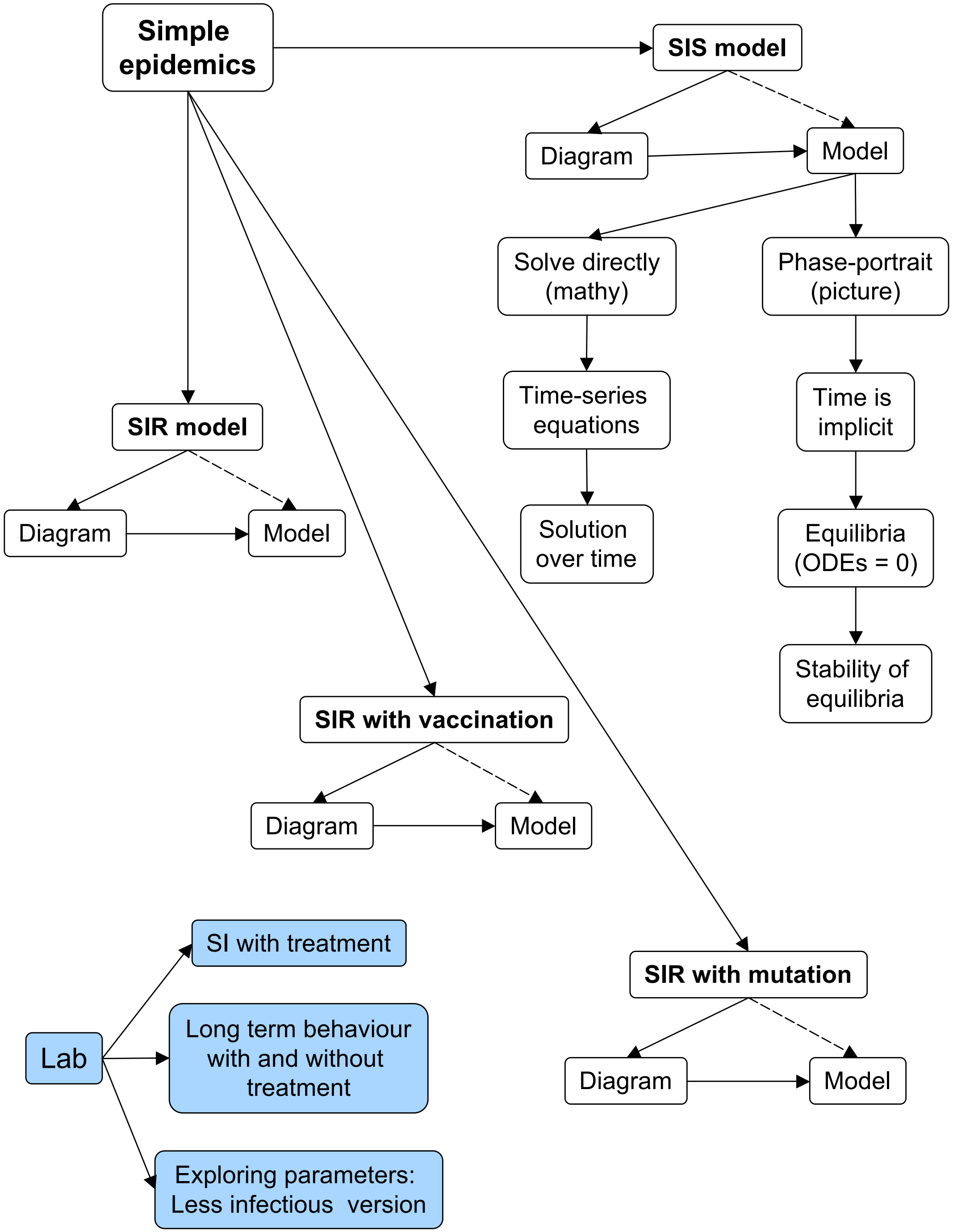
Simple epidemic models

- Construct ODE (Ordinary Differential Equation) models
- Relationship between the diagram and the equations
- Alter models to include other factors.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Ordinary Differential Equations(ODEs)

- ODEs deal with populations, not individuals
- We assume the population is well-mixed
- We keep track of the inflow and the outflow.

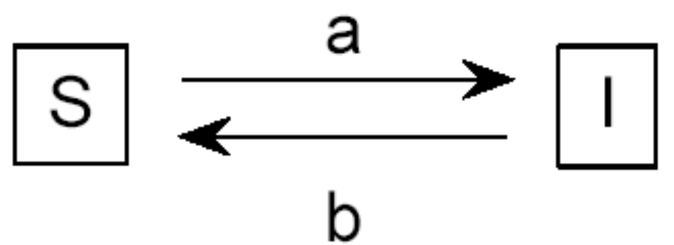
ODEs = Ordinary Differential Equations

SIS epidemic

- Susceptible→Infected→Susceptible
- You get sick, then recover, but without immunity
- E.g. the common cold.

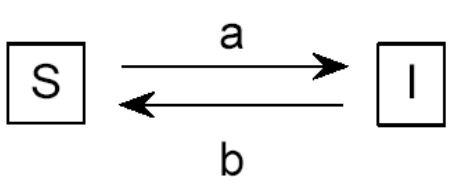
Diagram

- Susceptibles become infected at rate *a*
- Infecteds recover at rate *b.*



SIS equations

- Becoming infected depends on contact between Susceptibles and Infecteds *(aSI)*
- Recovery is at a constant rate, proportional to number of Infecteds *(b).*



$$\frac{dS}{dt} = bI - aSI$$
$$\frac{dI}{dt} = aSI - bI$$

a = infection rate
b = recovery rate

Total population is constant

- Add equations together
- $N=S+I$ (total population)
- $dN/dt=0$ → N is a constant.

$$\frac{dS}{dt}+\frac{dI}{dt}=bI-aSI+aSI-bI$$

$$\frac{dN}{dt}=0$$

$S' = bI-aSI$
$I' = aSI-bI$

S = *Susceptible*
I = *Infected*

Solving directly

- Since $N=S+I$, this means $S = N-I$

$$\frac{dI}{dt} = a(N-I)I - bI$$
$$= (aN - b - aI)I$$

$I' = aSI-bI$

- Let $A = aN-b$ be a constant

$$\frac{dI}{dt} = (A - aI)I\,.$$

S = Susceptible
I = Infected
a = infection rate
b = recovery rate

Separate the variables

- Put the *I*'s on one side and the *t*'s on the other

  (including *dI* and *dt*)

$$\frac{dI}{(A - aI)I} = dt\,.$$

I = Infected
a = infection rate
A = aN-b (constant)
N = total pop.
b = recovery rate

Time series solution

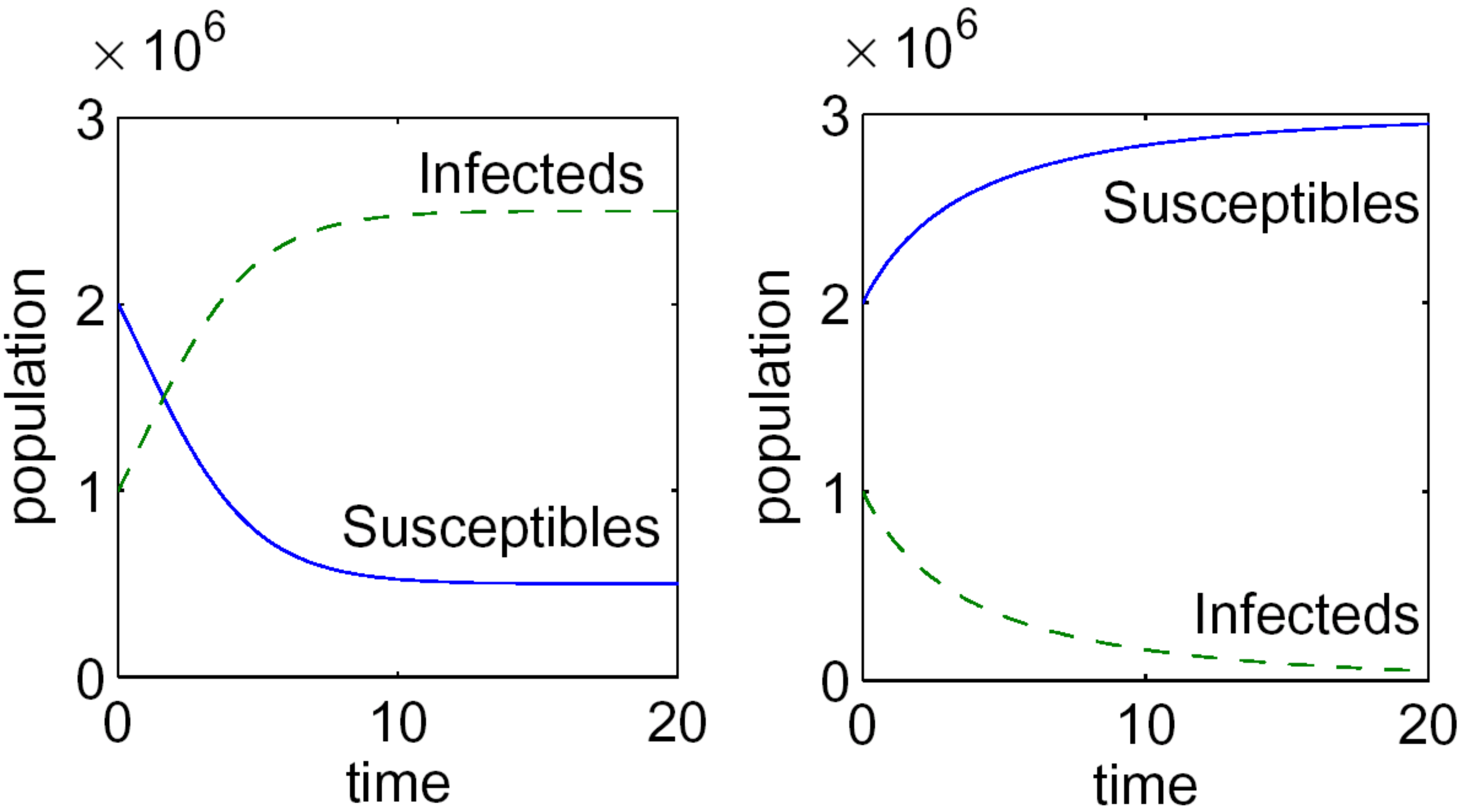
- Rearrange using partial fractions
- Integrate
- Use initial condition $I(0)=I_0$

S = Susceptible
I = Infected
a = infection rate
N = total pop.
b = recovery rate

$$I = \frac{(aN-b)I_0 e^{(aN-b)t}}{(aN-b)+aI_0[e^{(aN-b)t}-1]}$$

$$S = N - \frac{(aN-b)I_0 e^{(aN-b)t}}{(aN-b)+aI_0[e^{(aN-b)t}-1]}.$$

(See Epidemic Notes)



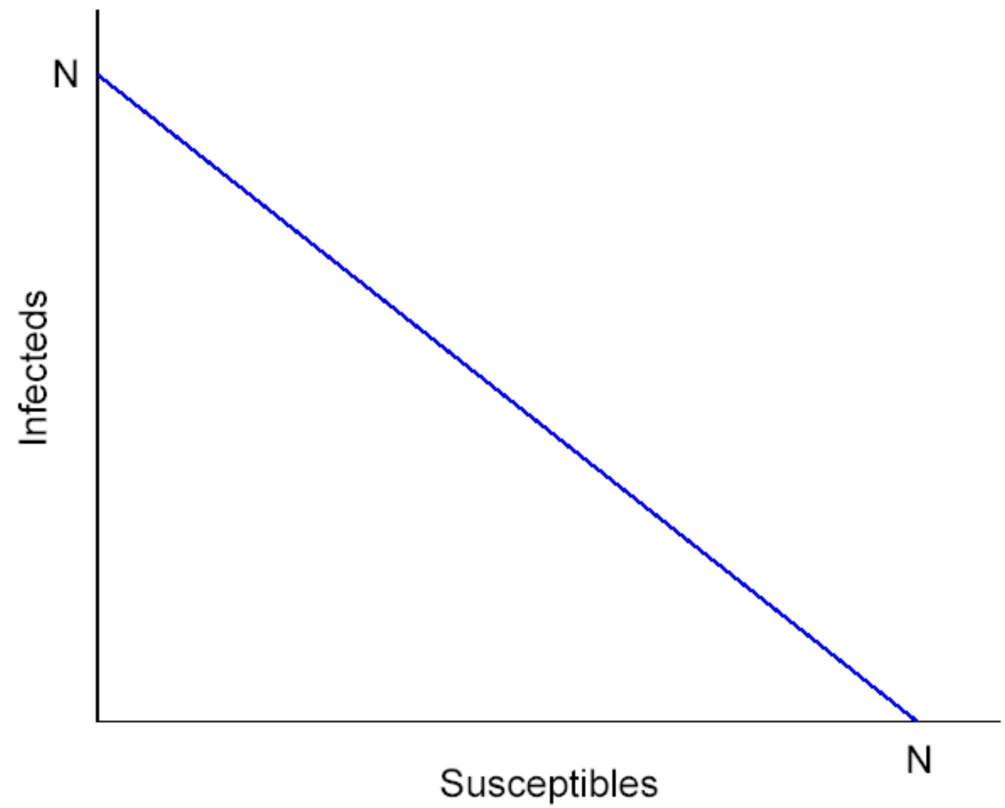
b=0.1 b=0.7

b = recovery rate
a= infection rate
N = population

(a=0.2 and N=3x10^6)

Phase portraits

- Since $N = S+I$,
  $I = -S+N$
- This is a straight line in I and S
- Time is implicit.



S = Susceptible
I = Infected
N = total pop.

Equilibrium points

- Equilibria occur when derivatives are zero:

$$(b - aS)I \quad = \quad 0 \qquad \text{when } S = \frac{b}{a} \text{ or when } I = 0$$

$$(aS - b)I \quad = \quad 0 \qquad \text{when } S = \frac{b}{a} \text{ or when } I = 0$$

$$\text{(We'll call } \tfrac{b}{a} \text{ `}p\text{'.)}$$

S = Susceptible
I = Infected
a = infection rate
b = recovery rate

S' = bI-aSI
I' = aSI-bI

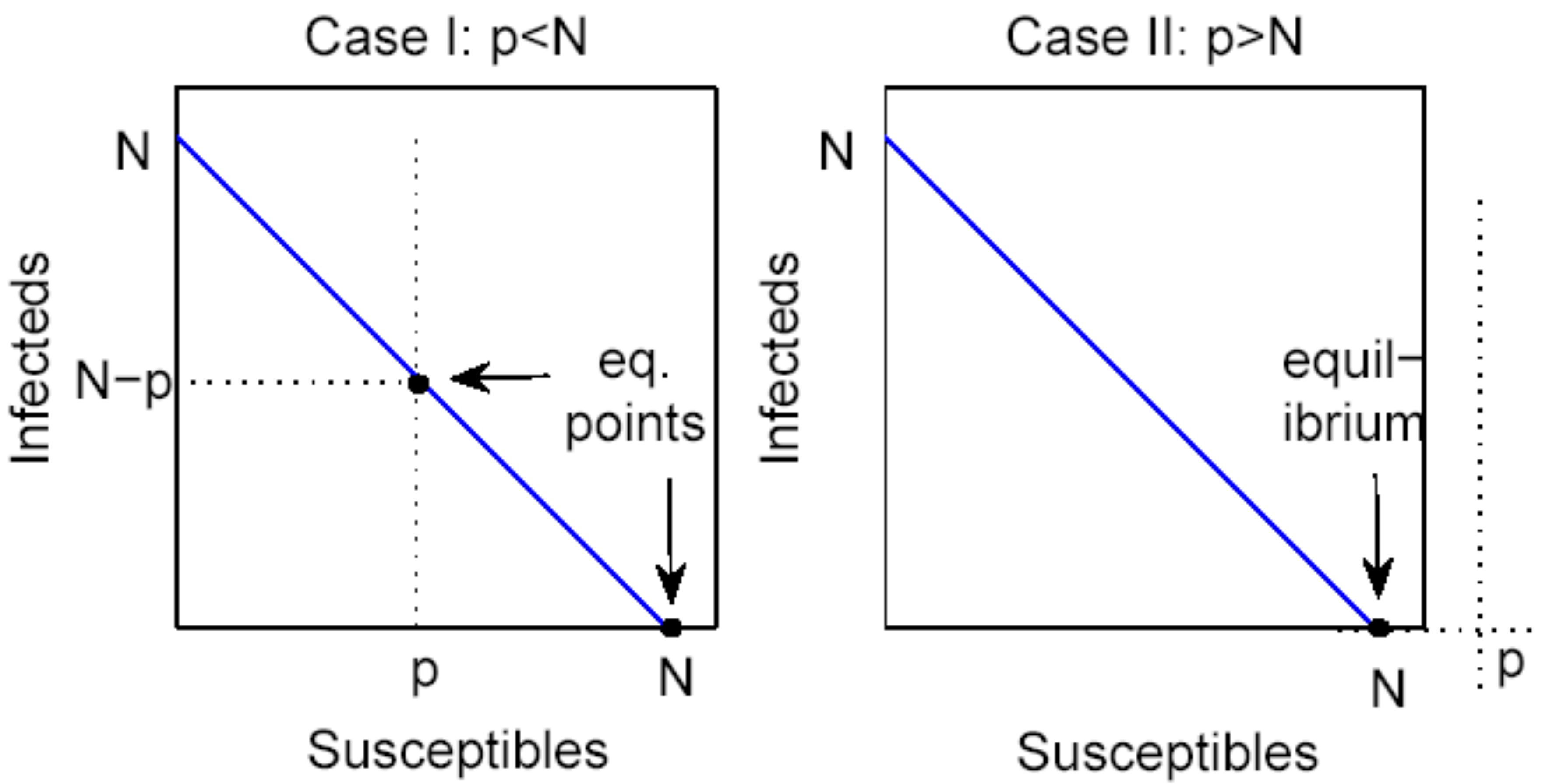
Two equilibria

- Thus our equilibrium points are

$$
\begin{aligned}
 & (\bar{S}, \bar{I}) &= & \quad (p, N - p) \\
\text{or} \quad & (\bar{S}, \bar{I}) &= & \quad (N, 0)
\end{aligned}
$$

- The latter always exists, the former is only biologically reasonable if $p<N$.

S = Susceptible *I = Infected*
N = total pop. *p = b / a*
b = recovery rate *a = infection rate*



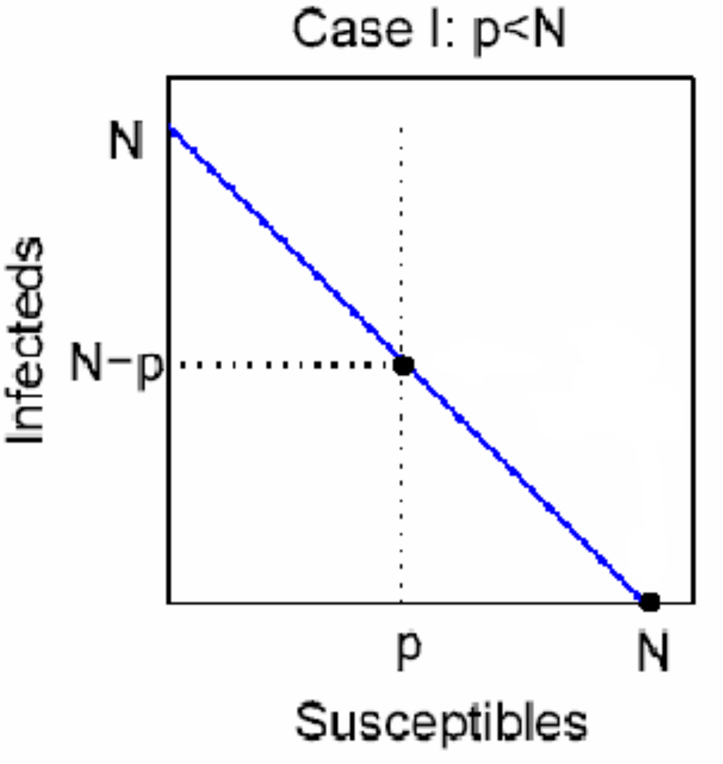
N = total pop. *p = b / a*
b = recovery rate *a = infection rate*

Stability

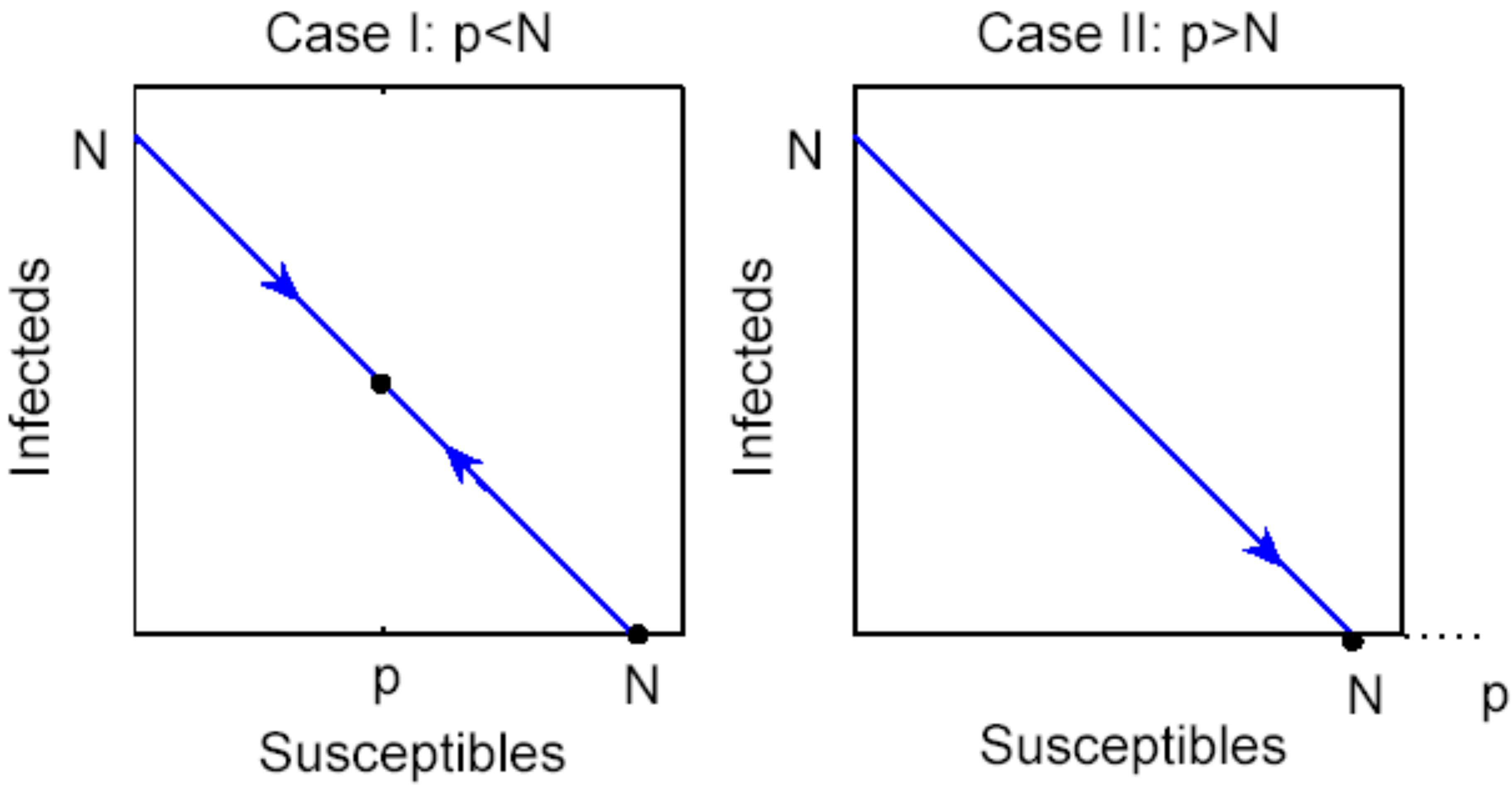
$$S' = a(p-S)I$$

$$I' = a(S-p)I$$

- $S < p \rightarrow S' > 0, I' < 0$
- $S > p \rightarrow S' < 0, I' > 0$



S = Susceptible	I = Infected
N = total pop.	p = b / a
b = recovery rate	a = infection rate

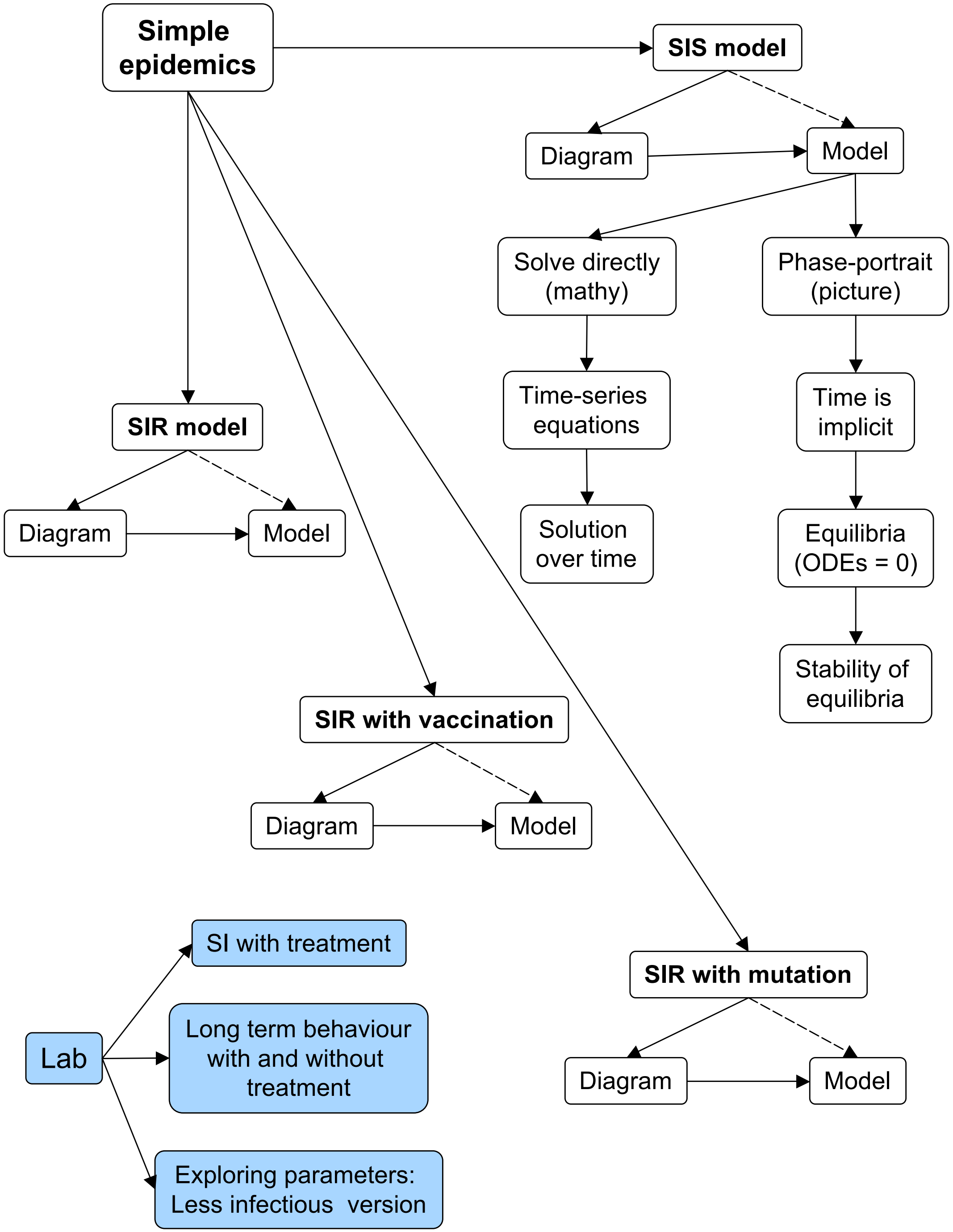


N = total pop.　　*p = b / a*
b = recovery rate　　*a = infection rate*

Stability implications

- When $p=b/a>N$, the recovery rate is high, so infecteds recover quickly and the population moves to a population of susceptibles
- When $p=b/a<N$, the infection rate is high and the infection stabilises at an endemic equilibrium.

N = total pop.
b = recovery rate
a = infection rate



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SIR epidemics

- Susceptible→Infected→Removed
- Removed can be recovered, immune, or dead.

a = infection rate
b = recovery rate

SIR equations

- Becoming infected depends on contact between Susceptibles and Infecteds *(aSI)*
- Recovery is at a constant rate, proportional to number of Infecteds *(b).*

$$\frac{dS}{dt} = -aSI$$

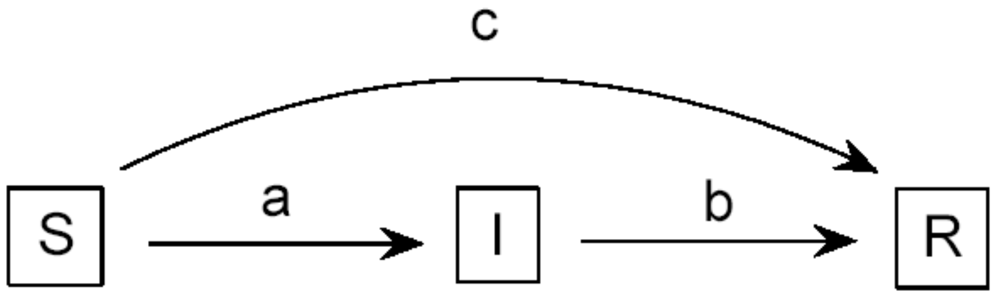
$$\frac{dI}{dt} = aSI - bI$$

$$\frac{dR}{dt} = bI$$

a = infection rate
b = recovery rate

SIR with vaccination

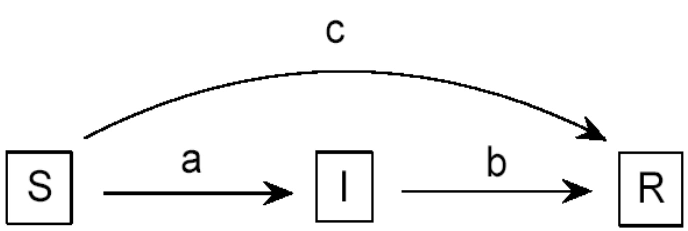
- A vaccine sends some Susceptibles directly to the Recovered (immune) state
- *N=S+I+R.*



S = Susceptible *I = Infected*
R = Recovered *a = infection rate*
b = recovery rate *c = vaccination rate*

Vaccination equations

- Vaccination is assumed to be a fixed number of shots per time period (*c*).

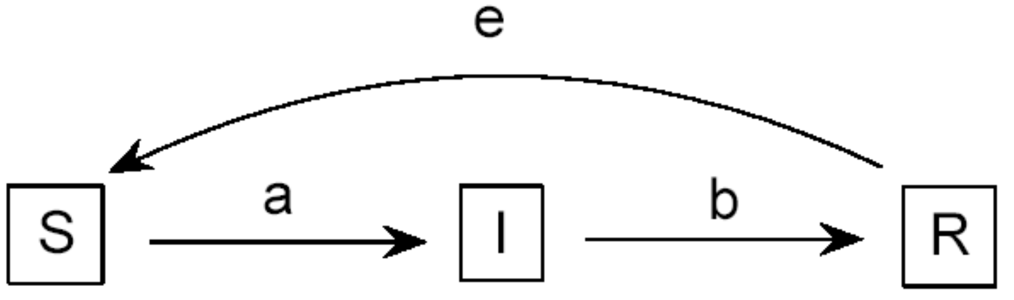


$$\frac{dS}{dt} = -aSI - c$$
$$\frac{dI}{dt} = aSI - bI$$
$$\frac{dR}{dt} = bI + c$$

S = Susceptible
I = Infected
R = Recovered
a = infection rate
b = recovery rate
c = vaccination rate

SIR with mutation

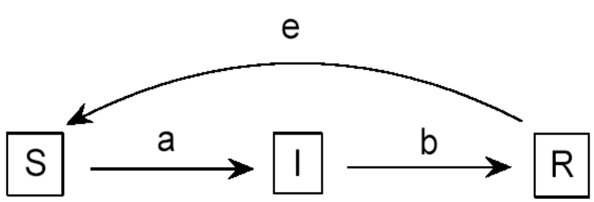
- If the virus mutates, Recovereds lose their immunity.



S = Susceptible *I = Infected*
R = Recovered *a = infection rate*
b = recovery rate *e = mutation rate*

Mutation equations

- A time-delay *T* allows a 'grace period' before people are susceptible again
- They become susceptible at a rate *(e)* depending on their status at time *t-T*.



$$\frac{dS(t)}{dt} = -aS(t)I(t) + eR(t-T)$$

$$\frac{dI(t)}{dt} = aS(t)I(t) - bI(t)$$

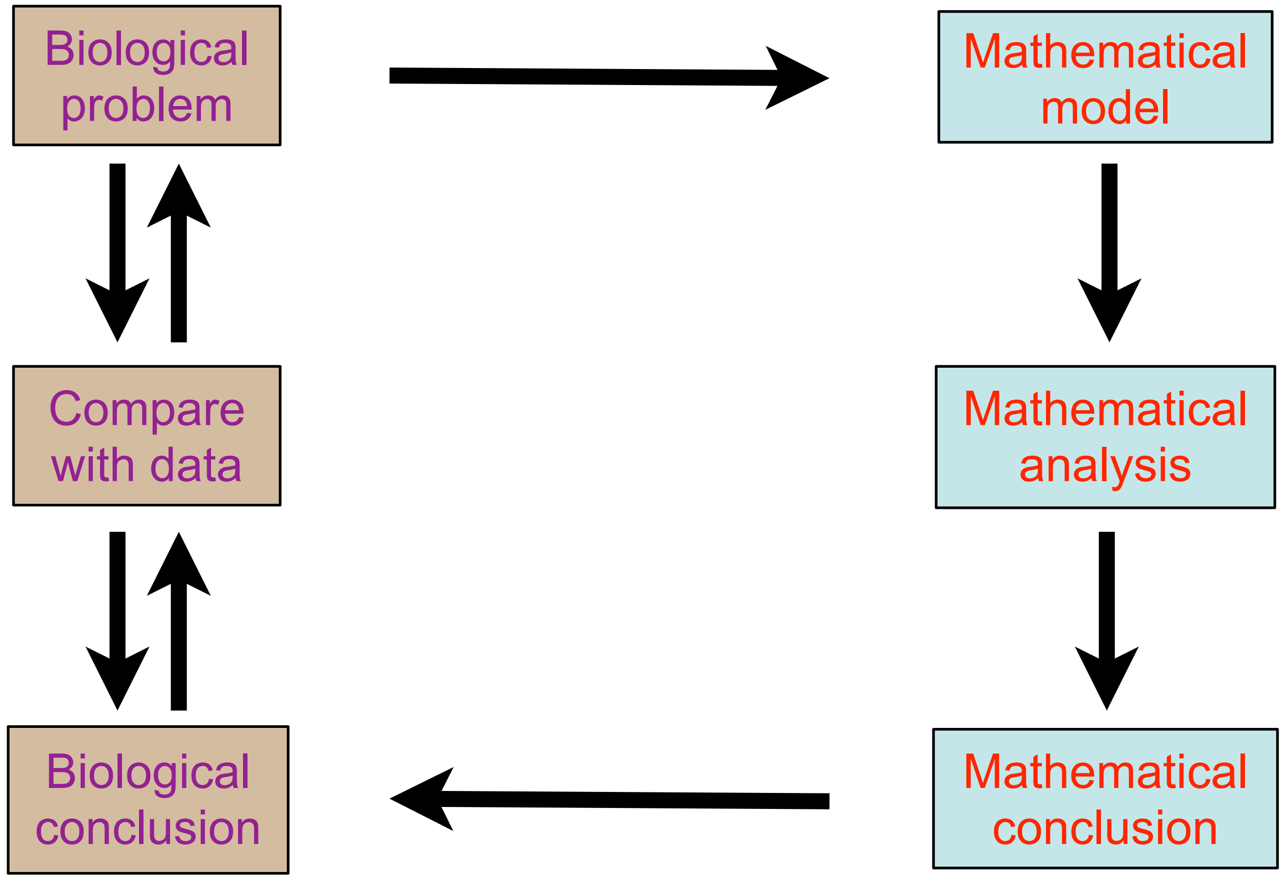
$$\frac{dR(t)}{dt} = bI(t) - eR(t-T)$$

S = Susceptible
I = Infected
R = Recovered
a = infection rate
b = recovery rate

Delay Differential Equations

- These are called delay-differential equations
- They are harder to analyse than ordinary differential equations, but are often more realistic.

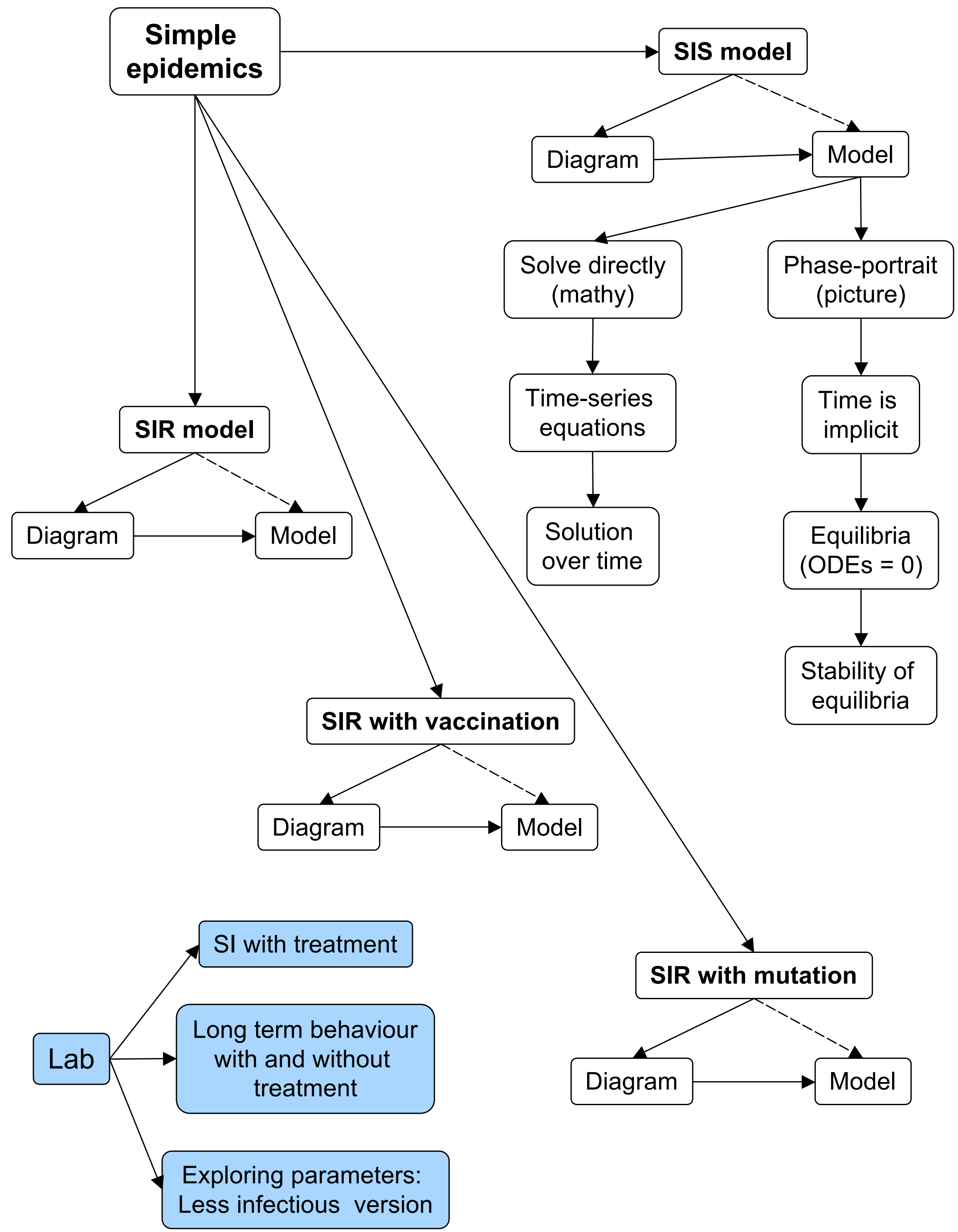
Using math to solve real problems



Summary

- From simple assumptions, we can make models that might be simple, or might be complicated
- Mathematical modelling is like map-making
- We need to decide which factors are important and which we can safely ignore

"All models are wrong... but some are useful"
- George Box.



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