Engineering Sciences 242r: Beams, Plates and Shells

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Contents

NONLINEAR CURVED BEAM THEORY

Nonlinear strain-displacement relations (exact)

A curved beam, or rod, is a one dimensional entity in the following formulation. Exact strain-displacement relations will be derived and then these will be approximated in several ways as appropriate for specific applications. The process and the several types of approximations illustrate parallel aspects of plate and shell theories. The centerline (later identified with the location of the neutral bending axis) in the undeformed and deformed states is depicted in Fig. A1. Let s be the distance measured along the centerline in the undeformed state and $\bar{s}$ the distance in the deformed state.

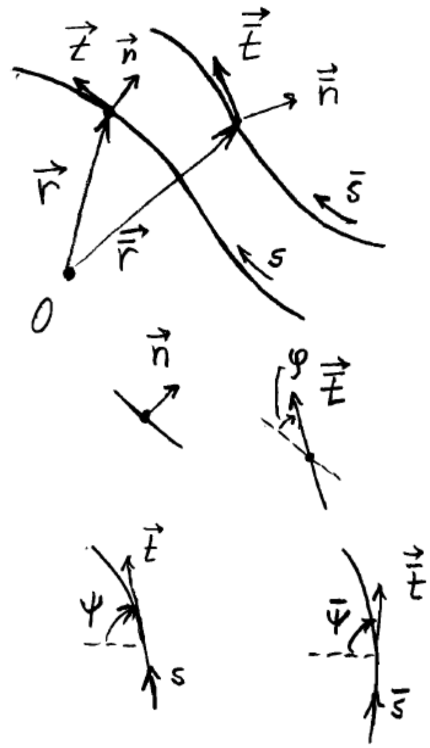


Fig. A1

With reference to Fig. A1, use the over-bar to denote quantities in the deformed state. The *displacements*, $w(s)$ and $v(s)$, are normal and tangent to the undeformed beam and are regarded as functions of distance along the undeformed beam:

$$\vec{\bar{r}} = \vec{r} + w(s)\vec{n} + v(s)\vec{t}, \qquad \vec{t} = \frac{d\vec{r}}{ds}, \quad \vec{n}\bullet\vec{t} = 0, \ \vec{n} \text{ is "on the right"}$$

Define the *rotation* φ as

$$\sin\varphi = \vec{\bar{t}}\bullet\vec{n} = -\vec{\bar{n}}\bullet\vec{t} \qquad (\cos\varphi = \vec{t}\bullet\vec{\bar{t}} = \vec{n}\bullet\vec{\bar{n}})$$

$$\vec{\bar{t}} = \frac{d\vec{\bar{r}}}{d\bar{s}} = \frac{d\vec{\bar{r}}}{ds}\frac{ds}{d\bar{s}}, \quad \frac{d\vec{\bar{r}}}{ds} = \vec{t}\left(1 + \frac{dv}{ds} + \frac{w}{R}\right) + \vec{n}\left(\frac{dw}{ds} - \frac{v}{R}\right) \equiv \vec{t}\,(1+e) + \vec{n}\beta$$

where the *radius of curvature* of the undeformed centerline is $\dfrac{1}{R(s)} = -\dfrac{d\psi}{ds}$,

$$\frac{d\vec{t}}{ds} = -\frac{1}{R}\vec{n}, \quad \frac{d\vec{n}}{ds} = \frac{1}{R}\vec{t}$$

and

$$e=\frac{dv}{ds}+\frac{w}{R},\ \ \beta=\frac{dw}{ds}-\frac{v}{R}$$

Note that

$$\sin\varphi=\vec{\bar{t}}\bullet\vec{n}=\beta\frac{ds}{d\bar{s}}$$

Define the *stretching strain* as

$$\varepsilon=\frac{d\bar{s}}{ds}-1=\sqrt{\frac{d\vec{\bar{r}}}{ds}\bullet\frac{d\vec{\bar{r}}}{ds}}-1=\sqrt{\left[\left(1+e\right)^2+\beta^2\right]}-1$$

The change in curvature $\Delta\kappa$ is

$$\Delta\kappa=\frac{1}{R}-\frac{1}{\bar{R}}\ \text{ with }\ \frac{1}{R}=-\frac{d\psi}{ds},\ \ \frac{1}{\bar{R}}=-\frac{d\bar{\psi}}{d\bar{s}}=-\frac{d\psi}{d\bar{s}}-\frac{d\varphi}{d\bar{s}}=-\frac{ds}{d\bar{s}}\left(\frac{d\psi}{ds}+\frac{d\varphi}{ds}\right)$$

Thus,

$$\Delta\kappa=\frac{ds}{d\bar{s}}\left(\frac{d\psi}{ds}+\frac{d\varphi}{ds}\right)-\frac{d\psi}{ds}=\left(1+\varepsilon\right)^{-1}\left[\frac{d\varphi}{ds}-\varepsilon\frac{d\psi}{ds}\right]=\left(1+\varepsilon\right)^{-1}\left[\frac{d\varphi}{ds}+\frac{\varepsilon}{R}\right]$$

This completes the nonlinear strain-displacement relations, except it is useful to define an alternative stretching strain measure, η, analogous to the Lagrangian measure in 3D elasticity:

$$\eta\equiv\frac{1}{2}\left[\left(\frac{d\bar{s}}{ds}\right)^2-1\right]=\frac{1}{2}\varepsilon(2+\varepsilon)$$

$$\eta=\frac{1}{2}\left[\frac{d\vec{\bar{r}}}{ds}\bullet\frac{d\vec{\bar{r}}}{ds}-1\right]=e+\frac{1}{2}e^2+\frac{1}{2}\beta^2$$

A) Summary of exact equations for finite strain and finite rotation

With $\quad e=\frac{dv}{ds}+\frac{w}{R},\ \ \beta=\frac{dw}{ds}-\frac{v}{R}$,

$$\sin\varphi=\frac{\beta}{1+\varepsilon},\quad \varepsilon=\frac{d\bar{s}}{ds}-1,\quad \eta=\varepsilon(1+\frac{1}{2}\varepsilon)=e+\frac{1}{2}e^2+\frac{1}{2}\beta^2,\quad \Delta\kappa=\frac{1}{1+\varepsilon}\left[\frac{d\varphi}{ds}+\frac{\varepsilon}{R}\right]$$

B) Inextensional strains and finite rotations (exact inextensional theory)

This theory reduces to the elastica for initially straight members. With e and β defined in A),

$$\varepsilon=0\Rightarrow\ \ \eta=0\ \Rightarrow\ \sin\varphi=\beta,\quad e+\frac{1}{2}e^2+\frac{1}{2}\beta^2=0,\quad \Delta\kappa=\frac{d\varphi}{ds}$$

C) Small strains and finite rotations

Assume $|\varepsilon| << 1$, which implies that $|\eta| << 1$ and $\varepsilon \cong \eta$. With e and β defined as above, use $|\varepsilon| << 1$ to obtain the following approximations:

$$\sin\varphi = \beta, \ \ \varepsilon = e + \frac{1}{2}e^2 + \frac{1}{2}\beta^2, \ \ \Delta\kappa = \left[\frac{d\varphi}{ds} + \frac{\varepsilon}{R}\right]$$

D) Small strains and moderate rotations

This is an important class of theories because it allows rigorous nonlinear buckling analyses for linear elastic beams, columns and rings. Moreover, some of the most widely used plate (von Karman theory) and shell theories (DMV theory) are derived under the assumption of small strains and moderate rotations. As in C), assume $|\varepsilon| << 1$. For moderate rotations we require $\varphi^2 << 1$, which in turn implies that $\beta^2 << 1$. (Note that this is less restrictive than $|\varphi| << 1$.) These assumptions imply, $e^2 << |e|$ and $\varphi \cong \beta$. Thus, the equations for this theory are

$$e = \frac{dv}{ds} + \frac{w}{R}, \ \ \varphi = \frac{dw}{ds} - \frac{v}{R}, \ \ \varepsilon = e + \frac{1}{2}\varphi^2, \quad \Delta\kappa = \left[\frac{d\varphi}{ds} + \frac{\varepsilon}{R}\right]$$

E) Small strains and small rotations (linear theory)

The linearized set of equations from A) or, equivalently, from D) are

$$\varepsilon = e = \frac{dv}{ds} + \frac{w}{R}, \quad \varphi = \frac{dw}{ds} - \frac{v}{R}, \quad \Delta\kappa = \left[\frac{d\varphi}{ds} + \frac{\varepsilon}{R}\right]$$

Homework Problem1: Vanishing of the strain measures under rigid body displacements

(i) For a rigid body translation, $\bar{\vec{r}} = \vec{r} + \vec{U}$, where $\vec{U}$ is a constant vector in the plane, show that $\varepsilon = 0$ and $\Delta\kappa$ =0 for all the five cases above.

(ii) Consider a rigid rotation about O in Fig. A1, $\bar{r}_i = \ell_{ij} r_j = r_i + (\ell_{ij} - \delta_{ji}) r_j$, where θ is the rotation about the normal to the plane and the non-zero ℓ_{ij} are $\ell_{11} = \cos\theta$, $\ell_{12} = -\sin\theta$, $\ell_{21} = \sin\theta$ and $\ell_{22} = \cos\theta$. Show that $\varepsilon = 0$ and $\Delta\kappa$ =0 for A, B and C above. For D & E, determine the dependence of ε and $\Delta\kappa$ on small θ, i.e. determine the lowest order dependence on small θ.

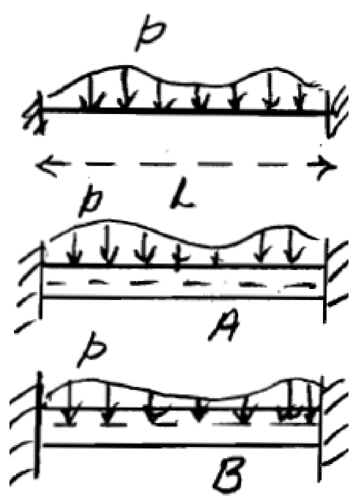
First order constitutive relations for small strain, linear elastic behavior (no restrictions on rotations other than the strains due to bending are small)

There are many possibilities that could be considered for constitutive laws. For example, if one were interested in applications to rubber materials that undergo large strains one would have to consider nonlinear finite strain constitutive models. Where the bending moment (or moment/length), M, and the stretching force (or force/length), F, are related to the two strain quantities through and energy density function, $\Phi(\Delta\kappa,\varepsilon)$ by

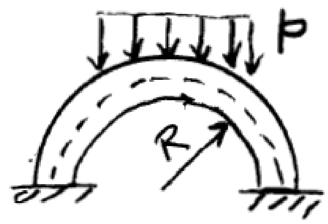
$$M=\frac{\partial\Phi}{\partial\Delta\kappa},\; F=\frac{\partial\Phi}{\partial\varepsilon}$$

However, almost exclusively in these notes, we will be concerned with applications relevant to structural materials which by their nature have strains due to stretching and bending that are small if linear elastic material is assumed, which it will. (It should be noted, however, that the strain-displacement relations and the equilibrium equations derived later are applicable when plastic straining occurs, but we will not consider plasticity in these notes.)

Digression: constitutive laws for first order beam theories



As background consider the engineering theory for straight wide plates and beams which is a first order theory. For wide plates, the constitutive relation is $M=B\Delta\kappa$ with $B=\bar{E}I$ where $\bar{E}=E/(1-\nu^2)$, $I=h^3/12$ and h is the thickness. The ode governing a uniform plate subject to a transverse load/length, $p(s)$, is $Bd^4w/d^4s=p$. This is a first order theory based on the deflection of the centerline. Consider a clamped beam of length L shown in the figure for the beam at the top and for a 2D plane strain elasticity problem at the bottom. Beam theory treats p as being applied along the centerline, and it cannot distinguish between loads applied along the top of the beam or along the centerline, for example, as in the case of the 2D problem. With energy/length $\Phi_A=M_A\Delta\kappa_A/2$ and $\Phi_B=M_B\Delta\kappa_B/2$ of the two 2D problems, typically, $(\Phi_A-\Phi_B)/\Phi_A=O(h/L)$. That is, depending on details that beam theory cannot capture, there are inherent relative differences on the order of h/L. With $\Phi=M\Delta\kappa/2$ from beam theory, it is also true that $(\Phi_A-\Phi)/\Phi_A=O(h/L)$.



For a curved beam another length enters, R --see example for a circular ring in the figure. The load induces both a change in curvature, $\Delta\kappa$, and a stretching strain, ε. Let $\Phi = \bar{E}I\Delta\kappa^2/2 + \bar{E}h\varepsilon^2/2$ be a measure of the energy/length. Plane strain, 2D elasticity solutions for annular regions such as that to the right differ by relative order h/R for problems that would be modeled identically by curved beam theory, e.g. the pressure acting on top vs. on the bottom of the beam. That is, for two such problems, A & B, $(\Phi_A - \Phi_B)/\Phi_A = O(h/R)$. This is an inherent error expected of curved beam theory. We will make good use of this result.

End of digression

For problems limited to small strains and linear stress-strain response, let B and S be the bending and stretching stiffness of the beam, respectively. These may be a function os s. (For the wide plate $B = Eh^3/[12(1-\nu^2)]$ and $S = Eh/(1-\nu^2)$.) Recall the standard constitutive equation for a *straight beam or rod* with uniform isotropic elastic properties across its cross-section under combined bending and stretching:

$$\Phi_A = \frac{1}{2}B\Delta\kappa^2 + \frac{1}{2}S\,\varepsilon^2; \quad M = \frac{\partial\Phi_A}{\partial\Delta\kappa} = B\Delta\kappa, \quad F = \frac{\partial\Phi_A}{\partial\varepsilon} = S\,\varepsilon$$

For plane strain deformations of a curved or flat plate, M and F are the moment and force per length perpendicular to the plane. *The constitutive relation above coincides exactly with the results from 3D linear elasticity for pure bending and stretching of a uniform flat plane in plane strain.* Moreover, the variation of the strain component parallel to the middle surface through the thickness of the plate is

$$\varepsilon_{11} = \varepsilon - y\Delta\kappa$$

where y is the distance from the middle surface.

One possibility for curved beam theory is to employ Φ_A as the energy/length. However, consider the following alternative. Instead of using $(\Delta\kappa, \varepsilon)$ as the pair of strain measures, use (K, ε) where $K = \Delta\kappa - \varepsilon/R$. (Knowledge of (K, ε) provides $(\Delta\kappa, \varepsilon)$ and vise versa, and the difference between the two sets vanishes for a flat plate.) The advantage of the alternative bending strain measure K is the simplified bending strain relation:

$$K = \frac{d\varphi}{ds}$$

Now, what if we were to use the following constitutive relation?

$$\Phi_B = \frac{1}{2} B K^2 + \frac{1}{2} S\, \varepsilon^2; \quad M = \frac{\partial \Phi_B}{\partial K} = B K, \quad F = \frac{\partial \Phi_B}{\partial \varepsilon} = S\, \varepsilon$$

For flat plates this is the same as that based on Φ_A. It is important to realize that a curved beam is, in fact, a 2D or 3D entity although we will model it as being 1D. The first order theory we are in the process of deriving is a 1D theory based solely on the deformation of the neutral bending axis. An important argument of W.T. Koiter that carries over to shell theory is that for a *first order curved beam or plate theory* (i.e. a theory that is valid to lowest order in h/R and/or h/L), either Φ_A or Φ_B may be used because the error intrinsic to any first order theory is of the order Φ_A-Φ_B. We will use Φ_B since it results in "nicer" equations, but solutions based on Φ_A would be equally valid. The analog will emerge in first order shell theory.

The details of Koiter's argument for curved plates are as follows. With $B = k^2 S$ where k is a radius of gyration and with $\varepsilon_b = k\Delta\kappa$ as a measure of the strain due to bending at a distance k from the neutral axis, then one can easily show that

$$\Phi_A = \frac{1}{2} S\left(\varepsilon_b{}^2 + \varepsilon^2\right) \text{ and } \Phi_B = \frac{1}{2} S\left(\varepsilon_b{}^2 + \varepsilon^2 - 2\varepsilon_b \varepsilon \frac{k}{R} + \varepsilon^2 \left(\frac{k}{R}\right)^2\right),$$

such that the relative difference between the two is

$$\frac{\Phi_A - \Phi_B}{\Phi_A} = \frac{k}{R}\left[\frac{2\varepsilon_b \varepsilon + \varepsilon^2 (k/R)}{\varepsilon_b{}^2 + \varepsilon^2}\right] = O\left(\frac{k}{R}\right)$$

Exact solutions from 2D plane strain elasticity for thin circular annular regions subject to various loadings show that differences of this order cannot be avoided if one considers the whole range of problems of interest for a first order theory (see the digression). That is, a specific problem might be more accurately represented by one or the other of these two constitute models, but if one considers all possible problems one is as accurate as the other and we are free to choose the one we like. The choice based on $\Phi \equiv \Phi_B$ and K gives the "nicest" set of equations, and this will be our choice.

Principle of Virtual Work (PVW) and Equilibrium Equations

We will illustrate the PVW and derive equilibrium equations for theory D for small strains and moderate rotations which has

$$\varepsilon = e + \frac{1}{2}\varphi^2, \quad K = \frac{d\varphi}{ds}, \quad e = \frac{dv}{ds} + \frac{w}{R}, \quad \varphi = \frac{dw}{ds} - \frac{v}{R}$$

Note that the only nonlinearity is the φ^2 term in ε. (This theory reduces to the 1D version of von Karman nonlinear plate theory for wide plates.) Virtual strains and displacements are related by

$$\delta\varepsilon = \delta e + \varphi\delta\varphi, \quad \delta K = \frac{d\delta\varphi}{ds}, \delta e = \frac{d\delta v}{ds} + \frac{\delta w}{R}, \quad \delta\varphi = \frac{d\delta w}{ds} - \frac{\delta v}{R}$$

Define the Principle of Virtual Work (PVW) in terms of the internal virtual work (IVW) and external virtual work (EVW) for a curved beam extending from 0 to L as

$$IVW \equiv \int_0^L \left(M\delta K + F\delta\varepsilon\right)ds$$

$$EVW \equiv \int_0^L \left(p_n\delta w + p_t\delta v\right)ds + \left[P_n\delta w + m\delta\varphi + P_t\delta v\right]_0^L$$

The PVW requires IVW=EVW to hold for all admissible virtual displacements δw and δv. The following illustrates the standard process for generating equilibrium equations and boundary conditions

$$IVW = \int_0^L \left(M\frac{d}{ds}\left(\frac{d\delta w}{ds} - \frac{\delta v}{R}\right) + F\left(\frac{d\delta v}{ds} + \frac{\delta w}{R} + \varphi\left(\frac{d\delta w}{ds} - \frac{\delta v}{R}\right)\right)\right)ds$$

Integrate by parts to obtain

$$IVW = \int_0^L \left[\left(\frac{d^2M}{d^2s} + \frac{F}{R} - \frac{d(F\varphi)}{ds}\right)\delta w + \left(\frac{1}{R}\frac{dM}{ds} - \frac{dF}{ds} - \frac{F\varphi}{R}\right)\delta v\right]ds$$

$$+\left[\left[M\frac{d\delta w}{ds} + \left(-\frac{dM}{ds} + F\varphi\right)\delta w + \left(-\frac{M}{R} + F\right)\delta v\right]\right]_0^L$$

Enforcing IVW=EVW for all admissible δw and δv gives the equilibrium equations

$$\frac{d^2M}{d^2s} + \frac{F}{R} - \frac{d(F\varphi)}{ds} = p_n \quad \text{and} \quad \frac{1}{R}\frac{dM}{ds} - \frac{dF}{ds} - \frac{F\varphi}{R} = p_t$$

and the conditions at the ends of the interval

$$\left[M\frac{d\delta w}{ds} + \left(-\frac{dM}{ds} + F\varphi\right)\delta w + \left(-\frac{M}{R} + F\right)\delta v\right]_0^L = \left[P_n\delta w + m\frac{d\delta w}{ds} + (P_t - \frac{m}{R})\delta v\right]_0^L$$

Thus, at either of the end the boundary conditions involve the specification of

$$M = m \quad \text{or} \quad \frac{dw}{ds} \quad ; \text{and} \quad -\frac{dM}{ds} + F\varphi = P_n \quad \text{or} \quad w; \quad \text{and} \quad \frac{M}{R} + F = \frac{m}{R} + P_t \quad \text{or} \quad v.$$

The distributed loads we have defined are defined as load/length of the undeformed beam and they act in the directions defined by the undeformed normal and tangent vectors, $\vec{n}$ and $\vec{t}$. They are called *dead loads*. A pressure loading, is an example of a *live load*, and it is defined a load/length in the deformed state and acts parallel to $\vec{\bar{n}}$. We will treat pressure loads later.

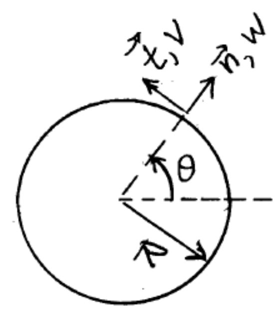
Small strain-moderate rotation equations for circular rings

With reference to the figure, let $s = R\theta$ where R is the radius of the ring and θ is the angular measure in the undeformed state measure counter clockwise. For a linear material, $M = BK$ and $F = S\varepsilon$. Let $(\ )' \equiv d(\ )/d\theta$ and note that $\varphi = (w' - v)/R$ and $e = (v' + w)/R$. The equilibrium equations expressed in terms of e and φ are

$$B\varphi''' + SR^2\left(e + \tfrac{1}{2}\varphi^2\right) - SR^2\left[\left(e + \tfrac{1}{2}\varphi^2\right)\varphi\right]' = R^3 p_n$$

$$B\varphi'' - SR^2\left(e + \tfrac{1}{2}\varphi^2\right)' - SR^2\left(e + \tfrac{1}{2}\varphi^2\right)\varphi = R^3 p_t$$

Boundary conditions depend on whether the ring is complete and how it is supported. We will use this set of equations to investigate buckling of a ring under dead pressure load.



These equations also apply to the dynamic behavior of a circular ring if one invokes D'Alembert's Principle and takes $p_n = -\rho\ddot{w}$ and $p_t = -\rho\ddot{v}$ where $(\dot{\ }) \equiv \partial(\ )/\partial t$ and ρ is the mass/unit length of the undeformed ring.

Linear vibrations of a circular ring

Linearize the above equations to obtain

$$B\varphi''' + SR^2 e = -\rho R^3\ddot{w} \quad \text{and} \quad B\varphi'' - SR^2 e' = -\rho R^3\ddot{v}$$

noting that e and φ depending linearly on $w(\theta, t)$ and $v(\theta, t)$.

Homework Problem #2: Vibration frequencies and modes of a complete ring

Consider solutions to the above equations of the form

$w = w_0 \cos n\theta \sin \omega t$ and $v = v_0 \sin n\theta \sin \omega t$

where $n = 0,1,2,3,....$ and ω is the unknown vibration frequency which will depend on n. Show that this is an eigenvalue problem and show that ω satisfies

$$\bar{\omega}^4 - \bar{\omega}^2 \left[n^4 + n^2 \left(1 + (R/r)^2\right) + (R/r)^2 \right] + n^2 \left(n^2 - 1\right)^2 (R/r)^2 = 0$$

where $\bar{\omega} = \sqrt{\dfrac{\rho R^4}{B}}\omega$ and $r = \sqrt{\dfrac{B}{S}}$ (for a uniform beam of thickness h, $r = h/(2\sqrt{3})$ and is called the radius of gyration of the section). Note that, in general, there are two frequencies for each n. Compute all the vibration modes and frequencies for n =0 to 10 for the case $r/R = 20$. Comment on the single mode for n =0. Comment on the mode for n =1 (be alert for a rigid body mode). Sketch the two modes for n =2 and comment on why the frequency of one of the modes is so much higher than the other. **Hint:** In carrying out this problem you will probably find it useful to calculate the relative amplitudes of the eigenmodes both for, w_0 and v_0, and for e_0 and φ_0.

Classical buckling of circular ring under uniform radial pressure (dead pressure)

The ring is subject to a uniform dead pressure loading with $p_n \equiv -p$ and $p_t = 0$. The ring and the loading are axisymmetric and it is easy to see that the following simple solution exactly satisfies the nonlinear coupled equations for small strains and moderate rotations:

$$w = w_0 = -\frac{pR^2}{S}, \; v = 0, \; e = e_0 = \frac{w_0}{R}, \;\; \varphi = 0$$

We conduct a buckling analysis analogous to the "classical" bucking analysis of a straight column. The question asked is the following: "Is there a critical value of pressure, denoted by p_C, at which a solution emerges other than the simple axisymmetric solution given above.

The following is a bifurcation analysis. For specified p, perturb w and v about $w = w_0$ and $v = 0$ with μ as the perturbation parameter. With

$$w = w_0 + \mu w_1 + \ldots, \quad v = \mu v_1 + \ldots,$$

one finds

$$\varphi = \mu\varphi_1 + \ldots = \mu\left(w_1' - v_1\right)/R + \ldots, \quad e = e_0 + \mu e_1 + \ldots = e_0 + \mu\left(v_1' + w_1\right)/R + \ldots$$

Substitute into the full nonlinear equations and linearize with respect to μ:

$$B\varphi_1''' + SR^2 e_1 + pR^3\varphi_1' = 0$$

$$B\varphi_1'' - SR^2 e_1' + pR^3\varphi_1 = 0$$

Eliminate e_1 from the above two equations to obtain

$$B\left(\varphi_1'''' + \varphi_1''\right) + pR^3\left(\varphi_1'' + \varphi_1\right) = 0$$

This is an eigenvalue problem with p as the eigenvalue. Look for eigenmodes of the form: $\varphi_1 = \sin n\theta$ (or $\varphi_1 = \cos n\theta$) for $n = 1, 2, 3, \ldots$ to find the eigenvalues:

$$p_n = \frac{B}{R^3} n^2$$

Note that $e_1' = (R/S)\left[B\varphi_1''/R^3 + p\varphi_1\right] = 0$ for all n. Solve for w_1^0 and v_1^0 where $w_1 = w_1^0 \cos n\theta$ and $v_1 = v_1^0 \sin n\theta$ given $e_1 = (v_1' + w_1)/R$ and $\varphi_1 = (w_1' - v_1)/R$ to obtain

$$v_1^0 n + w_1^0 = 0 \quad \text{and} \quad v_1^0 + w_1^0 n = -R$$

For $n = 1$ there is no solution. The lowest eigenvalue which is identified with the buckling pressure, p_C, occurs for $n = 2$ with

$$p_C \equiv p_2 = \frac{4B}{R^3}, \quad w_1^0 = -2R/3, \; v_1^0 = R/3 \qquad \textbf{\textit{(Buckling under dead pressure)}}$$

Note that $e_1 = 0$ and, thus, the buckling mode is *inextensional* to first order in μ.

Classical buckling of circular ring under uniform radial pressure (live pressure)

Under live pressure, the load/length in the deformed state is $-p\bar{\vec{n}}$ and the external virtual work is $EVW = -\int p\bar{\vec{n}} \bullet \left(\delta w\vec{n} + \delta v\vec{t}\right) d\bar{s}$. From page 2,

$$\bar{\vec{t}} = \frac{ds}{d\bar{s}}\left((1+e)\vec{t} + \beta\vec{n}\right) \text{ and, thus, } \quad \bar{\vec{n}} = \frac{ds}{d\bar{s}}\left(-\beta\vec{t} + (1+e)\vec{n}\right)$$

Now it is straightforward to see that

$$EVW = -\int p\left[(1+e)\delta w - \beta\delta v\right]ds$$

When this EVW is used in place of that used earlier for dead loads and with $\beta = \varphi$ for moderate rotations, one obtains for the circular ring under live pressure

$$B\varphi''' + SR^2\left(e + \tfrac{1}{2}\varphi^2\right) - SR^2\left[\left(e + \tfrac{1}{2}\varphi^2\right)\varphi\right]' = -R^3(1+e)p \quad \text{(replace rhs by } -R^3 p\text{)}$$

$$B\varphi'' - SR^2\left(e + \tfrac{1}{2}\varphi^2\right)' - SR^2\left(e + \tfrac{1}{2}\varphi^2\right)\varphi = R^3\varphi p$$

One additional approximation can be made consistent with the fact that we have already neglected terms like e compared to unity: replace the right hand side of the first equation by $-R^3 p$.

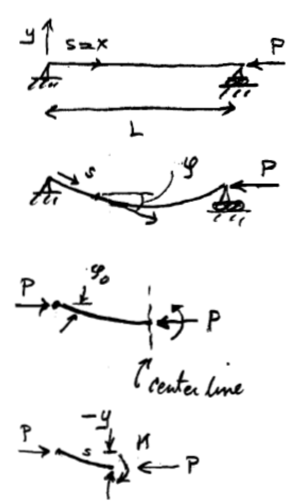
Homework problem #3: The buckling pressure for a circular ring under live pressure

(i) Show that the pre-buckling axisymmetric solution is the same as that for dead pressure.

(ii) Show that the critical buckling pressure (the lowest eigenvalue) is $p_C = 3B/R^3$ and the associated buckling mode is $\varphi = \varphi_0 \cos n\theta$ with $n = 2$.

The buckling pressure under *live pressure* is 33% lower than that under *dead pressure*.

Euler's elastica—Axial buckling of an inextensional uniform straight column



Consider the initially straight column in the figure. It is pinned at the ends ($M = 0$, $w = 0$, $v(0) = 0$) with a horizontal load P applied at the right end. The column is modeled as inextensional ($\varepsilon = 0$), and the theory B) for arbitrary large rotations is employed such that $M = Bd\varphi/ds$ where B is constant. Let $\bar{x}(s)$ and $\bar{y}(s)$ denote the location of points in the deformed state in the rectangular coordinate system shown. Note that $d\bar{x}/ds = \cos\varphi$ and $d\bar{y}/ds = -\sin\varphi$. Moment equilibrium about the left end (see figure) requires $M = \bar{y}(s)P$.

(*Comment*: Alternatively, you could obtain this equilibrium equation from the PVW.) Then note,

$$\frac{dM}{ds} = \frac{d\overline{y}}{ds} P = -\sin\varphi \, P \;\Rightarrow\; B\varphi'' + P\sin\varphi = 0 \;\; (0 \le s \le L), \;\; \varphi'(0) = \varphi'(L) = 0$$

where $(\ )' \equiv d(\ )/ds$. The equation is the same as that for finite oscillations of a pendulum if s is regarded as time. It has solutions that can be expressed in terms of elliptic functions, as will be seen below.

A first integral is readily noted:

$$B\varphi'' + P\sin\varphi = 0 \Rightarrow \left(\frac{1}{2}B\varphi'^2 - P\cos\varphi\right)' = 0 \Rightarrow \frac{1}{2}B\varphi'^2 - P\cos\varphi = C$$

Let $\varphi_0 \equiv \varphi(0)$ (which is unknown at this point). Then noting that $\varphi'(0) = 0$,

$$\frac{1}{2}B\varphi'^2 - P\cos\varphi = -P\cos\varphi_0 \Rightarrow \varphi'^2 = \left(\frac{2P}{B}\right)\left(\cos\varphi - \cos\varphi_0\right)$$
$$\Rightarrow \varphi' = \pm\sqrt{2B/P}\sqrt{\cos\varphi - \cos\varphi_0}$$

Let's look for solutions such as those depicted in the figure with $\varphi_0 > 0$ which are symmetric with respect to the center of the column such that for $0 \le s \le L/2$, $\varphi \ge 0$ and $\varphi' \le 0$ with $\varphi(L/2) = 0$. Then,

$$\sqrt{\frac{2P}{B}}\, s = \int_{\varphi}^{\varphi_0} \frac{d\varphi}{\sqrt{\cos\varphi - \cos\varphi_0}}$$

Since $\varphi(L/2) = 0$, it follows that

$$\sqrt{\frac{2P}{B}}\,\frac{L}{2} = \int_0^{\varphi_0} \frac{d\varphi}{\sqrt{\cos\varphi - \cos\varphi_0}} = \sqrt{2}E_1\left(\sin(\varphi_0/2)\right)$$

This is the relation between P and φ_0 -- E_1 is the complete elliptic integral of the first kind, but the integral is as easy to use for numerical evaluation. Note that for small φ_0,

$$\sqrt{\frac{2P}{B}}\,\frac{L}{2} \to \int_0^{\varphi_0} \frac{d\varphi}{\sqrt{(\varphi_0{}^2 - \varphi^2)/2}} = \frac{\pi}{\sqrt{2}}$$

Thus, the buckling load (the lowest load such that the column is not straight—also called the bifurcation load) is $P_C = \pi^2 B/L^2$; this is the famous Euler load.

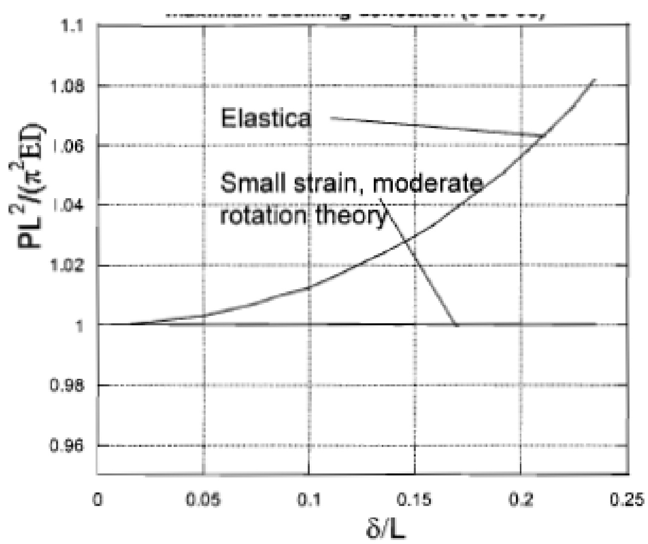
All other details of the deformed shape can be computed from the above equations. In particular, the deflection at the center of the column, $\delta \equiv -\bar{y}(L/2)$ is given by

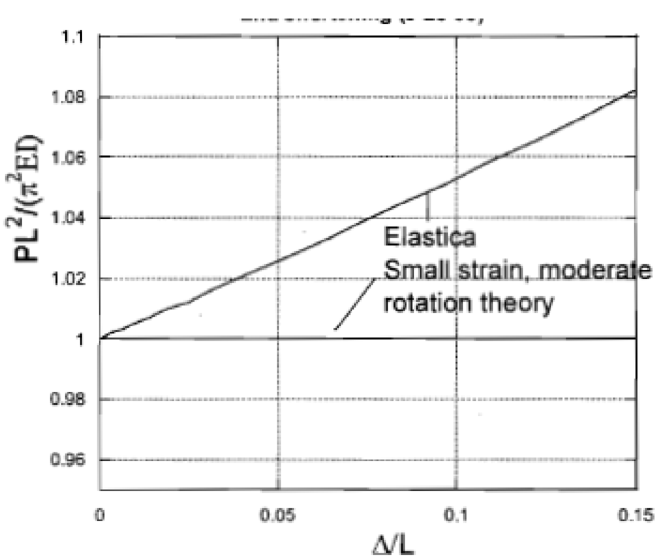
$$\frac{\delta}{L} = \frac{1}{L}\sqrt{\frac{B}{2P}}\int_0^{\varphi_0} \frac{\sin\varphi d\varphi}{\sqrt{\cos\varphi - \cos\varphi_0}}$$

And the horizontal displacement through which the end load works, $\Delta = L - \bar{x}(L)$ is

$$\frac{\Delta}{L} = 1 - \frac{2}{L}\int_0^{L/2} \frac{d\bar{x}}{ds}ds = 1 - \frac{2}{L}\sqrt{\frac{B}{2P}}\int_0^{\varphi_0} \frac{\cos\varphi d\varphi}{\sqrt{\cos\varphi - \cos\varphi_0}}$$

These relations are plotted in the following figure where they are compared with results based on small strain/moderate rotation theory (Homework Problem #4).





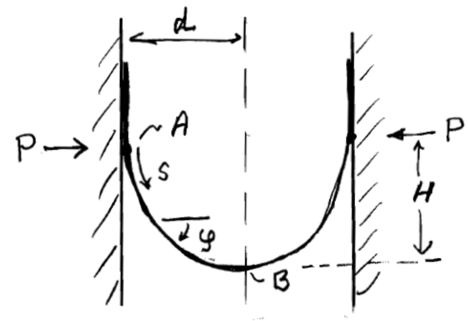
Homework Problem #4: Column buckling using small strain/moderate rotation theory

Consider the problem above within the contest of small strain/moderate rotation theory (the equations are listed on page 8 with $R = \infty$). Make no other approximations. Here are a few hints. Show that F is independent of $s = x$ and is therefore given by $F = -P$. The equation for $w(x)$ is an eigenvalue problem which gives the result $P = \pi^2 B / L^2$ for all values of δ / L where δ is the deflection at the center of the beam. Then, note that

$$\Delta \equiv -u(L) = -\int_0^L u' dx = \int_0^L \left(-\varepsilon + \tfrac{1}{2} w'^2 \right) dx$$

Determine results analogous to those in the figure above for Δ / L and δ / L. Recall that this theory is the same as von Karman plate theory for 1D problems.

Homework Problem #5: Spaepen's Elastica problem



Consider an initially straight rod that is squeezed between two platens by imposing d as shown in the figure. Equal and opposite forces P arise. Frans Spaepen used this as a test configuration to create a region of high curvature at B where a highly local material instability occurred once the curvature became large enough. The question of interest is the relation of the curvature at B, κ_B, to d. Assume finite rotation/inextentional strain theory and limit consideration to symmetric deflections about B. Note that the point of contact at A is not known in advance—its determination is part of the problem. Denote the length of the rod between A and B by $L/2$ (this is also unknown). The boundary conditions are

$$\varphi(0) = \pi / 2, \quad \varphi'(0) = 0, \quad \varphi(L/2) = 0$$

This would appear to over specify the problem, but the extra condition allows L to be determined. The condition $\varphi'(0) = 0$ follows from the fact that the rod is straight above A, and M is necessarily continuous across A (why?). Show that

$$\sqrt{\frac{P}{2B}}L = \int_0^{\pi/2}\frac{d\varphi}{\sqrt{\cos\varphi}} = 2.622\,,\quad \sqrt{\frac{2P}{B}}d = \int_0^{\pi/2}\sqrt{\cos\varphi}d\varphi = 1.198\,,$$

$$\sqrt{\frac{2P}{B}}H = \int_0^{\pi/2}\frac{\sin\varphi d\varphi}{\sqrt{\cos\varphi}} = 2\,,\quad \kappa(s) = \sqrt{\frac{2P\cos\varphi}{B}}\,,\quad \kappa_B = \sqrt{\frac{2P}{B}} = \frac{1.198}{d}$$

Homework Problem #6: Non-uniform pressure distribution acting on a circular ring

Consider an unsupported complete circular ring of radius R, uniform bending stiffness B and uniform stretching stiffness S. (As is typical for a thin ring, $B = k^2 S$ with $k/R << 1$.) The ring is loaded by a normal pressure distribution, $p_n = q\cos n\theta$ where n is an integer. Using linear ring theory, determine $w(\theta)$ and $v(\theta)$ for $n = 0, 2, 3, 4, \ldots$. Hint: guess the form for $w(\theta)$ and $v(\theta)$ with due regard for the phase.

(i) Why is there no solution for $n = 1$?

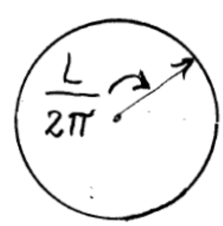
(ii) Determine the displacements $\delta_A = w(0) + w(\pi)$ and $\delta_B = w(\pi/2) + w(3\pi/2)$ for all n other than 1. Why is the result for $n = 0$ special?

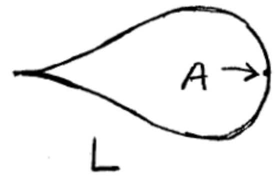
(iii) For each n, compute the bending energy, $\int_0^{2\pi}\frac{1}{2}BK^2 Rd\theta$, and the stretching energy, $\int_0^{2\pi}\frac{1}{2}S\varepsilon^2 Rd\theta$. Compare them by taking their ratio (use $B = k^2 S$) and remark on the difference between $n = 0$ and the other n.

(iv) Given you have the solution for $p_n = q_n\cos n\theta$ (and, therefore, also for $p_n = g_n \sin n\theta$), describe in words how you would produce the solution for any equilibrated normal pressure distribution, $p(\theta)$.

Homework Problem #6A—Two elastica problems

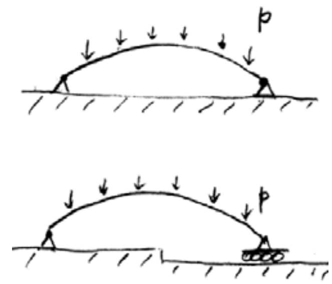
Both the deformed beams in the figure are straight with length L in the undeformed state. The top beam is bent into a circle and the ends are then welded such that the slope (and curvature) is continuous. The bottom beam is bent into the shape shown such that at the left the two ends meet with the ends tangent to each other as shown and then welded. Using the theory for the elastica, determine and compare the bending moment in the circular hoop and the bending moment at A in the loop.





Homework Problem #6B: Two circular arch problems (Messy solution!!)

The two circular arches in the figure to the right are identical. Each has modulus E, thickness t, width b perpendicular to the plane, radius of curvature R, and each has a curved length L in the undeformed state (the support span is $2R\sin(L/2R)$. Both are simply supported at each end $(M=0, w=0)$ and horizontal component of displacement of both is zero at the left end ($U_x = 0$). The upper arch has $U_x = 0$ at the right end, while the lower arch has no resistance to horizontal force at the right end $(F_x = 0)$. Using linear curved beam theory, determine the vertical deflection at the center of the beam.



(i) Show that for $L/\sqrt{Rt} << 1$, the center deflections of the two beams are essentially the same, and comment on why this is.

(ii) Show that for $L/\sqrt{Rt} >> 1$, the center deflections of the two beams are very different with the upper arch undergoing much less deflection—it is acting as an arch, while the lower beam is supporting the load by bending.