Contact Geometry of the Visual Cortex

Matilde Marcolli and Doris Tsao

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Geometry of Neuroscience

References for this lecture

- Jean Petitot, *Neurogéométrie de la Vision*, Les Éditions de l'École Polytechnique, 2008
- John B. Entyre, *Introductory Lectures on Contact Geometry*, arXiv:math/0111118
- W.C. Hoffman, *The visual cortex is a contact bundle*, Applied Mathematics and Computation, 32 (1989) 137–167.
- O. Ben-Shahar, S. Zucker, *Geometrical Computations Explain Projection Patterns of Long-Range Horizontal Connections in Visual Cortex*, Neural Computation, 16, 3 (2004) 445–476
- Alessandro Sarti, Giovanna Citti, Jean Petitot, *Functional geometry of the horizontal connectivity in the primary visual cortex*, Journal of Physiology - Paris 103 (2009) 3–45

Columnar Structure

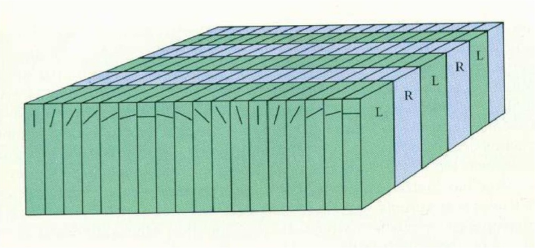
• another type of geometric structure present in visual cortex V1

• Hubel–Wiesel: columnar structures in V1: neurons sensitive to orientation record data (z, ℓ)

- z = a position on the retina
- ℓ = a line in the plane

• local product structure

$$\pi : \mathcal{R} \times \mathbb{P}^1 \twoheadrightarrow \mathcal{R}$$



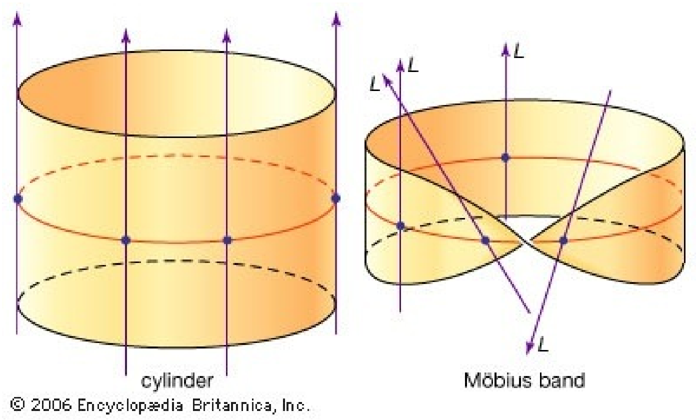
Fiber bundles

• topological space (or smooth differentiable manifold) E with base B and fiber F with

- surjection $\pi : E \twoheadrightarrow B$
- fibers $E_x = \pi^{-1}(x) \simeq F$ for all $x \in B$
- open covering $\mathcal{U} = \{U_\alpha\}$ of B such that $\pi^{-1}(U_\alpha) \simeq U_\alpha \times F$ with π restricted to $\pi^{-1}(U_\alpha)$ projection $(x, s) \mapsto x$ on $U_\alpha \times F$

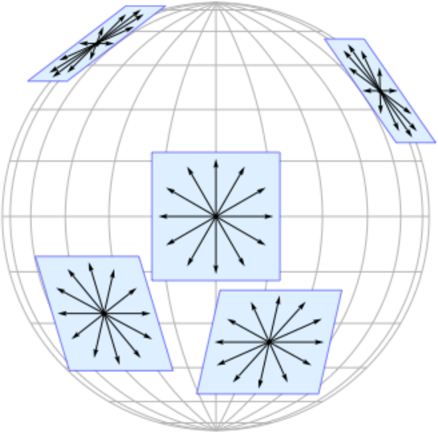
• sections $s : B \to E$ with $\pi \circ s = id$; locally on U_α

$$s|_{U_\alpha}(x) = (x, s_\alpha(x)), \quad \text{with } s_\alpha : U_\alpha \to F$$



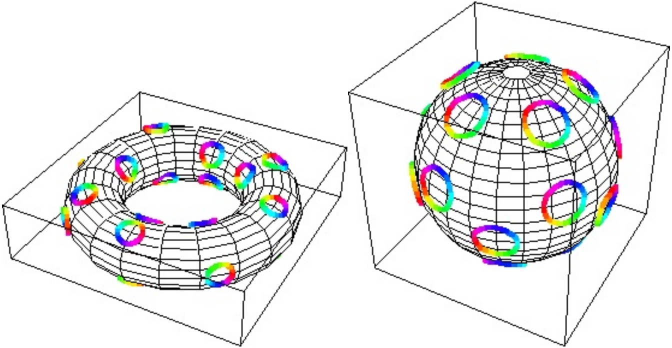
trivial and nontrivial $\mathbb{R}$-bundles over S^1

Tangent bundle TM of a smooth manifold M



Tangent bundle on a 2-sphere

- model of V1: bundle $\mathcal{E}$ with base $\mathcal{R}$ the retinal surface, fiber $\mathbb{P}^1$ the set of lines in the plane
- topologically $\mathbb{P}^1(\mathbb{R}) = S^1$ (circle) so locally V1 product $\mathbb{R}^2 \times S^1$
- circle bundle over a 2-dimensional surface



- We will see this leads to a geometric models of V1 based on Contact Geometry

Contact Geometry on 3-dimensional manifolds

- plane field ξ on 3-manifold M: subbundle of tangent bundle TM such that $\xi_x = T_x M \cap \xi$ is 2-dimensional subspace for all $x \in M$
- Example: $M = \Sigma \times S^1$ product of a 2-dimensional surface Σ and a circle S^1, then $\xi_{(x,\theta)} = T_x\Sigma \subset T_{(x,\theta)}M$ is a plane field
- real 1-form α on M determines at each point $x \in M$ a linear map

$$\alpha_x : T_x M \to \mathbb{R}$$

Kernel $\ker(\alpha_x)$ is either a plane or all of $T_x M$
if $\ker(\alpha_x) \neq T_x M$ for all $x \in M$ then $\xi = \ker(\alpha)$ is a plane field

- all plane fields locally given by $\xi = \ker(\alpha)$ for some 1-form α
- Example: $M = \Sigma \times S^1$ as above: $\xi = \ker(\alpha)$ with $\alpha = d\theta$

• plane field $\xi = \ker(\alpha)$ on 3-manifold M is contact structure iff

$$\alpha \wedge d\alpha \neq 0$$

equivalent condition $d\alpha|_{\xi} \neq 0$

• Standard Example: $M = \mathbb{R}^3$ with

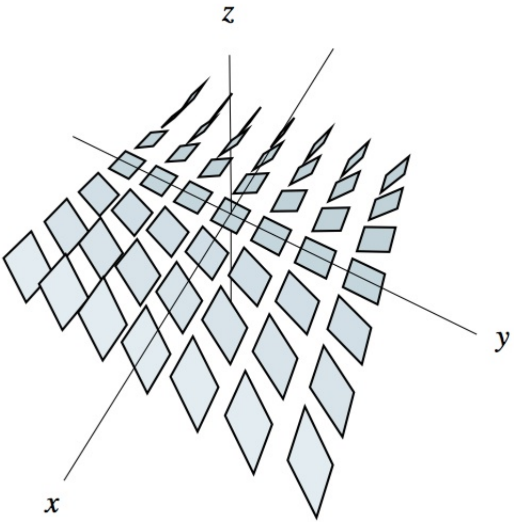
$$\alpha = dz + xdy$$

so $d\alpha = dx \wedge dy$ and $\alpha \wedge d\alpha = dz \wedge dx \wedge dy \neq 0$

• at a point (x, y, z) contact plane $\xi_{(x,y,z)}$ spanned by basis

$$\{\frac{\partial}{\partial x}, x\frac{\partial}{\partial z} - \frac{\partial}{\partial y}\}$$

• geometry of contact plane field ξ: when $x = 0$ (yz-plane) contact plane horizontal; at $(1, 0, 0)$ spanned by $\frac{\partial}{\partial x}, \frac{\partial}{\partial z} - \frac{\partial}{\partial y}$, tangent to x-axis, but tilted 45% clockwise, etc. start at origin and move along x-axis, plane keeps twisting clockwise



the standard contact structure on $\mathbb{R}^3$

Darboux's Theorem

- locally all contact structures look like the standard one
- (M, ξ) and (N, η) contact 3-manifolds, contactomorphism is diffeomorphism $f : M \to N$ such that $f_*(\xi) = \eta$; in terms of 1-forms $f^*(\alpha_\eta) = h\,\alpha_\xi$ for some non-zero $h : M \to \mathbb{R}$
- (M, ξ) contact 3-manifold, point $x \in M$, there are neighborhoods $\mathcal{N}$ of x and $\mathcal{U}$ of $(0,0,0)$ in $\mathbb{R}^3$ and contactomorphism

$$f : (\mathcal{N}, \xi|_{\mathcal{N}}) \to (\mathcal{U}, \xi_0|_{\mathcal{U}})$$

with ξ_0 the standard contact structure on $\mathbb{R}^3$

Example: contact structure on sphere S^3

- $f(x_1, y_1, x_2, y_2) = x_1^2 + y_1^2 + x_2^2 + y_2^2$ with $S^3 = f^{-1}(1) \subset \mathbb{R}^4$
- tangent spaces $T_{(x_1,y_1,x_2,y_2)} S^3 = \ker df_{(x_1,y_1,x_2,y_2)} = \ker(2x_1 dx_1 + 2y_1 dy_1 + 2x_2 dx_2 + 2y_2 dy_2)$
- identify $\mathbb{R}^4 = \mathbb{C}^2$ with complex structure $Jx_i = y_i$ and $Jy_i = -x_i$

$$J \frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}, \qquad J \frac{\partial}{\partial y_i} = -\frac{\partial}{\partial x_i}$$

- contact structure on S^3

$$\alpha = (x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2)|_{S^3}$$

$$\alpha \wedge d\alpha \neq 0$$

$\xi = \ker(\alpha)$ contact planes

- contact planes $\xi = \ker(\alpha)$ on S^3 are set of complex tangencies

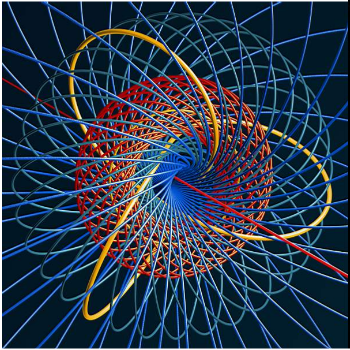
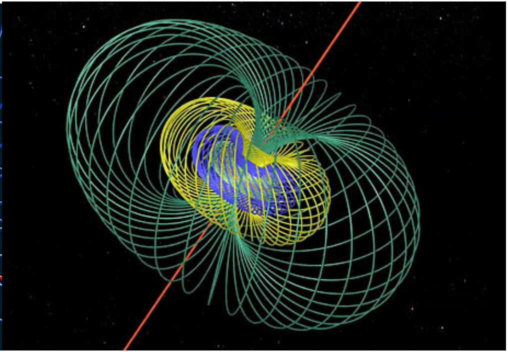
$$\xi = T_{(x_1,y_1,x_2,y_2)} S^3 \cap J(T_{(x_1,y_1,x_2,y_2)} S^3)$$

- 1-form α and complex structure:

$$\alpha = (df \circ J)|_{S^3}$$

- plane field $\xi = \ker(\alpha)$ orthogonal to the Hopf vector field

$$\dot{x}_1 = -y_1, \quad \dot{y}_1 = x_1, \quad \dot{x}_2 = -y_2, \quad \dot{y}_2 = x_2$$

Hopf vector field and Hopf fibration of S^3

Contact Structures and Complex Manifolds

• X complex manifold $\dim_{\mathbb{C}}(X) = 2$ with boundary ∂X, with $\dim_{\mathbb{R}} \partial X = 3$, and complex structure J on TX; function ϕ near boundary with $\partial X = \phi^{-1}(0)$ (collar neighborhood of boundary)

• complex tangencies

$$\ker(d\phi \circ J)$$

contact structure iff $d(d\phi \circ J)$ non-degenerate 2-form on planes ξ

• contact structure is fillable if obtained in this way

• Lutz–Martinet theorem: all 3-manifolds admit a contact structure (not always fillable)

Contact Geometry and Symplectic Geometry

• X real 4-dimensional manifold (or more generally even dimensional); symplectic structure on X: closed 2-form ω such that $\omega \wedge \omega \neq 0$ (or in dimension $2n$ form $\wedge^n \omega \neq 0$)

• Darboux's Theorem for symplectic forms: locally $\omega = dp \wedge dq$ (like a cotangent bundle)

• (X, ω) symplectic filling of contact 3-manifold (M, ξ) if $\partial X = M$ and $\omega|_\xi \neq 0$ area form on contact planes

• fillability by complex manifold special case: $\omega = d(d\phi \circ J)$ is symplectic

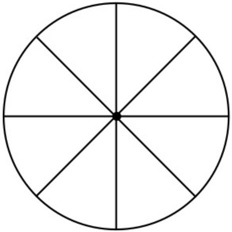
• not all contact structures are fillable by symplectic structures: if a contact structure is symplectically fillable then it is tight

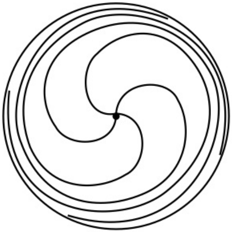
[Note: can always extend to symplectic on cylinder $X = M \times \mathbb{R}$ with $\omega = d\alpha + \alpha \wedge dt$ but not $M = \partial X$]

Tight and Overtwisted Contact Structures

• characteristic foliation: embedded oriented surface Σ in contact 3-manifold (M, ξ), lines $\ell_x = \xi_x \cap T_x\Sigma$ except at singular points where intersection is all $T_x\Sigma$; obtain foliation $\mathcal{F}_{\xi,\Sigma}$ of Σ with singular points

• overtwisted contact structure if $\exists$ embedded disk D with characteristic foliation $\mathcal{F}_{\xi,D}$ homeomorphic to either

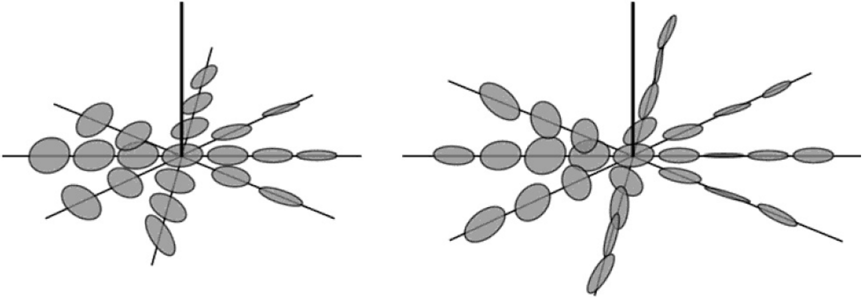




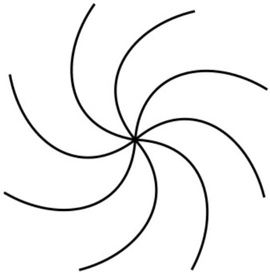
• tight contact structure: contains no overtwisted disk

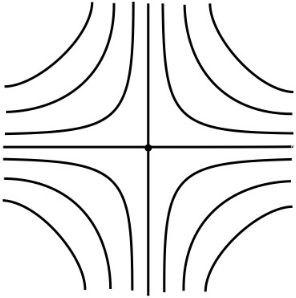
Examples

- tight: standard polar coordinates (r, θ, z) contact structure $\xi = \ker(dz + r^2 d\theta)$
- overtwisted: $\xi = \ker(\cos(r)\, dz + r \sin(r) d\theta)$, the overtwisted property sees the fact that contact planes $dz/d\theta = -r \tan(r)$ become vertical and twist over periodically (fig on the right)
- overtwisted disk $\{z = r^2 \,:\, 0 \leq r \leq \pi/2\}$



Generic singularities of the characteristic foliation





Some facts about contact structures and 3-manifolds

(Eliashberg, Gromov, Entyre, Honda, Bennequin, etc.)

- All 3-manifolds admit contact structures
- Some 3-manifolds do not admit any tight contact structure (though most of them do)
- If a contact structure is symplectically fillable then it is tight
- contact plane field ξ has an Euler class $e(\xi) \in H^2(M, \mathbb{Z})$: if tight then genus bound

$$|e(\xi)[\Sigma]| \leq -\chi(\Sigma)$$

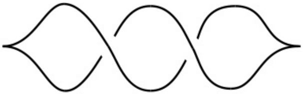
  if $\Sigma \neq S^2$ and zero otherwise (key idea: express in terms of singular points of the characteristic foliation, Poincaré–Hopf)

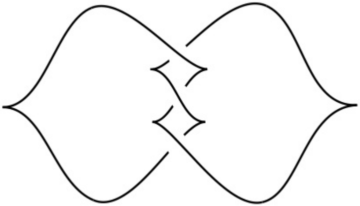
Legendrian knots

• knots $S^1 \hookrightarrow M$ in contact 3-manifold (M, ξ) such that curve always tangent to contact planes ξ

• every knot in a contact manifold can be continuously approximated by a Legendrian knot

• in standard contact structure in $\mathbb{R}^3$ with $\xi = \ker(dz + xdy)$ front projection (in yz-plane) looks like these



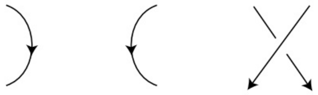


• invariants of Legendrian knots used to study contact manifolds (see Bennequin invariants, etc.)

Transverse knots

• knots $S^1 \hookrightarrow M$ in contact 3-manifold (M, ξ) such that curve always *transverse* to the contact planes ξ

• for standard contact structure projections of transverse knots in the xz-planes cannot have segments like

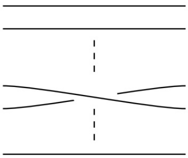


because $z'(t) - y(t)x'(t) > 0$ along a tranverse knot and vertical tangency would have $x' = 0$ and $z' < 0$, while second case $y(t)$ bounded by slope $z'(t)/x'(t)$ in xz-plane

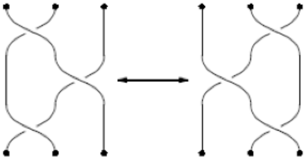
• any transverse knot in the standard contact structure is transversely isotopic to a closed braid

Braids: braid group

$$B_n = \langle \sigma_1, \ldots, \sigma_n \,|\, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i, \ |i-j| \geq 2 \rangle$$



A generator σ_i for the braid group B_n



braid group relations

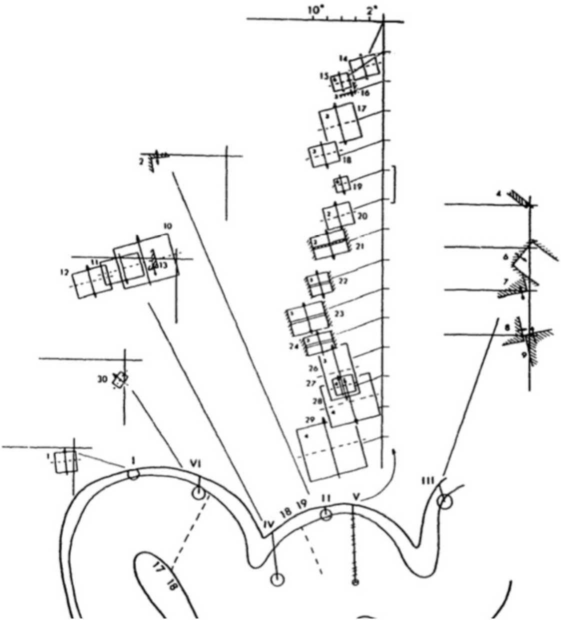
Visual Cortex as Contact Bundle

• W.C. Hoffman, *The visual cortex is a contact bundle*, Applied Mathematics and Computation, 32 (1989) 137–167

• Hubel–Wiesel microcolumns in columnar structure of V1 cortex exhibit both directional and areal response: model directional-areal response fields as contact planes directions

• "orientation response" refers to directionally sensitive response field of a single cortical neuron

• microelectrodes penetration measurements of directional and area response of neurons in the cat visual cortex show contact planes (Hubel, Wiesel)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I
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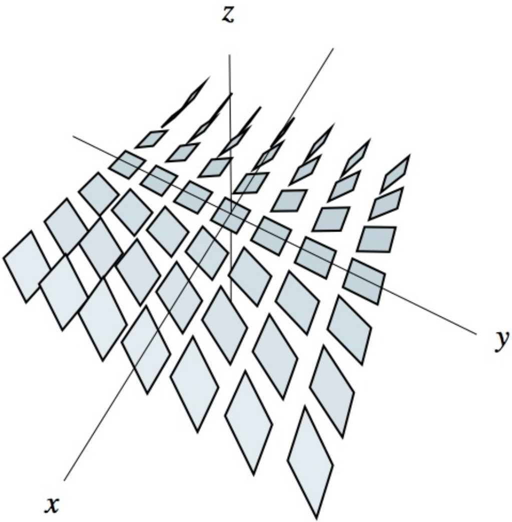
z
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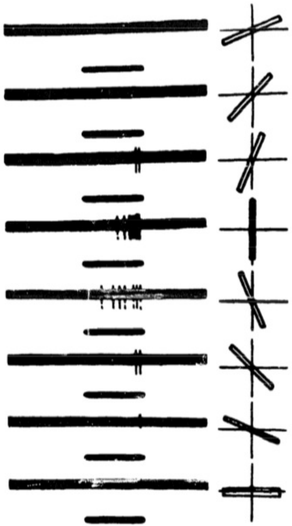
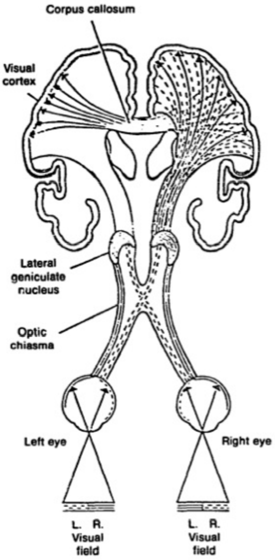


Fig. 8. A typical orientation response field (ORF) in the visual cortex: the neuronal firing rate response to shining a rectangular $1^\circ \times 8^\circ$ slit of light on the receptive field of a neuron whose "orientation" (i.e., directional) response is maximal in the vertical direction.

Visual Pathways



visual pathways from the retina to the visual cortex

Visual pathways and Connections on Fiber Bundles

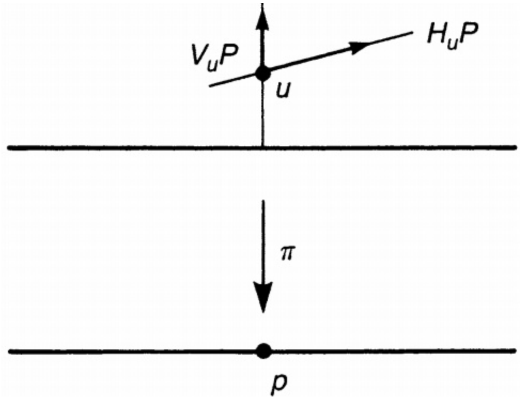
• *paths* (visual contours) are *lifted* along visual pathways from the retina to the visual cortex

• patterns of "constancies" are detected (shape, size, motion, color, etc.), then higher forms (areas 18 and 19 of the human visual cortex)

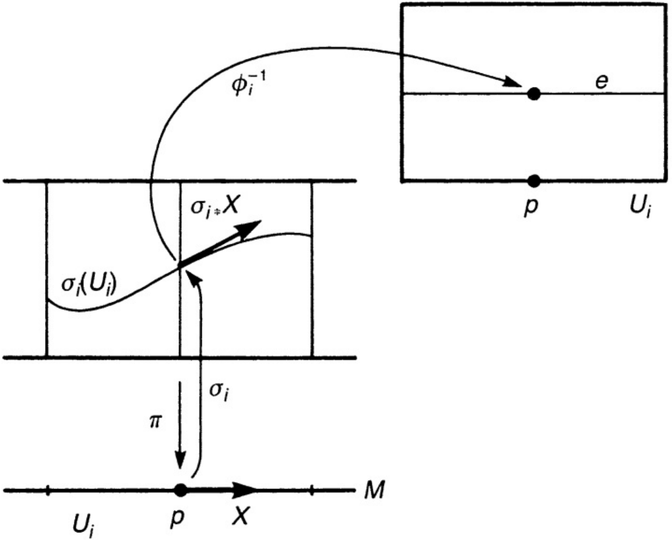
• *path lifting property* (from retina to cortex); geometrically path lifting from base $\mathcal{R}$ to total space of fibration $\mathcal{F}$ with fiber $\mathbb{P}^1(\mathbb{R})$

$$\mathbb{P}^1(\mathbb{R}) \hookrightarrow \mathcal{F} \stackrel{\pi}{\twoheadrightarrow} \mathcal{R}$$

• lifting a path along projection of a fibration: need to choose a horizontal direction at each point in the total space of the fibration (there is always a well defined vertical direction): a connection determines the choice of a horizontal direction

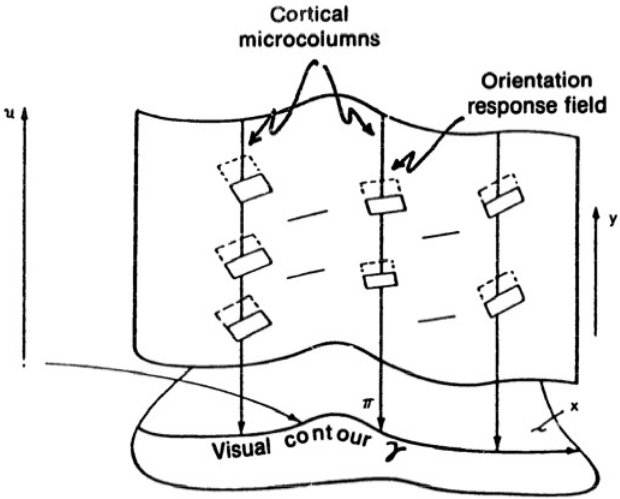


horizontal and vertical subspaces in the tangent space of a fibration



trivialization defined by local sections

(from Nakahara, *Geometry, Topology, and Physics*, CRC Press, 2003)



path lifting to the visual cortex (Hoffman)

Connection 1-form and Contact Planes

• connections and 1-forms: view a connection as a splitting of exact sequence

$$T\mathbb{P}^1 \to T\mathcal{F} \xrightarrow{\pi_*} T\mathcal{R}$$

of tangent spaces of fibration: choice of horizontal direction at each point; achieved by a 1-form α (scalar valued because circle bundle $\mathbb{P}^1(\mathbb{R}) \simeq S^1$) while vertical direction is $V = \ker(\pi_*)$

• Geometric Model: orientation response fields (ORFs) are contact planes $\xi = \ker(\alpha)$ determined by the connection 1-form α that performs the path lifting from the retina to the visual cortex

Question

- when lifting a path from retina to visual cortex get a path everywhere transversal to contact planes
- lift of a closed path in general not a closed path: endpoints lie on the same fiber of the fibration, but not necessarily the same point
- if obtain closed path, this can be knotted in the contact 3-manifold (transverse knot)
- when does this happen? what is the significance of knottedness? role of transverse and Legendrian knots in the visual cortex contact bundle?

Horizontal Connectivity in the Primary Visual Cortex

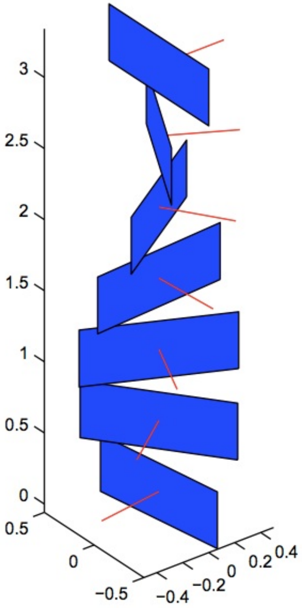
- Alessandro Sarti, Giovanna Citti, Jean Petitot, *Functional geometry of the horizontal connectivity in the primary visual cortex*, Journal of Physiology - Paris 103 (2009) 3–45
- on product $\mathcal{F} = \mathcal{R} \times \mathbb{P}^1(\mathbb{R})$ where $\mathcal{R} \simeq \mathbb{R}^2$ coordinates (x, y) and $\mathbb{P}^1(\mathbb{R}) \simeq S^1$ coordinate θ

$$\alpha = -\sin(\theta)dx + \cos(\theta)dy$$

is a contact form

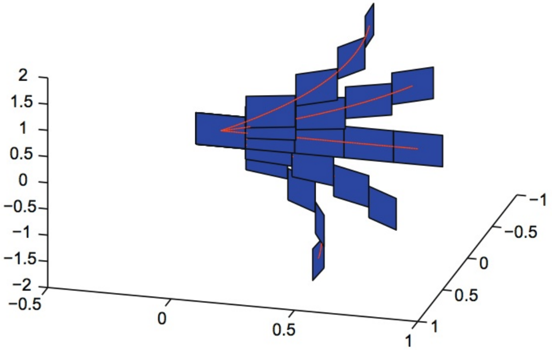
$$d\alpha = (\cos(\theta)dx + \sin(\theta)dy) \wedge d\theta, \quad \alpha \wedge d\alpha = -dx \wedge dy \wedge d\theta \neq 0$$

- contact planes spanned by $(\cos(\theta), \sin(\theta), 0)$ and $(0, 0, 1)$



The contact planes at every point, and the orthogonal vector X_3.

• the 1-form α relates local tangent vectors (in lift of retinal image) and forms integral curves, either along contact planes (Legendrian) or transverse: mechanism responsible for creating regular and illusory contours



integral curves along the contact planes

Scale Variable

• an additional scale variable $\sigma \in \mathbb{R}_+$: think of the visual field information recorded in the lift to the visual cortex not as a delta function but as a smeared distribution with Gaussian parameter σ (Gabor frames)

• when $\sigma \to 0$ recover geometric picture described above with integral curves

• geometric space $\mathcal{X} = \mathbb{R}^2 \times S^1 \times \mathbb{R}_+$, coordinates (x, y, θ, σ)

• 2-form on $\mathcal{X}$: scale $\alpha \mapsto \sigma^{-1}\alpha$

$$\omega = d(\sigma^{-1}\alpha) = \sigma^{-1} d\alpha + \sigma^{-2}\alpha \wedge d\sigma$$

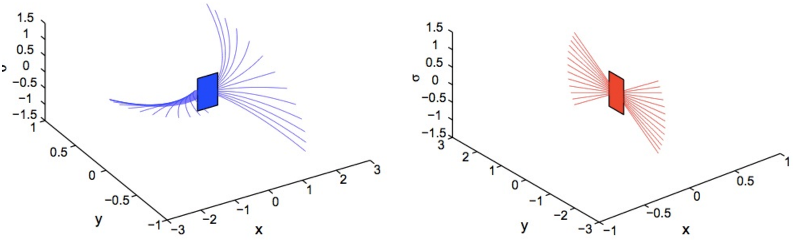
symplectic $\omega \wedge \omega = 2\sigma^{-3} d\alpha \wedge \alpha \wedge d\sigma = 2\sigma^{-3} dx \wedge dy \wedge d\theta \wedge d\sigma$

• not symplectically filling: blowing up at $\sigma \to 0$, don't have $\omega|_\xi$ at boundary, but $d\alpha + \alpha \wedge d\sigma$ would be

• $\omega = \sigma^{-1}\omega_1 \wedge \omega_2 + \sigma^{-2}\omega_3 \wedge \omega_4$ with ω_i 1-form dual to vector field X_i, corresponding vector fields

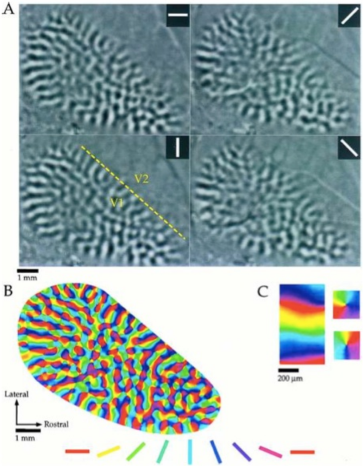
$$X_1 = \cos(\theta)\partial_x + \sin(\theta)\partial_y, \qquad X_2 = \partial_\theta,$$
$$X_3 = -\sin(\theta)\partial_x + \cos(\theta)\partial_y, \quad X_4 = \partial_\sigma$$

• for small σ predominant X_1X_2 contact planes; for large σ predominant X_3X_4-planes

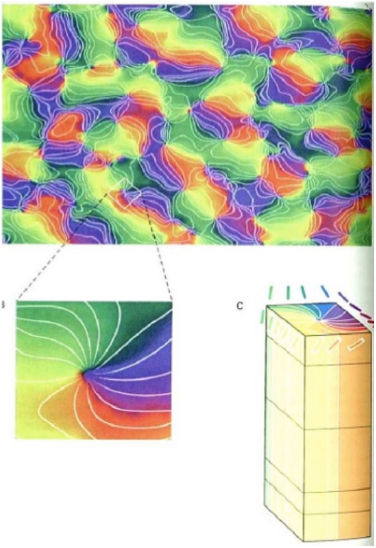


integral curves in the X_1X_2-planes and in the X_3X_4-planes

Pinwheel Structure in the Visual Cortex



V1 cortex of tupaya tree shrew: different orientations coded by colors
zoom in on regular and singular points (Petitot)



isoorientation (isochromatic) lines in the V1 cortex (Petitot)

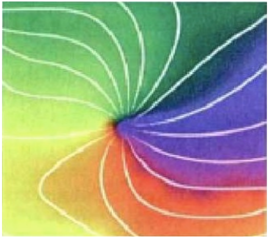
• given a section $\sigma : \mathcal{R} \to \mathcal{F}$ of the fibration

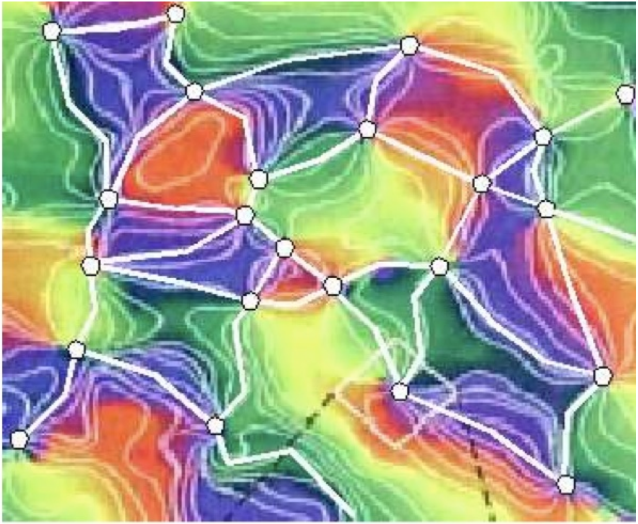
$$\mathbb{P}^1(\mathbb{R}) \hookrightarrow \mathcal{F} \stackrel{\pi}{\twoheadrightarrow} \mathcal{R}$$

determines a surface $\Sigma = \sigma(\mathcal{R}) \subset \mathcal{F}$

• isoorientation curves are canonical foliation $\ell_x = \xi_x \cap T_x\Sigma$ for this surface

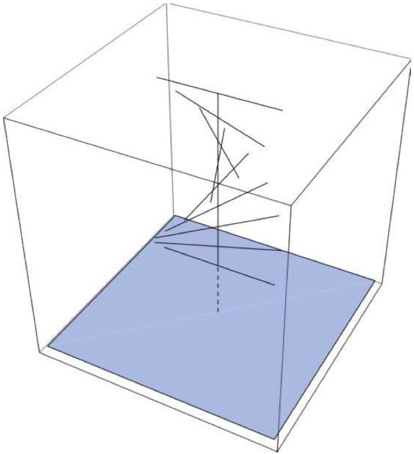
• pinwheels in Σ are overtwisted disks on the canonical foliation





networks of pinwheels (Petitot)

- projected down to $\mathcal{R}$ with $\pi : \mathcal{F} \to \mathcal{R}$ have network of pinwheels on $\mathcal{R}$ via $\pi \circ \sigma = 1$ identification of Σ and $\mathcal{R}$
- fiber over each pinwheel point is $\mathbb{P}^1(\mathbb{R})$
- can view these fibers as (real) *blowup* of $\mathcal{R}$ at pinwheel points

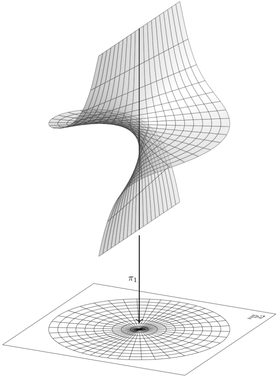


real blowup of $\mathbb{R}^2$ at a point (Petitot)

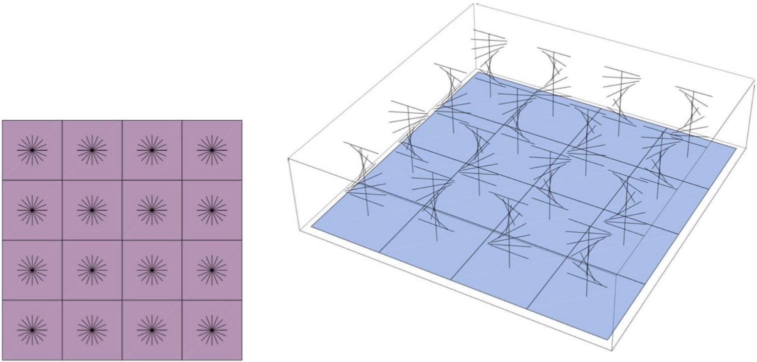
$$\mathrm{Bl}_p\mathbb{A}^2 = \{(x,y),[z:w] \,|\, xz + yw = 0\} \subset \mathbb{A}^2 \times \mathbb{P}^1$$

$$\mathrm{Bl}_p\mathbb{A}^2 = \{(q,\ell) \,|\, p,q \in \ell\}$$

for $p \neq q$ projection $\pi_1 : \mathrm{Bl}_p\mathbb{A}^2 \to \mathbb{A}^2$, $(q,\ell) \mapsto q$ isomorphism, because unique line ℓ through p and q, but over $p = q$ fiber is $\mathbb{P}^1$ set of all lines ℓ

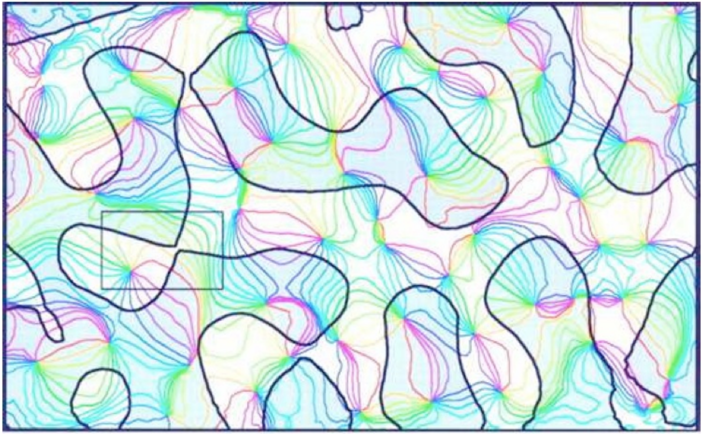


real blowup of $\mathbb{R}^2$ at a point (image by Charles Staats)



pinwheels in the base $\mathcal{R}$ and fibers (Petitot)

Observed relation between pinwheel structure and *ocular dominance domains*



pinwheels cut boundaries of ocular dominance domains transversely and nearly orthogonally (Petitot)