers [1,2,3], iterative methods have been used to implement constraints to model various information such as: the region of support in the spatial domain, the range of the CT numbers and nonnegativity of the image. The iterations consist of sequentially enforcing the constraints alternately in the projection domain and the space domain. Constraints may be implemented in the projection domain alone [4]; however, much information, such as complicated region of supports or nonnegativity, cannot be formulated in this approach.

A common problem with all of the published methods of transformation between the space and the projection domains is the lack of consistency of the transformation. That is, when a signal is transformed from one domain to the other and back again, with no modifications, the result is not the original signal. There is no practical solution for having exact consistency with any kind of transformation between the space and projection domains; however, it is possible to have consistency within a known and acceptable error bound. To have a consistent transformation is important solely because of having confidence in a reconstructed result, and as can be seen from the examples, it also improves the quality of the reconstruction.

Consistency of the Existing Methods

The iterative limited data reconstruction techniques can be classified in three main groups based on their formulations: filtered backprojection, direct Fourier reconstruction and algebraic reconstruction techniques (ART). The method reported in [1] transforms the signal from the projection domain to the space domain by using a standard filtered backprojection algorithm. It is advantageous to use the filtered backprojection technique because the actual CT hardware can be used in the implementation. The transformation from the space domain to the projection domain was done by numerical integration. The problem is that the two transformations are not inverses. In Fig.1 a 30x30 image is reconstructed using this method from 10 views with 30 detectors each, nonnegativity and region of support information. The reconstruction result becomes useless if the iteration is not stopped at the proper time. However, there is no rigorous rule to stop the iteration before the reconstruction starts to deteriorate.

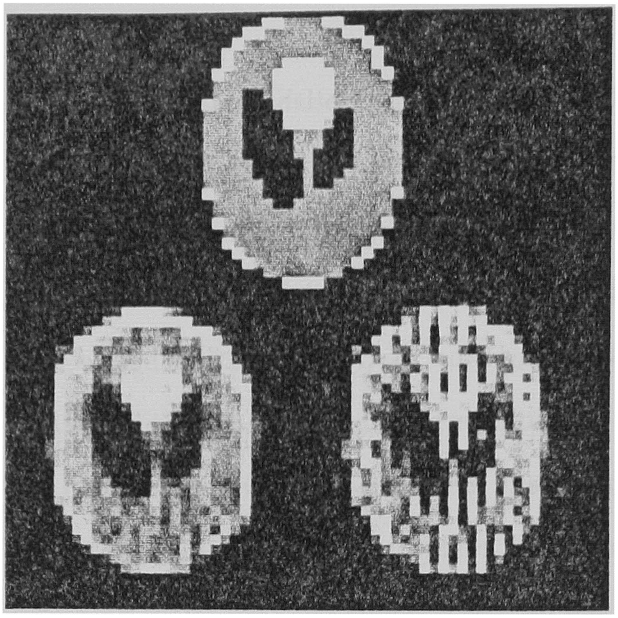


Figure 1 (Top) Simulated original picture. (Bottom, left) Reconstruction after 2 iterations with a standard filtered backprojection and numeric integration technique. (Bottom, right) Reconstruction after 10 iterations.

A consistent algorithm combining the projection data and a priori information may be developed using the set of reconstructions which are consistent with the given projection data and the filtered backprojection algorithm. This set can be written as:

$$S_c = \{f \mid R[f] = p\}$$

where, f is an image, p is the projection data, and $R[.]$ is a consistent reprojection operator. The consistent reprojection operator must be such that:

i. For all p's with no components in the null space of the filtered backprojection operator $B[.]$,

$$R[B[p]] = p$$

ii. For all f's with no components in the range space of $B[.]$

$$R[f] = 0$$

tion data and the filtered backprojection algorithm. This set can be written as:

$$S_c = \{f \mid R[f] = p\}$$

These two conditions require having the pseudo inverse of the operator $B[.]$ as the reprojection operator. The set S_c is convex (in fact a linear variety), and many other convex sets can be defined using other a priori information such as nonnegativity of the reconstruction, region of support, nonnegativity of the projections etc.. A solution then can be found by using the successive projections onto convex sets technique.

The projection of an image, f^0, onto S_c can be found by solving the following minimization problem:

$$\min_{f \in S_c} ||f - f^0||$$

and is given by:

$$f = B[p] + N[f^0]$$

where, $N[f^0]$ is the component of f^0 in the null space of $B^*[.]$. In [5] the importance of a priori information in determining the null space component of the reconstruction is emphasized. In this formulation, the effect of the a priori information is confined in the null space.

The method of successive projections onto convex sets converges to an element in the intersection of these sets. Thus, the solution must be a member of the intersection. In the above formulation, if p is the correct projection data, then the difference between the original image and the filtered backprojection reconstruction must lie in the null space of $B^*[.]$. In order to test this, a consistent image/projection pair is obtained by computing the projections of the image given in Fig.1(top), which is constant within each pixel. Fig.2 displays the component of the original image consisting of the filtered backprojection result and the null space component. The nonzero difference between this image and the original image indicates that to reconstruct the original image the projection data must be modified. This contradicts the initial assumption about the correctness of the projection data. The reason for this contradiction is the error introduced by the filtered backprojection algorithm. It is possible to estimate a region around the correct projection data in order to compensate for the reconstruction noise, however, the reconstruction noise is image dependent and very difficult to analyze [6].

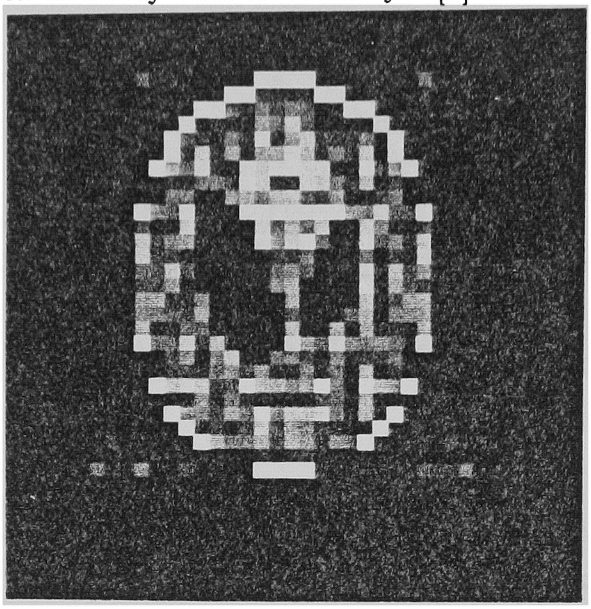


Figure 2 The closest feasible image to the original image.

The second group of techniques uses a direct Fourier inversion approach [2]. In these techniques the projections are interpolated from the polar grid to a rectangular grid in frequency domain. The iteration is done by enforcing the constraints in the frequency domain and space domain. The transformation from these two domains is done by the DFT and is consistent. However, because of the non-bandlimited nature of the reconstructed image, the interpolation cannot be done error free even if all of the available projections are used in the reconstruction process. Although it is possible to define an approximate consistency for this techniques through an analysis of the interpolation error, this approach will not be considered in this paper.

The third group of techniques employ (ART). In these techniques, the projection process is modeled as:

techniques the projections are interpolated from the polar grid to a rectangular grid in frequency domain. The iteration is done by enforcing the constraints in the frequency domain and space domain. The transformation from these two domains is done by the

where, the matrix A contains the weights of a numeric integration algorithm which is used to compute the line integrals of the sampled image [7]. In most of the applications of ART the image is assumed to be constant within each pixel. This assumption may be reasonable for a small pixel size which increases the dimensionality of the problem. The integration error is the only inconsistency in ART formulation and it is possible to obtain an approximate consistency by a proper model for the integration error. The importance of having tolerances for the computed projections in order to stop the ART iterations at an optimal point has been previously reported in [10,11,12]. However, there is no rigorous model for these tolerances.

A Consistent Iteration

The projections of an image are the values of its line or strip integrals over ray paths. These integrals must be computed using a sampled version of the image. In the classical approach, the image is assumed to be constant within the pixels and the integral is given by the summation of the intensities of the pixels on the ray path multiplied by the length of the rays inside these pixels. This is nothing but a rectangular approximation for the required integral. Because the original image is not constant in the pixels, this approach will introduce errors. Using different pixel shapes such as defining each ray path as a pixel [8], may be a solution for this problem however, because of the intersecting structure and large sizes of these pixels, this approach makes it very difficult to apply certain a priori information.

Higher order integration routines may be used to reduce the integration error [9]. It should be noted that, irrespective of the order of integration the result will be erroneous. This is because of the fact that the original image is not a differentiable function. The same reason also makes the classical error bounds for the computed integral too large to be useful. Clearly, in order to have a consistent formulation these errors have to be estimated. A possible approach for this is to use a probabilistic error model.

The largest errors occur at those points where the rays crosses a large intensity difference. The largest possible error magnitude at such points is equal to the ray length in the pixel multiplied with the intensity difference. However, the actual amount of error depends on the image and its position on the grid. The number of such errors on a particular ray can be modeled as a Poisson process. The parameter of the Poisson density is equal to the expected intensity transitions in a unit ray length and most of the time this can be estimated from a priori information about the object. The magnitude of each error is assumed to be a zero mean normal random variable with standard deviation adjusted to assign a small probability to the largest possible error. Finally, for a ray of length l, the total integration error has a generalized Poisson density which is given by:

$$f_e(e) = \sum_{k=1}^{\infty} \frac{(\lambda l)^k}{k!} \frac{\exp\{-\lambda l - e^2 / 2k\sigma^2\}}{\sqrt{2\pi k}\sigma}$$

Using the confidence levels for the integration error, the reconstruction problem can be formulated as a projection onto convex sets problem where the convex sets are defined using the projection data and a priori information. Specifically, a set defined by a single projection is:

$$S_{pi} = \{f \mid \lvert \langle a_i f \rangle - p_i \rvert \leq \delta_i \}$$

where, a_i's contain the weights of the integration algorithm and δ_i's are selected so that the probability of having the original solution inside the set is at a selected level such as 99%. This level is essentially a measure for the consistency of the formulation. The probability of having the original picture in the intersection of these sets will be determined by this level. Note that this does not take into account the correlation between projections which may be used to reduce the size of the sets. Another important point is to allow enough projections to be outside of the bounds because even if each set has a very high probability of containing the original signal this probability is not very high for the intersection set. For example, if each set contains the solution with a 0.99 probability and there are 300 sets, the probability of having the solution in the intersection set will be less than 5%. On the other hand, the probability of having the solution in the intersection of any 295 sets is larger than 91%. This tolerance is also useful for handling outliers.

Fig.3. displays the reconstructions obtained by using the sets defined using 300 actual projections (i.e. theoretically computed integrals), the nonnegativity and the region of support information. The original image is inside each of these sets with 99% probability. The reconstruction displayed in Fig. 3.a is obtained in 5 iterations and is in the intersection of 295 sets defined by the projections and the sets defined by nonnegativity and region of support information. The result displayed in Fig.3.b is obtained after 50 iterations are performed. The loss of quality is obvious.

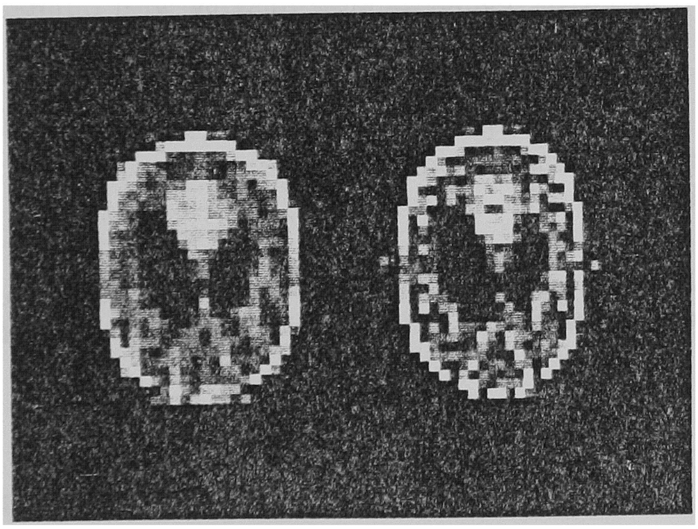
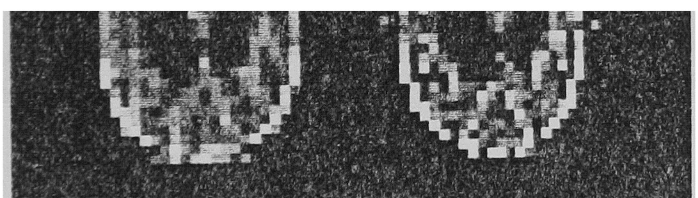


Figure 3 a. A consistent reconstruction at 91% level. b. Reconstruction obtained if the iterations are continued.



Summary

The consistency of an iterative reconstruction method is important in explaining the relation between the recorded data, a priori information and the reconstruction result. The difficulty in obtaining a consistent iteration based on the filtered backprojection method is demonstrated. ART are studied in order to obtain a consistent formulation through modeling the integration error involved. It is observed that, a consistent algorithm has a better performance in improving the result of a limited data reconstruction problem.

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