Continued Fractions

Notes for a short course at the Ithaca High School Senior Math Seminar

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Contents

1 Introduction

Continued Fractions are important in many branches of mathematics. They arise naturally in long division and in the theory of approximation to real numbers by rationals. These objects that are related to number theory help us find good approximations for real life constants.

1.1 Euclid's GCD algorithm

Given two positive integers, this algorithm computes the greatest common divisor (gcd) of the two numbers.

Algorithm: Let the two positive integers be denoted by a and b.

1. If $a < b$, swap a and b.
2. Divide a by b and find remainder r. If $r = 0$, then the gcd is b.
3. If $r \neq 0$, then set $a = b$, $b = r$ and go back to step 1.

This algorithm terminates and we end up finding the gcd of the two numbers we started with.

Example:
Take $a = 43, b = 19$.

$$\begin{aligned} 43 &= \mathbf{2} \times 19 + 5 \\ 19 &= \mathbf{3} \times 5 + 4 \\ 5 &= \mathbf{1} \times 4 + 1 \\ 4 &= \mathbf{4} \times 1 + 0 \end{aligned}$$

Hence, by Euclid's algorithm, the gcd of 43 and 19 is 1.

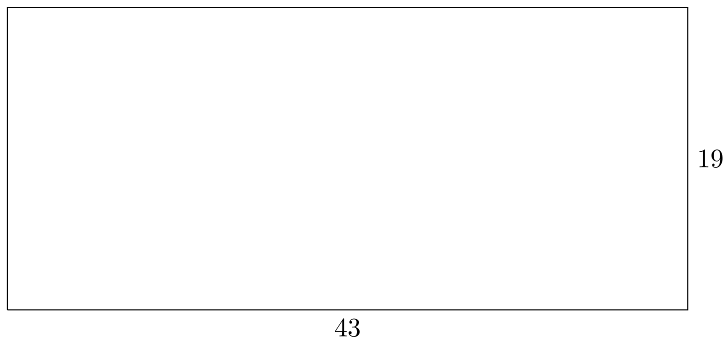
Observe that the quotient at each step of the algorithm has been highlighted. Using these numbers we can present the fraction $\frac{43}{19}$ in the following manner:

$$\frac{43}{19} = 2 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{4}}}$$

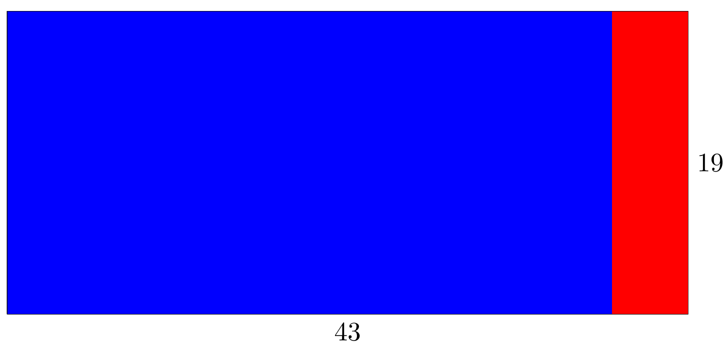
In general, it is true that given two positive integers, we can write the fraction in the above format by using the successive quotients obtained from Euclid's algorithm.

1.2 Pictorial Description

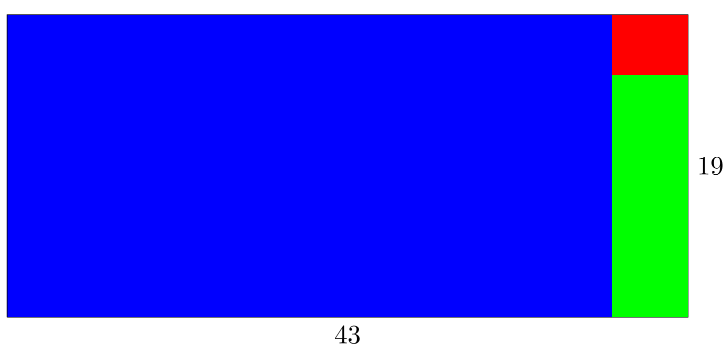
Lets look at the same example in a pictorial manner.
Consider a rectangle whose length is 43 units and whose width is 19 units.



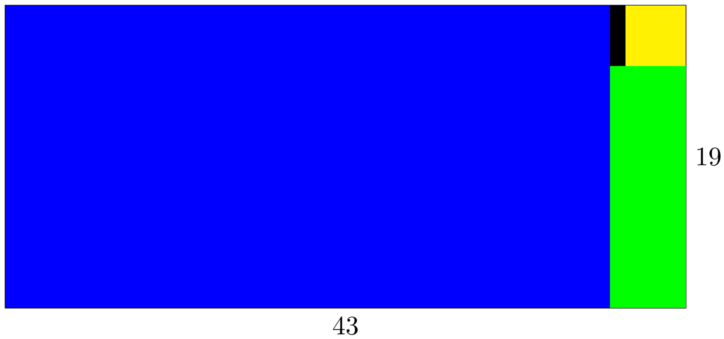
Divide it into squares of side length 19 units (coloured in blue) as shown:



We are left with a smaller rectangle of length 5 units and width 19 units (in red). Divide it further into squares of side length 5 units (in green).



This leaves us with a rectangular strip of length 5 units and width 4 units (in red). We continue this process of dividing the rectangle into squares of maximum possible side length. The number of squares in each step gives us precisely the successive quotients from the previous section.



2 (blue) squares of sidelength 19 units. **3** (green) squares of side length 5 units. **1** (yellow) square of side length 4 units. **4** (black) squares of side length 1 unit.
Thus,

$$\frac{43}{19} = 2 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{4}}}$$

1.3 Definitions

1.3.1 Simple Continued Fraction

Definition 1.1. *A **Simple Continued Fraction** is an expression of the form*

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + ...}}}$$

where a_i are non-negative integers, for $i > 0$ and a_0 can be any integer.

The above expression is cumbrous to write and is usually written in one of these two forms:

$$a_0 + \frac{1}{a_1+}\frac{1}{a_2+}\frac{1}{a_3+}$$

or using the list notation

$$[a_0, a_1, a_2, a_3, ...]$$

to mean the same thing as the continued fraction above.

Example: $\frac{43}{19} = [2, 3, 1, 4]$
In this notation, we have

$$[a_0] = \frac{a_0}{1}$$
$$[a_0, a_1] = a_0 + \frac{1}{a_1} = \frac{a_0 a_1 + 1}{a_1}$$
$$[a_0, a_1, a_2, ..., a_n] = a_0 + \frac{1}{[a_1, a_2, ..., a_n]} = [a_0, [a_1, a_2, ..., a_n]]$$

More generally, we have

$$[a_0, a_1, a_2, ..., a_n] = [a_0, a_1, ...a_{m-1}, [a_m, a_{m+1}, ..., a_n]], \text{ for } 1 \leq m \leq n$$

1.3.2 Convergents

Definition 1.2. *We call* $[a_0, ..., a_m]$ *(for* $0 \leq m \leq n$*) the* m*th* ***convergent*** *to* $[a_0, ..., a_n]$.

In our example, the convergents are

$$2 = \frac{2}{1}$$

$$2 + \frac{1}{3} = \frac{7}{3}$$

$$2 + \cfrac{1}{3 + \cfrac{1}{1}} = \frac{9}{4}$$

$$2 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{4}}} = \frac{43}{19}$$

2 Properties of Continued Fractions

2.1 Finite Continued Fractions

2.1.1 Rational Numbers

Theorem 2.1. *Every rational number has a simple continued fraction expansion which is finite and every finite simple continued fraction expansion is a rational number.*

Proof. Suppose we start with a rational number, then Euclid's algorithm terminates in finitely many steps. This is because the successive reminders are strictly decreasing as they have to be less than the respective quotients. By construction, the successive quotients in Euclid's algorithm precisely gives us a simple continued fraction expansion for the rational number we started with.
Conversely, if we have a simple finite continued fraction expansion $[a_0, a_1, ..., a_n]$, then we can inductively see that $[a_0, a_1, ..., a_n] = [a_0, [a_1, ..., a_n]] = (a_0([a_1, ..., a_n]) + 1)/[a_1, ..., a_n]$. Hence, $[a_0, ..., a_n]$ is a rational number.

Q.E.D.

This theorem now says that we can continue working with finite simple continued fractions as long as we are only working with rational numbers. Henceforth, we will work with finite simple continued fractions until section 7 where we will deal with irrational numbers.

Exercise 2.2. *(i) Find a simple continued fraction expansion of* $\frac{13}{8}$.
(ii) Compute the gcd of $(13, 8)$ *using Euclid's algorithm.*
(iii) What are its convergents?
(iv) Write the continued fraction from part (i) in list notation.

2.1.2 Inverting a Fraction

Given a non-zero rational number, we simply interchange the numerator and denominator to get its reciprocal.

For example, the reciprocal of $\frac{43}{19}$ is $\frac{19}{43}$.

Now we describe how to find the reciprocal of a rational number if it is described as a simple continued fraction:

1. If the simple continued fraction has a 0 as its first number, then remove the 0.
2. If the simple continued fraction does not have 0 as its first number, then shift all the numbers to the right and place 0 as the first entry.

Examples:

$\frac{43}{19} = [2, 3, 1, 4] \implies \frac{19}{43} = [\mathbf{0}, 2, 3, 1, 4]$

$\frac{3}{7} = [0, 2, 3] \implies \frac{7}{3} = [2, 3]$

2.2 Multiple Continued Fractions

Given a rational number, we have seen one way of constructing a simple continued fraction (namely by Euclid's algorithm). But is it the only way of getting a simple continued fraction? In this section and the next few sections we will see that there is essentially a unique way to write a rational number as a simple continued fraction.

Theorem 2.3. *If x is representable by a simple continued fraction with an odd (even) number of convergents, it is also representable by one with an even (odd) number.*

Proof. If $a_n \geq 2$,

$$[a_0, a_1, ..., a_n] = [a_0, a_1, ..., a_n - 1, 1]$$

If $a_n = 1$,

$$[a_0, a_1, ..., a_{n-1}, 1] = [a_0, a_1, ..., a_{n-1} + 1], [1] = [0, 1]$$

Q.E.D.

Thus, the proof of this theorem says that there are atleast 2 ways of writing a simple continued fraction for a rational number.

1. A simple continued fraction ending with some $m > 1$ i.e. $[...., m]$.
2. A simple continued fraction ending with 1 i.e. replace the final m by $(m - 1) + 1/1$ to get $[...., m - 1, 1]$.

Examples:

$$[1, 2, 3, 4, 5] = [1, 2, 3, 4, 4, 1]$$

$$\frac{3}{2} = 1 + \frac{1}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1}}$$

2.3 Relations between convergents

In this section, we see some properties of the simple continued fractions in terms of the numerators and denominators appearing in the convergents.

Theorem 2.4. *If p_n and q_n are defined by*
$p_0 = a_0, p_1 = a_1a_0 + 1, p_n = a_np_{n-1} + p_{n-2}$ *for* $2 \leq n$
$q_0 = 1, q_1 = a_1, q_n = a_nq_{n-1} + q_{n-2}$ *for* $2 \leq n$*,*
then

$$[a_0, a_1, ..., a_n] = \frac{p_n}{q_n}.$$

Proof. The proof proceeds by induction. The base cases are seen to be true by the assumptions given for $n = 0, n = 1$. Let us assume the statement to be true for some m. Then

$$[a_0, a_1, ...a_{m-1}, a_m] = \frac{p_m}{q_m} = \frac{a_mp_{m-1} + p_{m-2}}{a_mq_{m-1} + q_{m-2}}$$

Hence, we get

$$\begin{aligned}
[a_0, a_1, ...a_{m-1}, a_m, a_{m+1}] &= [a_0, a_1, ..., a_{m-1}, a_m + \frac{1}{a_{m+1}}] \\
&= \frac{(a_m + \frac{1}{a_{m+1}})p_{m-1} + p_{m-2}}{(a_m + \frac{1}{a_{m+1}})q_{m-1} + q_{m-2}} \\
&= \frac{a_{m+1}(a_mp_{m-1} + p_{m-2}) + p_{m-1}}{a_{m+1}(a_mq_{m-1} + q_{m-2}) + q_{m-1}} \\
&= \frac{a_{m+1}p_m + p_{m-1}}{a_{m+1}q_m + q_{m-1}} \\
&= \frac{p_{m+1}}{q_{m+1}}
\end{aligned}$$

By the principle of mathematical induction, p_n and q_n are indeed defined by the recursive relation stated in the theorem.

Q.E.D.

It follows that the nth convergent is

$$\frac{p_n}{q_n} = \frac{a_np_{n-1} + p_{n-2}}{a_nq_{n-1} + q_{n-2}}$$

Theorem 2.5. *The numbers p_n and q_n satisfy*

$$p_nq_{n-1} - p_{n-1}q_n = (-1)^{n-1}$$

Proof. From the previous theorem, we have

$$\begin{aligned}
p_nq_{n-1} - p_{n-1}q_n &= (a_np_{n-1} + p_{n-2})q_{n-1} - p_{n-1}(a_nq_{n-1} + q_{n-2}) \\
&= -(p_{n-1}q_{n-2} - p_{n-2}q_{n-1})
\end{aligned}$$

Repeating this step with $n - 1, n - 2, ..., 2$ in place of n, gives us

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}(p_1 q_0 - p_0 q_1) = (-1)^{n-1}(1) = (-1)^{n-1}$$

Q.E.D.

Example:

$$\frac{225}{157} = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{5}}}}$$

Its convergents are

$$1 = \frac{1}{1}$$

$$1 + \frac{1}{2} = \frac{3}{2}$$

$$1 + \cfrac{1}{2 + \cfrac{1}{3}} = \frac{10}{7}$$

$$1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4}}} = \frac{43}{30}$$

$$1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{5}}}} = \frac{225}{157}$$

i.e. $\frac{1}{1}, \frac{3}{2}, \frac{10}{7}, \frac{43}{30}, \frac{225}{157}$
The numerators and denominators of these convergents do satisfy

$$(3)(1) - (1)(2) = (-1)^{1-1} = 1$$
$$(10)(2) - (3)(7) = (-1)^{2-1} = -1$$
$$(43)(7) - (10)(30) = (-1)^{3-1} = 1$$
$$(225)(30) - (43)(157) = (-1)^{4-1} = -1$$

Definition 2.6. *We call*

$$a'_m = [a_m, a_{m+1}, ..., a_n]$$

the mth ***complete quotient*** *of the continued fraction*

$$[a_0, a_1, ..., a_n]$$

Let $x = [a_0, a_1, ..., a_n]$. Then

$$x = a'_0 = \frac{a'_1 a_0 + 1}{a'_1} = \frac{a'_n p_{n-1} + p_{n-2}}{a'_n q_{n-1} + q_{n-2}}$$

This follows by the exact same steps as the proof of Theorem 2.4.

Theorem 2.7. *$a_m = [a'_m]$, the integral part of a'_m, except that $a_{n-1} = [a_{n-1}] - 1$ when $a_n = 1$.*

Proof. If $n = 0$, then $a_0 = a'_0 = [a'_0]$. If $n > 0$, then

$$a'_m = a_m + \frac{1}{a'_{m+1}} \text{ for } (0 \leq m \leq n-1).$$

Now

$$a'_{m+1} > 1 \text{ for } (0 \leq m \leq n-1)$$

except that $a'_{m+1} = 1$ when $m = n-1$ and $a_n = 1$.
This is because $a_1, a_2, ..., a_n$ are all non-negative integers and inductively one can see that the above statement is true.
Hence

$$a_m < a'_m < a_m + 1 \text{ for } (0 \leq m \leq n-1)$$

and

$$a_m = [a'_m] \text{ for } (0 \leq m \leq n-1)$$

except in the case specified. And in any case

$$a_n = a'_n = [a'_n]$$

Q.E.D.

2.4 Uniqueness of Continued Fractions

In this section we use all the properties seen in the above theorems to show that under some minor conditions, every rational number has a unique finite simple continued fraction.

Theorem 2.8. *If two simple continued fractions*

$$[a_0, a_1, ..., a_n], [b_0, b_1, ..., b_N]$$

have the same value x, and $a_n > 1, b_N > 1$, then $n = N$ and the fractions are identical.

Proof. By Theorem 2.7, $a_0 = b_0 =$ integral part of x. Let us assume that the first m terms in the continued fractions are identical.Then

$$x = [a_0, a_1, ..., a_{m-1}, a'_m] = [b_0, b_1, ..., b_{m-1}, b'_m]$$

If $m = 1$, then

$$a_0 + \frac{1}{a'_1} = b_0 + \frac{1}{b'_1}$$

which implies $a'_1 = b'_1$ and by Theorem 2.7, $a_1 = b_1$. If $m > 1$, then

$$\frac{a'_m p_{m-1} + p_{m-2}}{a'_m q_{m-1} + q_{m-2}} = \frac{b'_m p_{m-1} + p_{m-2}}{b'_m q_{m-1} + q_{m-2}}$$

$(a'_m - b'_m)(p_{m-1}qm - 2 - p_{m-2}q_{m-1}) = 0$. But $(p_{m-1}qm - 2 - p_{m-2}q_{m-1}) = (-1)^m$, by Theorem 2.5 and so $a'_m = b'_m$. By Theorem 2.7, $a_m = b_m$.
Suppose now, $n \leq N$, then we have shown that $a_m = b_m \forall m \leq n$. If $N > n$, then

$$\frac{p_n}{q_n} = [a_0, a_1, ..., a_n] = [a_0, a_1, ..., a_n, b_{n+1}, b_{n+2}, ..., b_N] = \frac{b'_{n+1}p_n + p_{n-1}}{b'_{n+1}q_n + q_{n-1}}$$
$$\implies p_n q_{n-1} - p_{n-1}q_n = 0$$

which contradicts Theorem 2.5. Hence, $n = N$ and the fractions are identical.

Q.E.D.

Theorem 2.3 and Theorem 2.8 together tell us that **there are exactly two ways of writing any rational number as a finite simple continued fraction**. They also tell us how to convert one simple continued fraction to the other. Since we already know how to obtain a simple continued fraction by using the Euclid's algorithm, this is essentially the only way to obtain a simple continued fraction.

Examples:

$$[7] = [6, 1]$$
$$[1, 2, 2, 2] = [1, 2, 2, 1, 1]$$
$$[0, 1, 2, 3] = [0, 1, 2, 2, 1]$$
$$[1, 1, 1, 1, 1] = [1, 1, 1, 2]$$
$$[1] = [0, 1]$$

3 Computing Continued Fractions

We would like to see different ways of computing finite simple continued fractions. We know that these correspond precisely to rational numbers. In the next few sections we see different ways of representing rational numbers and how to go from one form to another.

3.1 Continued Fraction Algorithm

This algorithm is very similar to Euclid's algorithm and works even for irrational numbers. In case we start with an irrational number, the algorithm won't terminate but it will give us a way of writing the number as an infinite simple continued fraction.

Algorithm: Let x be a real number. Let $x_0 = x$.

1. Set a_m to be the integral part of x_m.
2. Set ξ_m to be $x_m - a_m$.
3. If $\xi_m \neq 0$, set $\frac{1}{\xi_m}$ as x_{m+1} and go back to step 1 to compute a_{m+1}.
4. If $\xi_m = 0$, terminate this algorithm.

The non-zero ξ_m obtained by this algorithm gives us the $(m+1)$th complete quotients a'_{m+1}. During each iteration, observe that $0 \leq \xi_m < 1$ and hence $a_{m+1} \geq 1$. Also, by construction we have

$$x = [a_0, a'_1] = [a_0, a_1, a'_2] = ...$$

where $a_0, a_1, a_2, ...$ are integers with $a_1 > 0, a_2 > 0, ...$
Hence this indeed gives us the simple continued fraction for x.

The system of equations

$$\begin{aligned} x &= a_0 + \xi_0 \\ \frac{1}{\xi_0} &= a_1' = a_1 + \xi_1 \\ \frac{1}{\xi_1} &= a_2' = a_2 + \xi_2 \\ &\ldots \end{aligned}$$

is known as the continued fraction algorithm.

This algorithm also gives us a way of quickly computing the simple continued fraction by using a calculator. However, some rounding errors will creep in when calculating $1/\xi_m$. Once these intermediate fractions become close to 0, we stop the calculations and that would give us a good approxiamtion of the number we started with.

Lets see how this works with the help of an example.
Example:
Let $x = 2.875$
Its integral part is 2 and so the continued fraction starts as $[2, ...]$.
$2.875 - 2 = 0.875$
Calculate $1/0.875$ using a calculator to get 1.14285714285714. Its integral part is 1.
So we now have $[2, 1, ...]$.
$1.14285714285714 - 1 = 0.14285714285714$. Calculate $1/0.14285714285714$ to get 7.00000000000014 whose integral part is 7.
The continued fraction is now $[2, 1, 7, ...]$.
$7.00000000000014 - 7 = 0.00000000000014$ which is "almost" 0.
So, we terminate the algorithm here to get $[2, 1, 7]$. In fact, it is indeed true that

$$2.875 = 2 + \cfrac{1}{1 + \cfrac{1}{7}}$$

3.2 Decimal expansion

If a rational number is given as a fraction, then we know how to use Euclid's algorithm to get a simple continued fraction. If the number is given in terms of its decimal expansion, then the previous algorithm gives us a way of getting a simple continued fraction. Any rational number has a terminating or eventually repeating (periodic) decimal expansion.
If we have a number with terminating decimal expansion, then we can always represent it as a proper fraction by using a denominator which is a big enough power of 10. The power of 10 required is just the number of digits to the right of the decimal point.
For instance,

1.2 is 12/10
2.875 is 2875/1000
0.00075 is 75/100000

Since all such decimal expansions can be converted to fractions, we can now use Euclid's algorithm to express them as continued fractions.

Example:
$2.875 = 2875/1000$. Using Euclid's algorithm for $(2875, 1000)$ gives us

$$\begin{aligned} 2875 &= \mathbf{2} \times 1000 + 875 \\ 1000 &= \mathbf{1} \times 875 + 125 \\ 875 &= \mathbf{7} \times 125 \end{aligned}$$

So, $2.875 = [2, 1, 7]$. There is no need to reduce the fraction to lowest terms to use Euclid's algorithm as can be seen from the above example.

3.3 Converting from Continued Fractions

Given a finite simple continued fraction, we would now like to recover the rational number from it. The natural way to go about it, is to evaluate the continued fraction from the right-hand end, simplifying each part in turn

$$[2,3,1,4] = 2 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{4}}} = 2 + \cfrac{1}{3 + \cfrac{1}{5/4}} = 2 + \cfrac{1}{3 + \cfrac{4}{5}} = 2 + \frac{1}{19/5} = 2 + \frac{5}{19} = \frac{43}{19}$$

i.e. $[2,3,1,4] = [2,3,1+1/4] = [2,3,5/4] = [2,3+1/(5/4)] = [2,19/5] = [2+1/(19/5)] = [43/5]$

There is another way to evaluate the simple continued fraction by going from left to right. Given $[a_0, a_1, a_2, ...,]$, we can recursively compute its convergents $[a_0], [a_0, a_1], [a_0, a_1, a_2], ...$ from the previous convergent.
If p_m/q_m denotes the mth convergent then $p_m/q_m = (a_m p_{m-1} + p_{m-2})/(a_m q_{m-1} + q_{m-2})$. This is the content of Theorem 2.4.

Thus, we have

CF	a_0	a_1	a_2	a_3
Num	a_0	$a_1 \times a_0 + 1$	$a_2 \times (a_1 \times a_0 + 1) + a_0$	$a_3 \times (a_2 \times (a_1 \times a_0 + 1) + a_0) + (a_1 \times a_0 + 1)$
Den	1	$a_1 \times 1 + 0$	$a_2 \times a_1 + 1$	$a_3 \times (a_2 \times a_1 + 1) + a_1$

It is easier to see this in an example.

Example: $[1, 1, 1, 1]$

CF	1	1	1	1
Num	1	2	3	5
Den	1	1	2	3

From this table we see that $[1, 1, 1, 1] = \frac{5}{3}$.

Remark: Observe that the convergents in this example are ratios of Fibonacci numbers! In general it is true that if the list notation of the continued fraction contains only 1s, then the convergents that appear are ratios of consecutive Fibonacci numbers.

4 Properties of Convergents

In this section, we will continue to use the notation that p_m is the numerator and q_m is the denominator of the mth convergent to the simple continued fraction $[a_0, a_1, a_2, ..., a_n]$.

4.1 Monotone Properties

Let x be given by $[a_0, a_1, ..., a_n]$ and let x_m denote the mth convergent p_m/q_m. Then we have the following theorems

Theorem 4.1. *The even convergents x_{2m} increase strictly with m, while the odd convergents x_{2m+1} decrease strictly.*

Proof. By Theorem 2.4 and Theorem 2.5, we get

$$\begin{aligned}
x_m - x_{m-2} &= \frac{p_m}{q_m} - \frac{p_{m-2}}{q_{m-2}} \\
&= \frac{p_m}{q_m} - \frac{p_{m-1}}{q_{m-1}} + \frac{p_{m-1}}{q_{m-1}} - \frac{p_{m-2}}{q_{m-2}} \\
&= \frac{(-1)^{m-1}}{q_m q_{m-1}} + \frac{(-1)^{m-2}}{q_{m-1} q_{m-2}} \\
&= \frac{(-1)^m (q_m - q_{m-2})}{q_m q_{m-2} q_{m-1}} \\
&= \frac{(-1)^m a_m q_{m-1}}{q_m q_{m-2} q_{m-1}} \\
&= \frac{(-1)^m a_m}{q_m q_{m-2}}
\end{aligned}$$

Since a_m, q_{m-2}, q_m are positive integers, this difference has a sign $(-1)^m$. Hence, the even convergents increase strictly while the odd convergents decrease strictly.

Q.E.D.

Remark: Thus, we have this picture:

$$x_0 < x_2 < x_4 < x_6 < x_8 < ...$$
$$x_1 > x_3 > x_5 > x_7 > x_9 > ...$$

Theorem 4.2. *Every odd convergent is greater than any even convergent.*

Proof. This proof follows by a similar argument as the previous theorem. Theorem 2.5 tells us that $x_m - x_{m-1}$ has the sign $(-1)^{m-1}$. So every odd convergent is greater than its predecessor and its successor. i.e.

$$x_{2m+1} > x_{2m} \text{ and } x_{2m+1} > x_{2m+2} \text{ for all } m$$

If there is some r such that $x_{2m+1} \leq x_{2r}$, then by Theorem 4.1 either $x_{2m+1} \leq x_{2r} < x_{2m}$ or $x_{2r+1} < x_{2m+1} \leq x_{2r}$ depending on whether $r < m$ or $r > m$. In either case this contradicts the fact that every odd convergent is greater than its predecessor.

Q.E.D.

Remark:
If you want to think in terms of examples, then all we are saying in the above proof is that if we want to show $x_3 > x_8$ then we simply use the chain of inequalities $x_3 > x_5 > x_7 > x_8$. Similarly, if we want to show $x_7 > x_2$, we use the chain of inequalities $x_7 > x_6 > x_4 > x_2$.

Theorem 4.3. *The value of the continued fraction is greater than that of any of its even convergents and less than that of any of its odd convergents(except that it is equal to the last convergent).*

Proof. The value of the continued fraction is the last convergent i.e the nth convergent. If n is even, then it is the greatest of the even convergents by Theorem 4.1 and less than all odd convergents by Theorem 4.2. Simliarly, if n is odd, then it is the least of the odd convergents by Theorem 4.1 and greater than all even convergents by Theorem 4.2. Hence, the value of the convergent is between the even and odd convergents.

Q.E.D.

Looking back at some of the examples we have seen, we can quickly check the validity of these theorems in those specific examples. In doing so, we also begin to get an idea of why convergents to a continued fraction are called so. At this point, it would be instructive to go back to the examples and plot all their convergents on the real line to develop a geometric picture of what is happening (i.e. the convergents to x must oscillate around x).

4.2 Best Approximations

In this section we see how close the convergents are to the number that we started with. The next few theorems try to answer the following question:

What is the best approximation to a given number with small denominators?

For instance, Archimedes found that π is approximately $\frac{22}{7}$. This is a simple and good approximation whose error is less that 0.002. For all fractions with denominators less than 10, this is the fraction with the least error. More generally, it is possible to find such approximations for any number.

Definition 4.4. *The rational number p/q is the* ***best approximation*** *to a real number x if the distance from p/q to x on the real line is less than the distance from any other rational number to x (with denominator less than or equal to q).*

Theorem 4.5. *The convergents to a simple continued fraction are in their lowest terms.*

Proof. If d divides p_m and q_m for some m, then d divides $p_{m+1}q_m - p_m q_{m+1}$ which is $(-1)^m$ by Theorem 2.5. So, p_m and q_m cannot have any common divisor other than ± 1 which implies all the convergents are in their lowest terms.

Q.E.D.

Theorem 4.6. *The denominators of the convergents satisfy the following inequalities*

$$q_n \geq n, \text{ with strict inequality when } n > 3.$$

Proof. $q_0 = 1, q_1 = a_1 \geq 1$. For $n \geq 2$,

$$q_n = a_n q_{n-1} + q_{n-2} \geq q_{n-1} + 1$$

and inductively we see that $q_n \geq n$. For $n > 3$,

$$q_n \geq q_{n-1} + q_{n-2} > q_{n-1} + 1 \geq n$$

and hence $q_n > n$.

Q.E.D.

Theorem 4.7. *Every simple continued fraction can be written as an alternating sum in the following manner:*

$$[a_0, a_1, ..., a_n] = a_0 + \frac{1}{q_1 q_0} - \frac{1}{q_2 q_1} + ... + (-1)^{n-1} \frac{1}{q_n q_{n-1}}$$

Proof. Observe that for any m, the mth convergent $\frac{p_m}{q_m}$ can be written as

$$\frac{p_m}{q_m} = \frac{p_m}{q_m} - \frac{p_{m-1}}{q_{m-1}} + \frac{p_{m-1}}{q_{m-1}} - \frac{p_{m-2}}{q_{m-2}} + ... + \frac{p_1}{q_1} - \frac{p_0}{q_0} + \frac{p_0}{q_0}$$

Using Theorem 2.5 and setting $m = n$ gives us the desired result.

Q.E.D.

Theorem 4.8. *For any number x with convergents $\frac{p_m}{q_m}$,*

$$\left| x - \frac{p_m}{q_m} \right| < \frac{1}{q_{m+1} q_m}$$

Proof. We will sketch the idea of the proof here. There are two way to see why the inequality holds.
The first proof uses the previous theorem. Notice that x can be written as

$$x = c_0 - c_1 - c_2 - ...$$

where $c_0 = a_0$ and $c_m = \frac{p_{m-1}}{q_{m-1}} - \frac{p_m}{q_m}$. These c_m become smaller and smaller and x minus the first m terms is less than the value of c_{m+1}. This will imply the inequality. Try to compute these c_m for some explicit example to see why this result is true.
The second proof is purely algebraic and uses complete quotients.

$$x = \frac{a'_{m+1} p_m + p_{m-1}}{a'_{m+1} q_m + q_{m-1}}$$

and so

$$x - \frac{p_m}{q_m} = -\frac{p_m q_{m-1} - p_{m-1} q_m}{q_m (a'_{m+1} q_m + q_{m-1})} = \frac{(-1)^m}{q_m (a'_{m+1} q_m + q_{m-1})} = \frac{(-1)^m}{q_m q'_{m+1}}$$

where $q'_m = a'_m q_{m-1} + q_{m-2}$. Since a_m is the integral part of a'_m, $q_{m+1} < q'_{m+1}$ and this gives us the desired inequality.

Q.E.D.

Theorem 4.9. $\frac{p_m}{q_m}$ *is the best approximation to x with denominator $\leq q_m$.*

Proof. The even convergents are increasing and the odd convergents are decreasing with x lying in between them. $q_m > q_{m-1}$ implies that $\dfrac{p_{m-1}}{q_{m-1}}$ and $\dfrac{p_m}{q_m}$ are two convergents with denominator less than or equal to q_m. Suppose $\dfrac{p}{q}$ is the best approximation to x with denominator less than or equal to q_m, then it has to lie between $\dfrac{p_m}{q_m}$ and $\dfrac{p_{m-1}}{q_{m-1}}$.

$$\left|\frac{p}{q} - \frac{p_{m-1}}{q_{m-1}}\right| = \left|\frac{pq_{m-1} - qp_{m-1}}{qq_{m-1}}\right| \geq \frac{1}{qq_{m-1}}$$

$$\left|\frac{p}{q} - \frac{p_{m-1}}{q_{m-1}}\right| \leq \left|\frac{p_m}{q_m} - \frac{p_{m-1}}{q_{m-1}}\right| = \frac{1}{q_m q_{m-1}}$$

From these two inequalites we get

$$\frac{1}{qq_{m-1}} \leq \left|\frac{p}{q} - \frac{p_{m-1}}{q_{m-1}}\right| \leq \frac{1}{q_m q_{m-1}}$$

but we know that $q \leq q_m$. Therefore, equality must hold everywhere which implies $\dfrac{p}{q} = \dfrac{p_m}{q_m}$. Hence, $\dfrac{p_m}{q_m}$ is the best approximation to x with denominator less than or equal to q_m.

Q.E.D.

This answers the question that was posed at the beginning of this section! If we want to find best approximations to a given number, then we calculate its simple continued fraction and look at its convergents. Amongst those, we pick a convergent with a huge denominator and that would give a very good approximation. If we want to get a good approximation directly from the list notation of a continued fraction, rather than compute its convergents then the idea would be to stop evaluating a continued fraction right before a large entry. i.e. If a_m is a very big number, then $[a_0, a_1, ..., a_{m-1}]$ will give a very good approximation of the number. The appendix has some approximations to π using what we have discussed here.

All convergents to x are best approximations to x but these are not all the best approximations! Consider the following example:
Example:

$$x = \frac{7}{38} = [0, 5, 2, 3]$$
$$\text{Convergents: } \frac{0}{1}, \frac{1}{5}, \frac{2}{11}, \frac{7}{38}$$

If we look at all fractions with denominators less than or equal to 28, then $\dfrac{5}{27}$ is the best approximation to x but it is not one of the convergents.

$$\frac{5}{27} = [0, 5, 2, 2]$$

This leads us to the following two questions:
1. What are all the best approximations?
2. How are they related to the convergents?

Lemma 4.10. *If* $b, d > 0$*, then* $\frac{a}{b} \leq \frac{c}{d} \implies \frac{a}{b} \leq \frac{a+c}{b+d} \leq \frac{c}{d}$

Proof. $\frac{a}{b} \leq \frac{c}{d} \implies ad - bc \leq 0 \implies ad \leq bc$

$ab + ad \leq ab + bc \implies \frac{a}{b} \leq \frac{a+c}{b+d}$

$ad + cd \leq bc + cd \implies \frac{a+c}{b+d} \leq \frac{c}{d}$

Q.E.D.

Definition 4.11. *A* ***semi-convergent*** *or* ***secondary convergent*** *to* x *is a number of the form* $(p_k + rp_{k+1})/(q_k + rq_{k+1})$ *where* p_k/q_k *and* p_{k+1}/q_{k+1} *are two consecutive convergents to* $x = [a_0, a_1, a_2, ...]$ *and* r *is an integer between* 0 *and* a_k.

Note in particular, that the convergents to x are also semi-convergents.

Theorem 4.12. *If* x *is any real number and* a/b *is not a semi-convergent to* x*, then* a/b *is not the best approximation to* x *with denominator less than or equal to* b.

Proof. Lets assume for simplicity that $a/b < x$. Since a/b is not a semi-convergent, it should lie between two convergents of the form p_k/q_k and p_{k+2}/q_{k+2} for some k. Lets say for simplicity that $p_k/q_k < p_{k+1}/q_{k+1}$ (a similar proof holds if $p_k/q_k > p_{k+1}/q_{k+1}$). Then by Lemma 4.10,

$$\frac{p_k}{q_k} < \frac{p_k + p_{k+1}}{q_k + q_{k+1}} < \frac{p_k + 2p_{k+1}}{q_k + 2q_{k+1}} < ... < \frac{p_k + a_k p_{k+1}}{q_k + a_k q_{k+1}} = \frac{p_{k+2}}{q_{k+2}}$$

Therefore,

$$\frac{p_k + rp_{k+1}}{q_k + rq_{k+1}} < \frac{a}{b} < \frac{p_k + (r+1)p_{k+1}}{q_k + (r+1)q_{k+1}} \text{ for some } 0 \leq r < a_k.$$

$$\begin{aligned} \frac{1}{b(q_k + rq_{k+1})} &\leq \left| \frac{a}{b} - \frac{p_k + rp_{k+1}}{q_k + rq_{k+1}} \right| \\ &< \left| \frac{p_k + (r+1)p_{k+1}}{q_k + (r+1)q_{k+1}} - \frac{p_k + rp_{k+1}}{q_k + rq_{k+1}} \right| \\ &= \frac{1}{((r+1)q_{k+1} + q_k)(rq_{k+1} + q_k)} \end{aligned}$$

$$\implies \frac{1}{b} \leq \frac{1}{q_k + (r+1)q_{k+1}} \leq \frac{1}{q_{k+2}} \implies q_{k+2} \leq b$$

This says that $\frac{a}{b}$ is not the best approximation to x since $\frac{p_{k+2}}{q_{k+2}}$ is closer to x than $\frac{a}{b}$ with denominator less than b.
An analogous argument holds when $a/b > x$.

Q.E.D.

This tells us that the only candidates for the best approximations are the semi-convergents. In fact, there is a precise statement which says which semi-convergents are the best approximations which we will not state here. Roughly speaking, about half the semi-convergents between p_k/q_k and p_{k+2}/q_{k+2} which are closest to p_{k+2}/q_{k+2} are the best approximations to x.

Going back to our example, 5/27 is a semi-convergent to 7/38 but not a convergent. All the best approximations to $7/38 = [0, 5, 2, 3]$ are **[0]**, $[0, 3]$, $[0, 4]$, **[0,5]**, **[0,5,2]**, $[0, 5, 2, 2]$, **[0,5,2,3]** (and the highlighted terms are the convergents).

When talking about best approximations we used the distance between a fraction p/q and x as a measure of how well it approximated x. However, if the denominator increases, then we should expect better approximations to have smaller distance from x. To take this into account, one could consider the product of the denominator and the distance between the fraction from x as measure of how well it approximates x (i.e. $q\left|x - p/q\right| = |qx - p|$).

Definition 4.13. *A fraction p/q is a* ***best approximation of the second kind*** *to a real number x if for every fraction a/b with denominator less than or equal to q, we have $|qx - p| < |bx - a|$.*

Theorem 4.14. *The convergents to a real number x are precisely all the best approximations of the second kind to x.*

Proof. We will only prove part of the theorem here by showing that any fraction which is not a convergent cannot be a best approximation of the second kind to x.

Let us assume that a/b is a best approximation of the second kind to x and its not a convergent to x. We'll further assume that $a/b < x$ and it lies between two convergents p_k/q_k and p_{k+2}/q_{k+2} for some k where $p_k/q_k < p_{k+1}/q_{k+1}$ (these assumptions are simply to make the proof easier to write and the proof can be modified for all cases).

$$\begin{aligned}\frac{1}{bq_k} &\leq \left|\frac{a}{b} - \frac{p_k}{q_k}\right| \\ &< \left|\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k}\right| \\ &= \frac{1}{q_k q_{k+1}}\end{aligned}$$

$$\implies \frac{1}{bq_k} < \frac{1}{q_k q_{k+1}} \implies q_{k+1} < b$$

And we also have

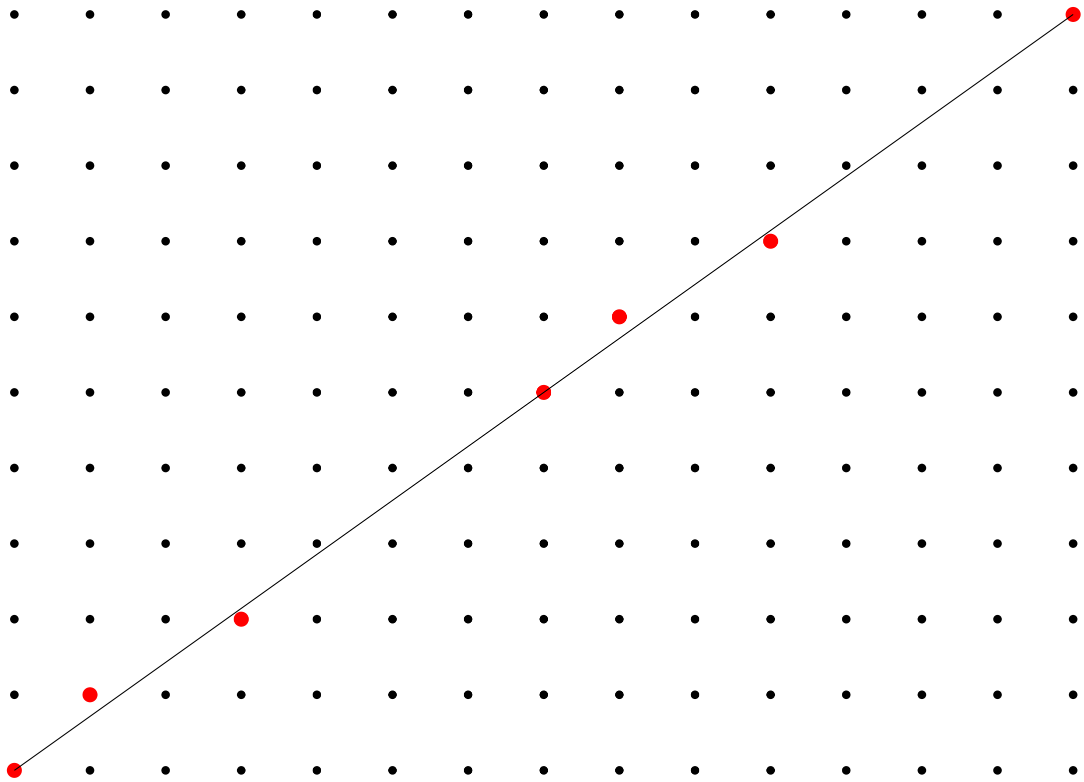
$$\begin{aligned}
|q_{k+1}x - p_{k+1}| &= q_{k+1}\left|x - \frac{p_{k+1}}{q_{k+1}}\right| \\
&\leq q_{k+1}\left|\frac{p_{k+2}}{q_{k+2}} - \frac{p_{k+1}}{q_{k+1}}\right| \\
&= \frac{1}{q_{k+2}} \\
&= b\frac{1}{bq_{k+2}} \\
&\leq b\left|\frac{p_{k+2}}{q_{k+2}} - \frac{a}{b}\right| \\
&\leq b\left|x - \frac{a}{b}\right| \\
&= |bx - a|
\end{aligned}$$

Since $\frac{p_{k+1}}{q_{k+1}}$ has a denominator less than b and $|q_{k+1}x - p_{k+1}| \leq |bx - a|$, $\frac{a}{b}$ cannot be a best approximation of the second kind to x.

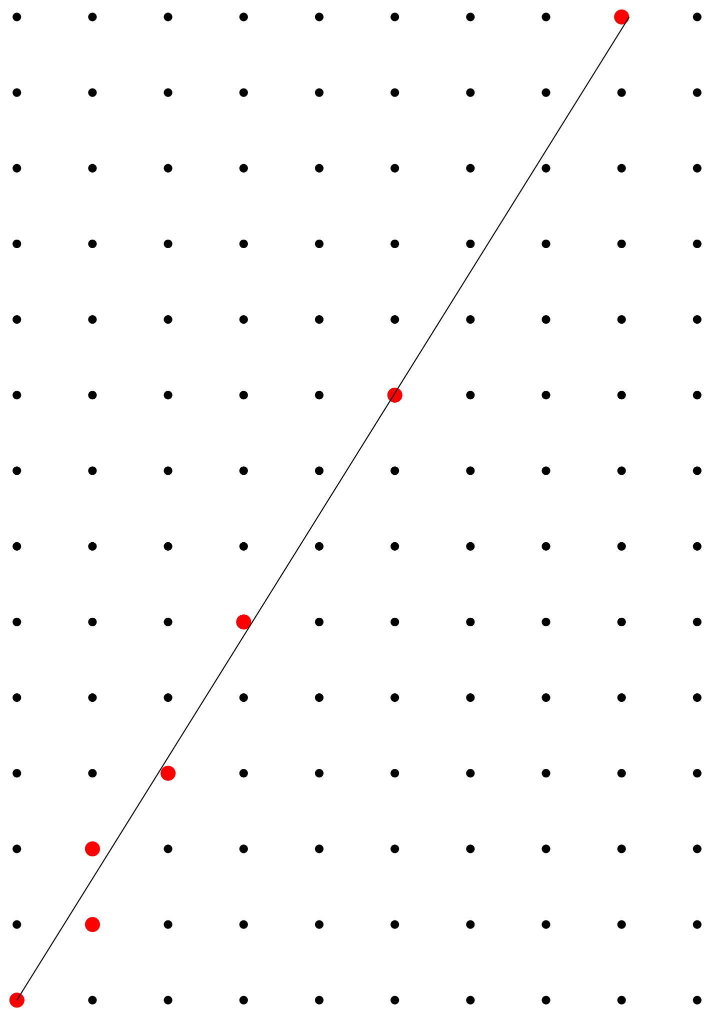
Q.E.D.

4.3 Geometric Interpretation

Look at the coordinate plane where all points with integer coordinates are marked. That would give us a lattice on the plane. Fix some α and draw the line $y = \alpha x$ on the coordinate plane. If α is rational, then it would intersect the lattice in infinitely many points. If α is irrational, then it will not intersect the lattice at any point other than the origin.



For example, lets look at what happens when α is $\frac{5}{7} = [0, 1, 2, 2]$.
The line segment joins $(0, 0)$ to $(14, 10)$ and the red points are $(0, 0), (1, 1), (3, 2), (7, 5), (8, 6), (10, 7), (14, 10)$.
The convergents to $\frac{5}{7}$ are $0, \frac{1}{1}, \frac{2}{3}, \frac{5}{7}$.



Simlarly, we can consider the case when α is $\frac{(1+\sqrt{5})}{2}$. This is called the golden ratio. The red points are $(0,0),(1,1),(1,2),(2,3),(3,5),(5,8),(8,13)$. The convergents to $\frac{(1+\sqrt{5})}{2}$ are $\frac{1}{1},\frac{2}{1},\frac{3}{2},\frac{5}{3},\frac{8}{5},\frac{13}{8},\ldots$.

Theorem 4.15. *The closest points in the lattice to the line $y=\alpha x$ are in one-one correspondence with the convergents to the continued fraction of α. Let these points be labelled as $A_m=(q_m,p_m)$. The point is above the line, if m is odd, and below otherwise.*

Proof. The proof follows from the observation that the distance of a point (b,a) to the line $y=\alpha x$ is $\frac{|b\alpha-a|}{\sqrt{1+\alpha^2}}$ which is a constant $(\sqrt{1+\alpha^2})$ times the measure of whether $\frac{a}{b}$ is a best approximation of the second kind to α.

Q.E.D.

Looking at the two examples where the graph has been drawn, we can see that this is indeed the case. The convergents to α have been highlighted in both cases.

Given just the points on the lattice closest to the line, it is also possible to recover the continued fraction as described by the following theorem:

Theorem 4.16. *Let $[a_0, a_1, ..., a_n]$ be the continued fraction for α and let the points A_m be the marked points in the lattice as before. Then a_m is the integral distance between the points A_m and A_{m+2}.*

Here, the integral distance between two points refers to the number of points on the line segment joining the two points minus 1.
These two theorems give us a geometric way of going back and forth between continued fractions and approximations.

5 Quadratic Equations

5.1 Examples

Lets start with some quadratic equation and try to naively get a continued fraction for its roots.
Example:
For instance, lets take

$$x^2 - 5x - 1 = 0$$

Instead of computing its roots using the quadratic formula, lets do the following steps:

$$x^2 = 5x + 1$$
$$x = 5 + 1/x$$

Since, we have x appearing on the right hand side, lets make the substitution again to get

$$x = 5 + 1/x = 5 + \frac{1}{5 + 1/x}$$

Does this look familiar? By continuing this process, we seem to get the infinite continued fraction $[5, 5, 5, ...]$.

Example: Consider

$$x^2 + x = 1$$

We try to do something similar to the previous example.

$$x^2 + x = 1$$
$$x(x + 1) = 1$$
$$x = \frac{1}{(1 + x)}$$
$$x = \cfrac{1}{1 + \cfrac{1}{1 + x}}$$

Again, by continuing this process we get $[0, 1, 1, 1, ...]$.

Example:

$$x^2 - 2x = 1$$
$$x^2 = 2x + 1$$
$$x = 2 + 1/x$$
$$x = 2 + \cfrac{1}{2 + \cfrac{1}{2 + 1/x}}$$

This gives us the continued fraction $[2, 2, 2, 2, 2, ...]$. This number is called the silver ratio (analogous to the golden ratio). The golden ratio is the continued fraction $[1, 1, 1, 1, 1, ...]$ and it has the worst possible approximations by rational numbers since the only numbers appearing in its continued fraction expansion is 1. In this sense, the silver ratio is the second worst number to be approximated by rational numbers.
We can use this example to get a continued fraction for the root of the next quadratic expression.

Example:

$$x^2 - 2 = 0$$

Setting, $y = x + 1$, we get

$$(y - 1)^2 - 2 = 0$$
$$y^2 - 2y - 1 = 0$$
$$y^2 - 2y = 1$$

From the previous example, we know that $y = [2, 2, 2, ...]$. So $x = \sqrt{2} = [1, 2, 2, 2, ...]$.
Given any quadratic equation, we can play around with the equation and see if we can recursively construct a continued fraction. There seems to be some pattern in all these examples.

1. The roots of all these quadratic equations can be expressed as an infinite continued fraction.
2. The numbers that appear in the continued fraction repeat after a while.

This will be formalised in the next section.

5.2 Periodic Continued Fractions

Definition 5.1. *A **periodic continued fraction** is an infinite continued fraction in which*

$$a_l = a_{l+k}$$

for a fixed positive k and all $l \geq L$.The set of partial quotients

$$a_L, a_{L+1}, ..., a_{L+k-1}$$

*is called the **period**, and the continued fraction may be written*

$$[a_0, a_1, ..., a_{L-1}, \overline{a_L, a_{L+1}, ..., a_{L+k-1}}].$$

Theorem 5.2. *A periodic continued fraction is a quadratic surd, i.e. an irrational root of a quadratic equation with integral coefficients.*

Proof. Using the notation set up in the definition, we have

$$\begin{aligned} a'_L &= [a_L, a_{L+1}, ..., a_{L+k-1}, a_L, a_{L+1}, ...] \\ &= [a_L, a_{L+1}, ..., a_{L+k-1}, a'_L] \\ &= \frac{p' a'_L + p''}{q' a'_L + q''} \end{aligned}$$

where p''/q'' and p'/q' are the last two convergents to $[a_L, a_{L+1}, ..., a_{L+k-1}]$.
This gives us the equation

$$q' a'^2_L + (q'' - p')a'_L - p'' = 0$$

We know that

$$x = \frac{p_{L-1} a'_L + p_{L-2}}{q_{L-1} a'_L + q_{L-2}}$$

which gives us

$$a'_L = \frac{p_{L-2} - q_{L-2} x}{q_{L-1} x - pL - 1}$$

Substituting this value back in the quadratic equation and clearing the denominators gives us a quadratic equation in x with integer coefficients.

Q.E.D.

The converse of the theorem is also true.

Theorem 5.3. *The continued fraction which represents a quadratic surd is periodic.*

This theorem will not be proved here. A proof of this theorem can be found in the references. This theorem is sometimes called the Continued Fraction Theorem.

5.3 Fibonacci Numbers

Lets start with this quadratic equation

$$x^2 - x - 1 = 0$$

and try to do simlar computations like in the examples before.

$$\begin{aligned} x^2 &= x + 1 \\ x &= 1 + 1/x \\ x &= 1 + \frac{1}{1 + 1/x} \end{aligned}$$

This gives us the periodic continued fraction $[1, 1, 1, ...]$. We saw a truncated version of this continued fraction while discussing an example for computing the numerators and denominators explicitly for convergents to a continued fraction.

Directly using the quadratic formula tells us that $(1 + \sqrt{5})/2$ is the positive root of this quadratic equation. This is precisely the golden ratio which we saw as an example when we looked at a geometric interpretation of convergents.

Definition 5.4. ***Fibonacci Sequence*** *is an integral sequence defined by*

$$F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$$

The first few Fibonacci numbers are $1, 1, 2, 3, 5, 8, 13, ...$

The closed form expression for the nth Fibonacci number is

$$F_n = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}}$$

where ϕ is the golden ratio. Thus we see that continued fractions connects the golden ratio to the Fibonacci sequence. The convergents to the continued fraction of the golden ratio are precisely the ratio of consecutive Fibonacci numbers.

Exercise 5.5. *Prove that any two consecutive Fibonacci numbers are relatively prime.*

Exercise 5.6. *Prove Cassini's identity.* ($F_{n-1}F_{n+1} - F_n^2 = (-1)^n$)

5.4 Pell Numbers

Definition 5.7. *The Pell numbers (like the Fibonacci sequence) are defined by this recurrence relation*

$$P_0 = 0, P_1 = 1, P_n = 2P_{n-1} + P_{n-2}$$

The first few Pell numbers are $0, 1, 2, 5, 12, 29, 70, ...$

The closed form expression for the nth Pell number is

$$P_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}$$

These numbers are the denominators of the convergents to $\sqrt{2} = [1, 2, 2, 2, ...]$. If we started with $L_0 = 2, L_1 = 2$ and constructed a sequence with the same recurrence relation as the Pell numbers then the sequence that we get is twice the numerators of the convergents to $\sqrt{2}$.

6 Other Simple Continued Fractions

In the previous section, when we were looking at quadratic equations, we didn't look at a negative root. We only considered the positive root and constructed its continued fraction. Now we will see what happens for continued fractions of negative numbers.

6.1 Negative numbers

The natural question to ask is if we need to allow negative numbers in the denominators of continued fractions to express a negative number as a continued fraction. Recall that, by definition, if $[a_0, a_1, a_2, ...,]$ is a simple continued fraction, then $a_1 > 0, a_2 > 0, ...$ One naive thing to do is to negate every number appearing in the continued fraction expansion. i.e If $x = [a_0, a_1, a_2, ...]$, then $-x = [-a_0, -a_1, -a_2, ...]$. In doing so, we get negative numbers in the denominators of the continued fraction.
It turns out that every negative number can be expressed with only $a_0 < 0$ and all other $a_1, a_2, ...$ being positive integers.

Let x be a negative number. Let W be the integral part and F be the fractional part of $-x$.

$$\begin{aligned} -x &= W + F, 0 < F < 1 \\ x &= -W - F \\ x &= (-W-1) + (1-F) \end{aligned}$$

Notice that $0 < (1-F) < 1$. Hence we can take the simple continued fraction of $(1-F)$, say $[a_0, a_1, ...]$ and add $(-W-1)$ to get a simple continued fraction for x. By doing so, we get

$$x = [a_0 - W - 1, a_1, a_2, ...]$$

with $a_1 > 0, a_2 > 0, ...$
If F was 0, then the continued fraction of x is x itself.

Example:
Let $x = -17/12$
$-x = 17/12 = 1 + 5/12 \implies x = (-1-1) + (1 - 5/12) = -2 + 7/12$
$7/12 = [0, 1, 1, 2, 2] \implies -17/12 = [-2, 1, 1, 2, 2]$
In this expression, the only negative term in the continued fraction expression is the first term, namely -2.

Given a continued fraction expression of x, is there a way of directly obtaining a continued fraction of $-x$? Yes, there is a neat way of going from one to the other!

Let $x = [a_0, a_1, a_2, a_3...] = [a_0, a_1, [a_2, a_3, ...]]$

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{[a_2, a_3, ...]}}$$

$$-x = -a_0 - \cfrac{1}{a_1 + \cfrac{1}{[a_2, a_3, ...]}} = -a_0 - 1 + \cfrac{1}{1 + \cfrac{1}{a_1 - 1 + \cfrac{1}{[a_2, a_3, ...]}}}$$

Therefore, $x = [a_0, a_1, a_2, a_3, ...] \implies -x = [-a_0 - 1, 1, a_1 - 1, a_2, a_3, ...]$

Example:
$x = 17/12 = [1, 2, 2, 2]$
$-x = -17/12 = [-1-1, 1, 2-1, 2, 2] = [-2, 1, 1, 2, 2]$

If we were to instead start with $x = -17/12$ we would do the following:
Example:
$x = -17/12 = [-2, 1, 1, 2, 2]$
$-x = 17/12 = [-(-2) - 1, 1, 1 - 1, 1, 2, 2] = [1, 1, 0, 1, 2, 2]$

The continued fraction for $17/12$ obtained by starting from the continued fraction of $-17/12$ doesn't quite give us the required continued fraction!

The only problem with the above method is that a_1 could be 1 which would make $a_1 - 1$ as 0. There is a simple trick to elminate 0 from the continued fraction expression. Whenever there is a 0, just add the previous term and the next term into a single entry and omit the 0. If 0 is the last term, then simply omit 0 and the previous term i.e.

$$[a_0, a_1, ..., a_m, 0, a_{m+1}, a_{m+2}..] = [a_0, a_1, ..., a_m + a_{m+1}, a_{m+2}, ...]$$
$$[a_0, a_1, ..., a_{n-2}, a_{n-1}, 0] = [a_0, a_1, ..., a_{n-2}]$$

Example:
$x = 21/8 = [2, 1, 1, 1, 2]$
$-x = -21/8 = [-2-1, 1, 1-1, 1, 1, 2] = [-3, 1, 0, 1, 1, 2] = [-3, 1+1, 1, 2] = [-3, 2, 1, 2]$

6.2 Subtraction in Continued Fractions

Recall that a simple continued fraction can be presented as $[a_0, a_1, ...]$ with $a_0, a_1, ...$ being integers and $a_1 > 0, a_2 > 0, ...$ Lagrange in the Appendices to his translation of Euler's Elements of Algebra points out that it is superflous to also have subtraction in the continued fraction expression. As a first step, if we have a continued fraction with subtraction in some denominator, we can replace it with a continued fraction with only addition and possibly negative terms. i.e

$$a - \frac{1}{b+c} = a + \frac{1}{-b-c}$$

Here, a, b, c can be numbers, convergents or complete quotients.
After doing so, we need a way of removing negative numbers from a continued fraction expression. With a little bit of algebra, we can see that this is also possible.

$$a_m + \cfrac{1}{-a_{m+1} + \cfrac{1}{[a_{m+2}, a_{m+3}, ...]}} = (a_m - 1) + \cfrac{1}{1 + \cfrac{1}{(a_{m+1} - 1) - \cfrac{1}{[a_{m+2}, a_{m+3}, ...]}}}$$

In other words,

$$[a_0, ..., a_m, -a_{m+1}, [a_{m+2}, a_{m+3}, ...]] = [a_0, ..., a_m - 1, 1, a_{m+1} - 1, -[a_{m+2}, a_{m+3}, ...]]$$
$$[a_0, ..., a_m, -a_{m+1}] = [a_0, ..., a_m - 1, 1, a_{m+1} - 1]$$

In this process, if a 0 appears then we know how to get rid of it by following the procedure from the previous section. Observe that removing a negative number might introduce further negative numbers that appear later on in the continued fraction. We can inductively start from the left and remove negative numbers from left to right.

Example:
$[2, -3, 4] = [2-1, 1, 3-1, -4]$
$[1, 1, 2, -4] = [1, 1, 2-1, 1, 4-1]$
$[1, 1, 2-1, 1, 4-1] = [1, 1, 1, 1, 3]$
We can indeed check that

$$[2, -3, 4] = 2 + \cfrac{1}{-3 + \cfrac{1}{4}} = \frac{18}{11} = [1, 1, 1, 1, 3] = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3}}}}$$

The next example illustrates all the rules discussed in this section.

Example:

$-30/13 = [-3, 1, 2, 4]$
$[-3, 1, 2, 4] = -[3, -1, -2, -4]$ We want the first entry to be positive.
$-[3, -1, 2, 4] = -[2, 1, 0, -[-2, -4]]$ This is the rule to eliminate a negative sign.
$-[2, 1, 0, -[-2, -4]] = -[2, 1, 0, 2, 4]$ Negative sign distributes to each entry.
$-[2, 1, 0, 2, 4] = -[2, 1 + 2, 4]$ Rule for removing 0.
$-[2, 3, 4] = [-3, 1, 2, 4]$ Rule for negating a continued fraction.

6.3 Difficulties with negative terms

There are many practical difficulties if we allow negative numbers in continued fractions.

1. Arbitrary length of a continued fraction may reduce to 0
   There are many continued fractions one can construct that just reduce to 0. For instance $[1, -1]$ and $[1, 1, 1, -1, 3]$ are two such examples. Try to construct other continued fractions for 0.

2. Later terms might collapse a continued fraction.
   Since there are many ways of writing 0 as a continued fraction, these may be part of a bigger continued fraction which makes it irrelevant when converting a continued fraction to a real number.

3. Uniqueness result of continued fractions no longer holds.
   It won't make sense any more to talk about "the" continued fraction expression of a real number.

4. Convergents may be infinite.
   For instance, the convergents to $[1, -2, 1, -2, 1, -2]$ are $1, 1/2, 0, \infty, 1, 1/2$.

5. Convergents may not alternate between larger and smaller numbers and we won't have a nice picture as before.
   The above example $[1, -2, 1, -2, 1, -2]$ demonstrates this.

6. Convergents may not converge.
   Find the convergents to $[0, \overline{2, -2}]$. The word "convergent" isn't appropriate if we allow negative terms.

7. Negative values can anyway be written without negative numbers(except the first convergent).
   This was the content of the beginning of this section where an explicit rule was discussed to still have $a_1 > 0, a_2 > 0, ...$

All these reasons tell us why we should stick to working with simple continued fractions.

6.4 General Continued Fraction

Definition 6.1. *The **General Continued Fraction** is a simple continued fraction in which the numerators can be any positive interger (not necessarily* 1*).*

Example:

$$\sqrt{6} = 2 + \cfrac{2}{4 + \cfrac{2}{4 + \cfrac{2}{4 + ...}}}$$

Working with a general continued fraction instead of a simple continued fraction does not give us any new information but in some cases it is easier to explicitly compute a general continued fraction rather than a simple continued fraction. There are some examples given in the Appendix.

Finding a general continued fraction of $\sqrt{n}$:
Let a be any number with $a^2 \leq n$.

$$\begin{aligned}\sqrt{n} &= a + x \\ n &= a^2 + 2ax + x^2 \\ n - a^2 &= x(2a + x) \\ x &= \frac{n - a^2}{2a + x}\end{aligned}$$

$$\sqrt{n} = a + x = a + \frac{n - a^2}{2a + x} = a + \cfrac{n - a^2}{2a + \cfrac{n - a^2}{2a + x}} = ...$$

Setting $n = 6, a = 2$ gives us the previous example. Taking $a = 1$ instead gives us

$$\sqrt{6} = 1 + \cfrac{5}{2 + \cfrac{5}{2 + \cfrac{5}{2 + ...}}}$$

Example:
Consider the quadratic equation $x^2 - x - 2 = 0$.

$$\begin{aligned}x^2 - x - 2 &= 0 \\ x^2 &= x + 2 \\ x &= 1 + \frac{2}{x} \\ x = 1 + \frac{2}{x} &= 1 + \cfrac{2}{1 + \cfrac{2}{x}} \\ x &= 1 + \cfrac{2}{1 + \cfrac{2}{1 + \cfrac{2}{1 + ...}}}\end{aligned}$$

What if we wanted a simple continued fraction for x instead of the above general continued fraction? We have already seen that the quadratic surds are precisely all the periodic continued fractions. However, the solution to this quadratic equation is in fact rational. More precisely

$$\begin{aligned}x^2 - x - 2 &= 0 \\ (x - 2)(x + 1) &= 0 \\ x &= 2, x = -1\end{aligned}$$

Notice that $x = 2$ is the only positive solution to the quadratic equation. Thus we have obtained a general continued fraction for 2. The only simple continued fractions for 2 are [2] and [1, 1].

$$2 = 1 + \cfrac{2}{1 + \cfrac{2}{1 + \cfrac{2}{1 + ...}}}$$

7 Irrational Numbers

Finite simple continued fractions correspond to rational numbers and every rational number has a finite simple continued fraction expression. In the previous sections we have been loosely using $[a_0, a_1, ...]$ without formally defining what this means. In this section we will see how the infinite simple continued fractions correspond to irrational numbers and what their properties are.

7.1 Definition

One of the main interests of continued fractions is in its application to the representation of irrationals.
Suppose $a_0, a_1, a_2, ...$ is a sequence of integers with $a_1 > 0, a_2 > 0, ...$ so that

$$x_n = [a_0, a_1, ..., a_n]$$

is a simple continued fraction of a rational number x_n for every n. If these x_n tend to a limit x when $n \to \infty$, then we say that $[a_0, a_1, ...]$ is x and we write

$$x = [a_0, a_1, a_2, ...]$$

Theorem 7.1. *If $a_0, a_1, a_2, ...$ is a sequence of integers with $a_1 > 0, a_2 > 0, ...$ then $x_n = [a_0, a_1, ..., a_n]$ tends to a limit x when $n \to \infty$.*

Proof. We have

$$x_m = \frac{p_m}{q_m} = [a_0, a_1, ..., a_m]$$

Note that x_m is also a convergent to x_n if $m \leq n$. By Theorem 4.1 and Theorem 4.2, the even convergents are increasing and less than x_1. Simlarly, the odd convergents are decreasing and greater than x_0. Hence, the even convergents will have a limit and the odd convergents will have a limit.
Theorem 2.5 and Theorem 4.5 tell us that

$$\frac{p_{2n-1}}{q_{2n-1}} - \frac{p_{2n}}{q_{2n}} = \frac{1}{q_{2n}q_{2n-1}} \leq \frac{1}{2n(2n-1)}$$

which tends to 0 as n tends to ∞. This implies that the limit for the odd and even convergents are the same and hence x_n will tend to a limit x as n tends to ∞.

Q.E.D.

In particular, this means that we can always talk about $[a_0, a_1, a_2, ...]$ as it is a well-defined real number.

7.2 Properties

In this section, we see some properties of these infinite simple continued fractions.

Theorem 7.2. *An infinite simple continued fraction is less than any of its odd convergents and greater than any of its even convergents.*

Proof. In the proof of Theorem 7.1, we see that the even convergents increase to the limit and the odd convergents decrease to the limit. This shows that the value of the continued fraction is less than any of its odd convergents and greater than any of its even convergents.

Q.E.D.

Theorem 7.3. *If* $[a_0, a_1, ..] = x$*, then*

$$a_0 = \text{integral part of } x, \; a_m = \text{integral part of } a'_m.$$

Proof. This proof is analogous to the case of finite continued fraction.

$$a'_m = [a_m, a_{m+1}, ...] = \lim_{n\to\infty}[a_m, a_{m+1}, ..., a_n] = a_m + \frac{1}{a'_{m+1}}$$

In particular, $x = a'_0 = a_0 + \dfrac{1}{a'_1}$

$$a'_m > a_m, a'_{m+1} > a_{m+1} > 0, 0 < \frac{1}{a'_{m+1}} < 1$$

Hence, a_m = integral part of a'_m.

Q.E.D.

Theorem 7.4. *Two infinite simple continued fractions which have the same value are identical.*

Proof. By the previous theorem, the first term of both continued fractions is just the integral part of the value. We can inductively see that a_m is the integral part of a'_m and hence has to be the same in both infinite continued fraction expressions. Hence, the two infinite continued fractions must be identical.

Q.E.D.

Theorem 7.5. *Every irrational number can be expressed in just one way as an infinite simple continued fraction.*

Proof. The continued fraction algorithm produces a simple continued fraction for an irrational number. The algorithm does not terminate as we know that finite simple continued fractions correspond to rational numbers and hence we get an infinite continued fraction. By the previous theorem, this infinite simple continued fraction expression is unique.

Q.E.D.

7.3 Equivalent Numbers

Definition 7.6. *If ξ and η are two numbers such that*

$$\xi = \frac{a\eta + b}{c\eta + d}$$

*where a, b, c, d are integers such that $ad - bc = \pm 1$, then ξ is said to be **equivalent** to η.*

Theorem 7.7. *The above relation is an equivalence relation.*

Proof. Any number is equivalent to itself because if we set $a = 1, b = 0, c = 0, d = 1$ in the definition above, we get

$$\eta = \frac{1.\eta + 0}{0.\eta + 1}$$

with $ad - bc = 1$.
If ξ is equivalent to η with a, b, c, d as in the definition then

$$\eta = \frac{-d\xi + b}{c\xi - a}$$

with $(-d)(-a) - bc = ad - bc = \pm 1$ which implies η is equivalent to ξ.
If ξ is equivalent to η with

$$\xi = \frac{a\eta + b}{c\eta + d}, ad - bc = \pm 1$$

and η is equivalent to ω with

$$\eta = \frac{a'\omega + b'}{c'\omega + d'}, a'd' - b'c' = \pm 1$$

then

$$\xi = \frac{A\omega + B}{C\omega + D}$$

with $A = aa' + bc', B = ab' + bd', C = ca' + dc', D = cb' + dd', AD - BC = (ad - bc)(a'd' - b'c') = \pm 1$ which implies ξ is equivalent to ω.
Hence, this is indeed an equivalence relation.

Q.E.D.

Theorem 7.8. *Any two rational numbers are equivalent.*

Proof. We will show that any rational number is equivalent to 0 which would imply that any two rational numbers are equivalent. Let a/b be a non-zero rational number where a and b are coprime integers. Consider the numbers $a, 2a, 3a, ..., (b-1)a$. Since a and b are coprime, b does not divide any of these numbers. Let us assume that no number in this sequence leaves the remainder 1 when divided by b. Therefore, two of these numbers must leave the same remainder when divided by b.
Say $m_1 a$ and $m_2 a$ leave the same remainder when divided by b with $m_1 < m_2$. Then b divides $(m_2 - m_1)a$ which is a contradiction as $m_2 - m_1$ is less than b and $gcd(a, b) = 1$. Hence there exists integers s and t such that $as = bt + 1$.
Then

$$\frac{a}{b} = \frac{t.0 + a}{s.0 + b}$$

with $tb - sa = (-1)$. This is precisely the condition which says a/b is equivalent to 0.

Q.E.D.

In the above proof we have used the pigeonhole principle to provide a proof by contradiction. A more direct proof is explained in the remarks below.

Remarks:

1. The statement that there exists integers s and t such that $as - bt = 1$ when a and b are coprime is called Bezout's identity.

2. An alternative method to prove this using continued fractions is as follows:

Take the continued fraction expression for $\frac{a}{b}$. Say $\frac{a}{b} = [a_0, a_1, ..., a_n]$. Then $\frac{a}{b} = \frac{p_n}{q_n}$ which implies $a = p_n, b = q_n$ since a and b are coprime. Theorem 2.5 implies $aq_{n-1} - bp_{n-1} = (-1)^{n-1}$. Either $(s,t) = (q_{n-1}, p_{n-1})$ or $(s,t) = (-q_{n-1}, -p_{n-1})$ depending on whether n is odd or even.

Theorem 7.9. *Two irrational numbers ξ and η are equivalent if and only if*

$$\xi = [a_0, a_1, ..., a_m, c_0, c_1, c_2, ...],\ \eta = [b_0, b_1, ..., b_n, c_0, c_1, ...],$$

the sequence of quotients in ξ after the m-th being the same as the sequence in η after the n-th for some m and n.

Proof. Suppose ξ and η have continued fraction expressions as in the theorem, then let $\omega = [c_0, c_1, ...]$. Then

$$\xi = [a_0, a_1, ..., a_m, \omega] = \frac{p_m \omega + p_{m-1}}{q_m \omega + q_{m-1}}$$

with $p_m q_{m-1} - p_{m-1} q_m = (-1)^{(m-1)}$ and hence ξ is equivalent to ω. Simliarly, η is equivalent to ω which implies ξ is equivalent to η.

We will not prove the second part of the theorem here.

Q.E.D.

This says that two irrational numbers are equivalent precisely when their continued fraction expansions are eventually the same.

8 Appendix

8.1 Special constants

Here are some continued fraction expansions of some special numbers which have been known for a long time:

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + ...}}}}}}} = [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, ...]$$

The first few convergents of π are

$$[3] = \frac{3}{1} = 3$$
$$[3, 7] = \frac{22}{7} = 3.\overline{142857}$$
$$[3, 7, 15] = \frac{333}{106} = 3.14150943...$$
$$[3, 7, 15, 1] = \frac{355}{113} = 3.14159292...$$
$$[3, 7, 15, 1, 292] = \frac{103993}{33102} = 3.14159265...$$

This says that $\frac{355}{113}$ is the best approximation to π by a rational number with denominator ≤ 113. In fact, an easy way to remember this number is to write the first three odd numbers twice each, 113355, and the first three digits form the denominator while the next three digits form the numerator.
Here are a few more general continued fraction expressions

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + ...}}}}$$

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{3 + \cfrac{2^2}{5 + \cfrac{3^2}{7 + \cfrac{4^2}{9 + ...}}}}$$

$$\pi = 3 + \cfrac{1^2}{6 + \cfrac{3^2}{6 + \cfrac{5^2}{6 + \cfrac{7^2}{6 + ...}}}}$$

$$\frac{\pi}{2} = 1 + \cfrac{1}{1 + \cfrac{1 \times 2}{1 + \cfrac{2 \times 3}{1 + \cfrac{3 \times 4}{1 + ...}}}}$$

$$e - 1 = 1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + ...}}}}$$

$$e - 1 = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + ...}}}}$$

8.2 Applications

1. Along the lines of Bezout's identity, we can use continued fractions to find integer solutions for the equation $ax+by=c$, where a, b, c are integers. If the greatest common divisor of a and b does not divide c, then there are no integer solutions.

   Example: $2x + 4y = 7$ has no integer solutions.

   We might as well assume, a, b are coprime. Then, Bezout's identity says there exist integers s, t such that $as - bt = 1$. Hence, $x = cs, y = -ct$ is a solution to $ax + by = c$. In fact, these integers have been explicitly constructed using continued fractions.

2. To show that the real numbers are uncountable, we typically use Cantor's diagonalization argument on the decimal expansion of real numbers. However, the argument is not very clean because the decimal expansion of a real number is not unique.

   Example: $0.\bar{9} = 1$

   Instead of the decimal expansion, we can use the continued fraction expressions.

   Cantor's diagonalization argument says that the infinite continued fractions are uncountable. Since the set of infinite continued fractions is in bijection with the set of irrational numbers, this says that the irrational numbers are uncountable. The finite continued fractions are in 2-1 correspondence with the set of rational numbers and hence the set of rational numbers is countable.

   There are a lot of applications of continued fractions since they give good approximations to real numbers. Here are some of them:

3. They are used in constructing clockwork models of the solar system (orrery). On a related subject, one can construct simple calendars which give a better approximation to a solar year than the current calendar we have (where leap years are omitted).

4. Phyllotaxis is the study of the arrangement of leaves (or any such botanical unit) around an axis or a stem. The numbers arising from such arrangements are very closely connected to simple continued fractions and their properties.

5. Continued fractions are used to study solutions of Diophantine equations. A fundamental solution for Pell's equation which is a particular kind of Diophantine equation

can be computed using continued fractions. There are algorithms known to compute these fundamental solutions using continued fractions.

6. In the study of rational tangles and knots, there is a correspondence with continued fractions. These are especially useful to distinguish rational tangles.

7. There are some trignometric formulae which are known to have simple expressions (involving square roots). Such expressions can be computed for many other angles by studying continued fractions. This heavily uses the idea that periodic continued fractions correspond to quadratic surds.

8. The periodic continued fractions are used in continued fraction factorization method (CFRAC) to provide an algorithm for (large) integer factorization. Further work has been done in this area using the properties of continued fractions.

9 Project Ideas

9.1 Cantor Set

Georg Cantor described a subset of $[0,1]$ interval which is now called the Cantor set (denoted by $\mathcal{C}$). It is constructed iteratively by removing the middle third from a set of line segments starting from $[0,1]$. The first step is to remove $(\frac{1}{3},\frac{2}{3})$ to get $[0,\frac{1}{3}]\cup[\frac{2}{3},1]$. The second step is to remove the middle thirds from the two line segments obtained in the previous step to get $[0,\frac{1}{9}]\cup[\frac{2}{9},\frac{1}{3}]\cup[\frac{2}{3},\frac{7}{9}]\cup[\frac{8}{9},1]$. We keep removing the middle thirds and whatever set we end up with is called the Cantor set.
We know that any number in $[0,2]$ can be written as a sum of two numbers from $[0,1]$. After the first step, observe that any number in $[0,2]$ can still be written as a sum of two numbers from $[0,\frac{1}{3}]\cup[\frac{2}{3},1]$. Does this hold true at every step of the process?
Lets fix a postive integer N and look at all continued fractions with bounded terms. Instead of $[0,2]$, can we now write some other interval as a sum of two numbers from this set?

- The primary goal of this project is to understand the Cantor set and what happens to $\mathcal{C}+\mathcal{C}$.
- The secondary goal is to relate this to continued fractions with bounded terms. i.e Expressions of the form

$$a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+...}}}$$

  where $a_0,a_1,a_2,...$ are bounded above by some fixed integer N.

9.2 Pell's equation

$x^2-dy^2=1$ is called Pell's equation. This equation was connected with the name of John Pell by Euler in a letter (10 August 1730) to Goldbach because Euler believed Pell responsible for a solution technique. A more general equation is $x^2-dy^2=N$ where N is some fixed integer and d is an integer greater than 1 that is not a perfect square. The objective is to find integer solutions of these equations.
For instance, consider the equation $x^2-3y^2=1$. The continued fraction of $\sqrt{3}$ is

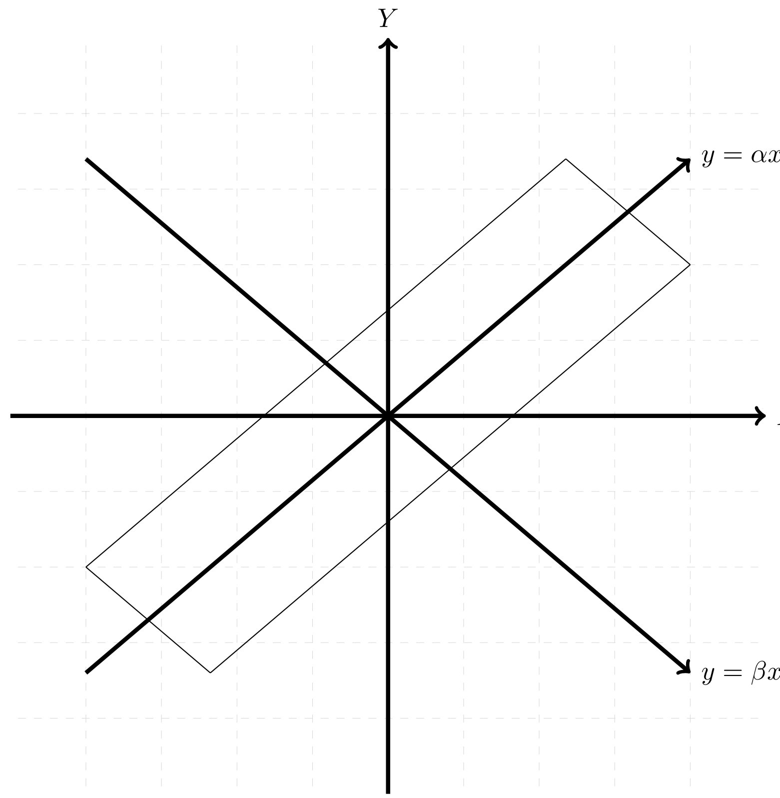
$$1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+...}}}}=[1,\overline{1,2}]$$

Observe that the convergents of this continued fraction are $\frac{1}{1}, \frac{2}{1}, \frac{5}{3}, \frac{7}{4}, \frac{19}{11}, \frac{26}{15}, ...$ and the integer solutions of Pell's equation are $(2,1),(7,4),(26,15),...$ which are the numerators and denominators of some of the convergents. Does this pattern hold in general? If so, which convergents give us solutions of Pell's equation?

- The primary goal is to relate the integer solutions to convergents of an appropriate continued fraction.
- The secondary goal is to relate the length of periodic continued fractions to solutions of the above equations when N is negative.

9.3 Empty parallelogram theorem

Let α and β be two different irrational numbers and consider the parallelogram in the plane with sides parallel to the lines $y = \alpha x$ and $y = \beta x$ and with center at the origin. What can we say about the area of such a parallelogram if we impose the condition that it shouldn't contain any lattice points apart from the origin? (A lattice point is a point on the plane with integer coordinates.)
Is there a value such that for any such parallelogram with area greater than this value, the parallelogram definitely contains a lattice point? Interestingly, it turns out that there is a lower bound for such a value that is independent of α and β which involves the golden ratio. The goal of this project is to understand the proof of this theorem using continued fractions. This project is similar to the topic of computing convergents of a continued fraction (of α) by seeing which lattice points occur closest to the line $y = \alpha x$.



9.4 Markov triples

A Markov triple (m_1, m_2, m_3) consists of three positive integers such that

$$m_1^2 + m_2^2 + m_3^2 = 3m_1m_2m_3$$

To each Markov triple there is a way of associating a quadratic form

$$ax^2 + bxy + cy^2$$

where a, b, c are integers that depend on m_1, m_2, m_3. A Markov number is an integer that belongs to at least one Markov triple. Two Markov triples are said to be adjacent to each other if two of their Markov numbers are the same. For example, $(5, 1, 2)$ is a Markov triple and its neighbors are $(1, 1, 2), (5, 29, 2), (5, 1, 13)$.

The goal is to start with a Markov triple and iteratively construct Markov triples by constructing the neighbors at each step. Certain periodic continued fractions appear as roots of quadratic forms associated to Markov triples. The secondary goal is to see what the roots are for the chain of Markov triples.

9.5 Calendars

In this project, the goal is to understand how to construct a calendar based on approximations of the time period of planet (using continued fractions). Are there better calendars than the one that we currenly use? If so, how better are they and why aren't we using them instead? A related topic is to understand how to choose gear sizes to construct an orrery if there is a practical constraint on the number of teeth on the gears. This problem is similar in flavor to the above problem in that it only involves integer arithmetic.

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