CONTINUUM MECHANICS
- Introduction to tensors

Attila KOSSA

Budapest University of Technology and Economics
Department of Applied Mechanics

2016.09.13.

Geometrical meaning of the scalar (or dot) product

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\varphi \tag{1}$$

where φ is the angle between the tips of $\mathbf{a}$ and $\mathbf{b}$, whereas $|\mathbf{a}|$ and $|\mathbf{b}|$ represent the length of $\mathbf{a}$ and $\mathbf{b}$. Vectors $\mathbf{a}$ and $\mathbf{b}$ are *orthogonal* (or *perpendicular* to each other) if their scalar product is zero, i.e. $\mathbf{a} \cdot \mathbf{b} = 0$. Obviously we can observe that $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$.

Geometrical meaning of the cross (or vector) product

$$\mathbf{a} \times \mathbf{b} = (|\mathbf{a}|\,|\mathbf{b}|\,\sin\varphi)\,\mathbf{e} \tag{2}$$

where $\mathbf{e}$ is a *unit vector* perpendicular to the plane spanned by vectors $\mathbf{a}$ and $\mathbf{b}$. Rotating $\mathbf{a}$ about $\mathbf{e}$ with positive angle φ carries $\mathbf{a}$ to $\mathbf{b}$. $\mathbf{a}$ and $\mathbf{b}$ are parallel if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$. It follows that $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.

Projection

Let the projection of vector $\mathbf{a}$ along the direction designated by the *unit vector* $\mathbf{e}$ be denoted by $\mathbf{a_e}$. Then

$$\mathbf{a_e} = (\mathbf{a} \cdot \mathbf{e})\,\mathbf{e} \tag{3}$$

Cartesian basis

A *Cartesian basis* defined by three mutually perpendicular vectors, $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, with the following properties:

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = 0, \qquad \mathbf{e}_1 \cdot \mathbf{e}_3 = 0, \qquad \mathbf{e}_2 \cdot \mathbf{e}_3 = 0, \tag{4}$$

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3, \qquad \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1, \qquad \mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2. \tag{5}$$

$\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are unit vectors. A Cartesian *coordinate frame* is defined by its origin O together with the right-handed orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

Component representation

Any vector $\mathbf{a}$ can be uniquely defined with the *linear combination* of the basis vectors ($\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$) as

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3, \tag{6}$$

where the *components* (a_1, a_2 and a_3) are real numbers. The components of $\mathbf{a}$ along the bases are obtained by calculating the projections

$$a_1 = \mathbf{a}\cdot\mathbf{e}_1, \qquad a_2 = \mathbf{a}\cdot\mathbf{e}_2, \qquad a_3 = \mathbf{a}\cdot\mathbf{e}_3. \tag{7}$$

Arranging the components into a 3×1 column matrix we arrive at the matrix representation of vector $\mathbf{a}$ as

$$[\mathbf{a}] = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}. \tag{8}$$

Obviously, the components of a vector $\mathbf{a}$ in other Cartesian basis will be different numbers.

Index notation I

Consider the component representation of vector **a**:

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = \sum_{i=1}^{3} a_i\mathbf{e}_i \tag{9}$$

In order to abbreviate (or simplify) the expression we can adopt the *Einstein's summation convention*: if and index appears twice in a term, then a sum must be applied over that index. Consequently, vector **a** can be given as

$$\mathbf{a} = \sum_{i=1}^{3} a_i\mathbf{e}_i = a_i\mathbf{e}_i. \tag{10}$$

Index notation II

$$\mathbf{a} = \sum_{i=1}^{3} a_i \mathbf{e}_i = a_i \mathbf{e}_i. \tag{11}$$

The index used to represent the sum is called *dummy index*. Replacing the index i in the above expression does not affect the final result, thus we can use any symbol:

$$a_i \mathbf{e}_i = a_b \mathbf{e}_b = a_M \mathbf{e}_M = a_\beta \mathbf{e}_\beta \qquad \text{etc.} \tag{12}$$

Any other index in an equation is a *free index*.

Kronecker delta symbol

The *Kronecker delta* symbol can be used to represents the components of the 3×3 identity matrix $[\mathbf{I}]$ as

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \tag{13}$$

Therefore the identity matrix can be written as

$$[\mathbf{I}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix}. \tag{14}$$

In addition, the Kronecker delta symbol represents the scalar product of the orthonormal basis:

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}. \tag{15}$$

Permutation symbol I

The *permutation symbol* is also called as *alternating symbol* or *Levi–Civita symbol*. It can be imagined as a symbol which represents 27 numbers (either 0, 1 or −1) depending on the value of the indices:

$$\epsilon_{ijk} = \begin{cases} 1 & \text{for even permutation of } ijk \\ -1 & \text{for odd permutation of } ijk \\ 0 & \text{if there is a repeated index} \end{cases} \tag{16}$$

Consequently

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1, \tag{17}$$

$$\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1, \tag{18}$$

$$\epsilon_{111} = \epsilon_{122} = \epsilon_{113} = ... = 0. \tag{19}$$

Permutation symbol II

The cross product of the basis vectors can be easily expressed using the permutation symbol as

$$\mathbf{e}_i \times \mathbf{e}_j = \epsilon_{ijk}\mathbf{e}_k, \tag{20}$$

$$\mathbf{e}_1 \times \mathbf{e}_2 = \epsilon_{123}\mathbf{e}_3 = \mathbf{e}_3, \tag{21}$$

$$\mathbf{e}_2 \times \mathbf{e}_3 = \epsilon_{231}\mathbf{e}_1 = \mathbf{e}_1, \tag{22}$$

$$\mathbf{e}_3 \times \mathbf{e}_1 = \epsilon_{312}\mathbf{e}_2 = \mathbf{e}_2. \tag{23}$$

Scalar product

The scalar product of vectors $\mathbf{a} = a_i\mathbf{e}_i$ and $\mathbf{b} = b_j\mathbf{e}_j$ is calculated as

$$\mathbf{a} \cdot \mathbf{b} = (a_i\mathbf{e}_i) \cdot (b_j\mathbf{e}_j) = a_i b_j \mathbf{e}_i \cdot \mathbf{e}_j = a_i b_j \delta_{ij} = a_i b_i, \tag{24}$$

$$a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3. \tag{25}$$

Observe the *replacement property* of δ_{ij}: If δ_{ij} appears in a term, where i (or j) is a dummy index, then it can be changed to j (or i) and δ_{ij} can be removed from the term. For example:

$$a_i b_j \delta_{ij} = a_i b_i = a_j b_j, \tag{26}$$

$$\sigma_{ab}\delta_{bk} = \sigma_{ak}, \tag{27}$$

$$c_{ijk}\delta_{jr}\delta_{sk} = c_{irk}\delta_{sk} = c_{irs}. \tag{28}$$

Cross product I

The cross product of vectors $\mathbf{a} = a_i \mathbf{e}_i$ and $\mathbf{b} = b_j \mathbf{e}_j$ is calculated as

$$\mathbf{a} \times \mathbf{b} = (a_i \mathbf{e}_i) \times (b_j \mathbf{e}_j) = a_i b_j \mathbf{e}_i \times \mathbf{e}_j = \underbrace{a_i b_j \epsilon_{ijk}}_{c_k} \mathbf{e}_k, \tag{29}$$

where (20) was applied. Using the summation convention it can be clearly concluded that $a_i b_j \epsilon_{ijk}$ is a quantity having only one index, namely k. We can denote this new quantity with c_k for simplicity, which is nothing else just the component of the new vector resulting from the cross product.

Cross product II

Therefore

$$\mathbf{a} \times \mathbf{b} = \underbrace{a_i b_j \epsilon_{ijk}}_{c_k} \mathbf{e}_k = c_k \mathbf{e}_k = \mathbf{c}, \tag{30}$$

where

$$c_1 = a_i b_j \epsilon_{ij1} = a_2 b_3 \epsilon_{231} + a_3 b_2 \epsilon_{321} = a_2 b_3 - a_3 b_2, \tag{31}$$

$$c_2 = a_i b_j \epsilon_{ij2} = a_3 b_1 \epsilon_{312} + a_1 b_3 \epsilon_{132} = a_3 b_1 - a_1 b_3, \tag{32}$$

$$c_3 = a_i b_j \epsilon_{ij3} = a_1 b_2 \epsilon_{123} + a_2 b_1 \epsilon_{213} = a_1 b_2 - a_2 b_1, \tag{33}$$

$$[\mathbf{c}] = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}. \tag{34}$$

Cross product III

We get the same result using the classical method to compute the cross product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, \tag{35}$$

$$\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2)\,\mathbf{e}_1 + (a_3 b_1 - a_1 b_3)\,\mathbf{e}_2 + (a_1 b_2 - a_2 b_1)\,\mathbf{e}_3. \tag{36}$$

Triple scalar product I

Geometrically, the *triple scalar product* between vectors **a**, **b** and **c** defines the volume of a paralelepiped spanned by **a**, **b** and **c** forming a right-handed system. Let

$$\mathbf{a} = a_i \mathbf{e}_i, \qquad \mathbf{b} = b_j \mathbf{e}_j, \qquad \mathbf{c} = c_m \mathbf{e}_m. \tag{37}$$

Then

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (a_i b_j \epsilon_{ijk} \mathbf{e}_k) \cdot (c_m \mathbf{e}_m) = a_i b_j \epsilon_{ijk} c_m (\mathbf{e}_k \cdot \mathbf{e}_m), \tag{38}$$

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = a_i b_j \epsilon_{ijk} c_m \delta_{km} = a_i b_j c_k \epsilon_{ijk}. \tag{39}$$

It can be verified that

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}. \tag{40}$$

Triple scalar product II

The triple scalar product can be calculated using determinant as

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \begin{vmatrix} [\mathbf{a}], [\mathbf{b}], [\mathbf{c}] \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \tag{41}$$

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = a_1 (b_2 c_3 - b_3 c_2) + b_1 (a_3 c_2 - a_2 c_3) + c_1 (a_2 b_3 - a_3 b_2). \tag{42}$$

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = a_1 b_j c_k \epsilon_{1jk} + b_1 a_i c_k \epsilon_{i1k} + c_1 a_i b_j \epsilon_{ij1}. \tag{43}$$

Triple vector product

Let

$$\mathbf{a} = a_q \mathbf{e}_q, \qquad \mathbf{b} = b_i \mathbf{e}_i, \qquad \mathbf{c} = c_j \mathbf{e}_j. \tag{44}$$

Then the triple vector product is obtained as

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (a_q \mathbf{e}_q) \times (b_i c_j \epsilon_{ijk} \mathbf{e}_k) = a_q b_i c_j \epsilon_{ijk} (\mathbf{e}_q \times \mathbf{e}_k), \tag{45}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \underbrace{a_q b_i c_j \epsilon_{ijk} \epsilon_{qkp}}_{d_p} \mathbf{e}_p. \tag{46}$$

Thus, the matrix representation of the resulting vector is

$$[\mathbf{a} \times (\mathbf{b} \times \mathbf{c})] = \begin{bmatrix} a_q b_i c_j \epsilon_{ijk} \epsilon_{qk1} \\ a_q b_i c_j \epsilon_{ijk} \epsilon_{qk2} \\ a_q b_i c_j \epsilon_{ijk} \epsilon_{qk3} \end{bmatrix} = \ldots \tag{47}$$

Epsilon-delta identities

The following useful identities can be easily verified:

$$\delta_{aa} = 3, \quad (48)$$

$$\epsilon_{abc}\epsilon_{abc} = 6, \quad (49)$$

$$\epsilon_{abm}\epsilon_{adm} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}, \quad (50)$$

$$\epsilon_{acd}\epsilon_{bcd} = 2\delta_{ab}. \quad (51)$$

Definition

A second-order tensor $\boldsymbol{\sigma}$ can be imagined as a *linear operator*. Applying $\boldsymbol{\sigma}$ on a vector $\mathbf{n}$ generates a new vector $\boldsymbol{\rho}$:

$$\boldsymbol{\rho} = \boldsymbol{\sigma}\mathbf{n}, \tag{52}$$

thus it defines a *linear transformation*. In hand-written notes we use double underline to indicate second-order tensors. Thus, the expression above can be written as

$$\underline{\rho} = \underline{\underline{\sigma}}\ \underline{\mathrm{n}}. \tag{53}$$

The second-order *identity tensor* $\mathbf{I}$ and the second order *zero tensor* $\mathbf{0}$ have the properties

$$\mathbf{In} = \mathbf{n}, \qquad \mathbf{0n} = \mathbf{0}. \tag{54}$$

The projection (3) can be expressed using second-order tensor $\mathbf{P}$: Acting $\mathbf{P}$ on $\mathbf{a}$ generates a new vector $\mathbf{a_e}$.

Representation in a coordinate frame

The second-order tensor $\boldsymbol{\sigma}$ has *nine components* in a given coordinate frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. The components σ_{ij} are computed by

$$\sigma_{ij} = \mathbf{e}_i \cdot (\boldsymbol{\sigma} \mathbf{e}_j) \,. \tag{55}$$

The matrix representation of $\boldsymbol{\sigma}$ in a given coordinate frame is

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} . \tag{56}$$

Dyadic product of two vectors

The matrix representation of the *dyadic* (or *tensor* or *direct*) product of vector **a** and **b** is

$$[\mathbf{a} \otimes \mathbf{b}] = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}, \tag{57}$$

$$[\mathbf{a} \otimes \mathbf{b}] = [\mathbf{a}] [\mathbf{b}]^T = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}. \tag{58}$$

The ij-th component of the resulting second-order tensor is $a_i b_j$. It can be seen that

$$(\mathbf{a} \otimes \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \cdot \mathbf{c})\, \mathbf{a}. \tag{59}$$

Representation of second-order tensors with dyads

The second-order tensor $\mathbf{A}$ can be written as the linear combination of the *dyads* formed by the basis vectors:

$$\mathbf{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j. \tag{60}$$

Thus, the identity tensor can be written as

$$\mathbf{I} = \delta_{ij}\mathbf{e}_i \otimes \mathbf{e}_j \tag{61}$$

The dyad $\mathbf{a}\otimes\mathbf{b}$ is a 2nd-order tensor, but not all 2nd-order tensor can be written as a dyadic product of two vectors! In general, a 2nd-order tensor has 9 components, whereas a dyad has only 6 components (2×3)

Indical notation I

Consider the equation

$$\mathbf{a} = \mathbf{b} + \mathbf{M}\mathbf{c}. \tag{62}$$

Its matrix representation is

$$[\mathbf{a}] = [\mathbf{b}] + [\mathbf{M}]\,[\mathbf{c}]\,, \tag{63}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}. \tag{64}$$

It can be formulated using *indical notation* as

$$a_i = b_i + M_{ij} c_j. \tag{65}$$

Indical notation II

- It should be observed that the same free index must appear in every term of an equation.
- The indical notation is an order-independent representation. In matrix notation the order of the multiplication cannot be changed, however, in the indical notation (using the summation convention) the terms can be rearranged without altering the result. Example: $\mathbf{Ab} \neq \mathbf{bA}$, but $A_{ij}b_j = b_jA_{ij}$.
- "*The essence of the Einstein summation notation is to create a set of notational defaults so that the summation sign and the range of the subscripts do not need to be written explicitly in each expression.*"
- "*It is a collection of time-saving conventions. After an initial investment of time, it converts difficult problems into problems with workable solutions. It does not make easy problem easier, however.*"

Trace

The *trace* of the second-order tensor $\mathbf{A}$ is

$$\mathrm{tr}\mathbf{A} = \mathrm{tr}\,[\mathbf{A}] = A_{11} + A_{22} + A_{33} = A_{ii}. \tag{66}$$

It is an *invariant quantity*.

Determinant

The *determinant* of the second-order tensor $\mathbf{A}$ is

$$\det\mathbf{A} = \det\left[\mathbf{A}\right], \tag{67}$$

$$\begin{aligned}\det\mathbf{A} &= A_{11}\left(A_{22}A_{33} - A_{23}A_{32}\right) - A_{12}\left(A_{21}A_{33} - A_{23}A_{31}\right) \\ &\quad + A_{13}\left(A_{21}A_{32} - A_{22}A_{31}\right), \end{aligned} \tag{68}$$

$$\det\mathbf{A} = \epsilon_{ijk}A_{1i}A_{2j}A_{3k}, \tag{69}$$

It is an *invariant quantity*. $\mathbf{A}$ is *singular* when $\det\mathbf{A} = 0$.
Useful relation

$$\det\left(\mathbf{AB}\right) = \det\mathbf{A}\det\mathbf{B}, \qquad \det\mathbf{A}^T = \det\mathbf{A} \tag{70}$$

Double contraction

The *double contraction* (or *double-dot product*) between 2nd-order tensors $\mathbf{A}$ and $\mathbf{B}$ is defined as

$$\mathbf{A} : \mathbf{B} = A_{ij} B_{ij} = A_{11} B_{11} + A_{12} B_{12} + \ldots, \tag{71}$$

$$\mathbf{A} : \mathbf{B} = \mathrm{tr}\left(\mathbf{A}^T \mathbf{B}\right) = \mathrm{tr}\left(\mathbf{B}^T \mathbf{A}\right) = \mathrm{tr}\left(\mathbf{A}\mathbf{B}^T\right) = \mathrm{tr}\left(\mathbf{B}\mathbf{A}^T\right). \tag{72}$$

Thus, the trace of $\mathbf{A}$ can be written as

$$\mathrm{tr}\mathbf{A} = \mathbf{I} : \mathbf{A} = \left(\delta_{ij} \mathbf{e}_i \otimes \mathbf{e}_j\right) : \left(A_{mn} \mathbf{e}_m \otimes \mathbf{e}_n\right) = \delta_{ij} A_{mn} \delta_{im} \delta_{jn_{ij}}, \tag{73}$$

$$\mathrm{tr}\mathbf{A} = \delta_{ij} A_{ij} = A_{ii} = A_{11} + A_{22} + A_{33}. \tag{74}$$

Norm

The norm of the second-order tensor **A** is calculated as

$$\|\mathbf{A}\| = \sqrt{\mathbf{A} : \mathbf{A}} \geq 0. \tag{75}$$

Symmetric and skew-symmetric parts I

The following identity holds for the *transpose* of **A**:

$$(\mathbf{Au}) \cdot \mathbf{v} = \left(\mathbf{uA}^T\right) \cdot \mathbf{v}. \tag{76}$$

A can be decomposed into the sum of a *symmetric* and a *skew-symmetric* parts as

$$\mathbf{A} = \mathbf{A}_{\text{symm}} + \mathbf{A}_{\text{skew}}, \tag{77}$$

$$\mathbf{A}_{\text{symm}} = \text{symm}\,(\mathbf{A}) = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^T\right), \tag{78}$$

$$\mathbf{A}_{\text{skew}} = \text{skew}\,(\mathbf{A}) = \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^T\right). \tag{79}$$

Symmetric and skew-symmetric parts II

Thus

$$\mathbf{A}_{\text{symm}} = \mathbf{A}_{\text{symm}}^{T}, \qquad \mathbf{A}_{\text{skew}} = -\mathbf{A}_{\text{skew}}^{T}. \tag{80}$$

$$\left(\mathbf{A}_{\text{symm}}\right)_{ab} = \left(\mathbf{A}_{\text{symm}}\right)_{ba} \tag{81}$$

$$\left(\mathbf{A}_{\text{skew}}\right)_{ab} = -\left(\mathbf{A}_{\text{skew}}\right)_{ba} \tag{82}$$

Symmetric part has 6, whereas the skew-symmetric part has 3 independent components.

Symmetric and skew-symmetric parts III

A skew-symmetric tensor $\mathbf{W}$ behaves like a vector. The following relation can be easily verified:

$$\mathbf{W}\mathbf{u} = \boldsymbol{\omega} \times \mathbf{u}, \tag{83}$$

$$[\mathbf{W}] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \implies [\boldsymbol{\omega}] = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}, \tag{84}$$

$$|\boldsymbol{\omega}| = \frac{1}{\sqrt{2}} \|\mathbf{W}\|. \tag{85}$$

Inverse

The inverse $\mathbf{A}^{-1}$ of $\mathbf{A}$ is defined as

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}. \tag{86}$$

A necessary and sufficient condition for the existence of $\mathbf{A}^{-1}$ is that $\det\mathbf{A} \neq 0$.

For invertible tensors $\mathbf{A}$ and $\mathbf{B}$:

$$(\mathbf{A}\mathbf{B})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}, \tag{87}$$

$$(k\mathbf{A})^{-1} = \frac{1}{k}\mathbf{A}^{-1}, \tag{88}$$

$$\left(\mathbf{A}^T\right)^{-1} = \left(\mathbf{A}^{-1}\right)^T = \mathbf{A}^{-T}, \tag{89}$$

$$\det\left(\mathbf{A}^{-1}\right) = \frac{1}{\det\mathbf{A}}. \tag{90}$$

Orthogonal tensor

A tensor $\mathbf{Q}$ is said to be *orthogonal* if

$$\mathbf{Q}^T = \mathbf{Q}^{-1}, \tag{91}$$

$$\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I} \tag{92}$$

proper orthogonal	$\det\mathbf{Q} = 1$
inproper orthogonal	$\det\mathbf{Q} = -1$

Proper orthogonal tensors represent *rotation*, whereas *inproper* orthogonal tensors represent *reflection*.

Definiteness

For all $\mathbf{v} \neq 0$:

Positive semi-definite	$\mathbf{v} \cdot \mathbf{A}\mathbf{v} \geq 0$
Positive definite	$\mathbf{v} \cdot \mathbf{A}\mathbf{v} > 0$
Negative semi-definite	$\mathbf{v} \cdot \mathbf{A}\mathbf{v} \leq 0$
Negative definite	$\mathbf{v} \cdot \mathbf{A}\mathbf{v} < 0$

Change of basis I

Let the bases vectors of two Cartesian coordinate system (having the same origin) be denoted by

$$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \qquad \text{and} \qquad \widetilde{\mathbf{e}}_1, \widetilde{\mathbf{e}}_2, \widetilde{\mathbf{e}}_3. \tag{93}$$

Then, a vector $\mathbf{a}$ can be written as

$$\mathbf{a} = a_i \mathbf{e}_i \equiv \widetilde{a}_j \widetilde{\mathbf{e}}_j, \tag{94}$$

where the components a_i and $\widetilde{a}_j$ are obviously different. Denote Q_{ij} the scalar products between the two bases as

$$Q_{ij} = \mathbf{e}_i \cdot \widetilde{\mathbf{e}}_j. \tag{95}$$

Change of basis II

Then

$$\widetilde{\mathbf{e}}_1 = (\mathbf{e}_1 \cdot \widetilde{\mathbf{e}}_1)\,\mathbf{e}_1 + (\mathbf{e}_2 \cdot \widetilde{\mathbf{e}}_1)\,\mathbf{e}_2 + (\mathbf{e}_3 \cdot \widetilde{\mathbf{e}}_1)\,\mathbf{e}_3, \tag{96}$$

$$\widetilde{\mathbf{e}}_2 = (\mathbf{e}_1 \cdot \widetilde{\mathbf{e}}_2)\,\mathbf{e}_1 + (\mathbf{e}_2 \cdot \widetilde{\mathbf{e}}_2)\,\mathbf{e}_2 + (\mathbf{e}_3 \cdot \widetilde{\mathbf{e}}_2)\,\mathbf{e}_3, \tag{97}$$

$$\widetilde{\mathbf{e}}_3 = (\mathbf{e}_1 \cdot \widetilde{\mathbf{e}}_3)\,\mathbf{e}_1 + (\mathbf{e}_2 \cdot \widetilde{\mathbf{e}}_3)\,\mathbf{e}_2 + (\mathbf{e}_3 \cdot \widetilde{\mathbf{e}}_3)\,\mathbf{e}_3. \tag{98}$$

Thus

$$\widetilde{\mathbf{e}}_j = Q_{ij}\mathbf{e}_i \qquad \text{and} \qquad \mathbf{e}_i = Q_{ij}\widetilde{\mathbf{e}}_j. \tag{99}$$

Combining (94) and (99) gives

$$a_i\mathbf{e}_i = \widetilde{a}_j Q_{ij}\mathbf{e}_i \qquad \text{and} \qquad a_i Q_{ij}\widetilde{\mathbf{e}}_j = \widetilde{a}_j\widetilde{\mathbf{e}}_j. \tag{100}$$

Thus

$$a_i = Q_{ij}\widetilde{a}_j \qquad \text{and} \qquad \widetilde{a}_j = a_i Q_{ij}. \tag{101}$$

Change of basis III

It can be more easily expressed using matrix notation as

$$[\mathbf{a}] = [\mathbf{Q}]\,[\widetilde{\mathbf{a}}] \qquad \text{and} \qquad [\widetilde{\mathbf{a}}] = [\mathbf{Q}]^T\,[\mathbf{a}]\,, \tag{102}$$

where $\mathbf{Q}$ contains the angle cosines as

$$[\mathbf{Q}] = \begin{bmatrix} \mathbf{e}_1\cdot\widetilde{\mathbf{e}}_2 & \mathbf{e}_1\cdot\widetilde{\mathbf{e}}_2 & \mathbf{e}_1\cdot\widetilde{\mathbf{e}}_3 \\ \mathbf{e}_2\cdot\widetilde{\mathbf{e}}_2 & \mathbf{e}_2\cdot\widetilde{\mathbf{e}}_2 & \mathbf{e}_2\cdot\widetilde{\mathbf{e}}_3 \\ \mathbf{e}_3\cdot\widetilde{\mathbf{e}}_2 & \mathbf{e}_3\cdot\widetilde{\mathbf{e}}_2 & \mathbf{e}_3\cdot\widetilde{\mathbf{e}}_3 \end{bmatrix}. \tag{103}$$

For 2nd-order tensor $\mathbf{A}$:

$$[\mathbf{A}] = [\mathbf{Q}]\left[\widetilde{\mathbf{A}}\right][\mathbf{Q}]^T \qquad \text{and} \qquad \left[\widetilde{\mathbf{A}}\right] = [\mathbf{Q}]^T\,[\mathbf{A}]\,[\mathbf{Q}]\,. \tag{104}$$

Deviatoric and spherical parts

Every tensor A can be decomposed into a *deviatoric* and *spherical* part as

$$\mathbf{A} = \operatorname{dev}(\mathbf{A}) + \operatorname{sph}(\mathbf{A}), \tag{105}$$

where

$$\operatorname{sph}(\mathbf{A}) = p\mathbf{I} = \left(\frac{1}{3}\operatorname{tr}\mathbf{A}\right)\mathbf{I}, \tag{106}$$

$$\operatorname{dev}(\mathbf{A}) = \mathbf{A} - \left(\frac{1}{3}\operatorname{tr}\mathbf{A}\right)\mathbf{I}. \tag{107}$$

Eigenvalues, eigenvectors I

An *eigenpair* of a 2nd-order tensor $\mathbf{A}$ mean a scalar λ_i and unit vector $\mathbf{n}_i$ satisfying

$$\mathbf{A}\mathbf{n}_i = \lambda_i \mathbf{n}_i \qquad \text{with} \qquad i = 1, 2, 3. \tag{108}$$

λ_i are the *eigenvalues* (or principal values), whereas $\mathbf{n}_i$ denote the *normalized eigenvectors* (principal directions, principal axes).

The eigenvalues are the roots of the *characteristic cubic equation*

$$\det(\mathbf{A} - \lambda_i \mathbf{I}) = 0. \tag{109}$$

Eigenvalues, eigenvectors II

The eigenvectors are defined by the linear homogeneous equations

$$(\mathbf{A} - \lambda_i \mathbf{I})\,\mathbf{n}_i = \mathbf{0}. \tag{110}$$

- Eigenvalues of *symmetric* **A** are *reals*.
- Eigenvalues of *positive definite symmetric* **A** are *strictly positive*.
- Eigenvectors of *symmetric* **A** form *mutually orthogonal basis*.

Spectral decomposition

Any symmetric 2nd-order tensor $\mathbf{A}$ can be represented by its eigenvalues λ_i and eigenvectors $\mathbf{n}_i$ as

$$\mathbf{A} = \sum_{i=1}^{3} \lambda_i \left(\mathbf{n}_i \otimes \mathbf{n}_i\right) = \sum_{i=1}^{3} \lambda_i \mathbf{m}_i, \tag{111}$$

where $\mathbf{m}_i = \mathbf{n}_i \otimes \mathbf{n}_i$ is the *basis tensor* (or projection tensor).
Matrix representation of $\mathbf{A}$ in the coordinate system formed by its eigenvectors $\mathbf{n}_i$ is

$$[\mathbf{A}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}. \tag{112}$$

Principal invariants

The *principal scalar invariants* of the 2nd-order tensor **A** are

$$I_1 = \mathrm{tr}\mathbf{A} = \lambda_1 + \lambda_2 + \lambda_3, \quad (113)$$

$$I_2 = \frac{1}{2}\left((\mathrm{tr}\mathbf{A})^2 - \mathrm{tr}\left(\mathbf{A}^2\right)\right) = \mathrm{tr}\left(\mathbf{A}^{-1}\right)\det\mathbf{A} = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3, \quad (114)$$

$$I_3 = \det\mathbf{A} = \lambda_1\lambda_2\lambda_3. \quad (115)$$

Cayley-Hamilton theorem

The *Cayley-Hamilton theorem* states that the 2nd-order tensor $\mathbf{A}$ satisfies its characteristic equation. Thus

$$\mathbf{A}^3 - I_1\mathbf{A}^2 + I_2\mathbf{A} - I_3\mathbf{I} = \mathbf{0}, \tag{116}$$

where I_1, I_2 and I_3 are the principal invariants of $\mathbf{A}$.

Indical notation

Commas in the subscript mean that a partial derivative is to be applied. The index after the comma represents partial derivatives with respect to the default arguments, which are usually the coordinates x_1, x_2 and x_3.

$$u_{i,j} = \frac{\partial u_i}{\partial x_j}. \tag{117}$$

Example:

$$a_i u_{i,j} = a_i \frac{\partial u_i}{\partial x_j} = a_1 \frac{\partial u_1}{\partial x_j} + a_2 \frac{\partial u_2}{\partial x_j} + a_3 \frac{\partial u_3}{\partial x_j}. \tag{118}$$

Nabla operator

The *nabla operator* (or *del operator* or *vector-differential operator*) is defined as

$$\nabla(\bullet) = \frac{\partial(\bullet)}{\partial x_i}\mathbf{e}_i, \tag{119}$$

$$\nabla(\bullet) = \frac{\partial(\bullet)}{\partial x_1}\mathbf{e}_1 + \frac{\partial(\bullet)}{\partial x_2}\mathbf{e}_2 + \frac{\partial(\bullet)}{\partial x_3}\mathbf{e}_3. \tag{120}$$

Gradient of a scalar field

The *gradient* of the smooth *scalar* field $T(\mathbf{x})$ (or $T(x_i)$ or $T(x_1, x_2, x_3)$) is the vector field

$$\mathrm{grad}T = \nabla T = \frac{\partial T}{\partial x_i}\mathbf{e}_i = \frac{\partial T}{\partial x_1}\mathbf{e}_1 + \frac{\partial T}{\partial x_2}\mathbf{e}_2 + \frac{\partial T}{\partial x_3}\mathbf{e}_3 \tag{121}$$

Indical notation:

$$T_{,i} = \frac{\partial T}{\partial x_i} \tag{122}$$

The matrix representation of the resulting vector:

$$[\mathrm{grad}T] = \begin{bmatrix} T_{,1} \\ T_{,2} \\ T_{,3} \end{bmatrix} \tag{123}$$

Directional derivative

- $T(\mathbf{x}) = \text{constant}$ denotes level surface
- The normal to the surface is $\text{grad}T$
- The unit normal is $\mathbf{n} = \frac{\text{grad}T}{|\text{grad}T|}$
- The directional derivative of T at $\mathbf{x}$ in the direction of a normalized vector $\mathbf{u}$ is $(\text{grad}T) \cdot \mathbf{u}$
- It takes the maximum (minimum) when $\mathbf{u} = \mathbf{n}$ ($\mathbf{u} = -\mathbf{n}$)
- The particular directional derivative $(\text{grad}T) \cdot \mathbf{n} = |\text{grad}T|$ is called as normal derivative

Gradient of a vector field

The gradient of the vector field $\mathbf{u}(\mathbf{x})$ is the second-order tensor field

$$\text{grad}\mathbf{u} = \nabla\mathbf{u} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \mathbf{u} \otimes \nabla = \frac{\partial u_i}{\partial x_j}\mathbf{e}_i \otimes \mathbf{e}_j \tag{124}$$

Matrix representation:

$$[\text{grad}\mathbf{u}] = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & u_{2,2} & u_{2,3} \\ u_{3,1} & u_{2,2} & u_{3,3} \end{bmatrix} \tag{125}$$

Transposed gradient:

$$\text{grad}^T\mathbf{u} = \nabla \otimes \mathbf{u} = \frac{\partial u_i}{\partial x_j}\mathbf{e}_j \otimes \mathbf{e}_i \tag{126}$$

Gradient of a 2nd-order tensor field

The gradient of the second-order tensor field $\mathbf{A}$ is the third-order tensor field

$$\begin{aligned} \text{grad}\mathbf{A} = \mathbf{A} \otimes \nabla &= (A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j) \otimes \left(\frac{\partial (\bullet)}{\partial x_k} \mathbf{e}_k \right) && (127) \\ &= \frac{\partial A_{ij}}{\partial x_k} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k = A_{ij,k}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k && (128) \end{aligned}$$

Thus, the ijk-th component is

$$(\text{grad}\mathbf{A})_{ijk} = A_{ij,k} \tag{129}$$

Divergence of a scalar field

Meaningless

Divergence of a vector field

The divergence of the vector field $\mathbf{a} = a_i \mathbf{e}_i$is the scalar field

$$\mathrm{div}\mathbf{a} = \nabla \cdot \mathbf{a} = \left(\frac{\partial\left(\bullet\right)}{\partial x_j} \mathbf{e}_j \right) \cdot \left(a_i \mathbf{e}_i\right) = \frac{\partial a_i}{\partial x_j} \delta_{ij} = \frac{\partial a_i}{\partial x_i} = a_{i,i} \qquad (130)$$

$$\mathrm{div}\mathbf{a} = \mathrm{tr}\left(\mathrm{grad}\mathbf{a}\right) \qquad (131)$$

If $\mathrm{div}\mathbf{a} = 0$ then $\mathbf{a}$ is said to be *solenoidal* (or *divergence-free* or *incompressible*).

Divergence of a 2nd-order tensor field

The divergence of the 2nd-order tensor field $\boldsymbol{\sigma}$ is the vector field

$$\text{div}\boldsymbol{\sigma} = \boldsymbol{\sigma} \cdot \nabla = (\sigma_{ij}\mathbf{e}_i \otimes \mathbf{e}_j) \cdot \left(\frac{\partial (\bullet)}{\partial x_k} \mathbf{e}_k \right) = \frac{\partial \sigma_{ij}}{\partial x_k} \delta_{jk} \mathbf{e}_i = \sigma_{ij,j} \mathbf{e}_i \tag{132}$$

Thus, the matrix representation is

$$[\text{div}\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_{1j,j} \\ \sigma_{2j,j} \\ \sigma_{3j,j} \end{bmatrix} = \begin{bmatrix} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} \\ \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} \\ \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} \end{bmatrix} \tag{133}$$

Curl of a scalar field

Meaningless

Curl of a vector field

The curl of the vector field $\mathbf{a} = a_j \mathbf{e}_j$ is the vector field

$$\text{curl}\mathbf{a} = \text{rot}\mathbf{a} = \nabla \times \mathbf{a} = \left(\frac{\partial}{\partial x_i} \mathbf{e}_i \right) \times (a_j \mathbf{e}_j) \tag{134}$$

$$= \frac{\partial a_j}{\partial x_i} \epsilon_{ijk} \mathbf{e}_k = a_{j,i} \epsilon_{ijk} \mathbf{e}_k \tag{135}$$

$$[curl\mathbf{a}] = \begin{bmatrix} a_{3,2} - a_{2,3} \\ a_{1,3} - a_{3,1} \\ a_{2,1} - a_{1,2} \end{bmatrix} \tag{136}$$

If $\text{curl}\mathbf{a} = 0$ then the vector field is *irrotational* (or *conservative* or *curl-free*).

If $\mathbf{a}$ can be expressed as $\mathbf{a} = \text{grad}\phi$, where ϕ is the potential of $\mathbf{a}$, then $\mathbf{a}$ is irrotational, beacause of the identity $\text{curl}\,(\text{grad}\phi) = \mathbf{0}$.

Curl of a 2nd-order tensor field

The curl of the 2nd-order tensor field $\mathbf{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ is the 2nd-order tensor field

$$\text{curl}\mathbf{A} = \nabla \times \mathbf{A} = \left(\frac{\partial}{\partial x_k}\mathbf{e}_k\right) \times (A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j) \tag{137}$$

$$= \frac{\partial A_{ij}}{\partial x_k}\epsilon_{kim}\mathbf{e}_m \otimes \mathbf{e}_j = A_{ij,k}\epsilon_{kim}\mathbf{e}_m \otimes \mathbf{e}_j \tag{138}$$

$$[\text{curl}\mathbf{A}] = \begin{bmatrix} A_{i1,k}\epsilon_{ki1} & A_{i2,k}\epsilon_{ki1} & A_{i3,k}\epsilon_{ki1} \\ A_{i1,k}\epsilon_{ki1} & A_{i2,k}\epsilon_{ki1} & A_{i3,k}\epsilon_{ki1} \\ A_{i1,k}\epsilon_{ki1} & A_{i2,k}\epsilon_{ki1} & A_{i3,k}\epsilon_{ki1} \end{bmatrix} \tag{139}$$

where $A_{i1,k}\epsilon_{ki1} = A_{31,2} - A_{21,3} = \frac{\partial A_{31}}{\partial x_2} - \frac{\partial A_{21}}{\partial x_3}$ for instance.

Laplacian of a scalar field

The Laplacian operator is defined as

$$\triangle(\bullet) = \nabla^2(\bullet) = \nabla(\bullet)\cdot\nabla(\bullet) = \left(\frac{\partial(\bullet)}{\partial x_i}\mathbf{e}_i\right)\cdot\left(\frac{\partial(\bullet)}{\partial x_j}\mathbf{e}_j\right) \tag{140}$$

$$\triangle(\bullet) = \frac{\partial^2(\bullet)}{\partial x_i \partial x_j}\mathbf{e}_i\cdot\mathbf{e}_j = \frac{\partial^2(\bullet)}{\partial x_i \partial x_j}\delta_{ij} = \frac{\partial^2(\bullet)}{\partial x_i \partial x_i} \tag{141}$$

$$\triangle(\bullet) = \frac{\partial^2(\bullet)}{\partial x_i^2} = \frac{\partial^2(\bullet)}{\partial x_1^2} + \frac{\partial^2(\bullet)}{\partial x_2^2} + \frac{\partial^2(\bullet)}{\partial x_3^2} \tag{142}$$

The Laplacian of a scalar field T is the scalar field

$$\triangle T = \frac{\partial^2 T}{\partial x_1^2} + \frac{\partial^2 T}{\partial x_2^2} + \frac{\partial^2 T}{\partial x_3^2} = T_{,ii} \tag{143}$$

Laplacian of a vector field

The Laplacian of a vector field $\mathbf{u} = u_i \mathbf{e}_i$ is the vector field

$$\triangle \mathbf{u} = \frac{\partial^2 (u_i \mathbf{e}_i)}{\partial x_j^2} = (u_{i,j})_{,j}\, \mathbf{e}_i = u_{i,jj} \mathbf{e}_i \qquad (144)$$

$$[\triangle \mathbf{u}] = \begin{bmatrix} \frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_2^2} \\ \frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_2^2} \\ \frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} + \frac{\partial^2 u_3}{\partial x_2^2} \end{bmatrix} \qquad (145)$$