Continuum Mechanics

Volume II of Lecture Notes on the Mechanics of Solids

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Please send corrections, suggestions and comments to *abeyaratne.vol.2@gmail.com*

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Dedicated to
Pods and Nangi
for their gifts of love and presence.

NOTE TO READER

I had hoped to finalize this second set of notes an year or two after publishing Volume I of this series back in 2007. However I have been distracted by various other interesting tasks and it has sat on a back-burner. Since I continue to receive email requests for this second set of notes, I am now making Volume II available *even though it is not as yet complete*. In addition, it has been "cleaned-up" at a far more rushed pace than I would have liked.

In the future, I hope to sufficiently edit my notes on Viscoelastic Fluids and Micromechanical Models of Viscoelastic Fluids so that they may be added to this volume; and if I ever get around to it, a chapter on the mechanical response of materials that are affected by electromagnetic fields.

I would be most grateful if the reader would please inform me of any errors in the notes by emailing me at *abeyaratne.vol.2@gmail.com*.

PREFACE

During the period 1986 - 2008, the Department of Mechanical Engineering at MIT offered a series of graduate level subjects on the Mechanics of Solids and Structures that included:

2.071: Mechanics of Solid Materials,
2.072: Mechanics of Continuous Media,
2.074: Solid Mechanics: Elasticity,
2.073: Solid Mechanics: Plasticity and Inelastic Deformation,
2.075: Advanced Mechanical Behavior of Materials,
2.080: Structural Mechanics,
2.094: Finite Element Analysis of Solids and Fluids,
2.095: Molecular Modeling and Simulation for Mechanics, and
2.099: Computational Mechanics of Materials.

Over the years, I have had the opportunity to regularly teach the second and third of these subjects, 2.072 and 2.074 (formerly known as 2.083), and the current four volumes are comprised of the lecture notes I developed for them. First drafts of these notes were produced in 1987 (Volumes I and IV) and 1988 (Volumes II) and they have been corrected, refined and expanded on every subsequent occasion that I taught these classes. The material in the current presentation is still *meant to be a set of lecture notes, not a text book*. It has been organized as follows:

Volume I: A Brief Review of Some Mathematical Preliminaries

Volume II: Continuum Mechanics

Volume III: A Brief Introduction to Finite Elasticity

Volume IV: Elasticity

This is Volume II.

My appreciation for mechanics was nucleated by Professors Douglas Amarasekara and Munidasa Ranaweera of the (then) University of Ceylon, and was subsequently shaped and grew substantially under the influence of Professors James K. Knowles and Eli Sternberg of the California Institute of Technology. I have been most fortunate to have had the opportunity to apprentice under these inspiring and distinctive scholars.

I would especially like to acknowledge the innumerable illuminating and stimulating interactions with my mentor, colleague and friend the late Jim Knowles. His influence on

me cannot be overstated.

I am also indebted to the many MIT students who have given me enormous fulfillment and joy to be part of their education.

I am deeply grateful for, and to, Curtis Almquist SSJE, friend and companion.

My understanding of elasticity as well as these notes have benefitted greatly from many useful conversations with Kaushik Bhattacharya, Janet Blume, Eliot Fried, Morton E. Gurtin, Richard D. James, Stelios Kyriakides, David M. Parks, Phoebus Rosakis, Stewart Silling and Nicolas Triantafyllidis, which I gratefully acknowledge.

Volume I of these notes provides a collection of essential definitions, results, and illustrative examples, designed to review those aspects of mathematics that will be encountered in the subsequent volumes. It is most certainly *not* meant to be a source for learning these topics for the first time. The treatment is concise, selective and limited in scope. For example, Linear Algebra is a far richer subject than the treatment in Volume I, which is limited to real 3-dimensional Euclidean vector spaces.

The topics covered in Volumes II and III are largely those one would expect to see covered in such a set of lecture notes. Personal taste has led me to include a few special (but still well-known) topics. Examples of these include sections on the statistical mechanical theory of polymer chains and the lattice theory of crystalline solids in the discussion of constitutive relations in Volume II, as well as several initial-boundary value problems designed to illustrate various nonlinear phenomena also in Volume II; and sections on the so-called Eshelby problem and the effective behavior of two-phase materials in Volume III.

There are a number of Worked Examples and Exercises at the end of each chapter which are an essential part of the notes. Many of these examples provide more details; or the proof of a result that had been quoted previously in the text; or illustrates a general concept; or establishes a result that will be used subsequently (possibly in a later volume).

The content of these notes are entirely classical, in the best sense of the word, and none of the material here is original. I have drawn on a number of sources over the years as I prepared my lectures. I cannot recall every source I have used but certainly they include those listed at the end of each chapter. In a more general sense the broad approach and philosophy taken has been influenced by:

Volume I: A Brief Review of Some Mathematical Preliminaries

I.M. Gelfand and S.V. Fomin, *Calculus of Variations*, Prentice Hall, 1963.

J.K. Knowles, *Linear Vector Spaces and Cartesian Tensors*, Oxford University Press, New York, 1997.

Volume II: Continuum Mechanics

P. Chadwick, *Continuum Mechanics: Concise Theory and Problems*, Dover,1999.

J.L. Ericksen, *Introduction to the Thermodynamics of Solids*, Chapman and Hall, 1991.

M.E. Gurtin, *An Introduction to Continuum Mechanics*, Academic Press, 1981.

M.E. Gurtin, E. Fried and L. Anand, *The Mechanics and Thermodynamics of Continua*, Cambridge University Press, 2010.

J. K. Knowles and E. Sternberg, *(Unpublished) Lecture Notes for AM136: Finite Elasticity*, California Institute of Technology, Pasadena, CA 1978.

C. Truesdell and W. Noll, The nonlinear field theories of mechanics, in *Handbüch der Physik*, Edited by S. Flügge, Volume III/3, Springer, 1965.

Volume III: Elasticity

M.E. Gurtin, The linear theory of elasticity, in *Mechanics of Solids - Volume II*, edited by C. Truesdell, Springer-Verlag, 1984.

J. K. Knowles, *(Unpublished) Lecture Notes for AM135: Elasticity*, California Institute of Technology, Pasadena, CA, 1976.

A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity*, Dover, 1944.

S. P. Timoshenko and J.N. Goodier, *Theory of Elasticity*, McGraw-Hill, 1987.

The following notation will be used in Volume II though there will be some lapses (for reasons of tradition): Greek letters will denote real numbers; lowercase boldface Latin letters will denote vectors; and uppercase boldface Latin letters will denote linear transformations. Thus, for example, α, β, γ... will denote scalars (real numbers); $\mathbf{x}, \mathbf{y}, \mathbf{z}, ...$ will denote vectors; and $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, ...$ will denote linear transformations. In particular, "$\mathbf{o}$" will denote the null vector while "$\mathbf{0}$" will denote the null linear transformation.

One result of this notational convention is that we will *not* use the uppercase bold letter $\mathbf{X}$ to denote the position vector of a particle in the reference configuration. Instead we use the lowercase boldface letters $\mathbf{x}$ and $\mathbf{y}$ to denote the positions of a particle in the reference and current configurations.

Contents

Chapter 1

Some Preliminary Notions

In this preliminary chapter we introduce certain basic notions that underly the continuum theory of materials. These concepts are essential ingredients of continuum modeling, though sometimes they are used implicitly without much discussion. We shall devote some attention to these notions in this chapter since that will allow for greater clarity in subsequent chapters.

For example we frequently speak of an "isotropic material". Does this mean that the material copper, for example, is isotropic? Suppose we have a particular piece of copper that *is* isotropic in a given configuration, and we deform it, will it still be isotropic? What is isotropy a property of? The material, the body, or the configuration? Speaking of which, what is the difference between a body, a configuration, and a region of space occupied by a body (and is it important to distinguish between them)? ... Often we will want to consider some physical property (e.g. the internal energy) associated with a part of a body (i.e. a definite set of particles of the body). As the body moves through space and this part occupies different regions of space at different times and the value of this property changes with time, it may be important to be precise about the fact that this property is assigned to a *fixed set of particles* comprising the part of the body and not the *changing region of space* that it occupies. ... Or, consider the propagation of a wavefront. Consider a point on the wavefront and a particle of the body, both of which happen to be located at the same point in space at a given instant. However these are distinct entities and at the next instant of time this same point on the wavefront and particle of the body would no longer be co-located in space. Thus in particular, the velocity of the point of the wavefront is different from the velocity of the particle of the body, even though they are located at the same point in space at the current instant.

The concepts introduced in this chapter aim to clarify such issues. We will not be pedantic about these subtleties. Rather, we shall make use of the framework and terminology introduced in this chapter only when it helps avoid confusion. The reader is encouraged to pay special attention to the *distinctions* between the different concepts introduced here. These concepts include the notions of

a body,

a configuration of the body,

a reference configuration of the body,

the region occupied by the body in some configuration,

a particle (or material point),

the location of a particle in some configuration,

a deformation,

a motion,

Eulerian and Lagrangian descriptions of a physical quantity,

Eulerian and Lagrangian spatial derivatives, and

Eulerian and Lagrangian time derivatives (including the material time derivative).

1.1 Bodies and Configurations.

Our aim is to develop a framework for studying how "objects that occur in nature" respond to the application of forces or other external stimuli. In order to do this, we must construct mathematical idealizations (i.e. mathematical models) of the "objects" and the "stimuli". Specifically, with regard to the "objects", we must model their geometric and constitutive character.

We shall use the term "body" to be a mathematical abstraction of an "object that occurs in nature". A *body* $\mathcal{B}$ is composed of a set of *particles*[1] p (or *material points*). In a given *configuration* of the body, each particle is located at some definite point $\mathbf{y}$ in three-dimensional space. The set of all the points in space, corresponding to the locations of all the particles, is the *region* $\mathcal{R}$ occupied by the body in that configuration. A particular body, composed of a particular set of particles, can adopt different configurations under the action

[1]A particle in continuum mechanics is different to what we refer to as a particle in classical mechanics. For example, a particle in classical mechanics has a mass $m > 0$, while a particle in continuum mechanics is not endowed with a property called mass.

of different stimuli (forces, heating etc.) and therefore occupy different regions of space under different conditions. Note the distinction between the *body*, a *configuration* of the body, and the *region* the body occupies in that configuration; we make these distinctions rigorous in what follows. Similarly note the distinction between a *particle* and the *position* in space it occupies in some configuration.

In order to appreciate the difference between a configuration and the region occupied in that configuration, consider the following example: suppose that a body, in a certain configuration, occupies a circular cylindrical region of space. If the object is "twisted" about its axis (as in torsion), it continues to occupy this same (circular cylindrical) region of space. Thus the region occupied by the body has not changed even though we would say that the body is in a different "configuration".

More formally, in continuum mechanics a *body* $\mathcal{B}$ is a collection of elements which can be put into one-to-one correspondence with some *region* $\mathcal{R}$ of Euclidean point space[2]. An element $p \in \mathcal{B}$ is called a *particle* (or *material point*). Thus, given a body $\mathcal{B}$, there is necessarily a mapping χ that takes particles $p \in \mathcal{B}$ into their geometric locations $\mathbf{y} \in \mathcal{R}$ in three-dimensional Euclidean space:

$$\boldsymbol{y} = \chi(p) \quad \text{where} \quad p \in \mathcal{B},\ \boldsymbol{y} \in \mathcal{R}. \tag{1.1}$$

The mapping χ is called a *configuration* of the body $\mathcal{B}$; $\boldsymbol{y}$ is the *position* occupied by the particle p in the configuration χ; and $\mathcal{R}$ is the *region* occupied by the body in the configuration χ. Often, we write $\mathcal{R} = \chi(\mathcal{B})$.

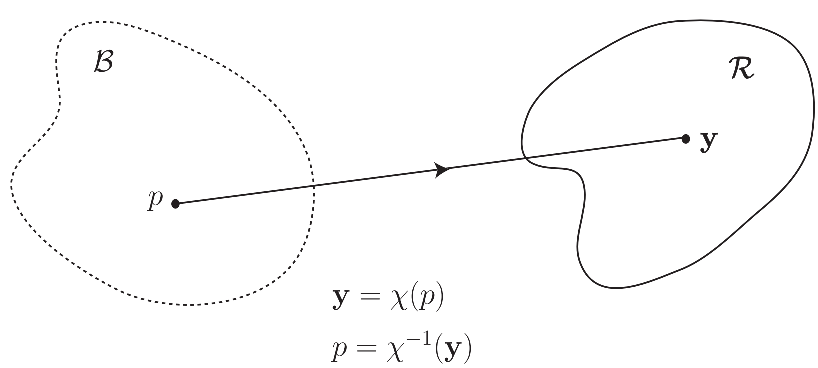

Figure 1.1: A body $\mathcal{B}$ that occupies a region $\mathcal{R}$ in a configuration χ. A particle $p \in \mathcal{B}$ is located at the position $\mathbf{y} \in \mathcal{R}$ where $\mathbf{y} = \chi(p)$. $\mathcal{B}$ is a mathematical abstraction. $\mathcal{R}$ is a region in three-dimensional Euclidean space.

[2]Recall that a "region" is an open *connected* set. Thus a single particle p does not constitute a body.

Since a configuration χ provides a *one-to-one* mapping between the particles p and positions $\mathbf{y}$, there is necessarily an inverse mapping χ^{-1} from $\mathcal{R} \to \mathcal{B}$:

$$p = \chi^{-1}(\boldsymbol{y}) \quad \text{where} \quad p \in \mathcal{B},\ \boldsymbol{y} \in \mathcal{R}. \tag{1.2}$$

Observe that bodies and particles, in the terminology used here, refer to "abstract" entities. Bodies are available to us through their configurations. Actual geometric measurements can be made on the place occupied by a particle or the region occupied by a body.

1.2 Reference Configuration.

In order to identify a particle of a body, we must label the particles. The abstract particle label p, while perfectly acceptable in principle and intuitively clear, is not convenient for carrying out calculations. It is more convenient to pick some arbitrary configuration of the body, say χ_{ref}, and use the (unique) position $\mathbf{x} = \chi_{\text{ref}}(p)$ of a particle in that configuration to label it instead. Such a configuration χ_{ref} is called a reference configuration of the body. It simply provides a convenient way in which to label the particles of a body. The particles are now labeled by $\mathbf{x}$ instead of p.

A second reason for considering a reference configuration is the following: we can study the geometric characteristics of a configuration χ by studying the geometric properties of the points occupying the region $\mathcal{R} = \chi(\mathcal{B})$. This is adequate for modeling certain materials (such as many fluids) where the behavior of the material depends only on the characteristics of the configuration *currently* occupied by the body. In describing most solids however one often needs to know the *changes* in geometric characteristics between one configuration and another configuration (e.g. the change in length, the change in angle etc.). In order to describe the change in a geometric quantity one must necessarily consider (at least) *two* configurations of the body: the configuration that one wishes to analyze, and a *reference configuration* relative to which the changes are to be measured.

Let χ_{ref} and χ be two configurations of a body $\mathcal{B}$ and let $\mathcal{R}_{\text{ref}}$ and $\mathcal{R}$ denote the regions occupied by the body $\mathcal{B}$ in these two configuration; see Figure 1.2. The mappings χ_{ref} and χ take $p \to \mathbf{x}$ and $p \to \mathbf{y}$, and likewise $\mathcal{B} \to \mathcal{R}_{\text{ref}}$ and $\mathcal{B} \to \mathcal{R}$:

$$\boldsymbol{x} = \chi_{\text{ref}}(p), \qquad \boldsymbol{y} = \chi(p). \tag{1.3}$$

Here $p \in \mathcal{B},\ \mathbf{x} \in \mathcal{R}_{\text{ref}}$ and $\mathbf{y} \in \mathcal{R}$. Thus $\mathbf{x}$ and $\mathbf{y}$ are the positions of particle p in the two configurations under consideration.

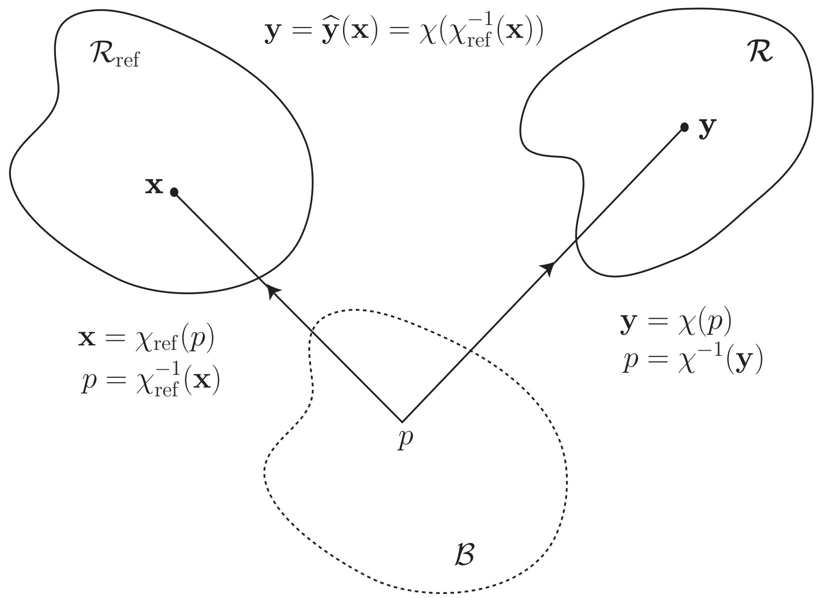

Figure 1.2: A body $\mathcal{B}$ that occupies a region $\mathcal{R}$ in a configuration χ, and another region $\mathcal{R}_{\text{ref}}$ in a second configuration χ_{ref}. A particle $p \in \mathcal{B}$ is located at $\mathbf{y} = \chi(p) \in \mathcal{R}$ in configuration χ, and at $\mathbf{x} = \chi_{\text{ref}}(p) \in \mathcal{R}_{\text{ref}}$ in configuration χ_{ref}. The mapping of $\mathcal{R}_{\text{ref}} \to \mathcal{R}$ is described by the deformation $\mathbf{y} = \widehat{\mathbf{y}}(\mathbf{x}) = \chi(\chi_{\text{ref}}^{-1}(\mathbf{x}))$.

This induces a mapping $\mathbf{y} = \widehat{\mathbf{y}}(\mathbf{x})$ from $\mathcal{R}_{\text{ref}} \to \mathcal{R}$:

$$\boldsymbol{y} = \widehat{\boldsymbol{y}}(\boldsymbol{x}) \stackrel{\text{def}}{=} \chi(\chi_{\text{ref}}^{-1}(\boldsymbol{x})), \qquad \mathbf{x} \in \mathcal{R}_{\text{ref}}, \mathbf{y} \in \mathcal{R}; \tag{1.4}$$

$\widehat{\boldsymbol{y}}$ is called a *deformation* of the body from the reference configuration χ_{ref}.

Frequently one picks a particular convenient (usually fixed) reference configuration χ_{ref} and studies deformations of the body relative to that configuration. This particular configuration need only be one that the body *can* sustain, not necessarily one that is actually sustained in the setting being analyzed. The choice of reference configuration is arbitrary in principle (and is usually chosen for reasons of convenience). Note that the function $\widehat{\boldsymbol{y}}$ in (1.4) depends on the choice of reference configuration.

When working with a single fixed reference configuration, as we will most often do, one can dispense with talking about the body $\mathcal{B}$, a configuration χ and the particle p, and work directly with the region $\mathcal{R}_{\text{ref}}$, the deformation $\mathbf{y}(\mathbf{x})$ and the position $\mathbf{x}$.

However, even when working with a single fixed reference configuration, sometimes, when introducing a new concept, for reasons of clarity we shall start by using p, $\mathcal{B}$ etc. before switching to $\mathbf{x}$, $\mathcal{R}_{\text{ref}}$ etc.

There will be occasions when we *must* consider more than one reference configuration; an example of this will be our analysis of material symmetry. In such circumstances one can avoid confusion by framing the analysis in terms the body $\mathcal{B}$, the reference configurations χ_1, χ_2 etc.

1.3 Description of Physical Quantities: Spatial and Referential (or Eulerian and Lagrangian) forms.

There are essentially two types of physical characteristics associated with a body. The first, such as temperature, is associated with *individual particles* of the body; the second, such as mass and energy, are associated with "*parts of the body*". One sometimes refers to these as intensive and extensive characteristics respectively. In this and the next few sections we will be concerned with properties of the former type; we shall consider the latter type of properties in Section 1.8.

First consider a characteristic such as the temperature of a particle. The temperature θ of particle p in the configuration χ is given by[3]

$$\theta = \theta_*(p) \tag{1.5}$$

where the function $\theta_*(p)$ is defined for all $p \in \mathcal{B}$. Such a description, though completely rigorous and well-defined, is not especially useful for carrying out calculations since a particle is an abstract entity. It is more useful to describe the temperature by a function of particle position by trading p for $\mathbf{y}$ by using $\mathbf{y} = \chi(p)$:

$$\theta = \overline{\theta}(\boldsymbol{y}) \stackrel{\text{def}}{=} \theta_*\left(\chi^{-1}(\mathbf{y})\right). \tag{1.6}$$

The function $\overline{\theta}(\mathbf{y})$ is defined for all $\mathbf{y} \in \mathcal{R}$. The functions θ_* and $\overline{\theta}$ both describe temperature: $\theta_*(p)$ is the temperature of the particle p while $\overline{\theta}(\mathbf{y})$ is the temperature of the particle located at $\mathbf{y}$. When p and $\mathbf{y}$ are related by $\mathbf{y} = \chi(p)$, the two functions $\overline{\theta}$ and θ_* have the same value since they both refer to the temperature of the same particle in the same configuration and they are related by $(1.6)_2$. One usually refers to the representation (1.5) which deals directly with the abstract particles as a *material description*; the representation (1.6) which deals

[3]Even though it is cumbersome to do so, in order to clearly distinguish three different characterizations of temperature from each other, we use the notation $\theta_*(\cdot), \overline{\theta}(\cdot)$ and $\widehat{\theta}(\cdot)$ to describe three distinct but related functions defined on $\mathcal{B}, \mathcal{R}$ and $\mathcal{R}_{\text{ref}}$ respectively.

with the positions of the particles in the deformed configuration, (the configuration in which the physical quantity is being characterized,) is called the *Eulerian or spatial description.*

If a reference configuration has been introduced we can label a particle by its position $\mathbf{x} = \chi_{\text{ref}}(p)$ in that configuration, and this in turn allows us to describe physical quantities in Lagrangian form. Consider again the temperature of a particle as given in (1.5). We can trade p for $\mathbf{x}$ using $\mathbf{x} = \chi_{\text{ref}}(p)$ to describe the temperature in *Lagrangian or referential form* by

$$\theta = \widehat{\theta}(\boldsymbol{x}) \stackrel{\text{def}}{=} \theta_*(\chi_{\text{ref}}^{-1}(\mathbf{x})). \tag{1.7}$$

The function $\widehat{\theta}$ is defined for all $\boldsymbol{x} \in \mathcal{R}_{ref}$. The referential description $\widehat{\theta}(\mathbf{x})$ can also be generated from the spatial description through

$$\widehat{\theta}(\mathbf{x}) = \overline{\theta}(\widehat{\mathbf{y}}(\mathbf{x})). \tag{1.8}$$

It is essential to emphasize that the function $\widehat{\theta}$ does *not* give the temperature of a particle in the reference configuration; rather, $\widehat{\theta}(\mathbf{x})$ is the temperature *in the deformed configuration* of the particle located at $\mathbf{x}$ in the reference configuration.

A physical field that is, for example, described by a function defined on $\mathcal{R}$ and expressed as a function of $\mathbf{y}$, can just as easily be expressed through a function defined on $\mathcal{R}_{\text{ref}}$ and expressed as a function of $\mathbf{x}$; and vice versa. For example in the chapter on stress we will encounter two 2-tensors $\mathbf{T}$ and $\mathbf{S}$ called the Cauchy stress and the first Piola-Kirchhoff stress. It is customary to express $\mathbf{T}$ as a function of $\mathbf{y} \in \mathcal{R}$ and $\mathbf{S}$ as a function of $\mathbf{x} \in \mathcal{R}_{\text{ref}}$: $\mathbf{T}(\mathbf{y})$ and $\mathbf{S}(\mathbf{x})$. This is because certain calculations simplify when done in this way. However they both refer to stress at a particle in a deformed configuration where in one case the particle is labeled by its position in the deformed configuration and in the other by its position in the reference configuration. In fact, by making use of the deformation $\mathbf{y} = \widehat{\mathbf{y}}(\mathbf{x})$ we can write $\mathbf{T}$ as a function of $\mathbf{x}$: $\widehat{\mathbf{T}}(\mathbf{x}) = \mathbf{T}(\widehat{\mathbf{y}}(\mathbf{x}))$, and likewise $\mathbf{S}$ as a function of $\mathbf{y}$: $\overline{\mathbf{S}}(\mathbf{y}) = \mathbf{S}(\widehat{\mathbf{y}}^{-1}(\mathbf{y}))$, if we so need to.

1.4 Eulerian and Lagrangian Spatial Derivatives.

To be specific, consider again the temperature field θ in the body in a configuration χ. We can express this either in Lagrangian form

$$\theta = \widehat{\theta}(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathcal{R}_{\text{ref}}, \tag{1.9}$$

or in Eulerian form

$$\theta = \overline{\theta}(\boldsymbol{y}), \quad \boldsymbol{y} \in \mathcal{R}. \tag{1.10}$$

Both of these expressions give the temperature of a particle in the deformed configuration where the only distinction is in the labeling of the particle. These two functions are related by (1.8).

It is cumbersome to write the decorative symbols, i.e., the "hats" and the "bars", all the time and we would prefer to write $\theta(\boldsymbol{x})$ and $\theta(\boldsymbol{y})$. If such a notation is adopted one must be particularly attentive and continuously use the context to decide which function one means.

Suppose, for example, that we wish to compute the gradient of the temperature field. If we write this as $\boldsymbol{\nabla}\theta$ we would not know if we were referring to the Lagrangian spatial gradient

$$\boldsymbol{\nabla}\widehat{\theta}(\boldsymbol{x}) \qquad \text{which has components} \quad \frac{\partial \widehat{\theta}}{\partial x_i}(\boldsymbol{x}), \tag{1.11}$$

or to the Eulerian spatial gradient

$$\boldsymbol{\nabla}\overline{\theta}(\boldsymbol{y}) \qquad \text{which has components} \quad \frac{\partial \overline{\theta}}{\partial y_i}(\boldsymbol{y}). \tag{1.12}$$

In order to avoid this confusion we use the notation Grad θ and grad θ instead of $\boldsymbol{\nabla}\theta$ where

$$\text{Grad } \theta = \boldsymbol{\nabla}\widehat{\theta}(\boldsymbol{x}), \qquad \text{and} \qquad \text{grad } \theta = \boldsymbol{\nabla}\overline{\theta}(\boldsymbol{y}). \tag{1.13}$$

The gradient of the particular vector field $\widehat{\mathbf{y}}(\mathbf{x})$, the deformation, is denoted by $\mathbf{F}(\mathbf{x})$ and is known as the *deformation gradient tensor*:

$$\mathbf{F}(\mathbf{x}) = \text{Grad } \widehat{\mathbf{y}}(\mathbf{x}) \qquad \text{with components} \quad F_{ij} = \frac{\partial \widehat{y}_i}{\partial x_j}(\mathbf{x}). \tag{1.14}$$

It plays a central role in describing the kinematics of a body.

The symbols *Div* and *div*, and *Curl* and *curl* are used similarly.

In order to relate Grad θ to grad θ we merely need to differentiate (1.8) with respect to $\boldsymbol{x}$ using the chain rule. This gives

$$\frac{\partial \widehat{\theta}}{\partial x_i} = \frac{\partial \overline{\theta}}{\partial y_j}\frac{\partial \widehat{y}_j}{\partial x_i} = \frac{\partial \overline{\theta}}{\partial y_j}F_{ji} = F_{ji}\frac{\partial \overline{\theta}}{\partial y_j} \tag{1.15}$$

where summation over the repeated index j is taken for granted. This can be written as

$$\text{Grad}\,\theta = \boldsymbol{F}^T \text{ grad}\,\theta. \tag{1.16}$$

Similarly, if $\boldsymbol{w}$ is any vector field, one can show that

$$\text{Grad}\,\boldsymbol{w} = (\text{grad}\,\boldsymbol{w})\boldsymbol{F}, \tag{1.17}$$

and for any tensor field $\mathbf{T}$, that

$$\text{Div}\,\boldsymbol{T} = J\,\text{div}\,(J^{-1}\boldsymbol{F}\boldsymbol{T}) \qquad \text{where} \quad J = \det\mathbf{F}. \tag{1.18}$$

1.5 Motion of a Body.

A motion of a body is a one-parameter family of configurations $\chi(p,t)$ where the parameter t is time:

$$\boldsymbol{y} = \chi(p,t), \qquad p \in \mathcal{B}, \quad t_0 \ \leq\ t\ \leq\ t_1. \tag{1.19}$$

This motion takes place over the time interval $[t_0, t_1]$. The body occupies a time-dependent region $\mathcal{R}_t = \chi(\mathcal{B}, t)$ during the motion, and the vector $\boldsymbol{y} \in \mathcal{R}_t$ is the position occupied by the particle p at time t during the motion χ. For each particle p, (1.19) describes the equation of a curve in three-dimensional space which is the path of this particle.

Next consider the *velocity* and *acceleration* of a particle, defined as the rate of change of position and velocity respectively of that particular particle:

$$\mathbf{v} = \mathbf{v}_*(p,t) = \frac{\partial\chi}{\partial t}(p,t) \quad \text{and} \quad \mathbf{a} = \mathbf{a}_*(p,t) = \frac{\partial^2\chi}{\partial t^2}(p,t). \tag{1.20}$$

Since a particle p is only available to us through its location $\mathbf{y}$, it is convenient to express the velocity and acceleration as functions of $\mathbf{y}$ and t (rather than p and t). This is readily done by using $p = \chi^{-1}(\mathbf{y},t)$ to eliminate p in favor of $\mathbf{y}$ in (1.20) leading to the velocity and acceleration fields $\overline{\mathbf{v}}(\mathbf{y},t)$ and $\overline{\mathbf{a}}(\mathbf{y},t)$:

$$\begin{aligned} \mathbf{v} &= \overline{\mathbf{v}}(\mathbf{y},t) = \mathbf{v}_*(p,t)\Big|_{p=\chi^{-1}(\mathbf{y},t)} = \mathbf{v}_*(\chi^{-1}(\mathbf{y},t),t), \\ \mathbf{a} &= \overline{\mathbf{a}}(\mathbf{y},t) = \mathbf{a}_*(p,t)\Big|_{p=\chi^{-1}(\mathbf{y},t)} = \mathbf{a}_*(\chi^{-1}(\mathbf{y},t),t), \end{aligned} \tag{1.21}$$

where $\mathbf{v}_*(p,t)$ and $\mathbf{a}_*(p,t)$ are given by (1.20).

It is worth emphasizing that the velocity and acceleration of a particle can be defined *without* the need to speak of a reference configuration.

If a reference configuration χ_{ref} has been introduced and $\mathbf{x} = \chi_{\text{ref}}(p)$ is the position of a particle in that configuration, we can describe the motion alternatively by

$$\mathbf{y} = \widehat{\mathbf{y}}(\mathbf{x}, t) \stackrel{\text{def}}{=} \chi(\chi_{\text{ref}}^{-1}(\mathbf{x}), t). \tag{1.22}$$

The particle velocity in Lagrangian form is given by

$$\mathbf{v} = \widehat{\mathbf{v}}(\mathbf{x}, t) = \mathbf{v}_*(\chi_{\text{ref}}^{-1}(\mathbf{x}), t) \tag{1.23}$$

or equivalently by

$$\mathbf{v} = \widehat{\mathbf{v}}(\mathbf{x}, t) = \overline{\mathbf{v}}(\widehat{\mathbf{y}}(\mathbf{x}, t), t). \tag{1.24}$$

Similar expressions for the acceleration can be written. The function $\widehat{\mathbf{v}}(\mathbf{x}, t)$ does *not* of course give the velocity of a particle in the reference configuration but rather the velocity at time t of the particle which is associated with the position $\mathbf{x}$ in the reference configuration.

Sometimes, the reference configuration is chosen to be the configuration of the body at the initial instant, i.e., $\chi_{\text{ref}}(p) = \chi(p, t_0)$, in which case $\mathbf{x} = \widehat{\mathbf{y}}(\mathbf{x}, t_0)$.

1.6 Eulerian and Lagrangian Time Derivatives.

To be specific, consider again the temperature field θ in the body *at time t*. As noted previously, it is cumbersome to write the decorative symbols, i.e., the "hats" and the "bars" over the Eulerian and Lagrangian representations $\overline{\theta}(\mathbf{y}, t)$ and $\widehat{\theta}(\mathbf{x}, t)$ and so we sometimes write both these functions as $\theta(\boldsymbol{y}, t)$ and $\theta(\boldsymbol{x}, t)$ being attentive when we do so.

For example consider the time derivative of θ. If we write this simply as $\partial\theta/\partial t$ we would not know whether we were referring to the Lagrangian or the Eulerian derivatives

$$\frac{\partial\widehat{\theta}}{\partial t}(\boldsymbol{x}, t) \qquad \text{or} \qquad \frac{\partial\overline{\theta}}{\partial t}(\boldsymbol{y}, t) \tag{1.25}$$

respectively. To avoid confusion we therefore use the notation $\dot{\theta}$ and θ' instead of $\partial\theta/\partial t$ where

$$\dot{\theta} = \frac{\partial\widehat{\theta}}{\partial t}(\boldsymbol{x}, t) \qquad \text{and} \qquad \theta' = \frac{\partial\overline{\theta}}{\partial t}(\boldsymbol{y}, t). \tag{1.26}$$

$\dot{\theta}$ is called the *material time derivative*[4] (since in calculating the time derivative we are keeping the particle, identified by $\mathbf{x}$, fixed).

[4]In fluid mechanics this is often denoted by $D\theta/Dt$.

We can relate $\dot{\theta}$ to θ' by differentiating $\widehat{\theta}(\mathbf{x},t) = \overline{\theta}(\widehat{\mathbf{y}}(\mathbf{x},t),t)$ with respect to t and using the chain rule. This gives

$$\frac{\partial\widehat{\theta}}{\partial t} = \frac{\partial\overline{\theta}}{\partial y_i}\frac{\partial\widehat{y}_i}{\partial t} + \frac{\partial\overline{\theta}}{\partial t} \tag{1.27}$$

or

$$\dot{\theta} = \theta' + (\operatorname{grad}\theta)\cdot\boldsymbol{v} \tag{1.28}$$

where $\mathbf{v}$ is the velocity. Similarly, for any vector field $\boldsymbol{w}$ one can show that

$$\dot{\boldsymbol{w}} = \boldsymbol{w}' + (\operatorname{grad}\boldsymbol{w})\boldsymbol{v} \tag{1.29}$$

where $\operatorname{grad}\boldsymbol{w}$ is a tensor. Unless explicitly stated otherwise, we shall *always use an over dot to denote the material time derivative.*

1.7 A Part of a Body.

We say that $\mathcal{P}$ is a *part* of the body $\mathcal{B}$ if (i) $\mathcal{P}\subset\mathcal{B}$ and (ii) $\mathcal{P}$ itself is a body, i.e. there is a configuration χ of $\mathcal{B}$ such that $\chi(\mathcal{P})$ is a region. (Note therefore that a single particle p does not constitute a part of the body.)

If $\mathcal{R}_t$ and $\mathcal{D}_t$ are the respective regions occupied at time t by a body $\mathcal{B}$ and a part of it $\mathcal{P}$ during a motion, then $\mathcal{D}_t\subset\mathcal{R}_t$.

As the body undergoes a motion, the region $\mathcal{D}_t = \chi(\mathcal{P},t)$ that is occupied by a part of the body will evolve with time. Note that even though the region $\mathcal{D}_t$ changes with time, the set of particles associated with it does not change with time. The region $\mathcal{D}_t$ is always associated with the same part $\mathcal{P}$ of the body. Such a region, which is always associated with the same set of particles, is called a *material region.* In subsequent chapters when we consider the "global balance principles of continuum thermomechanics", such as momentum or energy balance, they will always be applied to a material region (or equivalently to a part of the body). Note that the region occupied by $\mathcal{P}$ in the reference configuration, $\mathcal{D}_{\text{ref}} = \chi_{\text{ref}}(\mathcal{P})$, does not vary with time.

Next consider a surface $\mathcal{S}_t$ that moves in space through $\mathcal{R}_t$. One possibility is that this surface, even though it moves, is always associated with the same set of particles (so that it "moves with the body".) This would be the case for example of the surface corresponding to the interface between two perfectly bonded parts in a composite material. Such a surface is called a material surface since it is associated with the same particles at all times. A second possibility is that the surface is not associated with the same set of particles, as is the case for example for a wave front propagating through the material. The wave front is associated with different particles at different times as it sweeps through $\mathcal{R}_t$. Such a surface is not a material surface. Note that the surface $\mathcal{S}_{\rm ref}$, which is the pre-image of $\mathcal{S}_t$ in the reference configuration, does not vary with time for a material surface but *does* vary with time for a non-material surface.

In general, a time dependent family of curves $\mathcal{C}_t$, surfaces $\mathcal{S}_t$ and regions $\mathcal{D}_t$ are said to represent, respectively, a *material curve*, a *material surface* and a *material region* if they are associated with the *same set of particles* at all times.

1.8 Extensive Properties and their Densities.

In the previous sections we considered physical properties such as temperature that were associated with individual particles of the body. Certain other physical properties in continuum physics (such as for example mass, energy and entropy) are associated with parts of the body and not with individual particles.

Consider an arbitrary part $\mathcal{P}$ of a body $\mathcal{B}$ that undergoes a motion χ. As usual, the regions of space occupied by $\mathcal{P}$ and $\mathcal{B}$ at time t during this motion are denoted by $\chi(\mathcal{P}, t)$ and $\chi(\mathcal{B}, t)$ respectively, and the location of the particle p is $\mathbf{y} = \chi(p, t)$.

We say that Ω is an *extensive physical property* of the body if there is a function $\Omega(\cdot, t; \chi)$ defined on the set of all parts $\mathcal{P}$ of $\mathcal{B}$ which is such that

(i)
$$\Omega(\mathcal{P}_1 \cup \mathcal{P}_2, t; \chi) = \Omega(\mathcal{P}_1, t; \chi) + \Omega(\mathcal{P}_2, t; \chi) \tag{1.30}$$
for all arbitrary disjoint parts $\mathcal{P}_1$ and $\mathcal{P}_2$ (which simply states that the value of the property Ω associated with two disjoint parts is the sum of the individual values for each of those parts), and

(ii)
$$\Omega(\mathcal{P}, t; \chi) \to 0 \quad \text{as the volume of } \chi(\mathcal{P}, t) \to 0. \tag{1.31}$$

Under these circumstance there exists a *density* $\omega(p, t; \chi)$ such that

$$\Omega(\mathcal{P}, t; \chi) = \int_{\mathcal{P}} \omega(p, t; \chi)\, dp. \tag{1.32}$$

Thus, we have the property $\Omega(\mathcal{P}, t; \chi)$ associated with parts $\mathcal{P}$ of the body and its density $\omega(p, t; \chi)$ associated with particles p of the body, e.g. the energy of $\mathcal{P}$ and the energy density at p.

It is more convenient to trade the particle p for its position $\mathbf{y}$ using $p = \chi^{-1}(\mathbf{y}, t)$ and work with the (Eulerian or spatial) density function $\overline{\omega}(\mathbf{y}, t; \chi)$ in terms of which

$$\Omega(\mathcal{P}, t; \chi) = \int_{\mathcal{D}_t} \overline{\omega}(\mathbf{y}, t; \chi)\, \mathrm{d}V_y.$$

Any physical property associated in such a way with all parts of a body has an associated density; for example the mass m, internal energy e, and the entropy H have corresponding mass[5], internal energy and entropy densities which we will denote by ρ, ε and η.

References:

1. C. Truesdell, *The Elements of Continuum Mechanics,* Lecture 1, Springer-Verlag, NewYork, 1966.

2. R.W. Ogden, *Non-Linear Elastic Deformations*, §§2.1.2 and 2.1.3, Dover, 1997.

3. C. Truesdell, *A First Course in Rational Continuum Mechanics*, §§1 to 4 and §7 of Chapter 1 and §§1–4 of Chapter 2, Academic Press, New York 1977.

[5] In the particular case of mass, one has the added feature that $m(\mathcal{P}) > 0$ whence $\rho(\mathbf{y}, t) > 0$.

Chapter 2

Kinematics: Deformation

In this chapter we shall consider various *geometric issues* concerning the deformation of a body. At this stage we will not address the *causes* of the deformation, such as the applied loading or the temperature changes, nor will we discuss the characteristics of the material of which the body is composed, assuming only that it can be described as a continuum. Our focus will be on purely geometric issues[1].

A roadmap of this chapter is as follows: in Section 2.1 we describe the notion of a deformation. In Section 2.2 we introduce the central ingredient needed for describing the deformation of an entire neighborhood of a particle – the deformation gradient tensor. Some particular homogeneous deformations such as pure stretch, uniaxial extension and simple shear are presented in Section 2.3. We then consider in Section 2.4 an infinitesimal curve, surface and region in the reference configuration and examine their images in the deformed configuration where the image and pre-image in each case is associated with the same set of particles. A rigid deformation is described in Section 2.5. The decomposition of a general deformation gradient tensor into the product of a rigid rotation and a pure stretch is presented in Section 2.6. Section 2.7 introduces the notion of strain, and finally we consider the linearization of the prior results in Section 2.8.

[1] It is worth mentioning that in developing a continuum theory for a material, the appropriate kinematic description of the body is not totally independent of, say, the nature of the forces. For example, in describing the interaction between particles in a dielectric material subjected to an electric field, one has to allow for internal forces *and internal couples* between every pair of points in the body. This in turn requires that the kinematics allow for independent displacement *and rotation* fields in the body. In general, the kinematics and the forces must be *conjugate* to each other in order to construct a self-consistent theory. This will be made more clear in subsequent chapters.

2.1 Deformation

In this chapter we will primarily be concerned with how the geometric characteristics of one configuration of the body (the "deformed" or "current" configuration) *differ* from those of some other configuration of the body (an "undeformed" or "reference" configuration). Thus we consider two configurations in which the body occupies the respective regions[2] $\mathcal{R}$ and $\mathcal{R}_0$. The corresponding position vectors of a generic particle are $\mathbf{y} \in \mathcal{R}$ and $\mathbf{x} \in \mathcal{R}_0$. In this chapter we shall consider one fixed reference configuration and therefore we can uniquely identify a particle by its position $\mathbf{x}$ in that configuration. The *deformation* of the body from the reference configuration to the deformed configuration is described by a mapping

$$\mathbf{y} = \widehat{\mathbf{y}}(\mathbf{x}) \tag{2.1}$$

which takes $\mathcal{R}_0 \to \mathcal{R}$. We use the "hat" over $\mathbf{y}$ in order to distinguish the *function* $\widehat{\mathbf{y}}(\cdot)$ from its value $\mathbf{y}$. As we progress through these notes, we will most often omit the "hat" unless the context does not make clear whether we are referring to $\widehat{\mathbf{y}}$ or $\mathbf{y}$, and/or it is essential to emphasize the distinction.

The *displacement* vector field $\widehat{\mathbf{u}}(\mathbf{x})$ is defined on $\mathcal{R}_0$ by

$$\widehat{\mathbf{u}}(\mathbf{x}) = \widehat{\mathbf{y}}(\mathbf{x}) - \mathbf{x}; \tag{2.2}$$

see Figure 2.1. In order to fully characterize the deformed configuration of the body one must specify the function $\widehat{\mathbf{y}}$ (or equivalently $\widehat{\mathbf{u}}$) at *every* particle of the body, i.e. on the entire domain $\mathcal{R}_0$.

We impose the physical requirements that (a) a single particle[3] $\mathbf{x}$ will not split into two particles and occupy two locations $\mathbf{y}^{(1)}$ and $\mathbf{y}^{(2)}$, and that (b) two particles $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ will not coalesce into a single particle and occupy one location $\mathbf{y}$. This implies that (2.1) must be a one-to-one mapping. Consequently there exists a one-to-one inverse deformation

$$\mathbf{x} = \overline{\mathbf{x}}(\mathbf{y}) \tag{2.3}$$

that carries $\mathcal{R} \to \mathcal{R}_0$. Since (2.3) is the inverse of (2.1), it follows that

$$\overline{\mathbf{x}}\left(\widehat{\mathbf{y}}(\mathbf{x})\right) = \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathcal{R}_0, \qquad \widehat{\mathbf{y}}\left(\overline{\mathbf{x}}(\mathbf{y})\right) = \mathbf{y} \quad \text{for all } \mathbf{y} \in \mathcal{R}. \tag{2.4}$$

[2]In Chapter 1 we denoted the region occupied by the body in the reference configuration by $\mathcal{R}_{\text{ref}}$. Here, we call it $\mathcal{R}_0$.

[3]Whenever there is no confusion in doing so, we shall use more convenient but less precise language such as "the particle $\mathbf{x}$" rather than "the particle p located at $\mathbf{x}$ in the reference configuration".

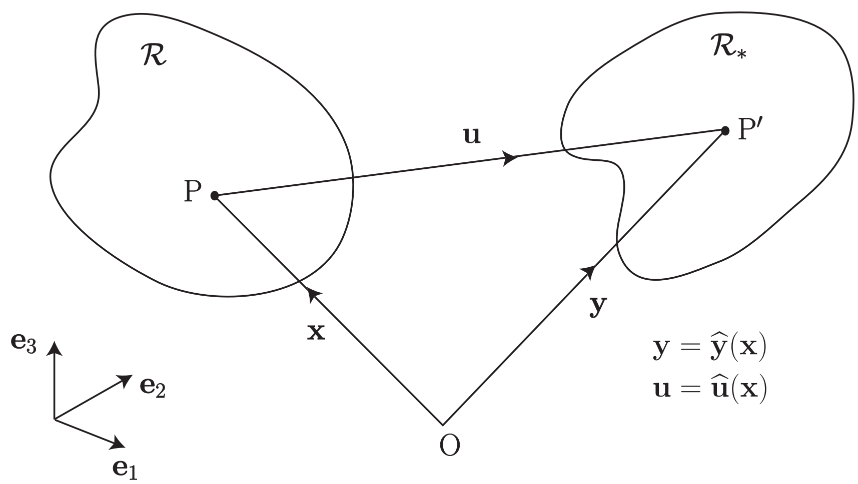

Figure 2.1: The respective regions $\mathcal{R}_0$ and $\mathcal{R}$ occupied by a body in a reference configuration and a deformed configuration; the position vectors of a generic particle in these two configurations are denoted by $\mathbf{x}$ and $\mathbf{y}$. The displacement of this particle is $\mathbf{u}$.

Unless explicitly stated otherwise, we will assume that $\widehat{\mathbf{y}}(\mathbf{x})$ and $\overline{\mathbf{x}}(\mathbf{y})$ are "smooth", or more specifically that they may each be differentiated at least twice, and that these derivatives are continuous on the relevant regions:

$$\widehat{\mathbf{y}} \in C^2(\mathcal{R}_0), \qquad \overline{\mathbf{x}} \in C^2(\mathcal{R}). \tag{2.5}$$

We will relax these requirements occasionally. For example, if we consider a "dislocation" it will be necessary to allow the displacement field to be discontinuous across a surface in the body; and if we consider a "two-phase composite material", we must allow the gradient of the displacement field to be discontinuous across the interface between the different materials.

Finally, consider a fixed right-handed orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. When we refer to components of vector and tensor quantities, it will always be with respect to this basis. In particular, the components of $\mathbf{x}$ and $\mathbf{y}$ in this basis are $x_i = \mathbf{x} \cdot \mathbf{e}_i$ and $y_i = \mathbf{y} \cdot \mathbf{e}_i,\ i = 1, 2, 3$. In terms of its components, equation (2.1) reads

$$y_i = y_i(x_1, x_2, x_3). \tag{2.6}$$

See Problems 2.1 and 2.2.

2.2 Deformation Gradient Tensor. Deformation in the Neighborhood of a Particle.

Let $\mathbf{x}$ denote the position of a generic particle of the body in the reference configuration. Questions that we may want to ask, such as what is the state of stress at this particle? will the material fracture at this particle? and so on, depend not only on the deformation at $\mathbf{x}$ but also on the deformation of *all* particles in a small neighborhood of $\mathbf{x}$. Thus, the deformation in the entire *neighborhood* of a generic particle plays a crucial role in this subject and we now focus on this. Thus we imagine a small ball of material centered at $\mathbf{x}$ and ask what happens to this ball as a result of the deformation. Intuitively, we expect the deformation of the ball (i.e. the local deformation near $\mathbf{x}$,) to consist of a combination of a rigid translation, a rigid rotation and a "straining", notions that we shall make precise in what follows. The so-called *deformation gradient tensor* at a generic particle $\mathbf{x}$ is defined by

$$\mathbf{F}(\mathbf{x}) = \operatorname{Grad} \mathbf{y}(\mathbf{x}). \tag{2.7}$$

This is the principal entity used to study the deformation in the immediate neighborhood of $\mathbf{x}$. The deformation gradient $\mathbf{F}(\mathbf{x})$ is a 2-tensor field and its components

$$F_{ij}(\mathbf{x}) = \frac{\partial y_i(\mathbf{x})}{\partial x_j} \tag{2.8}$$

correspond to the elements of a 3×3 matrix field $[F(\mathbf{x})]$.

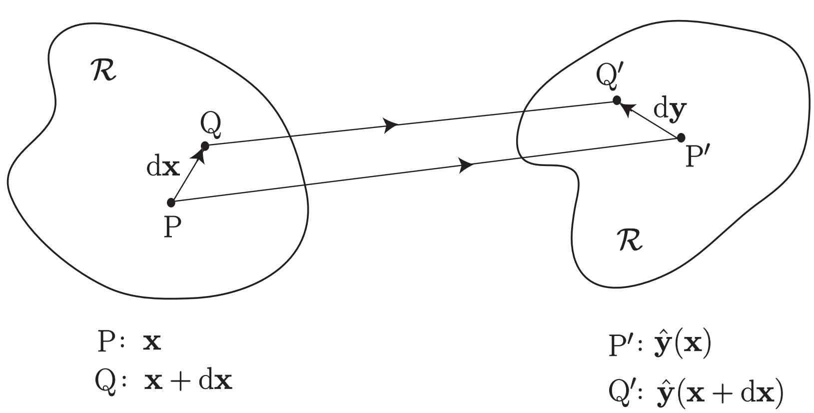

Figure 2.2: An infinitesimal material fiber in the reference and deformed configurations.

Consider two particles p and q located at $\mathbf{x}$ and $\mathbf{x}+d\mathbf{x}$ in the reference configuration; their locations are depicted by P and Q in Figure 2.2. The infinitesimal *material* fiber[4] joining these two particles is $d\mathbf{x}$. In the deformed configuration these two particles are located at $\mathbf{y}(\mathbf{x})$ and $\mathbf{y}(\mathbf{x}+d\mathbf{x})$ respectively, and the deformed image of this infinitesimal material fiber is described by the vector

$$d\mathbf{y} = \mathbf{y}(\mathbf{x}+d\mathbf{x}) - \mathbf{y}(\mathbf{x}). \tag{2.9}$$

Since p and q are neighboring particles we can approximate this expression for small $|\mathrm{d}\mathbf{x}|$ by the Taylor expansion

$$d\mathbf{y} = \big(\mathrm{Grad}\ \mathbf{y}\big)d\mathbf{x} + \mathrm{O}(|\mathrm{d}\mathbf{x}|^2) = \mathbf{F}\ \mathrm{d}\mathbf{x} + \mathrm{O}(|\mathrm{d}\mathbf{x}|^2), \tag{2.10}$$

which we can formally write as

$$d\mathbf{y} = \mathbf{F}d\mathbf{x}, \tag{2.11}$$

or in terms of components as

$$dy_i = F_{ij}\, dx_j \qquad \text{or} \quad \{y\} \;=\; [F]\,\{x\}. \tag{2.12}$$

Note that this approximation does *not* assume that the deformation or deformation gradient is small; only that the two particles p and q are close to each other.

Thus $\mathbf{F}$ carries an infinitesimal undeformed material fiber $d\mathbf{x}$ into its location $d\mathbf{y}$ in the deformed configuration.

In physically realizable deformations we expect that (a) a single fiber $d\mathbf{x}$ will not split into two fibers $d\mathbf{y}^{(1)}$ and $d\mathbf{y}^{(2)}$, and (b) that two fibers $d\mathbf{x}^{(1)}$ and $d\mathbf{x}^{(2)}$ will not coalesce into a single fiber $d\mathbf{y}$. This means that (2.11) must be a one-to-one relation between $d\mathbf{x}$ and $d\mathbf{y}$ and thus that $\mathbf{F}$ must be *non-singular*. Thus in particular the *Jacobian* determinant, J, must not vanish:

$$J \;=\; \det\mathbf{F} \neq 0. \tag{2.13}$$

Next, consider three linearly independent material fibers $d\mathbf{x}^{(i)}, i = 1,2,3$, as shown in Figure 2.3. The deformation carries these fibers into the three locations $d\mathbf{y}^{(i)} = \mathbf{F}d\mathbf{x}^{(i)}, i = 1,2,3$. A deformation *preserves orientation* if every right-handed triplet of fibers $\{d\mathbf{x}^{(1)}, d\mathbf{x}^{(2)}, d\mathbf{x}^{(3)}\}$ is carried into a right-handed triplet of fibers $\{d\mathbf{y}^{(1)}, d\mathbf{y}^{(2)}, d\mathbf{y}^{(3)}\}$, i.e. the deformation is orientation preserving if every triplet of fibers for which $(d\mathbf{x}^{(1)} \times d\mathbf{x}^{(2)}) \cdot d\mathbf{x}^{(3)} > 0$ is carried into a triplet of fibers for which $(d\mathbf{y}^{(1)} \times d\mathbf{y}^{(2)}) \cdot d\mathbf{y}^{(3)} > 0$. By using an identity established in one of the worked examples in Chapter 3 of Volume I, it follows that

[4]The notion of a *material* curve was explained at the end of Section 1.7: the fiber here being a material fiber implies that PQ and $P'Q'$ are associated with the same set of particles.

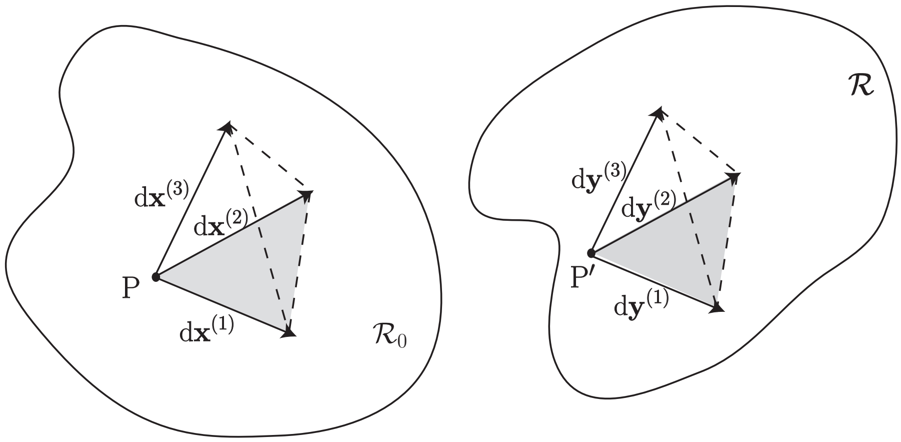

Figure 2.3: An orientation preserving deformation: the right-handed triplet of infinitesimal material fibers $\mathrm{d}\mathbf{x}^{(1)}, \mathrm{d}\mathbf{x}^{(2)}, \mathrm{d}\mathbf{x}^{(3)}$ are carried into a right-handed triplet of fibers $\mathrm{d}\mathbf{y}^{(1)}, \mathrm{d}\mathbf{y}^{(2)}, \mathrm{d}\mathbf{y}^{(3)}$.

$(d\mathbf{y}^{(1)} \times d\mathbf{y}^{(2)}) \cdot d\mathbf{y}^{(3)} = (\mathbf{F}d\mathbf{x}^{(1)} \times \mathbf{F}d\mathbf{x}^{(2)}) \cdot \mathbf{F}d\mathbf{x}^{(3)} = (\det \mathbf{F})\ (d\mathbf{x}^{(1)} \times d\mathbf{x}^{(2)}) \cdot d\mathbf{x}^{(3)}$. Consequently orientation is preserved if and only if

$$J \; = \; \det \mathbf{F} > 0. \tag{2.14}$$

In these notes we will only consider orientation-preserving deformations[5].

The deformation of a generic particle $\mathbf{x} + d\mathbf{x}$ in the neighborhood of particle $\mathbf{x}$ can be written formally as

$$\mathbf{y}(\mathbf{x} + \mathrm{d}\mathbf{x}) = \mathbf{y}(\mathbf{x}) + \mathbf{F}d\mathbf{x}. \tag{2.15}$$

Therefore in order to characterize the deformation of the entire neighborhood of $\mathbf{x}$ we must know both the deformation $\mathbf{y}(\mathbf{x})$ and the deformation gradient tensor $\mathbf{F}(\mathbf{x})$ at $\mathbf{x}$; $\mathbf{y}(\mathbf{x})$ characterizes the translation of that neighborhood while $\mathbf{F}(\mathbf{x})$ characterizes both the rotation and the "strain" at $\mathbf{x}$ as we shall see below.

A deformation $\mathbf{y}(\mathbf{x})$ is said to be *homogeneous* if the deformation gradient tensor is constant on the entire region $\mathcal{R}_0$. Thus, a homogeneous deformation is characterized by

$$\mathbf{y}(\mathbf{x}) = \mathbf{F}\mathbf{x} + \boldsymbol{b} \tag{2.16}$$

where $\mathbf{F}$ is a constant tensor and $\boldsymbol{b}$ is a constant vector. It is easy to verify that a set of points which lie on a straight line/plane/ellipsoid in the reference configuration will continue

[5]Some deformations that do not preserve orientation are of physical interest, e.g. the turning of a tennis ball inside out.

to lie on a straight line/plane/ellipsoid in the deformed configuration if the deformation is homogeneous.

2.3 Some Special Deformations.

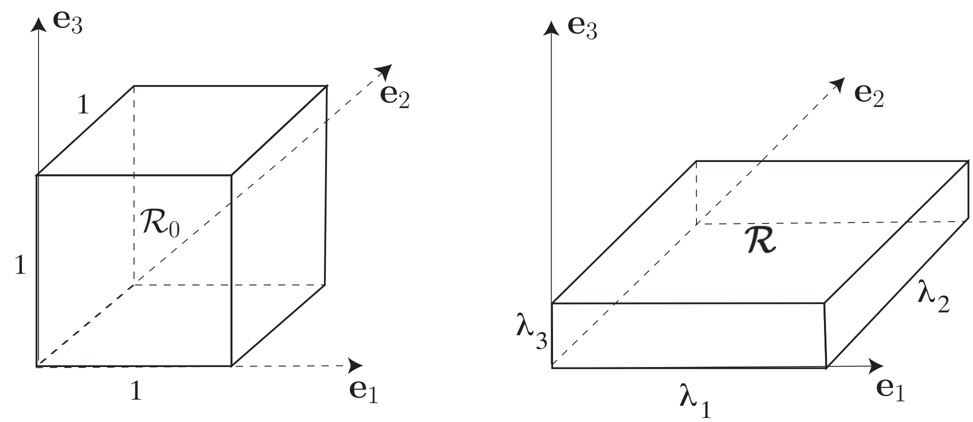

Figure 2.4: Pure homogeneous stretching of a cube. A unit cube in the reference configuration is carried into an orthorhombic region of dimensions $\lambda_1 \times \lambda_2 \times \lambda_3$.

Consider a body that occupies a unit cube in a reference configuration. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a fixed orthonormal basis with the basis vectors aligned with the edges of the cube; see Figure 2.4. Consider a *pure homogeneous stretching* of the cube,

$$\mathbf{y} = \mathbf{F}\mathbf{x} \qquad \text{where} \quad \mathbf{F} = \lambda_1 \mathbf{e}_1 \otimes \mathbf{e}_1 \; + \; \lambda_2 \mathbf{e}_2 \otimes \mathbf{e}_2 \; + \; \lambda_3 \mathbf{e}_3 \otimes \mathbf{e}_3, \tag{2.17}$$

where the three $\lambda_i^\prime s$ are positive constants. In terms of components in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, this deformation reads

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} . \tag{2.18}$$

The $1 \times 1 \times 1$ undeformed cube is mapped by this deformation into a $\lambda_1 \times \lambda_2 \times \lambda_3$ orthorhombic region $\mathcal{R}$ as shown in Figure 2.4. The volume of the deformed region is $\lambda_1\lambda_2\lambda_3$. The positive constants λ_1, λ_2 and λ_3 here represent the ratios by which the three edges of the cube *stretch* in the respective directions $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Any material fiber that was parallel to an edge of the cube in the reference configuration simply undergoes a stretch and no rotation under this deformation. However this is not in general true of any other material fiber – e.g. one oriented along a diagonal of a face of the cube – which will undergo both a length change and a rotation.

The deformation (2.17) is a *pure dilatation* in the special case

$$\lambda_1 = \lambda_2 = \lambda_3$$

in which event $\mathbf{F} = \lambda_1 \mathbf{I}$. The volume of the deformed region is λ_1^3.

If the deformation is *isochoric*, i.e. if the volume does not change, then $\lambda_1, \lambda_2, \lambda_3$ must be such that

$$\lambda_1 \lambda_2 \lambda_3 = 1. \tag{2.19}$$

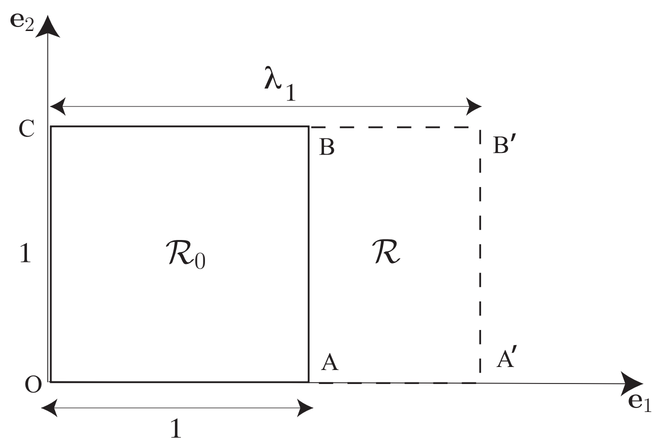

Figure 2.5: Uniaxial stretch in the $\mathbf{e}_1$-direction. A unit cube in the reference configuration is carried into a $\lambda_1 \times 1 \times 1$ tetragonal region $\mathcal{R}$ in the deformed configuration.

If $\lambda_2 = \lambda_3 = 1$, then the body undergoes a *uniaxial stretch* in the $\mathbf{e}_1$-direction (and no stretch in the $\mathbf{e}_2$ and $\mathbf{e}_3$ directions); see Figure 2.5. In this case

$$\mathbf{F} = \lambda_1 \mathbf{e}_1 \otimes \mathbf{e}_1 \ + \ \mathbf{e}_2 \otimes \mathbf{e}_2 \ + \ \mathbf{e}_3 \otimes \mathbf{e}_3, = \mathbf{I} + (\lambda_1 - 1)\mathbf{e}_1 \otimes \mathbf{e}_1.$$

If $\lambda_1 > 1$ the deformation is an elongation, whereas if $\lambda_1 < 1$ it is a contraction. (The terms "tensile" and "compressive" refer to stress not deformation.) More generally the deformation $\mathbf{y} = \mathbf{F}\mathbf{x}$ where

$$\mathbf{F} = \mathbf{I} + (\lambda - 1)\mathbf{n} \otimes \mathbf{n}, \qquad |\mathbf{n}| = 1, \tag{2.20}$$

represents a uniaxial stretch in the direction $\mathbf{n}$.

The cube is said to be subjected to a *simple shearing deformation* if

$$\mathbf{y} = \mathbf{F}\mathbf{x} \qquad \text{where } \mathbf{F} = \mathbf{I} + k\ \mathbf{e}_1 \otimes \mathbf{e}_2$$

and k is a constant. In terms of components in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, this deformation reads

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{2.21}$$

The simple shear deformation carries the cube into the sheared region $\mathcal{R}$ as shown in Figure 2.6. Observe that the displacement field here is given by $\mathbf{u}(\mathbf{x}) = \mathbf{y}(\mathbf{x}) - \mathbf{x} = \mathbf{F}\mathbf{x} - \mathbf{x} = k(\mathbf{e}_1 \otimes \mathbf{e}_2)\mathbf{x} = kx_2\,\mathbf{e}_1$. Thus each plane $x_2 =$ constant is displaced rigidly in the x_1-direction, the amount of the displacement depending linearly on the value of x_2. One refers to a plane $x_2 =$ constant as a *shearing (or glide) plane*, the x_1-direction as the *shearing direction* and k is called the *amount of shear*. One can readily verify that $\det\big(\mathbf{I} + k\,\mathbf{e}_1 \otimes \mathbf{e}_2\big) = 1$ wherefore a simple shear automatically preserves volume; (that $\det \mathbf{F}$ is a measure of volume change is discussed in Section 2.4.3).

More generally the deformation $\mathbf{y} = \mathbf{F}\mathbf{x}$ where

$$\mathbf{F} = \mathbf{I} + k\mathbf{m} \otimes \mathbf{n}, \qquad |\mathbf{m}| = |\mathbf{n}| = 1, \quad \mathbf{m} \cdot \mathbf{n} = 0, \tag{2.22}$$

represents a simple shear whose glide plane normal and shear direction are $\mathbf{n}$ and $\mathbf{m}$ respectively.

If

$$\lambda_3 = 1,$$

equation (2.17) describes a *plane deformation* in the $1,2$-plane (i.e. stretching occurs only in the $1,2$-plane; fibers in the $\mathbf{e}_3$-direction remain unstretched); and a *plane equi-biaxial stretch* in the $1,2$-plane if

$$\lambda_1 = \lambda_2, \quad \lambda_3 = 1.$$

If the material fibers in the direction defined by some unit vector $\mathbf{m}_0$ in the reference configuration remain *inextensible*, then $\mathbf{m}_0$ and its deformed image $\mathbf{F}\mathbf{m}_0$ must have the same length: $|\mathbf{F}\mathbf{m}_0| = |\mathbf{m}_0| = 1$ which holds if and only if

$$\mathbf{F}\mathbf{m}_0 \cdot \mathbf{F}\mathbf{m}_0 = \mathbf{F}^T\mathbf{F}\mathbf{m}_0 \cdot \mathbf{m}_0 = 1.$$

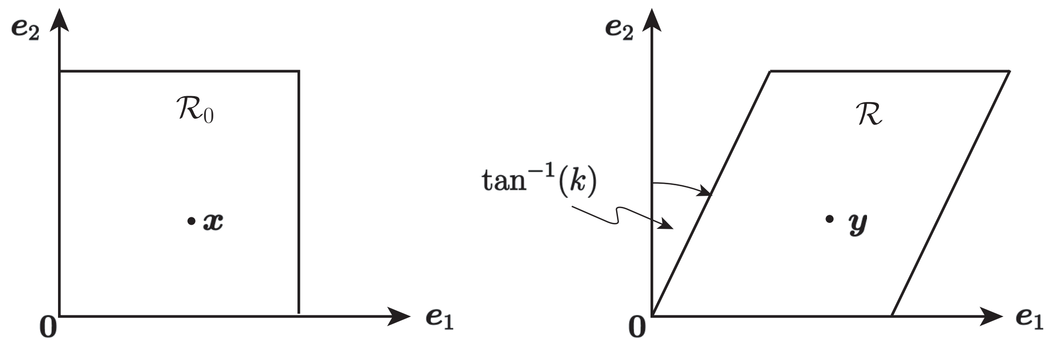

Figure 2.6: Simple shear of a cube. Each plane x_2 = constant undergoes a displacement in the x_1-direction by the amount kx_2.

For example, if $\mathbf{m}_0 = \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2$, we see by direct substitution that λ_1, λ_2 must obey the constraint equation

$$\lambda_1^2\cos^2\theta + \lambda_2^2\sin^2\theta = 1.$$

Given $\mathbf{m}_0$, this restricts $\mathbf{F}$.

We can now consider combinations of deformations, each of which is homogeneous. For example consider a deformation $\mathbf{y} = \mathbf{F}_1\mathbf{F}_2\mathbf{x}$ where $\mathbf{F}_1 = \mathbf{I} + \alpha\mathbf{a}\otimes\mathbf{a}$, $\mathbf{F}_2 = \mathbf{I} + k\mathbf{m}\otimes\mathbf{n}$, the vectors $\mathbf{a}, \mathbf{m}, \mathbf{n}$ have unit length, and $\mathbf{m}\cdot\mathbf{n} = 0$. This represents a simple shearing of the body (with amount of shear k, glide plane normal $\mathbf{n}$ and shear direction $\mathbf{m}$) in which $\mathbf{x} \to \mathbf{F}_2\mathbf{x}$, followed by a uniaxial stretching (in the $\mathbf{a}$ direction) in which $\mathbf{F}_2\mathbf{x} \to \mathbf{F}_1(\mathbf{F}_2\mathbf{x})$; see Figure 2.7 for an illustration of the case $\mathbf{a} = \mathbf{n} = \mathbf{e}_2, \mathbf{m} = \mathbf{e}_1$.

The preceding deformations were all homogeneous in the sense that they were all of the special form $\mathbf{y} = \mathbf{F}\mathbf{x}$ where $\mathbf{F}$ was a constant tensor. Most deformations $\mathbf{y} = \mathbf{y}(\mathbf{x})$ are not of this form. A simple example of an inhomogeneous deformation is

$$\left.\begin{aligned} y_1 &= x_1\cos\beta x_3 - x_2\sin\beta x_3,\\ y_2 &= x_1\sin\beta x_3 + x_2\cos\beta x_3,\\ y_3 &= x_3. \end{aligned}\right\}$$

This can be shown to represent a torsional deformation about the $\mathbf{e}_3$-axis in which each plane x_3 = constant rotates by an angle βx_3. The matrix of components of the deformation

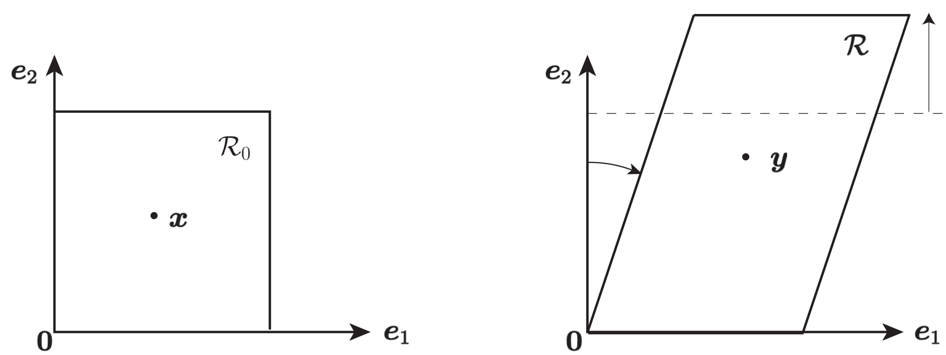

Figure 2.7: A unit cube subjected to a simple shear (with glide plane normal $\mathbf{e}_2$) followed by a uniaxial stretch in the direction $\mathbf{e}_2$.

gradient tensor associated with this deformation is

$$[F] = \left[\frac{\partial y_i}{\partial x_j}\right] = \begin{pmatrix} \cos\beta x_3 & -\sin\beta x_3 & -\beta x_1 \sin\beta x_3 - \beta x_2 \cos\beta x_3 \\ \sin\beta x_3 & \cos\beta x_3 & \beta x_1 \cos\beta x_3 - \beta x_2 \sin\beta x_3 \\ 0 & 0 & 1 \end{pmatrix};$$

observe that the components F_{ij} of the deformation gradient tensor here depend of (x_1, x_2, x_3).

See Problems 2.3 and 2.4.

2.4 Transformation of Length, Orientation, Angle, Volume and Area.

As shown by (2.15), the deformation gradient tensor $\mathbf{F}(\mathbf{x})$ characterizes all geometric changes *in the neighborhood* of the particle $\mathbf{x}$. We now examine the deformation of an infinitesimal material fiber, infinitesimal material surface and an infinitesimal material region. Specifically, we calculate quantities such as the *local* [6] change in length, angle, volume and area in terms of $\mathbf{F}(\mathbf{x})$. The change in length is related to the notion of fiber stretch (or strain), the change in angle is related to the notion of shear strain and the change in volume is related to the

[6] i.e. the geometric changes of *infinitesimally small* line, area and volume elements at $\mathbf{x}$.

notion of volumetric (or dilatational) strain – notions that we will encounter shortly and play an important role in this subject. The change in area is indispensable when calculating the true stress on a surface.

2.4.1 Change of Length and Orientation.

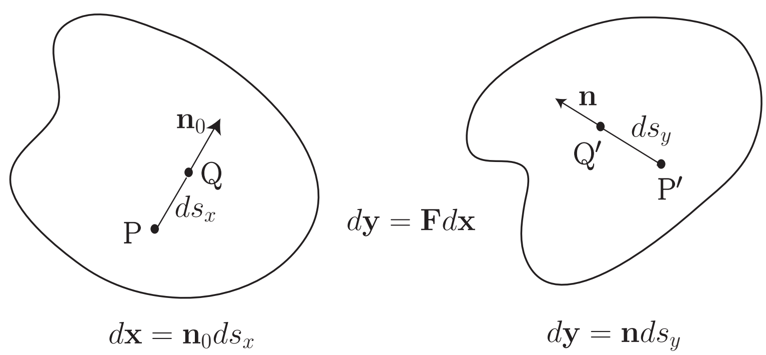

Figure 2.8: An infinitesimal material fiber: in the reference configuration it has length ds_x and orientation $\mathbf{n}_0$; in the deformed configuration it has length ds_y and orientation $\mathbf{n}$.

Suppose that we are given a material fiber that has length ds_x and orientation $\mathbf{n}_0$ in the reference configuration: $d\mathbf{x} = (ds_x)\mathbf{n}_0$. We want to calculate its length and orientation in the deformed configuration.

If the image of this fiber in the deformed configuration has length ds_y and orientation $\mathbf{n}$, then $d\mathbf{y} = (ds_y)\mathbf{n}$. Since $d\mathbf{y}$ and $d\mathbf{x}$ are related by $d\mathbf{y} = \mathbf{F}d\mathbf{x}$, it follows that

$$(ds_y)\mathbf{n} = (ds_x)\mathbf{F}\mathbf{n}_0. \tag{2.23}$$

Thus the deformed length of the fiber is

$$ds_y = |d\mathbf{y}| = |\mathbf{F}d\mathbf{x}| = ds_x|\mathbf{F}\mathbf{n}_0|. \tag{2.24}$$

The *stretch ratio* λ at the particle $\mathbf{x}$ in the direction $\mathbf{n}_0$ is defined as the ratio

$$\lambda = ds_y/ds_x \tag{2.25}$$

and so

$$\lambda = |\mathbf{F}\mathbf{n}_0|. \tag{2.26}$$

This gives the stretch ratio $\lambda = \lambda(\mathbf{n}_0) = |\mathbf{F}\mathbf{n}_0|$ of any fiber that was in the $\mathbf{n}_0$-direction in the reference configuration. You might want to ask the question, among all fibers of all orientations at $\mathbf{x}$, which has the maximum stretch ratio?

The orientation $\mathbf{n}$ of this fiber in the deformed configuration is found from (2.23) to be

$$\mathbf{n} = \frac{\mathbf{F}\mathbf{n}_0}{|\mathbf{F}\mathbf{n}_0|}. \tag{2.27}$$

2.4.2 Change of Angle.

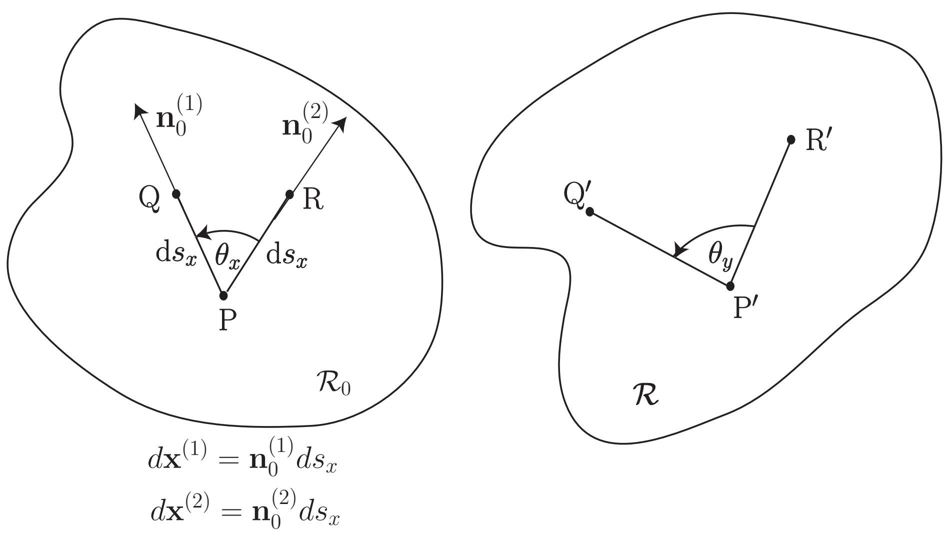

Figure 2.9: Two infinitesimal material fibers. In the reference configurations they have equal length ds_x and directions $\mathbf{n}_0^{(1)}$ and $\mathbf{n}_0^{(2)}$.

Suppose that we are given two fibers $d\mathbf{x}^{(1)}$ and $d\mathbf{x}^{(2)}$ in the reference configuration as shown in Figure 2.9. They both have the same length ds_x and they are oriented in the respective directions $\mathbf{n}_0^{(1)}$ and $\mathbf{n}_0^{(2)}$ where $\mathbf{n}_0^{(1)}$ and $\mathbf{n}_0^{(2)}$ are unit vectors: $d\mathbf{x}^{(1)} = ds_x\mathbf{n}_0^{(1)}$ and $d\mathbf{x}^{(2)} = ds_x\mathbf{n}_0^{(2)}$. Let θ_x denote the angle between them: $\cos\theta_x = \mathbf{n}_0^{(1)} \cdot \mathbf{n}_0^{(2)}$. We want to determine the angle between them in the deformed configuration.

In the deformed configuration these two fibers are characterized by $\mathbf{F}d\mathbf{x}^{(1)}$ and $\mathbf{F}d\mathbf{x}^{(2)}$. By definition of the scalar product of two vectors $\mathbf{F}d\mathbf{x}^{(1)} \cdot \mathbf{F}d\mathbf{x}^{(2)} = |\mathbf{F}d\mathbf{x}^{(1)}||\mathbf{F}d\mathbf{x}^{(2)}| \cos\theta_y$ and so the angle θ_y between them is found from

$$\cos\theta_y = \frac{\mathbf{F}d\mathbf{x}^{(1)}}{|\mathbf{F}d\mathbf{x}^{(1)}|} \cdot \frac{\mathbf{F}d\mathbf{x}^{(2)}}{|\mathbf{F}d\mathbf{x}^{(2)}|} = \frac{\mathbf{F}\mathbf{n}_0^{(1)} \cdot \mathbf{F}\mathbf{n}_0^{(2)}}{|\mathbf{F}\mathbf{n}_0^{(1)}||\mathbf{F}\mathbf{n}_0^{(2)}|}\ . \tag{2.28}$$

The decrease in angle $\gamma = \theta_x - \theta_y$ is the shear associated with the directions $\mathbf{n}_0^{(1)}, \mathbf{n}_0^{(2)}$: $\gamma = \gamma(\mathbf{n}_0^{(1)}, \mathbf{n}_0^{(2)})$. One can show that $\gamma \neq \pi/2$; (see Section 25 of Truesdell and Toupin). You might want to ask the question, among all pairs of fibers at $\mathbf{x}$, which pair suffers the maximum change in angle, i.e. maximum shear?

2.4.3 Change of Volume.

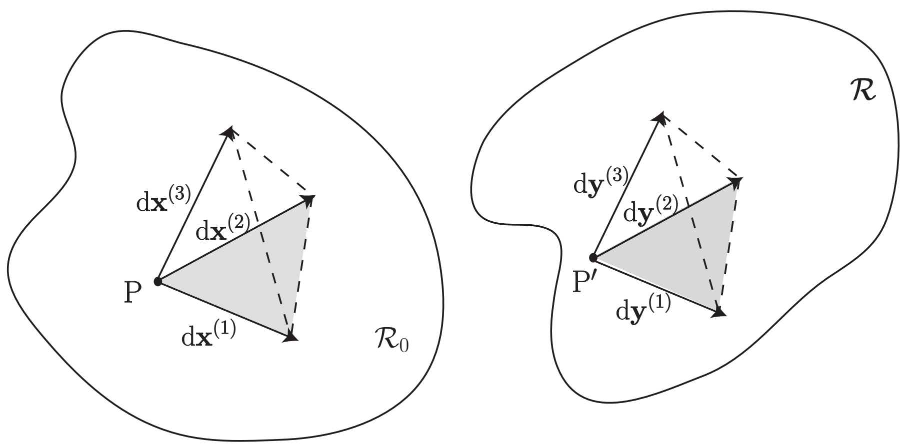

Figure 2.10: Three infinitesimal material fibers defining a tetrahedral region. The volumes of the tetrahedrons in the reference and deformed configurations are dV_x and dV_y respectively.

Next, consider three linearly independent material fibers $d\mathbf{x}^{(i)}, i = 1, 2, 3$, as shown in Figure 2.10. By geometry, the volume of the tetrahedron formed by these three fibers is

$$dV_x = \left| \frac{1}{6}(d\mathbf{x}^{(1)} \times d\mathbf{x}^{(2)}) \cdot d\mathbf{x}^{(3)} \right|;$$

see the related worked example in Chapter 2 of Volume I. The deformation carries these fibers into the three fibers $d\mathbf{y}^{(i)} = \mathbf{F} d\mathbf{x}^{(i)}$. The volume of the deformed tetrahedron is

$$\begin{aligned} dV_y &= |\tfrac{1}{6}(d\mathbf{y}^{(1)} \times d\mathbf{y}^{(2)}) \cdot d\mathbf{y}^{(3)}| = |\tfrac{1}{6}(\mathbf{F}d\mathbf{x}^{(1)} \times \mathbf{F}d\mathbf{x}^{(2)}) \cdot \mathbf{F}d\mathbf{x}^{(3)}| \\ &= |\det \mathbf{F}|\, |\tfrac{1}{6}(d\mathbf{x}^{(1)} \times d\mathbf{x}^{(2)}) \cdot d\mathbf{x}^{(3)}| = \det \mathbf{F}\; dV_x, \end{aligned}$$

where in the penultimate step we have used the identity noted just above (2.14) and the fact that $\det \mathbf{F} > 0$. Thus the volumes of a differential volume element in the reference and deformed configurations are related by

$$dV_y = J\; dV_x \qquad \text{where} \quad \mathrm{J} = \det \mathbf{F}. \tag{2.29}$$

Observe from this that a deformation preserves the volume of *every* infinitesimal volume element if and only if

$$J(\mathbf{x}) = 1 \qquad \text{at all} \quad \mathbf{x} \in \mathcal{R}_0. \tag{2.30}$$

Such a deformation is said to be isochoric or locally volume preserving.

An incompressible *material* is a material that can *only* undergo isochoric deformations.

2.4.4 Change of Area.

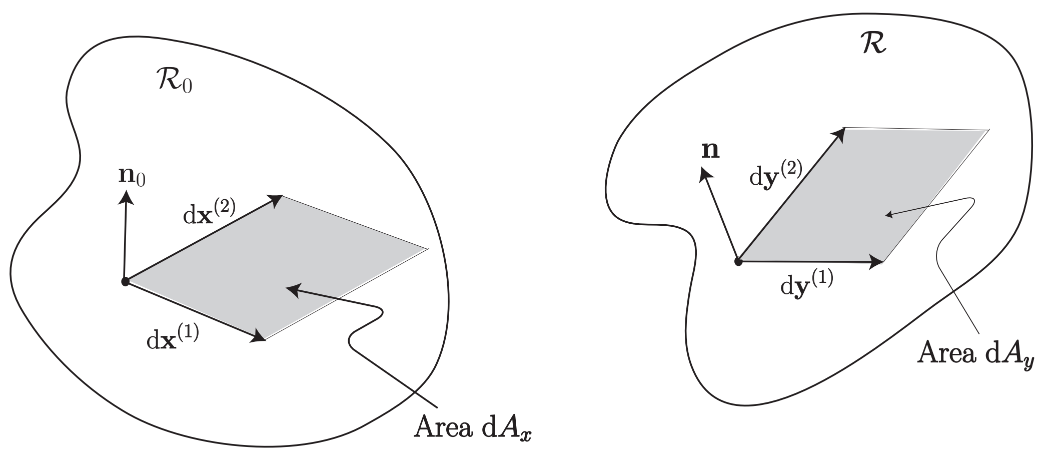

Figure 2.11: Two infinitesimal material fibers defining a parallelogram.

Next we consider the relationship between two area elements in the reference and deformed configurations. Consider the area element in the reference configuration defined by the fibers $d\mathbf{x}^{(1)}$ and $d\mathbf{x}^{(2)}$ as shown in Figure 2.11. Suppose that its area is $\mathrm{d}A_x$ and that $\mathbf{n}_0$ is a unit normal to this plane. Then, from the definition of the vector product,

$$d\mathbf{x}^{(1)} \times d\mathbf{x}^{(2)} = \mathrm{d}A_x \, \mathbf{n}_0. \tag{2.31}$$

Similarly if $\mathrm{d}A_y$ and $\mathbf{n}$ are the area and the unit normal, respectively, to the surface defined in the deformed configuration by $d\mathbf{y}^{(1)}$ and $d\mathbf{y}^{(2)}$, then

$$d\mathbf{y}^{(1)} \times d\mathbf{y}^{(2)} = \mathrm{d}A_y \, \mathbf{n}. \tag{2.32}$$

It is worth emphasizing that the surfaces under consideration (shown shaded in Figure 2.11) are composed of the same particles, i.e. they are "material" surfaces. Note that $\mathbf{n}_0$ and

$\mathbf{n}$ are defined by the fact that they are normal to these material surface elements. Since $d\mathbf{y}^{(i)} = \mathbf{F}d\mathbf{x}^{(i)}$, (2.32) can be written as

$$\mathbf{F}d\mathbf{x}^{(1)} \times \mathbf{F}d\mathbf{x}^{(2)} = \mathrm{d}A_y\,\mathbf{n}. \tag{2.33}$$

Then, by using an algebraic result from the relevant worked example in Chapter 3 of Volume I, and combining (2.31) with (2.33) we find that

$$\mathrm{d}A_y\,\mathbf{n} = \mathrm{d}A_x\,J\,\mathbf{F}^{-T}\,\mathbf{n}_0\ . \tag{2.34}$$

This relates the vector areas $\mathrm{d}A_y\mathbf{n}$ and $\mathrm{d}A_x\mathbf{n}_0$. By taking the magnitude of this vector equation we find that the areas dA_y and dA_x are related by

$$\mathrm{d}A_y = \mathrm{d}A_x\,J\,|\mathbf{F}^{-T}\,\mathbf{n}_0|; \tag{2.35}$$

On using (2.35) in (2.34) we find that the unit normal vectors $\mathbf{n}_0$ and $\mathbf{n}$ are related by

$$\mathbf{n} = \frac{\mathbf{F}^{-T}\mathbf{n}_0}{|\mathbf{F}^{-T}\mathbf{n}_0|}. \tag{2.36}$$

Observe that $\mathbf{n}$ is not in general parallel to $\mathbf{F}\mathbf{n}_0$ indicating that a material fiber in the direction characterized by $\mathbf{n}_0$ is not mapped into a fiber in the direction $\mathbf{n}$. As noted previously, $\mathbf{n}_0$ and $\mathbf{n}$ are defined by the fact that they are normal to the material surface elements being considered; not by the fact that one is the image of the other under the deformation. The particles that lie along the fiber $\mathbf{n}_0$ are mapped by $\mathbf{F}$ into a fiber that is in the direction of $\mathbf{F}\mathbf{n}_0$ which is *not* generally perpendicular to the plane defined by $d\mathbf{y}^{(1)}$ and $d\mathbf{y}^{(2)}$.

See Problems 2.5 - 2.10.

2.5 Rigid Deformation.

We now consider the special case of a *rigid deformation*. A deformation is said to be rigid if the distance between all pairs of particles is preserved under the deformation, i.e. if the distance $|\mathbf{z} - \mathbf{x}|$ between any two particles $\mathbf{x}$ and $\mathbf{z}$ in the reference configuration equals the distance $|\mathbf{y}(\mathbf{z}) - \mathbf{y}(\mathbf{x})|$ between them in the deformed configuration:

$$|\mathbf{y}(\mathbf{z})-\mathbf{y}(\mathbf{x})|^2 = \Big[y_i(\mathbf{z})-y_i(\mathbf{x})\Big]\Big[y_i(\mathbf{z})-y_i(\mathbf{x})\Big] = (z_i-x_i)(z_i-x_i) \qquad \text{for all } \mathbf{x},\mathbf{z} \in \mathcal{R}_0. \tag{2.37}$$

Since (2.37) holds for all $\mathbf{x}$, we may take its derivative with respect to x_j to get

$$-2F_{ij}(\mathbf{x})\,(y_i(\mathbf{z}) - y_i(\mathbf{x})) = -2(z_j - x_j) \qquad \text{for all } \mathbf{x}, \mathbf{z} \in \mathcal{R}_0, \tag{2.38}$$

where $F_{ij}(\mathbf{x}) = \partial y_i(\mathbf{x})/\partial x_j$ are the components of the deformation gradient tensor. Since (2.38) holds for all $\mathbf{z}$ we may take its derivative with respect to z_k to obtain $F_{ij}(\mathbf{x})F_{ik}(\mathbf{z}) = \delta_{jk}$, i.e.

$$\mathbf{F}^T(\mathbf{x})\mathbf{F}(\boldsymbol{z}) = \mathbf{1} \qquad \text{for all } \boldsymbol{x}, \mathbf{z} \in \mathcal{R}_0. \tag{2.39}$$

Finally, since (2.39) holds for all $\mathbf{x}$ and all $\mathbf{z}$, we can take $\mathbf{x} = \mathbf{z}$ in (2.39) to get

$$\mathbf{F}^T(\mathbf{x})\mathbf{F}(\mathbf{x}) = \mathbf{I} \qquad \text{for all } \mathbf{x} \in \mathcal{R}_0. \tag{2.40}$$

Thus we conclude that $\mathbf{F}(\mathbf{x})$ is an orthogonal tensor at each $\mathbf{x}$. In fact, since $\det \boldsymbol{F} > 0$, it is proper orthogonal and therefore represents a rotation.

The (possible) dependence of $\mathbf{F}$ on $\mathbf{x}$ implies that $\mathbf{F}$ might be a different proper orthogonal tensor at different points $\mathbf{x}$ in the body. However, returning to (2.39), multiplying both sides of it by $\mathbf{F}(\mathbf{x})$ and recalling that $\mathbf{F}$ is orthogonal gives

$$\mathbf{F}(\mathbf{z}) = \mathbf{F}(\mathbf{x}) \qquad \text{at all } \mathbf{x}, \mathbf{z} \in \mathcal{R}_0; \tag{2.41}$$

(2.41) implies that $\mathbf{F}(\mathbf{x})$ is a *constant* tensor.

In conclusion, the deformation gradient tensor associated with a rigid deformation is a constant rotation tensor. Thus at all $\mathbf{x} \in \mathcal{R}_0$ we can denote $\mathbf{F}(\mathbf{x}) = \mathbf{Q}$ where $\mathbf{Q}$ is a constant proper orthogonal tensor. Thus necessarily a rigid deformation has the form

$$\mathbf{y} = \mathbf{y}(\mathbf{x}) = \mathbf{Q}\mathbf{x} + \boldsymbol{b} \tag{2.42}$$

where $\mathbf{Q}$ is a constant rotation tensor and $\boldsymbol{b}$ is a constant vector. Conversely it is easy to verify that (2.42) satisfies (2.37).

A rigid *material* (or rigid body) is a material that can *only* undergo rigid deformations.

One can readily verify from (2.42) and the results of the previous section that in a rigid deformation the length of every fiber remains unchanged; the angle between every two fibers remains unchanged; the volume of any differential element remains unchanged; and the unit vectors $\mathbf{n}_0$ and $\mathbf{n}$ normal to a surface in the reference and deformed configurations are simply related by $\mathbf{n} = \mathbf{Q}\mathbf{n}_0$.

2.6 Decomposition of Deformation Gradient Tensor into a Rotation and a Stretch.

As mentioned repeatedly above, the deformation gradient tensor $\mathbf{F}(\mathbf{x})$ completely characterizes the deformation in the vicinity of the particle $\mathbf{x}$. Part of this deformation is a rigid rotation, the rest is a "distortion" or "strain". The central question is "which part of $\mathbf{F}$ is the rotation and which part is the strain?" The answer to this is provided by the polar decomposition theorem discussed in Chapter 2 of Volume I. According to this theorem, every nonsingular tensor $\mathbf{F}$ with positive determinant can be written uniquely as the product of a proper orthogonal tensor $\mathbf{R}$ and a symmetric positive definite tensor $\mathbf{U}$ as

$$\mathbf{F} = \mathbf{R}\,\mathbf{U}; \tag{2.43}$$

$\mathbf{R}$ represents the rotational part of $\mathbf{F}$ while $\mathbf{U}$ represents the part that is not a rotation. It is readily seen from (2.43) that $\mathbf{U}$ is given by the positive definite square root

$$\mathbf{U} = \sqrt{\mathbf{F}^T\mathbf{F}} \tag{2.44}$$

so that $\mathbf{R}$ is then given by

$$\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}. \tag{2.45}$$

Since a generic undeformed material fiber is carried by the deformation from $d\mathbf{x} \to d\mathbf{y} = \mathbf{F}\,d\mathbf{x}$, we can write the relationship between the two fibers as

$$d\mathbf{y} = \mathbf{R}\,(\mathbf{U}d\mathbf{x}). \tag{2.46}$$

This allows us to view the deformation of the fiber in two-steps: first, the fiber $d\mathbf{x}$ is taken by the deformation to $\mathbf{U}\,d\mathbf{x}$, and then, it is rotated rigidly by $\mathbf{R}$: $\mathrm{d}\mathbf{x} \to \mathbf{U}\mathrm{d}\mathbf{x} \to \mathbf{R}(\mathbf{U}\mathrm{d}\mathbf{x}) = \mathrm{d}\mathbf{y}$.

The essential property of $\mathbf{U}$ is that it is symmetric and positive definite. This allows us to physically interpret $\mathbf{U}$ as follows: since $\mathbf{U}$ is symmetric, it has three real eigenvalues λ_1, λ_2 and λ_3, and a corresponding triplet of orthonormal eigenvectors $\mathbf{r}_1, \mathbf{r}_2$ and $\mathbf{r}_3$. Since $\mathbf{U}$ is positive definite, all three eigenvalues are positive. Thus the matrix of components of $\mathbf{U}$ in the principal basis $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ is

$$[U] = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \qquad\qquad \lambda_i > 0. \tag{2.47}$$

If the components of d$\mathbf{x}$ in this principal basis are

$$\{dx\} = \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix} \qquad \text{then} \qquad [U]\,\{dx\} = \begin{pmatrix} \lambda_1 dx_1 \\ \lambda_2 dx_2 \\ \lambda_3 dx_3 \end{pmatrix}.$$

Thus when d$\mathbf{x} \to \mathbf{U}d\mathbf{x}$, *the fiber* d$\mathbf{x}$ *is stretched by the tensor* $\mathbf{U}$ *in the principal directions of* $\mathbf{U}$ *by amounts given by the corresponding eigenvalues of* $\mathbf{U}$. The tensor $\mathbf{U}$ is called the *right stretch tensor.*

The stretched fiber $\mathbf{U}\,d\mathbf{x}$ is now taken by the rigid rotation $\mathbf{R}$ from $\mathbf{U}d\mathbf{x} \to \mathbf{R}(\mathbf{U}d\mathbf{x})$. Note that in general, the fiber $d\mathbf{x}$ will rotate while it undergoes the stretching deformation $d\mathbf{x} \to \mathbf{U}d\mathbf{x}$, since $d\mathbf{x}$ is not necessarily parallel to $\mathbf{U}d\mathbf{x}$; however this is not a rigid rotation since the length of the fiber also changes.

The alternative version of the polar decomposition theorem (Chapter 2 of Volume I) provides a second representation for $\mathbf{F}$. According to this part of the theorem, every nonsingular tensor $\mathbf{F}$ with positive determinant can be written uniquely as the product of a symmetric positive definite tensor $\mathbf{V}$ with a proper orthogonal tensor $\mathbf{R}$ as

$$\mathbf{F} = \mathbf{V}\mathbf{R}; \tag{2.48}$$

the tensor $\mathbf{R}$ here is identical to that in the preceding representation and represents the rotational part of $\mathbf{F}$. It is readily seen from (2.48) that $\mathbf{V}$ is given by

$$\mathbf{V} = \sqrt{\mathbf{F}\mathbf{F}^T} \tag{2.49}$$

and that $\mathbf{R}$ is given by

$$\mathbf{R} = \mathbf{V}^{-1}\mathbf{F}. \tag{2.50}$$

Since $\mathbf{R} = \mathbf{V}^{-1}\mathbf{F} = \mathbf{F}\mathbf{U}^{-1}$ it follows that $\mathbf{V} = \mathbf{F}\mathbf{U}\mathbf{F}^{-1}$.

A generic undeformed fiber $d\mathbf{x}$ can therefore alternatively be related to its image $d\mathbf{y}$ in the deformed configuration by

$$d\mathbf{y} = \mathbf{V}\,(\mathbf{R}\,d\mathbf{x}), \tag{2.51}$$

and so we can view the deformation of the fiber as first, a rigid rotation from $d\mathbf{x}$ to $\mathbf{R}\,d\mathbf{x}$, followed by a stretching by $\mathbf{V}$. Since $\mathbf{V}$ is symmetric and positive definite, all three of its eigenvalues, λ_1, λ_2 and λ_3 are real and positive; moreover the corresponding eigenvectors form an orthonormal basis $\{\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3\}$ – a principal basis of $\mathbf{V}$. Thus the deformation can alternatively be viewed as, first, a rigid rotation of the fiber by the tensor $\mathbf{R}$ followed by stretching in the principal directions of $\mathbf{V}$: $d\mathbf{x} \to \mathbf{R}d\mathbf{x} \to \mathbf{V}(\mathbf{R}d\mathbf{x})$. The tensor $\mathbf{V}$ is called the *left stretch tensor*.

It is easy to show that the eigenvalues λ_1, λ_2 and λ_3 of $\mathbf{U}$ are identical to those of $\mathbf{V}$. Moreover one can show that the eigenvectors by $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ of $\mathbf{U}$ are related to the eigenvectors $\{\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3\}$ of $\mathbf{V}$ by $\boldsymbol{\ell}_i = \mathbf{R}\mathbf{r}_i,\ i = 1,2,3$. The common eigenvalues of $\mathbf{U}$ and $\mathbf{V}$, are known as the *principal stretches* associated with the deformation (at $\mathbf{x}$). The stretch tensors $\mathbf{U}$ and $\mathbf{V}$ can be expressed in terms of their eigenvectors and eigenvalues as

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{r}_i \otimes \mathbf{r}_i, \qquad \mathbf{V} = \sum_{i=1}^{3} \lambda_i \boldsymbol{\ell}_i \otimes \boldsymbol{\ell}_i; \tag{2.52}$$

see Section 2.2 of Volume I. As shown in one of the worked examples in Chapter 2 of Volume I, we also have the representations

$$\mathbf{F} = \sum_{i=1}^{3} \lambda_i \boldsymbol{\ell}_i \otimes \mathbf{r}_i, \qquad \mathbf{R} = \sum_{i=1}^{3} \boldsymbol{\ell}_i \otimes \mathbf{r}_i. \tag{2.53}$$

The expressions (2.24), (2.27), (2.28), (2.29) and (2.35) describe changes in length, orientation, angle, volume and area in terms of the deformation gradient tensor $\mathbf{F}$. Since a rotation does not change length, angle, area and volume we expect that these equations (except for the one for orientation) should be independent of the rotation tensor $\mathbf{R}$ in the polar decomposition. By using $\mathbf{F} = \mathbf{R}\mathbf{U}$ in (2.24), (2.28), (2.29) and (2.35) it is readily seen that they can be expressed in terms of $\mathbf{U}$ as

$$\left.\begin{aligned} ds_y &= ds_x\, \sqrt{\mathbf{U}^2\mathbf{n}_0 \cdot \mathbf{n}_0}, \\ \cos\theta_y &= \frac{\mathbf{U}^2\mathbf{n}_0^{(1)} \cdot \mathbf{n}_0^{(2)}}{\sqrt{\mathbf{U}^2\mathbf{n}_0^{(1)} \cdot \mathbf{n}_0^{(1)}}\, \sqrt{\mathbf{U}^2\boldsymbol{n}_0^{(2)} \cdot \mathbf{n}_0^{(2)}}}, \\ \mathrm{d}V_y &= \mathrm{d}V_x\ \det\mathbf{U}, \\ \mathrm{d}A_y &= \mathrm{d}A_x\ (\det\mathbf{U})|\mathbf{U}^{-1}\mathbf{n}_0|, \end{aligned}\right\} \tag{2.54}$$

which emphasizes the fact that these changes depend only on the stretch tensor $\mathbf{U}$ and not the rotational part $\mathbf{R}$ of the deformation gradient tensor[7]. The formula (2.27) for the change of orientation of a fiber takes the form

$$\mathbf{n} = \mathbf{R}\, \frac{\mathbf{U}\mathbf{n}_0}{|\mathbf{U}\mathbf{n}_0|}, \tag{2.55}$$

[7]Recall that for any tensor $\mathbf{A}$ and any two vectors $\mathbf{x}$ and $\mathbf{y}$, we have $\mathbf{A}\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{A}^T\mathbf{y}$.

which shows that the orientation of a fiber changes due to both stretching and rotation.

Observe that the expressions in (2.54) give us information about the deformed images of various geometric entities, given their pre-images in the reference configuration; for example, the right hand side of $(2.54)_1$ involves the orientation $\mathbf{n}_0$ of the fiber in the reference configuration; the right hand side of $(2.54)_2$ involves the orientations $\mathbf{n}_0^{(1)}$ and $\mathbf{n}_0^{(2)}$ of the two fibers in the reference configuration; and so on.

If instead, the geometric entities are given in the deformed configuration, and we want to determine the geometric properties of their pre-images in the reference configuration, these can be readily calculated in terms of the left stretch tensor $\mathbf{V}$. Consider, for example, a fiber which in the deformed configuration has length ds_y and orientation $\mathbf{n}$. Then its length ds_x in the reference configuration can be calculated as follows:

$$ds_x = |d\mathbf{x}| = |\mathbf{F}^{-1}d\mathbf{y}| = |\mathbf{R}^{-1}\mathbf{V}^{-1}d\mathbf{y}| = |\mathbf{V}^{-1}d\mathbf{y}| = ds_y\,|\mathbf{V}^{-1}\mathbf{n}|\,. \tag{2.56}$$

Similarly, if two fibers $ds_y\mathbf{n}^{(1)}$ and $ds_y\mathbf{n}^{(2)}$ in the deformed configuration are given and they subtend an angle θ_y, then the angle θ_x that their pre-images subtend in the reference configuration is given by

$$\cos\theta_x = \frac{\mathbf{V}^{-2}\mathbf{n}^{(1)}\cdot\mathbf{n}^{(2)}}{\sqrt{\mathbf{V}^{-2}\boldsymbol{n}^{(1)}\cdot\mathbf{n}^{(1)}}\,\sqrt{\mathbf{V}^{-2}\mathbf{n}^{(2)}\cdot\mathbf{n}^{(2)}}}\,. \tag{2.57}$$

Similarly an expression for the volume dV_x in the reference configuration of a differential volume element can be calculated in terms of the volume dV_y in the deformed configuration and the stretch tensor $\mathbf{V}$; and likewise an expression for the area dA_x in the reference configuration of a differential area element can be calculated in terms of the area dA_y and unit normal $\mathbf{n}$ in the deformed configuration and the stretch tensor $\mathbf{V}$.

Thus we see that the left stretch tensor $\mathbf{V}$ allows us to compute geometric quantities in the reference configuration in terms of their images in the deformed configuration; and that similarly the right stretch tensor $\mathbf{U}$ allows us to compute geometric quantities in the deformed configuration in terms of their pre-images in the reference configuration. In this sense we can view $\mathbf{U}$ and $\mathbf{V}$ as, respectively, Lagrangian and Eulerian stretch tensors.

Remark: It is quite tedious to calculate the tensors $\mathbf{U} = (\mathbf{F}^T\mathbf{F})^{1/2}$ and $\mathbf{V} = (\mathbf{F}\mathbf{F}^T)^{1/2}$. However, since there is a one-to-one relation between $\mathbf{U}$ and $\mathbf{U}^2$, and similarly between $\mathbf{V}$ and $\mathbf{V}^2$, we can just as well use $\mathbf{U}^2$ and $\mathbf{V}^2$ as our measures of stretch; these are usually denoted by $\mathbf{C}$ and $\mathbf{B}$:

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2, \qquad \mathbf{B} = \mathbf{F}\mathbf{F}^T = \mathbf{V}^2, \tag{2.58}$$

and are referred to as the *right* and *left Cauchy–Green deformation tensors* respectively. Note that the eigenvalues of $\mathbf{C}$ and $\mathbf{B}$ are $\lambda_1^2,\ \lambda_2^2$ and λ_3^2, where λ_i are the principal stretches, and that the eigenvectors of $\mathbf{C}$ and $\mathbf{B}$ are the same as those of $\mathbf{U}$ and $\mathbf{V}$ respectively. The two Cauchy-Green tensors admit the spectral representations

$$\mathbf{C} = \sum_{i=1}^{3} \lambda_i^2\,(\mathbf{r}_i \otimes \mathbf{r}_i), \qquad \mathbf{B} = \sum_{i=1}^{3} \lambda_i^2\,(\boldsymbol{\ell}_i \otimes \boldsymbol{\ell}_i). \tag{2.59}$$

The particular scalar-valued functions of $\mathbf{C}$

$$I_1(\mathbf{C}) = \mathrm{tr}\,\mathbf{C}, \qquad I_2(\mathbf{C}) = \frac{1}{2}\Big[\mathrm{tr}\,\mathbf{C}^2 \ - \ \big(\mathrm{tr}\,\mathbf{C}\big)^2\Big], \qquad I_3(\mathbf{C}) = \det\mathbf{C}, \tag{2.60}$$

are called the principal scalar invariants of $\mathbf{C}$. It can be readily verified that these functions have the property that for each symmetric tensor $\mathbf{C}$,

$$I_i(\mathbf{C}) = I_i(\mathbf{Q}\mathbf{C}\mathbf{Q}^T), \qquad i = 1, 2, 3, \tag{2.61}$$

for all orthogonal tensors $\mathbf{Q}$. They are invariant scalar-valued functions in this sense. Finally, it can be shown that they satisfy the identity

$$\det(\mathbf{C} - \mu\mathbf{I}) = -\mu^3 + I_1(\mathbf{C})\,\mu^2 - I_2(\mathbf{C})\,\mu + I_3(\mathbf{C})$$

for all scalars μ.

The principal scalar invariants can be written in terms of the principal stretches as

$$I_1(\mathbf{C}) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \qquad I_2(\mathbf{C}) = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2, \qquad I_3(\mathbf{C}) = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{2.62}$$

The principal scalar invariants of $\mathbf{B}$ and $\mathbf{C}$ coincide:

$$I_i(\mathbf{C}) = I_i(\mathbf{B}), \quad i = 1, 2, 3.$$

See Problem 2.11.

2.7 Strain.

It is clear that $\mathbf{U}$ and $\mathbf{V}$ are the essential ingredients that characterize the non-rigid part of the deformation. If "the body is not deformed", i.e. the deformed configuration happens to coincide with the reference configuration, the deformation is given by $\mathbf{y}(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \mathcal{R}_0$, and therefore $\mathbf{F} = \mathbf{I}$ and $\mathbf{U} = \mathbf{V} = \mathbf{I}$. Thus the stretch equals the identity $\mathbf{I}$ in the reference configuration. "Strain" on the other hand customarily vanishes in the reference configuration. Thus strain is simply an alternative measure for the non-rigid part of the deformation chosen such that it vanishes in the reference configuration. This is the only essential difference between stretch and strain. Thus for example we could take $\mathbf{U} - \mathbf{I}$ for the strain where $\mathbf{U}$ is the stretch.

Various measures of Lagrangian strain and Eulerian strain are used in the literature, examples of which we shall describe below. It should be pointed out that continuum theory does not prefer[8] one strain measure over another; each is a one-to-one function of the stretch tensor and so all strain measures are equivalent. In fact, one does not even have to introduce the notion of strain and the theory could be based entirely on the stretch tensors $\mathbf{U}$ and $\mathbf{V}$.

The various measures of *Lagrangian strain* used in the literature are all related to the stretch $\mathbf{U}$ in a one-to-one manner. Examples include the Green strain, the generalized Green strain and the Hencky (or logarithmic) strain, defined by the respective expressions

$$\frac{1}{2}(\mathbf{U}^2 - \mathbf{I}), \qquad \frac{1}{m}(\mathbf{U}^m - \mathbf{I}) \quad \text{and} \qquad \ln \mathbf{U}, \tag{2.63}$$

where m is a non-zero integer[9]. The principal directions of each of these strain tensors are the same as those of $\mathbf{U}$, i.e. $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$; the associated principal strains are

$$\tfrac{1}{2}(\lambda_i^2 - 1), \qquad \tfrac{1}{m}(\lambda_i^m - 1), \qquad \text{and} \qquad \ln \lambda_i \tag{2.64}$$

respectively.

Similarly, various measures of *Eulerian strain* are used in the literature, all of them being related to the stretch $\mathbf{V}$ in a one-to-one manner. Examples include the Almansi strain, the generalized Almansi strain and the logarithmic strain, defined by the respective expressions

$$\frac{1}{2}(\mathbf{I} - \mathbf{V}^{-2}), \qquad \frac{1}{m}(\mathbf{V}^m - \mathbf{I}), \quad \text{and} \qquad \ln \mathbf{V} \tag{2.65}$$

[8]It sometimes happens that the constitutive description of a particular material takes an especially simple form when one particular strain measure is used in its characterization, while a different strain measure might lead to a simple constitutive description for some other material. This might then lead to a preference for one strain measure over another for a particular material.

[9]Recall that the logarithm of the symmetric positive definite tensor $\mathbf{U}$ is defined by $\ln \mathbf{U} = \sum_{i=1}^{3} \ln \lambda_i \, (\mathbf{r}_i \otimes \mathbf{r}_i)$ where λ_i and $\mathbf{r}_i$ are eigenvalues and eigenvectors of $\mathbf{U}$.

where m is a non-zero integer. The principal directions of each of these strain tensors are the same as those of $\mathbf{V}$, i.e. $\{\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3\}$.

The preceding examples may be *unified and generalized* as follows: Let $e(\cdot)$ be a(ny) scalar valued function that is defined on $(0, \infty)$ such that

$$\begin{aligned} \text{a)} \quad e(1) &= 0, \\ \text{b)} \quad e'(1) &= 1, \\ \text{c)} \quad e'(\lambda) &> 0 \quad \text{for all} \quad \lambda > 0. \end{aligned} \tag{2.66}$$

Then, one can define the Lagrangian strain tensor $\mathbf{E}(\mathbf{U})$ to be the tensor with eigenvectors $\mathbf{r}_i$ and corresponding eigenvalues $e(\lambda_i)$, i.e.

$$\mathbf{E} = \sum_{i=1}^{3} e(\lambda_i)(\mathbf{r}_i \otimes \mathbf{r}_i). \tag{2.67}$$

The condition $(2.66)_1$ ensures that $\mathbf{E} = \mathbf{O}$ if the deformed configuration coincides with the reference configuration. As we shall see shortly, condition $(2.66)_2$ ensures that $\mathbf{E}(\mathbf{U})$ linearizes to the classical infinitesimal strain tensor[10]. Condition $(2.66)_3$ ensures that each principal strain $e(\lambda_i)$ increases monotonically with the corresponding principal stretch λ_i; note that, necessarily, the principal strain is positive for extensions ($\lambda_i > 1$) and negative for contractions ($\lambda_i < 1$).

Observe that all of these Lagrangian strain tensors are symmetric. Their diagonal components E_{11}, E_{22} and E_{33} are known as the *normal* components of strain, while the off-diagonal components E_{12}, E_{23} and E_{31} are the *shear* components of strain. Since $\mathbf{E}$ is symmetric, it follows that it has a *principal basis* which is in fact $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$; in this basis, its matrix of components is diagonal; the shear components of strain vanish in this basis and the normal components are the *principal strains*.

A generalized Eulerian strain tensor can be defined analogously.

The *Green strain* tensor $\mathbf{E}$ is defined by

$$\mathbf{E} = \frac{1}{2}(\mathbf{U}^2 - \mathbf{I}) = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}) = \frac{1}{2}\Big(\text{Grad}\,\mathbf{u} + (\text{Grad}\,\mathbf{u})^T + (\text{Grad}\,\mathbf{u})^T\text{Grad}\,\mathbf{u}\Big), \tag{2.68}$$

where $\mathbf{u}$ is the displacement vector. It has components

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right). \tag{2.69}$$

[10]Note that for values of λ close to unity, equations $(2.66)_{1,2}$ leads to $e(\lambda) \approx \lambda - 1 = (ds_y - ds_x)/ds_x$ which is the familiar definition of normal strain in infinitesimal deformation theory.

The expression $(2.54)_1$ for change in length of a fiber can be written in terms of the Green strain as

$$ds_y = ds_x \sqrt{1 + 2\mathbf{E}\mathbf{n}_0 \cdot \mathbf{n}_0}, \tag{2.70}$$

which allows us to calculate the change in length of the fiber, per unit reference length, as

$$\frac{ds_y - ds_x}{ds_x} = \sqrt{1 + 2\mathbf{E}\mathbf{n}_0 \cdot \mathbf{n}_0} \ - \ 1. \tag{2.71}$$

Equation (2.71) characterizes the relative elongation, or normal strain, in the *arbitrary* fiber direction $\mathbf{n}_0$. As a special case, consider a fiber which is oriented in the direction $\mathbf{n}_0 = \mathbf{e}_1$. Then (2.71) yields

$$\frac{ds_y - ds_x}{ds_x} = \sqrt{1 + 2E_{11}} \ - \ 1; \tag{2.72}$$

observe that this expression only involves the strain E_{11}. Thus in general, the normal components of strain E_{11}, E_{22} and E_{33} characterize length changes in the respective coordinate directions x_1, x_2 and x_3. Similarly, the expression $(2.54)_2$ for the change in angle between two fibers can be written in terms of the Green strain as

$$\cos\theta_y = \frac{(\mathbf{I} + 2\mathbf{E})\mathbf{n}_0^{(1)} \cdot \mathbf{n}_0^{(2)}}{\sqrt{(\mathbf{I} + 2\mathbf{E})\mathbf{n}_0^{(1)} \cdot \mathbf{n}_0^{(1)}}\ \sqrt{(\mathbf{I} + 2\mathbf{E})\mathbf{n}_0^{(2)} \cdot \mathbf{n}_0^{(2)}}}. \tag{2.73}$$

As a special case take $\mathbf{n}_0^{(1)} = \mathbf{e}_1$ and $\mathbf{n}_0^{(2)} = \mathbf{e}_2$ so that in the reference configuration the two fibers are oriented in the x_1- and x_2-directions. Then the preceding equation simplifies to

$$\cos\theta_y = \frac{2E_{12}}{\sqrt{(1 + 2E_{11})}\ \sqrt{(1 + 2E_{22})}} \tag{2.74}$$

showing that the angle between these two fibers in the deformed configuration depends on the shear strain E_{12} *and* the normal strains E_{11} and E_{22}.

See Problems 2.12 and 2.13.

2.8 Linearization.

The *displacement* $\mathbf{u}(\mathbf{x})$ at a particle $\mathbf{x}$ is defined by

$$\mathbf{u}(\mathbf{x}) = \mathbf{y}(\mathbf{x}) - \mathbf{x}, \tag{2.75}$$

and the *displacement gradient tensor*

$$\mathbf{H}(\mathbf{x}) = \text{Grad}\, \mathbf{u}(\mathbf{x}) \tag{2.76}$$

has components

$$H_{ij} = \frac{\partial u_i}{\partial x_j}. \tag{2.77}$$

From (2.2), (2.76) it follows that the displacement gradient $\mathbf{H}$ and the deformation gradient $\mathbf{F}$ are related by

$$\mathbf{H} = \mathbf{F} - \mathbf{I}. \tag{2.78}$$

Since the various kinematic quantities encountered previously, such as the stretches $\mathbf{U}, \mathbf{V}$, the rotation $\mathbf{R}$ and the strain $\mathbf{E}$, were expressed in terms of the deformation gradient tensor $\mathbf{F}$, they can all be represented instead in terms of the displacement gradient tensor $\mathbf{H}$. In many physical circumstances the displacement gradient $\mathbf{H}$ is "small". This will be made precise below, and our goal in this section is to derive approximations for $\mathbf{U}, \mathbf{V}, \mathbf{R}, \mathbf{E}$ etc. in this special case.

To this end we note three preliminary algebraic results. First, recall from Volume I, that the norm (or magnitude) of a tensor $\mathbf{A}$ is defined as

$$|\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}} \;=\; \sqrt{\text{trace}(\mathbf{A}^T\mathbf{A})}. \tag{2.79}$$

Observe that (i) $|\mathbf{A}| > 0$ for all $\mathbf{A} \neq \mathbf{0}$; that (ii) in terms of the components A_{ij} of $\mathbf{A}$ in any orthonormal basis,

$$|\mathbf{A}| = (A_{11}^2 + A_{12}^2 + A_{13}^2 + A_{21}^2 + \cdots + A_{33}^2)^{1/2}, \tag{2.80}$$

and therefore that (iii) if $|\mathbf{A}| \to 0$ then each component $A_{ij} \to 0$ as well.

Second, let $\mathbf{Z}(\mathbf{H})$ be a function that is defined for all 2-tensors $\mathbf{H}$ and whose values are also 2-tensors. We say that $\mathbf{Z}(\mathbf{H}) = O(|\mathbf{H}|^n)$ as $|\mathbf{H}| \to 0$ if there exists a number $\alpha > 0$ such that $|\mathbf{Z}(\mathbf{H})| < \alpha|\mathbf{H}|^n$ as $|\mathbf{H}| \to 0$.

And third, if $\mathbf{A}$ is any symmetric tensor, and m is a real number, then

$$(\mathbf{I} + \mathbf{A})^m = \mathbf{I} + m\mathbf{A} + \mathbf{B} \qquad \text{where} \qquad |\mathbf{B}| = O(|\mathbf{A}|^2) \quad \text{as} \quad |\mathbf{A}| \to 0 \tag{2.81}$$

which can be readily established in a principal basis of $\mathbf{A}$.

We are now in a position to linearize our preceding kinematic quantities in the special case when $\mathbf{H} = \operatorname{Grad}\mathbf{u} = \mathbf{F} - \mathbf{I}$ is small. To this end we set

$$|\mathbf{H}| = \epsilon \tag{2.82}$$

and conclude that as $\epsilon \to 0$,

$$\begin{aligned}
\mathbf{U}^2 &= \mathbf{F}^T\mathbf{F} &&= \mathbf{I} + \mathbf{H} + \mathbf{H}^T + O(\epsilon^2), \\
\mathbf{V}^2 &= \mathbf{F}\mathbf{F}^T &&= \mathbf{I} + \mathbf{H} + \mathbf{H}^T + O(\epsilon^2), \\
\mathbf{U} &= \sqrt{\mathbf{U}^2} &&= \mathbf{I} + \tfrac{1}{2}(\boldsymbol{H} + \mathbf{H}^T) + O(\epsilon^2), \\
\mathbf{V} &= \sqrt{\mathbf{V}^2} &&= \mathbf{I} + \tfrac{1}{2}(\boldsymbol{H} + \mathbf{H}^T) + O(\epsilon^2), \\
\mathbf{R} &= \mathbf{F}\mathbf{U}^{-1} &&= \mathbf{I} + \tfrac{1}{2}(\mathbf{H} - \mathbf{H}^T) + O(\epsilon^2),
\end{aligned} \tag{2.83}$$

where we have used (2.81) in deriving the last three equations here. Using the properties of $e(\cdot)$, we can linearize the Lagrangian strain tensor (2.67) to get

$$\mathbf{E}(\mathbf{U}) = \sum_i e(\lambda_i)\mathbf{r}_i \otimes \mathbf{r}_i = \sum_i (\lambda_i - 1)\mathbf{r}_i \otimes \boldsymbol{r}_i + O\left(\epsilon^2\right). \tag{2.84}$$

Finally, we define two 2-tensors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\omega}$ by

$$\boldsymbol{\varepsilon} \stackrel{\text{def}}{=} \tfrac{1}{2}(\mathbf{H} + \mathbf{H}^T) \qquad \text{and} \qquad \boldsymbol{\omega} \stackrel{\text{def}}{=} \tfrac{1}{2}(\mathbf{H} - \mathbf{H}^{\mathrm{T}}), \tag{2.85}$$

which in component form reads

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad \omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right). \tag{2.86}$$

Observe that the stretch tensors can be approximated as

$$\mathbf{U} = \mathbf{I} + \boldsymbol{\varepsilon} + O\left(\epsilon^2\right), \qquad \mathbf{V} = \mathbf{I} + \boldsymbol{\varepsilon} + O\left(\epsilon^2\right), \tag{2.87}$$

that the general Lagrangian strain tensor $\mathbf{E}$ can be approximated as

$$\mathbf{E} = \boldsymbol{\varepsilon} + O\left(\epsilon^2\right), \tag{2.88}$$

and that the rotation tensor $\mathbf{R}$ can be approximated as

$$\mathbf{R} = \mathbf{I} + \boldsymbol{\omega} + O\left(\epsilon^2\right). \tag{2.89}$$

The tensors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\omega}$ are known as the *infinitesimal strain tensor* and the *infinitesimal rotation tensor* respectively and play a central role in the theory of solids undergoing infinitesimal deformations. Note that if ε_i is an eigenvalue of $\boldsymbol{\varepsilon}$ then

$$\lambda_i = 1 + \varepsilon_i + O(\epsilon^2). \tag{2.90}$$

Remark: It is useful to observe from (2.11), (2.78) and (2.85) that a fiber $d\mathbf{x}$ in the reference configuration and its image $d\mathbf{y}$ in the deformed configuration are related by

$$d\mathbf{y} = d\mathbf{x} + \boldsymbol{\varepsilon}\, d\mathbf{x} + \boldsymbol{\omega}\, d\mathbf{x} + O(\epsilon^2), \tag{2.91}$$

which expresses the fact that, in the linearized theory, the local deformation can be *additively* decomposed into a strain and a rotation. This is in contrast to the multiplicative decomposition $\mathrm{d}\mathbf{y} = \mathbf{RU}\mathrm{d}\mathbf{x}$ for a finite deformation.

Remark: The linearized versions of equations (2.72) and (2.74) read

$$\varepsilon_{11} \approx \frac{ds_y - ds_x}{ds_x}, \qquad \varepsilon_{12} \approx \frac{1}{2}\cos\theta_y \approx \frac{1}{2}(\pi/2 - \theta_y). \tag{2.92}$$

It follows from this that when the deformation is infinitesimal, the normal strain component ε_{11} represents the change in length per reference length of a fiber that was in the x_1-direction in the reference configuration; and that the shear strain component ε_{12} represents one half the decrease in angle between two fibers that were in the x_1- and x_2-directions in the reference configuration.

Remark: There are certain physical circumstances in which one wants to carry out a different linearization (i.e., linearization based on the smallness of some other quantity, not Grad $\mathbf{u}$). For example, consider rolling up a sheet of paper. The rolled-up configuration is the deformed configuration; the flat one is the reference one. In this situation one has large rotations $\mathbf{R}$ but small strains $\mathbf{U}-\mathbf{I}$. Thus one might wish to linearize based on the assumption that $\mathbf{U}-\mathbf{I}$ is small (but leave $\mathbf{R}$ arbitrary). Note that under these conditions $\mathbf{H}$ may not be small.

See Problem 2.14.

2.9 Worked Examples and Exercises.

Problem 2.1. A body occupies a hollow circular cylindrical region $\mathcal{R}_0$ in a reference configuration where $\mathcal{R}_0$ has inner radius a, outer radius b and length L: $\mathcal{R}_0 = \{(x_1, x_2, x_3) : a < (x_1^2 + x_2^2)^{1/2} < b,\ 0 < x_3 < L\}$.

All components of vectors are taken with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ shown in the figure. A particle located at (x_1, x_2, x_3) in the reference configuration is carried to the location (y_1, y_2, y_3) by the deformation

$$\left.\begin{aligned} y_1 &= f(r)\Big[x_1 \cos\phi(x_3) - x_2 \sin\phi(x_3)\Big], \\ y_2 &= f(r)\Big[x_2 \cos\phi(x_3) + x_1 \sin\phi(x_3)\Big], \\ y_3 &= \lambda x_3; \end{aligned}\right\} \qquad (a)$$

where $r = (x_1^2 + x_2^2)^{1/2}$. Here $\phi(x_3)$ is a given smooth function defined on $(0, L)$ and $\lambda > 0$ is a constant. Describe the physical nature of this deformation.

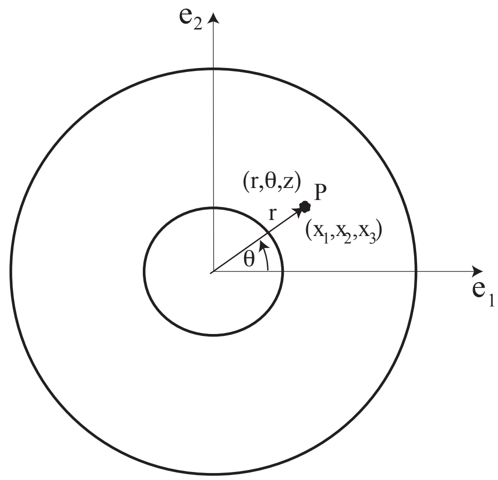

Figure 2.12: Cross-section of the region $\mathcal{R}_0$ occupied by the body in a reference configuration: a hollow circular cylinder of inner radius a, outer radius b (and length L).

<u>*Solution*</u>: First, from $(a)_3$ it is clear that this deformation describes a uniform stretching of the cylinder in the x_3-direction. Thus in particular, the deformed length of the cylinder is λL.

The cylindrical polar coordinates (r, θ, z) of a particle in the reference configuration are related to the rectangular cartesian coordinates (x_1, x_2, x_3) by

$$x_1 = r\cos\theta, \qquad x_2 = r\sin\theta, \qquad x_3 = z. \qquad (b)$$

On substituting (b) into (a) and using a standard trigonometric identity one gets

$$y_1 = rf(r)\cos\Big(\theta + \phi(z)\Big), \qquad y_2 = rf(r)\sin\Big(\theta + \phi(z)\Big), \qquad y_3 = \lambda z. \qquad (c)$$

Observe from (c) that

$$y_1^2 + y_2^2 = \Big(rf(r)\Big)^2.$$

Therefore the particles that lie on any circle $r = c =$ constant in the reference configuration are carried by the deformation onto a circle in the deformed configuration of radius $cf(c)$. Thus the cylinder undergoes a radial expansion (if $f(c) > 1$) or radial contraction (if $f(c) < 1$). The function $f(r)$ is related to the radial deformation of the cylinder.

Finally observe from (c) that

$$\frac{y_2}{y_1} = \tan\Big(\theta + \phi(z)\Big). \qquad (d)$$

Therefore the points that lie on any radial straight line $\theta = c =$ constant, $z =$ constant in the reference configuration are carried by the deformation onto the radial straight line defined by (d). Therefore cross-sections of the cylinder are twisted by this deformation. According to (d), the cross-section at $z = c =$ constant is rotated by the angle $\phi(c)$.

Problem 2.2. Bending of a slab.

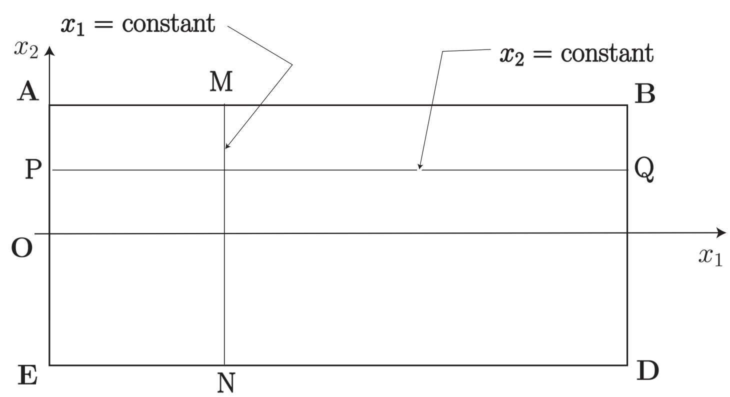

Figure 2.13: Slab-like region $\mathcal{R}_0 = \{(x_1, x_2, x_3) \mid 0 \leq x_1 \leq L, -h \leq x_2 \leq h, -b/2 \leq x_3 \leq b/2\}$ occupied by a body in the reference configuration.

Consider a body that occupies the slab-like region $\mathcal{R}_0 = \{(x_1, x_2, x_3) \mid 0 \leq x_1 \leq L, -h \leq x_2 \leq h, -b/2 \leq x_3 \leq b/2\}$ in a reference configuration. Figure 2.13 shows a side view of this slab looking down the x_3-axis. The slab is subjected to a deformation that carries it into the region $\mathcal{R}$ shown in Figure 2.14. More specifically, the body is subjected to a deformation that has the following properties:

i) The displacement vector of every particle is parallel to the (x_1, x_2)-plane.

ii) Every plane $x_3 =$ constant in $\mathcal{R}_0$ deforms identically. Consequently one can treat this as a two-dimensional problem and work on the (x_1, x_2)-plane.

iii) Each straight line $x_1 =$ constant is carried into a straight line in the deformed configuration, (e.g. AE $\to$ A′E′, MN $\to$ M′N′, BD $\to$ B′D′ etc.); moreover, the family of such straight lines corresponding to the various values of x_1 all pass through the same point $(y_1, y_2) = (0, \gamma L)$, see Figure 2.14.

iv) Each straight line $x_2 =$ constant is deformed into a circular arc centered at $(0, \gamma L)$ as shown in Figure 2.14, (e.g. AB $\to$ A′B′, PQ $\to$ P′Q′, OC $\to$ O′C′ etc.).

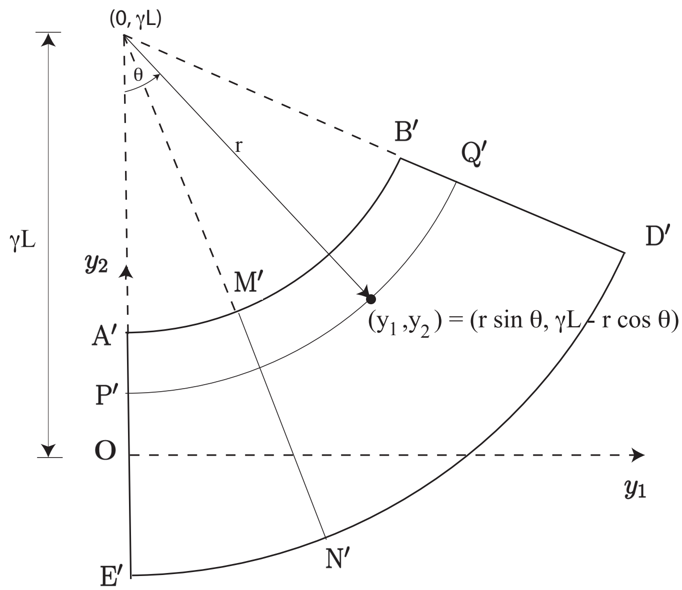

Figure 2.14: The region occupied by the deformed body. The points P′, Q′, M′, N′, etc. are the images in the deformed configuration of the points P, Q, M, N, etc. in the reference configuration. Vertical straight lines, e.g. MN, in the reference configuration are mapped into straight lines, e.g. M′N′, that pass through the point $(0, \gamma L)$. Horizontal straight lines in the reference configuration, e.g. PQ, are carried into circular arcs, e.g. P′Q′

Determine the mathematical characterization, $y_i = y_i(x_1, x_2, x_3)$, of this deformation.

<u>*Solution*</u>: Since no particle has a displacement component in the $\mathbf{e}_3$-direction it follows that $u_3(x_1, x_2, x_3)$ $= 0$ and therefore that

$$y_3(x_1, x_2, x_3) = x_3 \qquad \text{for all} \quad (\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3) \in \mathcal{R}_0.$$

Moreover, since every plane $x_3 =$ constant deforms identically, it follows that $y_1(x_1, x_2, x_3)$ and $y_2(x_1, x_2, x_3)$ are independent of x_3:

$$y_1 = y_1(x_1, x_2), \quad y_2 = y_2(x_1, x_2) \qquad \text{for all} \quad (\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3) \in \mathcal{R}_0, \tag{a}$$

where $y_1(x_1, x_2)$ and $y_2(x_1, x_2)$ are to be determined.

Given the shape of the domain $\mathcal{R}$ it seems natural to consider polar coordinates (r, θ) centered at the point $(0, \gamma L)$ to describe the deformed geometry. Thus instead of the representation (a) we can write equivalently write

$$y_1 = r(x_1, x_2) \sin \theta(x_1, x_2), \qquad y_2 = \gamma L - r(x_1, x_2) \cos \theta(x_1, x_2),$$

where $r(x_1, x_2)$ and $\theta(x_1, x_2)$ are to be determined.

Since a straight line $x_1 =$ constant maps into a straight line $\theta =$ constant it follows that $\theta(x_1, x_2)$ cannot depend on x_2 and so

$$\theta = \theta(x_1).$$

Next, since a straight line $x_2 =$ constant maps into the arc of a circle $r =$ constant it follows that $r(x_1, x_2)$ cannot depend on x_1 and so

$$r = r(x_2).$$

Thus in summary the deformation from $(x_1, x_2) \to (y_1, y_2)$ described in the problem is characterized by

$$y_1 = r(x_2)\sin\theta(x_1), \qquad y_2 = \gamma L - r(x_2)\cos\theta(x_1) \qquad (a)$$

where $r(x_1) > 0$ and $\theta(x_2) \in [0, 2\pi)$ are arbitrary functions.

Problem 2.3. [Truesdell and Toupin]

(a) Show that an arbitrary homogeneous deformation can be decomposed into the product of a simple shear, a uniaxial extension normal to the plane of shear, a pure dilatation, and a rotation.

(b) Show that an arbitrary homogeneous deformation can be decomposed into the product of three simple shears on mutually orthogonal planes, a pure dilatation, and a rotation.

Problem 2.4.

a. Let $\mathbf{y} = \mathbf{F}_1\mathbf{x}$ and $\mathbf{y} = \mathbf{F}_2\mathbf{x}$ be two arbitrary homogeneous deformations. Suppose that the deformation $\mathbf{y} = \mathbf{F}_1\mathbf{F}_2\mathbf{x}$ is a simple shear. Then, is the deformation $\mathbf{y} = \mathbf{F}_2\mathbf{F}_1\mathbf{x}$ also a simple shear? If it is a simple shear (either in general or under special circumstances), what is the associated amount of shear, glide plane normal and direction of shear?

b. Under what conditions (if any) are two simple shears commutative? That is, suppose that $\mathbf{y} = \mathbf{F}_1\mathbf{x}$ and $\mathbf{y} = \mathbf{F}_2\mathbf{x}$ represent two (distinct) simple shear deformations. Then consider the two deformations $\mathbf{y} = \mathbf{F}_1\mathbf{F}_2\mathbf{x}$ and $\mathbf{y} = \mathbf{F}_2\mathbf{F}_1\mathbf{x}$ which arise by sequentially applying the preceding simple shears. The question asks you to determine the conditions under which the deformations $\mathbf{y} = \mathbf{F}_1\mathbf{F}_2\mathbf{x}$ and $\mathbf{y} = \mathbf{F}_2\mathbf{F}_1\mathbf{x}$ are identical.

Problem 2.5. Suppose that the region $\mathcal{R}_0$ occupied by a body in a reference configuration is a unit cube. The body undergoes the homogeneous deformation $\mathbf{y} = \mathbf{F}\mathbf{x}$ described by

$$y_1 = \lambda_1 x_1, \qquad y_2 = \lambda_2 x_2, \qquad y_3 = \lambda_3 x_3,$$

where the components here have been taken with respect to an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ that is aligned with the axes of the cube. Derive relationships between the principal stretches in each of the following cases:

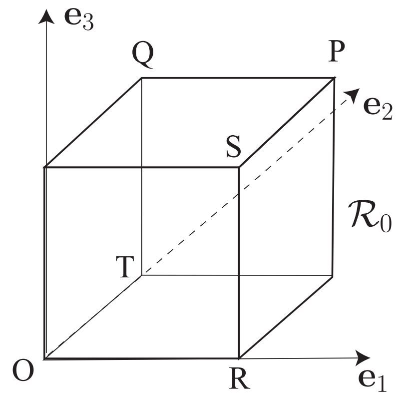

Figure 2.15: Unit cube $\mathcal{R}_0$ occupied by a body in its reference configuration.

(a) The body is composed of an incompressible material.

(b) The length of the fiber OP remains unchanged by the deformation.

(c) The angle between the fibers OP and QR remains unchanged by the deformation.

(d) The area of the plane $RSQT$ remains unchanged by the deformation.

(e) The orientation of the plane $RSQT$ remains unchanged by the deformation.

Solution: All components of vectors and tensors will be taken with respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. In particular the matrix of components of the deformation gradient tensor is

$$[F] = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \qquad \text{and} \qquad J = \det[F] = \lambda_1\lambda_2\lambda_3.$$

(a) In general, volume elements in the reference and deformed configurations are related by $dV_y = JdV_x$. If the material is incompressible then $dV_y = dV_x$ and so $J = 1$. Thus if the material is incompressible, the deformation must be such that

$$\lambda_1\lambda_2\lambda_3 = 1.$$

(b) In general, a fiber $ds_x\mathbf{n}_0$ in the reference configuration has length $ds_y = ds_x|\mathbf{F}\mathbf{n}_0|$ in the deformed configuration. Thus if the fiber does not change length, then $ds_x = ds_y$ and so $|\mathbf{F}\mathbf{n}_0| = 1$. Here, the fiber $\overrightarrow{\mathrm{OP}}$ can be expressed as

$$\overrightarrow{\mathrm{OP}} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 = \sqrt{3}\,\mathbf{n}_0$$

where the unit vector $\mathbf{n}_0$ defines the direction of $\overrightarrow{\mathrm{OP}}$ and is given by

$$\mathbf{n}_0 = \frac{\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3}{\sqrt{3}}.$$

Thus

$$[F]\{n_0\} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{Bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{Bmatrix} = \begin{Bmatrix} \lambda_1/\sqrt{3} \\ \lambda_2/\sqrt{3} \\ \lambda_3/\sqrt{3} \end{Bmatrix}.$$

Since the fiber $\vec{\mathrm{OP}}$ does not change length we must have $|\mathbf{F}\mathbf{n}_0| = 1$, and so the deformation must be such that

$$\frac{\lambda_1^2}{3} + \frac{\lambda_2^2}{3} + \frac{\lambda_3^2}{3} = 1.$$

(c) In general, the angle θ_x between two material fibers that are in the directions of the unit vectors $\mathbf{n}_0^{(1)}$ and $\mathbf{n}_0^{(2)}$ in the reference configuration is given by $\cos\theta_x = \mathbf{n}_0^{(1)} \cdot \mathbf{n}_0^{(2)}$. The angle θ_y in the deformed configuration between these same two fibers is given by $\cos\theta_y = (\mathbf{F}\mathbf{n}_0^{(1)}/|\mathbf{F}\mathbf{n}_0^{(1)}|) \cdot (\mathbf{F}\mathbf{n}_0^{(2)}/|\mathbf{F}\mathbf{n}_0^{(2)}|)$. Thus if the angle remains unchanged by the deformation we must have

$$\frac{\mathbf{F}\mathbf{n}_0^{(1)}}{|\mathbf{F}\mathbf{n}_0^{(1)}|} \cdot \frac{\mathbf{F}\mathbf{n}_0^{(2)}}{|\mathbf{F}\mathbf{n}_0^{(2)}|} \;=\; \mathbf{n}_0^{(1)} \cdot \mathbf{n}_0^{(2)} \; .$$

Here, the material fibers $\vec{\mathrm{OP}}$ and $\vec{\mathrm{QR}}$ are described by

$$\vec{\mathrm{OP}} \;= \sqrt{3}\,\mathbf{n}_0^{(1)} \quad \text{where the unit vector } \mathbf{n}_0^{(1)} = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3}, \text{ and}$$

$$\vec{\mathrm{QR}} \;= \sqrt{3}\,\mathbf{n}_0^{(2)} \quad \text{where the unit vector } \mathbf{n}_0^{(2)} = (\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3)/\sqrt{3}.$$

Thus

$$\cos\theta_x = \mathbf{n}_0^{(1)} \cdot \mathbf{n}_0^{(2)} = \Big(1.1 + 1.(-1) + 1.(-1)\Big)\frac{1}{3} = -\frac{1}{3}.$$

Moreover, from the preceding we have

$$[F]\{n_0^{(1)}\} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{Bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{Bmatrix} = \begin{Bmatrix} \lambda_1/\sqrt{3} \\ \lambda_2/\sqrt{3} \\ \lambda_3/\sqrt{3} \end{Bmatrix}, \qquad |\mathbf{F}\mathbf{n}_0^{(1)}| = \sqrt{\frac{\lambda_1^2}{3} + \frac{\lambda_2^2}{3} + \frac{\lambda_3^2}{3}},$$

and

$$[F]\{n_0^{(2)}\} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{Bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ -1/\sqrt{3} \end{Bmatrix} = \begin{Bmatrix} \lambda_1/\sqrt{3} \\ -\lambda_2/\sqrt{3} \\ -\lambda_3/\sqrt{3} \end{Bmatrix}, \qquad |\mathbf{F}\mathbf{n}_0^{(2)}| = \sqrt{\frac{\lambda_1^2}{3} + \frac{\lambda_2^2}{3} + \frac{\lambda_3^2}{3}}.$$

Thus

$$\cos\theta_y = \frac{\mathbf{F}\mathbf{n}_0^{(1)}}{|\mathbf{F}\mathbf{n}_0^{(1)}|} \cdot \frac{\mathbf{F}\mathbf{n}_0^{(2)}}{|\mathbf{F}\mathbf{n}_0^{(2)}|} = \Big(\frac{\lambda_1^2}{3} - \frac{\lambda_2^2}{3} - \frac{\lambda_3^2}{3}\Big)\Big/\Big(\frac{\lambda_1^2}{3} + \frac{\lambda_2^2}{3} + \frac{\lambda_3^2}{3}\Big).$$

Since $\theta_x = \theta_y$ it follows that $\cos\theta_y = \cos\theta_x = -1/3$ and therefore that

$$\frac{\big(\lambda_1^2 - \lambda_2^2 - \lambda_3^2\big)}{\big(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\big)} \;=\; -\frac{1}{3},$$

i.e.,

$$2\lambda_1^2 = \lambda_2^2 + \lambda_3^2.$$

Thus if the angle between $\vec{\mathrm{OP}}$ and $\vec{\mathrm{QR}}$ is to remain unchanged, the deformation must satisfy the above restriction.

(d) In general, differential elements of area in the reference and deformed configurations are related by $dA_y = J|\mathbf{F}^{-T}\mathbf{m}_0|\, dA_x$. If a particular area element remains unchanged, $dA_y = dA_x$, the deformation must be much that $J|\mathbf{F}^{-T}\mathbf{m}_0| = 1$. Here, a unit vector normal to the plane RSQT is

$$\mathbf{m}_0 = \frac{1}{\sqrt{2}}\big(\mathbf{e}_1 + \mathbf{e}_2\big).$$

Since the matrix of components $[F]$ is diagonal we have $[F] = [F]^T$. Moreover $[F]^{-1}$ is also diagonal and its components are the reciprocals of the components of $[F]$. Thus

$$[F]^{-T}\{m_0\} = \begin{pmatrix} 1/\lambda_1 & 0 & 0 \\ 0 & 1/\lambda_2 & 0 \\ 0 & 0 & 1/\lambda_3 \end{pmatrix} \begin{Bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1/(\lambda_1\sqrt{2}) \\ 1/(\lambda_2\sqrt{2}) \\ 0 \end{Bmatrix}.$$

Thus the area of RSQT is preserved if $J|\mathbf{F}^{-T}\mathbf{m}_0| = 1$, i.e.

$$\lambda_1\lambda_2\lambda_3\left(\frac{1}{2\lambda_1^2} + \frac{1}{2\lambda_2^2}\right)^{1/2} = 1.$$

(e) If the orientation of an area element does not change in a deformation, then $\mathbf{m}_0 = \mathbf{m}$ where $\mathbf{m}_0$ and $\mathbf{m}$ are unit normal vectors to the reference and deformed area elements respectively. In general, $\mathbf{m} = \mathbf{F}^{-T}\mathbf{m}_0/|\mathbf{F}^{-T}\mathbf{m}_0|$, and so if the orientation does not change, we must have $\mathbf{F}^{-T}\mathbf{m}_0 = |\mathbf{F}^{-T}\mathbf{m}_0|\mathbf{m}_0$. In the specific case of the plane RSQT, we can substitute the preceding expressions for $\mathbf{F}$ and $\mathbf{m}$ into this equation and simplify the result to get

$$\lambda_1 = \lambda_2$$

which is precisely what one would expect intuitively.

Problem 2.6. (*a*) Under what conditions does the orientation of a material fiber remain unchanged (invariant) in a given deformation?

(*b*) The region $\mathcal{R}_0$ occupied by a body in a reference configuration is a unit cube. The orthonormal basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are aligned with the edges of the cube. Consider the following *volume preserving* deformation:

$$\mathbf{y} = \big(\lambda_1\mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3\big)\big(\mathbf{I} + k\,\mathbf{e}_1 \otimes \mathbf{e}_2\big)\mathbf{x}, \qquad \lambda_1 \neq 1, \lambda_2 \neq 1, k \neq 0.$$

Describe the physical nature of this deformation, and thus, based on your intuition, list the invariant directions.

(c) Now show mathematically that there are exactly three material directions whose orientations remain invariant in this deformation and determine these directions.

<u>Solution</u>: Consider a material fiber that in the reference configuration is oriented in the direction of the unit vector $\mathbf{m}_0$. In the deformed configuration we know that this fiber will be in the direction of the vector $\mathbf{F}\mathbf{m}_0$. Thus if the direction of this fiber remains unchanged $\mathbf{m}_0$ is parallel to $\mathbf{F}\mathbf{m}_0$ and so for some scalar μ we must have

$$\mathbf{F}\mathbf{m}_0 = \mu\mathbf{m}_0.$$

Remark: This states that $\mathbf{m}_0$ is an eigenvector of $\mathbf{F}$. Since $\mathbf{F}$ is not symmetric in general, it may not have a full complement of real eigenvalues and eigenvectors. In three dimensional space, $\mathbf{F}$ has three eigenvalues. If one eigenvalue is complex, its complex conjugate is also an eigenvalue. Thus complex eigenvalues occur in pairs. Therefore $\mathbf{F}$, in three dimensions, has either zero or two complex eigenvalues; or equivalently it has either one or three real eigenvalues. Thus an arbitrary deformation gradient tensor $\mathbf{F}$ will have (in general) either one or three directions that remain invariant. There maybe more than this many directions if the real eigenvalues are repeated.

The given deformation has the form $\mathbf{y} = \mathbf{F}_2\mathbf{F}_1\mathbf{x}$ and so it can be viewed in two steps. First $\mathbf{x} \to \mathbf{F}_1\mathbf{x}$ and then $\mathbf{F}_1\mathbf{x} \to \mathbf{F}_2(\mathbf{F}_1\mathbf{x})$. The tensor $\big(\mathbf{I} + k\,\mathbf{e}_1 \otimes \mathbf{e}_2\big)$, i.e. $\mathbf{F}_1$, represents a *simple shear* in the x_1, x_2-plane with the direction of shearing being $\mathbf{e}_1$. The tensor $\big(\lambda_1\mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3\big)$, i.e. $\mathbf{F}_2$, represents a *biaxial stretching* in the $\mathbf{e}_1$- and $\mathbf{e}_2$-directions. (Note that $\lambda_3 = 1$.) Thus the given deformation is the composition of these two deformations both of which are entirely in the $\mathbf{e}_1, \mathbf{e}_2$-plane.

Since particles have zero displacement in the $\mathbf{e}_3$-direction, we see geometrically that any material fiber that is in the $\mathbf{e}_3$-direction in the reference configuration will remain in the $\mathbf{e}_3$-direction in the deformed configuration. Thus $\mathbf{e}_3$ would be an invariant direction.

Next consider a fiber that is in the $\mathbf{e}_1$ direction in the reference configuration. The simple shear will simply slide this fiber in the $\mathbf{e}_1$-direction. The biaxial stretch with stretch and translate this fiber without rotation. Thus any material fiber that is in the $\mathbf{e}_1$-direction in the reference configuration will remain in the $\mathbf{e}_1$-direction in the deformed configuration. Thus $\mathbf{e}_1$ would also be an invariant direction.

Since both $\mathbf{e}_1$ and $\mathbf{e}_3$ are distinct eigenvectors of $\mathbf{F}$, it must have two corresponding real eigenvalues. Thus the third eigenvalue of $\mathbf{F}$ must also be real and the corresponding eigenvector will be a third direction that remains invariant in this deformation. It is not easy to determine this direction intuitively.

We now proceed to calculate the invariant directions of $\mathbf{F}$ mathematically. First we note that

$$[F] = [F_2][F_1] = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 & k\lambda_1 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We are told that this deformation is volume preserving whence $\det \mathbf{F} = 1$. This implies that $\lambda_1\lambda_2 = 1$. In order to simplify the notation we can therefore set $\lambda_1 = \lambda$ and $\lambda_2 = \lambda^{-1}$ whence

$$[F] \;=\; \begin{pmatrix} \lambda & k\lambda & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In order to find the orientation preserving directions we find the eigenvalues of $\mathbf{F}$ by solving $\det(\mathbf{F}-\mu\mathbf{I}) = 0$ for μ, and then finding the associated eigenvectors $\mathbf{m}_0$ from $\mathbf{F}\mathbf{m}_0 = \mu\mathbf{m}_0$. Thus we first solve

$$\det(\mathbf{F}-\mu\mathbf{I}) = \det\begin{pmatrix} \lambda-\mu & k\lambda & 0 \\ 0 & \lambda^{-1}-\mu & 0 \\ 0 & 0 & 1-\mu \end{pmatrix} \;=\; (1-\mu)(\lambda-\mu)(\lambda^{-1}-\mu) = 0$$

to find the three eigenvalues $\mu_1 = 1$, $\mu_2 = \lambda$ and $\mu_3 = \lambda^{-1}$. The corresponding eigenvectors are then found from $\mathbf{F}\mathbf{m}_0^{(i)} = \mu_i \mathbf{m}_0^{(i)},\ i = 1, 2, 3$:

$$\{m_0^{(1)}\} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}, \quad \{m_0^{(2)}\} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}, \quad \{m_0^{(3)}\} = \begin{Bmatrix} -k\lambda \\ \lambda-\lambda^{-1} \\ 0 \end{Bmatrix}.$$

The previous geometric discussion had already told us that fibers in the directions $\mathbf{m}_0^{(1)}$ and $\mathbf{m}_0^{(2)}$ do not change their orientation. We now know that fibers in the direction $\mathbf{m}_0^{(3)}$ also preserve their orientation.

Problem 2.7. [Chadwick] An *incompressible* body is reinforced by embedding two families of straight *inextensible* fibers in it. The fiber directions $\mathbf{m}_0$ in the reference configuration are given by $m_1^0 = \cos\theta$, $m_2^0 = \pm\sin\theta$, $m_3^0 = 0$ where θ is a constant $0 < \theta < \pi/2$. This body is subjected to the homogeneous deformation

$$y_1 = \lambda_1 x_1, \qquad y_2 = \lambda_2 x_2, \qquad y_3 = \lambda_3 x_3.$$

In view of the internal geometric constraints of the material, show that the only deformations (of the above form) that this material can sustain are those that have

$$\lambda_1^2 \cos^2\theta + \lambda_2^2 \sin^2\theta = 1, \qquad \lambda_1\lambda_2\lambda_3 = 1.$$

Hence show that the thickness of the sheet in the x_3-direction cannot be made arbitrarily small, and that in particular, the ratio between the deformed and undeformed thicknesses must be $\geq \sin 2\theta$. When this contraction is achieved, show that the two families of fibers are orthogonal in the deformed configuration.

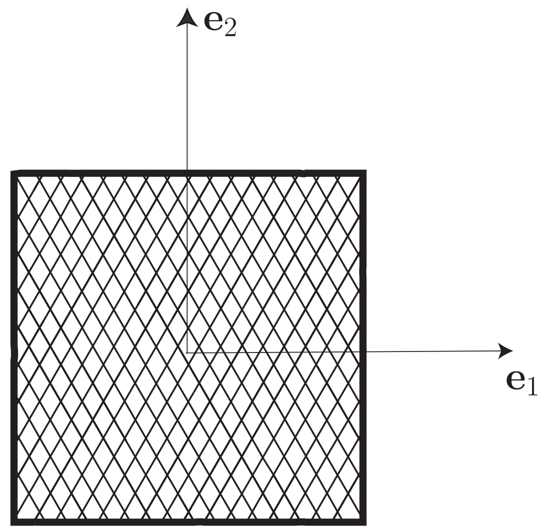

Figure 2.16: Square slab reinforced with two families of inextensible fibers at $\pm\theta$.

<u>*Solution*</u>: The fibers are oriented in the directions $\mathbf{m}_0^{\pm} = \cos\theta\,\mathbf{e}_1 \pm \sin\theta\,\mathbf{e}_2$. Thus the components of the deformed images of $\mathbf{m}_0^{\pm}$ are given by

$$[F]\{m_0^{\pm}\} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{Bmatrix} \cos\theta \\ \pm\sin\theta \\ 0 \end{Bmatrix} = \begin{Bmatrix} \lambda_1\cos\theta \\ \pm\lambda_2\sin\theta \\ 0 \end{Bmatrix}.$$

Since material fibers that lie in the directions $\mathbf{m}_0^{\pm}$ are inextensible we must have $|\mathbf{F}\mathbf{m}_0^{\pm}| = |\mathbf{m}_0^{\pm}|$, i.e.

$$\lambda_1^2\cos^2\theta + \lambda_2^2\sin^2\theta = 1. \qquad (a)$$

In addition, since the material is incompressible we have $\det\mathbf{F} = \lambda_1\lambda_2\lambda_3 = 1$:

$$\lambda_3 = \frac{1}{\lambda_1\lambda_2}. \qquad (b)$$

Equation (a) tells us that the (positive) stretches λ_1 and λ_2 cannot take arbitrary values because they must always must lie on the ellipse defined by (a). Thus in particular we see that $0 < \lambda_1 < 1/\cos\theta$ and $0 < \lambda_2 < 1/\sin\theta$. Thus the sheet *cannot* be extended by arbitrary amounts in the $\mathbf{e}_1$- and $\mathbf{e}_2$-directions.

We wish to determine whether the constraints (a) and (b) place any similar restriction on λ_3. To this end, note that equation (a) can be written equivalently as

$$(\lambda_1\cos\theta - \lambda_2\sin\theta)^2 + \lambda_1\lambda_2\sin 2\theta = 1$$

which in turn can be rearranged to yield

$$\frac{1}{\lambda_1\lambda_2} = \sin 2\theta + \frac{(\lambda_1\cos\theta - \lambda_2\sin\theta)^2}{\lambda_1\lambda_2}.$$

Since the stretches are necessarily positive, the second term on the right hand side is non-negative and so

$$\frac{1}{\lambda_1 \lambda_2} \geq \sin 2\theta.$$

On combining this with (b) yields $\lambda_3 \geq \sin 2\theta$. This shows that the ratio of the deformed thickness to the undeformed thickness of the sheet cannot be made any smaller than $\sin 2\theta$.

Problem 2.8. Show that a plane isochoric deformation is equivalent to a simple shear followed by a rotation.

<u>Solution</u>: Consider a homogeneous, planar, isochoric deformation. Since the deformation is homogeneous it has the form $\mathbf{y} = \mathbf{F}\mathbf{x}$ where $\mathbf{F}$ is a constant tensor whose determinant is unity since the deformation is volume preserving. Without loss of generality we can omit the rotational part of $\mathbf{F}$ and consider a deformation $\mathbf{y} = \mathbf{U}\mathbf{x}$ where $\mathbf{U}$ is symmetric and positive definite. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal principal basis of $\mathbf{U}$ with corresponding principal stretches λ_1, λ_2 and $\lambda_3 = 1$. All components of vectors and tensors will be taken with respect to this basis. Since the deformation is isochoric, $\det \mathbf{U} = \lambda_1 \lambda_2 = 1$ and so we can set $\lambda_1 = \lambda, \lambda_2 = \lambda^{-1}$. Thus the components of the stretch tensor can be written as

$$[U] = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $\mathbf{a}$ and $\mathbf{n}$ be vectors, and let $\mathbf{R}$ be a tensor, whose components are given by

$$\{a\} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \quad \{n\} = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}, \quad [R] = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $\mathbf{a}$ and $\mathbf{n}$ are unit vectors and that $\mathbf{a} \cdot \mathbf{n} = 0$. Moreover observe that $\mathbf{R}$ is a proper orthogonal tensor.

One can now show by straightforward algebraic substitution that

$$\mathbf{U} \;=\; \mathbf{R}(\mathbf{I} + k\mathbf{a} \otimes \mathbf{n}),$$

provided we take

$$k = \lambda^{-1} - \lambda, \qquad \tan\theta = \lambda, \qquad \tan\varphi = \frac{1}{2}(\lambda - \lambda^{-1}).$$

Here $\mathbf{R}$ represents a rotation and $\mathbf{I} + k\mathbf{a} \otimes \mathbf{n}$ represents a simple shear.

Thus the given deformation can be written as $\mathbf{y} = \mathbf{F}\mathbf{x} = (\lambda\, \mathbf{e}_1 \otimes \mathbf{e}_1 \,+\, \lambda^{-1}\mathbf{e}_2 \otimes \mathbf{e}_2)\mathbf{x} = \mathbf{R}(\mathbf{I} + k\mathbf{a} \otimes \mathbf{n})\mathbf{x}$, and so we conclude from that a plane isochoric deformation is equivalent to a simple shear followed by a rotation.

Problem 2.9. Consider the deformation

$$\mathbf{y} = \big(\mathbf{I} + \alpha \mathbf{a} \otimes \mathbf{a}\big)\big(\mathbf{I} + k \mathbf{m} \otimes \mathbf{n}\big)\mathbf{x}$$

which represents a simple shearing of a body followed by a uniaxial stretching; here the vectors $\mathbf{a}, \mathbf{m}, \mathbf{n}$ have unit length and $\mathbf{m} \cdot \mathbf{n} = 0$.

A plane $\mathcal{P}$ in the reference region $\mathcal{R}$ is said to remain *undistorted* by a deformation if the distance between every pair of particles *on* $\mathcal{P}$ is the same in the reference and deformed configurations. Not all deformations have an undistorted plane.

Under what conditions (on $\alpha, k, \mathbf{a}, \mathbf{m}, \mathbf{n}$) does the preceding deformation have an undistorted plane?

Problem 2.10. A plane $\mathcal{P}$ in the reference region $\mathcal{R}$ is said to remain *undistorted* by a deformation if the distance between every pair of particles *on* $\mathcal{P}$ is the same in the reference and deformed configurations. In certain deformations there are no undistorted planes while in others there are. In certain problems in Materials Science, it is sometimes important to find all such undistorted planes in a given deformation.

Show that the homogeneous deformation $\mathbf{y} = \mathbf{F}\mathbf{x}$ has an undistorted plane if and only if all three of the following conditions hold: (a) $\lambda_1 = 1$, (b) $\lambda_2 \leq 1$, (c) $\lambda_3 \geq 1$, where the λ_i's are the principal stretches.

Show also that if the deformation $\mathbf{y} = \mathbf{F}\mathbf{x}$ has one undistorted plane then it necessarily has at least two undistorted planes.

Problem 2.11. Calculate the rotation tensor $\mathbf{R}$ and the stretch tensors $\mathbf{U}$ and $\mathbf{V}$ in the polar decomposition of the deformation gradient tensor $\mathbf{F}$ in simple shear. Describe, using a sketch, how $\mathbf{x} \to \mathbf{U}\mathbf{x} \to \mathbf{R}\mathbf{U}\mathbf{x}$ and how $\mathbf{x} \to \mathbf{R}\mathbf{x} \to \mathbf{V}\mathbf{R}\mathbf{x}$.

Solution: Consider a body that occupies a cubic domain $\mathcal{R}_0$ in a reference configuration. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a fixed orthonormal basis where the three unit vectors are aligned with the edges of the cube. The cube is subjected to the *simple shearing deformation*

$$\mathbf{y} = (\mathbf{1} + k\ \mathbf{e}_1 \otimes \mathbf{e}_2)\mathbf{x},$$

where k is a constant. This carries the cube into the sheared region $\mathcal{R}$ shown in Figure 2.6. In terms of components with respect to the given basis[11], this can be expressed as

$$y_1 = x_1 + kx_2, \quad y_2 = x_2, \quad y_3 = x_3. \qquad (a)$$

Observe that the displacement field $\mathbf{u}(\mathbf{x}) = \mathbf{y}(\mathbf{x}) - \mathbf{x}$ has components

$$u_1 = kx_2, \quad u_2 = 0, \quad u_3 = 0.$$

[11] Unless explicitly stated otherwise, all components are taken with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

Thus each plane x_2 = constant is displaced rigidly in the x_1-direction, the amount of the displacement depending linearly on the value of x_2. One refers to a plane x_2 = constant as a *shearing (or glide) plane* and the x_1-direction as the *shearing direction.*

It follows from (a) that the matrix of components of the deformation gradient tensor $\mathbf{F}$ is

$$[F] = \left[\frac{\partial y_i}{\partial x_j}\right] = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $\det \mathbf{F} = 1$ so that a simple shear deformation is locally volume preserving (isochoric).

The components of the Cauchy-Green tensor $\mathbf{U}^2$ are

$$[U^2] = [F]^T[F] = \begin{pmatrix} 1 & k & 0 \\ k & 1+k^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and its eigenvalues are given by the roots λ^2 of the equation

$$\det[\mathbf{U}^2 - \lambda^2\mathbf{I}] \;=\; \det \begin{pmatrix} 1-\lambda^2 & k & 0 \\ k & 1+k^2-\lambda^2 & 0 \\ 0 & 0 & 1-\lambda^2 \end{pmatrix} = 0$$

which simplifies to

$$(1-\lambda^2)(\lambda^4 - (2+k^2)\lambda^2 + 1) = 0.$$

The roots of this equation, i.e. the eigenvalues of $\mathbf{U}^2$, are

$$\lambda_1^2 = \frac{2+k^2+k\sqrt{k^2+4}}{2} \;(\geq 1), \qquad \lambda_2^2 = \frac{2+k^2-k\sqrt{k^2+4}}{2} \;(\leq 1), \qquad \lambda_3^2 = 1.$$

The corresponding eigenvectors of $\mathbf{U}^2$ are given by $\xi_1\mathbf{e}_1 + \xi_2\mathbf{e}_2 + \xi_3\mathbf{e}_3$ where

$$\begin{pmatrix} 1-\lambda^2 & k & 0 \\ k & 1+k^2-\lambda^2 & 0 \\ 0 & 0 & 1-\lambda^2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

For each $\lambda = \lambda_i$ this can be solved for ξ_1, ξ_2, ξ_3 thus leading to the corresponding eigenvector $\mathbf{r}_i$:

$$\left.\begin{aligned} \mathbf{r}_1 &= \frac{1}{\sqrt{1+\lambda_1^2}}\mathbf{e}_1 + \frac{\lambda_1}{\sqrt{1+\lambda_1^2}}\mathbf{e}_2 &&= \cos\theta_r\ \mathbf{e}_1 + \sin\theta_r\ \mathbf{e}_2, \\ \mathbf{r}_2 &= -\frac{\lambda_1}{1+\lambda_1^2}\mathbf{e}_1 + \frac{1}{\sqrt{1+\lambda_1^2}}\mathbf{e}_2 &&= -\sin\theta_r\ \mathbf{e}_1 + \cos\theta_r\ \mathbf{e}_2, \\ \mathbf{r}_3 &= \mathbf{e}_3, \end{aligned}\right\}$$

where we have set

$$\cos\theta_r = \frac{1}{\sqrt{1+\lambda_1^2}}, \qquad \sin\theta_r = \frac{\lambda_1}{\sqrt{1+\lambda_1^2}},$$

or equivalently

$$\tan 2\theta_r = -\frac{2}{k}, \qquad \frac{\pi}{4} \leq \theta_r < \frac{\pi}{2}.$$

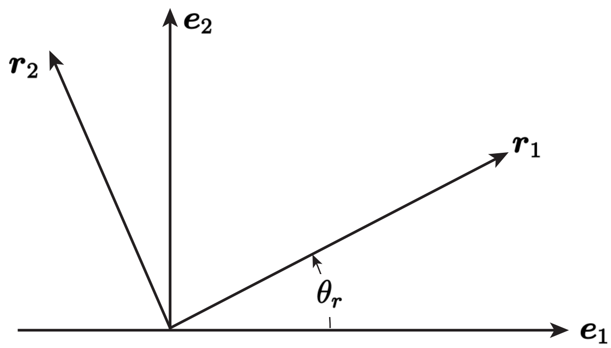

Figure 2.17: Principal directions $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ of the (right) Lagrangian stretch tensor $\mathbf{U}$.

The angle θ_r has the significance shown in Figure 2.17.

Now consider the stretch tensor $\mathbf{U}$ itself. Its eigenvectors are the same as those of $\mathbf{U}^2$ while its eigenvalues are $\lambda_1, \lambda_2, \lambda_3$:

$$\lambda_1 = \frac{k + \sqrt{k^2+4}}{2} \ (\geq 1), \qquad \lambda_2 = \frac{\sqrt{k^2+4} - k}{2} \ (\leq 1), \qquad \lambda_3 = 1.$$

The matrix of components of $\mathbf{U}$ in its principal basis $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ is

$$[U'] \ = \ \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

In order to find $[U]$, the matrix of components of $\mathbf{U}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, we use the usual tensor transformation rule[12]

$$[U'] = [Q][U][Q]^T$$

where the elements of the rotation matrix $[R]$ that relates the two bases is given by $Q_{ij} = \mathbf{r}_i \cdot \mathbf{e}_j$, i.e.

$$[Q] = \begin{pmatrix} \cos\theta_r & \sin\theta_r & 0 \\ -\sin\theta_r & \cos\theta_r & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Combining the three preceding equations and solving for $[U]$ leads to

$$[U] = \frac{1}{\sqrt{4+k^2}} \begin{pmatrix} 2 & k & 0 \\ k & 2+k^2 & 0 \\ 0 & 0 & \sqrt{4+k^2} \end{pmatrix}.$$

The components of the rotation tensor $\mathbf{R}$ in the polar decomposition of $\mathbf{F}$ can now be calculated using $\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}$ which leads to

$$[R] = \frac{1}{\sqrt{4+k^2}} \begin{pmatrix} 2 & k & 0 \\ -k & 2 & 0 \\ 0 & 0 & \sqrt{4+k^2} \end{pmatrix}.$$

[12]See Section 3.5 of Volume 1.

Finally, by using $\mathbf{V} = \mathbf{F}\mathbf{R}^T$ one can show that the matrix of components of the left stretch tensor $\mathbf{V}$ are

$$[V] = \frac{1}{\sqrt{4+k^2}} \begin{pmatrix} 2+k^2 & k & 0 \\ k & 2 & 0 \\ 0 & 0 & \sqrt{4+k^2} \end{pmatrix}.$$

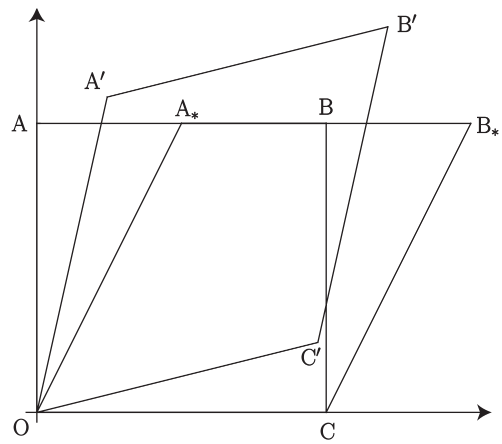

Figure 2.18: Simple shear deformation viewed in two steps: $\mathbf{y} = \mathbf{F}\mathbf{x} = \mathbf{R}(\mathbf{U}\mathbf{x})$. The pure stretch $\mathbf{x} \to \mathbf{U}\mathbf{x}$ takes the region $OABC \to O'A'B'C'$ and the subsequent rotation $\mathbf{U}\mathbf{x} \to \mathbf{R}(\mathbf{U}\mathbf{x})$ takes $OA'B'C' \to OA_*B_*C$.

We may now visualize the simple shear deformation (a) in two steps as follows:

$$\mathbf{y} = \mathbf{F}\mathbf{x} = \mathbf{R}\mathbf{U}\mathbf{x} = \mathbf{R}(\mathbf{U}\mathbf{x}).$$

First, the deformation $\mathbf{x} \to \mathbf{U}\mathbf{x}$ stretches the square $OABC$ in Figure 2.18 by the amounts λ_1, λ_2 in the principal directions $\mathbf{r}_1, \mathbf{r}_2$ leading to the region $OA'B'C'$. This is then followed by the deformation $\mathbf{U}\mathbf{x} \to \mathbf{R}(\mathbf{U}\mathbf{x})$ which rigidly rotates $OA'B'C'$ into the region OA_*B_*C which is the region occupied by the deformed body.

Alternatively we may visualize the simple shear deformation (a) in the two steps $\mathbf{y} = \mathbf{F}\mathbf{x} = \mathbf{V}\mathbf{R}\mathbf{x} = \mathbf{V}(\mathbf{R}\mathbf{x})$. The first step $\mathbf{x} \to \mathbf{R}\mathbf{x}$ rigidly rotates the square $OABC$ in Figure 2.19 into the square $OA''B''C''$. Second, this is followed by the deformation $\mathbf{R}\mathbf{x} \to \mathbf{V}(\mathbf{R}\mathbf{x})$ which stretches $OA''B''C''$ by the amounts λ_1, λ_2 in the principal directions $\boldsymbol{\ell}_1, \boldsymbol{\ell}_2$ to get the region OA_*B_*C occupied by the deformed body.

Problem 2.12. Consider a body which occupies a region $\mathcal{R}_0$ in a reference configuration and a region $\mathcal{R}$ in a deformed configuration. The deformation which takes $\mathcal{R}_0 \to \mathcal{R}$ is homogeneous:

$$\mathbf{y} = \mathbf{F}\mathbf{x}, \qquad \mathbf{F} = \text{ constant}.$$

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal basis.

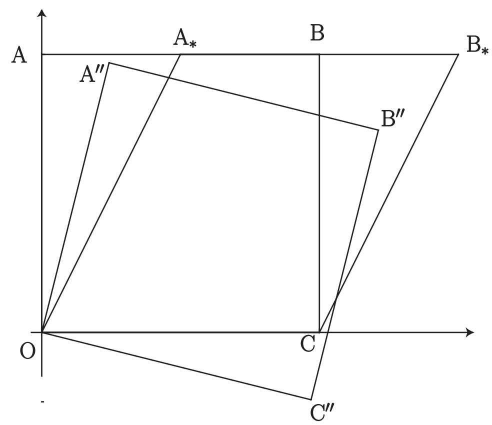

Figure 2.19: Simple shear deformation viewed in two steps: $\mathbf{y} = \mathbf{F}\mathbf{x} = \mathbf{V}(\mathbf{R}\mathbf{x})$. The rotation $\mathbf{x} \to \mathbf{R}\mathbf{x}$ takes the region $OABC \to OA''B''C''$ and the pure stretch $\mathbf{R}\mathbf{x} \to \mathbf{V}(\mathbf{R}\mathbf{x})$ takes $OA''B''C'' \to OA_*B_*C$.

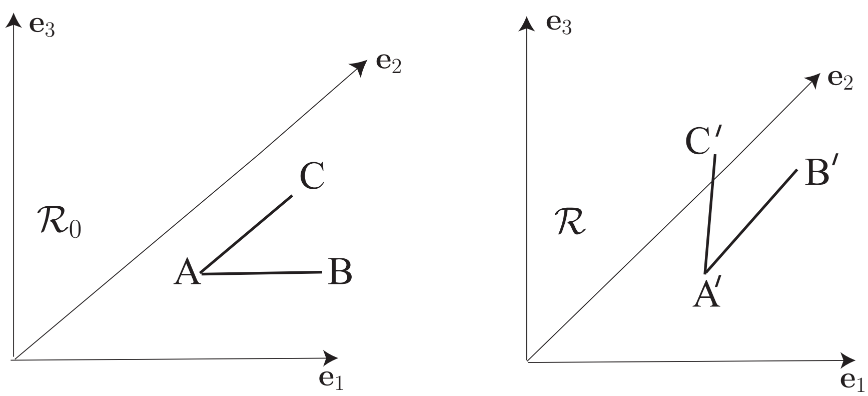

Figure 2.20: Two material fibers AB and AC in the reference configuration are carried into $A'B'$ and $A'C'$ respectively by a deformation.

Consider two material fibers AB and AC which, in the reference configuration, have equal length s_0, and are oriented in the respective directions $\mathbf{e}_1$ and $\mathbf{e}_2$. In the current configuration these fibers are described by $A'B'$ and $A'C'$.

Suppose that the lengths s_1 and s_2 of the fibers $A'B'$ and $A'C'$ in the current configuration have been measured. Suppose that the angle between these fibers in the current configuration has also been measured and is given by $\pi/2 - \phi$.

Calculate the strain components E_{11}, E_{22} and E_{12} in terms of s_0, s_1, s_2 and ϕ where the strain tensor $\mathbf{E}$ is the Green strain tensor

$$\mathbf{E} = (1/2)(\mathbf{U}^2 - \mathbf{I}).$$

Linearize your answer in the case of an infinitesimal deformation.

Solution: Set $ds_x = s_0, ds_y = s_1$ and $\mathbf{n}_0 = \mathbf{e}_1$ in (2.70) to get

$$s_1 = s_0\sqrt{1+2E_{11}}$$

and therefore

$$E_{11} = \frac{1}{2}\left[\left(\frac{s_1}{s_0}\right)^2 - 1\right]. \qquad (a)$$

Similarly setting $ds_x = s_0, ds_y = s_2$ and $\mathbf{n}_0 = \mathbf{e}_2$ in (2.70) yields

$$E_{22} = \frac{1}{2}\left[\left(\frac{s_2}{s_0}\right)^2 - 1\right]. \qquad (b)$$

Next, set $\theta_y = \pi/2 - \phi$, $\mathbf{n}_0^{(1)} = \mathbf{e}_1$ and $\mathbf{n}_0^{(2)} = \mathbf{e}_2$ in (2.73) to get

$$\sin\phi = \frac{2E_{12}}{\sqrt{1+2E_{11}}\sqrt{1+2E_{22}}}$$

whence

$$E_{12} = \frac{1}{2}\frac{s_1}{s_0}\frac{s_2}{s_0}\sin\phi. \qquad (c)$$

For an infinitesimal deformation, we set $s_1 = s_0 + \Delta s_1$, substitute this into (a) and approximate the result for small $\Delta s_1/s_0$. This leads to

$$E_{11} = \frac{s_1 - s_0}{s_0}$$

with the error being quadratic. Similarly one finds

$$E_{22} = \frac{s_2 - s_0}{s_0}$$

to linear accuracy. Substituting $s_1 = s_0 + \Delta s_1$ and $s_2 = s_0 + \Delta s_2$ into (c) and approximating the result for small $\phi, \Delta s_1/s_0$ and $\Delta s_2/s_0$ leads to

$$E_{12} = \frac{1}{2}\phi$$

where the error is quadratic.

Problem 2.13. Consider a simple shear

$$y_1 = x_1 + kx_2, \qquad y_2 = x_2, \qquad y_3 = x_3,$$

where the components have been taken with respect to a fixed orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Calculate the components of the Lagrangian logarithmic strain tensor $\mathbf{E} = \ln \mathbf{U}$ with respect to this same basis.

Solution: In an earlier Example we worked out the details of the polar decomposition of the deformation gradient tensor $\mathbf{F}$ for a simple shear. From those results, the eigenvalues of the right stretch tensor $\mathbf{U}$ were

$$\lambda_1 = \frac{\sqrt{k^2+4}+k}{2}, \qquad \lambda_2 = \frac{\sqrt{k^2+4}-k}{2}, \qquad \lambda_3 = 1, \qquad (a)$$

and the corresponding eigenvectors were

$$\mathbf{r}_1 = \cos\theta_r\,\mathbf{e}_1 + \sin\theta_r\,\mathbf{e}_2, \qquad \mathbf{r}_2 = -\sin\theta_r\,\mathbf{e}_1 + \cos\theta_r\,\mathbf{e}_2, \qquad \mathbf{r}_3 = \mathbf{e}_3, \tag{b}$$

where

$$\cos\theta_r = \frac{1}{\sqrt{1+\lambda_1^2}}, \qquad \sin\theta_r = \frac{\lambda_1}{\sqrt{1+\lambda_1^2}}. \tag{c}$$

Since the Lagrangian logarithmic strain tensor is given by

$$\ln\mathbf{U} = \ln\lambda_1\,\mathbf{r}_1\otimes\mathbf{r}_1 + \ln\lambda_2\,\mathbf{r}_2\otimes\mathbf{r}_2 + \ln\lambda_3\,\mathbf{r}_3\otimes\mathbf{r}_3,$$

we substitute (b) into this and expand the result to get

$$\begin{aligned}\ln\mathbf{U} \;=\;& (\cos^2\theta_r\ln\lambda_1 + \sin^2\theta_r\ln\lambda_2)\mathbf{e}_1\otimes\mathbf{e}_1 + (\ln\lambda_1 - \ln\lambda_2)\sin\theta_r\cos\theta_r\,\mathbf{e}_1\otimes\mathbf{e}_2 \\ & + (\ln\lambda_1 - \ln\lambda_2)\sin\theta_r\cos\theta_r\,\mathbf{e}_2\otimes\mathbf{e}_1 + (\sin^2\theta_r\ln\lambda_1 + \cos^2\theta_r\ln\lambda_2)\mathbf{e}_2\otimes\mathbf{e}_2.\end{aligned}$$

The coefficient of $\mathbf{e}_i\otimes\mathbf{e}_j$ in this equation is the i,j-component of the tensor $\ln\mathbf{U}$ in the basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$. Expressions for the λ's and θ_r in terms of the amount of shear k are given above in (a) and (c).

Problem 2.14. Consider a body that undergoes a *simple shearing* $\mathbf{y} = (\mathbf{I} + k\;\mathbf{e}_1\otimes\mathbf{e}_2)\mathbf{x}$ where k is a constant and $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ is a fixed orthonormal basis. Calculate

a. The components of the deformation gradient tensor $\mathbf{F}$, the displacement gradient tensor $\mathbf{H}$, the Green strain tensor $\mathbf{E}$, the infinitesimal strain tensor $\boldsymbol{\varepsilon}$, and the infinitesimal rotation tensor $\boldsymbol{\omega}$.

b. Discuss the distinction between $\mathbf{E}$ and $\boldsymbol{\varepsilon}$.

Solution: In terms of components with respect to the given basis, this deformation can be expressed as

$$y_1 = x_1 + kx_2, \quad y_2 = x_2, \quad y_3 = x_3.$$

The deformation gradient tensor $\mathbf{F} = \operatorname{Grad}\mathbf{y}$ associated with this deformation has components

$$[F] = \left[\frac{\partial y_i}{\partial x_j}\right] = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Observe that the associated displacement field $\mathbf{u}(\mathbf{x}) = \mathbf{y}(\mathbf{x}) - \mathbf{x}$ has components

$$u_1 = kx_2, \quad u_2 = 0, \quad u_3 = 0. \tag{a}$$

Thus each plane x_2 = constant is displaced rigidly in the x_1-direction, the amount of the displacement depending linearly on the value of x_2. One refers to a plane x_2 = constant as a *shearing (or glide) plane*, the x_1-direction as the *shearing direction* and k is called the *amount of shear*.

The displacement gradient tensor $\mathbf{H} = \text{Grad}\,\mathbf{u}$ associated with the displacement field (a) has components

$$[H] = \left[\frac{\partial u_i}{\partial x_j}\right] = \begin{pmatrix} 0 & k & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad (b)$$

The components

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right),$$

of the Green strain tensor specialize in this case to

$$[E] = \begin{pmatrix} 0 & k/2 & 0 \\ k/2 & k^2/2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

while the components

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

of the infinitesimal strain tensor specialize to

$$[\varepsilon] = \begin{pmatrix} 0 & k/2 & 0 \\ k/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Similarly the components

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

of the infinitesimal rotation tensor specialize in simple shear to

$$[\omega] = \begin{pmatrix} 0 & k/2 & 0 \\ -k/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We now make some observations. First observe that $\varepsilon_{22} = 0$ and $E_{22} = k^2/2$. Both these strain components are concerned with the change in length of a material fiber that, in the reference configuration, was in the $\mathbf{e}_2$-direction. Since $E_{22} \neq 0$ it follows from (2.72) that this material fiber undergoes a change in length. On the other hand since $\varepsilon_{22} = 0$ it follows from (2.92) that this material fiber does not change its length in an infinitesimal deformation; or more precisely, that any changes in length of this fiber are $O(\epsilon^2)$. It can be seen geometrically from Figure 2.6 that a fiber which, in the reference configuration was in the $\mathbf{e}_2$-direction, does undergo a change in length due to the deformation. This is consistent with the fact that $E_{22} \neq 0$. We see from (b) that the components of the displacement gradient are infinitesimal when $k << 1$. Note that $E_{22} = k^2/2 = O(k^2)$ as $k \to 0$ which shows that the change in length of this fiber is $O(k^2)$

Similarly when the k^2 terms are omitted from the rotation tensor $\mathbf{R}$ one can show that $\mathbf{R} = \mathbf{I} + \boldsymbol{\omega} + O(k^2)$; (see subsequent example on the polar decomposition of the deformation gradient tensor in simple shear for an expression for $\mathbf{R}$).

Problem 2.15. A body undergoes a deformation in which the deformation gradient tensor field is given by $\mathbf{Q}(\mathbf{x})$ where $\mathbf{Q}$ is proper orthogonal at each point $\mathbf{x} \in \mathcal{R}_0$. Show that $\mathbf{Q}$ must necessarily be independent of $\mathbf{x}$. Thus, if the body undergoes a rigid motion in the vicinity of each particle $\mathbf{x}$, in fact all particles necessarily undergo the same rigid motion.

Problem 2.16. *Compatibility.* Let $\mathcal{R}_0$ be the region occupied by a body in a reference configuration. Suppose that we are given a deformation $\widehat{\mathbf{y}}(\mathbf{x})$ which is defined and three-times continuously differentiable on $\mathcal{R}_0$. Let $\mathbf{C}(\mathbf{x}) = \mathbf{U}^2(\mathbf{x}) = \mathbf{F}^T(\mathbf{x})\mathbf{F}(\mathbf{x})$ be the associated right Cauchy-Green deformation tensor field. Prove that necessarily,

$$R_{ijkm} = 0 \qquad \text{on } \mathcal{R}_0, \tag{2.93}$$

where

$$\begin{aligned} R_{ijkm} &\stackrel{\text{def}}{=} \Gamma_{jmi,k} - \Gamma_{jki,m} + C^{-1}_{pq}(\Gamma_{rkp}\Gamma_{imq} - \Gamma_{jmp}\Gamma_{ikq}), \\ \Gamma_{ijk} &\stackrel{\text{def}}{=} 1/2(C_{jk,i} + C_{ik,j} - C_{ij,k}). \end{aligned}$$

Remark: Conversely, suppose that you are given a function $\mathbf{C}(\mathbf{x})$ which is defined and twice-continuously differentiable on $\mathcal{R}_0$, and whose value at each $\mathbf{x}$ is a symmetric positive definite tensor. Suppose further that $\mathcal{R}_0$ is a simply-connected region. Then one can show that, if $\mathbf{C}(\mathbf{x})$ satisfies (2.93), then there is a deformation $\widehat{\mathbf{y}}(\mathbf{x})$ such that the given $\mathbf{C}(\mathbf{x})$ is the associated right Cauchy-Green deformation tensor field.

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Chapter 3

Kinematics: Motion

In Chapter 2 we examined a single configuration of a body. Here we consider a time-dependent sequence of configurations, i.e. we study the motion of a body during which it occupies different regions of space at each instant of time. The reader is referred back to Chapter 1 for a discussion of some basic notions underlying a motion.

An outline of the material in this chapter is as follows: in Section 3.1 we characterize the motion of a body with respect to a fixed reference configuration. In Section 3.2 we examine the special case of a rigid motion. The velocity gradient tensor is the key ingredient for studying the time rate of change of geometric characteristics in the neighborhood of a particle; it is introduced in Section 3.3. In that section we also decompose the velocity gradient tensor into the stretching tensor and the spin tensor. In Section 3.4 we consider an infinitesimal material curve, surface and region in the current configuration and calculate the time rate of change of length, angle, area and volume. Formulae for these can be expressed solely in terms of the stretching tensor. However the rate at which the orientation of an infinitesimal fiber changes depends on both the stretching and spin tensors. In Section 3.5 we consider an arbitrary motion, and study it using the current configuration as the reference configuration. Thus the reference configuration in this analysis is time-dependent. We show that in this framework, the time rate of change of the stretch tensors and rotation tensor in the polar decomposition equal the stretching tensor and the spin tensor respectively. This is not true if the reference configuration had been, for example, fixed. In Section 3.7 we calculate the time rate of change of the volume integral of some field quantity where the integration is taken over the current (time-dependent) region occupied by a part of the body. Similar transport equations are also presented for integrals over material surfaces and material curves. In Section 3.8 we introduce the concept of material frame indifference (or objectivity or observer independence). The notions of the convected and co-rotational time derivatives of a vector and tensor field are introduced in Section 3.9. And finally in Section 3.10 we linearize the prior results.

3.1 Motion.

With respect to a fixed reference configuration, a *motion* can be characterized by

$$\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{x}, t) \qquad \mathbf{x} \in \mathcal{R}_0, \quad \mathbf{y} \in \mathcal{R}_t, \quad t \in [t_0, t_1], \tag{3.1}$$

where $\mathbf{x}$ is the location of a generic particle in the reference configuration, $\mathbf{y}$ is its location at time t, and $[t_0, t_1]$ is the time interval over which the motion takes place. The body occupies a region $\mathcal{R}_0$ in the reference configuration and a region $\mathcal{R}_t$ at time t.

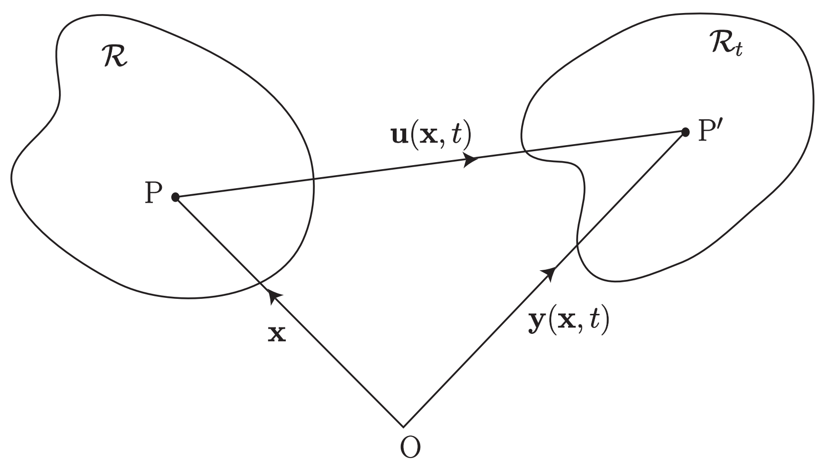

Figure 3.1: The respective regions $\mathcal{R}_0$ and $\mathcal{R}_t$ occupied by a body in a reference configuration and at time t during a motion $\mathbf{y} = \mathbf{y}(\mathbf{x}, t)$. The position of a generic particle in the reference configuration and at time t are denoted by $\mathbf{x}$ and $\mathbf{y}(\mathbf{x}, t)$ respectively. The displacement is given by $\mathbf{u}(\mathbf{x}, t)$.

It is *not* necessary that the reference configuration be occupied by the body at any time during the motion. If it so happens that the body occupies the reference configuration at the initial instant of time, then we may refer to it as the *initial configuration* of the body. In this case,

$$\mathbf{x} = \mathbf{y}(\mathbf{x}, t_0), \qquad \mathbf{F}(\mathbf{x}, t_0) = \mathbf{I} \qquad \text{for all } \mathbf{x} \in \mathcal{R}_0. \tag{3.2}$$

The *velocity* and *acceleration* of a particle are given by

$$\boldsymbol{v}(\boldsymbol{x},t) = \frac{\partial \boldsymbol{y}}{\partial t}(\boldsymbol{x},t) = \dot{\mathbf{y}}, \qquad \boldsymbol{a}(\boldsymbol{x},t) = \frac{\partial \boldsymbol{v}}{\partial t}(\boldsymbol{x},t) = \dot{\boldsymbol{v}}. \tag{3.3}$$

Recall from Chapter 1 that an over dot denotes the material time derivative, i.e. the time rate of change at a fixed particle $\mathbf{x}$.

Most of our preceding discussion on deformations carries over to motions in an obvious manner. In particular, we have the deformation gradient tensor $\boldsymbol{F}(\boldsymbol{x},t)$, the Jacobian determinant $J(\boldsymbol{x},t)$, the proper orthogonal (rotation) tensor $\boldsymbol{R}(\boldsymbol{x},t)$ and the symmetric positive definite (stretch) tensors $\boldsymbol{U}(\boldsymbol{x},t)$, $\boldsymbol{V}(\boldsymbol{x},t)$:

$$\boldsymbol{F}(\boldsymbol{x},t) = \operatorname{Grad} \boldsymbol{y}(\boldsymbol{x},\mathrm{t}), \tag{3.4}$$

$$J(\boldsymbol{x},t) = \det \boldsymbol{F}(\boldsymbol{x},t) > 0, \tag{3.5}$$

$$\boldsymbol{F}(\boldsymbol{x},t) = \boldsymbol{R}(\boldsymbol{x},t)\boldsymbol{U}(\boldsymbol{x},t) = \boldsymbol{V}(\boldsymbol{x},t)\boldsymbol{R}(\boldsymbol{x},t). \tag{3.6}$$

The principal stretches $\lambda_1(\boldsymbol{x},t), \lambda_2(\boldsymbol{x},t), \lambda_3(\boldsymbol{x},t)$ are the common eigenvalues of $\boldsymbol{U}(\mathrm{x},t)$ and $\boldsymbol{V}(\mathrm{x},t)$; the corresponding eigenvectors $\boldsymbol{r}_1(\boldsymbol{x},t)$, $\boldsymbol{r}_2(\boldsymbol{x},t)$, $\boldsymbol{r}_3(\boldsymbol{x},t)$ and $\boldsymbol{\ell}_1(\boldsymbol{x},t)$, $\boldsymbol{\ell}_2(\boldsymbol{x},t)$, $\boldsymbol{\ell}_3(\boldsymbol{x},t)$ are the principal directions of the Lagrangian and Eulerian stretch tensors, and so on.

3.2 Rigid Motions.

A rigid deformation is characterized by (2.42). Thus a rigid motion is characterized by

$$\boldsymbol{y} = \boldsymbol{Q}(t)\boldsymbol{x} + \boldsymbol{b}(t) \tag{3.7}$$

where the proper orthogonal tensor $\mathbf{Q}(t)$ and the vector $\mathbf{b}(t)$ denote the rotational and translational parts of the motion at time t. Differentiating (3.7) with respect to t yields the velocity field (in Lagrangian form):

$$\boldsymbol{v}(\boldsymbol{x},t) = \dot{\boldsymbol{Q}}(t)\boldsymbol{x} + \dot{\boldsymbol{b}}(t). \tag{3.8}$$

In Classical Mechanics of rigid bodies, the velocity of a rigid body is written in a quite different form to (3.8), and in particular involves an angular velocity vector. In the remainder of this section we shall show how (3.8) can be written in that familiar form.

We first express the velocity field (3.8) in Eulerian form by trading $\mathbf{x}$ for $\mathbf{y}$. To this end we solve (3.7) for $\mathbf{x}$ which leads to

$$\boldsymbol{x} = \boldsymbol{Q}^T\boldsymbol{y} - \boldsymbol{Q}^T\boldsymbol{b}, \tag{3.9}$$

and substitute (3.9) into (3.8) to obtain

$$\boldsymbol{v}(\boldsymbol{y}, t) = \boldsymbol{\Omega}(t)\boldsymbol{y} + \boldsymbol{c}(t) \tag{3.10}$$

where we have set $\boldsymbol{\Omega} = \dot{\boldsymbol{Q}}\boldsymbol{Q}^T$ and $\boldsymbol{c} = \dot{\boldsymbol{b}} - \dot{\boldsymbol{Q}}\boldsymbol{Q}^T\boldsymbol{b}$.

In order to further simplify the velocity field (3.10), we first recall that $\boldsymbol{Q}(t)$ is an orthogonal tensor and therefore $\boldsymbol{Q}(t)\boldsymbol{Q}^T(t) = \boldsymbol{I}$ at each instant t. Differentiating this with respect to t yields

$$\dot{\boldsymbol{Q}}\boldsymbol{Q}^T + \boldsymbol{Q}\dot{\boldsymbol{Q}}^T = \mathbf{0} \tag{3.11}$$

or

$$\boldsymbol{\Omega} = -\boldsymbol{\Omega}^T. \tag{3.12}$$

Thus $\boldsymbol{\Omega}(t)$ is skew symmetric at each instant t.

Finally recall from Chapter 2 of Volume I that, given any skew symmetric tensor $\boldsymbol{\Omega}$ there exists a vector $\boldsymbol{w}$ such that

$$\boldsymbol{\Omega}\boldsymbol{p} = \boldsymbol{w} \times \boldsymbol{p} \qquad \text{for all vectors } \boldsymbol{p}. \tag{3.13}$$

By using this result in (3.10), it follows that the velocity field (3.10) can be written equivalently as

$$\boldsymbol{v}(\boldsymbol{y}, t) = \boldsymbol{w}(t) \times \boldsymbol{y} + \boldsymbol{c}(t); \tag{3.14}$$

$\boldsymbol{w}(t)$ is called the *angular velocity vector* of the rigid motion. Note from (3.10) and (3.14) that

$$\boldsymbol{\Omega} = \operatorname{grad} \boldsymbol{v} \qquad \text{and} \qquad \boldsymbol{w} = \frac{1}{2}\operatorname{curl} \boldsymbol{v}.$$

3.3 Velocity Gradient, Stretching and Spin Tensors.

When studying a deformation, we wanted to calculate the changes in various geometric quantities such as length, area, etc. and found that the deformation gradient tensor $\mathbf{F}$ was the key ingredient needed for doing so. Now we wish to study the *rate of change of* *various geometric quantities* and will find that the velocity gradient tensor $\boldsymbol{L}$ is the essential ingredient for this. The Lagrangian velocity field $\widehat{\mathbf{v}}(\mathbf{x},t)$ can be expressed in Eulerian form $\overline{\mathbf{v}}(\mathbf{y},t)$ by using the inverse motion $\mathbf{x} = \overline{\mathbf{x}}(\mathbf{y},t)$ to trade the $\mathbf{x}$ for $\mathbf{y}$:

$$\mathbf{v} = \overline{\mathbf{v}}(\mathbf{y},t) = \widehat{\mathbf{v}}(\overline{\mathbf{x}}(\mathbf{y},t),t). \tag{3.15}$$

The *velocity gradient tensor* is defined as the (spatial) gradient of the velocity field:

$$\boldsymbol{L}(\boldsymbol{y},t) = \mathrm{grad}\,\boldsymbol{v}(\boldsymbol{y},t) = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{y}}(\mathbf{y},t) \tag{3.16}$$

where we have now omitted the "overline" on $\mathbf{v}$. In terms of components in an orthonormal basis,

$$L_{ij} = \frac{\partial v_i}{\partial y_j}. \tag{3.17}$$

Note the useful fact that

$$\mathrm{tr}\ \mathbf{L} = \mathrm{div}\ \mathbf{v}. \tag{3.18}$$

On using (3.4), $(3.3)_1$ and (1.17) we find

$$\dot{\boldsymbol{F}} \;=\; \frac{\partial}{\partial t}\left(\mathrm{Grad}\,\boldsymbol{y}(\boldsymbol{x},t)\right) = \mathrm{Grad}\,\left(\frac{\partial \boldsymbol{y}}{\partial t}(\boldsymbol{x},t)\right) = \mathrm{Grad}\,\boldsymbol{v} = (\mathrm{grad}\,\boldsymbol{v})\boldsymbol{F} = \boldsymbol{L}\boldsymbol{F}. \tag{3.19}$$

This leads to the following useful expression for $\mathbf{L}$:

$$\boldsymbol{L} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1}. \tag{3.20}$$

The *stretching tensor* (or *rate of deformation tensor*) $\boldsymbol{D}$ and the *spin tensor* $\boldsymbol{W}$ are defined as the symmetric and skew-symmetric parts of $\boldsymbol{L}$:

$$\boldsymbol{D} = \tfrac{1}{2}(\boldsymbol{L} + \boldsymbol{L}^T), \qquad\qquad \boldsymbol{W} = \tfrac{1}{2}(\boldsymbol{L} - \boldsymbol{L}^T); \tag{3.21}$$

clearly the stretching tensor is symmetric, the spin tensor is skew-symmetric, and their sum equals the velocity gradient tensor:

$$\boldsymbol{D} = \boldsymbol{D}^T \qquad \boldsymbol{W} = -\boldsymbol{W}^T \qquad \text{and} \quad \boldsymbol{L} = \boldsymbol{D} + \boldsymbol{W}. \tag{3.22}$$

Note from this that

$$\mathrm{tr}\,\mathbf{D} = \mathrm{tr}\,\mathbf{L}, \qquad \mathrm{tr}\,\mathbf{W} = 0. \tag{3.23}$$

Observe from (3.16) and (3.21) that

$$\boldsymbol{D} = \tfrac{1}{2}\left(\mathrm{grad}\,\boldsymbol{v} + (\mathrm{grad}\,\boldsymbol{v})^T\right), \qquad\qquad \boldsymbol{W} = \tfrac{1}{2}\left(\mathrm{grad}\,\boldsymbol{v} - (\mathrm{grad}\,\boldsymbol{v})^T\right), \tag{3.24}$$

or in terms of their components,

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial y_j} + \frac{\partial v_j}{\partial y_i}\right), \qquad W_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial y_j} - \frac{\partial v_j}{\partial y_i}\right). \tag{3.25}$$

Since $\mathbf{W}$ is skew symmetric, there exists a unique axial vector $\mathbf{w}$ such that $\mathbf{W}\mathbf{p} = \mathbf{w} \times \mathbf{p}$ for all vectors $\mathbf{p}$. One can show that

$$\mathbf{w} = \frac{1}{2}\,\mathrm{curl}\,\boldsymbol{v} \tag{3.26}$$

which is the local angular velocity at a particle. The vorticity $\boldsymbol{\omega}$ is defined as

$$\boldsymbol{\omega} = 2\mathbf{w} = \mathrm{curl}\,\boldsymbol{v}. \tag{3.27}$$

Note from (3.10), (3.16) and (3.21), that in the special case of a rigid motion

$$\boldsymbol{L}(\boldsymbol{y}, t) = \boldsymbol{\Omega}(t), \qquad \boldsymbol{D}(\boldsymbol{y}, t) = \boldsymbol{O}, \qquad \boldsymbol{W}(\boldsymbol{y}, t) = \boldsymbol{\Omega}(t). \tag{3.28}$$

The tensors[1] $\boldsymbol{D}$ and $\boldsymbol{W}$ play an important role in the kinematics of motions. We turn next to interpreting $\boldsymbol{D}$ and $\boldsymbol{W}$.

3.4 Rate of Change of Length, Orientation, and Volume.

3.4.1 Rate of Change of Length and Orientation.

Let $d\boldsymbol{x}$ be a material fiber through the particle $\boldsymbol{x}$ in the reference configuration, and let $d\boldsymbol{y}$ be its image at time t. Then $d\mathbf{y} = d\mathbf{y}(\mathbf{x}, t; d\mathbf{x}) = \boldsymbol{F}(\boldsymbol{x}, t)d\boldsymbol{x}$. Differentiating this with respect to t and using (3.20) gives $(d\boldsymbol{y})^{\cdot} = \dot{\boldsymbol{F}}d\boldsymbol{x} = (\boldsymbol{L}\boldsymbol{F})d\boldsymbol{x} = \boldsymbol{L}(\boldsymbol{F}d\boldsymbol{x}) = \boldsymbol{L}d\boldsymbol{y}$. Thus

$$(d\boldsymbol{y})^{\cdot} = \boldsymbol{L}d\boldsymbol{y} \tag{3.29}$$

and so $\boldsymbol{L}$ characterizes the rate of change of length and orientation of a material fiber in the current configuration.

[1] It is worth remarking that there is no simple relation between the stretching tensor $\mathbf{D}$ and time rates of change of the stretch tensors, i.e. $\dot{\mathbf{U}}$ and $\dot{\mathbf{V}}$; nor is there a simple relation between the spin tensor $\mathbf{W}$ and the time rate of change of the rotation tensor, i.e. $\dot{\mathbf{R}}$. The relations between these quantities is established in Problem 3.4. However see also Section 3.5.

From this basic equation we can separately calculate the rate of change of length and the rate of change of orientation of the fiber. To this end, let

$$d\boldsymbol{y} = \boldsymbol{n}\, ds \tag{3.30}$$

where $ds(t)$ is the current length of the fiber and the unit vector $\boldsymbol{n}(t)$ is its current orientation. Taking the material time derivative of $ds^2 = d\boldsymbol{y} \cdot d\boldsymbol{y}$ and using (3.29) gives $ds(ds)^{\cdot} = d\boldsymbol{y} \cdot (d\boldsymbol{y})^{\cdot} = \boldsymbol{L}d\boldsymbol{y} \cdot d\boldsymbol{y}$ which further simplifies on using (3.30) to $(ds)^{\cdot}/ds = \boldsymbol{L}\boldsymbol{n} \cdot \boldsymbol{n}$. On using the fact that $\boldsymbol{L} = \boldsymbol{D} + \boldsymbol{W}$, and furthermore recalling from Chapter 2 of Volume I that $\boldsymbol{W}\boldsymbol{n} \cdot \boldsymbol{n} = 0$ since $\boldsymbol{W}$ is skew-symmetric, we can write the preceding equation as

$$\frac{1}{ds}(ds)^{\cdot} = \boldsymbol{D}\boldsymbol{n} \cdot \boldsymbol{n}. \tag{3.31}$$

Consequently, we conclude that the stretching tensor $\boldsymbol{D}$ characterizes the rate of change of length.

In particular, we can now interpret the diagonal components of $[D]$ as follows: by definition $D_{ij} = \mathbf{D}\mathbf{e}_i \cdot \mathbf{e}_j$. First pick $\boldsymbol{n} = \mathbf{e}_i$. Then (3.31) gives

$$D_{ii} = \frac{1}{ds}(ds)^{\cdot} \qquad \text{(no sum on i)}. \tag{3.32}$$

Therefore for each $i = 1, 2, 3$, the component D_{ii} of the stretching tensor measures the rate of change of length per unit length of a material fiber that is *currently* in the $\mathbf{e}_i$-direction. Next, suppose that we pick $\boldsymbol{n} = \boldsymbol{l}_1$ where $\boldsymbol{l}_1$ is a principal vector of the Eulerian stretch tensor $\boldsymbol{V}$. Recall from our discussion of deformations that in this special case $ds = \lambda_1 ds_0$ where ds_0 is the length of the fiber in the reference configuration and λ_1 is the principal stretch in direction $\boldsymbol{l}_1$. Now, (3.32) gives

$$D_{11} = \frac{\dot{\lambda}_1}{\lambda_1} = (\ln \lambda_1)^{\cdot}. \tag{3.33}$$

Note that D_{11} here is the 11-component of $\boldsymbol{D}$ in the special basis $\{\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3\}$ (whereas D_{11} in (3.32) is the 11-component of $\mathbf{D}$ in any arbitrary orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$).

Finally we consider the rate of change of orientation. From (3.29) and (3.30) we have $(\boldsymbol{n}\, ds)^{\cdot} = \boldsymbol{L}\boldsymbol{n}\, ds$ which when expanded out reads $\dot{\boldsymbol{n}} = \boldsymbol{L}\boldsymbol{n} - \boldsymbol{n}(ds)^{\cdot}/ds$. Since $(ds)^{\cdot}/ds = \mathbf{n}\cdot\mathbf{L}\mathbf{n}$ this can be written as

$$\dot{\mathbf{n}} = \mathbf{L}\mathbf{n} - \big(\mathbf{n} \cdot \mathbf{L}\mathbf{n}\big)\, \mathbf{n}. \tag{3.34}$$

Finally, on using $\boldsymbol{L} = \boldsymbol{D} + \boldsymbol{W}$ this simplifies to

$$\dot{\boldsymbol{n}} = \boldsymbol{W}\boldsymbol{n} + \boldsymbol{D}\boldsymbol{n} - (\boldsymbol{D}\boldsymbol{n} \cdot \boldsymbol{n})\boldsymbol{n}. \tag{3.35}$$

Observe first that in a rigid motion, since $\boldsymbol{D} = \boldsymbol{0}$, (3.35) specializes to $\dot{\boldsymbol{n}} = \boldsymbol{W}\boldsymbol{n}$; thus in this special case the vector $\boldsymbol{n}$ rotates with the spin $\boldsymbol{W}$. In a general motion however, we see from (3.35) that $\boldsymbol{D}\boldsymbol{n} - (\boldsymbol{D}\boldsymbol{n}\cdot\boldsymbol{n})\boldsymbol{n}$ represents an additional motion superposed on the rigid spin $\boldsymbol{W}$.

Since $\boldsymbol{D}$ is symmetric it has three real eigenvalues δ_i and a corresponding set of orthonormal eigenvectors $\boldsymbol{d}_i$:

$$\boldsymbol{D}\boldsymbol{d}_i = \delta_i \boldsymbol{d}_i, \qquad i = 1,2,3, \qquad \text{(no summation over } i). \tag{3.36}$$

Observe from (3.36) that $\boldsymbol{D}\boldsymbol{d}_i = (\boldsymbol{D}\boldsymbol{d}_i \cdot \boldsymbol{d}_i)\mathbf{d}_i$ so that if we pick $\boldsymbol{n} = \boldsymbol{d}_i$ in (3.35), we get

$$\dot{\boldsymbol{d}}_i = \boldsymbol{W}\boldsymbol{d}_i. \tag{3.37}$$

Thus a material fiber which *happens to lie along a principal axis of* $\boldsymbol{D}$ *at time* t rotates at an angular velocity $\boldsymbol{W}$. This motivates the use of the term spin to describe $\boldsymbol{W}$. It is worth emphasizing that as the orthonormal triplet of vectors $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ corresponding to a principal basis of $\mathbf{D}$ rotates with time, *different* material fibers are aligned with these directions in general. In terms of the axial vector $\mathbf{w}$ associated with the skew symmetric tensor $\mathbf{W}$, we have

$$\dot{\mathbf{d}}_i = \mathbf{w} \times \mathbf{d}_i;$$

see (3.26).

3.4.2 Rate of Change of Angle.

Consider two material fibers $d\boldsymbol{x}^{(1)}, d\boldsymbol{x}^{(2)}$ in the reference configuration whose images at time t are $d\boldsymbol{y}^{(1)} = ds_1\ \boldsymbol{n}^{(1)}$ and $d\boldsymbol{y}^{(2)} = ds_2\ \boldsymbol{n}^{(2)}$. The angle θ between these fibers at time t is given by

$$\cos\theta = \boldsymbol{n}^{(1)}\cdot\boldsymbol{n}^{(2)}. \tag{3.38}$$

Differentiating (3.38) with respect to t and using (3.35) gives

$$\begin{aligned} -\sin\theta\,\dot{\theta} &= \dot{\boldsymbol{n}}^{(1)}\cdot\boldsymbol{n}^{(2)} + \boldsymbol{n}^{(1)}\cdot\dot{\boldsymbol{n}}^{(2)} \\ &= \left\{\boldsymbol{W}\boldsymbol{n}^{(1)} + \boldsymbol{D}\boldsymbol{n}^{(1)} - (\boldsymbol{D}\boldsymbol{n}^{(1)}\cdot\boldsymbol{n}^{(1)})\boldsymbol{n}^{(1)}\right\}\cdot\boldsymbol{n}^{(2)} \\ &\quad + \left\{\boldsymbol{W}\boldsymbol{n}^{(2)} + \boldsymbol{D}\boldsymbol{n}^{(2)} - (\boldsymbol{D}\boldsymbol{n}^{(2)}\cdot\boldsymbol{n}^{(2)})\boldsymbol{n}^{(2)}\right\}\cdot\boldsymbol{n}^{(1)} \end{aligned} \tag{3.39}$$

and therefore

$$\dot{\theta} = \frac{-2\boldsymbol{D}\boldsymbol{n}^{(1)}\cdot\boldsymbol{n}^{(2)} + \{\boldsymbol{D}\boldsymbol{n}^{(1)}\cdot\boldsymbol{n}^{(1)} + \boldsymbol{D}\boldsymbol{n}^{(2)}\cdot\boldsymbol{n}^{(2)}\}(\boldsymbol{n}^{(1)}\cdot\boldsymbol{n}^{(2)})}{\sin\theta}. \tag{3.40}$$

Consequently we see that the rate of change of angle between two material fibers can be characterized in terms of the stretch tensor $\boldsymbol{D}$.

Specifically if we choose $\boldsymbol{n}^{(1)} = \boldsymbol{e}_1$ and $\boldsymbol{n}^{(2)} = \boldsymbol{e}_2$ then (3.40) specializes to

$$D_{12} = -\tfrac{1}{2}\dot{\theta}. \tag{3.41}$$

Thus the off-diagonal component D_{12} of $\mathbf{D}$ equals $-1/2$ the rate of change of the angle between two material fibers that currently happen to be oriented in the $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ directions.

3.4.3 Rate of Change of Volume.

Consider a material volume dV_x in the reference configuration, and let dV_y be the volume occupied by this same collection of particles in the current configuration. Recall that $dV_y = J\, dV_x$ where $J = \det \boldsymbol{F}$. Recall the following standard formula for differentiating the determinant (see Volume I),

$$\frac{d}{dt}\big(\det \mathbf{F}\big) = (\det \mathbf{F})\ \mathrm{tr}\left(\frac{\mathrm{d}\mathbf{F}}{\mathrm{dt}}\mathbf{F}^{-1}\right). \tag{3.42}$$

On using this we get

$$\dot{J} = \ \det \mathbf{F}\ \mathrm{tr}\,(\dot{\boldsymbol{F}}\boldsymbol{F}^{-1}) = \mathrm{J}\ \mathrm{tr}\,\mathbf{L} \tag{3.43}$$

where we have used (3.20). Since $\mathrm{tr}\,\boldsymbol{D} = \mathrm{tr}\,\boldsymbol{L}$, we can write this equivalently as

$$\dot{J} = J\,\mathrm{tr}\,\boldsymbol{D} = J\,\mathrm{div}\ \boldsymbol{v} \tag{3.44}$$

where in the last step we have used (3.17). Thus

$$\frac{(dV_y)^{\cdot}}{dV_y} = \frac{\dot{J}}{J} = \mathrm{div}\ \boldsymbol{v}. \tag{3.45}$$

Note that in the particular case of an isochoric (i.e. locally volume preserving) motion,

$$\mathrm{div}\ \boldsymbol{v}(\boldsymbol{y}, t) = \mathbf{0}. \tag{3.46}$$

3.4.4 Rate of Change of Area and Orientation.

Consider a material surface which at time t has area dA_y and unit normal $\mathbf{n}$. We leave it as an exercise to the reader to show that the rate of change of area and the rate of rotation of the unit normal are given by

$$(dA_y)^{\cdot} = \big(\mathrm{tr}\,\mathbf{L} - \mathbf{n}\cdot\mathbf{L}\mathbf{n}\big)\,dA_y, \tag{3.47}$$

$$\dot{\mathbf{n}} = \big(\mathbf{n} \cdot \mathbf{L}\mathbf{n}\big)\,\mathbf{n} - \mathbf{L}^T\mathbf{n}. \tag{3.48}$$

Question: Why is (3.34) different to (3.48)?

Example: Simple Shearing Motion.

Consider a unit cube undergoing the simple shearing motion

$$y_1 = x_1 + k(t)x_2, \qquad y_2 = x_2, \qquad y_3 = x_3. \tag{3.49}$$

We wish to calculate the components D_{ij} of the stretching tensor and discuss the significance of the various components.

Before carrying out a mathematical calculation using (3.49), let's think about the physics of the motion. Consider first a material fiber which currently is in the $\boldsymbol{e}_1$-direction; clearly its length doesn't change with time, and so $D_{11} = 0$. Similarly we must have $D_{33} = 0$. Next consider a material fiber that happens to be currently aligned with the $\mathbf{e}_2$-direction. Note that this fiber is *not* always aligned with the $\mathbf{e}_2$-direction. In particular, observe that its length has been contracting (in the past) and is about to begin extending (in the future). Thus *instantaneously* its length is not changing and so $D_{22} = 0$. Finally, consider a pair of fibers that are currently in the $\mathbf{e}_1$- and $\mathbf{e}_3$-directions. Clearly the angle between them is not changing. Thus $D_{13} = D_{31} = 0$. Similarly $D_{23} = D_{32} = 0$. Thus the only non-zero components of $\mathbf{D}$ are D_{12} and $D_{21}(= D_{12})$.

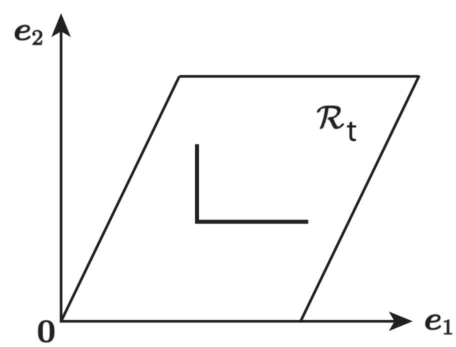

Figure 3.2: The region occupied by a cube undergoing simple shear. The figure is drawn at a particular instant t and shows two material fibers, one of which happens to be horizontal and the other vertical, at this particular instant.

Now we carry out a formal calculation of D_{ij}. Differentiating (3.49) gives

$$v_1 = \dot{k}x_2, \quad v_2 = 0, \quad v_3 = 0, \tag{3.50}$$

which, by using (3.49) can be expressed in the Eulerian form

$$v_1 = \dot{k}y_2, \quad v_2 = 0, \quad v_3 = 0. \tag{3.51}$$

The components of the velocity gradient tensor can be calculated from (3.51) to be

$$[L] = \left[\frac{\partial v_i}{\partial y_j}\right] = \begin{pmatrix} 0 & \dot{k} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.52}$$

and thus the components of the stretching tensor and spin tensor, $[D] = ([L] + [L]^T)/2$ and $[W] = ([L] - [L]^T)/2$, are

$$[D] = \begin{pmatrix} 0 & \dot{k}/2 & 0 \\ \dot{k}/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad [W] = \begin{pmatrix} 0 & \dot{k}/2 & 0 \\ -\dot{k}/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.53}$$

From (3.53) we can find the principal values and principal directions of $\boldsymbol{D}$ to be

$$\delta_1 = \dot{k}/2, \qquad \delta_2 = -\dot{k}/2, \qquad \delta_3 = 0, \tag{3.54}$$

$$\boldsymbol{d}_1 = \boldsymbol{e}_1/\sqrt{2} + \boldsymbol{e}_2/\sqrt{2}, \qquad \boldsymbol{d}_2 = -\boldsymbol{e}_1/\sqrt{2} + \boldsymbol{e}_2/\sqrt{2}, \qquad \boldsymbol{d}_3 = \boldsymbol{e}_3. \tag{3.55}$$

Observe that in this particular motion the principal directions $\boldsymbol{d}_i$ happen to be time independent. This is usually not the case.

Remark: We leave it to the reader to reconcile the apparent contradiction between the formula (3.37), and the formulas (3.53), (3.55) together with the fact that the $\boldsymbol{d}_i$'s are time independent.

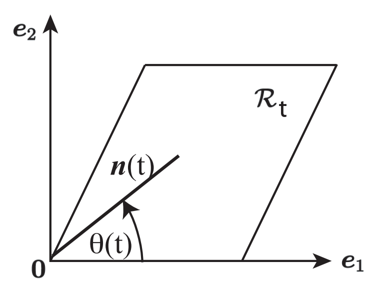

Figure 3.3: The region $\mathcal{R}_t$ occupied by a cube undergoing simple shear. The figure shows a material fiber in the direction $\mathbf{n}(t)$.

Finally let us explicitly evaluate the formula (3.35) for determining the rate of change of direction of a material fiber. Let $\boldsymbol{n}$ be the *current* orientation of a material fiber and suppose that

$$\boldsymbol{n}(t) = \cos\theta(t)\ \boldsymbol{e}_1 + \sin\theta(t)\ \boldsymbol{e}_2. \tag{3.56}$$

Then from (3.56) and (3.53) one readily finds that

$$\boldsymbol{W}\boldsymbol{n} = \frac{\dot{k}}{2}(\sin\theta\ \boldsymbol{e}_1 - \cos\theta\ \boldsymbol{e}_2), \tag{3.57}$$

and from (3.53) and (3.56) one obtains

$$\boldsymbol{D}\boldsymbol{n} - (\boldsymbol{D}\boldsymbol{n}\cdot\boldsymbol{n})\boldsymbol{n} = -\frac{\dot{k}}{2}\cos 2\theta(\sin\theta\boldsymbol{e}_1 - \cos\theta\boldsymbol{e}_2). \tag{3.58}$$

From (3.35), we see that the (3.57) and (3.58) represent the two parts of $\dot{\boldsymbol{n}}$, the first due to spin and the second due to stretching. Observe that each component is in the $\mathbf{e}_1, \mathbf{e}_2$-plane. On combining them we have

$$\dot{\boldsymbol{n}} = \frac{\dot{k}}{2}(1-\cos 2\theta)(\sin\theta\boldsymbol{e}_1 - \cos\theta\boldsymbol{e}_2) = \dot{k}\sin^2\theta(\sin\theta\boldsymbol{e}_1 - \cos\theta\boldsymbol{e}_2). \tag{3.59}$$

On the other hand, differentiating (3.56) gives $\dot{\mathbf{n}} = (-\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2)\dot{\theta}$ and so comparing this with (3.59) shows that $\dot{\theta} = -\dot{k}\sin^2\theta$.

3.5 Current Configuration as Reference Configuration.

In our analysis thus far we have always worked with respect to one fixed reference configuration. It is not necessary to do so and we could, just as well, use a time-dependent reference configuration. A useful special case of this is to take the current configuration as the time-dependent reference configuration. In this section we formulate such a description of a motion and establish some useful results. In particular we show (in a certain precise sense) that the time rates of change of the stretch and rotation tensors of the polar decomposition equal the stretching and spin tensors; see footnote 1 in this chapter.

Consider a body $\mathcal{B}$. In some time-independent reference configuration, an arbitrary particle $p \in \mathcal{B}$ is located at $\mathbf{x}$; and the body occupies a region $\mathcal{R}_0$. We consider a motion of this body over a time interval $\tau_0 \le \tau \le \tau_1$. (In this section we let τ and t denote *an arbitrary instant of time* and *the current instant of time* respectively.) Suppose that this motion is described, *with respect to the aforementioned reference configuration*, by the mapping

$$\mathbf{z} = \widehat{\mathbf{y}}(\mathbf{x},\tau), \qquad \mathbf{x}\in\mathcal{R}_0, \quad \tau_0 \le \tau \le \tau_1, \tag{3.60}$$

where $\mathbf{z} \in \mathcal{R}_\tau$ is the location of particle p at time τ and (3.60) maps $\mathcal{R}_0 \to \mathcal{R}_\tau$; see Figure 3.4.

Suppose that t is the "current instant of time" and that the position of particle p at this instant is $\mathbf{y}$:

$$\mathbf{y} = \widehat{\mathbf{y}}(\mathbf{x},t), \qquad \mathbf{x}\in\mathcal{R}_0. \tag{3.61}$$

Note that $\mathbf{z} = \mathbf{y}$ when $\tau = t$. If the configuration at time t (the "current configuration") is used as a (time-dependent) reference configuration, then the motion at hand can be described

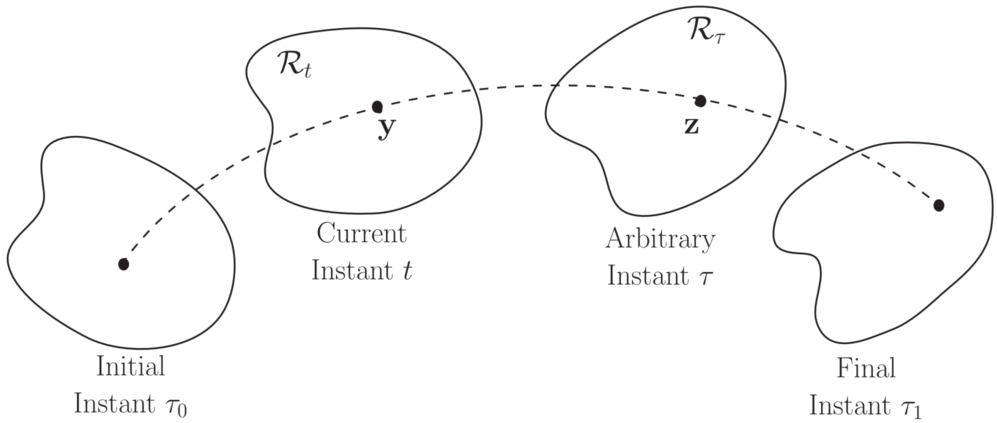

Figure 3.4: A motion on a time interval $\tau_0 \leq \tau \leq \tau_1$ is described, with respect to a time-independent reference configuration (not shown), by $\mathbf{z} = \widehat{\mathbf{y}}(\mathbf{x}, \tau)$; here the position of a particle at an *arbitrary instant of time* τ is $\mathbf{z}$, and its position in the reference configuration is $\mathbf{x}$. The *current instant of time* is denoted by t and at $\tau = t$ this particle is located at $\mathbf{y}$. This same motion, when described with the current configuration taken to be the reference configuration, $\mathbf{y} \to \mathbf{z}$, is described by $\mathbf{z} = \widehat{\mathbf{y}}_t(\mathbf{y}, \tau)$; see equation (3.62).

with respect to this second reference configuration by

$$\mathbf{z} = \widehat{\mathbf{y}}(\widehat{\mathbf{y}}^{-1}(\mathbf{y}, t), \tau) \stackrel{\text{def}}{=} \widehat{\mathbf{y}}_t(\mathbf{y}, \tau), \tag{3.62}$$

where $\widehat{\mathbf{y}}^{-1}(\cdot, t)$ is the inverse of $\widehat{\mathbf{y}}(\cdot, t)$ at each t. Equations (3.60) - (3.62) lead to the identity

$$\widehat{\mathbf{y}}(\mathbf{x}, \tau) = \widehat{\mathbf{y}}_t(\widehat{\mathbf{y}}(\mathbf{x}, t), \tau). \tag{3.63}$$

Perhaps it is worth remarking that *the subscript t does not represent differentiation* with respect to t; we could have just as well written, say, $\widehat{\mathbf{y}}(\mathbf{y}, \tau; t)$ instead of $\widehat{\mathbf{y}}_t(\mathbf{y}, \tau)$.

By (3.60), the deformation gradient tensor of the configuration at time τ, with respect to the time-independent reference configuration, is

$$\mathbf{F}(\mathbf{x}, \tau) = \underset{x}{\nabla} \widehat{\mathbf{y}}(\mathbf{x}, \tau). \tag{3.64}$$

Similarly by $(3.62)_2$, the deformation gradient tensor of the configuration at time τ, with respect to the time-dependent reference configuration, is

$$\mathbf{F}_t(\mathbf{y}, \tau) = \underset{y}{\nabla} \widehat{\mathbf{y}}_t(\mathbf{y}, \tau). \tag{3.65}$$

We first show that *at the current instant t, the deformation gradient tensor with respect to the current configuration, and the stretch and rotation tensors related to it by its polar* *decomposition, all equal the identity.* To show this we begin by taking the gradient of the identity (3.63) with respect to $\mathbf{x}$ which leads to

$$\nabla_x \widehat{\mathbf{y}}(\mathbf{x},\tau) = \nabla_y \widehat{\mathbf{y}}_t(\mathbf{y},\tau)\ \nabla_x \widehat{\mathbf{y}}(\mathbf{x},t) \tag{3.66}$$

where we have used the chain rule. Thus by (3.64) – (3.66):

$$\mathbf{F}(\mathbf{x},\tau) = \mathbf{F}_t(\mathbf{y},\tau)\mathbf{F}(\mathbf{x},t). \tag{3.67}$$

Now set $\tau = t$ in (3.67) to get

$$\mathbf{F}_t(\mathbf{y},t) = \mathbf{I}. \tag{3.68}$$

Thus at the current instant t, the deformation gradient tensor with respect to the current configuration is the identity. On using the polar decomposition of $\mathbf{F}_t(\mathbf{y},t)$ we get

$$\mathbf{F}_t(\mathbf{y},t) = \mathbf{R}_t(\mathbf{y},t)\mathbf{U}_t(\mathbf{y},t) = \mathbf{I}.$$

Certainly $\mathbf{R}_t(\mathbf{y},t) = \mathbf{U}_t(\mathbf{y},t) = \mathbf{I}$ is one choice for the rotation and stretch tensors that are consistent with this. By the uniqueness of the polar decomposition, this is in fact the only choice for $\mathbf{R}_t(\mathbf{y},t)$ and $\mathbf{U}_t(\mathbf{y},t)$. Thus

$$\mathbf{R}_t(\mathbf{y},t) = \mathbf{I}, \qquad \mathbf{U}_t(\mathbf{y},t) = \mathbf{I}. \tag{3.69}$$

Next we show that *at the current instant t, the time rate of change of the deformation gradient, stretch and rotation tensors all with respect to the current configuration, equal the velocity gradient, stretching and spin tensors respectively.* Differentiating (3.67) with respect to time[2] τ gives

$$\dot{\mathbf{F}}(\mathbf{x},\tau) = \frac{\partial}{\partial\tau}\mathbf{F}_t(\mathbf{y},\tau)\ \mathbf{F}(\mathbf{x},t). \tag{3.70}$$

On using $\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}$, see (3.20), we get

$$\mathbf{L}(\mathbf{z},\tau)\mathbf{F}(\mathbf{x},\tau) = \frac{\partial}{\partial\tau}\mathbf{F}_t(\mathbf{y},\tau)\ \mathbf{F}(\mathbf{x},t)$$

where $\mathbf{L}$ is the velocity gradient tensor associated with the motion $\mathbf{z} = \widehat{\mathbf{y}}(\mathbf{x},\tau)$. Setting $\tau = t$ in this leads to the first result stated above, viz.

$$\mathbf{L}(\mathbf{y},t) = \frac{\partial}{\partial\tau}\mathbf{F}_t(\mathbf{y},\tau)\Big|_{\tau=t} \tag{3.71}$$

[2]Here and in the rest of this section, when we differentiate with respect to time τ we shall do so at a fixed particle and a fixed current instant, i.e. we shall hold $\mathbf{x},\mathbf{y}$ and t fixed.

where we have made use of the fact that $\mathbf{z} = \mathbf{y}$ when $\tau = t$. Next we use the polar decomposition $\mathbf{F}_t(\mathbf{y},\tau) = \mathbf{R}_t(\mathbf{y},\tau)\mathbf{U}_t(\mathbf{y},\tau)$ in (3.71) to get

$$\mathbf{L}(\mathbf{y},\tau) = \frac{\partial}{\partial\tau}\Big(\mathbf{R}_t(\mathbf{y},\tau)\mathbf{U}_t(\mathbf{y},\tau)\Big) \;=\; \left(\frac{\partial}{\partial\tau}\mathbf{R}_t(\mathbf{y},\tau)\right)\mathbf{U}_t(\mathbf{y},\tau) + \mathbf{R}_t(\mathbf{y},\tau)\left(\frac{\partial}{\partial\tau}\mathbf{U}_t(\mathbf{y},\tau)\right).$$

Setting $\tau = t$ in this and using the fact that $\mathbf{U}_t(\mathbf{y},t) = \mathbf{R}_t(\mathbf{y},t) = \mathbf{I}$ leads to

$$\mathbf{L}(\mathbf{y},t) = \frac{\partial}{\partial\tau}\mathbf{R}_t(\mathbf{y},\tau)\bigg|_{\tau=t} + \frac{\partial}{\partial\tau}\mathbf{U}_t(\mathbf{y},\tau)\bigg|_{\tau=t}. \tag{3.72}$$

However $\mathbf{R}_t(\mathbf{y},\tau)$ is orthogonal:

$$\mathbf{R}_t(\mathbf{y},\tau)\mathbf{R}_t^T(\mathbf{y},\tau) = \mathbf{I}.$$

Differentiating this with respect to τ gives

$$\left(\frac{\partial}{\partial\tau}\mathbf{R}_t(\mathbf{y},\tau)\right)\mathbf{R}_t^T(\mathbf{y},\tau) + \mathbf{R}_t(\mathbf{y},\tau)\left(\frac{\partial}{\partial\tau}\mathbf{R}_t^T(\mathbf{y},\tau)\right) = \mathbf{0},$$

which, upon setting $\tau = t$ and using $\mathbf{R}_t(\mathbf{y},t) = \mathbf{I}$, simplifies to

$$\frac{\partial}{\partial\tau}\mathbf{R}_t(\mathbf{y},\tau)\bigg|_{\tau=t} + \frac{\partial}{\partial\tau}\mathbf{R}_t^T(\mathbf{y},\tau)\bigg|_{\tau=t} = \mathbf{0}.$$

Therefore

$$\frac{\partial}{\partial\tau}\mathbf{R}_t(\mathbf{y},\tau)\bigg|_{\tau=t}$$

is skew-symmetric. Consequently (3.72) and the definition of the stretching tensor, $\mathbf{D} = (\mathbf{L}+\mathbf{L}^T)/2$, gives

$$\mathbf{D}(\mathbf{y},t) = \frac{\partial}{\partial\tau}\mathbf{U}_t(\mathbf{y},\tau)\bigg|_{\tau=t}; \tag{3.73}$$

similarly (3.72) and the definition of the spin tensor, $\mathbf{W} = (\mathbf{L}-\mathbf{L}^T)/2$, gives

$$\mathbf{W}(\mathbf{y},t) = \frac{\partial}{\partial\tau}\mathbf{R}_t(\mathbf{y},\tau)\bigg|_{\tau=t},$$

which are the other two results stated at the beginning of this paragraph. Thus when the reference configuration is taken to be the current configuration the stretching tensor $\mathbf{D}$ coincides with the time rate of change of the stretch tensor $\mathbf{U}$; and the spin tensor $\mathbf{W}$ coincides with the time rate of change of the rotation tensor $\mathbf{R}$. The relations between these quantities with respect to a time-independent reference configuration are established in Problem 3.4.

3.6 Worked Examples and Exercises

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Problem 3.1. The velocity field in a certain deforming continuum is known to be given by

$$\mathbf{v} = -\alpha y_2 \mathbf{e}_1 \ + \ \alpha y_1 \mathbf{e}_2 \ + \ \beta \mathbf{e}_3$$

where α and β are constants. Calculate the acceleration of a particle during this motion.

<u>*Solution*</u>: Working directly, we take the (material) time derivative of the given velocity field which leads to the desired result

$$\dot{\mathbf{v}} = -\alpha \dot{y}_2 \mathbf{e}_1 \ + \ \alpha \dot{y}_1 \mathbf{e}_2 = -\alpha v_2 \mathbf{e}_1 \ + \ \alpha v_1 \mathbf{e}_2 = -\alpha(\alpha y_1)\mathbf{e}_1 \ + \ \alpha(-\alpha y_2)\mathbf{e}_2 = -\alpha^2 y_1 \mathbf{e}_1 \ - \ \alpha^2 y_2 \mathbf{e}_2.$$

Alternatively we could use the formula $\dot{\mathbf{v}} = \big(\text{grad}\,\mathbf{v}\big)\mathbf{v} + \partial \mathbf{v}(\mathbf{y}, t)/\partial t$ to get the same result

$$\dot{\mathbf{v}} = \big(-\alpha \mathbf{e}_1 \otimes \mathbf{e}_2 + \alpha \mathbf{e}_2 \otimes \mathbf{e}_1\big)(-\alpha y_2 \mathbf{e}_1 \ + \ \alpha y_1 \mathbf{e}_2 \ + \ \beta \mathbf{e}_3) = -\alpha^2 y_1 \mathbf{e}_1 \ - \ \alpha^2 y_2 \mathbf{e}_2.$$

Problem 3.2. A fluid undergoes the motion

$$y_1 = (\cos at)x_1 - (\sin at)x_2, \qquad y_2 = (\sin at)x_1 + (\cos at)x_2, \qquad y_3 = x_3 \ + \ bt,$$

for $t \geq 0$ where a and b are constants and $-\infty < x_i < \infty, i = 1, 2, 3$. Observe that (y_1, y_2, y_3) are the current coordinates of the particle that at time $t = 0$ was located at (x_1, x_2, x_3).

Determine the path (curve) followed by the particle which is at $(1, 0, 1)$ at time $t = 2\pi/a$. What is the qualitative character of this curve?

Calculate the quantities div $\dot{\mathbf{v}}$ and $(\text{div}\,\mathbf{v})^{\cdot}$.

If the temperature of the particle which is located at (y_1, y_2, y_3) at time t is given by

$$\theta(y_1, y_2, y_3, t) = y_1 y_2 y_3 \ e^{-t},$$

calculate $\dot{\theta}$ and Grad θ.

Problem 3.3. The spatial (Eulerian) representation of the velocity field associated with a particular motion of a fluid is

$$v_1 = -\alpha y_2, \qquad v_2 = \alpha y_1, \qquad v_3 = \beta.$$

The reference configuration coincides with the configuration occupied by the body at the initial instant whence $\mathbf{x} = \hat{\mathbf{y}}(\mathbf{x}, 0)$. Determine the path of a generic particle for times $t > 0$.

Solution: We are given that $\dot{y}_1 = -\alpha y_2$, $\dot{y}_2 = \alpha y_1$. Combining these leads to $\ddot{y}_1 + \alpha^2 y_1 = 0$, $\ddot{y}_2 + \alpha^2 y_2 = 0$, and by solving them we find that necessarily

$$\begin{aligned} y_1 &= C_1(x_1,x_2,x_3)\sin\alpha t \ + \ D_1(x_1,x_2,x_3)\cos\alpha t, \\ y_2 &= C_2(x_1,x_2,x_3)\sin\alpha t \ + \ D_2(x_1,x_2,x_3)\cos\alpha t. \end{aligned}$$

In order to ensure that these expressions satisfy the given equations we substitute them back into $\dot{y}_1 = -\alpha y_2, \dot{y}_2 = \alpha y_1$ which shows that we must have $C_1 = -D_2, C_2 = D_1$. Thus we conclude that

$$\begin{aligned} y_1 &= C_1(x_1,x_2,x_3)\sin\alpha t \ + \ C_2(x_1,x_2,x_3)\cos\alpha t, \\ y_2 &= C_2(x_1,x_2,x_3)\sin\alpha t \ - \ C_1(x_1,x_2,x_3)\cos\alpha t. \end{aligned}$$

On integrating the third equation $\dot{y}_3 = \beta$ we obtain

$$y_3 = \beta t + C_3(x_1,x_2,x_3).$$

Next we use the fact that the reference configuration coincides with the configuration occupied by the body at the initial instant and so set $t = 0$ and $(y_1,y_2,y_3) = (x_1,x_2,x_3)$. This shows that $C_1 = -x_2, C_2 = x_1, C_3 = x_3$. Even though these expressions for C_i were derived by setting $t = 0$, since $C_i = C_i(x_1,x_2,x_3)$ are independent of t, they hold for all t. Thus we conclude that

$$\left.\begin{aligned} y_1 &= -x_2\sin\alpha t \ + \ x_1\cos\alpha t, \\ y_2 &= x_1\sin\alpha t \ + \ x_2\cos\alpha t, \\ y_3 &= x_3 + \beta t. \end{aligned}\right\}$$

For each fixed particle, i.e. fixed (x_1,x_2,x_3), this provides a parametric description $y_1 = y_1(t), y_2 = y_2(t), y_3 = y_3(t)$ of the path followed by the particle during the flow.

Problem 3.4. How is the *stretch* tensor $\mathbf{U}$ related to the *stretching* tensor $\mathbf{D}$? Similarly, how is the *rotation* tensor $\mathbf{R}$ related to the *spin* tensor $\mathbf{W}$?

Solution: Recall that $\mathbf{F} = \mathbf{RU}$ where $\mathbf{R}$ is orthogonal and $\mathbf{U}$ is symmetric positive definite. Differentiating this and substituting the result into the formula $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ leads to

$$\mathbf{L} = \dot{\mathbf{R}}\mathbf{U}\mathbf{F}^{-1} + \mathbf{R}\dot{\mathbf{U}}\mathbf{F}^{-1}.$$

Substituting $\mathbf{F}^{-1} = \mathbf{U}^{-1}\mathbf{R}^T$ into this leads to

$$\mathbf{L} = \dot{\mathbf{R}}\mathbf{R}^T + \mathbf{R}\dot{\mathbf{U}}\mathbf{U}^{-1}\mathbf{R}^T.$$

Since $\mathbf{R}$ is orthogonal, $\mathbf{R}\mathbf{R}^T = \mathbf{I}$ which when differentiated leads to $\dot{\mathbf{R}}\mathbf{R}^T = -\mathbf{R}\dot{\mathbf{R}}^T$ (i.e. $\dot{\mathbf{R}}\mathbf{R}^T$ is skew symmetric). Using this together with the preceding equation for $\mathbf{L}$ and $\mathbf{D} = (\mathbf{L} + \mathbf{L}^T)/2$, $\mathbf{W} = (\mathbf{L} - \mathbf{L}^T)/2$ leads to

$$\left.\begin{aligned} \mathbf{D} &= \tfrac{1}{2}\mathbf{R}\big(\dot{\mathbf{U}}\mathbf{U}^{-1} \ + \ \mathbf{U}^{-1}\dot{\mathbf{U}}\big)\mathbf{R}^T, \\ \mathbf{W} &= \dot{\mathbf{R}}\mathbf{R}^T \ + \tfrac{1}{2}\mathbf{R}\big(\dot{\mathbf{U}}\mathbf{U}^{-1} \ - \ \mathbf{U}^{-1}\dot{\mathbf{U}}\big)\mathbf{R}^T. \end{aligned}\right\}$$

Remark: Note that if the reference configuration happens to coincide with the current configuration then $\mathbf{F} = \mathbf{I}$ and so $\mathbf{R} = \mathbf{U} = \mathbf{I}$, whence the preceding equations specialize to

$$\mathbf{D} = \dot{\mathbf{U}}, \qquad \mathbf{W} = \dot{\mathbf{R}}.$$

Problem 3.5. Let $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ be the right Cauchy-Green tensor and let $\overset{(n)}{\mathbf{C}}$ be its nth material time derivative, $n = 1, 2, \ldots$. The *nth Rivlin-Ericksen tensor* $\mathbf{A}_n$ is defined by

$$\mathbf{A}_n = \mathbf{F}^{-T} \overset{(n)}{\mathbf{C}} \mathbf{F}^{-1}. \tag{3.74}$$

Show that

$$\mathbf{A}_1 = 2\mathbf{D} \qquad \text{and} \qquad \mathbf{A}_{n+1} = \dot{\mathbf{A}}_n + \mathbf{A}_n\mathbf{L} + \mathbf{L}^T\mathbf{A}_n. \tag{3.75}$$

<u>*Solution*</u>: Setting $n = 1$ in definition (3.74) gives

$$\mathbf{A}_1 = \mathbf{F}^{-T}\dot{\mathbf{C}}\mathbf{F}^{-1} = \mathbf{F}^{-T}(\mathbf{F}^T\mathbf{F})^{\cdot}\,\mathbf{F}^{-1} = \mathbf{F}^{-T}(\dot{\mathbf{F}}^T\mathbf{F} + \mathbf{F}^T\dot{\mathbf{F}})\mathbf{F}^{-1}.$$

Since $\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}$ we have

$$\dot{\mathbf{F}}^T\mathbf{F} + \mathbf{F}^T\dot{\mathbf{F}} = (\mathbf{L}\mathbf{F})^T\mathbf{F} + \mathbf{F}^T\mathbf{L}\mathbf{F} = \mathbf{F}^T\mathbf{L}^T\mathbf{F} + \mathbf{F}^T\mathbf{L}\mathbf{F} = \mathbf{F}^T(\mathbf{L} + \mathbf{L}^T)\mathbf{F} = 2\mathbf{F}^T\mathbf{D}\mathbf{F}.$$

Combining the preceding two equations leads to

$$\mathbf{A}_1 = 2\mathbf{D}$$

which establishes $(3.75)_1$.

Next, again from definition (3.74) and $\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}$ we have

$$\begin{aligned}
\mathbf{A}_{n+1} &= \mathbf{F}^{-T} \overset{(n+1)}{\mathbf{C}} \mathbf{F}^{-1} = \mathbf{F}^{-T}\big(\overset{(n)}{\mathbf{C}} \big)^{\cdot}\mathbf{F}^{-1} = \mathbf{F}^{-T}(\mathbf{F}^T\mathbf{A}_n\mathbf{F})^{\cdot}\,\mathbf{F}^{-1} \\
&= \mathbf{F}^{-T}\dot{\mathbf{F}}^T\mathbf{A}_n\mathbf{F}\mathbf{F}^{-1} + \mathbf{F}^{-T}\mathbf{F}^T\dot{\mathbf{A}}_n\mathbf{F}\mathbf{F}^{-1} + \mathbf{F}^{-T}\mathbf{F}^T\mathbf{A}_n\dot{\mathbf{F}}\mathbf{F}^{-1} \\
&= \mathbf{F}^{-T}\mathbf{F}^T\mathbf{L}^T\mathbf{A}_n\mathbf{F}\mathbf{F}^{-1} + \dot{\mathbf{A}}_n + \mathbf{F}^{-T}\mathbf{F}^T\mathbf{A}_n\mathbf{L}\mathbf{F}\mathbf{F}^{-1} \\
&= \mathbf{L}^T\mathbf{A}_n + \dot{\mathbf{A}}_n + \mathbf{A}_n\mathbf{L}
\end{aligned}$$

which establishes $(3.75)_2$.

Problem 3.6. Show that a motion is rigid if and only if the stretching tensor $\mathbf{D}$ vanishes at all points and all times.

<u>*Solution*</u>: In term of components in a fixed basis, we are given that $D_{ij}(\mathbf{y}, t) = \frac{1}{2}(\partial v_i/\partial y_j + \partial v_j/\partial y_i) = 0$ at all $\mathbf{y} \in \mathcal{R}_t$. Thus

$$\frac{\partial v_i}{\partial y_j} = -\frac{\partial v_j}{\partial y_i} \tag{a}$$

It follows from this that

$$\frac{\partial^2 v_i}{\partial y_j \partial y_k} = \frac{\partial^2 v_i}{\partial y_k \partial y_j} = -\frac{\partial^2 v_j}{\partial y_k \partial y_i} = -\frac{\partial^2 v_j}{\partial y_i \partial y_k} = \frac{\partial^2 v_k}{\partial y_i \partial y_j} = \frac{\partial^2 v_k}{\partial y_j \partial y_i} = -\frac{\partial^2 v_i}{\partial y_j \partial y_k};$$

here, in the first, third and fifth steps we have simply changed the order of differentiation while in the second, fourth and sixth steps we have used equation (a). Thus,

$$\frac{\partial^2 v_i}{\partial y_j \partial y_k} = 0 \qquad \text{at all } \mathbf{y} \in \mathcal{R}_t.$$

Integrating once leads to

$$\frac{\partial v_i}{\partial y_k} = C_{ik}(t) \qquad \text{at all } \mathbf{y} \in \mathcal{R}_t, \tag{b}$$

while integrating again leads to

$$v_i(\mathbf{y}, t) = \sum_{k=1}^{3} C_{ik}(t) y_k \ + \ c_i(t)$$

or equivalently

$$\mathbf{v}(\mathbf{y}, t) = \mathbf{C}(t)\mathbf{y} + \mathbf{c}(t) \qquad \text{at all } \mathbf{y} \in \mathcal{R}_t. \tag{c}$$

Observe from (a) and (b) that $\mathbf{C}(t)$ is a skew symmetric tensor. Thus by comparing (c) with (3.10) we conclude that if $\mathbf{D}(\mathbf{y}, t) = \mathbf{0}$ then the motion is a rigid body motion.

The converse can be verified readily by substituting (c) into the formula $\mathbf{D} = \frac{1}{2}(\text{grad}\, \mathbf{v} + (\text{grad}\, \mathbf{v})^T)$.

Problem 3.7. Consider a simple shearing motion

$$y_1 = x_1 + k(t) x_2, \qquad y_2 = x_2, \qquad y_3 = x_3,$$

where the components have been taken with respect to a fixed orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Let $\mathbf{E} = \ln \mathbf{U}$ be the Lagrangian logarithmic strain tensor. Calculate the components of the following "strain-rate tensors" with respect to this same basis:

(a) the (material time derivative) strain-rate tensor $\dot{\mathbf{E}}$,

(a) the co-rotational strain-rate tensor $\overset{\circ}{\mathbf{E}} \overset{\text{def}}{=} \dot{\mathbf{E}} - \mathbf{W}\mathbf{E} + \mathbf{E}\mathbf{W}$;

here $\mathbf{W}$ is the spin tensor. The co-rotational rate is also called the Jaumann rate. It represents the rate of change relative to a basis rotating with the local body spin $\mathbf{W}$.

Problem 3.8. Show that the two (vector) kinematic jump conditions at a singular surface, (6.25) and (6.30), are equivalent to the one (tensor) jump condition

$$[\![\mathbf{v}]\!] \otimes \mathbf{n}_0 \ + \ [\![\mathbf{F}]\!]\, V_0 = \mathbf{o} \tag{3.76}$$

provided $V_0 \neq 0$.

<u>*Solution*</u>: First suppose that (3.76) holds. Operating both sides of it on the unit normal $\mathbf{n}_0$ leads to

$$[\![\mathbf{v}]\!] \ + \ [\![\mathbf{F}\mathbf{n}_0]\!]\, V_0 = \mathbf{o}, \tag{6.30}$$

because $([\![\mathbf{v}]\!] \otimes \mathbf{n}_0)\mathbf{n}_0 = \mathbf{v}(\mathbf{n}_0 \cdot \mathbf{n}_0) = \mathbf{v}$. Operating both sides of (3.76) on any unit vector ℓ that is tangent to $\mathcal{S}_0(t)$ gives

$$[\![\ \mathbf{F}\ell]\!] = \mathbf{o} \tag{6.25}$$

provided $V_0 \neq 0$ because $([\![\mathbf{v}]\!] \otimes \mathbf{n}_0)\boldsymbol{\ell} = \mathbf{v}(\mathbf{n}_0 \cdot \boldsymbol{\ell}) = \mathbf{0}$. Thus if (3.76) holds then so do (6.25) and (6.30).

Conversely, suppose that (6.25) and (6.30) hold. For an arbitrary vector $\mathbf{g}$ we have the following sequence of results:

$$V_0[\![\mathbf{F}]\!]\mathbf{g} = V_0[\![\mathbf{F}]\!](\mathbf{n}_0 \cdot \mathbf{g})\mathbf{n}_0 = -[\![\mathbf{v}]\!](\mathbf{n}_0 \cdot \mathbf{g}) = -([\![\mathbf{v}]\!] \otimes \mathbf{n}_0)\mathbf{g};$$

in the first step we have used the fact that by (6.25), when $[\![\mathbf{F}]\!]$ operates on the component of $\mathbf{g}$ that is tangential to the surface one gets the null vector and so $[\![\mathbf{F}]\!]$ operating on $\mathbf{g}$ is identical to $[\![\mathbf{F}]\!]$ operating on the component of $\mathbf{g}$ that is normal to $\mathcal{S}_0$; in the second step we have used (6.30); and in the third step we have used the vector identity $(\mathbf{a} \otimes \mathbf{b})\mathbf{c} = (\mathbf{c} \cdot \mathbf{b})\mathbf{a}$. Therefore

$$\Big(V_0[\![\mathbf{F}]\!] + ([\![\mathbf{v}]\!] \otimes \mathbf{n}_0)\Big)\mathbf{g} = \mathbf{0}$$

for any vector $\mathbf{g}$ and so (3.76) holds.

Problem 3.9. In this problem we study the motion of a propagating surface in a body. In contrast to the singular surfaces studied in Section 6.1, here we assume that the motion $\widehat{\mathbf{y}}(\mathbf{x}, t)$ is smooth everywhere and at all times. We leave it as an exercise to the reader to modify the analysis in the present problem to the class of singular surfaces studied previously and to compare the results thus obtained with those in Section 6.1.

Let $\mathcal{S}_t$ be a surface defined by

$$\mathcal{S}_t: \ \varphi(\mathbf{y}, t) = 0$$

that is contained within the region $\mathcal{R}_t$ occupied by a body at the current instant. Its pre-image $\mathcal{S}_0(t)$ in the reference configuration is the surface, contained within the region $\mathcal{R}_0$, defined by

$$\mathcal{S}_0(t): \ \widehat{\varphi}(\mathbf{x}, t) = 0$$

where $\widehat{\varphi}(\mathbf{x}, t) = \varphi(\widehat{\mathbf{y}}(\mathbf{x}, t), t)$ and $\mathbf{y} = \widehat{\mathbf{y}}(\mathbf{x}, t)$ is the motion of the body assumed to be smooth. Note that at each instant t, the particles p associated with $\mathcal{S}_0(t)$ and $\mathcal{S}_t$ are identical because $\mathbf{x}$ and $\mathbf{y}$ are related through the motion $\mathbf{y} = \widehat{\mathbf{y}}(\mathbf{x}, t)$.

As time progresses, the surface $\mathcal{S}_t$ propagates within the region $\mathcal{R}_t$ and in general, different particles p are associated with it at different times. The surface $\mathcal{S}_0(t)$ will propagate correspondingly within the region $\mathcal{R}_0$.

Determine

(i) the conditions under which this surface is a material surface, and

(ii) the relation between the speeds V and V_0 with which the respective surfaces $\mathcal{S}_t$ and $\mathcal{S}_0(t)$ propagate in directions normal to themselves.

Solution:

(i) In the special case of a *material surface*, the same set of particles is associated with the surface $\mathcal{S}_t$ at all times, and therefore this must be true of $\mathcal{S}_0(t)$ as well. Thus $\mathcal{S}_0(t)$ must be time independent and remain stationary in $\mathcal{R}_0$. Two examples of material surfaces are (i) the boundary of the body and (ii) an interface between two bodies that are joined together at that interface. Thus for a material surface, $\widehat{\varphi}(\mathbf{x}, t)$ is independent of t:

$$\widehat{\varphi}(\mathbf{x}) = \varphi(\widehat{\mathbf{y}}(\mathbf{x}, t), t).$$

Differentiating this with respect to time at fixed $\mathbf{x}$ gives

$$\dot{\varphi} = \mathbf{v} \cdot \text{grad}\ \varphi + \partial\varphi/\partial t = 0,$$

i.e. if $\varphi(\mathbf{y}, t) = 0$ is a material surface then its material time derivative must vanish. This is necessary and sufficient for $\varphi = 0$ to be a material surface.

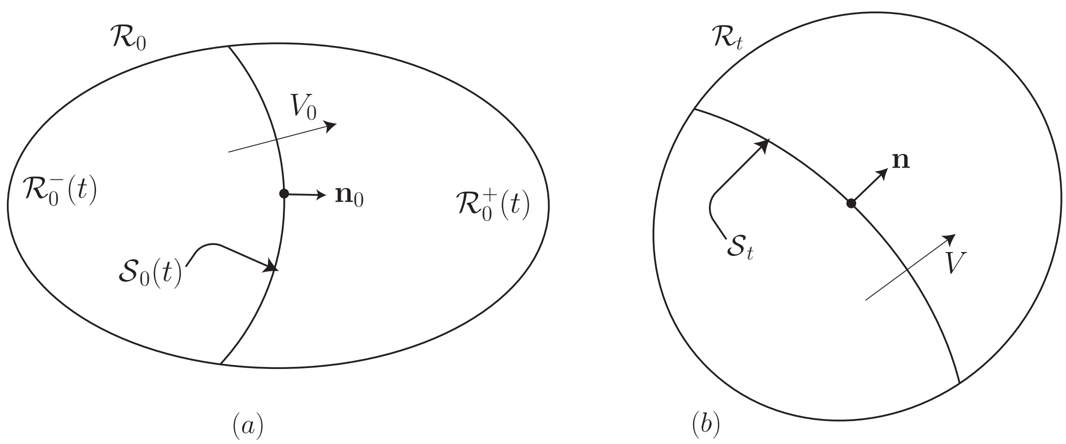

Figure 3.5: (a) The region $\mathcal{R}_0$ and a surface $\mathcal{S}_0(t) : \widehat{\varphi}(\mathbf{x}, t) = 0$ in the reference configuration, and (b) the corresponding region $\mathcal{R}_t(t)$ and surface $\mathcal{S}_t : \varphi(\mathbf{y}, t) = 0$ in the current configuration. The velocity of a point on $\mathcal{S}_t$ in the direction of the normal $\mathbf{n}$ is V. Likewise, the velocity of a point on $\mathcal{S}_0(t)$ in the direction of the normal $\mathbf{n}_0$ is V_0.

(ii) Note that for a non-material surface, the motion of $\mathcal{S}_t$ is not necessarily related to the motion of the body. The propagation speed of the surface described by $\varphi(\mathbf{y}, t) = 0$, in the direction of the unit normal vector

$$\mathbf{n} = \frac{\text{grad}\,\varphi}{|\text{grad}\,\varphi|}, \tag{3.77}$$

is[3]

$$V = -\frac{\varphi'}{|\text{grad}\,\varphi|} \tag{3.78}$$

[3]In the particular case when $\mathcal{S}_t$ is a *material surface*, since $\mathcal{S}_t$ is attached to the same set of particles

where $\varphi' = \partial\varphi(\mathbf{y},t)/\partial t$. Similarly the propagation speed of the surface $\widehat{\varphi}(\mathbf{x},t) = 0$ in the direction of its unit normal vector

$$\mathbf{n}_0 = \frac{\operatorname{Grad}\varphi}{|\operatorname{Grad}\varphi|}, \tag{3.79}$$

is

$$V_0 = -\frac{\dot{\varphi}}{|\operatorname{Grad}\varphi|}, \tag{3.80}$$

where $\dot{\varphi} = \partial\widehat{\varphi}(\mathbf{x},t)/\partial t$. However the surfaces $\mathcal{S}_t$ and $\mathcal{S}_0(t)$ are related through the motion by $\widehat{\varphi}(\mathbf{x},t) = \varphi(\widehat{\mathbf{y}}(\mathbf{x},t),t)$, and it follows from this that

$$\operatorname{Grad}\varphi = \mathbf{F}^T\Big(\operatorname{grad}\varphi\Big), \qquad \dot{\varphi} = \operatorname{grad}\varphi\cdot\mathbf{v} + \varphi'. \tag{3.81}$$

Therefore from (3.77) - (3.81) we find the following relations between the unit normals $\mathbf{n}_0$ and $\mathbf{n}$, and the speeds V_0 and V:

$$\mathbf{n}_0 = \frac{|\operatorname{grad}\varphi|}{|\operatorname{Grad}\varphi|}\,\mathbf{F}^T\mathbf{n}, \qquad V_0 = \frac{|\operatorname{grad}\varphi|}{|\operatorname{Grad}\varphi|}\,(V - \mathbf{v}\cdot\mathbf{n}). \tag{3.82}$$

Problem 3.10. In a certain motion of a body the evolving closed surface

$$\varphi(y_1,y_2,y_3,t) = y_1^2 + (1+t^2)y_2^2 + y_3^2 - 2ty_1y_2 - 1 = 0$$

is a *material surface*, i.e. even though this surface changes with time the *particles* that lie on it are the same at all times. Determine an example of a motion for which this would be true.

<u>*Solution*</u>: The surface of interest is defined by

$$\varphi = y_1^2 + (1+t^2)y_2^2 + y_3^2 - 2ty_1y_2 - 1 = 0. \tag{a}$$

Since it is a material surface, we must have $\dot{\varphi} = 0$: i.e.

$$\dot{\varphi} = -2(y_1 - ty_2)y_2 \;+\; 2(y_1 - ty_2)v_1 + \big[2(1+t^2)y_2 - 2ty_1\big]v_2 + 2y_3v_3 = 0 \tag{b}$$

where

$$\dot{y_1} = v_1, \qquad \dot{y_2} = v_2, \qquad \dot{y_3} = v_3. \tag{c}$$

If the reference configuration coincides with the initial configuration then $\hat{\mathbf{y}}(\mathbf{x},0) = \mathbf{x}$, i.e.

$$\hat{y}_1(x_1,x_2,x_3,0) = x_1, \qquad \hat{y}_2(x_1,x_2,x_3,0) = x_2, \qquad \hat{y}_3(x_1,x_2,x_3,0) = x_3. \tag{d}$$

Equations $(a) - (d)$ do not determine the motion $y_i = \hat{y}_i(x_1,x_2,x_3,t)$ uniquely but do impose restrictions on it.

We now find a particular motion that is consistent with $(a) - (d)$. Suppose that the velocity field is unidirectional so that $v_2 = v_3 = 0$ throughout the body at all times. Using this and integrating $(c)_{2,3}$ gives

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p at all times, the propagation speed of the surface $\mathcal{S}_t$ at $(\mathbf{y},t)$ equals the particle velocity at $(\mathbf{y},t)$ in the $\mathbf{n}$-direction. Thus the propagation speed of a material surface obeys $V = \mathbf{v}\cdot\mathbf{n}$.

$\hat{y}_2(x_1,x_2,x_3,t) = \alpha(x_1,x_2,x_3)$, $\hat{y}_3(x_1,x_2,x_3,t) = \beta(x_1,x_2,x_3)$; in view of $(d)_{2,3}$, this in turn requires that $\alpha = x_2$, $\beta = x_3$. Thus we have

$$\begin{aligned}\hat{y}_2(x_1,x_2,x_3,t) &= x_2,\\ \hat{y}_3(x_1,x_2,x_3,t) &= x_3.\end{aligned} \qquad (e)$$

Since $v_2 = v_3 = 0$, equation (b) simplifies to $v_1 = y_2$ which in turn reduces to $\dot{y}_1 = x_2$ in view of $(e)_1$. Integrating this leads to $\hat{y}_1(x_1,x_2,x_3,t) = tx_2 + \gamma(x_1,x_2,x_3)$, which in view of $(d)_1$ yields $\gamma = x_1$. Thus

$$\hat{y}_1(x_1,x_2,x_3,t) = tx_2 + x_1. \qquad (f)$$

The motion of the body is thus described by $(e), (f)$ which is seen to be a simple shear.

Remark: When the reference configuration coincides with the initial configuration, the pre-image of the surface $\varphi(\mathbf{y},t) = 0$ is found by setting $t = 0$ and $\mathbf{y} = \mathbf{x}$: $\varphi(\mathbf{x},0) = 0$. This shows that the pre-image of the particular surface described by (a) is the sphere $x_1^2 + x_2^2 + x_3^2 = 1$.

Problem 3.11. Find necessary and sufficient conditions that ensure that a curve is a material curve.

Solution: See Section 75 of Truesdell and Toupin.

Problem 3.12. Show that the left stretch tensor $\mathbf{V}_t$ in the polar decomposition $\mathbf{F}_t = \mathbf{V}_t\mathbf{R}_t$ obeys the relationship

$$\mathbf{D}(\mathbf{y},t) = \left.\frac{\partial}{\partial\tau}\mathbf{V}_t(\mathbf{y},\tau)\right|_{\tau=t},$$

where we have used the notation and setting introduced in Section 3.5.This shows that, at the current instant t, the time rate of change of the right stretch tensor with respect to the current configuration, equals the stretching and spin tensors, cf. equation (3.73).

Problem 3.13. Show that the nth Rivlin-Ericksen tensor $\mathbf{A}_n$ defined in Problem 3.5 obeys

$$\mathbf{A}_n(\mathbf{y},t) = \left.\frac{\partial^n}{\partial\tau^n}\mathbf{C}_t(\mathbf{y},\tau)\right|_{\tau=t} \qquad (3.83)$$

where we have used the notation and setting introduced in Section 3.5.

Solution: A straightforward calculation using the Cauchy Green tensors $\mathbf{C}(\mathbf{x},\tau) = \mathbf{F}^T(\mathbf{x},\tau)\mathbf{F}(\mathbf{x},\tau)$, $\mathbf{C}_t(\mathbf{y},\tau) = \mathbf{F}_t^T(\mathbf{y},\tau)\mathbf{F}_t(\mathbf{y},\tau)$ and equation (3.67) shows that

$$\mathbf{C}(\mathbf{x},\tau) = \mathbf{F}^T(\mathbf{x},\tau)\mathbf{F}(\mathbf{x},\tau) = \mathbf{F}^T(\mathbf{x},t)\mathbf{F}_t^T(\mathbf{y},\tau)\mathbf{F}_t(\mathbf{y},\tau)\mathbf{F}(\mathbf{x},t) = \mathbf{F}^T(\mathbf{x},t)\mathbf{C}_t(\mathbf{y},\tau)\mathbf{F}(\mathbf{x},t).$$

Differentiating $\mathbf{C}(\mathbf{x},\tau) = \mathbf{F}^T(\mathbf{x},t)\mathbf{C}_t(\mathbf{y},\tau)\mathbf{F}(\mathbf{x},t)$ n times with respect to τ leads to

$$\overset{(n)}{\mathbf{C}}(\mathbf{x},\tau) = \mathbf{F}^T(\mathbf{x},t)\frac{\partial^n}{\partial\tau^n}\mathbf{C}_t(\mathbf{y},\tau)\mathbf{F}(\mathbf{x},t).$$

On setting $\tau = t$ in here we get

$$\overset{(n)}{\mathbf{C}}(\mathbf{x}, t) = \mathbf{F}^T(\mathbf{x}, t)\, \frac{\partial^n}{\partial \tau^n}\, \mathbf{C}_t(\mathbf{y}, \tau)\bigg|_{\tau = t} \mathbf{F}(\mathbf{x}, t).$$

Combining this with the definition of the nth Rivlin Ericksen tensor given in Problem 3.5 leads to (3.83).

Problem 3.14. Consider the particular motion of a continuum where the deformation gradient tensor at a generic particle is given by

$$\mathbf{F}(t) = \mathbf{Q}(t)\, \mathrm{e}^{tk\mathbf{N}_0}$$

where $\mathbf{Q}(t)$ is orthogonal at each t, the scalar k is constant, and the constant tensor $\mathbf{N}_0$ has the properties

$$|\mathbf{N}_0| = 1, \qquad \mathrm{tr}\, \mathbf{N}_0 = 0.$$

Using the properties of the exponential tensor from Section 3.6 of Volume I, or otherwise, show that $\det \mathbf{F}(t) = 1$ and that the eigenvalues of $\mathbf{C}_t(\tau) = \mathbf{F}_t^T(\tau)\mathbf{F}_t(\tau)$, i.e. the principal relative stretch tensor, are constant during this motion.

Remark: Consider the particular case where $\mathbf{N}_0 = \mathbf{e}_1 \otimes \mathbf{e}_2$ for an arbitrary pair of mutually orthogonal unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$. Observe that $\mathbf{N}_0^2 = \mathbf{0}$ for this $\mathbf{N}_0$, from which follows that $\mathbf{N}_0^n = \mathbf{0}$ for $n = 2, 3, \ldots$. The series representation of the exponential tensor then simplifies to

$$\mathrm{e}^{tk\mathbf{N}_0} = \sum_{n=0}^{\infty} \frac{t^n k^n}{n!}\, \mathbf{N}_0^n = \mathbf{I} + kt\mathbf{N}_0$$

and therefore the given deformation gradient tensor specializes to $\mathbf{F}(t) = \mathbf{Q}(t)\big(\mathbf{I} + kt\, \mathbf{e}_1 \otimes \mathbf{e}_2\big)$. This describes a simple shear followed by a rotation.

3.7 Transport Equations.

Let $\beta(p, t)$ be some scalar property, say the internal energy density, at the particle p at time t during some motion. Then the total internal energy of a part $\mathcal{P}$ is given by the integral of β over the part $\mathcal{P}$. If we wish to calculate the time rate of change of the internal energy associated with part $\mathcal{P}$ during the motion we must evaluate

$$\frac{d}{dt} \int_{\mathcal{P}} \beta(p, t)\, dp.$$

Though this is conceptually easy to state, a particle, a body and a part of a body are abstract notions (see Chapter 1) and so we cannot get very far with evaluating the above expression.

Therefore we transform $p \to \mathbf{y}$ and $\mathcal{P} \to \mathcal{D}_t$ using the motion $\mathbf{y} = \chi(p,t)$ in which case the preceding integral can be written in the equivalent form

$$\frac{d}{dt}\int_{\mathcal{D}_t} \beta(\mathbf{y},t)dV$$

where $\beta(\mathbf{y},t)$ is the internal energy per unit current volume at the particle that is located at $\mathbf{y}$ at time t, and $\mathcal{D}_t$ is the region occupied by $\mathcal{P}$ at time t, during this motion. Recall that by definition, a part $\mathcal{P}$ consists of a particular fixed set of particles. The particles associated with $\mathcal{P}$ do not change with time, and so we are studying the internal energy of the same set of particles during the motion. The region $\mathcal{D}_t$ that $\mathcal{P}$ occupies however does vary with time. In this section we derive formulae for evaluating terms such as the preceding one where the only challenge is that we want to calculate the time derivative of an integral that is taken over a region that changes with time.

Let the fields $\beta(\mathbf{y},t)$ and $\mathbf{b}(\mathbf{y},t)$ characterize a scalar and vector property associated with a motion of the body. For example $\mathbf{b}(\mathbf{y},t)$ might be the linear momentum density (i.e. linear momentum per unit volume) of the particle located at $\mathbf{y}$ at time t. Suppose that $\mathcal{D}_t$ is a material subregion of $\mathcal{R}_t$. Then one can show that

$$\left.\begin{aligned} \frac{d}{dt}\int_{\mathcal{D}_t} \beta\, dV_y &= \int_{\mathcal{D}_t} \left(\dot{\beta} + \beta \operatorname{div}\mathbf{v}\right) dV_y, \\ \frac{d}{dt}\int_{\mathcal{D}_t} \mathbf{b}\, dV_y &= \int_{\mathcal{D}_t} \left(\dot{\mathbf{b}} + \mathbf{b} \operatorname{div}\mathbf{v}\right) dV_y; \end{aligned}\right\} \tag{3.84}$$

if $\mathcal{S}_t$ is a material surface[4] in $\mathcal{R}_t$, one can show that

$$\left.\begin{aligned} \frac{d}{dt}\int_{\mathcal{S}_t} \beta\mathbf{n}\, dA_y &= \int_{\mathcal{S}_t} \left[\left(\dot{\beta} + \beta \operatorname{div}\mathbf{v}\right)\mathbf{n} - \beta\mathbf{L}^T\mathbf{n}\right] dA_y, \\ \frac{d}{dt}\int_{\mathcal{S}_t} \mathbf{b}\cdot\mathbf{n}\, dA_y &= \int_{\mathcal{S}_t} \left(\dot{\mathbf{b}} + \mathbf{b} \operatorname{div}\mathbf{v} - \mathbf{L}\mathbf{b}\right)\cdot\mathbf{n}\, dA_y; \end{aligned}\right\} \tag{3.85}$$

and if $\mathcal{C}_t$ is a material curve in $\mathcal{R}_t$, one can show that

$$\left.\begin{aligned} \frac{d}{dt}\int_{\mathcal{C}_t} \beta\, d\mathbf{y} &= \int_{\mathcal{C}_t} \left(\dot{\beta}\, d\mathbf{y} + \beta\, \mathbf{L}d\mathbf{y}\right), \\ \frac{d}{dt}\int_{\mathcal{C}_t} \mathbf{b}\cdot d\mathbf{y} &= \int_{\mathcal{C}_t} \left(\dot{\mathbf{b}} + \mathbf{L}^T\mathbf{b}\right)\cdot d\mathbf{y}. \end{aligned}\right\} \tag{3.86}$$

In order to establish these relations one first transforms the time dependent domains $\mathcal{D}_t$, $\mathcal{S}_t$ or $\mathcal{C}_t$ on the left hand sides of these equations to their respective (time independent) images

[4]Recall from Section 1.7 that the terms material region, material surface and material curve refer to a region $\mathcal{D}_t$, surface $\mathcal{S}_t$ and curve $\mathcal{C}_t$ which consist of the same particles of the body at all times.

in the reference configuration $\mathcal{D}_0$, $\mathcal{S}_0$ or $\mathcal{C}_0$ by using $dV_y = JdV_x$, $\mathbf{n}\, dA_y = J\mathbf{F}^{-T}\mathbf{n}_0\, dA_x$ and $d\mathbf{y} = \mathbf{F}d\mathbf{x}$ respectively. The pre-images $\mathcal{D}_0$, $\mathcal{S}_0$ and $\mathcal{C}_0$ are time independent because we are considering a fixed set of particles. Since the integrals are now taken over time independent domains, the time derivatives that are outside the integrals can now be taken inside. After simplification, one uses the relations $dV_y = JdV_x$ etc. in reverse to now transform the referential domains back to the current domains thus leading to the desired results. For example, in order to establish $(3.84)_2$ we proceed as follows:

$$\begin{aligned}
\frac{d}{dt}\int_{\mathcal{D}_t} \mathbf{b}(\mathbf{y},t)\ dV_y &= \frac{d}{dt}\int_{\mathcal{D}_0} \mathbf{b}(\mathbf{x},t)\ J(\mathbf{x},t)\, dV_x \\
&= \int_{\mathcal{D}_0} \frac{\partial}{\partial t}\Big(\mathbf{b}(\mathbf{x},t)\, J(\mathbf{x},t)\Big)\ dV_x \\
&= \int_{\mathcal{D}_0} \Big(\dot{\mathbf{b}}\, J + \mathbf{b}\dot{J}\Big)\ dV_x \\
&= \int_{\mathcal{D}_0} \Big(\dot{\mathbf{b}}\, J + \mathbf{b}\, J\, \mathrm{div}\, \mathbf{v}\Big)\ \mathrm{dV_x} \\
&= \int_{\mathcal{D}_t} \Big(\dot{\mathbf{b}} + \mathbf{b}\ \mathrm{div}\, \mathbf{v}\Big)\, dV_y,
\end{aligned} \tag{3.87}$$

where in the first step we have transformed $\mathcal{D}_t \to \mathcal{D}_0$ while in the last step we have transformed $\mathcal{D}_0 \to \mathcal{D}_t$ both by using $dV_y = JdV_x$. In the preceding calculation we have used the formulae $\dot{\mathbf{F}} = \mathbf{LF}$ and $\dot{J} = J\, \mathrm{trace}\mathbf{L} = \mathrm{J}\, \mathrm{trace}(\dot{\mathbf{F}}\mathbf{F}^{-1}) = \mathrm{J}\, \mathrm{div}\, \mathbf{v}$.

Similarly, to establish $(3.85)_1$ we proceed as follows:

$$\begin{aligned}
\frac{d}{dt}\int_{\mathcal{S}_t} \beta(\mathbf{y},t)\ \mathbf{n}\ dA_y &= \frac{d}{dt}\int_{\mathcal{S}_0} \beta(\mathbf{x},t)\ J\mathbf{F}^{-T}\mathbf{n}_0\ dA_x \\
&= \int_{\mathcal{S}_0} \frac{\partial}{\partial t}\Big(\beta\, J\mathbf{F}^{-T}\Big)\, \mathbf{n}_0\ dA_x \\
&= \int_{\mathcal{S}_0} \Big(\dot{\beta}\ J\mathbf{F}^{-T}\ +\ \beta\ \dot{J}\mathbf{F}^{-T}\ +\ \beta\ J(\mathbf{F}^{-T})^{\cdot}\ \Big)\, \mathbf{n}_0\ dA_x \\
&= \int_{\mathcal{S}_0} \Big(\dot{\beta}\, J\mathbf{F}^{-T}\ +\ \beta\, J\,(\mathrm{div}\, \mathbf{v})\, \mathbf{F}^{-T}\ +\ \beta\, J\big(-\mathbf{F}^{-T}(\dot{\mathbf{F}})^T\mathbf{F}^{-T}\big)\ \Big)\, \mathbf{n}_0\ dA_x \\
&= \int_{\mathcal{S}_t} \Big[\Big(\dot{\beta} + \beta\, \mathrm{div}\, \mathbf{v}\Big)\, \mathbf{I} - \beta\mathbf{L}^T\Big]\ \mathbf{n}\ dA_y,
\end{aligned} \tag{3.88}$$

where we have used $\dot{\mathbf{F}} = \mathbf{LF}$, $\dot{J} = J\, \mathrm{div}\, \mathbf{v}$, $\mathbf{n}\, dA_y = J\mathbf{F}^{-T}\mathbf{n}_0\, dA_x$, as well as the formula $(\mathbf{F}^{-T})^{\cdot} = -\mathbf{F}^{-T}\, (\dot{\mathbf{F}})^T\, \mathbf{F}^{-T}$ which results from differentiating $\mathbf{F}^{-T}\mathbf{F}^T = \mathbf{I}$ with respect to time.

The remaining transport equations can be established similarly.

The transport formulae (3.84) can be written in the alternative useful forms

$$\left.\begin{aligned} \frac{d}{dt}\int_{\mathcal{D}_t} \beta \, dV_y &= \int_{\mathcal{D}_t} \frac{\partial \beta}{\partial t}\, dV_y + \int_{\partial \mathcal{D}_t} \beta \, (\mathbf{v}\cdot\mathbf{n})\, dA_y, \\ \frac{d}{dt}\int_{\mathcal{D}_t} \mathbf{b} \, dV_y &= \int_{\mathcal{D}_t} \frac{\partial \mathbf{b}}{\partial t} dV_y + \int_{\partial \mathcal{D}_t} \mathbf{b} \, (\mathbf{v}\cdot\mathbf{n})\, dA_y; \end{aligned}\right\} \tag{3.89}$$

here $\mathbf{n}$ is a unit outward normal on the boundary $\partial\mathcal{D}_t$ and therefore $\mathbf{v}\cdot\mathbf{n}$ is the outward normal particle speed. Note that the last terms in each equation above can be interpreted as the flux of the quantity β or $\mathbf{b}$ at the surface $\partial\mathcal{D}_t$.

3.8 Change of Observer. Objective Physical Quantities.

An *observer* $\mathcal{O}$ (or frame of reference) determines the place $\mathbf{y}$ and the instant t at which an *event* occurs. (One sometimes speaks of an observer as being a rigid body moving in 3-dimensional Euclidean space carrying a clock.) Only the ratios of distances (and not the distances themselves), and the ratios of time intervals (and not the intervals themselves), can be experienced by an observer.

Certain physical quantities must be independent of the observer and in this section we develop the framework necessary for studying this issue. Observer independence will play an indispensable role in our subsequent consideration of constitutive relationships where we shall require them to be independent of the observer.

Let $\mathcal{O}$ and $\mathcal{O}^*$ be two observers. Suppose that the same event is characterized by $(\mathbf{y}, t)$ and $(\mathbf{y}^*, t^*)$ by these two observers. The *two observers are said to be equivalent* if they agree on the distance between every two points, agree on orientation (i.e. right-handedness or left-handedness), agree on the time interval between every two instants, and agree on the sense of time (i.e. the order in which two instants occur). It is possible to show that these four requirements hold if and only if $(\mathbf{y}, t)$ and $(\mathbf{y}^*, t^*)$ are related by

$$\mathbf{y}^* = \mathbf{Q}(t)\mathbf{y} + \mathbf{c}(t), \qquad t^* = t + a, \tag{3.90}$$

where a is an arbitrary constant, $\mathbf{c}(t)$ is an arbitrary time-dependent vector, and $\mathbf{Q}(t)$ is an arbitrary time-dependent proper orthogonal tensor. Alternatively given one observer $\mathcal{O}$, we can view (3.90) as describing an *observer transformation* from $\mathcal{O}$ to an equivalent observer $\mathcal{O}^*$.

In general, the characterization of a physical quantity associated with a motion depends on the observer. It is important to distinguish between physical quantities that depend intrinsically on the observer and those that don't. A physical quantity is said to be *observer independent* – or *objective* (or *frame indifferent*) – if it is invariant under all observer transformations. The importance of whether a physical quantity is objective or not will become apparent when we consider constitutive relationships in what follows.

Specifically, suppose that a certain physical quantity is denoted by the scalar field $\beta(\mathbf{y}, t)$ by observer $\mathcal{O}$, and by $\beta^*(\mathbf{y}^*, t^*)$ by observer $\mathcal{O}^*$. Then we say that this scalar physical quantity is observer independent or objective if

$$\beta^*(\mathbf{y}^*, t^*) = \beta(\mathbf{y}, t) \tag{3.91}$$

for all equivalent observers $\mathcal{O}$, $\mathcal{O}^*$, i.e. when $\mathbf{y}, \mathbf{y}^*, t, t^*$ are related by (3.90). Next, consider a physical quantity that is denoted by the vector field $\mathbf{b}(\mathbf{y}, t)$ by observer $\mathcal{O}$, and by $\mathbf{b}^*(\mathbf{y}^*, t^*)$ by observer $\mathcal{O}^*$. Then we say that this vector physical quantity is objective if

$$\mathbf{b}^*(\mathbf{y}^*, t^*) = \mathbf{Q}(t)\mathbf{b}(\mathbf{y}, t) \tag{3.92}$$

for all equivalent observers $\mathcal{O}$, $\mathcal{O}^*$, i.e. when $\mathbf{y}, \mathbf{y}^*, t, t^*$ are related by (3.90). Finally, consider a physical quantity that is denoted by the tensor field $\mathbf{B}(\mathbf{y}, t)$ by observer $\mathcal{O}$, and by $\mathbf{B}^*(\mathbf{y}^*, t^*)$ by observer $\mathcal{O}^*$. Then we say that this tensor physical quantity is objective if

$$\mathbf{B}^*(\mathbf{y}^*, t^*) = \mathbf{Q}(t)\mathbf{B}(\mathbf{y}, t)\mathbf{Q}^T(t) \tag{3.93}$$

for all equivalent observers $\mathcal{O}$, $\mathcal{O}^*$, i.e. when $\mathbf{y}, \mathbf{y}^*, t, t^*$ are related by (3.90).

Physically, equation (3.92) ensures that the components associated with an objective vector quantity are unaffected by an observer transformation. To see this, consider an orthonormal basis $\mathrm{X} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and a second orthonormal basis $\mathrm{X}^* = \{\mathbf{Q}\mathbf{e}_1, \mathbf{Q}\mathbf{e}_2, \mathbf{Q}\mathbf{e}_3\}$. See Figure 3.6. Suppose that an observer-independent physical quantity is denoted by the vector $\mathbf{b}$ by observer $\mathcal{O}$. The components of $\mathbf{b}$ in the basis X are $b_i = \mathbf{b} \cdot \mathbf{e}_i$. If a second observer $\mathcal{O}^*$ represents this same physical quantity by the vector $\mathbf{b}^*$ then the components of $\mathbf{b}^*$ in the basis X^* are $b_i^* = \mathbf{b}^* \cdot \mathbf{Q}\mathbf{e}_i = \mathbf{Q}\mathbf{b} \cdot \mathbf{Q}\mathbf{e}_i = \mathbf{b} \cdot \mathbf{e}_i = b_i$, i.e. the components of $\mathbf{b}^*$ in the basis X^* are identical to the components of $\mathbf{b}$ in the basis X.

Similarly (3.93) ensures that the components associated with an objective tensor quantity are unaffected by an observer transformation. Suppose that an observer-independent physical quantity is denoted by the tensor $\mathbf{B}$ by observer $\mathcal{O}$. The components of $\mathbf{B}$ in the basis X

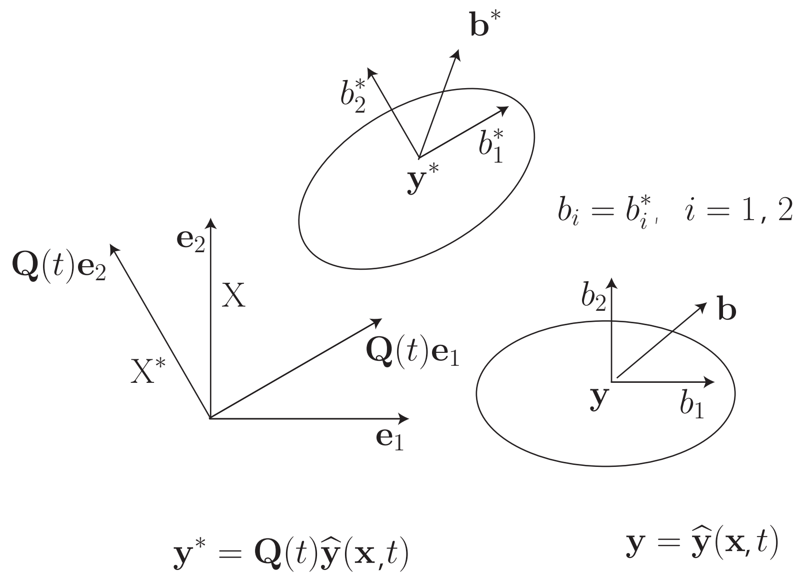

Figure 3.6: The figure is drawn for the special case $a = 0$, $\mathbf{c} = \mathbf{o}$. At the instant depicted, the two observers see the same particle $\mathbf{x}$ to be located at $\mathbf{y}$ and $\mathbf{y}^* = \mathbf{Q}\mathbf{y}$ respectively. Two bases $\mathsf{X} = \{\mathbf{e}_1, \mathbf{e}_2\}$ and $\mathsf{X}^* = \{\mathbf{Q}\mathbf{e}_1^*, \mathbf{Q}\mathbf{e}_2^*\}$ are shown. If $\mathbf{b}(\mathbf{y}, t)$ is an objective vector quantity then the components of $\mathbf{b}$ in X equals the components of $\mathbf{b}^*$ in X^*: $\quad b_i = b_i^*$.

are given by $B_{ij} = \mathbf{e}_i \cdot \mathbf{B}\mathbf{e}_j$. If a second observer $\mathcal{O}^*$ represents this same physical quantity by the tensor $\mathbf{B}^*$ then the components of $\mathbf{B}^*$ in the basis X* are $B^*_{ij} = \mathbf{Q}\mathbf{e}_i \cdot \mathbf{B}^*(\mathbf{Q}\mathbf{e}_j) = \mathbf{Q}\mathbf{e}_i \cdot (\mathbf{Q}\mathbf{B}\mathbf{Q}^T)(\mathbf{Q}\mathbf{e}_j) = \mathbf{e}_i \cdot \mathbf{B}\mathbf{e}_j = B_{ij}$, i.e. the components of $\mathbf{B}^*$ in the basis X* are identical to the components of $\mathbf{B}$ in the basis X.

Since kinematic quantities such as the deformation gradient tensor $\mathbf{F}$ and the stretching tensor $\mathbf{D}$ are purely geometric in nature, one can simply verify whether they are objective or not. On the other hand for mechanical or thermodynamic quantities such as stress and energy, one has to postulate whether they should be objective or not based on the underlying physics.

We end this section by examining whether the following kinematic quantities are objective: particle velocity $\mathbf{v}$, particle acceleration $\mathbf{a}$, the deformation gradient tensor $\mathbf{F}$, the right stretch tensor $\mathbf{U}$, the velocity gradient tensor $\mathbf{L}$, the spin tensor $\mathbf{W}$, the left stretch tensor $\mathbf{V}$, and the stretching tensor $\mathbf{D}$. We will find that the first six are not objective while the last two are objective.

To establish the preceding claims consider a motion described by

$$\mathbf{y} = \mathbf{y}(\mathbf{x}, t) \tag{3.94}$$

by observer $\mathcal{O}$. An equivalent observer $\mathcal{O}^*$ describes this motion as

$$\mathbf{y}^* = \mathbf{Q}(t^* - a)\mathbf{y}(\mathbf{x}, t^* - a) + \mathbf{c}(t^* - a) \stackrel{\text{def}}{=} \mathbf{y}^*(\mathbf{x}, t^*) \tag{3.95}$$

where we have used $\mathbf{y}^* = \mathbf{Q}\mathbf{y} + \mathbf{c}, t^* = t + a$. The particle velocity is given by

$$\mathbf{v}^*(\mathbf{y}^*, t^*) = \frac{\partial \mathbf{y}^*}{\partial t^*} = \dot{\mathbf{Q}}(t)\mathbf{y} + \mathbf{Q}(t)\mathbf{v}(\mathbf{y}, t) + \dot{\mathbf{c}}(t) \tag{3.96}$$

where $\mathbf{y}^* = \mathbf{Q}\mathbf{y} + \mathbf{c}, t^* = t + a$; because of the terms $\dot{\mathbf{Q}}$ and $\dot{\mathbf{c}}$ in this expression we see that in general $\mathbf{v}^* \neq \mathbf{Q}\mathbf{v}$ and so the particle velocity is not an objective quantity. Differentiating this again leads to

$$\mathbf{a}^*(\mathbf{y}^*, t^*) = \ddot{\mathbf{Q}}(t)\mathbf{y} + 2\dot{\mathbf{Q}}(t)\mathbf{v}(\mathbf{y}, t) + \mathbf{Q}(t)\mathbf{a}(\mathbf{y}, t) + \ddot{\mathbf{c}}(t) \tag{3.97}$$

which shows that in general $\mathbf{a}^* \neq \mathbf{Q}\mathbf{a}$ and so the particle acceleration is also not objective.

Observe that the expression (3.97) relating the acceleration vectors $\mathbf{a}^*$ and $\mathbf{a}$ *does* reduce to the objectivity relation $\mathbf{a}^* = \mathbf{Q}\mathbf{a}$ if (and only if) $\mathbf{Q}(t)$ is a constant orthogonal tensor and $\dot{\mathbf{c}}$ is a constant vector. In this particular case the two motions $\mathbf{y}(\mathbf{x}, t)$ and $\mathbf{y}^*(\mathbf{x}, t) = \mathbf{Q}\mathbf{y}(\mathbf{x}, t) + \mathbf{c}$ are said to be related by a *Galilean Transformation*. The acceleration is therefore objective under Galilean transformations. This is an important observation since the equations of motion (to follow in later chapters) involve the acceleration and are only valid relative to an inertial frame.

Next consider the deformation gradient tensor. From (3.95) we have

$$\mathbf{F}^*(\mathbf{x}, t^*) = \frac{\partial \mathbf{y}^*}{\partial \mathbf{x}} = \mathbf{Q}(t)\mathbf{F}(\mathbf{x}, t) \tag{3.98}$$

and so $\mathbf{F}^* \neq \mathbf{Q}\mathbf{F}\mathbf{Q}^T$. Thus the deformation gradient tensor is not objective. By the polar decomposition theorem we have $\mathbf{F}^* = \mathbf{R}^*\mathbf{U}^* = \mathbf{Q}\mathbf{F} = \mathbf{Q}\mathbf{R}\mathbf{U}$. Since the polar decomposition is unique it follows that necessarily $\mathbf{R}^* = \mathbf{Q}\mathbf{R}$ and

$$\mathbf{U}^* = \mathbf{U}. \tag{3.99}$$

Thus the right stretch tensor $\mathbf{U}$ is also not objective. On the other hand from the alternative version of the polar decomposition theorem we have $\mathbf{F}^* = \mathbf{V}^*\mathbf{R}^* = \mathbf{Q}\mathbf{V}\mathbf{R} = \mathbf{Q}\mathbf{V}\mathbf{Q}^T\mathbf{R}^*$ and so

$$\mathbf{V}^* = \mathbf{Q}\mathbf{V}\mathbf{Q}^T. \tag{3.100}$$

Thus the left stretch tensor $\mathbf{V}$ is objective.

Next, differentiating (3.96) gives

$$\mathbf{L}_* = \frac{\partial \mathbf{v}^*}{\partial \mathbf{y}^*} = \mathbf{Q}\dot{\mathbf{F}}\mathbf{F}^{-1}\mathbf{Q}^T + \dot{\mathbf{Q}}\mathbf{Q}^T = \mathbf{Q}\mathbf{L}\mathbf{Q}^T + \boldsymbol{\Omega} \tag{3.101}$$

whence we see that the velocity gradient tensor is not objective in general since $\mathbf{L}^* \neq \mathbf{QLQ}$; here $\boldsymbol{\Omega} = \dot{\mathbf{Q}}\mathbf{Q}^T$ is skew symmetric. Turning finally to the stretching tensor we have

$$\mathbf{D}^* = \frac{1}{2}(\mathbf{L}_* + \mathbf{L}_*^T) = \frac{1}{2}(\mathbf{QLQ}^T + \mathbf{QL}^T\mathbf{Q}^T) = \mathbf{QDQ}^T \tag{3.102}$$

since $\boldsymbol{\Omega} = -\boldsymbol{\Omega}^T$. Therefore $\mathbf{D}$ is objective.

Similarly on finds that

$$\mathbf{W}^* = \mathbf{QWQ}^T + \boldsymbol{\Omega}.$$

Remark: Instead of the preceding phrasing in terms of two equivalent observers, the notion of objectivity (material frame indifference) can be alternatively phrased in terms of a single observer who considers two motions $\mathbf{y}(\mathbf{x}, t)$ and $\mathbf{y}^*(\mathbf{x}, t^*)$ related by

$$\mathbf{y}^*(\mathbf{x}, t^*) = \mathbf{Q}(t^* - a)\mathbf{y}(\mathbf{x}, t^* - a) + \mathbf{c}(t^* - a); \tag{3.103}$$

here a is a constant, and at each instant t, $\mathbf{Q}(t)$ is an arbitrary proper orthogonal tensor and $\mathbf{c}(t)$ is an arbitrary vector. In most of the subsequent chapters it is sufficient to consider the special case $a = 0$ and $\mathbf{c} = \mathbf{o}$ in which case

$$\mathbf{y}^*(\mathbf{x}, t) = \mathbf{Q}(t)\mathbf{y}(\mathbf{x}, t). \tag{3.104}$$

We shall phrase our discussion of material frame indifference (objectivity) in subsequent sections in terms of two motions related by (3.103) or (3.104) rather than in terms of the earlier phrasing in terms of equivalent observers.

3.9 Convecting and Co-Rotating Bases and Rates.

Convecting Bases and Rates: Consider three material fibers passing through a point $\mathbf{y}$ at time t. Suppose that the vectors $\mathbf{f}_1(\mathbf{y}, t), \mathbf{f}_2(\mathbf{y}, t)$ and $\mathbf{f}_3(\mathbf{y}, t)$ are tangent to these material fibers and suppose further that this triplet of vectors is linearly independent. Then

$\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ forms a basis. If these vectors are attached to the fibers, they will move with, or *convect* with, the body. From (3.29) it follows that

$$\dot{\mathbf{f}}_i = \mathbf{L}\mathbf{f}_i \tag{3.105}$$

It can be shown that if $\{\mathbf{f}_1(\mathbf{y}, t), \mathbf{f}_2(\mathbf{y}, t), \mathbf{f}_3(\mathbf{y}, t)\}$ is a (convecting) basis at one instant t, then it forms a basis for all time.

Next, let $\mathbf{a}(\mathbf{y}, t)$ be an arbitrary vector field with components a_i in the basis $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$:

$$a_i = \mathbf{f}_i \cdot \mathbf{a}. \tag{3.106}$$

On taking the material time derivative of both sides of this we get

$$\overset{\triangle}{a}_i = \dot{\mathbf{f}}_i \cdot \mathbf{a} + \mathbf{f}_i \cdot \dot{\mathbf{a}} = \mathbf{L}\mathbf{f}_i \cdot \mathbf{a} + \mathbf{f}_i \cdot \mathbf{L}\mathbf{f}_i = \mathbf{f}_i \cdot (\dot{\mathbf{a}} + \mathbf{L}^T\mathbf{a}). \tag{3.107}$$

Here we have denoted the material time derivative of a_i by $\overset{\triangle}{a}_i$ rather than $\dot{a}_i$. Had we called it $\dot{a}_i$, one might assume that $\dot{a}_i$ are the components of the vector $\dot{\mathbf{a}}$, which they are not. It follows from (3.107) that $\overset{\triangle}{a}_i$ are the components of the vector

$$\overset{\triangle}{\mathbf{a}} = \dot{\mathbf{a}} + \mathbf{L}^T\mathbf{a} \tag{3.108}$$

which is often referred to as the convected time derivative of $\mathbf{a}$.

Similarly, let $\mathbf{A}(\mathbf{y}, t)$ be an arbitrary tensor field with components A_{ij} in the basis $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$:

$$A_{ij} = \mathbf{f}_i \cdot \mathbf{A}\mathbf{f}_j. \tag{3.109}$$

On taking the material time derivative of both sides of this we get

$$\overset{\triangle}{A}_{ij} = \dot{\mathbf{f}}_i \cdot \mathbf{A}\mathbf{f}_j + \mathbf{f}_i \cdot \dot{\mathbf{A}}\mathbf{f}_j + \mathbf{f}_i \cdot \mathbf{A}\dot{\mathbf{f}}_j = \mathbf{L}\mathbf{f}_i \cdot \mathbf{A}\mathbf{f}_j + \mathbf{f}_i \cdot \dot{\mathbf{A}}\mathbf{f}_j + \mathbf{f}_i \cdot \mathbf{A}\mathbf{L}\mathbf{f}_j = \mathbf{f}_i \cdot (\dot{\mathbf{A}} + \mathbf{L}^T\mathbf{A} + \mathbf{A}\mathbf{L})\mathbf{f}_j. \tag{3.110}$$

As before, we have denoted the material time derivative of A_{ij} by $\overset{\triangle}{A}_{ij}$ rather than $\dot{A}_{ij}$. Had we called it $\dot{A}_{ij}$, one might assume that $\dot{A}_{ij}$ are the components of the tensor $\dot{\mathbf{A}}$ which they are not. It follows from (3.110) that $\overset{\triangle}{A}_{ij}$ are the components of the tensor

$$\overset{\triangle}{\mathbf{A}} = \dot{\mathbf{A}} + \mathbf{A}\mathbf{L} + \mathbf{L}^T\mathbf{A} \tag{3.111}$$

which is often referred to as the convected time derivative of $\mathbf{A}$.

Co-Rotating Bases and Rates: Consider a triplet of orthonormal vectors $\mathbf{f}_1(\mathbf{y},t)$, $\mathbf{f}_2(\mathbf{y},t)$ and $\mathbf{f}_3(\mathbf{y},t)$ that forms a basis at some instant of time. We say that these vectors spin with the body, or *co-rotate* with the body if

$$\dot{\mathbf{f}}_i = \mathbf{W}\mathbf{f}_i \tag{3.112}$$

where $\mathbf{W}$ is the spin tensor; see (3.21). It can be shown that if $\{\mathbf{f}_1(\mathbf{y},t), \mathbf{f}_2(\mathbf{y},t), \mathbf{f}_3(\mathbf{y},t)\}$ is a (co-rotating) orthonormal basis at one instant t, then it forms an orthonormal basis for all time. A calculation entirely analogous to the preceding can be carried out to show that, if a_i are the components of a vector in this basis, then the material time derivative of a_i, denoted by $\overset{\circ}{a}_i$, are the components of the vector

$$\overset{\circ}{\mathbf{a}} = \dot{\mathbf{a}} - \mathbf{W}\mathbf{a} \tag{3.113}$$

referred to as the co-rotational time derivative of $\mathbf{a}$. Likewise, if A_{ij} are the components of a tensor in this basis, then the material time derivative of A_{ij}, denoted by $\overset{\circ}{A}_{ij}$, are the components of the tensor

$$\overset{\circ}{\mathbf{A}} = \dot{\mathbf{A}} + \mathbf{A}\mathbf{W} - \mathbf{W}\mathbf{A} \tag{3.114}$$

referred to as the co-rotational time derivative of $\mathbf{A}$. In writing both of the preceding equations we have used the fact that $\mathbf{W} = -\mathbf{W}^T$.

Problem 3.18 asks you to show that the material time derivatives $\dot{\mathbf{a}}$ and $\dot{\mathbf{A}}$ of an objective vector $\mathbf{a}$ and an objective tensor $\mathbf{A}$ are not objective in general, but that their convected and co-rotational rates of change are objective.

3.10 Linearization.

We now restrict attention to motions which are "infinitesimal" in the sense that the displacement gradient tensor is small, $\epsilon = \mathbf{H} = |\mathrm{Grad}\,\mathbf{u}| \ll 1$. Linearized expressions for the stretch tensors, the rotation tensor and the strain tensor where derived previously in Section 2.8. Differentiating the expressions (2.85), (2.76) that define the *infinitesimal strain tensor* $\boldsymbol{\varepsilon}$ and the *infinitesimal rotation tensor* $\boldsymbol{\omega}$ with respect to t at fixed $\mathbf{x}$ gives

$$\dot{\boldsymbol{\varepsilon}} = \frac{1}{2}(\mathrm{Grad}\,\mathbf{v} \;+\; (\mathrm{Grad}\,\mathbf{v})^{\mathrm{T}}), \qquad \dot{\boldsymbol{\omega}} = \frac{1}{2}(\mathrm{Grad}\,\mathbf{v} \;-\; (\mathrm{Grad}\,\mathbf{v})^{\mathrm{T}}). \tag{3.115}$$

Recall that the velocity gradient tensor $\mathbf{L}$, the strain-rate tensor $\mathbf{D}$ and the spin tensor $\mathbf{W}$ are defined by

$$\mathbf{L} = \mathrm{grad}\,\mathbf{v}, \qquad \mathbf{D} = \frac{1}{2}(\mathrm{grad}\,\mathbf{v} + (\mathrm{grad}\,\mathbf{v})^{\mathrm{T}}), \qquad \mathbf{W} = \frac{1}{2}(\mathrm{grad}\,\mathbf{v} - (\mathrm{grad}\,\mathbf{v})^{\mathrm{T}}). \tag{3.116}$$

When the motion is infinitesimal, it is readily seen that $\text{Grad}\,\mathbf{v} = \text{grad}\,\mathbf{v} + \mathrm{O}(\epsilon^2)$. Therefore

$$\mathbf{D} = \dot{\boldsymbol{\varepsilon}} + O(\epsilon^2), \qquad \mathbf{W} = \dot{\boldsymbol{\omega}} + O(\epsilon^2), \tag{3.117}$$

or in component form

$$L_{ij} = \frac{\partial v_i}{\partial x_j}, \qquad \dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \qquad \dot{\omega}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right), \tag{3.118}$$

3.11 Worked Examples and Exercises.

Problem 3.15. Show that the transport formulae (3.84) can be written in the alternative forms

$$\begin{aligned} \frac{d}{dt}\int_{\mathcal{D}_t} \beta \, dV_y &= \int_{\mathcal{D}_t} \frac{\partial \beta}{\partial t}\, dV_y + \int_{\partial\mathcal{D}_t} \beta\,(\mathbf{v}\cdot\mathbf{n})\, dA_y, \\ \frac{d}{dt}\int_{\mathcal{D}_t} \mathbf{b} \, dV_y &= \int_{\mathcal{D}_t} \frac{\partial \mathbf{b}}{\partial t} dV_y + \int_{\partial\mathcal{D}_t} \mathbf{b}\,(\mathbf{v}\cdot\mathbf{n})\, dA_y. \end{aligned} \tag{3.119}$$

Here $\mathbf{n}$ is a unit outward normal on the boundary $\partial\mathcal{D}_t$ and therefore $\mathbf{v}\cdot\mathbf{n}$ is the outward normal particle speed. Note that the last terms in each equation above can be interpreted as the flux of the quantity β or $\mathbf{b}$ on the surface $\partial\mathcal{D}_t$.

Problem 3.16. Let $\mathcal{S}_t$ be a surface contained within the region $\mathcal{R}_t$ occupied by the body and let Γ_t be the closed curve which forms the boundary of $\mathcal{S}_t$. The *circulation associated with the curve* Γ_t is defined by

$$C(\Gamma_t) = \int_{\Gamma_t} \mathbf{v}\cdot d\mathbf{y}. \tag{3.120}$$

A flow is said to be circulation preserving if $C(\Gamma_t)$ is time independent for every closed curve Γ_t in $\mathcal{R}_t$. Calculate the *vorticity*[5] $\boldsymbol{\omega}(\mathbf{x},t)$ at a generic particle of the body at time t in terms of the vorticity $\boldsymbol{\omega}(\mathbf{x},0)$ at the initial instant. (Assume for simplicity that the reference configuration of the body coincides with the initial configuration of the body.)

<u>*Solution*</u>: On using Stokes' Theorem (e.g. see Volume 1, Section 5.2) and $\boldsymbol{\omega} = \text{curl}\,\mathbf{v}$ we get

$$C(\Gamma_t) = \int_{\Gamma_t} \mathbf{v}\cdot d\mathbf{y} = \int_{\mathcal{S}_t} (\text{curl}\,\mathbf{v})\cdot\mathbf{n}\, dA_y = \int_{\mathcal{S}_t} \boldsymbol{\omega}\cdot\mathbf{n}\, dA_y.$$

Differentiating this with respect to time and using the transport equation $(3.85)_2$ yields

$$\frac{d}{dt}\, C(\Gamma_t) = \frac{d}{dt}\int_{\mathcal{S}_t} \boldsymbol{\omega}\cdot\mathbf{n}\; dA_y = \int_{\mathcal{S}_t} \big(\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\,\text{trace}\,\mathbf{L} - \mathbf{L}\boldsymbol{\omega}\big)\cdot\mathbf{n}\, dA_y$$

[5] Recall that $\boldsymbol{\omega} = \text{curl}\,\mathbf{v}$.

If the flow is circulation preserving

$$\frac{d}{dt}\,C(\Gamma_t) = 0$$

for all curves $\Gamma_t \in \mathcal{R}_t$. Thus we conclude that

$$\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\,\mathrm{trace}\,\mathbf{L} - \mathbf{L}\boldsymbol{\omega} = \mathbf{0}.$$

Pre-operating on this with $J\mathbf{F}^{-1}$ leads to

$$J\mathbf{F}^{-1}\dot{\boldsymbol{\omega}} + J\mathbf{F}^{-1}\boldsymbol{\omega}\,\mathrm{trace}\,\mathbf{L} - J\mathbf{F}^{-1}\mathbf{L}\boldsymbol{\omega} = \mathbf{0}.$$

Recall from (3.43) that $\dot{J} = J\,\mathrm{trace}\,\mathbf{L}$. Furthermore, differentiate $\mathbf{F}\mathbf{F}^{-1} = \mathbf{I}$ with respect to t and use $\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}$ to obtain $\left(\mathbf{F}^{-1}\right)^{\cdot} = -\mathbf{F}^{-1}\mathbf{L}$. Using these in the preceding equation allow us to write it as

$$J\mathbf{F}^{-1}\dot{\boldsymbol{\omega}} + \dot{J}\mathbf{F}^{-1}\boldsymbol{\omega} + J(\mathbf{F}^{-1})^{\cdot}\boldsymbol{\omega} = \mathbf{0}$$

or equivalently as

$$\left(J\mathbf{F}^{-1}\boldsymbol{\omega}\right)^{\cdot} = \mathbf{0}.$$

Thus $J\mathbf{F}^{-1}\boldsymbol{\omega}$ remains constant at each particle and so in particular

$$J(\mathbf{x},t)\mathbf{F}^{-1}(\mathbf{x},t)\boldsymbol{\omega}(\mathbf{x},t) = J(\mathbf{x},0)\mathbf{F}^{-1}(\mathbf{x},0)\boldsymbol{\omega}(\mathbf{x},0).$$

If the reference configuration coincides with the initial configuration then $\mathbf{F}(\mathbf{x},0) = \mathbf{I}$ and $J(\mathbf{x},0) = 1$. Thus the preceding equation simplifies to

$$J(\mathbf{x},t)\mathbf{F}^{-1}(\mathbf{x},t)\boldsymbol{\omega}(\mathbf{x},t) = \boldsymbol{\omega}(\mathbf{x},0)$$

which yields

$$\boldsymbol{\omega}(\mathbf{x},t) = \frac{1}{J(\mathbf{x},t)}\mathbf{F}(\mathbf{x},t)\boldsymbol{\omega}(\mathbf{x},0).$$

Problem 3.17. Show that the (oriented) unit normal vector to a surface in the deformed configuration is objective.

<u>*Solution*</u>: By (2.36) we have that the two unit normal vectors are given by

$$\mathbf{n} = \frac{\mathbf{F}^{-T}\mathbf{n}_0}{|\mathbf{F}^{-T}\mathbf{n}_0|}, \qquad \text{and} \qquad \mathbf{n}_* = \frac{\mathbf{F}_*^{-T}\mathbf{n}_0}{|\mathbf{F}_*^{-T}\mathbf{n}_0|}.$$

Moreover, we have from (3.98) that

$$\mathbf{F}_* = \mathbf{Q}\mathbf{F}.$$

Since $\mathbf{Q}$ is orthogonal, $\mathbf{Q}^{-1} = \mathbf{Q}^T$ and $|\mathbf{Q}\mathbf{q}| = |\mathbf{q}|$ for any vector $\mathbf{q}$. Combining these leads to

$$\mathbf{n}_* = \frac{\mathbf{F}_*^{-T}\mathbf{n}_0}{|\mathbf{F}_*^{-T}\mathbf{n}_0|} = \frac{(\mathbf{Q}\mathbf{F})^{-T}\mathbf{n}_0}{|(\mathbf{Q}\mathbf{F})^{-T}\mathbf{n}_0|} = \frac{\mathbf{Q}\mathbf{F}^{-T}\mathbf{n}_0}{|\mathbf{Q}\mathbf{F}^{-T}\mathbf{n}_0|} = \mathbf{Q}\frac{\mathbf{F}^{-T}\mathbf{n}_0}{|\mathbf{F}^{-T}\mathbf{n}_0|} = \mathbf{Q}\mathbf{n}.$$

Thus the unit normal vector $\mathbf{n}$ is objective.

Problem 3.18. Suppose that $\alpha(\mathbf{y},t), \mathbf{a}(\mathbf{y},t)$ and $\mathbf{A}(\mathbf{y},t)$ are objective scalar, vector and tensor fields respectively.

a) Show that $\dot{\alpha}$ is also objective, but that $\dot{\mathbf{a}}$ and $\dot{\mathbf{A}}$ are not objective (in general).

b) If $\dot{\mathbf{a}}$ is objective, show that necessarily $\mathbf{a} = \mathbf{o}$.

c) If $\dot{\mathbf{A}}$ is objective show that $\mathbf{A} = \beta\,\mathbf{I}$ for some scalar field $\beta(\mathbf{y},t)$.

d) Show that the so-called *convected rates of change* of $\mathbf{a}$ and $\mathbf{A}$ are always objective, where the convected rates of change are defined by

$$\overset{\triangle}{\mathbf{a}} = \dot{\mathbf{a}} + \mathbf{L}^T\mathbf{a} \qquad \text{and} \qquad \overset{\triangle}{\mathbf{A}} = \dot{\mathbf{A}} + \mathbf{L}^{\mathrm{T}}\mathbf{A} + \mathbf{A}\mathbf{L}.$$

e) Which of the following rates-of-change of $\mathbf{A}$ are objective?

$$\begin{aligned}
\overset{\triangle}{\mathbf{A}} &\overset{\text{def}}{=} \dot{\mathbf{A}} + \mathbf{L}^T\mathbf{A} + \mathbf{A}\mathbf{L},\\
\overset{\triangledown}{\mathbf{A}} &\overset{\text{def}}{=} \dot{\mathbf{A}} - \mathbf{L}\mathbf{A} - \mathbf{A}\mathbf{L}^T,\\
\overset{\circ}{\mathbf{A}} &\overset{\text{def}}{=} \dot{\mathbf{A}} - \mathbf{W}\mathbf{A} + \mathbf{A}\mathbf{W},\\
\overset{\square}{\mathbf{A}} &\overset{\text{def}}{=} \tfrac{1}{2}(\overset{\triangle}{\mathbf{A}} - \overset{\triangledown}{\mathbf{A}}) = \mathbf{D}\mathbf{A} + \mathbf{A}\mathbf{D}.
\end{aligned}$$

The tensor $\overset{\circ}{\mathbf{A}}$ is called the *co-rotational rate of change* of $\mathbf{A}$.

<u>*Solution*</u>: We will establish the second result in part (d). Since $\mathbf{A}$ is objective

$$\mathbf{A}_* = \mathbf{Q}\mathbf{A}\mathbf{Q}^T; \qquad (i)$$

by definition, the convected derivatives of $\mathbf{A}$ in the two motions are

$$\overset{\triangle}{\mathbf{A}} = \dot{\mathbf{A}} + \mathbf{L}^T\mathbf{A} + \mathbf{A}\mathbf{L}, \qquad (ii)$$

$$\overset{\triangle}{\mathbf{A}}_* = \dot{\mathbf{A}}_* + \mathbf{L}_*^T\mathbf{A}_* + \mathbf{A}_*\mathbf{L}_*; \qquad (iii)$$

from (3.101) we have

$$\mathbf{L}_* = \mathbf{Q}\mathbf{L}\mathbf{Q}^T + \dot{\mathbf{Q}}\mathbf{Q}^T; \qquad (iv)$$

differentiating the orthogonality condition $\mathbf{Q}\mathbf{Q}^T = \mathbf{I}$ yields

$$\mathbf{Q}\dot{\mathbf{Q}}^T = -\dot{\mathbf{Q}}\mathbf{Q}^T; \qquad (v)$$

and differentiating (i) gives

$$\dot{\mathbf{A}}_* = \dot{\mathbf{Q}}\mathbf{A}\mathbf{Q}^T + \mathbf{Q}\dot{\mathbf{A}}\mathbf{Q}^T + \mathbf{Q}\mathbf{A}\dot{\mathbf{Q}}^T. \qquad (vi)$$

We now take equation (iii) and eliminate the terms $\dot{\mathbf{A}}_*, \mathbf{L}_*$ and $\mathbf{A}_*$ on its right hand side by substituting from $(vi), (iv)$ and (i) respectively. Finally we use (v) to simplify the result further eventually ending up with

$$\overset{\triangle}{\mathbf{A}}_* = \mathbf{Q}\dot{\mathbf{A}}\mathbf{Q}^T + \mathbf{Q}\mathbf{L}^T\mathbf{A}\mathbf{Q}^T + \mathbf{Q}\mathbf{A}\mathbf{L}\mathbf{Q}^T = \mathbf{Q}\,\overset{\triangle}{\mathbf{A}}\,\mathbf{Q}^T$$

from which we conclude that the convected derivative $\overset{\triangle}{\mathbf{A}}$ is objective.

Problem 3.19. Show that the Rivlin-Ericksen tensors $\mathbf{A}_n(\mathbf{y}, t)$ defined in one of the preceding examples are objective.

Problem 3.20. *Is the relative deformation gradient tensor objective?* Let $\mathbf{F}_t(\tau)$ be the deformation gradient tensor of the configuration at time τ with the configuration at time t being taken as the reference configuration; see Section 3.5. Determine the relation between the relative deformation gradient tensors $\mathbf{F}_t(\tau)$ and $\mathbf{F}^*_t(\tau)$ in two motions χ and $\chi^* = \mathbf{Q}\chi$ where $\mathbf{Q}(t)$ is a rotation tensor at each t.

Solution: Let $\mathbf{F}(t), \mathbf{F}^*(t)$ and $\mathbf{F}_t(\tau), \mathbf{F}^*_t(\tau)$ denote the deformation gradient tensors and relative deformation gradient tensors in the two motions. Recall from Section 3.5 that in the motion χ

$$\mathbf{F}_t(\tau) = \mathbf{F}(\tau)\mathbf{F}^{-1}(t), \tag{a}$$

and therefore similarly in the motion χ^*,

$$\mathbf{F}^*_t(\tau) = \mathbf{F}^*(\tau)\,\overset{*}{\mathbf{F}}{}^{-1}(t). \tag{b}$$

Knowing from (3.98) that the deformation gradient tensors in the two motions are related by

$$\mathbf{F}^*(t) = \mathbf{Q}(t)\mathbf{F}(t), \tag{c}$$

we can determine the relation between $\mathbf{F}^*_t(\tau)$ and $\mathbf{F}_t(\tau)$ as follows:

$$\begin{aligned}
\mathbf{F}^*_t(\tau) &= \mathbf{F}^*(\tau)\,\overset{*}{\mathbf{F}}{}^{-1}(t), \\
&= \mathbf{Q}(\tau)\mathbf{F}(\tau)\mathbf{F}^{-1}(t)\mathbf{Q}^T(t), \\
&= \mathbf{Q}(\tau)\mathbf{F}_t(\tau)\mathbf{Q}^T(t),
\end{aligned}$$

where in the first, second and third steps we have used, respectively, (b), (c) and (a). Therefore the relative deformation gradient tensors are related by

$$\mathbf{F}^*_t(\tau) = \mathbf{Q}(\tau)\mathbf{F}_t(\tau)\mathbf{Q}^T(t).$$

Note that the argument of $\mathbf{Q}$ in its two occurrences here are different, t in one case and τ in the other. Thus the relative deformation gradient tensor is not objective.

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Chapter 4

Mechanical Balance Laws and Field Equations

In this chapter we shall consider the application of the fundamental principles of mechanics – viz., the conservation of mass and the mechanical principles of linear and angular momentum balance – to a continuum. In the next chapter we shall turn to the thermodynamic principles corresponding to the first and second laws of thermodynamics[1] .

A roadmap of this chapter is as follows: in Section 4.1 we introduce the notions of a global balance law and a local field equation in general terms. Then in Section 4.2 we present the balance law and field equation associated with the principle of mass balance. Next in Section 4.3 we introduce the notion of force, more specifically body force and traction, and discuss various attributes of them. Section 4.4 states the principles of global balance of linear and angular momentum. From it we deduce the notion of stress and discuss it in Section 4.5. Section 4.6 is devoted to deriving the field equations associated with momentum balance. The principal stresses and principal directions are discussed in Section 4.7. The analysis and discussion up to this point is carried out entirely in the current configuration. We now turn in Section 4.8 to reformulating the preceding analysis with respect to a reference configuration. In particular the concept of the first Piola Kirchhoff stress tensor is introduced and discussed. Section 4.9 considers the rate at which stresses do work – the stress power – and discusses the notion of work-conjugate stress-strain pairs. Finally in Section 4.10 the preceding results are linearized.

[1]If electromagnetic effects are important then we would add Maxwell's equations to this list.

4.1 Introduction

Consider an arbitrary motion χ of a body $\mathcal{B}$ that takes a particle $p \in \mathcal{B}$ to the location $\mathbf{y} = \chi(p, t)$ at time t. Let $\mathcal{R}_t = \chi(\mathcal{B}, t)$ be the region occupied by $\mathcal{B}$ at time t and let $\mathcal{D}_t = \chi(\mathcal{P}, t)$ be the region occupied by some part $\mathcal{P}$ of the body. The motion takes place over a time interval[2] $[t_0, t_1]$.

Suppose that $\Omega(\mathcal{P}, t; \chi)$ is the value of some extensive physical property associated with the part $\mathcal{P}$ at time t (e.g. the internal energy). As discussed in Section 1.8, under suitable assumptions there exists a density $\omega(\mathbf{y}, t; \chi)$ of this property (e.g. the internal energy per unit volume in the current configuration) such that

$$\Omega(\mathcal{P}, t; \chi) = \int_{\mathcal{D}_t} \omega(\mathbf{y}, t; \chi)\, \mathrm{d}V_y.$$

The density ω depends on the position $\mathbf{y}$ (or particle p). For simplicity of writing we shall generally omit the fact that the density depends on the motion and simply write $\omega(\mathbf{y}, t)$.

In this chapter the quantities Ω that we shall consider are the mass, linear momentum and angular momentum of $\mathcal{P}$, while in the next chapter we shall also consider the energy and entropy. These have the respective representations

$$\int_{\mathcal{D}_t} \rho\, dV_y, \quad \int_{\mathcal{D}_t} \rho\mathbf{v}\, dV_y, \quad \int_{\mathcal{D}_t} \mathbf{y} \times \rho\mathbf{v}\, dV_y, \quad \int_{\mathcal{D}_t} \Big(\rho\varepsilon + \frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v}\Big)\, dV_y, \quad \int_{\mathcal{D}_t} \rho\eta\, dV_y; \tag{4.1}$$

here $\rho(\mathbf{y}, t)$ is the mass density, $\mathbf{v}(\mathbf{y}, t)$ is the particle velocity, $\varepsilon(\mathbf{y}, t)$ is the specific internal energy (or internal energy density), and $\eta(\mathbf{y}, t)$ is the specific entropy (or entropy density). Note from the preceding expressions that $\rho\varepsilon$, for example, represents the internal energy per unit *volume* (where volume dV_y is measured in the current configuration); consequently ε itself represents the internal energy per unit *mass*.

The physical principles of interest to us concern the time rate of change of these various physical quantities, and all but the second law of thermodynamics have the form,

$$\int_{\mathcal{D}_t} \beta\, dV_y \;+\; \int_{\partial\mathcal{D}_t} \zeta\, dA_y = \frac{d}{dt}\int_{\mathcal{D}_t} \omega\, \mathrm{d}V_y \tag{4.2}$$

where β represents the bulk (volumetric) generation of Ω at points within the region $\mathcal{D}_t$ and is often called the "source", while ζ represents the generation of Ω at points on the surface

[2]When we have an equation that holds for all time in this interval, we shall, for simplicity, say that it holds "for all time".

(boundary) $\partial\mathcal{D}_t$ and is often called the "flux". The second law of thermodynamics also has the form (4.2) but with an inequality replacing the equality. Equation (4.2) is to hold for *all* parts of the body or equivalently for all subregions $\mathcal{D}_t \subset \mathcal{R}_t$.

An equation of the form (4.2) is known as a *global balance law*; a "balance law" since it describes how the rate of increase of the amount of Ω in $\mathcal{P}$ is balanced by the generation of Ω at the particles in the bulk of $\mathcal{P}$ plus the flux of Ω across the particles on the boundary of $\mathcal{P}$; and "global" because it holds for parts $\mathcal{P}$ of the body rather than locally at each particle p of the body. It is worth pointing out that the balance law (4.2) is *applied to a fixed set of particles of the body* (i.e. the set of particles that comprise the part $\mathcal{P}$). Even though the domain $\mathcal{D}_t$ varies with time, it always contains the same set of particles[3].

While a global balance law is a clear way in which to state a basic physical principle, in order to carry out calculations it is much more convenient to work with an equivalent statement that holds at each point $\mathbf{y} \in \mathcal{R}_t$. Such an equivalent statement is called a *(local) field equation.*

In order to derive a field equation from a balance law, the standard procedure is to first show or observe that the surface field ζ in (4.2) depends on the unit normal vector to $\partial\mathcal{D}_t$ in a convenient form, and to then use the divergence theorem[4] to convert that surface integral into a volume integral over $\mathcal{D}_t$. Next, the time derivative on the right hand side is taken inside the integral using one of the transport equations from Section 3.7. Now the entire equation has the form of a single integral over the region $\mathcal{D}_t$ that vanishes for all subregions $\mathcal{D}_t$. Thus by localization[5] we conclude that the integrand itself must vanish at each point $\mathbf{y}$, thus leading to the field equation associated with the balance law (4.2).

All fields will be assumed to be smooth in this and the next chapter. Specifically, we shall assume that the particle velocity $\mathbf{v}(\mathbf{y}, t)$, the mass density $\rho(\mathbf{y}, t)$, the Cauchy stress $\mathbf{T}(\mathbf{y}, t)$, the heat flux $\mathbf{q}(\mathbf{y}, t)$, the specific internal energy $\varepsilon(\mathbf{y}, t)$, the temperature $\theta(\mathbf{y}, t)$, and the specific entropy $\eta(\mathbf{y}, t)$ are all continuously differentiable jointly in position and time. The body force density $\mathbf{b}(\mathbf{y}, t)$ and the heat supply $r(\mathbf{y}, t)$ are assumed to be continuous in position and time.

Finally we remark that the discussion of the balance laws and field equations that follow immediately below will be carried out entirely in the current configuration. We will not even

[3] An interesting class of problems that requires one to rethink this formulation pertains to problems involving growth, e.g. the growth of a tumor, where particles are added to the tumor.

[4] See Chapter 5.2 of Volume I.

[5] See Section 5.3 of Volume I.

refer to a reference configuration. However for reasons of convenience, we shall consider an alternative (equivalent) formulation with respect to a reference configuration later in this chapter.

4.2 Conservation of Mass.

Given any part $\mathcal{P}$ of a body, its mass is a positive, scalar valued property, whose dimension is independent of length and time, that we denote by $m(\mathcal{P}, t; \chi)$. In terms of the *mass density* $\rho(\mathbf{y}, t; \chi)\,(> 0)$

$$m(\mathcal{P}, t; \chi) = \int_{\mathcal{D}_t} \rho(\mathbf{y}, t; \chi)\, \mathrm{d}V_y; \tag{4.3}$$

see Section 1.8.

The *conservation of mass* states that the mass of any part $\mathcal{P}$ does not depend on the motion or time: i.e. $m(\mathcal{P}, t; \chi) = m(\mathcal{P})$. Note that ρ continues to be configuration and time dependent even though $m(\mathcal{P})$ is not. Since $m(\mathcal{P})$ is time independent, we differentiate (4.3) with respect to t to get the *balance law*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{D_t} \rho(\mathbf{y}, t; \chi)\, \mathrm{d}V_y = 0 \tag{4.4}$$

which must hold for *all* sub-regions $D_t \subset R_t$. For simplicity of notation, from hereon we shall omit explicitly displaying the motion χ in the mass density and simply write $\rho(\mathbf{y}, t)$.

On using the transport equation $(3.84)_1$ this becomes

$$\int_{D_t} (\dot{\rho} + \rho \,\mathrm{div}\, \mathbf{v})\, \mathrm{d}V_y = 0\ . \tag{4.5}$$

Since this holds for all $D_t \subset R_t$, and since the integrand is continuous, it can be localized[6] to get the *field equation*

$$\dot{\rho} + \rho \,\mathrm{div}\, \mathbf{v} = 0 \tag{4.6}$$

which holds for all $\mathbf{y} \in \mathcal{R}_t$ and all t. Recall that the over dot represents the material time derivative.

The steps in the preceding calculation can be reversed to obtain (4.4) from (4.6). Thus the field equation (4.6) and balance law (4.4) are equivalent.

[6]See Chapter 5.3 of Volume I.

A useful result: In what follows we shall be repeatedly using one of the transport equations established in Section 3.7. However in each case the integrand will involve the mass density ρ multiplying a smooth scalar- or vector-valued field ϕ. In this event the transport equations simplify further and we make a note of this here before proceeding further. By using either of the transport equations in (3.84)

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{D_t} \rho\, \phi\, \mathrm{d}V_y = \int_{D_t} \big[(\rho\phi)^{\cdot} + \rho\phi\, \mathrm{div}\, \mathbf{v}\big]\, \mathrm{d}V_y = \int_{D_t} \big[\rho\dot{\phi} + (\dot{\rho} + \rho\, \mathrm{div}\, \mathbf{v})\phi\big]\, \mathrm{d}V_y. \tag{4.7}$$

The term in parentheses on the right hand side vanishes by the balance of mass field equation (4.6) and so we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{D_t} \rho\, \phi\, \mathrm{d}V_y = \int_{D_t} \rho\, \dot{\phi}\, \mathrm{d}V_y\ . \tag{4.8}$$

Equation (4.8) will be used frequently in what follows. Note that in deriving it we have *not* ignored the fact that $\mathcal{D}_t$ and ρ are time dependent even though the end result appears to suggest this.

4.3 Force.

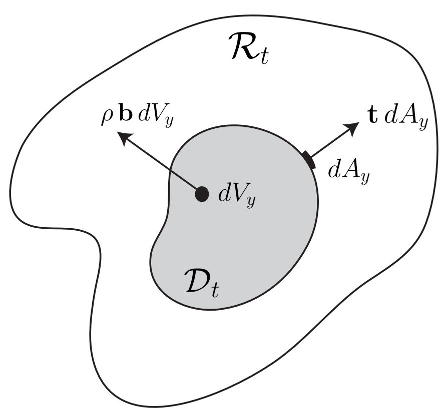


Figure 4.1: Forces on the part $\mathcal{P}$: the traction $\mathbf{t}$ is a force per unit area acting at points on the boundary $\partial\mathcal{D}_t$ of the subregion, and the body force $\mathbf{b}$ is a force per unit mass acting at points in the interior of $\mathcal{D}_t$.

We now turn our attention to the forces that act on an arbitrary part $\mathcal{P}$ of the body at time t. For simplicity, we will sometimes refer to "the forces that act on the region $\mathcal{D}_t$" rather than (more correctly) the part $\mathcal{P}$. These forces are most conveniently described in terms of entities that act on the region $\mathcal{D}_t$ occupied by $\mathcal{P}$ *in the current configuration.* As depicted in Figure 4.1 we assume that there are two types of *forces*: body forces — that act at each point in the interior of the region D_t, and contact forces (or tractions) — that act at points on the boundary of D_t and represent forces due to contact between $\mathcal{P}$ and the rest of the body $\mathcal{B} - \mathcal{P}$ across the surface $\partial\mathcal{D}_t$[7]. The body force density is a force per unit volume (or mass), while the contact force density is a force per unit surface area; see Figure 4.1.

In order to characterize a force, we must specify how it contributes to (*i*) the resultant force, (*ii*) the resultant moment about a point, and (*iii*) how it does work. Let $\mathbf{b}$ denote the *body force per unit mass.* This is distributed over the interior of $\mathcal{D}_t$. The resultant body force, the resultant moment of the body forces about a fixed point O, and the rate at which the body forces do work are taken to be

$$\int_{\mathcal{D}_t} \rho\mathbf{b}\,\mathrm{d}V_y, \qquad \int_{\mathcal{D}_t} \mathbf{y} \times \rho\mathbf{b}\,\mathrm{d}V_y, \qquad \int_{\mathcal{D}_t} \rho\mathbf{b}\cdot\mathbf{v}\,\mathrm{d}V_y \ , \tag{4.9}$$

respectively, where $\mathbf{y}$ is position, $\mathbf{v}$ is particle velocity and ρ is the mass density in the current configuration. Similarly, let $\mathbf{t}$ denote the *contact force per unit area (or traction).* This is distributed over the boundary of $\mathcal{D}_t$. The resultant contact force, the resultant moment of the contact forces about a fixed point O, and the rate at which the contact forces do work are taken to be

$$\int_{\partial\mathcal{D}_t} \mathbf{t}\,\mathrm{d}A_y \ , \qquad \int_{\partial\mathcal{D}_t} \mathbf{y} \times \mathbf{t}\,\mathrm{d}A_y \ , \qquad \int_{\partial\mathcal{D}_t} \mathbf{t}\cdot\mathbf{v}\,\mathrm{d}A_y \ , \tag{4.10}$$

respectively. Note that $\mathbf{t}$ represents a force per unit *current* area. On summing the above expressions we conclude that the *resultant force* on $\mathcal{P}$ is

$$\int_{\mathcal{D}_t} \rho\mathbf{b}\,\mathrm{d}V_y + \int_{\partial\mathcal{D}_t} \mathbf{t}\,\mathrm{d}A_y \ , \tag{4.11}$$

the *resultant moment* about O of the forces on $\mathcal{P}$ is

$$\int_{\mathcal{D}_t} \mathbf{y} \times \rho\mathbf{b}\,\mathrm{d}V_y + \int_{\partial\mathcal{D}_t} \mathbf{y} \times \mathbf{t}\,\mathrm{d}A_y \ , \tag{4.12}$$

and the *total rate of working* of the external forces on $\mathcal{P}$ is

$$\int_{\mathcal{D}_t} \rho\mathbf{b}\cdot\mathbf{v}\,\mathrm{d}V_y + \int_{\partial\mathcal{D}_t} \mathbf{t}\cdot\mathbf{v}\,\mathrm{d}A_y \ . \tag{4.13}$$

[7]If some part of $\partial\mathcal{D}_t$ coincides with a part of the boundary of the body $\partial\mathcal{R}_t$, then the contact force on that portion of the boundary is applied by agents outside of the body via contact with the body.

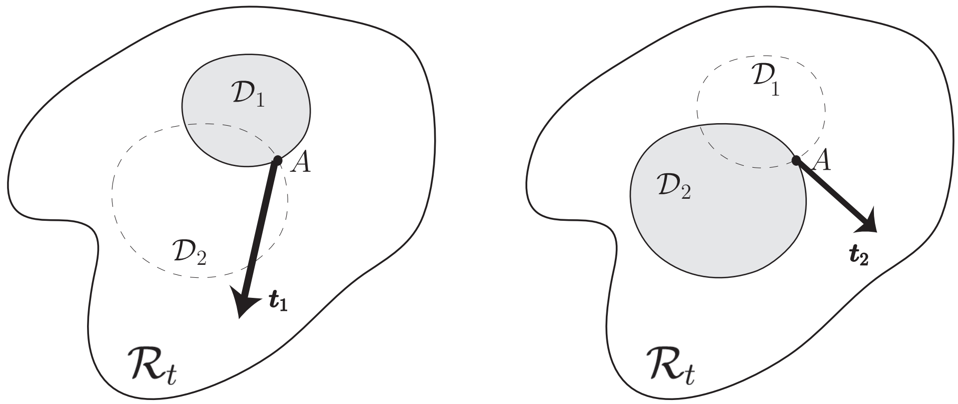


Figure 4.2: Regions $\mathcal{D}_1$ and $\mathcal{D}_2$ occupied by two different parts of the body with the point A common to both boundaries $\partial\mathcal{D}_1$ and $\partial\mathcal{D}_2$. The figure on the left has isolated $\mathcal{D}_1$ while that on the right has isolated $\mathcal{D}_2$. The traction $\mathbf{t}_1$ is applied on $\partial\mathcal{D}_1$ at A by the material outside $\mathcal{D}_1$. Similarly, the traction $\mathbf{t}_2$ is applied on $\partial\mathcal{D}_2$ at A by the material outside $\mathcal{D}_2$.

Remark (a): In order for the formulae (4.11) - (4.13) to be useful, we must specify the variables that these force densities depend on. We expect that in general the body force density may depend on both position $\mathbf{y}$ and time t, and so we assume that

$$\mathbf{b} = \mathbf{b}(\mathbf{y}, t). \tag{4.14}$$

Remark (b): We now turn to the traction $\mathbf{t}$. One might assume that the traction also depends only on the same variables as the body force, i.e. $\mathbf{t} = \mathbf{t}(\mathbf{y}, t)$. However some thought shows that this cannot be so. To see this, consider two parts of the body, $\mathcal{P}_1$ and $\mathcal{P}_2$, which occupy regions $\mathcal{D}_1$ and $\mathcal{D}_2$ at time t as shown in Figure 4.2. Let $\mathbf{y}_A$ be the position of a point that is common to both boundaries $\partial\mathcal{D}_1$ and $\partial\mathcal{D}_2$ as shown. In general, we expect that the contact force exerted on $\partial\mathcal{D}_1$ at $\mathbf{y}_A$ (by the material outside $\mathcal{D}_1$), to be different to the contact force exerted on $\partial\mathcal{D}_2$ at $\mathbf{y}_A$ (by the material outside $\mathcal{D}_2$). However if $\mathbf{t}$ is a function of $\mathbf{y}$ and t only, then it cannot capture this difference since both of these tractions would have the value $\mathbf{t}(\mathbf{y}_A, t)$. Thus the traction must depend on more than just the position and time. It must also depend on the specific surface under consideration as well: $\mathbf{t} = \mathbf{t}(\mathbf{y}, t, \partial D_t)$. To first order, a surface is described by its unit outward normal vector $\mathbf{n}$, and so we shall assume that

$$\mathbf{t} = \mathbf{t}(\mathbf{y}, t, \mathbf{n}). \tag{4.15}$$

Remark (c): The assumption (4.15) is known as *Cauchy's hypothesis*. In order to appreciate its limitations, consider two parts $\mathcal{P}_1$ and $\mathcal{P}_2$ of the body and suppose that at some time t

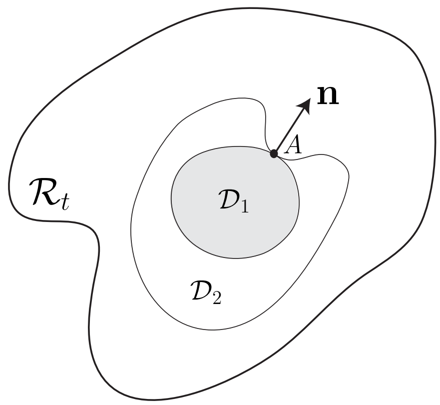


Figure 4.3: Regions $\mathcal{D}_1$ and $\mathcal{D}_2$ occupied by two distinct parts of the body. The point A is common to the boundaries of both these regions. Moreover, the unit outward normal vector at A to both boundaries $\partial\mathcal{D}_1$ and $\partial\mathcal{D}_2$ is $\mathbf{n}$.

they occupy regions $\mathcal{D}_1$ and $\mathcal{D}_2$ as shown in Figure 4.3. Note that the point A is common to both boundaries $\partial\mathcal{D}_1$ and $\partial\mathcal{D}_2$. Moreover, note that the unit outward normal vectors to $\partial\mathcal{D}_1$ and $\partial\mathcal{D}_2$ at A are the same. By Cauchy's hypothesis $\mathbf{t} = \mathbf{t}(\mathbf{y}, t, \mathbf{n})$, and so the traction on both surfaces $\partial\mathcal{D}_1$ and $\partial\mathcal{D}_2$ at A is the same; the traction does not, for example, depend on the curvature of the boundary when the Cauchy hypothesis is invoked.

Remark (d): It is worth emphasizing that the traction $\mathbf{t}(\mathbf{y}, t, \mathbf{n})$ denotes the force per unit area on $\partial\mathcal{D}_t$ applied *by* the part of the body which is outside $\mathcal{D}_t$ *on* the material inside $\mathcal{D}_t$. Often we speak of the side into which $\mathbf{n}$ points as the *positive side* of the surface (which is the outside of $\mathcal{D}_t$) and the side that $\mathbf{n}$ points away from as the *negative side* of the surface (which is the inside of $\mathcal{D}_t$). Then $\mathbf{t}(\mathbf{y}, t, \mathbf{n})$ is the force density applied by the positive side on the negative side. Consider for example a body which at some time t occupies the region $\mathcal{R}_t = \mathcal{D}_1 \cup \mathcal{D}_2$ shown in Figure 4.4: the cubic subregion $\mathcal{D}_1$ is occupied by part of the body and the rest of the body occupies $\mathcal{D}_2$. The figure on the left in Figure 4.4 has isolated $\mathcal{D}_1$ while that on the right has isolated $\mathcal{D}_2$. Consider the particle A whose position vector at time t is $\mathbf{y}_A$. Then in order to calculate the traction applied *by* $\mathcal{D}_2$ *on* $\mathcal{D}_1$ at A, we draw the unit normal to $\partial\mathcal{D}_1$ that points outward from $\mathcal{D}_1$. This is denoted by $\mathbf{n}$ in the figure on the left. Thus this traction is $\mathbf{t}(\mathbf{y}_A, t, \mathbf{n})$. On the other hand if we want to calculate the traction applied *by* $\mathcal{D}_1$ *on* $\mathcal{D}_2$ at A, we must draw the unit normal to $\partial\mathcal{D}_2$ that points outward from $\mathcal{D}_2$ which is $-\mathbf{n}$ in the figure on the right. Thus this traction is $\mathbf{t}(\mathbf{y}_A, t, -\mathbf{n})$.

Remark (e): The traction acts in a direction that is determined by the internal forces within the body and need *not* be normal to the surface. The component of traction that is normal

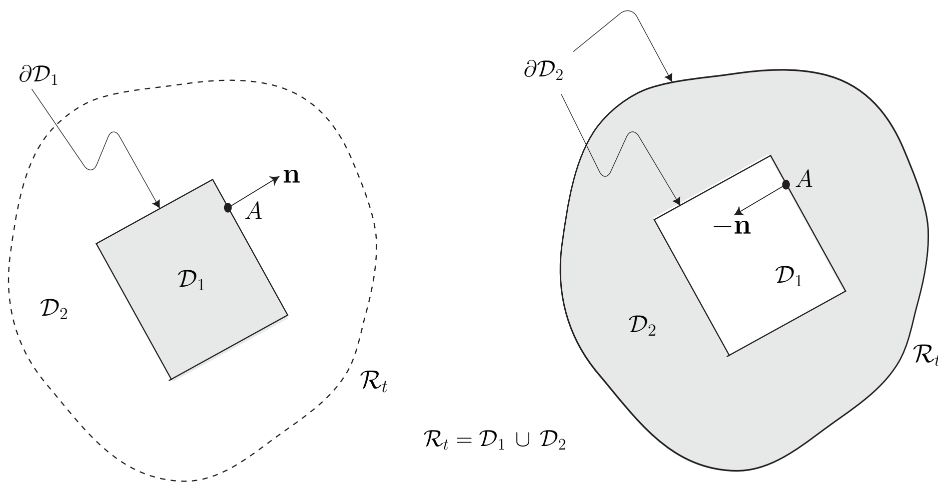


Figure 4.4: The region $\mathcal{R}_t$ occupied by a body in its current configuration is the union of the two subregions $\mathcal{D}_1$ and $\mathcal{D}_2$: $\mathcal{R}_t = \mathcal{D}_1 \cup \mathcal{D}_2$. The figure on the left has isolated $\mathcal{D}_1$ while that on the right has isolated $\mathcal{D}_2$. The unit normal vector to $\partial\mathcal{D}_1$ at A that points out of $\mathcal{D}_1$ is $\mathbf{n}$. The unit normal vector to $\partial\mathcal{D}_2$ at A that points out of $\mathcal{D}_2$ is $-\mathbf{n}$.

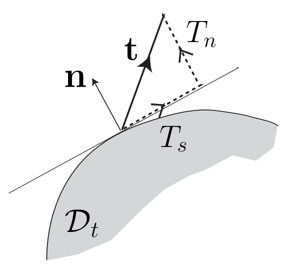


Figure 4.5: Components of the traction $\mathbf{t}$: normal stress T_n and resultant shear stress T_s .

to the surface $\partial\mathcal{D}_t$ is called the *normal stress* and we denote it by T_n:

$$T_n = \mathbf{t} \cdot \mathbf{n}\,; \tag{4.16}$$

see Figure 4.5. The tangential component of $\mathbf{t}$ is called the *resultant shear stress* and we denote it by T_s:

$$T_s = |\mathbf{t} - T_n\,\mathbf{n}| = \left(|\mathbf{t}|^2 - T_n^2\right)^{1/2} \quad . \tag{4.17}$$

Remark (f): The traction on the boundary $\partial\mathcal{R}_t$ of the region occupied by the *entire body* is applied by agents outside of the body through physical contact along the boundary. When formulating and solving a boundary-value problem, these tractions are boundary conditions.

Note that they act on the boundary $\partial\mathcal{R}_t$ in the current configuration whose location is in general *not known* a priori.

4.4 The Balance of Momentum Principles.

The *balance principle for linear momentum* postulates that in an inertial frame, the resultant force on any part of the body equals the rate of increase of its linear momentum:

$$\int_{\mathcal{D}_t} \rho\mathbf{b}\,\mathrm{d}V_y + \int_{\partial\mathcal{D}_t} \mathbf{t}\,\mathrm{d}A_y = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{D}_t} \rho\mathbf{v}\,\mathrm{d}V_y. \tag{4.18}$$

Similarly, the *balance principle for angular momentum* postulates that in an inertial frame, the resultant moment on any part of the body about a fixed point O equals the rate of increase of its angular momentum (about O):

$$\int_{\mathcal{D}_t} \mathbf{y}\times\rho\mathbf{b}\,\mathrm{d}V_y + \int_{\partial\mathcal{D}_t} \mathbf{y}\times\mathbf{t}\,\mathrm{d}A_y = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{D}_t} \mathbf{y}\times\rho\mathbf{v}\,\mathrm{d}V_y\ . \tag{4.19}$$

Both (4.18) and (4.19) must hold for every part of the body and therefore for all subregions $D_t \subset R_t$.

When the fields are smooth, by using (4.8) we can write the balance laws (4.18) and (4.19) in the equivalent forms

$$\int_{\mathcal{D}_t} \rho\mathbf{b}\,\mathrm{d}V_y + \int_{\partial\mathcal{D}_t} \mathbf{t}\,\mathrm{d}A_y = \int_{\mathcal{D}_t} \rho\dot{\mathbf{v}}\,\mathrm{d}V_y\ , \tag{4.20}$$

$$\int_{\mathcal{D}_t} \mathbf{y}\times\rho\mathbf{b}\,\mathrm{d}V_y + \int_{\partial\mathcal{D}_t} \mathbf{y}\times\mathbf{t}\,\mathrm{d}A_y = \int_{\mathcal{D}_t} \mathbf{y}\times\rho\dot{\mathbf{v}}\,\mathrm{d}V_y\ , \tag{4.21}$$

where $\dot{\mathbf{v}}$ is the acceleration of a particle. In deriving (4.21) we have used the fact that $\dot{\mathbf{y}}\times\mathbf{v} = \mathbf{v}\times\mathbf{v} = \mathbf{o}$.

4.5 A Consequence of Linear Momentum Balance: Stress.

We now explore some implications of the balance of linear momentum. The analysis in this section addresses the question of how the traction vector $\mathbf{t}(\mathbf{y}, t, \mathbf{n})$ depends on the unit normal vector $\mathbf{n}$. The position $\mathbf{y}$ and time t will play no central role in the discussion to

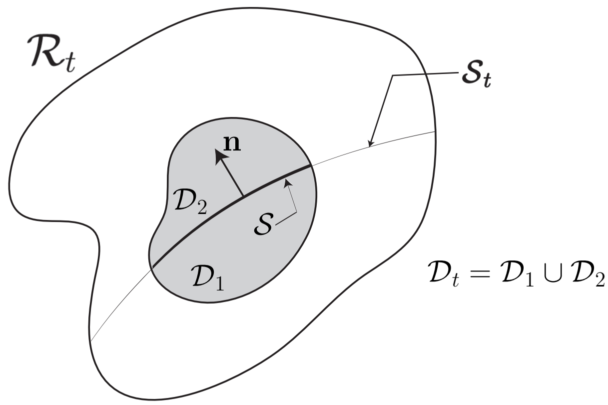


Figure 4.6: A surface $\mathcal{S}_t$ contained within $\mathcal{R}_t$. The sub-region $\mathcal{D}_t$ is intersected by this surface.

follow, and so it is convenient in this section to suppress these variables and write $\mathbf{t}(\mathbf{n})$ instead of $\mathbf{t}(\mathbf{y}, t, \mathbf{n})$.

First, consider the region $\mathcal{R}_t$ occupied by the body in its current configuration and let $\mathcal{S}_t$ be an arbitrary surface contained within it. Pick a sub-region $\mathcal{D}_t$ that is intersected by $\mathcal{S}_t$ and which is thus separated into regions $\mathcal{D}_1$ and $\mathcal{D}_2$: $\mathcal{D}_t = \mathcal{D}_1 \cup \mathcal{D}_1$; see Figure 4.6. $\mathcal{S}$ is the portion of $\mathcal{S}_t$ that is contained within $\mathcal{D}_t$ and is therefore the interface between $\mathcal{D}_1$ and $\mathcal{D}_2$. Note that the unit vector $\mathbf{n}$ on $\mathcal{S}$ shown in the figure is outward to $\mathcal{D}_1$ whereas $-\mathbf{n}$ is outward to $\mathcal{D}_2$. Thus when the balance of linear momentum (4.20) is applied to $\mathcal{D}_1$, the traction term will involve the integral of $\mathbf{t}(\mathbf{n})$ over $\mathcal{S}$, whereas when it is applied to $\mathcal{D}_2$, it will involve the integral of $\mathbf{t}(-\mathbf{n})$ over $\mathcal{S}$. We now apply the linear momentum principle (4.20) to each of the regions $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_t$ individually, and then subtract the first two of the resulting equations from the third. This leads to

$$\int_{\mathcal{S}} [\mathbf{t}(\mathbf{n}) + \mathbf{t}(-\mathbf{n})] \ dA_y \ = \ 0. \tag{4.22}$$

Since this must hold for arbitrary choices of $\mathcal{D}_t$, and therefore for arbitrary choices of $\mathcal{S}$, it follows by localization that the integrand must vanish at each point on $\mathcal{S}_t$. Thus we conclude that $\mathbf{t}(-\mathbf{n}) = -\mathbf{t}(\mathbf{n})$, or displaying all the variables,

$$\mathbf{t}(\mathbf{y}, t, -\mathbf{n}) = -\mathbf{t}(\mathbf{y}, t, \mathbf{n}) \tag{4.23}$$

for all unit vectors $\mathbf{n}$.

Observe that this is the analog for a continuum of Newton's third law for particles. It says that the traction exerted on the positive side of a surface by the negative side, is equal in magnitude and opposite in direction to the traction exerted on the negative side by the positive side. Note that this is a *consequence* of the balance of linear momentum and not a separate postulate.

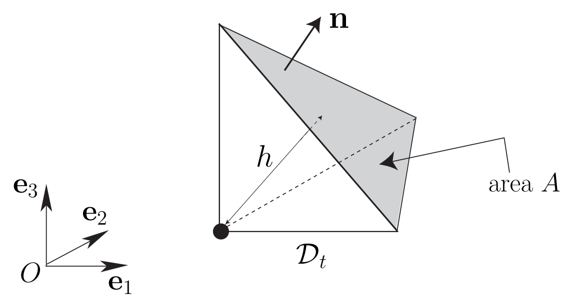


Figure 4.7: Tetrahedral subregion $\mathcal{D}_t$ of the body.

We now establish a second, critically important, consequence of the balance of linear momentum, namely that the traction vector $\mathbf{t}(\mathbf{n})$ depends *linearly* on the normal vector $\mathbf{n}$. This is called Cauchy's Theorem. In order to establish this, consider the tetrahedral subregion $\mathcal{D}_t$ shown in Figure 4.7 with three of its faces parallel to the coordinate planes. Observe that the unit outward normal vectors to the four faces of $\mathcal{D}_t$ are $\mathbf{n}$, $-\mathbf{e}_1$, $-\mathbf{e}_2$ and $-\mathbf{e}_3$. Moreover, if the area of the shaded face is A, one can readily show from geometry that the area, A_k, of the face normal to $\mathbf{e}_k$ is $n_k A$. Next, we apply the balance of linear momentum to this tetrahedron, and take the limit of the resulting equation as the height $h \to 0$ with the orientations of all the faces held fixed. One readily finds that in this limit the volumetric terms (which involve the body force and inertial terms) approach zero like h^3 whereas the area terms (which involve the traction) approach zero like h^2. Therefore only the area terms survive in this limit, and this leads to

$$A\mathbf{t}(\mathbf{n}) + A_1\mathbf{t}(-\mathbf{e}_1) + A_2\mathbf{t}(-\mathbf{e}_2) + A_3\mathbf{t}(-\mathbf{e}_3) = \mathbf{o}. \tag{4.24}$$

Because of (4.23) and $A_k = n_k A$, we can write this as

$$\mathbf{t}(\mathbf{n}) = n_1\mathbf{t}(\mathbf{e}_1) + n_2\mathbf{t}(\mathbf{e}_2) + n_3\mathbf{t}(\mathbf{e}_3) = \mathbf{t}(\mathbf{e}_j)n_j. \tag{4.25}$$

Now define the nine scalars T_{ij} by

$$T_{ij} = \mathbf{t}(\mathbf{e}_i) \cdot \mathbf{e}_j. \tag{4.26}$$

Note that this says that T_{ij} is the jth component of the traction on the plane normal to $\mathbf{e}_i$. Thus we can equivalently write

$$\mathbf{t}(\mathbf{e}_i) = T_{i1}\mathbf{e}_1 + T_{i2}\mathbf{e}_2 + T_{i3}\mathbf{e}_3 = T_{ij}\mathbf{e}_j. \tag{4.27}$$

Substituting (4.27) into (4.25) gives

$$\mathbf{t}(\mathbf{n}) = (T_{ji}\mathbf{e}_i)n_j = T_{ji}\mathbf{e}_i(\mathbf{e}_j \cdot \mathbf{n}) = T_{ji}(\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{n} = \Big(T_{ji}\,\mathbf{e}_i \otimes \mathbf{e}_j\Big)\mathbf{n}\,. \tag{4.28}$$

Next, let $\mathbf{T}$ be the second-order tensor whose components in the basis $\{\mathbf{e}_1,\, \mathbf{e}_2,\, \mathbf{e}_3\}$ are T_{ij}, i.e[8].

$$\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j. \tag{4.29}$$

Since this implies that $\mathbf{T}^T = T_{ij}\mathbf{e}_j \otimes \mathbf{e}_i$, it now follows that (4.28) can be written as $\mathbf{t}(\mathbf{n}) = \mathbf{T}^T\mathbf{n}$, or by writing out all the variables:

$$\mathbf{t}(\mathbf{y}, t, \mathbf{n}) = \mathbf{T}^T(\mathbf{y}, t)\mathbf{n}. \tag{4.30}$$

In terms of components we have

$$t_i(\mathbf{n}) = T_{ji}n_j, \qquad\qquad \{t\} = [T]^T\{n\}. \tag{4.31}$$

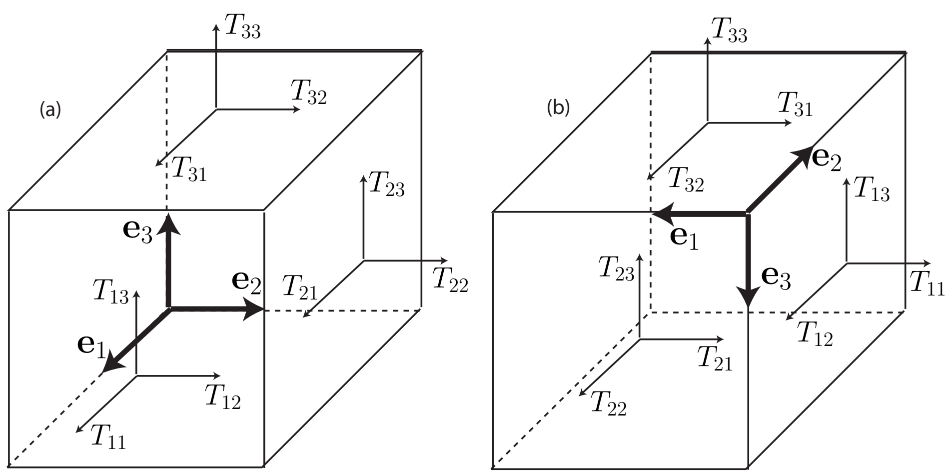


Figure 4.8: The figure shows two views of the same cubic region and the same basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Figure (a) shows the stress components T_{ij} acting on three faces of the body that are normal to the $+\mathbf{e}_1$, $+\mathbf{e}_2$ and $+\mathbf{e}_3$. Observe how the figure is consistent with $T_{ij} = \mathbf{t}(\mathbf{e}_i)\cdot(\mathbf{e}_j)$. Figure (b) shows the stress components T_{ij} acting on three faces normal to the $-\mathbf{e}_1$, $-\mathbf{e}_2$ and $-\mathbf{e}_3$ directions. Note in this case the consistency with $T_{ij} = \mathbf{t}(-\mathbf{e}_i) \cdot (-\mathbf{e}_j)$.

The tensor $\mathbf{T}(\mathbf{y}, t)$ is known as the *Cauchy stress tensor*. Note because of (4.27) that its component T_{ij} represents the j^{th} component of the force per unit area acting on a surface

[8]See Chapter 2.2 of Volume I

whose normal is in the i^{th} direction; this is illustrated in Figure 4.8(a). Note that the surface referenced here must be normal to the i^{th} direction *in the current configuration*; similarly the area referenced here refers to area in the current configuration. Note that $\mathbf{T}$ *does not depend on* the normal vector $\mathbf{n}$. Therefore we may speak of the stress at a point; in contrast, recall that when speaking of traction we must speak of the traction on a surface through a point. When $\mathbf{T}(\mathbf{y}, t)$ is known, equation (4.30) can be used to calculate the traction $\mathbf{t}(\mathbf{y}, t, \mathbf{n})$ on any plane through $\mathbf{y}$. The balance of angular momentum will show that $\mathbf{T}$ is symmetric.

Observe from (4.26), (4.23) that

$$T_{ij} = \mathbf{t}(-\mathbf{e}_i) \cdot (-\mathbf{e}_j). \tag{4.32}$$

This allows us to describe the traction on a face with unit outward normal $-\mathbf{e}_i$. In particular, as shown in Figure 4.8(b), the force per unit area in the $-\mathbf{e}_i$ direction acting on a surface with unit normal $-\mathbf{e}_j$ is T_{ij}.

Problems 4.1 - 4.4.

4.6 Field Equations Associated with the Momentum Balance Principles.

We now return to the global balance laws for linear and angular momentum introduced earlier in (4.20) and (4.21). We are now in a position to derive their local versions, versions that hold at each particle of the body. Consider first the *linear momentum principle* (4.20). Substituting (4.30) into it and then using the divergence theorem leads to

$$\int_{\mathcal{D}_t} \rho \dot{\mathbf{v}} \, \mathrm{d}V_y = \int_{\partial\mathcal{D}_t} \mathbf{T}^T \mathbf{n} \, \mathrm{d}A_y + \int_{\mathcal{D}_t} \rho \mathbf{b} \, \mathrm{d}V_y \ = \int_{\mathcal{D}_t} \operatorname{div} \mathbf{T}^T \, \mathrm{d}V_y + \int_{\mathcal{D}_t} \rho \mathbf{b} \, \mathrm{d}V_y \ , \tag{4.33}$$

and so,

$$\int_{\mathcal{D}_t} (\operatorname{div} \mathbf{T}^T + \rho \mathbf{b} - \rho \dot{\mathbf{v}}) \, \mathrm{d}V_y = 0 \ . \tag{4.34}$$

Since this must hold for all parts of the body and therefore for all choices of D_t, it can be localized to yield

$$\operatorname{div} \mathbf{T}^T + \rho \mathbf{b} = \rho \, \dot{\mathbf{v}} \tag{4.35}$$

at each $\mathbf{y} \in \mathcal{R}_t$ and all times t. In component form

$$\frac{\partial T_{ji}}{\partial y_j} + \rho b_i \;=\; \rho \dot{v}_i. \tag{4.36}$$

The *equation of motion* (4.35) is the field equation corresponding to linear momentum balance.

Conversely, when the equation of motion (4.35) and the traction-stress relation (4.30) hold, then necessarily the global linear momentum balance law (4.20) also holds.

We turn next to the *angular momentum principle* (4.21). Recall that for any two vectors $\mathbf{a}$ and $\mathbf{b}$, the ith component of the vector $\mathbf{a} \times \mathbf{b}$ is $e_{ijk}a_j b_k$ where e_{ijk} is the alternator. Thus we can write (4.21) in component form as

$$\int_{\partial \mathcal{D}_t} e_{ijk}\, y_j\, t_k \,\mathrm{d}A_y + \int_{\mathcal{D}_t} e_{ijk}\, y_j \rho b_k \,\mathrm{d}V_y = \int_{\mathcal{D}_t} e_{ijk}\, \rho y_j \dot{v}_k \, dV_y \;. \tag{4.37}$$

The term involving the traction can be simplified by first using the traction-stress relation (4.31), then using the divergence theorem and finally expanding the result. This leads to

$$\begin{aligned} \int_{\partial \mathcal{D}_t} e_{ijk}\, y_j\, t_k \,\mathrm{d}A_y \;&=\; \int_{\partial \mathcal{D}_t} e_{ijk}\, y_j\, T_{mk} n_m \,\mathrm{d}A_y = \int_{\mathcal{D}_t} e_{ijk}\, \frac{\partial (y_j\, T_{mk})}{\partial y_m} \,\mathrm{d}V_y \\ &=\; \int_{\mathcal{D}_t} e_{ijk} \left(\delta_{jm}\, T_{mk} + y_j\, \frac{\partial T_{mk}}{\partial y_m} \right) \,\mathrm{d}V_y \;. \end{aligned} \tag{4.38}$$

Substituting (4.38) into (4.37) and making use of the equation of motion (4.35) now leads to

$$\int_{\mathcal{D}_t} e_{ijk}\, T_{jk} \,\mathrm{d}V_y = 0 \;. \tag{4.39}$$

Since (4.39) must to hold for all choices of D_t, it follows that

$$e_{ijk}\, T_{jk} = 0 \tag{4.40}$$

at each point in $\mathcal{R}_t$. Multiplying (4.40) by e_{ipq} and then making use of the identity $e_{ijk}\, e_{ipq} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}$ simplifies this finally to

$$T_{ij} = T_{ji} \tag{4.41}$$

or equivalently

$$\mathbf{T} = \mathbf{T}^T \tag{4.42}$$

at each $\mathbf{y} \in \mathcal{R}_t$ and at all times. Therefore, the stress tensor $\mathbf{T}$ is *symmetric*. Equation (4.42) is a local consequence of the balance of angular momentum.

Conversely, when the symmetry condition (4.42), the equations of motion (4.35), and the traction-stress relation (4.30) all hold, then necessarily the global balance of angular momentum (4.21) also holds.

In summary, the global balance laws for linear and angular momentum hold if and only if the following local conditions hold at each point in the body and at each instant during the motion:

$$\left.\begin{aligned} \text{div}\,\mathbf{T} + \rho\mathbf{b} &= \rho\dot{\mathbf{v}}, \\ \mathbf{T} &= \mathbf{T}^T, \end{aligned}\right\} \tag{4.43}$$

with the traction on a surface related to the stress through

$$\mathbf{t} = \mathbf{T}\mathbf{n}. \tag{4.44}$$

In component form

$$\frac{\partial T_{ij}}{\partial y_j} + \rho b_i = \rho \dot{v}_i, \qquad T_{ij} = T_{ji}, \qquad t_i = T_{ij} n_j. \tag{4.45}$$

4.7 Principal Stresses.

Since $\mathbf{T}$ is symmetric, it has three real eigenvalues, T_1, T_2, T_3, and a set of three corresponding orthonormal eignevectors, $\boldsymbol{\nu}^{(1)}, \boldsymbol{\nu}^{(2)}, \boldsymbol{\nu}^{(3)}$:

$$\mathbf{T}\boldsymbol{\nu}^{(i)} = T_i\ \boldsymbol{\nu}^{(i)} \qquad\qquad \text{(no sum on i)}; \tag{4.46}$$

the eigenvalues T_i are called the *principal stresses* and the eigenvectors $\boldsymbol{\nu}^{(i)}$ define the *principal directions*. The triplet of vectors $\{\boldsymbol{\nu}^{(1)}, \boldsymbol{\nu}^{(2)}, \boldsymbol{\nu}^{(3)}\}$ defines an orthonormal basis referred to as a principal basis of stress. The matrix of stress components in this basis is diagonal and is given by

$$[T] = \begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_3 \end{pmatrix}. \tag{4.47}$$

In order to get some physical insight into the significance of the principal stresses and principal directions, consider an arbitrary point $\mathbf{y}$ in $\mathcal{R}_t$ and consider an arbitrary plane

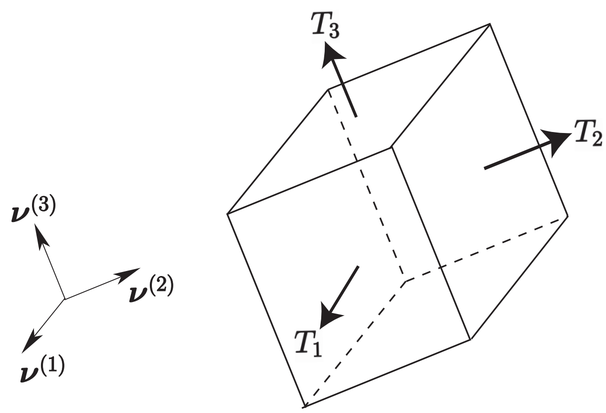


Figure 4.9: Principal stresses and principal directions.

through $\mathbf{y}$. Let $\mathbf{n}$ be the unit normal vector to this plane. The associated normal stress T_n is (see (4.16) and (4.44))

$$T_n(\mathbf{n}) = \mathbf{t} \cdot \mathbf{n} = \mathbf{Tn} \cdot \mathbf{n}. \tag{4.48}$$

A natural question to ask is, from among all planes passing through this point, on which plane is T_n largest? And on which one is it smallest? This requires one to consider $T_n(\mathbf{n})$ as a function of $\mathbf{n}$ and to find the specific vector(s) $\mathbf{n}$ at which it has its extrema. In one of the exercises at the end of this chapter it will be shown that the largest value of $T_n(\mathbf{n})$ is given by the largest principal stress, i.e. $\max\{T_1, T_2, T_3\}$; and that the corresponding principal direction defines the plane on which it acts.

Finally it is worth noting that the principal directions of the stress tensor $\mathbf{T}$ have no relationship, in general, to the principal directions of the stretch tensors $\mathbf{U}$ or $\mathbf{V}$. There may be a relationship between them for *particular materials*, but this is constitutive law dependent.

Remark: Consider a fixed orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and let T_{ij} be the components of $\mathbf{T}$ in this basis. The special case in which the components of stress in this basis has the form

$$\begin{pmatrix} T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is said to describe a state of *uniaxial stress* in the $\mathbf{e}_1$ direction. Note that we can write this as $\mathbf{T} = T\mathbf{e}_1 \otimes \mathbf{e}_1$. The special case in which $[T]$ has the form

$$\begin{pmatrix} T & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & T \end{pmatrix}$$

is said to describe a state of *equitriaxial stress* or a state of pure pressure $-T$. Note that we can write this as $\mathbf{T} = T\mathbf{I}$. Finally the special case in which $[T]$ has the form

$$\begin{pmatrix} T_{11} & T_{12} & 0 \\ T_{12} & T_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is said to describe a state of *plane stress* in the $\mathbf{e}_1, \mathbf{e}_2$ plane.

Problems 4.5 - 4.14.

In closing, it is worth repeating that we have had no need to refer to a reference configuration in the preceding discussion of the notions of traction and stress, or the balance laws and field equations of momentum. Though not conceptually necessary, sometimes it is convenient to introduce a reference configuration and to refer various quantities and equations to that configuration. We turn to this in the next section.

4.8 Formulation of Mechanical Principles with Respect to a Reference Configuration.

Consider an arbitrary motion $\mathbf{y} = \chi(p, t)$ of a body $\mathcal{B}$ and let $\mathcal{R}_t = \chi(\mathcal{B}, t)$ be the region occupied by the body at time t. Let $\mathcal{P}$ be some part of the body and let $\mathcal{D}_t = \chi(\mathcal{P}, t)$ be the region occupied by this part at time t. We now consider a reference configuration χ_{ref} of the body. Suppose that the particle p is located at $\mathbf{x} = \chi_{\text{ref}}(p)$ in this configuration. Let $\mathcal{R}_0 = \chi_{\text{ref}}(\mathcal{B})$ and $\mathcal{D}_0 = \chi_{\text{ref}}(\mathcal{P})$ be the respective regions occupied by the body $\mathcal{B}$ and the part $\mathcal{P}$ in this configuration. Note that the reference configuration need not coincide with a configuration that the body actually occupies during the motion; in this sense our use of the notation $\mathcal{D}_0$ (and $\mathcal{R}_0$) is misleading since the reader might assume that these regions refer to the initial configuration. Perhaps $\mathcal{D}_r$ or $\mathcal{D}_{\text{ref}}$ would be better.

A motion of the body maybe characterized by

$$\mathbf{y} = \mathbf{y}(\mathbf{x}, t) \tag{4.49}$$

where $\mathbf{x} \in \mathcal{R}_0, t \in [t_0, t_1]$ and $\mathbf{y} \in \mathcal{R}_t$. We assume that $\mathbf{y}(\mathbf{x}, t)$ is twice continuously differentiable jointly in position and time.

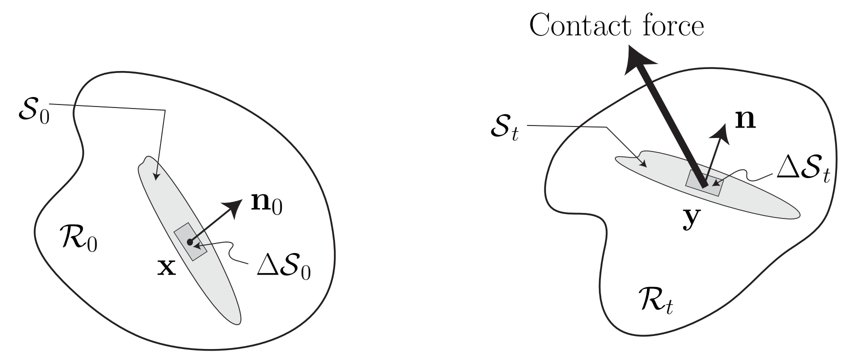
We first consider mass balance. We know from Chapter 2 that a (material) volume element dV_x has volume $dV_y = J dV_x$ at the current instant where $J = \det \mathbf{F}$. Therefore the mass density ρ_0 in the reference configuration is related to the mass density ρ in the current configuration by

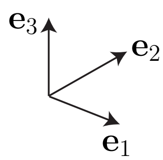
$$\rho_0 = \rho \, J \qquad \text{where} \quad J = \det \mathbf{F}. \tag{4.50}$$

Next we turn to momentum balance, traction and stress. We begin by addressing the following two issues:

(i) Recall first that the stress tensor $\mathbf{T}$ represents the force per unit *current* area, and that the traction $\mathbf{Tn}$ acts on the surface whose normal in the *current* configuration is $\mathbf{n}$. Even though the forces act on the current configuration of the body, it is still sometimes convenient to refer them to the geometry of the reference configuration. Often, the current configuration is not known a priori and is to be determined, while the reference configuration can be chosen at will. Therefore, it is sometimes more convenient to consider a different stress tensor, say $\mathbf{S}$, which represents force per unit *reference* area, and with the associated traction $\mathbf{Sn}_0$ acting on the surface which is normal to the direction $\mathbf{n}_0$ in the *reference* configuration. This is illustrated in Figure 4.10. Note that the forces act in the current configuration.

(ii) Next recall that our formulation has been in the current configuration, so that a field quantity such as, say, the stress $\mathbf{T}$, was taken to be a function of the *current* position $\mathbf{y}$ and time: $\mathbf{T} = \mathbf{T}(\mathbf{y}, t)$. Consequently the field equations, such as the equations of motion (4.43), hold at points $\mathbf{y} \in \mathcal{R}_t$. Sometimes it is more convenient to use the position in the reference configuration $\mathbf{x}$ instead of $\mathbf{y}$. To do this we could simply substitute the motion $\mathbf{y} = \mathbf{y}(\mathbf{x}, t)$ into the field, e.g. $\mathbf{T}(\mathbf{y}(\mathbf{x}, t), t)$ to get $\mathbf{T}(\mathbf{x}, t)$, and to then write the field equations in a form that holds at points $\mathbf{x} \in \mathcal{R}$ of the reference region. The field equations in terms of $\mathbf{T}(\mathbf{x}, t)$ turns out to have a complicated form, but, if written in terms of some other suitable stress tensor $\mathbf{S}(\mathbf{x}, t)$ has a simple form. We set ourself the task for finding such a stress tensor $\mathbf{S}$.





$$\begin{aligned} \text{Contact force} &= \mathbf{t} \times \text{Area of}(\Delta\mathcal{S}_t) \\ &= \mathbf{T}\mathbf{n} \times \text{Area of}(\Delta\mathcal{S}_t) \\ &= \mathbf{S}\mathbf{n}_0 \times \text{Area of}(\Delta\mathcal{S}_0) \\ &= \mathbf{s} \times \text{Area of}(\Delta\mathcal{S}_0) \end{aligned}$$

Figure 4.10: Surface $\mathcal{S}_t$ and surface element $\Delta\mathcal{S}_t$ in current configuration, and their images $\mathcal{S}_0$ and $\Delta\mathcal{S}_0$ in the reference configuration. Different (equivalent) ways for characterizing the contact force are shown. Note that the contact force acts on the current configuration.

Consider some fixed instant during the motion. Let $\mathcal{S}_t$ be a surface in $\mathcal{R}_t$ and let $\mathcal{S}_0$ be its image in the reference configuration; let $\mathbf{y}$ be a point on $\mathcal{S}_t$ and let $\mathbf{x}$ be its image on $\mathcal{S}_0$; let $\mathbf{n}$ be a unit normal vector to $\mathcal{S}_t$ at $\mathbf{y}$, and let $\mathbf{n}_0$ be the corresponding unit normal vector to $\mathcal{S}_0$ at $\mathbf{x}$; and finally, let $\Delta\mathcal{S}_t$ be an infinitesimal surface element on $\mathcal{S}_t$ at $\mathbf{y}$ whose area is $\mathrm{d}A_y$, and let $\Delta\mathcal{S}_0$ be its image in the reference configuration whose area is $\mathrm{d}A_x$. This is illustrated in Figure 4.10.

If $\mathbf{t}$ is the traction at $\mathbf{y}$ on $\mathcal{S}_t$, then the contact force on the surface element $\Delta\mathcal{S}_t$ is the product of this traction with the area $\mathrm{d}A_y$:

$$\text{The contact force on } \Delta\mathcal{S}_t \;=\; \mathbf{t}\,\mathrm{d}A_y \;=\; \mathbf{T}\mathbf{n}\,\mathrm{d}A_y. \tag{4.51}$$

Next, recall from (2.34) the geometric relation

$$\mathrm{d}A_y\,\mathbf{n} = \mathrm{d}A_x\, J\,\mathbf{F}^{-T}\,\mathbf{n}_0 \tag{4.52}$$

relating the area $\mathrm{d}A_y$ to the area $\mathrm{d}A_x$, and the unit normal $\mathbf{n}$ to the unit normal $\mathbf{n}_0$. Combining (4.52) with (4.51) gives

$$\text{The contact force on } \Delta\mathcal{S}_t \;=\; (J\,\mathbf{T}\,\mathbf{F}^{-T})\,\mathbf{n}_0\,\mathrm{d}A_x \tag{4.53}$$

It is natural therefore to define a tensor $\mathbf{S}$ and a vector $\mathbf{s}$ by the respective equations

$$\mathbf{S} = J\,\mathbf{T}\,\mathbf{F}^{-T} , \tag{4.54}$$

$$\mathbf{s} = \mathbf{S}\,\mathbf{n}_0 , \tag{4.55}$$

whence

$$\text{The contact force on } \Delta\mathcal{S}_t = \mathbf{t}\,\mathrm{d}A_y = \mathbf{Tn}\,\mathrm{d}A_y = \mathbf{s}\,\mathrm{d}A_x = \mathbf{Sn}_0\,\mathrm{d}A_x . \tag{4.56}$$

Thus, $\mathbf{s}$ is the contact force per unit *referential area*. Note that it acts on the surface element $\Delta\mathcal{S}_t$ in the current configuration. This is illustrated in Figure 4.10. The vector $\mathbf{s}$ is called the *first Piola-Kirchhoff traction vector* and the tensor $\mathbf{S}$ is called the *first Piola-Kirchhoff stress tensor*.

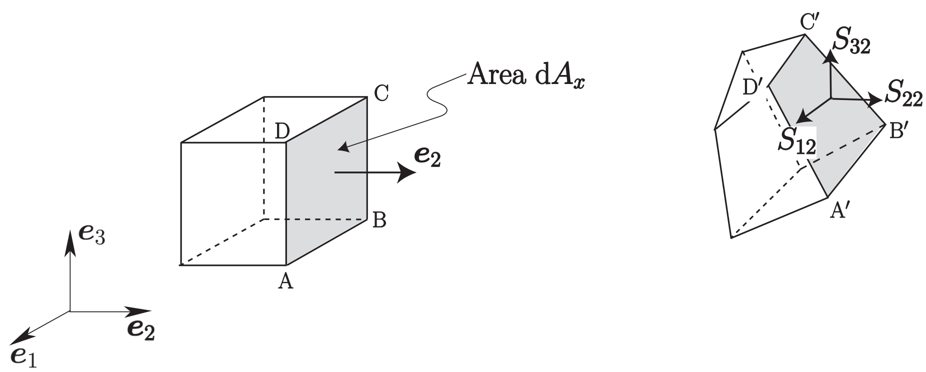


Figure 4.11: Physical significance of the components of the stress tensor $\mathbf{S}$: the shaded surface in the reference configuration is normal to $\mathbf{e}_2$ and has area dA_x. The i^{th} component of *force* acting on the image of this surface in the current configuration is $S_{i2} \times dA_x$.

The physical significance of the components of $\mathbf{S}$ can be deduced as follows. Recall first that the component S_{ij} of the tensor $\mathbf{S}$ in an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is defined by $S_{ij} = (\mathbf{S}\mathbf{e}_j) \cdot \mathbf{e}_i$. Next, taking $\mathbf{n}_0 = \mathbf{e}_j$ in (4.55) and taking the scalar product of the resulting equation with $\mathbf{e}_i$ gives $\mathbf{s}(\mathbf{e}_j) \cdot \mathbf{e}_i = \mathbf{S}\,\mathbf{e}_j \cdot \mathbf{e}_i$. By combining these it follows that

$$S_{ij} = \mathbf{S}\,\mathbf{e}_j \cdot \mathbf{e}_i = \mathbf{s}(\mathbf{e}_j) \cdot \mathbf{e}_i. \tag{4.57}$$

Therefore, S_{ij} is the i^{th} component of force per unit referential area acting on the surface which was normal to the j^{th} direction in the reference configuration. For example consider a surface element which was normal to $\mathbf{e}_2$ in the reference configuration; see Figure 4.11. Then, by taking $\mathbf{n}_0 = \mathbf{e}_2$ in (4.55), the contact force on this element is

$$\text{Contact force on } \Delta\mathcal{S}_t = \mathbf{t}\,\mathrm{d}A_y = \mathbf{s}\,\mathrm{d}A_x = \mathbf{S}\,\mathbf{e}_2\,\mathrm{d}A_x = (S_{12}\,\mathbf{e}_1 + S_{22}\,\mathbf{e}_2 + S_{32}\,\mathbf{e}_3)\,\mathrm{d}A_x. \tag{4.58}$$

This is illustrated in Figure 4.11. Note that the force we have calculated acts on the current configuration. The image of this surface in the reference configuration had unit normal $\mathbf{e}_1$. In order to determine the force, the appropriate component of $\mathbf{S}$ is multiplied by the area dA_x of the image of this surface element in the reference configuration.

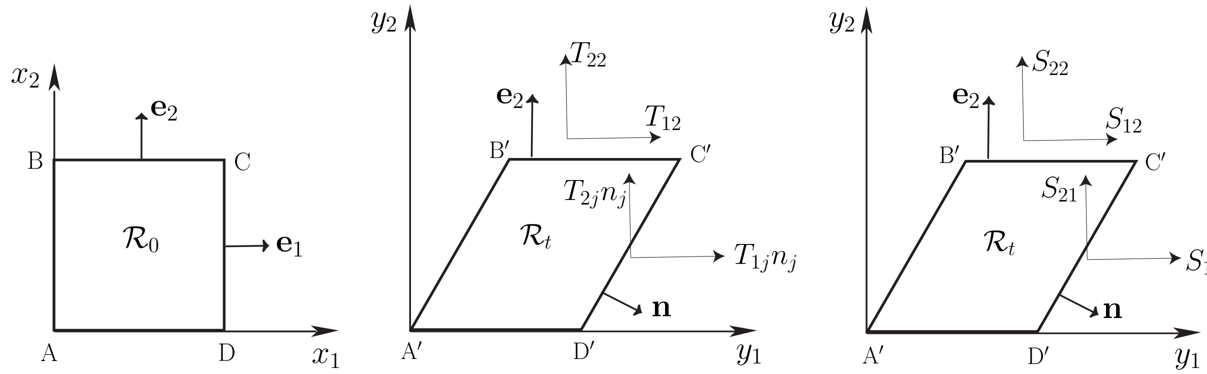


Figure 4.12: Simple shear. The middle and rightmost figures *both* show the region $\mathcal{R}_t$ occupied by the body at the same instant t. They depict the tractions on the faces $B'C'$ and $C'D'$ in two different, but equivalent, ways: the middle figure describes the traction in terms of the stress $\mathbf{T}$ while the rightmost figure describes them in terms of $\mathbf{S}$. The corresponding forces are given by multiplying each traction by either the current area of the relevant surface or its area in the reference configuration, depending on whether one is working with $\mathbf{S}$ or $\mathbf{T}$ respectively.

To illustrate this further, consider a simple shear deformation of a block as shown in Figure 4.12. The figure of the left shows the region $\mathcal{R}_0$, whereas the middle and right figures both show the region $\mathcal{R}_t$. The face $C'D'$ has a unit outward normal $\mathbf{n}$ and therefore

$$\text{Contact force on } C'D' = \mathbf{T}\mathbf{n}\times|C'D'| = [(T_{1j}n_j)\mathbf{e}_1 + (T_{2j}n_j)\mathbf{e}_2 + (T_{3j}n_j)\mathbf{e}_3]\times|C'D'|\,; \quad (4.59)$$

this is illustrated in the middle figure. Since the face CD, which is the image of $C'D'$, has a unit outward normal $\mathbf{e}_1$, we can *equivalently* write

$$\text{Contact force on } C'D' = \mathbf{S}\mathbf{e}_1 \times |CD| = [S_{11}\mathbf{e}_1 + S_{21}\mathbf{e}_2 + S_{31}\mathbf{e}_3] \times |CD|; \qquad (4.60)$$

this is illustrated in the right most figure. Similarly the unit outward normals to the face $B'C'$, and to its image BC, are both $\mathbf{e}_2$, and therefore we can write

$$\begin{aligned} \text{Contact force on } B'C' \quad &= \quad \mathbf{T}\mathbf{e}_2 \times |B'C'| = [T_{12}\mathbf{e}_1 + T_{22}\mathbf{e}_2 + T_{32}\mathbf{e}_3] \times |B'C'|, \\ &= \quad \mathbf{S}\mathbf{e}_2 \times |BC| = [S_{12}\mathbf{e}_1 + S_{22}\mathbf{e}_2 + S_{32}\mathbf{e}_3] \times |BC|; \end{aligned} \qquad (4.61)$$

these are also illustrated in Figure 4.12.

We now turn to the second point made at the beginning of this section. As mentioned there, any field quantity which is a function of the current position $\mathbf{y}$ and time t can be readily be converted to a field which depends on the reference position $\mathbf{x}$ and t by using the motion $\mathbf{y} = \mathbf{y}(\mathbf{x}, t)$ to change $\mathbf{y} \to \mathbf{x}$. Thus, for example, the Cauchy stress can be written as $\bar{\mathbf{T}}(\mathbf{x}, t) = \mathbf{T}(\mathbf{y}(\mathbf{x}, t), t)$ and so on. As mentioned there, one finds that making this change to the Cauchy stress does not lead to any advantages. It is however convenient to express $\mathbf{S}$ as a function of $\mathbf{x}$ and t, so that (4.54), (4.55) would (more precisely) read

$$\mathbf{S}(\mathbf{x}, t) = J(\mathbf{x}, t)\, \mathbf{T}(\mathbf{y}(\mathbf{x}, t), t)\, \mathbf{F}^{-T}(\mathbf{x}, t)\ , \tag{4.62}$$

$$\mathbf{s}(\mathbf{x}, \mathbf{n}_0, t) = \mathbf{S}(\mathbf{x}, t)\, \mathbf{n}_0\ . \tag{4.63}$$

Substituting (4.62) into (4.43) and simplifying shows that the equation of motion $(4.43)_1$ and the angular momentum equation $(4.43)_2$ can be written in the equivalent forms

$$\left.\begin{aligned} \operatorname{Div}\mathbf{S} + \rho_0\mathbf{b} &= \rho_0\dot{\mathbf{v}}, \\ \mathbf{S}\mathbf{F}^T &= \mathbf{F}\mathbf{S}^T, \end{aligned}\right\} \tag{4.64}$$

where these field equations must hold at every point $\mathbf{x} \in \mathcal{R}_0$ and every instant t; the traction on a surface continues to be related to the stress through

$$\mathbf{s} = \mathbf{S}\mathbf{n}_0. \tag{4.65}$$

In component form

$$\frac{\partial S_{ij}}{\partial x_j} + \rho_0 b_i = \rho_0 \dot{v}_i, \qquad S_{ik}F_{jk} = F_{ik}S_{jk}, \qquad s_i = S_{ij}n_j. \tag{4.66}$$

Note that in general, the stress tensor $\mathbf{S}$ is *not* symmetric. Consequently, among other things, this means that $\mathbf{S}$ may not have three real eigenvalues, so that we usually do not speak of the principal values of the first Piola-Kirchhoff stress tensor.

It is useful to construct the various terms of, say, the global balance of linear momentum in terms of these referential ingredients. Let $\mathcal{D}_0$ and $\mathcal{D}_t$ be the regions occupied by a part $\mathcal{P}$ of the body in the reference configuration and the current configuration respectively. From (4.56) the resultant contact force on $\mathcal{P}$ at time t is

$$= \int_{\partial\mathcal{D}_t} \mathbf{t}\, dA_y = \int_{\partial\mathcal{D}_t} \mathbf{T}\mathbf{n}\, dA_y = \int_{\partial\mathcal{D}_0} \mathbf{s}\, dA_x = \int_{\partial\mathcal{D}_0} \mathbf{S}\mathbf{n}_0\, dA_x.$$

Since ρ and ρ_0 are the respective mass densities in the reference and current configurations, and since $\mathbf{b}$ is the body force per unit mass, the resultant body force on $\mathcal{P}$ at time t is

$$= \int_{\mathcal{D}_t} \rho \mathbf{b}\, dV_y = \int_{\mathcal{D}_0} \rho_0 \mathbf{b}\, dV_x.$$

Similarly the linear momentum of $\mathcal{P}$ at time t is

$$= \int_{\mathcal{D}_t} \rho \mathbf{v}\, dV_y = \int_{\mathcal{D}_0} \rho_0 \mathbf{v}\, dV_x.$$

Consequently, the balance law for linear momentum (4.18) can be equivalently written as

$$\int_{\partial\mathcal{D}_0} \mathbf{s}\, \mathrm{d}A_x + \int_{\mathcal{D}_0} \rho_0\, \mathbf{b}\, \mathrm{d}V_x = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{D}_0} \rho_0\, \mathbf{v}\, \mathrm{d}V_x \tag{4.67}$$

which must hold for all subregions $\mathcal{D}_0$ of the region $\mathcal{R}_0$ occupied by the body in the reference configuration. The field equation $(4.64)_1$ corresponding to (4.67) can now be derived (alternatively) by using the divergence theorem on (4.67) and then localizing the result.

Similarly, the resultant moment of the contact force on $\mathcal{P}$ at time t is given by

$$= \int_{\partial\mathcal{D}_t} \mathbf{y} \times \mathbf{t}\, dA_y = \int_{\partial\mathcal{D}_t} \mathbf{y} \times \mathbf{T}\mathbf{n}\, dA_y = \int_{\partial\mathcal{D}_0} \mathbf{y}(\mathbf{x},t) \times \mathbf{s}\, dA_x = \int_{\partial\mathcal{D}_0} \mathbf{y}(\mathbf{x},t) \times \mathbf{S}\mathbf{n}_0\, dA_x$$

where we have again used (4.56). In this way one finds that the balance law for angular momentum (4.19) can be written equivalently as

$$\int_{\partial\mathcal{D}_0} \mathbf{y} \times \mathbf{S}\mathbf{n}_0\, dA_x + \int_{\mathcal{D}_0} \mathbf{y} \times \rho_0 \mathbf{b}\, dV_x = \frac{d}{dt} \int_{\mathcal{D}_0} \mathbf{y} \times \rho_0 \mathbf{v}\, dV_x\ . \tag{4.68}$$

Perhaps it is worth remarking that it is *not* $\mathbf{x} \times \mathbf{S}\mathbf{n}_0$ etc. that appear here but rather $\mathbf{y}(\mathbf{x},t) \times \mathbf{S}\mathbf{n}_0$ etc.

In the literature one encounters a number of stress measures some examples of which are

$\mathbf{T}$	Cauchy stress tensor
$J\mathbf{T}$	Kirchhoff stress tensor
$\mathbf{S} = J\mathbf{T}\mathbf{F}^{-T}$	First Piola – Kirchhoff stress tensor
$\mathbf{S}^{(2)} = J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T}$	Second Piola – Kirchhoff stress tensor
$\mathbf{S}^T = J\mathbf{F}^{-1}\mathbf{T}$	Nominal stress tensor
$\mathbf{S}^{(1)} = \frac{1}{2}\left(\mathbf{S}^T\mathbf{R} + \mathbf{R}^T\mathbf{S}\right)$	Biot stress tensor

Even though many of these stresses have no simple physical significance, they are sometimes useful in, say, carrying out computations.

Problems 4.15 - 4.18.

4.9 Stress Power.

4.9.1 A Work-Energy Identity.

Consider some part $\mathcal{P}$ of the body. We now derive a relationship between the rate of external work on $\mathcal{P}$, the rate of internal working within $\mathcal{P}$, and the kinetic energy of $\mathcal{P}$. This analysis, like everything else so far, is independent of constitutive relation and is valid for *all* materials. It should also be noted that the relationship derived here is *not* the first law of thermodynamics; it is a relationship that is entirely mechanical in character.

Let $p(\mathcal{P};t)$ denote the *rate at which external work* is being done on $\mathcal{P}$ at some instant during the motion. From (4.13),

$$p(\mathcal{P};t) = \int_{\partial\mathcal{D}_t} \mathbf{t}\cdot\mathbf{v}\,\mathrm{d}A_y + \int_{\mathcal{D}_t} \rho\mathbf{b}\cdot\mathbf{v}\,\mathrm{d}V_y\ . \tag{4.69}$$

It is convenient to work in terms of components in some fixed orthonormal basis. Then we have

$$\begin{aligned}
p(\mathcal{P};t) &= \int_{\partial\mathcal{D}_t} t_i v_i\,\mathrm{d}A_y + \int_{\mathcal{D}_t} \rho b_i v_i\,\mathrm{d}V_y \ = \int_{\partial\mathcal{D}_t} T_{ij}n_j v_i\,\mathrm{d}A_y + \int_{\mathcal{D}_t} \rho b_i v_i\,\mathrm{d}V_y \\
&= \int_{\mathcal{D}_t} \frac{\partial}{\partial y_j}\Big(T_{ij}v_i\Big)\,\mathrm{d}V_y + \int_{\mathcal{D}_t} \rho b_i v_i\,\mathrm{d}V_y \\
&= \int_{\mathcal{D}_t} \left[\frac{\partial T_{ij}}{\partial y_j}v_i + T_{ij}\frac{\partial v_i}{\partial y_j}\ +\ \rho b_i v_i\right]\mathrm{d}V_y = \int_{\mathcal{D}_t}\left[\rho v_i\dot{v}_i + T_{ij}\frac{\partial v_i}{\partial y_j}\right]\mathrm{d}V_y \\
&= \int_{\mathcal{D}_t}\left[\rho\frac{\partial}{\partial t}\left(\frac{1}{2}v_i v_i\right) + \frac{1}{2}T_{ij}\left(\frac{\partial v_i}{\partial y_j} + \frac{\partial v_j}{\partial y_i}\right)\right]\mathrm{d}V_y \\
&= \int_{\mathcal{D}_t}\left[\rho\frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) + T_{ij}D_{ij}\right]\mathrm{d}V_y = \int_{\mathcal{D}_t}\left[\rho\frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) + \mathbf{T}\cdot\mathbf{D}\right]\mathrm{d}V_y \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{D}_t}\frac{1}{2}\,\rho\,\mathbf{v}\cdot\mathbf{v}\,\mathrm{d}V_y + \int_{\mathcal{D}_t}\mathbf{T}\cdot\mathbf{D}\,\mathrm{d}V_y
\end{aligned} \tag{4.70}$$

In the first line we have used the relation between traction and stress; in the second line we have used the divergence theorem; in the third line we have used the equation of motion $(4.45)_1$; in the fourth line we have made use of the fact that the stress tensor is symmetric; in the fifth line we have introduced the stretching tensor $\mathbf{D}$ and used the formula for the inner product of two tensors; and in the final line we have used (4.8).

And thus, by combining (4.70) with (4.69) we find the work-energy relation

$$\int_{\partial\mathcal{D}_t} \mathbf{t}\cdot\mathbf{v}\,\mathrm{d}A_y + \int_{\mathcal{D}_t} \rho\mathbf{b}\cdot\mathbf{v}\,\mathrm{d}V_y = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{D}_t} \frac{1}{2}\,\rho\,\mathbf{v}\cdot\mathbf{v}\,\mathrm{d}V_y + \int_{\mathcal{D}_t} \mathbf{T}\cdot\mathbf{D}\,\mathrm{d}V_y\,. \tag{4.71}$$

Equation (4.71) states that the rate of external work on any part of the body (represented by the left-hand side) equals the rate of increase of kinetic energy of that part (represented by the first term on the right-hand side) and the rate of internal work within that part (represented by the second term on the right-hand side). Thus the last term of the right-hand side of (4.71) represents the rate of working by the internal stresses in $\mathcal{D}_t$ and is often called the *stress power*. In general, the stress power accounts for both stored and dissipated energy. The integral involving the stress power in (4.71) *cannot in general be written as the time derivative* of the volume integral of some scalar field.

One can readily show that the work-energy identity (4.71) can be expressed equivalently in the referential formulation by

$$\int_{\partial\mathcal{D}_0} \mathbf{s}\cdot\mathbf{v}\,\mathrm{d}A_x + \int_{\mathcal{D}_0} \rho_0\,\mathbf{b}\cdot\mathbf{v}\,\mathrm{d}V_x = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{D}_0} \frac{1}{2}\,\rho_0\,\mathbf{v}\cdot\mathbf{v}\,\mathrm{d}V_x + \int_{\mathcal{D}_0} \mathbf{S}\cdot\dot{\mathbf{F}}\,\mathrm{d}V_x \tag{4.72}$$

and that the *stress power density* – the stress power per unit reference volume – can be written as

$$\text{Stress power density } = \mathbf{S}\cdot\dot{\mathbf{F}} = J\,\mathbf{T}\cdot\mathbf{D} \tag{4.73}$$

where $J = \det\mathbf{F}$.

Later in Section 5.5 we will show that the stress power is an objective quantity.

4.9.2 Work Conjugate Stress-Strain Pairs.

Consider a body undergoing an arbitrary motion. Suppose that the stress power density can be expressed in the form $\mathbf{A}\cdot\dot{\mathbf{B}}$ where $\mathbf{B}$ is a strain measure (in the sense of Section 2.7) and the components of $\mathbf{A}$ have the dimensions of stress. Then we say that the stress $\mathbf{A}$ and the strain $\mathbf{B}$ are conjugate[9]. This conjugacy reflects a special relationship between these two measures.

For example, consider the family of Lagrangian strain tensors

$$\mathbf{E}^{(n)} = \frac{1}{n}\Big(\mathbf{U}^n - \mathbf{I}\Big), \qquad n \neq 0.$$

Can one find a corresponding family of stress tensors $\mathbf{S}^{(n)}$ such that the

$$\text{stress power density } = \mathbf{S}^{(n)} \cdot \dot{\mathbf{E}}^{(n)} \ ?$$

Consider the *case* $n = 2$. Since $\mathbf{U}^2 = \mathbf{F}^T\mathbf{F}$,

$$\dot{\mathbf{E}}^{(2)} = \frac{1}{2}\left(\dot{\mathbf{F}}^T\mathbf{F} + \mathbf{F}^T\dot{\mathbf{F}}\right).$$

Recalling that $\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}$ where $\mathbf{L}$ is the velocity gradient tensor and using this to eliminate $\dot{\mathbf{F}}$ yields

$$\dot{\mathbf{E}}^{(2)} = \frac{1}{2}\left(\mathbf{F}^T\mathbf{L}^T\mathbf{F} + \mathbf{F}^T\mathbf{L}\mathbf{F}\right) = \mathbf{F}^T\mathbf{D}\mathbf{F}$$

where $\mathbf{D} = (\mathbf{L}+\mathbf{L}^T)/2$ is the stretching tensor. So the tensor $\mathbf{S}^{(2)}$ that we seek must be such that

$$\mathbf{S}^{(2)} \cdot \mathbf{F}^T\mathbf{D}\mathbf{F} = J\mathbf{T}\cdot\mathbf{D}.$$

It follows from this that

$$\mathbf{S}^{(2)} = J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T} \quad \text{or equivalently} \quad \mathbf{S}^{(2)} = \mathbf{F}^{-1}\mathbf{S}$$

which is in fact the second Piola-Kirchhoff stress tensor.

The case of general n is treated in Chapter 3.5 of Ogden's book.

4.10 Linearization.

Finally we turn to motions that are infinitesimal in the sense that the displacement gradient is small: $|\text{Grad}\,\mathbf{u}| << 1$. Since $\mathbf{F} = \mathbf{I} + \text{Grad}\,\mathbf{u}$ and $J = \det\mathbf{F} = 1 + \text{Div}\,\mathbf{u} + O(|\text{Grad}\,\mathbf{u}|^2)$, it follows from $\mathbf{T} = J^{-1}\mathbf{S}\mathbf{F}^T$ that, to leading order,

$$\mathbf{S} \sim \mathbf{T}\ . \tag{4.74}$$

[9] Recall from (4.73) that the stress power density is given by $J\,\mathbf{T}\cdot\mathbf{D} = \mathbf{S}\cdot\dot{\mathbf{F}}$. However note that $\mathbf{F}$ is not a strain and $\mathbf{D}$ is not the material time derivative of a strain. Thus one usually does not refer to the quantities here as being conjugate.

Thus the 1st Piola-Kirchoff stress tensor and the Cauchy stress tensor do not differ in infinitesimal motions to leading order. For clarity we shall use the symbol $\boldsymbol{\sigma}$ for the stress under these circumstances. Similarly the mass density $\rho = \rho_0 + O(|\mathrm{Grad}\,\mathbf{u}|)$.

Thus for infinitesimal motions, the equation of motion $(4.64)_1$ reads, to leading order,

$$\mathrm{Div}\,\boldsymbol{\sigma} + \rho_0\,\mathbf{b} = \rho_0\,\dot{\mathbf{v}} \tag{4.75}$$

while the angular momentum requirement $(4.64)_2$ tells us that $\boldsymbol{\sigma}$ is symmetric:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T; \tag{4.76}$$

both of these equations must hold at each $\mathbf{x} \in \mathcal{R}_0$ and all times t. Note that these field equations hold on the region $\mathcal{R}_0$ occupied in the reference configuration. In formulating these various force requirements we do not need to distinguish between the reference and current geometries. Similarly, the traction-stress relation is

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}_0 \tag{4.77}$$

where we take $\mathbf{n}_0$ to be the unit normal in the undeformed configuration. Finally the stress component σ_{ij} is the ith component of force per unit area on the surface normal to $\mathbf{e}_j$.

Thus in conclusion, for infinitesimal deformations we will work with the stress tensor $\boldsymbol{\sigma}$ and we do not need to consider the deformed configuration in formulating any of the fundamental principles for stress. Reviewing the preceding material in this chapter, we see that, for example, we can interpret the stress components σ_{ij} as in Figure 4.8 with T_{ij} replaced by σ_{ij}; we do not need to address whether the planes shown lie in the reference or current configurations. Similarly in the example discussed in Section 4.5, the prismatic region described is that which is occupied by the body and we do not need to address whether that is in the undeformed or deformed configurations.

Problems 4.19 - 4.23.

4.11 Objectivity of Mechanical Quantities.

We shall consider the objectivity of both mechanical and thermodynamic quantities in Section 5.5.

4.12 Worked Examples and Exercises.

Unless explicitly told otherwise, neglect body forces and inertial effects.

Problem 4.1. Consider all planes that pass through a particular point $\mathbf{y}$. Let $\mathbf{n}$ denote a unit normal vector to such a plane and let $T_n(\mathbf{y}, \mathbf{n})$ be the "normal stress" on this plane:

$$T_n = \mathbf{t}(\mathbf{y}, \mathbf{n}) \cdot \mathbf{n}.$$

From among all planes through $\mathbf{y}$, on which one is T_n a maximum?

Problem 4.2.

Show that there is always a plane through any point $\mathbf{y}$ on which the resultant shear stress T_s is zero. Is there also, in general, a plane on which the normal stress T_n vanishes?

Problem 4.3. Let $\mathbf{n}$ be a unit vector which is equally inclined to the principal axes of $\mathbf{T}$. The plane normal to $\mathbf{n}$ is known as the *octahedral plane*. Calculate the normal stress and the resultant shear stress on the octahedral plane in terms of the principal stress components T_1, T_2, T_3.

Problem 4.4. All vector and tensor components referred to in this problem are components with respect to an arbitrary fixed basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

(a) Let $\mathbf{T}$ be a stress tensor with principal stresses T_1, T_2, T_3 and corresponding principal directions $\boldsymbol{\nu}^{(1)}, \boldsymbol{\nu}^{(2)}, \boldsymbol{\nu}^{(3)}$. Express T_{ij} in terms of the principal stresses and the components of the principal directions.

(b) Let $\mathbf{T}$ be a stress tensor which corresponds to a pure uniaxial stress T_o in a direction $\mathbf{m}$. Express T_{ij} in terms of T_o and the components of $\mathbf{m}$.

(c) Let $\mathbf{T}$ be a stress tensor which corresponds to the superposition of a uniaxial stress T_1 in the direction $\mathbf{e}_1$, a uniaxial stress T_2 in the direction $\mathbf{e}_2$, and a uniaxial stress T_3 in the direction $(\mathbf{e}_1 + \mathbf{e}_2)/\sqrt{2}$. Express T_{ij} in terms of T_1, T_2, T_3.

(d) Let $\mathbf{T}$ be a stress tensor which corresponds to a state of pure shear stress T with respect to the mutually orthogonal directions $\mathbf{m}, \mathbf{n}$. Express T_{ij} in terms of T and the components of $\mathbf{m}, \mathbf{n}$.

Solution:

(a) Let ν_{ij} be the jth component of the vector $\boldsymbol{\nu}^{(i)}$:

$$\boldsymbol{\nu}^{(i)} = \sum_{j=1}^{3} \nu_{ij}\mathbf{e}_j. \tag{a}$$

The scalars ν_{ij} can be calculated from $\nu_{ij} = \boldsymbol{\nu}^{(i)} \cdot \mathbf{e}_j$. We are given that

$$\mathbf{T} = \sum_{i=1}^{3} T_i \boldsymbol{\nu}^{(i)} \otimes \boldsymbol{\nu}^{(i)}. \tag{b}$$

Substituting (a) into (b) and simplifying leads to

$$\mathbf{T} = \sum_{i=1}^{3} T_i \left(\sum_{j=1}^{3} \nu_{ij}\mathbf{e}_j \right) \otimes \left(\sum_{k=1}^{3} \nu_{ik}\mathbf{e}_k \right) = \sum_{j=1}^{3}\sum_{k=1}^{3} \left(\sum_{i=1}^{3} T_i \nu_{ij}\nu_{ik} \right) \mathbf{e}_j \otimes \mathbf{e}_k.$$

Therefore the component T_{jk} of stress is given by

$$T_{jk} = \sum_{i=1}^{3} T_i \nu_{ij}\nu_{ik}.$$

(b) Let $m_i = \mathbf{m}\cdot\mathbf{e}_i$ be the ith component of $\mathbf{m}$:

$$\mathbf{m} = \sum_{i=1}^{3} m_i\mathbf{e}_i. \tag{c}$$

We are given that

$$\mathbf{T} = T_o \mathbf{m} \otimes \mathbf{m}. \tag{d}$$

Substituting (c) into (d) gives

$$\mathbf{T} = T_o \left(\sum_{i=1}^{3} m_i\mathbf{e}_i \right) \otimes \left(\sum_{j=1}^{3} m_j\mathbf{e}_j \right) = \sum_{i=1}^{3}\sum_{j=1}^{3} (T_o\, m_i m_j)\, \mathbf{e}_i \otimes \mathbf{e}_j$$

Therefore

$$T_{ij} = T_o\, m_i m_j.$$

(c) Here we are given that

$$\begin{aligned} \mathbf{T} &= T_1\mathbf{e}_1 \otimes \mathbf{e}_1 + T_2\mathbf{e}_2 \otimes \mathbf{e}_2 + \tfrac{T_3}{2}(\mathbf{e}_1 + \mathbf{e}_2) \otimes (\mathbf{e}_1 + \mathbf{e}_2) \\ &= \left(T_1 + \tfrac{T_3}{2}\right) \mathbf{e}_1 \otimes \mathbf{e}_1 + \left(T_2 + \tfrac{T_3}{2}\right) \mathbf{e}_2 \otimes \mathbf{e}_2 + \tfrac{T_3}{2} \left(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1\right). \end{aligned}$$

Thus

$$T_{11} = T_1 + \frac{T_3}{2}, \quad T_{22} = T_2 + \frac{T_3}{2}, \quad T_{12} = T_{21} = \frac{T_3}{2}, \quad T_{13} = T_{23} = T_{33} = 0.$$

(d) Let $m_i = \mathbf{m}\cdot\mathbf{e}_i$ and $n_i = \mathbf{n}\cdot\mathbf{e}_i$ be the ith components of $\mathbf{m}$ and $\mathbf{n}$:

$$\mathbf{m} = \sum_{i=1}^{3} m_i\mathbf{e}_i, \qquad \mathbf{n} = \sum_{i=1}^{3} n_i\mathbf{e}_i. \tag{e}$$

We are given that

$$\mathbf{T} = T(\mathbf{m} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{m}). \tag{f}$$

Substituting (e) into (f) gives

$$\mathbf{T} = T\left(\sum_{i=1}^{3} m_i \mathbf{e}_i\right) \otimes \left(\sum_{j=1}^{3} n_j \mathbf{e}_j\right) + T\left(\sum_{i=1}^{3} n_i \mathbf{e}_i\right) \otimes \left(\sum_{j=1}^{3} m_j \mathbf{e}_j\right) = \sum_{i=1}^{3}\sum_{j=1}^{3} T\Big(m_i n_j + m_j n_i\Big)\mathbf{e}_i \otimes \mathbf{e}_j.$$

Therefore

$$T_{ij} = T\Big(m_i n_j + m_j n_i\Big).$$

Problem 4.5. A submersible underwater vessel has the form of a sphere of (outer) radius r containing a cubic compartment whose dimensions are $a \times a \times a$. (Assume that these are the dimensions of the vessel in its pressurized configuration.) When submerged, the outer surface is subjected to a uniform pressure p_2 while the compartment pressure is p_1. Calculate the *mean stress* $\overline{\mathbf{T}}$ in the vessel, defined as

$$\overline{\mathbf{T}} = \frac{1}{\mathrm{vol}(\mathcal{R}_b)} \int_{R_b} \mathbf{T}\, \mathrm{d}V_y; \tag{a}$$

here $\mathcal{R}_b$ is the region of space occupied by the body and its volume is $\mathrm{vol}(\mathcal{R}_b) = 4\pi r^3/3 - a^3$.

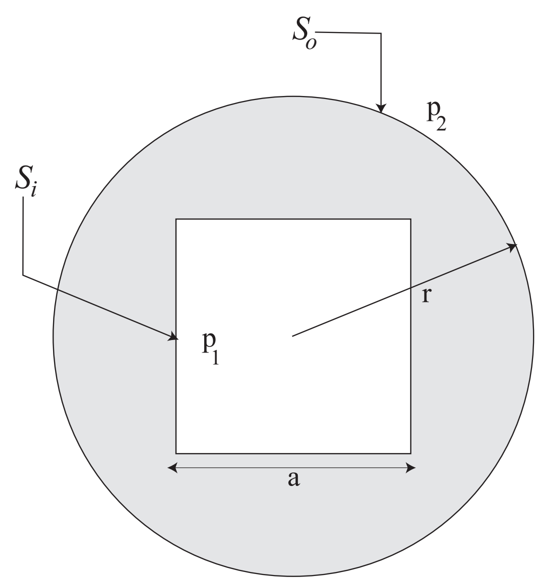


Figure 4.13: A spherical vessel of radius r with hollow cubic compartment of side a.

<u>*Solution*</u>: Since we are only given the loading on the boundary of the body in this problem, it appears that the mean stress in the body must depend solely on the traction on the boundary. If this is true, it is natural to wonder whether this is true for a body of more general shape. Therefore, before attacking the specific problem at hand, let us see what we can say about the mean stress more generally.

We would like to calculate a formula for the mean stress in the body in terms of the boundary tractions alone (if this is possible). If we were to calculate the integral of t_i over the entire boundary we would get zero because of equilibrium; moreover, since $\overline{T}_{ij}$ is a second order tensor (two subscripts) the otherside of the formula we want to derive should also involve a second order tensor (two subscripts). We can consider two possibility: the integral over the boundary of either $t_i t_j$ or $y_i t_j$. The former has the wrong dimensions (force squared) and so we consider the later. Based on this motivation we proceed as follows:

$$\begin{aligned}\textstyle\int_{\partial\mathcal{R}_b} y_i t_j \, dA_y &= \textstyle\int_{\partial\mathcal{R}_b} y_i T_{jk} n_k \, dA_y = \int_{\mathcal{R}_b} (y_i T_{jk})_{,k} \, dV_y = \int_{\mathcal{R}_b} (y_{i,k} T_{jk} + y_i T_{jk,k}) \, dV_y \\ &= \textstyle\int_{\mathcal{R}_b} (\delta_{ik} T_{jk}) \, dV_y = \int_{\mathcal{R}_b} T_{ij} \, dV_y.\end{aligned}$$

In this calculation we have made use of the traction-stress relation $t_i = T_{ij} n_j$, the divergence theorem, the equilibrium equation in the absence of body forces $T_{ij,j} = 0$ and the fact that $y_{i,j} = \partial y_i / \partial y_j = \delta_{ij}$. This shows that

$$\int_{\mathcal{R}_b} T_{ij} \, dV_y = \int_{\partial\mathcal{R}_b} y_i t_j \, dA_y \qquad \text{or equivalently} \qquad \int_{\mathcal{R}_b} \mathbf{T} \, dV_y = \int_{\partial\mathcal{R}_b} \mathbf{y} \otimes \mathbf{t} \, dA_y.$$

Therefore the mean Cauchy stress defined by (a) can be expressed solely in terms of the traction on the boundary by

$$\overline{\mathbf{T}} \; = \frac{1}{\text{vol}(R_b)} \int_{\partial R_b} \mathbf{y} \otimes \mathbf{t} \, \mathrm{d}A_y. \tag{b}$$

Now consider the specific problem at hand. The region $\mathcal{R}_b$ occupied by the body is contained between a closed outer spherical surface $\mathcal{S}_o$ and a closed inner cubic surface $\mathcal{S}_i$. Let $\mathcal{R}_c$ denote the region enclosed by the inner surface $\mathcal{S}_i$, i.e. the region occupied by the cubic cavity; therefore vol$(\mathcal{R}_c) = a^3$. Similarly let $\mathcal{R}_0 = \mathcal{R}_b + \mathcal{R}_c$ denote the region enclosed by the outer spherical surface $\mathcal{S}_o$ so that vol$(\mathcal{R}_o) = 4\pi r^3/3$.

We are given that $\mathbf{t} = -p_1 \mathbf{n}$ on $\mathcal{S}_i$ and $\mathbf{t} = -p_2 \mathbf{n}$ on $\mathcal{S}_o$. Therefore

$$\int_{\mathcal{S}_i} y_i t_j \, dA_y = -p_1 \int_{\mathcal{S}_i} y_i n_j \, dA_y = -p_1 \int_{\mathcal{R}_c} y_{i,j} \, dV_y = -p_1 \delta_{ij} \int_{\mathcal{R}_c} dV_y = -p_1 \ \text{vol}(\mathcal{R}_c) \, \delta_{ij}$$

where we have used the divergence theorem, and similarly,

$$\int_{\mathcal{S}_o} y_i t_j \, dA_y = -p_2 \ \text{vol}(\mathcal{R}_o) \, \delta_{ij}.$$

Since $\partial\mathcal{R}_b = \mathcal{S}_i \cup \mathcal{S}_o$ we can combine these with (b) to obtain

$$\overline{\mathbf{T}} = \frac{1}{\text{vol}(\mathcal{R}_b)} \Big(\int_{\mathcal{S}_o} \mathbf{y} \otimes \mathbf{t} \, dA_y + \int_{\mathcal{S}_i} \mathbf{y} \otimes \mathbf{t} \Big) \, dA_y = \frac{\Big(- p_1 \ \text{vol}(\mathcal{R}_\text{c}) \mathbf{I} - p_2 \ \text{vol}(\mathcal{R}_\text{o}) \mathbf{I} \Big)}{\text{vol}(\mathcal{R}_b)} \; .$$

Thus

$$\overline{\mathbf{T}} = \overline{\mathbf{T}} = -\frac{p_1(a^3) + p_2(4\pi r^3/3)}{4\pi r^3/3 - a^3} \ \mathbf{I}.$$

Problem 4.6. Suppose that $\mathcal{R}_t$, the region occupied by a certain body in its current configuration, is a right circular cylinder of length l and radius a:

$$\mathcal{R}_t = \{(y_1, y_2, y_3) \mid y_1^2 + y_2^2 \le a^2, \ \ -l \le y_3 \le 0\}.$$

Suppose that the matrix of components of the Cauchy stress tensor field in the cylinder is

$$[T] = \begin{pmatrix} 0 & 0 & -\alpha y_2 \\ 0 & 0 & \alpha y_1 \\ -\alpha y_2 & \alpha y_1 & \beta + \gamma y_1 + \delta y_2 \end{pmatrix} \qquad (a)$$

where α, β, γ and δ are constants. The components here (and throughout) have been taken with respect to an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ where $\mathbf{e}_3$ is aligned with the axis of the cylinder.

(i) Verify that this stress field satisfies the equilibrium equations in the absence of body forces.

(ii) Verify that the curved surface of the cylinder is traction-free.

(iii) Calculate the traction on the end $y_3 = 0$. Hence calculate the resultant force and couple acting on the cylinder at the end $y_3 = 0$. Hence show that the parameters α, β, γ and δ describe, respectively, twisting of the cylinder about the y_3-axis, pulling of the cylinder in the y_1-direction, bending of the cylinder about the y_2-axis, and bending of the cylinder about the y_1-axis.

(iv) Given a circular cylinder which is subjected to axial loading, twisting and bending, does it therefore follow that the stress field in the body has to be the stress field given in this problem statement?

(v) Calculate the principal components of stress at an arbitrary point in the body. Calculate the value of the largest normal stress in the cylinder.

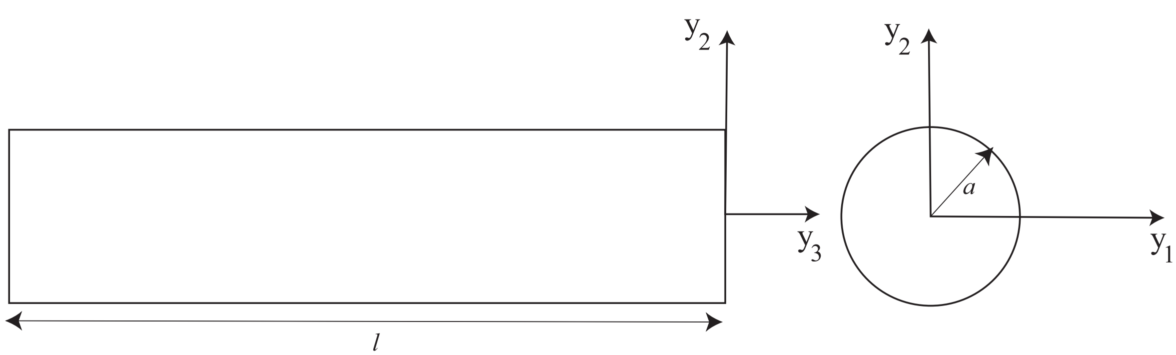


Figure 4.14: A right circular cylinder of length ℓ and radius a.

Solution:

(i) To check whether this stress field is in equilibrium we substitute the given stress field into the equilibrium equations $T_{ij,j} = 0$. It is easily seen that these equations hold trivially since each term in each equation

$$T_{11,1} + T_{12,2} + T_{13,3} = 0; \quad T_{21,1} + T_{22,2} + T_{23,3} = 0; \quad T_{31,1} + T_{32,2} + T_{33,3} = 0.$$

is identically zero.

(*ii*) The components of the unit outward normal vector $\mathbf{n}$ on the curved surface $y_1^2 + y_2^2 = a^2$ can be written as

$$n_1 = \cos\theta, \quad n_2 = \sin\theta, \quad n_3 = 0, \qquad 0 \le \theta < 2\pi,$$

and so the traction on this surface can be calculated using $\mathbf{t} = \mathbf{T}\mathbf{n}$. This first two of these equations hold trivially

$$t_1 = T_{11}n_1 + T_{12}n_2 + T_{13}n_3 = 0; \quad t_2 = T_{21}n_1 + T_{22}n_2 + T_{23}n_3 = 0.$$

The third equation simplifies as follows

$$t_3 = T_{31}n_1 + T_{32}n_2 + T_{33}n_3 = (-\alpha y_2)\cos\theta + (\alpha y_1)\sin\theta = (-\alpha a\sin\theta)\cos\theta + (\alpha a\cos\theta)\sin\theta = 0$$

where we have used the fact that $y_1 = a\cos\theta, y_2 = a\sin\theta$ at the "point θ" on the curved boundary.

(*iii*) The components of the unit outward normal vector $\mathbf{n}$ on the end $y_3 = 0$ are $n_1 = n_2 = 0, n_3 = 1$. Therefore the traction on this surface has components $t_i = T_{ij}n_j$:

$$t_1 = T_{13}n_3 = -\alpha y_2, \quad t_2 = T_{23}n_3 = \alpha y_1, \quad t_3 = T_{33}n_3 = \beta + \gamma y_1 + \delta y_2.$$

Let $\mathcal{S}_0$ denote the region occupied by the end at $y_3 = 0$: $\mathcal{S}_0 = \{(y_1, y_2, y_3) \mid y_3 = 0 \ y_1^2 + y_2^2 \le a^2 \ \}$. Then the resultant force on $\mathcal{S}_0$ is given by the integral of $\mathbf{t}$ over the surface $\mathcal{S}_0$. Therefore

$$\begin{aligned}
F_1 &= \int_{\mathcal{S}_0} t_1\, dA_y = \int_{\mathcal{S}_0} T_{1j}n_j\, dA_y = \int_{\mathcal{S}_0} T_{13}\, dA_y = \int_{\mathcal{S}_0} (-\alpha y_2)\, dA_y &&= 0,\\
F_2 &= \int_{\mathcal{S}_0} t_2\, dA_y = \int_{\mathcal{S}_0} T_{2j}n_j\, dA_y = \int_{\mathcal{S}_0} T_{23}\, dA_y = \int_{\mathcal{S}_0} (\alpha y_1)\, dA_y &&= 0\\
F_3 &= \int_{\mathcal{S}_0} t_3\, dA_y = \int_{\mathcal{S}_0} T_{3j}n_j\, dA_y = \int_{\mathcal{S}_0} T_{33}\, dA_y = \int_{\mathcal{S}_0} (\beta + \gamma y_1 + \delta y_2)\, dA_y &&= \beta\pi a^2.
\end{aligned}$$

Therefore the resultant force on this end is a pure axial force in the y_3-direction:

$$\mathbf{F} = \beta\pi a^2\ \mathbf{e}_3. \tag{b}$$

Turning next to the resultant moment on this face we recall that in general the resultant moment $\mathbf{M}$ is given by the integral of $\mathbf{y} \times \mathbf{t}$ over the surface $\mathcal{S}_0$. Therefore

$$\begin{aligned}
M_1 &= \int_{\mathcal{S}_0} y_2 t_3\, dA_y = \int_{\mathcal{S}_0} y_2(\beta + \gamma y_1 + \delta y_2)\, dA_y = \delta\frac{\pi a^4}{4},\\
M_2 &= \int_{\mathcal{S}_0} -y_1 t_3\, dA_y = \int_{\mathcal{S}_0} -y_1(\beta + \gamma y_1 + \delta y_2)\, dA_y = -\gamma\frac{\pi a^4}{4},\\
M_3 &= \int_{\mathcal{S}_0} (-y_2 t_1 + y_1 t_2)\, dA_y = \int_{\mathcal{S}_0} (\alpha y_2^2 + \alpha y_1^2)\, dA_y = \alpha\frac{\pi a^4}{2}.
\end{aligned}$$

Therefore the resultant moment on the face $y_3 = 0$ is

$$\mathbf{m} = \frac{\pi a^4}{4}\big(\delta\mathbf{e}_1 - \gamma\mathbf{e}_2 + 2\alpha\mathbf{e}_3\big). \tag{c}$$

Therefore from equations (b) and (c) we conclude that the parameters α, β, γ and δ describe, respectively, twisting of the cylinder about the y_3-axis, pulling of the cylinder in the y_1-direction, bending of the cylinder about the y_2-axis, and bending of the cylinder about the y_1-axis.

(iv) The given stress field is in equilibrium, maintains a traction free curved surface and is consistent with having a traction distribution on the ends $y_3 = 0$ and $y_3 = -\ell$ that result in axial loading, twisting and bending. *Remark:* Note that the traction distribution on the two ends arising from (a) has a very specific form. There are other traction distributions that can be applied on the two ends that also correspond to to axial loading, twisting and bending. The stress field in the body corresponding to such an alternative applied traction distribution on the two ends would differ from (a).

(v) The principal stresses at an arbitrary point in the body is given by the eigenvalues T of the given stress tensor:

$$\det[T - TI] = \det\begin{pmatrix} -T & 0 & -\alpha y_2 \\ 0 & -T & \alpha y_1 \\ -\alpha y_2 & \alpha y_1 & \beta + \gamma y_1 + \delta y_2 - T \end{pmatrix} = 0.$$

Expanding the determinant and solving the resulting cubic equation for T shows that the largest of the three roots is

$$T = \frac{1}{2}\Big\{\beta + \gamma y_1 + \delta y_2 + \sqrt{(\beta + \gamma y_1 + \delta y_2)^2 + 4\alpha^2(y_1^2 + y_2^2)}\Big\}.$$

This is the largest principal stress at a point (y_1, y_2, y_3). In order to find the maximum principal stress from among all points in the body, we need to maximize T as a function of y_1, y_2 on the circle $y_1^2 + y_2^2 \le a^2$.

Problem 4.7. Consider a material such as a polarized dielectric solid under the action of an electric field, where, in addition to a body force $\mathbf{b}(\mathbf{y}, t)$, there is also a *body couple* $\mathbf{c}(\mathbf{y}, t)$ per unit mass. Also, at any point $\mathbf{y}$ on a surface $\mathcal{S}_t$ suppose that there is, in addition to the contact force $\mathbf{t}(\mathbf{y}, t, \mathbf{n})$ a *contact couple* $\mathbf{m}(\mathbf{y}, t, \mathbf{n})$; here $\mathbf{n}$ is the unit outward normal vector at a point on a surface in the body and $\mathbf{c}$ is the couple applied by the material on the positive side of $\mathcal{S}_t$ on the material on the negative side. (The "positive side" of $\mathcal{S}_t$ is the side into which $\mathbf{n}$ points.)

Write down the global linear and angular momentum principles for this case. Show that, in addition to the stress tensor $\mathbf{T}$, there is also a *couple stress tensor* $\mathbf{Z}(\mathbf{y}, t)$ such that

$$\mathbf{m} = \mathbf{Z}\mathbf{n}.$$

Derive the local consequences of the momentum principles. Is the stress tensor $\mathbf{T}$ symmetric?

Solution: *Existence of couple stress tensor.* The additional couples that exist in the current setting have no effect on the forces and so the balance of linear momentum remains as is:

$$\int_{\partial\mathcal{D}_t} \mathbf{t}\, \mathrm{d}A_y = \int_{\mathcal{D}_t} \rho\dot{\mathbf{v}}\, \mathrm{d}V_y.$$

In the usual way this implies the existence of the Cauchy stress tensor $\mathbf{T}$ such that $\mathbf{t} = \mathbf{T}^T\mathbf{n}$. The couples do contribute to the resultant moment and so the balance of angular momentum reads

$$\int_{\partial\mathcal{D}_t} \mathbf{y} \times \mathbf{t}\, \mathrm{d}A_y + \int_{\partial\mathcal{D}_t} \mathbf{m}(\mathbf{y}, t, \mathbf{n})\, \mathrm{d}A_y + \int_{\mathcal{D}_t} \rho\mathbf{c}\, \mathrm{d}V_y = \int_{\mathcal{D}_t} \mathbf{y} \times \rho\dot{\mathbf{v}}\, \mathrm{d}V_y.$$

(In this and the next problem one could, and perhaps ought to, introduce a *new independent* kinematic field that represents the local rotation conjugate to the couples, which then contributes an additional term to the inertia; see for example Toupin.) Using the fact that $\mathbf{t} = \mathbf{T}^T\mathbf{n}$ and the divergence theorem allows us to convert the first surface integral into a volume integral. On applying this balance principle to a tetrahedral region and shrinking the region to a point, all the volume integrals vanish and only the contribution of $\mathbf{m}$ over the boundary remains. Then mimicing the steps we used to show the existence of stress $\mathbf{T}$ allows us to conclude that there exists a tensor $\mathbf{Z}(\mathbf{y}, t)$ that is independent of $\mathbf{n}$ such that

$$\mathbf{m}(\mathbf{y}, t, \mathbf{n}) = \mathbf{Z}(\mathbf{y}, t)\mathbf{n}.$$

Field equations. Since the linear momentum balance law and the traction-stress relation are the familiar ones the calculation in the notes goes through and we find

$$\operatorname{div} \mathbf{T}^T = \rho\dot{\mathbf{v}} \qquad \text{or in terms of components} \quad T_{ji,j} = \rho\dot{v}_i.$$

Next, substituting $\mathbf{t} = \mathbf{T}^T\mathbf{n}$ and $\mathbf{m} = \mathbf{Z}\mathbf{n}$ into the angular momentum balance above gives

$$\int_{\partial\mathcal{D}_t} \mathbf{y} \times \mathbf{T}^T\mathbf{n}\, \mathrm{d}A_y + \int_{\partial\mathcal{D}_t} \mathbf{Z}\mathbf{n}\, \mathrm{d}A_y + \int_{\mathcal{D}_t} \rho\mathbf{c}\, \mathrm{d}V_y = \int_{\mathcal{D}_t} \mathbf{y} \times \rho\dot{\mathbf{v}}\, \mathrm{d}V_y$$

or in terms of components

$$\int_{\partial\mathcal{D}_t} \varepsilon_{ijk} y_j T_{pk} n_p\, \mathrm{d}A_y + \int_{\partial\mathcal{D}_t} Z_{ip} n_p\, \mathrm{d}A_y + \int_{\mathcal{D}_t} \rho c_i\, \mathrm{d}V_y = \int \varepsilon_{ijk} y_j \rho\dot{v}_k\, \mathrm{d}V_y.$$

Using the divergence theorem to convert the area integrals to volume integrals and then localizing the resul in the familiar way leads

$$\varepsilon_{ijk}\delta_{jp}T_{pk} + \varepsilon_{ijk}y_jT_{pk,p} + Z_{ip,p} + \rho c_i - \varepsilon_{ijk}y_j\rho\dot{v}_k = 0.$$

which simplifies to

$$\varepsilon_{ijk}T_{jk} + Z_{ip,p} + \rho c_i = 0.$$

This can we written in an alternative, more illuminating form, by first multiplying it by ε_{ipq} and then using the familiar identity $\varepsilon_{ipq}\varepsilon_{ijk} = \delta_{pj}\delta_{qk} - \delta_{pk}\delta_{qj}$. This leads to

$$\left(\delta_{pj}\delta_{qk} - \delta_{pk}\delta_{qj}\right) T_{jk} + \varepsilon_{ipq}Z_{ij,j} + \rho\varepsilon_{ipq}c_i = 0$$

and finally to

$$T_{pq} - T_{qp} + \varepsilon_{ipq}Z_{ij,j} + \rho\varepsilon_{ipq}c_i = 0.$$

Observe that the Cauchy stress $\mathbf{T}$ is not symmetric and that the above equation provides an expression for $\mathbf{T} - \mathbf{T}^T$ in terms of the couple stress and body couple.

Problem 4.8. Write down an expression for the rate of working of the forces and couples in the setting of the previous example. Rewrite this in the form of a volume integral of the local power.

<u>Solution</u>: The rate of working of a couple is equal to the inner product of the couple with an appropriate measure of the work conjugate rotation-rate. We continue to work in the setting of classical kinematics. (In this and the previous problem one could, and perhaps ought to, introduce a *new independent* kinematic field that represents the local rotation and is work conjugate to the couples; see for example Toupin.)

We first calculate the rate of working of the traction:

$$\begin{aligned}
\textstyle\int_{\partial\mathcal{D}_t} t_i v_i \, \mathrm{d}A_y &= \textstyle\int_{\partial\mathcal{D}_t} T_{pi} v_i n_p \, \mathrm{d}A_y \;=\; \int_{\mathcal{D}_t} (T_{pi,p} v_i + T_{pi} v_{i,p}) \, \mathrm{d}V_y \\
&= \textstyle\int_{\mathcal{D}_t} \rho \dot{v}_i v_i \, \mathrm{d}V_y + \int_{\mathcal{D}_t} T_{pi} v_{i,p} \, \mathrm{d}V_y \\
&= \textstyle\int_{\mathcal{D}_t} \rho \dot{v}_i v_i \, \mathrm{d}V_y + \int_{\mathcal{D}_t} (T_{pi} D_{ip} + T_{pi} W_{ip}) \, \mathrm{d}V_y \\
&= \textstyle\int_{\mathcal{D}_t} \rho \dot{v}_i v_i \, \mathrm{d}V_y + \int_{\mathcal{D}_t} \mathbf{T}^T \cdot \mathbf{D} \, \mathrm{d}V_y + \int_{\mathcal{D}_t} T_{pi} W_{ip} \, \mathrm{d}V_y
\end{aligned}$$

where $\mathbf{D}$ and $\mathbf{W}$ are the stretching and spin tensors respectively. The integrand of the last term can be simplified as follows

$$\begin{aligned}
T_{pi} W_{ip} &= T_{pi} W_{ip} + T_{ip} W_{pi} \;=\; (T_{pq} - T_{qp}) W_{qp} \\
&= -\left\{ \varepsilon_{ipq} Z_{ij,j} + \rho \varepsilon_{ipq} c_i \right\} W_{qp}
\end{aligned}$$

Now integrating this over the region $\mathcal{D}_t$ gives

$$\begin{aligned}
\int_{\mathcal{D}_t} T_{pi} W_{ip} \, \mathrm{d}V_y &= -\int_{\mathcal{D}_t} \left\{ \varepsilon_{ipq} Z_{ij,j} W_{qp} + \rho \epsilon_{ipq} c_i W_{qp} \right\} \, \mathrm{d}V_y \\
&= -\int_{\partial\mathcal{D}_t} \varepsilon_{ipq} Z_{ij} W_{qp} n_j \, \mathrm{d}A_y - \int_{\mathcal{D}_t} \rho \varepsilon_{ipq} c_i W_{qp} \, \mathrm{d}V_y \\
&\quad + \int_{\mathcal{D}_t} \varepsilon_{ipq} Z_{ij} W_{qp,j} \, \mathrm{d}V_y
\end{aligned}$$

Finally we substitute this back into the first equation to get

$$\begin{aligned}
&\int_{\partial\mathcal{D}_t} t_i v_i \, \mathrm{d}A_y + \int_{\partial\mathcal{D}_t} \varepsilon_{ipq} m_i W_{qp} \, \mathrm{d}A_y + \int_{\mathcal{D}_t} \rho \varepsilon_{ipq} c_i W_{qp} \, \mathrm{d}V_y \\
&\qquad = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{D}_t} \frac{1}{2} \rho v_i v_i \, \mathrm{d}V_y + \int_{\mathcal{D}_t} \mathbf{T}^T \cdot \mathbf{D} \, \mathrm{d}V_y + \int_{\mathcal{D}_t} \varepsilon_{ipq} Z_{ij} W_{qp,j} \, \mathrm{d}V_y
\end{aligned} \tag{4.78}$$

This can be further simplified by using the relation between the spin tensor $\mathbf{W}$ and the angular velocity vector $\boldsymbol{\omega}$. The term

$$
\begin{aligned}
\int_{\partial\mathcal{D}_t} \varepsilon_{ipq}\, m_i\, W_{qp}\, \mathrm{d}A_y &= \int_{\partial\mathcal{D}_t} \varepsilon_{ipq}\, m_i\, \varepsilon_{pqk}\, \omega_k\, \mathrm{d}A_y \\
&= \int_{\partial\mathcal{D}_t} \varepsilon_{pqi}\, \varepsilon_{pqk}\, m_i\, \omega_k\, \mathrm{d}A_y \\
&= \int_{\partial\mathcal{D}_t} (\delta_{qq}\delta_{ik} - \delta_{qk}\delta_{iq})\, m_i\omega_k\, \mathrm{d}A_y \\
&= \int_{\partial\mathcal{D}_t} (3m_i\omega_i - m_q\omega_q)\, \mathrm{d}A_y \;=\; \int_{\partial\mathcal{D}_t} m_i\omega_i\, \mathrm{d}A_y
\end{aligned}
$$

The other term

$$
\int_{\mathcal{D}_t} \rho\, \varepsilon_{ipq}\, c_i\, W_{qp}\, \mathrm{d}V_y = 2\int_{\mathcal{D}_t} \rho\, c_i\, \omega_i\, \mathrm{d}V_y
$$

And finally the third term

$$
\begin{aligned}
\int_{\mathcal{D}_t} \varepsilon_{ipq}\, Z_{ij}\, W_{qp,j}\, \mathrm{d}V_y &= \int_{\mathcal{D}_t} \varepsilon_{ipq}\, Z_{ij}\, \varepsilon_{qpk}\, \omega_{k,j}\, \mathrm{d}V_y \\
&= \int_{\mathcal{D}_t} \varepsilon_{qip}\, \varepsilon_{qpk}\, Z_{ij}\, \omega_{k,j}\, \mathrm{d}V_y \\
&= \int_{\mathcal{D}_t} (\delta_{ip}\delta_{pk} - \delta_{ik}\delta_{pp})\; Z_{ij}\, \omega_{k,j}\, \mathrm{d}V_y \\
&= \int_{\mathcal{D}_t} \{Z_{ij}\omega_{i,j} - 3Z_{ij}\omega_{i,j}\}\; \mathrm{d}V_y \;=\; -2\int_{\mathcal{D}_t} Z_{ij}\, \omega_{ij}\, \mathrm{d}V_y
\end{aligned}
$$

On collecting these results we finally get

$$
\begin{aligned}
&\int_{\partial\mathcal{D}_t} t_i v_i\, \mathrm{d}A_y + \int_{\partial\mathcal{D}_t} m_i\omega_i\, \mathrm{d}A_y + 2\int_{\mathcal{D}_t} \rho\, c_i\, \omega_i\, \mathrm{d}V_y \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{D}_t} \frac{1}{2}\rho v_i v_i\, \mathrm{d}V_y + \int_{\mathcal{D}_t} \mathbf{T}^T \cdot \mathbf{D}\, \mathrm{d}V_y - 2\int_{\mathcal{D}_t} Z_{ij}\, \omega_{ij}\, \mathrm{d}V_y
\end{aligned}
$$

Problem 4.9. In the preceding example we encountered couple stresses, and specifically, showed that there is a *couple stress tensor* $\mathbf{Z}$.

a. Let $\mathbf{Z}_0$ be the referential version of $\mathbf{Z}$, i.e. the tensor analogous to what the first Piola-Kirchoff stress tensor $\mathbf{S}$ is to the Cauchy stress tensor $\mathbf{T}$. Derive a formula for $\mathbf{Z}_0$.

b. Derive the field equation corresponding to angular momentum balance in its referential form in terms of $\mathbf{Z}_0$.

Problem 4.10. In these notes we formulated and analyzed the basic balance laws by focussing attention on an arbitrary fixed part $\mathcal{P}$ of the body $\mathcal{B}$; our attention at all times was therefore on the same set of particles. Sometimes it is convenient to work with a *control volume* instead: i.e. a fixed region in space (with different particles entering and leaving as time progresses).

Let $\mathcal{R}_t$ be the region occupied by the body at the instant t. Let Π be a fixed region of space which is such that $\Pi \subset R_t$ for all times close to t.

Show that the balance laws for mass, linear momentum and anugular momentum as stated in class are *equivalent* to the following alternative statements for the control volume Π:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Pi} \rho \,\mathrm{d}V_y = -\int_{\partial\Pi} \rho \mathbf{v}\cdot\mathbf{n}\,\mathrm{d}A_y,$$

$$\int_{\partial\Pi} \mathbf{t}\,\mathrm{d}A_y + \int_{\Pi} \mathbf{b}\,\mathrm{d}V_y = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Pi} \rho\mathbf{v}\,\mathrm{d}V_y + \int_{\partial\Pi} (\rho\mathbf{v})\,\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}A_y,$$

$$\int_{\partial\Pi} \mathbf{y}\times\mathbf{t}\,\mathrm{d}A_y + \int_{\Pi} \mathbf{y}\times\mathbf{b}\,\mathrm{d}V_y = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Pi} \mathbf{y}\times\rho\mathbf{v}\,\mathrm{d}V_y + \int_{\partial\Pi} \mathbf{y}\times(\rho\mathbf{v})\,\mathbf{v}\cdot n\,\mathrm{d}A_y.$$

Problem 4.11. [Gurtin] In this problem you will establish the equations of motion for *an arbitrary (not necessarily rigid) body* $\mathcal{B}$ in terms of the *motion of its center of mass*. Let

$\overline{\mathbf{y}}(t)$ = the current location of the center of mass of $\mathcal{B}$,

$m(\mathcal{B})$ = the total mass of $\mathcal{B}$,

$\boldsymbol{\ell}(\mathcal{B},t)$ = the total linear momentum of $\mathcal{B}$ at time t,

$\boldsymbol{\alpha}(\mathcal{B},t)$ = the total angular momentum of $\mathcal{B}$ at time t about a fixed origin $\mathbf{o}$,

$\boldsymbol{\alpha}_{cm}(\mathcal{B},t)$ = the "spin angular momentum"; i.e., the moment about the center of mass of the linear momentum of the body relative to a frame moving with the center of mass, i.e.

$$\boldsymbol{\alpha}_{cm}(\mathcal{B},t) = \int_{R_t} (\mathbf{y}-\overline{\mathbf{y}})\times\rho(\mathbf{v}-\dot{\overline{\mathbf{y}}})\,dV_y\ ,$$

$\mathbf{f}(\mathcal{B},t)$ = the resultant force acting on $\mathcal{B}$ at time t,

$\boldsymbol{\tau}_{cm}(\mathcal{B},t)$ = the resultant moment about the center of mass acting on $\mathcal{B}$ at time t.

Show that

(*a*) $\boldsymbol{\ell}(\mathcal{B},t) = m(\mathcal{B})\,\dot{\overline{\mathbf{y}}}(t)$,

(b) $\dot{\boldsymbol{\alpha}}(\mathcal{B},t) = \dot{\boldsymbol{\alpha}}_{cm}(\mathcal{B},t) + \mathbf{y} \times \dot{\boldsymbol{\ell}}(\mathcal{B},t)$,

(c) $\boldsymbol{\tau}_{cm}(\mathcal{B},t) = \dot{\boldsymbol{\alpha}}_{cm}$.

Problem 4.12. Consider a *rigid body* $\mathcal{B}$ undergoing an arbitrary motion. Calculate the total linear momentum $\boldsymbol{\ell}(\mathcal{B},t)$, the total angular momentum $\boldsymbol{\alpha}(\mathcal{B},t)$, and the total kinetic energy $K(\mathcal{B},t)$ of $\mathcal{B}$ at any instant during the motion.

Problem 4.13. In this problem you will derive *Euler's equations of motion for a rigid body* $\mathcal{B}$. Refer to Problem 4.11 for the definitions of some of the symbols below. Let the angular velocity of the body during this motion be $\boldsymbol{\omega}(t)$.

(a) Show that

$$\boldsymbol{\alpha}_{cm}(\mathcal{B},t) = \mathbf{J}(\mathcal{B},t)\,\boldsymbol{\omega}(t)$$

where $\mathbf{J}$ is the inertia tensor of $\mathcal{B}$ relative to the center of mass, i.e.,

$$\mathbf{J}(\mathcal{B},t) = \int_{\mathcal{R}_t} \left((\mathbf{z}\cdot\mathbf{z})\,\mathbf{I} - \mathbf{z}\otimes\mathbf{z} \right)\, \rho\, dV_y$$

where

$$\mathbf{z}(\mathbf{y},t) = \mathbf{y} - \overline{\mathbf{y}}(t), \qquad \mathbf{y}(\mathbf{x},t) = \mathbf{Q}(t)\mathbf{x} + \mathbf{b}(t).$$

(b) Show that

$$\mathbf{J}(\mathcal{B},t) = \mathbf{Q}(t)\,\mathbf{J}(\mathcal{B},0)\,\mathbf{Q}^T(t)$$

and conclude from this that the components of $\mathbf{J}$ with respect to a frame that rotates with the body are independent of time.

(c) Since $\mathbf{J}$ is symmetric, there is a frame in which the component matrix of $\mathbf{J}$ is diagonal. Consider such a principal frame which rotates with the body. Let J_i be the diagonal components of $\mathbf{J}$ (i.e., the principal moments of inertia) and let ω_i be the components of $\boldsymbol{\omega}$ in this same frame. By calculating the components of $\boldsymbol{\alpha}_{cm}$ relative to this frame and using the results of the previous problems, show that

$$\begin{aligned}
(\tau_{cm})_1 &= J_1\,\dot{\omega}_1 + (J_3 - J_2)\,\omega_2\omega_3,\\
(\tau_{cm})_2 &= J_2\,\dot{\omega}_2 + (J_1 - J_3)\,\omega_1\omega_3,\\
(\tau_{cm})_3 &= J_3\,\dot{\omega}_3 + (J_2 - J_1)\,\omega_1\omega_2.
\end{aligned}$$

Problem 4.14. Consider a body that occupies a region $\mathcal{R}$ in a deformed configuration. It is in equilibrium under the action of applied surface tractions on its entire boundary $\partial\mathcal{R}$. Suppose that this traction is parallel to a fixed unit vector $\mathbf{a}$ at every point on ∂R (so that one might say that the *loading* is uniaxial). Show that the mean stress in the body, defined as

$$\overline{\mathbf{T}} = \frac{1}{\text{vol}\,(\mathcal{R})}\int_{\mathcal{R}} \mathbf{T}(\mathbf{y})\,dV_y\ ,$$

has the form $\overline{\mathbf{T}} = \tau\mathbf{a}\otimes\mathbf{a}$ for some constant τ. Therefore the mean stress is also uniaxial.

Problem 4.15. Consider a body that occupies a unit cube in a reference configuration:

$$\mathcal{R}_0 = \{(x_1, x_2, x_3)\ |\ 0 \le x_1 \le 1,\ 0 \le x_2 \le 1,\ 0 \le x_3 \le 1\}.$$

It is subjected to the following deformation:

$$y_1 = x_1 + kx_2, \quad y_2 = hx_2, \quad y_3 = x_3,$$

where k and h are constants. The corresponding components of the 1^{st}-Piola-Kirchhoff stress tensor are

$$[S] = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix}$$

where each S_{ij} is a constant. You may consider h, k and S_{ij} to be given.

Consider a surface $\mathcal{S}_0$ in the reference configuration that is characterized by $x_1 + x_2 = 1$. The deformation carries $\mathcal{S}_0 \to \mathcal{S}_*$.

a. Calculate the force (vector) which acts on $\mathcal{S}_*$.

b. Calculate the true (Cauchy) traction on $\mathcal{S}_*$.

c. Calculate the normal component of true (Cauchy) traction on $\mathcal{S}_*$.

Solution: The matrix of components of the deformation gradient tensor, $[F]$, are found by differentiating $y_1 = x_1 + kx_2,\ y_2 = hx_2,\ y_3 = x_3$ and recalling that $F_{ij} = \partial y_i/\partial x_j$. Its inverse and determinant can then be found from matrix algebra:

$$[F] = \begin{pmatrix} 1 & k & 0 \\ 0 & h & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad [F]^{-1} = \begin{pmatrix} 1 & -k/h & 0 \\ 0 & 1/h & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad J = \det[F] = h. \qquad (a)$$

Our writing will be simplified if we set

$$\ell = \Big[1 + (1-k)^2/h^2\Big]^{-1/2}. \qquad (b)$$

A unit normal vector $\mathbf{n}_0$ to the surface $\mathcal{S}_0$ is

$$\mathbf{n}_0 = (1/\sqrt{2})\mathbf{e}_1 + (1/\sqrt{2})\mathbf{e}_2. \tag{c}$$

Therefore a unit normal vector $\mathbf{n}$ to its deformed image $\mathcal{S}$ is

$$\mathbf{n} = \frac{\mathbf{F}^{-T}\mathbf{n}_0}{|\mathbf{F}^{-T}\mathbf{n}_0|} = \ell\,\mathbf{e}_1 + (1-k)h\,\mathbf{e}_2. \tag{d}$$

where we have used $(a)_2, (b), (c)$.

The areas A_x and A_y of the surfaces $\mathcal{S}_0$ and $\mathcal{S}$ are

$$A_x = \sqrt{2}, \qquad\qquad A_y = A_x J|\mathbf{F}^{-T}\mathbf{n}_0| = h/\ell. \tag{e}$$

(a) The resultant force on the deformed surface is given by

$$\text{force } = \mathbf{s}A_x = \mathbf{S}\mathbf{n}_0 A_x = (S_{11}+S_{12})\mathbf{e}_1 + (S_{21}+S_{22})\mathbf{e}_2 + (S_{31}+S_{32})\mathbf{e}_3.$$

(b) Since the resultant force $= \mathbf{s}A_x = \mathbf{t}A_y$ we find the Cauchy (true) traction to be

$$\mathbf{t} = \mathbf{s}A_x/A_y = (\ell/h)\Big[(S_{11}+S_{12})\mathbf{e}_1 + (S_{21}+S_{22})\mathbf{e}_2 + (S_{31}+S_{32})\mathbf{e}_3\Big]. \tag{f}$$

(c) The normal stress on the plane is given by (d) and (f) as

$$\mathbf{t}\cdot\mathbf{n} = \frac{1}{h\ell^2}\Big[S_{11} + S_{12} + (S_{21}+S_{22})(1-k)h\Big].$$

Problem 4.16. Consider a thin sheet which is subjected to a state of plane stress. Figure 4.15 shows (a plan view of) a material element which occupies an infinitesimal square $ABCD$ in the reference configuration and a region $A'B'C'D'$ in the current configuration. Draw a free body diagram of the region $A'B'C'D'$ and mark all forces which act on it (in terms of the components of the 1st Piola-Kirhhoff stress tensor $\mathbf{S}$), e.g. the force on one surface is $S_{11} \times \mathsf{A}rea$, while that on another is $(S_{11} + (\partial S_{11}/\partial x_1)\,\delta x_1) \times \mathsf{A}rea$, and so on.

By summing forces and moments, derive the corresponding equilibrium equations associated with force and moment balance.

Problem 4.17. Consider a state of plane stress and let (R, Θ, Z) be cylindrical coordinates in the reference state. Derive the equilibrium equations obeyed by the cylindrical components of the first Piola-Kirchhoff stress $\mathbf{S}(R, \Theta)$.

(NOTE: It is probably easiest to consider a free-body-diagram of an infinitesimal material volume and derive the equilibrium equations by summing forces. Since $S_{R\Theta} \neq S_{\Theta R}$ etc. you must be careful! You should verify that your answer reduces to the classical cylindrical coordinate equilibrium equations that you can find in any book in the special case when $\mathbf{S}$ is symmetrical.)

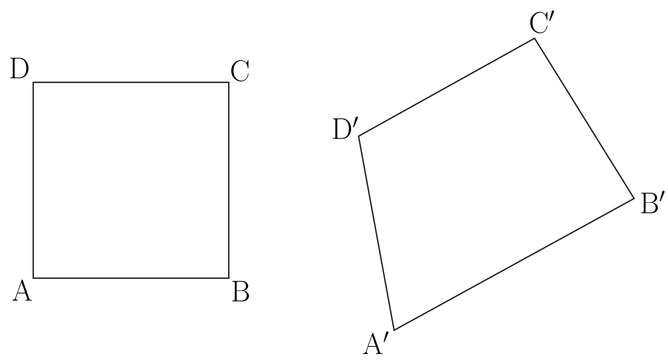


Figure 4.15: An infinitesimal material region $ABCD$ in the reference configuration and its image $A'B'C'D'$ in the deformed configuration.

Problem 4.18. Establish the *Principle of Virtual Work*, i.e. show that the equilibrium equations

$$S_{ij,j} + b_i = 0$$

hold *if and only if*

$$\int_{\partial\mathcal{R}_0} S_{ij} n_j w_i dA + \int_{\mathcal{R}_0} b_i w_i = \int_{\mathcal{R}_0} S_{ij} \gamma_{ij} dV$$

for *all arbitrary* smooth enough vector fields $\mathbf{w}(\mathbf{x})$. Here $\mathcal{R}_0$ is the region occupied by the body in the reference configuration, and $\gamma_{ij} = (1/2)(w_{i,j} + w_{j,i})$.

Remark: Note that $\mathbf{w}(\mathbf{x})$ is not necessarily the actual displacement field in the body; it is called a "virtual displacement".

Problem 4.19. Two symmetric tensors are said to be *coaxial* if their principal axes coincide. Prove that the Cauchy stress tensor $\mathbf{T}$ and the left Cauchy-Green tensor $\mathbf{B}$ are coaxial if and only if the second Piola-Kirchhoff tensor $\mathbf{S}^{(2)}$ is coaxial with the right Cauchy-Green strain tensor $\mathbf{C}$.

Problem 4.20. Consider the family of Lagrangian strain tensors

$$\mathbf{E}^{(n)} = \frac{1}{n}\Big(\mathbf{U}^n - \mathbf{I}\Big), \qquad n \neq 0.$$

For both $n = 1$ and $n = -2$ find the corresponding conjugate stress tensors $\mathbf{S}^{(n)}$ for which the stress power density is given by

$$\mathbf{S}^{(n)} \cdot \dot{\mathbf{E}}^{(n)}.$$

<u>*Solution*</u>: When $n = 1$ we have $\mathbf{E}^{(1)} = \mathbf{U} - \mathbf{I}$ and so $\dot{\mathbf{E}}^{(1)} = \dot{\mathbf{U}}$. The tensor $\mathbf{S}^{(1)}$ must be such that

$$p = \mathbf{S}^{(1)} \cdot \dot{\mathbf{U}}.$$

In Section 4.9.2 we found a tensor $\mathbf{S}^{(2)}$ such that

$$p = \mathbf{S}^{(2)} \cdot \dot{\mathbf{E}}^{(2)}.$$

On equating these two expressions for p we get

$$\mathbf{S}^{(1)} \cdot \dot{\mathbf{U}} = \mathbf{S}^{(2)} \cdot \dot{\mathbf{E}}^{(2)}.$$

Differentiating $\mathbf{E}^{(2)} = (\mathbf{U}\mathbf{U} - \mathbf{I})/2$ leads to $\dot{\mathbf{E}}^{(2)} = (\dot{\mathbf{U}}\mathbf{U} + \mathbf{U}\dot{\mathbf{U}})/2$. Substituting this into the preceding equation yields

$$\mathbf{S}^{(1)} \cdot \dot{\mathbf{U}} = \frac{1}{2}\mathbf{S}^{(2)} \cdot (\dot{\mathbf{U}}\mathbf{U} + \mathbf{U}\dot{\mathbf{U}}) = \frac{1}{2}\mathbf{S}^{(2)}\mathbf{U} \cdot \dot{\mathbf{U}} + \frac{1}{2}\mathbf{U}\mathbf{S}^{(2)} \cdot \dot{\mathbf{U}}$$

and therefore

$$\mathbf{S}^{(1)} = \frac{1}{2}\left(\mathbf{S}^{(2)}\mathbf{U} + \mathbf{U}\mathbf{S}^{(2)}\right).$$

Since in Section 4.9.2 we found that $\mathbf{S}^{(2)} = \mathbf{F}^{-1}\mathbf{S}$ we thus have the final result

$$\mathbf{S}^{(1)} = \frac{1}{2}\Big(\mathbf{F}^{-1}\mathbf{S}\mathbf{U} + \mathbf{R}^T\mathbf{S}\Big).$$

Since $\mathbf{S}\mathbf{F}^T = \mathbf{F}\mathbf{S}^T$ by angular momentum balance, this can be written as

$$\mathbf{S}^{(1)} = \frac{1}{2}\Big(\mathbf{S}^T\mathbf{R} + \mathbf{R}^T\mathbf{S}\Big).$$

Problem 4.21. In this exercise do not assume that the linear and angular momentum balance principles hold. Consider a continuum that is undergoing a quasi-static motion i.e. a time-dependent motion with inertia neglected. The power, i.e. the rate at which the surface and body forces do work on any part $\mathcal{P}$ of the body during a motion, is given by

$$\int_{\partial\mathcal{D}_0} \mathbf{S}\mathbf{n} \cdot \mathbf{v}\, dA_x \;+\; \int_{\mathcal{D}_0} \rho_o \mathbf{b} \cdot \mathbf{v}\, dV_x$$

where $\mathbf{v}(\mathbf{x}, t)$ is the velocity field associated with the motion and $\mathcal{D}_0$ is the region occupied by $\mathcal{P}$ in the reference configuration. The power should be frame indifferent. Derive necessary and sufficient conditions which ensure this.

Problem 4.22. Determine the stress that is conjugate to the (Lagrangian) logarithmic strain tensor $\ln \mathbf{U}$.

<u>*Solution*</u>: See the paper by A, Hoger, The stress conjugate to logarithmic strain, *International Journal of Solids and Structures*, **23**(1987), pp. 1645-1656.

Problem 4.23. Pick any Eulerian strain tensor of your choice. Find the stress tensor that is conjugate to it.

<u>Solution</u>: See the paper by Andrew Norris, Eulerian conjugate stress and strain, *J. Mech. Materials Struct.*, **3**(2008), pp. 243-260. In general finding stress tensors conjugate to Eulerian strains is much more difficult than the corresponding problem for Lagrangian strains.

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Chapter 5

Thermodynamic Balance Laws and Field Equations

We now consider the laws of thermodynamics as applied to a deformable continuum. In Section 5.1 we state the global balance law associated with the first law and derive the associated field equation (sometimes called the energy equation). The second law of thermodynamics in considered in Section 5.2 where, from the global inequality we derive the associated local inequality. The preceding analyses are carried out in the current configuration with no mention of a reference configuration. The discussion is now reframed with respect to a reference configuration in Section 5.3. Section 5.4 summarizes the results. The material frame indifference of the various quantities introduced in this and the preceding chapter, e.g. traction, heat flux, stress, etc., are studied in Section 5.5.

5.1 The First Law of Thermodynamics.

As before, let $\mathcal{P}$ be some part of a body $\mathcal{B}$ that is undergoing a motion $\mathbf{y} = \chi(p, t)$. Let $\mathcal{R}_t = \chi(\mathcal{B}, t)$ and $\mathcal{D}_t = \chi(\mathcal{P}, t)$ denote the respective regions occupied by $\mathcal{B}$ and $\mathcal{P}$ at time t. We suppose that the *heating* of $\mathcal{P}$ is due to two sources: (i) heat supply—which is the rate at which heat is generated/provided to all particles throughout $\mathcal{P}$, e.g. due to radiation or heat sources, and (ii) heat flux—which is the rate at which heat enters into $\mathcal{P}$ across the boundary ∂D_t.

Let r be the heat supply rate per unit mass and let h be the heat flux per unit current area. Both r and h represent heat supplied *to* the part $\mathcal{P}$ *from* the exterior of $\mathcal{P}$. Then the total rate of heating of $\mathcal{P}$ is

$$\int_{D_t} \rho r \, \mathrm{d}V_y + \int_{\partial D_t} h \, \mathrm{d}A_y \, . \tag{5.1}$$

We assume that the heat supply depends on position $\mathbf{y}$ and time t, and that the heat flux depends on the position $\mathbf{y}$, the time t and the surface ∂D_t. More specifically, we assume that h depends on ∂D_t only through the local unit normal vector. Thus we take

$$r = r(\mathbf{y}, t), \qquad h = h(\mathbf{y}, t, \mathbf{n}). \tag{5.2}$$

This is entirely analogous to our previous treatment of the mechanical quantities the body force $\mathbf{b}(\mathbf{y}, t)$ and the traction $\mathbf{t}(\mathbf{y}, t, \mathbf{n})$.

Turning next to the energy associated with $\mathcal{P}$, that portion of the energy which $\mathcal{P}$ possesses in addition to its kinetic energy is called the *internal energy*. Let $U(\mathcal{P}, t; \chi)$ denote the internal energy of the part $\mathcal{P}$ at time t during the motion χ. It follows from the discussion in Section 1.8 that there exists a density of internal energy, say $\varepsilon(\mathbf{y}, t)$, such that the total internal energy of $\mathcal{P}$ can be written as

$$U(\mathcal{P}, t; \chi) = \int_{\mathcal{D}_t} \rho\varepsilon(\mathbf{y}, t) \, \mathrm{d}V_y; \tag{5.3}$$

$\varepsilon(\mathbf{y}, t)$ is called the *specific internal energy* (per unit mass). For simplicity we have not displayed the dependency of ε on the motion χ and written $\varepsilon(\mathbf{y}, t)$ in place of $\varepsilon(\mathbf{y}, t; \chi)$.

The *first law of thermodynamics* states that at each instant during a motion, the sum of the rates of working and heating on any part $\mathcal{P}$ must equal the rate of increase of the total energy of $\mathcal{P}$ (which is comprised of the kinetic and internal energies), i.e.,

$$\int_{\partial D_t} \mathbf{t} \cdot \mathbf{v} \, \mathrm{d}A_y + \int_{D_t} \rho \, \mathbf{b} \cdot \mathbf{v} \, \mathrm{d}V_y + \int_{\partial D_t} h \, \mathrm{d}A_y + \int_{D_t} \rho r \, \mathrm{d}V_y$$

$$= \frac{\mathrm{d}}{\mathrm{d}t} \int_{D_t} \frac{1}{2} \rho \, \mathbf{v} \cdot \mathbf{v} \, \mathrm{d}V_y + \frac{\mathrm{d}}{\mathrm{d}t} \int_{D_t} \rho\varepsilon \, \mathrm{d}V_y \tag{5.4}$$

for all subregions D_t of $\mathcal{R}_t$. By using the work-energy identity (4.71), we can write this as

$$\int_{\partial D_t} h \, \mathrm{d}A_y + \int_{D_t} \rho r \, \mathrm{d}V_y + \int_{D_t} \mathbf{T} \cdot \mathbf{D} \, \mathrm{d}V_y = \frac{\mathrm{d}}{\mathrm{d}t} \int_{D_t} \rho\varepsilon \, \mathrm{d}V_y \, . \tag{5.5}$$

One can show that the first law of thermodynamics (5.5) implies that the heat flux $h(\mathbf{y}, t, \mathbf{n})$ depends linearly on the unit vector $\mathbf{n}$, i.e., that there is a vector $\mathbf{q}(\mathbf{y}, t)$ such that[1]

$$h(\mathbf{y}, t, \mathbf{n}) = \mathbf{q}(\mathbf{y}, t) \cdot \mathbf{n} \, . \tag{5.6}$$

[1] Frequently, authors take $h = -\mathbf{q} \cdot \mathbf{n}$.

The proof of this result is analogous to the proof of Cauchy's Theorem for traction/stress: one applies (5.5) to a region D_t in the form of a tetrahedron and then takes the limit as the tetrahedron shrinks to a point. The result (5.6) is known as Fourier's Theorem and the vector $\mathbf{q}$ is called the *heat flux vector*. Note that $\mathbf{q}(\mathbf{y},t)$ does not depend on the normal vector $\mathbf{n}$. If $\mathcal{S}_t$ is a surface in $\mathcal{D}_t$, and $\mathbf{n}$ is a unit normal vector at some point on $\mathcal{S}_t$, let us refer to the side into which $\mathbf{n}$ points as the positive side of $\mathcal{S}_t$. Then $\mathbf{q}$ represents the heat flux from the positive side to the negative side.

The physical significance of the components of $\mathbf{q}$ can be deduced as follows. (In this paragraph we shall write $h(\mathbf{n})$ instead of $h(\mathbf{y},t,\mathbf{n})$ since $\mathbf{y}$ and t play no role.) Recall that the component q_i of the vector $\mathbf{q}$ in an orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ is defined by $q_i = \mathbf{q}\cdot\mathbf{e}_i$. Thus taking $\mathbf{n} = \mathbf{e}_i$ in (5.6) gives $h(\mathbf{e}_i) = \mathbf{q}\cdot\mathbf{e}_i = q_i$. Therefore $q_i = h(\mathbf{e}_i)$ is the i^{th} component of heat flux per unit current area flowing across the surface that is normal to the i^{th} direction. Observe that $h(-\mathbf{e}_i) = -q_i$.

Next we derive the field equation associated with (5.5). On using (5.6) and (4.8) in (5.5), one gets

$$\int_{D_t} \mathbf{T}\cdot\mathbf{D}\,\mathrm{d}V_y + \int_{\partial D_t} \mathbf{q}\cdot\mathbf{n}\,\mathrm{d}A_y + \int_{D_t} \rho r\,\mathrm{d}V_y = \int_{D_t} \rho\dot{\varepsilon}\,\mathrm{d}V_y$$

which, on using the divergence theorem leads to

$$\int_{D_t} (\mathbf{T}\cdot\mathbf{D} + \operatorname{div}\mathbf{q} + \rho r - \rho\dot{\varepsilon})\,\mathrm{d}V_y = 0\ .$$

Since this must hold for all subregions D_t of $\mathcal{R}_t$ we can localize the result to obtain the *energy field equation*

$$\mathbf{T}\cdot\mathbf{D} + \operatorname{div}\mathbf{q} + \rho r = \rho\dot{\varepsilon}\ , \tag{5.7}$$

which must hold at all $\mathbf{y}\in\mathcal{R}_t$ and all times t.

Conversely, if the energy field equation (5.7) and the heat flux - heat flux vector relation (5.6) hold, then one can reverse the preceding steps and show that the global balance law (5.5) holds.

5.2 The Second Law of Thermodynamics.

Recall from the thermodynamics of quasi-static homogeneous processes that the first and second laws of thermodynamics are commonly written as

$$W + Q = \dot{U}, \qquad \frac{Q}{\theta} \le \dot{S}\ , \tag{5.8}$$

where W and Q are the rates of working and heating respectively, θ is the absolute temperature, and U and S are the internal energy and entropy respectively. The kinetic energy is ignored because the process is quasi-static, and no spatial variations are present in the system since it is taken to be homogeneous. Our task is to generalize $(5.8)_2$ to the current setting. In the previous section we generalized the first these: $W + Q = \dot{U}$.

We start with the supply of entropy to the body. As before, let $\mathcal{P}$ be some part of a body that is undergoing a motion $\mathbf{y} = \chi(p,t)$ and let $\mathcal{D}_t = \chi(\mathcal{P},t)$ denote the region occupied by $\mathcal{P}$ at time t. We suppose that the *supply of entropy* to $\mathcal{P}$ is due to two sources: (i) the bulk rate of entropy supply per unit mass, $\eta_b(\mathbf{y},t)$, which is the rate at which entropy is generated/provided to all particles throughout $\mathcal{P}$, and (ii) the rate of entropy flux, $\eta_s(\mathbf{y},t,\mathbf{n})$, that is distributed over the particles on the boundary $\partial\mathcal{D}_t$ and represents the rate at which entropy flows into $\mathcal{P}$ across the boundary ∂D_t. The total rate of entropy supplied to $\mathcal{P}$ is thus

$$\int_{\mathcal{D}_t} \rho\, \eta_b(\mathbf{y},t)\, dV_y \;+\; \int_{\partial\mathcal{D}_t} \eta_s(\mathbf{y},t,\mathbf{n})\, dA_y.$$

We now introduce the *absolute temperature* $\theta(\mathbf{y},t)$ that, by assumption[2], relates entropy and heat through

$$\eta_b(\mathbf{y},t) = \frac{r}{\theta}, \qquad \eta_s(\mathbf{y},t,\mathbf{n}) = \frac{\mathbf{q}\cdot\mathbf{n}}{\theta}$$

where $r(\mathbf{y},t)$ and $\mathbf{q}(\mathbf{y},t)$ are the bulk and surface heat supply rates. Note that the entropy supply and entropy flux vanish if and only if the heat supply and heat flux vanish. Thus the total rate at which entropy is supplied to the part $\mathcal{P}$ is

$$\int_{D_t} \frac{\rho r}{\theta}\, \mathrm{d}V_y \;+\; \int_{\partial D_t} \frac{\mathbf{q}\cdot\mathbf{n}}{\theta}\, \mathrm{d}A_y;$$

compare this with the left hand side of $(5.8)_2$.

Next consider the right hand side of $(5.8)_2$. Let $S(\mathcal{P},t;\chi)$ be the entropy of the part $\mathcal{P}$ at time t. From Section 1.8 it follows that there exists a density of entropy, say $\eta(\mathbf{y},t)$, such that the entropy of $\mathcal{P}$ can be written as

$$S(\mathcal{P},t;\chi) = \int_{\mathcal{D}_t} \rho\eta(\mathbf{y},t)\, \mathrm{d}V_y; \tag{5.9}$$

η is called the *specific entropy* (per unit mass). For simplicity we have not displayed the dependency of η on the motion χ: $\eta(\mathbf{y},t;\chi)$.

[2]For a more detailed discussion, see Müller and T. Ruggeri (1993).

The *Second Law of Thermodynamics* states that at each instant during a motion, the sum of the rates of entropy flux and entropy supply cannot exceed the rate of increase of the entropy of $\mathcal{P}$:

$$\int_{D_t} \frac{\rho r}{\theta}\, \mathrm{d}V_y + \int_{\partial D_t} \frac{\mathbf{q}\cdot\mathbf{n}}{\theta}\, \mathrm{d}A_y \leq \frac{\mathrm{d}}{\mathrm{d}t}\int_{D_t} \rho\, \eta\, \mathrm{d}V_y\ . \tag{5.10}$$

Thus the net *rate of entropy production*, i.e. the imbalance, is

$$\Gamma = \frac{\mathrm{d}}{\mathrm{d}t}\int_{D_t} \rho\, \eta\, \mathrm{d}V_y \ - \int_{D_t} \frac{\rho r}{\theta}\, \mathrm{d}V_y \ - \int_{\partial D_t} \frac{\mathbf{q}\cdot\mathbf{n}}{\theta}\, \mathrm{d}A_y \ \geq 0. \tag{5.11}$$

We can introduce the density γ of the entropy production rate so that

$$\Gamma = \int_{D_t} \rho\, \gamma\, \mathrm{d}V_y \ \geq 0 \qquad \text{where} \quad \gamma = \rho\, \dot{\eta} - \frac{\rho r}{\theta} - \operatorname{div}\left(\frac{\mathbf{q}}{\theta}\right). \tag{5.12}$$

The inequality (5.12) must hold for all subregions D_t of $\mathcal{R}_t$. In the now familiar way one can readily derive the field condition corresponding to the global entropy imbalance law (5.11) to be

$$\rho\gamma = \rho\, \dot{\eta} - \frac{\rho r}{\theta} - \operatorname{div}\left(\frac{\mathbf{q}}{\theta}\right) \geq 0 \tag{5.13}$$

which is to hold at all $\mathbf{y} \in \mathcal{R}_t$ and all times t.

Remark: The statement of the second law in the forms (5.10) is called the *Clausius-Duhem inequality*. It is a generalization of equation $(5.8)_2$. One can generalize $(5.8)_2$ in other ways, for example by requiring that some version of it hold at each particle, e.g. to postulate that

$$\rho\, \dot{\eta} \geq \frac{\operatorname{div}\mathbf{q}}{\theta} + \frac{\rho r}{\theta} \tag{5.14}$$

This is called the Clausius-Planck Inequality. Observe that the Clausius-Duhem inequality (5.13) and the Clausius-Planck inequality (5.14) are different in general, but coincide when the temperature field is homogeneous: $\theta(\mathbf{y}, t) = \theta(t)$.

As we shall see later, certain other thermodynamic fields can play important roles in various settings. The most common of these fields are the

$$\left.\begin{aligned}
\text{Helmholtz free energy per unit mass } \psi \quad &= \quad \varepsilon - \eta\, \theta,\\
\text{Enthalpy per unit mass} \quad &= \quad \varepsilon - \frac{\mathbf{S}\cdot\mathbf{F}}{\rho_0},\\
\text{Gibbs free energy per unit mass} \quad &= \quad \varepsilon - \eta\, \theta - \frac{\mathbf{S}\cdot\mathbf{F}}{\rho_0}.
\end{aligned}\right\} \tag{5.15}$$

5.3 Formulation of Thermodynamic Principles with Respect to a Reference Configuration.

Consider some fixed instant during the motion. Consider the geometric context associated with Figure 4.10. Here $\mathcal{S}_t$ is a surface in $\mathcal{R}_t$ and $\mathcal{S}_0$ is its image in the reference configuration. Let $\mathbf{y}$ be a point on $\mathcal{S}_t$ and let $\mathbf{x}$ be its image on $\mathcal{S}_0$; let $\mathbf{n}$ be a unit normal vector to $\mathcal{S}_t$ at $\mathbf{y}$, and let $\mathbf{n}_0$ be the corresponding unit normal vector to $\mathcal{S}_0$ at $\mathbf{x}$; and finally, let $\Delta\mathcal{S}_t$ be an infinitesimal surface element on $\mathcal{S}_t$ at $\mathbf{y}$ whose area is $\mathrm{d}A_y$, and let $\Delta\mathcal{S}_0$ be its image in the reference configuration whose area is $\mathrm{d}A_x$.

Let h be the heat flux at $\mathbf{y}$ on $\mathcal{S}_t$. Then the rate of heat flow across the surface element $\Delta\mathcal{S}_t$ is the product of this flux with the area $\mathrm{d}A_y$:

$$\text{The rate of heat flow across } \Delta\mathcal{S}_t \;=\; h\,\mathrm{d}A_y \;=\; \mathbf{q}\cdot\mathbf{n}\,\mathrm{d}A_y. \tag{5.16}$$

Next, recall from (2.34) the geometric relation

$$\mathrm{d}A_y\,\mathbf{n} = \mathrm{d}A_x\,J\,\mathbf{F}^{-T}\,\mathbf{n}_0 \tag{5.17}$$

relating the area $\mathrm{d}A_y$ to the area $\mathrm{d}A_x$, and the unit normal $\mathbf{n}$ to the unit normal $\mathbf{n}_0$. Combining (5.16) with (5.17) gives

$$\text{The rate of heat flow across } \Delta\mathcal{S}_t \;=\; \mathbf{q}\cdot(J\,\mathbf{F}^{-T})\,\mathbf{n}_0\,\mathrm{d}A_x = \; J\mathbf{F}^{-1}\mathbf{q}\cdot\;\mathbf{n}_0\,\mathrm{d}A_x\;. \tag{5.18}$$

It is natural therefore to define a vector $\mathbf{q}_0$ and a scalar h_0 by

$$\mathbf{q}_0 = J\,\mathbf{F}^{-1}\mathbf{q}, \qquad\qquad h_0 = \mathbf{q}_0\cdot\mathbf{n}_0\;. \tag{5.19}$$

It follows then from (5.16), (5.18) and (5.19) that

$$\text{The rate of heat flow across } \Delta\mathcal{S}_t \;= h\,dA_y \;=\; \mathbf{q}\cdot\mathbf{n}\,dA_y \;= h_0\,dA_x \;=\; \mathbf{q}_0\cdot\mathbf{n}_0\,dA_x. \tag{5.20}$$

Thus, h_0 is the heat flux per unit *referential area*. Note that it flows through the surface element $\Delta\mathcal{S}_t$ in the current configuration.

The physical significance of the components of $\mathbf{q}_0$ can be deduced as follows. Recall that the component q_i^0 of the vector $\mathbf{q}_0$ in an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is defined by $q_i^0 = \mathbf{q}_o\cdot\mathbf{e}_i$. Thus taking $\mathbf{n}_0 = \mathbf{e}_i$ in $(5.19)_2$ gives $h_0(\mathbf{e}_i) = \mathbf{q}_0\cdot\mathbf{e}_i = q_i^0$. Therefore $q_i^0 = h_0(\mathbf{e}_i)$ is the i^{th} component of heat flux per unit referential area flowing across the surface that was normal to the i^{th} direction in the reference configuration.

On introducing the referential heat flux vector $\mathbf{q}_0(\mathbf{x},t)$ we can write the global versions of the first and second laws of thermodynamics as

$$\begin{aligned}\int_{\partial D_0} \mathbf{s}\cdot\mathbf{v}\,\mathrm{d}A_x \quad &+ \quad \textstyle\int_{D_0} \rho_0\,\mathbf{b}\cdot\mathbf{v}\,\mathrm{d}V_x + \int_{\partial D_0} h_0\,\mathrm{d}A_x + \int_{D_0} \rho_0\,r\,\mathrm{d}V_X \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{D_0} \frac{1}{2}\rho_0\,\mathbf{v}\cdot\mathbf{v}\,\mathrm{d}V_x + \frac{\mathrm{d}}{\mathrm{d}t}\int_{D_0} \rho_0\,\varepsilon\,\mathrm{d}V_x\ ,\end{aligned} \tag{5.21}$$

and

$$\int_{D_0} \frac{\rho_0 r}{\theta}\,\mathrm{d}V_x + \int_{\partial D_0} \frac{\mathbf{q}_0\cdot\mathbf{n}_0}{\theta}\,\mathrm{d}A_x \le \frac{\mathrm{d}}{\mathrm{d}t}\int_{D_0} \rho_0\,\eta\,\mathrm{d}V_x\ , \tag{5.22}$$

respectively. These must hold for all subregions $\mathcal{D}_0$ of $\mathcal{R}_0$. The corresponding field conditions can be found in the usual way to be

$$\begin{aligned}\mathbf{S}\cdot\dot{\mathbf{F}} + \mathrm{Div}\,\mathbf{q}_0 + \rho_0 r &= \rho_0\,\dot{\varepsilon}, \\ \mathrm{Div}\left(\frac{\mathbf{q}_0}{\theta}\right) + \frac{\rho_0 r}{\theta} &\le \rho_o\,\dot{\eta},\end{aligned} \tag{5.23}$$

which hold at all $\mathbf{x}\in\mathcal{R}_0$ and all times t.

5.4 Summary.

The field equations associated with the balance of mass, linear momentum, angular momentum, energy, and the imbalance of entropy can be written as

$$\left.\begin{aligned}\dot{\rho} + \rho\,\mathrm{div}\,\mathbf{v} &= 0, \\ \mathrm{div}\,\mathbf{T} + \rho\mathbf{b} &= \rho\dot{\mathbf{v}}, \\ \mathbf{T} &= \mathbf{T}^T, \\ \mathbf{T}\cdot\mathbf{D} + \mathrm{div}\,\mathbf{q} + \rho r &= \rho\,\dot{\varepsilon}, \\ \mathrm{div}\left(\frac{\mathbf{q}}{\theta}\right) + \frac{\rho r}{\theta} &\le \rho\,\dot{\eta},\end{aligned}\right\} \tag{5.24}$$

which must hold at each $\mathbf{y} \in \mathcal{R}_t$ and all times t; and equivalently in the forms

$$\left.\begin{aligned} \rho_0 &= \rho J, \\ \mathrm{Div}\,\mathbf{S} + \rho_0\,\mathbf{b} &= \rho_0\,\dot{\mathbf{v}}, \\ \mathbf{S}\mathbf{F}^T &= \mathbf{F}\mathbf{S}^T, \\ \mathbf{S}\cdot\dot{\mathbf{F}} + \mathrm{Div}\,\mathbf{q}_0 + \rho_0 r &= \rho_0\,\dot{\varepsilon}, \\ \mathrm{Div}\left(\frac{\mathbf{q}_0}{\theta}\right) + \frac{\rho_0 r}{\theta} &\leq \rho_o\,\dot{\eta}, \end{aligned}\right\} \tag{5.25}$$

which must hold at each $\mathbf{x} \in \mathcal{R}_0$ and all times t.

It is useful for later purposes to rewrite the first and second laws of thermodynamics using the specific Helmholtz free-energy ψ in place of the specific internal energy ε where these quantities are related through

$$\psi = \varepsilon - \eta\,\theta; \tag{5.26}$$

recall (5.15). One simply substitutes $\varepsilon = \psi + \rho\theta$ into either $(5.24)_4$ or $(5.25)_4$ to obtain the alternative forms of the first law. We shall not record the results here. We note however that if the energy equation is now solved for the heat supply r, and the result used to eliminate r from the entropy inequality, one is led to

$$\rho\dot{\psi} - \mathbf{T}\cdot\mathbf{D} + \rho\eta\dot{\theta} - \mathbf{q}\cdot\frac{\mathrm{grad}\,\theta}{\theta} \leq 0, \tag{5.27}$$

or equivalently

$$\rho_0\dot{\psi} - \mathbf{S}\cdot\dot{\mathbf{F}} + \rho_0\eta\dot{\theta} - \mathbf{q}_0\cdot\frac{\mathrm{Grad}\,\theta}{\theta} \leq 0. \tag{5.28}$$

5.5 Objectivity of Thermomechanical Quantities.

The notion of material frame indifference, or objectivity, was introduced and discussed in Section 3.8. We adopt the description of this concept in the form described in the last remark of that section.

Let $\alpha, \mathbf{a}, \mathbf{A}$ be some scalar, vector and 2-tensor fields associated with a motion $\mathbf{y}(\mathbf{x}, t)$ and let $\alpha^*, \mathbf{a}^*, \mathbf{A}^*$ be the corresponding fields in the related motion $\mathbf{y}^*(\mathbf{x}, t) = \mathbf{Q}(t)\mathbf{y}(\mathbf{x}, t)$

where $\mathbf{Q}(t)$ is proper orthogonal at all t. Then these properties are said to be objective if

$$\alpha^*(\mathbf{y}^*, t^*) = \alpha(\mathbf{y}, t), \qquad \mathbf{a}^*(\mathbf{y}^*, t^*) = \mathbf{Q}(t)\mathbf{a}(\mathbf{y}, t), \qquad \mathbf{A}^*(\mathbf{y}^*, t^*) = \mathbf{Q}(t)\mathbf{A}(\mathbf{y}, t)\mathbf{Q}^T(t) \tag{5.29}$$

where[3]

$$\mathbf{y}^* = \mathbf{Q}(t)\mathbf{y}, \qquad t^* = t, \tag{5.30}$$

for all time dependent proper orthogonal tensors $\mathbf{Q}(t)$. Figure 5.1 depicts the regions $\mathcal{D}$ and $\mathcal{D}_*$ occupied by the same part $\mathcal{P}$ of a body in a pair of motions $\mathbf{y}(\mathbf{x}, t)$ and $\mathbf{Q}(t)\mathbf{y}(\mathbf{x}, t)$.

As we noted in Section 3.8, in the case of purely kinematic quantities such as velocity, deformation gradient, rate of stretching tensor etc., one can simply verify whether or not those quantities are objective. Not so with quantities that are not solely kinematic. *Here one has to postulate whether a particular physical quantity is objective or not based on the underlying physics.* When we discussed mass balance at the beginning of this chapter we introduced one new field, the mass density $\rho(\mathbf{y}, t)$; then we moved to the momentum principles and introduced two new fields, the traction $\mathbf{t}(\mathbf{y}, t, \mathbf{n})$ and the body force density $\mathbf{b}(y, t)$; next we turned to the first law of thermodynamics and introduced another three fields, the specific internal energy $\varepsilon(\mathbf{y}, t)$, the heat flux $h(\mathbf{y}, t, \mathbf{n})$ and the heat supply $r(\mathbf{y}, t)$; finally we considered the second law of thermodynamics and introduced the specific entropy $\eta(\mathbf{y}, t)$ and the temperature $\theta(\mathbf{y}, t)$. We now *postulate* that all of these fields must be objective.

$$\left.\begin{array}{c} \rho^* = \rho, \\[2ex] \mathbf{t}^* = \mathbf{Q}\mathbf{t}, \qquad \mathbf{b}^* = \mathbf{Q}\mathbf{b}, \\[2ex] \varepsilon^* = \varepsilon, \qquad h^* = h, \qquad r^* = r, \\[2ex] \eta^* = \eta, \qquad \theta^* = \theta. \end{array}\right\} \tag{5.31}$$

In order to understand the basis for these postulates[4], consider for example the traction. Figure 5.1 depicts the boundary surfaces $\partial\mathcal{D}_t$ and $\partial\mathcal{D}_t^*$ of some part of a body in the pair of motions $\mathbf{y}(\mathbf{x}, t)$ and $\mathbf{Q}(t)\mathbf{y}(\mathbf{x}, t)$. Note that the outward normal vectors at corresponding

[3] In general we should consider $\mathbf{y}^* = \mathbf{Q}\mathbf{y} + \mathbf{c}$, $t^* = t + a$ where $\mathbf{c}(t)$ is a time dependent vector and a is a constant. However for our considerations it is sufficient to consider the special case (5.30) where $\mathbf{c} = \mathbf{o}$, $a = 0$.

[4] In the case of the mass density, let ρ_0 denote the mass density in some reference configuration. Then mass balance tells us that $\rho_0 = \rho^* J^*$ and $\rho_0 = \rho J$. However we know from Section 3.8 that $\mathbf{F}^* = \mathbf{Q}\mathbf{F}$. Therefore $J = \det \mathbf{F} = \det \mathbf{F}^* = J^*$ and so it follows that $\rho = \rho^*$. Thus the objectivity of the mass density is automatic and is not in fact a postulate; it has been built into its definition.

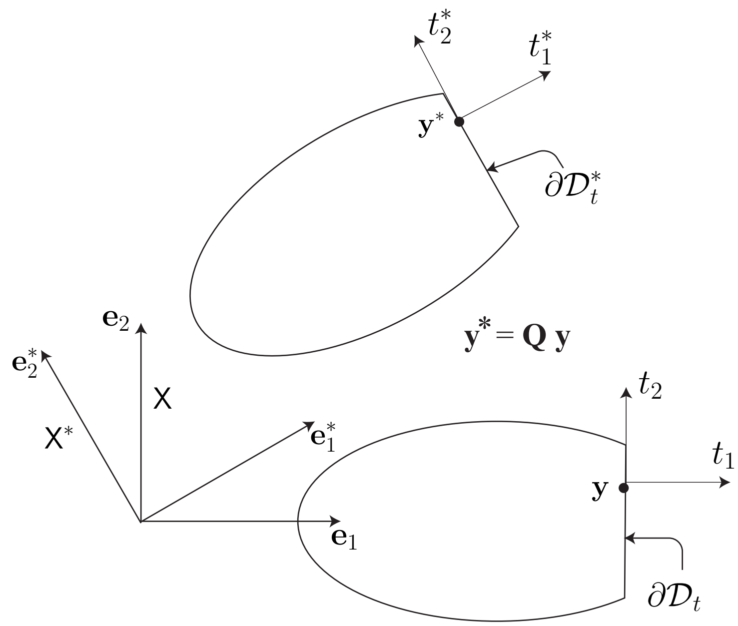

Figure 5.1: The figure shows the regions occupied by a body in two motions $\widehat{\mathbf{y}}(\mathbf{x},t)$ and $\mathbf{Q}(t)\widehat{\mathbf{y}}(\mathbf{x},t)$. At the instant depicted, the same particle is located at $\mathbf{y}$ and $\mathbf{y}^* = \mathbf{Q}\mathbf{y}$ in these two motions. The bases $\mathsf{X} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\mathsf{X}^* = \{\mathbf{e}_1^*, \mathbf{e}_2^*, \mathbf{e}_3^*\}$ are related by $\mathbf{e}_i^* = \mathbf{Q}\mathbf{e}_i$, $i = 1, 2, 3$. We postulate, based on physical expectation, that the traction components (t_1, t_2, t_3) should equal the traction components (t_1^*, t_2^*, t_3^*) shown in the figure, i.e. that the components of the traction vector $\mathbf{t}$ in the basis X should equal the components of the traction vector $\mathbf{t}^*$ in the basis X^*.

points on $\partial\mathcal{D}_t$ and ∂D_t^* are related by $\mathbf{n}^* = \mathbf{Q}\mathbf{n}$. On physical grounds, it is natural to postulate that the components of $\mathbf{t}$ in the basis X should equal the corresponding components of $\mathbf{t}^*$ in the basis X^* (see Figure 5.1). Thus we require that $t_i = t_i^*$ where $t_i = \mathbf{e}_i \cdot \mathbf{t}$ and $t_i^* = \mathbf{e}_i^* \cdot \mathbf{t}^*$. It can be readily shown now from $\mathbf{e}_i \cdot \mathbf{t} = \mathbf{e}_i^* \cdot \mathbf{t}^*$ and $\mathbf{e}_i^* = \mathbf{Q}\mathbf{e}_i$ that

$$\mathbf{t}^* = \mathbf{Q}\mathbf{t},$$

i.e. that the traction is an objective vector.

One can show as a consequence of (5.31) that the Cauchy stress tensor $\mathbf{T}$, heat flux vector $\mathbf{q}$ and stress power density $J\mathbf{T}\cdot\mathbf{D}$ are automatically objective:

$$\mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}, \qquad \mathbf{q}^* = \mathbf{Q}\mathbf{q}, \qquad J^*\mathbf{T}^*\cdot\mathbf{D}^* = J\mathbf{T}\cdot\mathbf{D}. \tag{5.32}$$

To show the first of these claims, consider the objectivity of the traction. As just noted, the unit outward normal vector $\mathbf{n}$ at a point on the boundary $\partial\mathcal{D}_t$ is objective:

$$\mathbf{n}^* = \mathbf{Q}\mathbf{n}. \tag{5.33}$$

Now consider the Cauchy stress tensors $\mathbf{T}$ and $\mathbf{T}^*$ which are related to the tractions by

$$\mathbf{t}^* = \mathbf{T}^*\mathbf{n}^*, \qquad \mathbf{t} = \mathbf{T}\mathbf{n}. \tag{5.34}$$

Substituting the first of $(5.31)_2$ and (5.33) into $(5.34)_1$ gives $\mathbf{Q}\mathbf{t} = \mathbf{T}^*\mathbf{Q}\mathbf{n}$ and now substituing $(5.34)_2$ into this to eliminate $\mathbf{t}$ yields $\mathbf{Q}\mathbf{T}\mathbf{n} = \mathbf{T}^*\mathbf{Q}\mathbf{n}$. Since $\mathbf{T}$ does not depend on the normal vector $\mathbf{n}$ and this result must hold for all unit vectors $\mathbf{n}$ it follows that $\mathbf{Q}\mathbf{T} = \mathbf{T}^*\mathbf{Q}$ or

$$\mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^T. \tag{5.35}$$

Thus we conclude that the Cauchy stress tensor is objective.

Similarly by using $h = \mathbf{q}\cdot\mathbf{n}$ and $h^* = \mathbf{q}^*\cdot\mathbf{n}^*$ one can show that the postulate $(5.31)_3$, i.e. that the heat flux h be objective, implies that the heat flux vector must be objective:

$$\mathbf{q}^* = \mathbf{Q}\mathbf{q}. \tag{5.36}$$

Since $J = \det\mathbf{F}, \mathbf{D}$ and $\mathbf{T}$ are each objective, it can be readily shown that the stress power per unit reference volume, $J\mathbf{T}\cdot\mathbf{D}$, is also objective:

$$J^*\mathbf{T}^*\cdot\mathbf{D}^* = J\mathbf{T}\cdot\mathbf{D}. \tag{5.37}$$

<u>*Remark*</u>: One can show that the first Piola-Kirchhoff stress tensor $\mathbf{S}$ and the Lagrangian heat flux vector $\mathbf{q}_0$ obey

$$\mathbf{S}^* = \mathbf{Q}\mathbf{S}, \qquad \mathbf{q}_0^* = \mathbf{q}_0, \tag{5.38}$$

and therefore are *not objective*, whereas the Piola-Kirchhoff traction $\mathbf{s} = \mathbf{S}\mathbf{n}_0$ and referential heat flux $h_0 = \mathbf{q}_0\cdot\mathbf{n}_0$ obey

$$\mathbf{s}^* = \mathbf{Q}\mathbf{s}, \qquad h_0^* = h_0, \tag{5.39}$$

and therefore are objective. In establishing the last pair of equations it is useful to note that the unit normal vector in the reference configuration is $\mathbf{n}_0$ to both observers.

Finally consider the time rate of change of stress, a quantity that is needed for the formulation of the constitutive relations for certain materials. Since the Cauchy stress is objective, taking the material time derivative of $\mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^T$ leads to $\dot{\mathbf{T}}^* = \mathbf{Q}\dot{\mathbf{T}}\mathbf{Q}^T + \dot{\mathbf{Q}}\mathbf{T}\mathbf{Q}^T + \mathbf{Q}\mathbf{T}\dot{\mathbf{Q}}^T$ and so $\dot{\mathbf{T}}$ is not objective (except under Gallilean transformations where

Q is time independent). However, as was shown in one of the problems in Section 3.6, the co-rotational derivative of **T** defined by

$$\overset{\triangle}{\mathbf{T}} = \dot{\mathbf{T}} + \mathbf{L}^T\mathbf{T} + \mathbf{T}\mathbf{L},$$

is objective. All of the following quantities, each of which has the dimension of stress rate, can be shown to be objective:

$$\begin{array}{lcll}
\overset{\triangle}{\mathbf{T}} & = & \dot{\mathbf{T}} + \mathbf{L}^T\mathbf{T} + \mathbf{T}\mathbf{L} & \text{Convected rate,} \\
\overset{\triangledown}{\mathbf{T}} & = & \dot{\mathbf{T}} - \mathbf{L}\mathbf{T} - \mathbf{T}\mathbf{L}^T & \text{Oldroyd rate,} \\
\overset{\circ}{\mathbf{T}} & = & \dot{\mathbf{T}} - \mathbf{W}\mathbf{T} + \mathbf{T}\mathbf{W} & \text{Co} - \text{rotational or Jaumann rate,} \\
\overset{\otimes}{\mathbf{T}} & = & \dot{\mathbf{T}} + \mathbf{T}\boldsymbol{\Omega} - \boldsymbol{\Omega}\mathbf{T} & \text{Green} - \text{Naghdi rate,} \\
\overset{\square}{\mathbf{T}} & = & \frac{1}{2}(\overset{\triangle}{\mathbf{T}} - \overset{\triangledown}{\mathbf{T}}) = \mathbf{D}\mathbf{T} + \mathbf{T}\mathbf{D}; &
\end{array}$$

here $\boldsymbol{\Omega} = \dot{\mathbf{R}}\mathbf{R}^T$ is the angular velocity tensor. See Chapter 3.9 for a discussion of some of these rates. These stress rates are encountered in certain constitutive relations.

The significance of the preceding discussion on objectivity will become apparent when we consider constitutive relations in what follows and require that they be independent of the observer. Roughly speaking, various physical fields such as the stress are related to the kinematic fields through constitutive equations, and every candidate constitutive law must be consistent with the preceding requirements arising from objectivity.

5.6 Worked Examples and Exercises.

Problem 5.1. Show that there is a (time-independent) additive degree of non-uniqueness in the specification of the specific internal energy and the specific entropy; i.e. show that the first and second laws of thermodynamics remain invariant under the transformations

$$\varepsilon(\mathbf{y},t) \;\to\; \varepsilon(\mathbf{y},t) + \varepsilon_0(\mathbf{y}), \qquad \eta(\mathbf{y},t) \;\to\; \eta(\mathbf{y},t) + \eta_0(\mathbf{y}),$$

where $\varepsilon_0(\mathbf{y})$ and $\eta_0(\mathbf{y})$ are arbitrary functions. (These functions would be chosen by picking a datum from which the internal energy and entropy are measured.)

Problem 5.2. Consider a thermo-mechanical process of a body subjected to the following mechanical boundary conditions:

$$\mathbf{Tn} = -p_0\,\mathbf{n} \quad \text{on } \partial\mathcal{D}_1, \qquad \mathbf{v} = \mathbf{o} \quad \text{on } \partial\mathcal{D}_2,$$

and the following thermal boundary conditions:

$$\theta = \theta_0 \quad \text{on } \partial\mathcal{D}_3, \qquad \mathbf{q}\cdot\mathbf{n} = 0 \quad \text{on } \partial\mathcal{D}_4;$$

here the pressure p_0 and temperature θ_0 are constants, and $\partial\mathcal{D}_1 \cup \partial\mathcal{D}_2 = \partial\mathcal{D}_3 \cup \partial\mathcal{D}_4 = \mathcal{D}_t$. Show that there is a Lyapunov function associated with any process of this body.

Problem 5.3. As we will see in a later chapter, for a general elastic material the Helmholtz free energy can be expressed as a function of the deformation gradient tensor and the temperature, $\psi = \widehat{\psi}(\mathbf{F}, \theta)$, and in addition,

$$\mathbf{S} = \rho_0 \widehat{\psi}_{\mathbf{F}}(\mathbf{F}, \theta), \qquad \eta = -\hat{\psi}_\theta(\mathbf{F}, \theta)$$

where the subscripts denote differentiation, e.g. $\widehat{\psi}_{\mathbf{F}} = \partial\widehat{\psi}/\partial\mathbf{F}$. Suppose further that Fourier's law of heat conduction $\mathbf{q}_0 = \mathbf{K}(\mathbf{F}, \theta)\,\mathrm{Grad}\,\theta$ holds. The referential heat conductivity tensor $\mathbf{K}$ may be assumed to be symmetric. Specialize the general field equations of thermo-mechanics to such a material. In particular derive a set of four partial differential equations that involve the three components $u_i(\mathbf{x}, t)$ of displacement and the temperature $\theta(\mathbf{x}, t)$.

Problem 5.4. Reconsider the setting of Problem 5.3 and consider a uni-axial motion of a body:

$$y_1 = x + u(x, t), \qquad y_2 = x_2, \qquad y_3 = x_3,$$

where we have set $x_1 = x$ and $u_1 = u$ for convenience. Specialize the general field equations of thermo-mechanics to such a motion.

Problem 5.5. Reconsider the general three-dimensional setting of Problem 5.3. Derive a formula for the specific heat at constant deformation gradient.

Problem 5.6. Reconsider the general three-dimensional setting of Problem 5.3 and calculate the rate of entropy production Γ.

Problem 5.7. Truesdell defines the "internal dissipation rate" (per unit mass) δ by

$$\delta = \theta\dot{\eta} - (1/\rho)(\mathrm{div}\mathbf{q} + \rho r).$$

Note that δ involves thermodynamic quantities only. Note also that the Clausius-Planck inequality (5.14) requires that $\delta \geq 0$. Finally observe that the density of the entropy production rate γ is related to δ by

$$\gamma = \frac{\delta}{\theta} + \frac{1}{\rho\theta^2}\,\mathbf{q}\cdot\mathrm{grad}\,\theta.$$

Gurtin, Fried and Anand refer to $\theta\gamma$ as the dissipation rate and the Clausius-Duhem inequality requires that this be non-negative.

(a) Show that one can alternatively write δ as

$$\delta = \mathbf{T}\cdot\mathbf{D}/\rho - (\dot{\psi} + \dot{\theta}\eta)$$

where $\psi = \varepsilon - \eta\theta$ is known as the Helmholtz free-energy.

(b) Consider an elastic fluid (i.e., a compressible, inviscid fluid). For such as material,

$$\begin{aligned} \mathbf{T} &= -p(v,\eta)\mathbf{I}, & \varepsilon &= \hat{\varepsilon}(v,\eta), \\ p &= -\frac{\partial\hat{\varepsilon}}{\partial v}(v,\eta) & \theta &= \frac{\partial\hat{\varepsilon}}{\partial\eta}(v,\eta) \end{aligned}$$

where $v = 1/\rho$ is the specific volume. Show that $\delta = 0$ for an elastic fluid.

(c) Next consider a general elastic material. In this case,

$$\begin{aligned} \mathbf{S} &= \widehat{\mathbf{S}}(\mathbf{F},\eta), & \varepsilon &= \hat{\varepsilon}(\mathbf{F},\eta), \\ \mathbf{S} &= \rho_0\frac{\partial\hat{\varepsilon}}{\partial\mathbf{F}}(\mathbf{F},\eta), & \theta &= \frac{\partial\hat{\varepsilon}}{\partial\eta}(\mathbf{F},\eta). \end{aligned}$$

(As we shall see later, this is an alternative equivalent characterization of an elastic material to that given in Problem 5.3.) Show that $\delta = 0$ for a general elastic material.

(d) Consider next a linearly viscous (compressible) fluid. For such a material,

$$\begin{aligned} \mathbf{T} &= -p(v,\eta)\mathbf{I} + \lambda(\mathrm{tr}D)\mathbf{1} + 2\mu\mathbf{D}, & \epsilon &= \hat{\varepsilon}(v,\eta), \\ p &= -\frac{\partial\hat{\varepsilon}}{\partial v}(v,\eta), & \theta &= \frac{\partial\hat{\varepsilon}}{\partial\eta}(v,\eta) \end{aligned}$$

where $v = 1/\rho$ is the specific volume and λ and μ depend at most on temperature. Derive an expression for the internal dissipation rate δ and show that it is positive semi-definite for every symmetric tensor $\mathbf{D}$ if and only if

$$\mu \geq 0, \qquad 3\lambda + 2\mu \geq 0.$$

(e) Does the second law of thermodynamics (in all cases) require that $\delta \geq 0$? Does it require it in the special case of a linearly viscous fluid?

Solution

(a) We use the energy equation

$$\mathbf{T}\cdot\mathbf{D} + \text{div}\,\mathbf{q} + \rho r = \rho\dot{\varepsilon}\ ,$$

to eliminate the heat flux $\mathbf{q}$ and heta supply r from the internal dissipation rate:

$$\delta = \theta\dot{\eta} - (1/\rho)(\text{div}\mathbf{q} + \rho r) = \theta\dot{\eta} - (1/\rho)(\rho\dot{\varepsilon} - \mathbf{T}\cdot\mathbf{D}) = \mathbf{T}\cdot\mathbf{D}/\rho + \theta\dot{\eta} - \dot{\varepsilon} \qquad (a)$$

Substituting $\varepsilon = \psi + \eta\theta$ into this leads to

$$\delta = \mathbf{T}\cdot\mathbf{D}/\rho + \theta\dot{\eta} - (\dot{\psi} + \theta\dot{\eta} + \dot{\theta}\eta) = \mathbf{T}\cdot\mathbf{D}/\rho - (\dot{\psi} + \dot{\theta}\eta).$$

(b) For an invicid compressible fluid we are given that

$$\mathbf{T} = -p(v,\eta)\mathbf{I}, \qquad \varepsilon = \hat{\varepsilon}(v,\eta), \qquad p = -\frac{\partial\hat{\varepsilon}}{\partial v}(v,\eta) \qquad \theta = \frac{\partial\hat{\varepsilon}}{\partial\eta}(v,\eta) \qquad (b)$$

where $v = 1/\rho$ is the specific volume. First we calculate $\mathbf{T}\cdot\mathbf{D}$:

$$\mathbf{T}\cdot\mathbf{D} = -p\mathbf{I}\cdot\mathbf{D} = -p\,\text{tr}\,\mathbf{D} = p\,\frac{\dot{\rho}}{\rho} \qquad (c)$$

where in the last step we have used mass balance $\dot{\rho} + \rho\text{tr}\,\mathbf{D} = 0$. Next we calculate $\dot{\varepsilon}$:

$$\dot{\varepsilon} = \frac{\partial\hat{\varepsilon}}{\partial v}\dot{v} + \frac{\partial\hat{\varepsilon}}{\partial\eta}\dot{\eta} = -p\dot{v} + \theta\dot{\eta} \qquad (d)$$

where we have used $(b)_{2,3}$. To evaluate the internal dissipation rate δ it is most convenient for us to use the form given in equation (a) and to substitute (c) and (d) into it:

$$\delta = \mathbf{T}\cdot\mathbf{D}/\rho + \theta\dot{\eta} - \dot{\varepsilon} = p\,\frac{\dot{\rho}}{\rho^2} + p\dot{v}.$$

Since $v = 1/\rho$ it follows that $\dot{v} = -\dot{\rho}/\rho^2$ and therefore $\delta = 0$.

(c) For a general elastic material we are given

$$\mathbf{S} = \widehat{\mathbf{S}}(\mathbf{F},\eta), \qquad \varepsilon = \hat{\varepsilon}(\mathbf{F},\eta), \qquad \mathbf{S} = \rho_0\frac{\partial\hat{\varepsilon}}{\partial\mathbf{F}}(\mathbf{F},\eta), \qquad \theta = \frac{\partial\hat{\varepsilon}}{\partial\eta}(\mathbf{F},\eta). \qquad (e)$$

First we calculate $\dot{\varepsilon}$:

$$\dot{\varepsilon} = \frac{\partial\hat{\varepsilon}}{\partial\mathbf{F}}\cdot\dot{\mathbf{F}} + \frac{\partial\hat{\varepsilon}}{\partial\eta}\dot{\eta} = \mathbf{S}\cdot\dot{\mathbf{F}}/\rho_0 + \theta\dot{\eta} \qquad (f)$$

where we have used $(e)_{3,4}$. Recall from our discussion on stress power, see (4.73) and (4.50), that

$$\mathbf{T}\cdot\mathbf{D}/\rho = \mathbf{S}\cdot\dot{\mathbf{F}}/\rho_0. \qquad (g)$$

To evaluate the internal dissipation rate δ it is most convenient for us to use the form given in equation (a) and to substitute (f) and (g) into it:

$$\delta = \mathbf{T}\cdot\mathbf{D}/\rho + \theta\dot{\eta} - \dot{\varepsilon} = \mathbf{S}\cdot\dot{\mathbf{F}}/\rho_0 + \theta\dot{\eta} - \Big(\mathbf{S}\cdot\dot{\mathbf{F}}/\rho_0 + \theta\dot{\eta}\Big) = 0.$$

(d) For a linearly viscous (compressible) fluid we are given that

$$\mathbf{T} = -p(v,\eta)\mathbf{1} + \lambda(\mathrm{tr}D)\mathbf{1} + 2\mu\mathbf{D}, \quad \epsilon = \hat{\varepsilon}(v,\eta), \quad p = -\frac{\partial\hat{\varepsilon}}{\partial v}(v,\eta), \quad \theta = \frac{\partial\hat{\varepsilon}}{\partial\eta}(v,\eta)$$

where $v = 1/\rho$ is the specific volume and λ and μ depend at most on temperature. First calculate $\mathbf{T}\cdot\mathbf{D}$:

$$\mathbf{T}\cdot\mathbf{D} = \Big(-p\mathbf{I} + \lambda(\mathrm{tr}D)\mathbf{I} + 2\mu\mathbf{D}\Big)\cdot\mathbf{D} = p\,\frac{\dot{\rho}}{\rho} + \lambda(\mathrm{tr}\,\mathbf{D})^2 + 2\mu\mathrm{tr}\,\mathbf{D}^2$$

where we have used $\mathbf{I}\cdot\mathbf{D} = \mathrm{tr}\,\mathbf{D},\ \mathbf{D}\cdot\mathbf{D} = \mathrm{tr}\,\mathbf{D}^2$ and that mass balance gives $\dot{\rho} + \rho\mathrm{tr}\,\mathbf{D} = 0$. Next we calculate $\dot{\varepsilon}$ and proceed as in part (b) to get

$$\rho\delta = \lambda(\mathrm{tr}\,\mathbf{D})^2 + 2\mu\mathrm{tr}\,\mathbf{D}^2. \qquad (h)$$

Note that a *sufficient* condition for δ to be nonnegative is for $\lambda \geq 0, \mu \geq 0$. To obtain a condition that is both necessary and sufficient we set $d = \mathrm{tr}\,\mathbf{D}/3$ and observe that

$$\mathrm{tr}\,(\mathbf{D} - d\mathbf{I})^2 = \mathrm{tr}\,\Big(\mathbf{D}^2 - 2d\mathbf{D} + d^2\mathbf{I}\Big) = \mathrm{tr}\,\mathbf{D}^2 - 2d\,\mathrm{tr}\,\mathbf{D} + 3d^2 = \mathrm{tr}\,\mathbf{D}^2 - \frac{2}{3}(\mathrm{tr}\,\mathbf{D})^2 + \frac{1}{3}(\mathrm{tr}\,\mathbf{D})^2$$

Therefore we can write the internal dissipation rate as

$$\rho\delta = \lambda\,(\mathrm{tr}\,\mathbf{D})^2 + 2\mu\,\mathrm{tr}\,\mathbf{D}^2 = \left(\lambda + \frac{2\mu}{3}\right)(\mathrm{tr}\,\mathbf{D})^2 + 2\mu\,\mathrm{tr}\,(\mathbf{D} - d\mathbf{I})^2 \geq 0 \qquad (j)$$

Therefore a sufficient condition for δ to be nonnegative is to have

$$\mu > 0, \qquad \lambda + \frac{2}{3}\mu > 0. \qquad (k)$$

Since (j) is to hold for all $\mathbf{D}$, if we pick $\mathbf{D} = d\mathbf{I}$ we see that $\lambda + 2\mu/3 \geq 0$ is necessary for $\delta \geq 0$. And if we pick any traceless $\mathbf{D}$ so that $d = 0$ we see that $\mu \geq 0$ is also necessary. Therefore the inequalities (k) are necessary and sufficient for the internal dissipation rate to be nonnegative.

(e) The entropy inequality arising from the Clausius Duhem inequality is

$$\rho\,\dot{\eta} \geq \mathrm{div}\left(\frac{\mathbf{q}}{\theta}\right) + \frac{\rho r}{\theta}.$$

Let γ denote the rate of entropy production per unit mass so that

$$\gamma = \dot{\eta} - \frac{1}{\rho}\mathrm{div}\left(\frac{\mathbf{q}}{\theta}\right) - \frac{r}{\theta} \geq 0.$$

On expanding $\mathrm{div}\,(\mathbf{q}/\theta)$ and using the definition of δ we can write

$$\gamma = \frac{\delta}{\theta} + \frac{\mathbf{q}\cdot\mathrm{grad}\,\theta}{\rho\theta^2}$$

Suppose heat cannot flow in the direction of increasing temperature, so that $\mathbf{q}\cdot\mathrm{grad}\,\theta \geq 0$. Then

$$\delta \geq 0\ \Rightarrow \gamma \geq 0 \qquad \text{though} \qquad \gamma \geq 0 \not\Rightarrow \delta \geq 0.$$

Here $\gamma \geq 0$ is the entropy inequality.

Problem 5.8. In these notes we formulated and analyzed the basic balance laws of thermodynamics by focussing attention on an arbitrary fixed part $\mathcal{P}$ of the body $\mathcal{B}$; our attention at all times was therefore on the same set of particles. Sometimes it is convenient to work with a *control volume* instead: i.e. a fixed region in space (with different particles entering and leaving as time progresses).

Let $\mathcal{R}_t$ be the region occupied by the body at the instant t. Let Π be a fixed region of space which is such that $\Pi \subset R_t$ for all times close to t.

Derive statements of the first and second laws of thermodynamics that are valid for a control volume Π.

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Chapter 6

Singular Surfaces and Jump Conditions

6.1 Introduction.

Our analyses of kinematics and balance laws in the preceding chapters assumed that the motion $\mathbf{y}(\mathbf{x}, t)$ and various other fields such as the stress $\mathbf{S}(\mathbf{x}, t)$ and heat flux $\mathbf{q}_0(\mathbf{x}, t)$ possessed certain degrees of smoothness. Though these smoothness requirements are often met, there are some circumstances of physical interest in which they are not.

Of particular interest are settings where the classical degree of smoothness is met everywhere except at one or more surfaces in the body: certain fields suffer finite jump discontinuities at such a surface but remain smooth everywhere else. We refer to such a surface generically as a "singular surface". A familiar example of a singular surface is the common interface between two bonded bodies. Other examples include problems involving impact loading of solids, transonic flows in gas dynamics, and phase transitions in solids where the singular surface corresponds to a wave front, a shock wave and a phase boundary respectively.

Consider the following Lagrangian framework for studying a singular surface: a body occupies a region $\mathcal{R}_0$ in a (time-independent) reference configuration. Let $\mathcal{S}_0(t)$ be a surface that moves through $\mathcal{R}_0$. Note the distinction between a point on the surface and a particle of the body, even if they both have the same location at some particular instant. Since $\mathcal{S}_0(t)$ moves through $\mathcal{R}_0$ it is associated with different particles of the body at different times and so is *not a material surface*. In the special case when $\mathcal{S}_0$ is time-independent, it

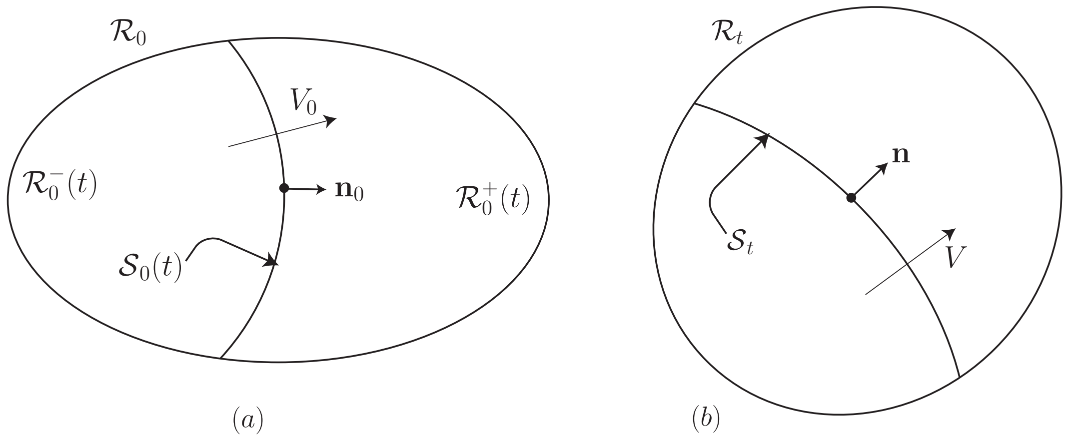

Figure 6.1: The (time-independent) region $\mathcal{R}_0$ occupied by a body in a reference configuration is separated into two (time-dependent) subregions $\mathcal{R}_0^{\pm}(t)$ by the moving singular surface $\mathcal{S}_0(t)$. The region $\mathcal{R}_t$ and surface $\mathcal{S}_t$ are corresponding images in the current configuration. The unit vector $\mathbf{n}_0$ is normal to $\mathcal{S}_0(t)$ and points into the region $\mathcal{R}_0^+$. It is $\mathcal{S}_t$ that one would observe in the laboratory.

is then associated with the same particles at all times. A shock wave is an example of a non-material singular surface, while the interface between two bonded bodies is a material singular surface.

Suppose that the surface $\mathcal{S}_0(t)$ separates $\mathcal{R}_0$ into two subregions $\mathcal{R}_0^+(t)$ and $\mathcal{R}_0^-(t)$ where $\mathcal{R}_0^+(t) \cup \mathcal{R}_0^-(t) = \mathcal{R}_0$; see Figure 6.1(a). Let $\mathbf{n}_0(\mathbf{x}, t)$ be a unit vector that is normal to $\mathcal{S}_0(t)$ at $\mathbf{x}$. The surface is oriented such that $\mathbf{n}_0$ points into the "positive region" $\mathcal{R}_0^+$. Let $V_0(\mathbf{x}, t)$ denote the propagation speed of a point of this surface in the direction $\mathbf{n}_0$ normal to the surface.

Consider a generic (scalar, vector or tensor) field $\alpha(\mathbf{x}, t)$ which is discontinuous across $\mathcal{S}_0(t)$ but whose limiting values at a point $\mathbf{x} \in \mathcal{S}_0(t)$ exist when $\mathbf{x}$ is approached from the plus and minus sides. These values are denoted by $\alpha^+(\mathbf{x}, t)$ and $\alpha^-(\mathbf{x}, t)$ respectively. The difference between α^+ and α^- is the *jump* in α at the point $\mathbf{x}$ which we denote by

$$[\![\, \alpha \,]\!] = \alpha^+ - \alpha^-. \tag{6.1}$$

One can have different types of singular surfaces depending on the quantities that are discontinuous across it. For example, at each instant t, the motion $\mathbf{y}(\cdot, t)$ maybe continuous but the velocity $\mathbf{v}(\cdot, t)$ may suffer a jump discontinuity across $\mathcal{S}_0$. Or the motion and velocity fields may both be continuous but the acceleration $\dot{\mathbf{v}}(\cdot, t)$ may suffer a jump discontinuity at $\mathcal{S}_0$. And so on.

In these notes we assume that the various fields possess the usual degree of smoothness on either side of $\mathcal{S}_0(t)$. We insist that the motion $\mathbf{y}(\mathbf{x}, t)$ be continuous on $\mathcal{R}_0$ but we allow the deformation gradient tensor $\mathbf{F}$, the particle velocity $\mathbf{v}$, the first Piola-Kirchhoff stress tensor $\mathbf{S}$, the referential heat flux vector $\mathbf{q}_0$, the specific internal energy ε, the specific entropy η and the temperature θ to suffer jump discontinuities across $\mathcal{S}_0$. This means they have limiting values[1] $\mathbf{F}^{\pm}$, $\mathbf{q}_0^{\pm}$, $\mathbf{v}^{\pm}$, $\mathbf{S}^{\pm}$, $\varepsilon^{\pm}$, $\eta^{\pm}$ and $\theta^{\pm}$ on $\mathcal{S}_0$. These limiting values are not entirely arbitrary since the continuity of the motion and the balance laws impose certain restrictions on them. These restrictions – the jump conditions – are what we wish to derive and study in this chapter. This class of singular surfaces describe, in particular, both a *shock wave* and a *phase boundary*.

Finally we remark that it is the image of $\mathcal{S}_0$ in the current configuration, say $\mathcal{S}_t$, that one would observe in the laboratory. See Figure 6.1(b).

6.2 Jump Conditions in 1-D Theory.

For simplicity we begin by carrying out our analysis in a purely one-dimensional theory. The three-dimensional case will be addressed in subsequent sections.

Accordingly we consider a one-dimensional continuum that occupies the interval $[0, L]$ of the x-axis in a reference configuration: $\mathcal{R}_0 = \{x : 0 \leq x \leq L\}$. During a motion of this body, the particle located at x in the reference configuration is carried at time t to the point y by the motion $y = y(x, t)$. The counterpart of a singular surface in this one dimensional setting is a *singular point* across which certain fields are allowed to have jump discontinuities.

We assume in these notes that the motion is continuous, with piecewise continuous first and second derivatives with respect to both spatial and time coordinates. The stretch λ and the particle velocity v are defined by $\lambda = y_x$ and $v = y_t$, where the subscripts x and t indicate partial derivatives. The degree of smoothness assumed allows λ and v to have jump discontinuities at a singular point.

Suppose that the motion involves a single singular point whose referential location at time t is $x = s_0(t) \in (0, L)$. The stretch $\lambda(x, t)$ and particle velocity $v(x, t)$ have limiting

[1] The body force $\mathbf{b}$ and heat supply r, both of which are prescribed externally, are taken to be continuous throughout $\mathcal{R}_0$.

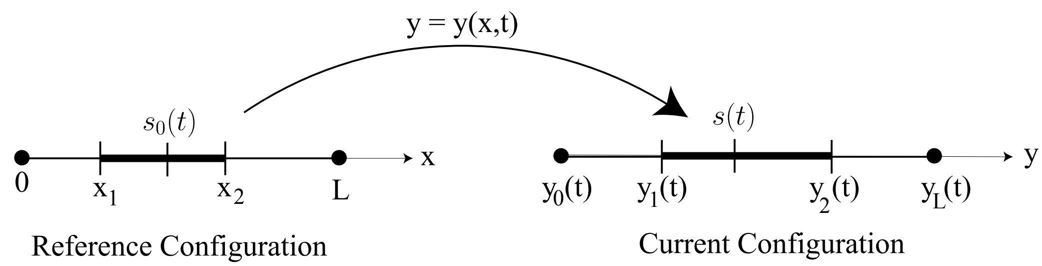

Figure 6.2: Reference and current configurations of a bar. The motion involves one singular point whose referential and spatial locations are $x = s_0(t)$ and $y = s(t)$ respectively.

values $\lambda^\pm$ and $v^\pm$ at $x = s_0$:

$$\lambda^\pm(t) = \lim_{\varepsilon\to 0} \lambda(s_0(t) \pm \varepsilon, t), \qquad v^\pm(t) = \lim_{\varepsilon\to 0} v(s_0(t) \pm \varepsilon, t), \tag{6.2}$$

By assumption $(\lambda^+, v^+) \neq (\lambda^-, v^-)$ reflecting the fact that the stretch and particle velocity suffer jump discontinuities at $x = s_0$.

We begin by considering the kinematics of the motion. Since the motion $y(x,t)$ is continuous everywhere, and therefore in particular at $x = s_0(t)$, we have

$$y(s_0(t)+, t) = y(s_0(t)-, t). \tag{6.3}$$

Differentiating this with respect to time and setting $\lambda = y_x$ and $v = y_t$ shows that

$$\lambda\big(s_0(t)+, t\big)\, \dot{s}_0(t) \;+\; v\big(s_0(t)+, t\big) \;=\; \lambda\big(s_0(t)-, t\big)\, \dot{s}_0(t) \;+\; v\big(s_0(t)-, t\big). \tag{6.4}$$

Thus the limiting values of the stretch $\lambda^\pm$ are related to the limiting values of the particle velocity $v^\pm$ by

$$\lambda^+ \dot{s}_0 + v^+ = \lambda^- \dot{s}_0 + v^-;$$

here $\dot{s}_0(t)$ is the (referential or Lagrangian) speed of the singular point. One is therefore led to the kinematic jump condition

$$[\![\ \lambda\ ]\!] \dot{s}_0 + [\![\ v\ ]\!] = 0 \tag{6.5}$$

which must hold at $x = s_0(t)$. This is the one-dimensional counterpart of the three-dimensional jump condition (6.30) to be established later in this chapter.

If one were to observe this body in a laboratory, it is the current configuration that one would observe, and so in particular, one would observe the singular point to be located at

$y = s(t) = y(s_0(t), t)$; see Figure 6.2. To calculate the velocity of the ("Eulerian") singular point we differentiate $s(t) = y(s_0(t), t)$ with respect to t to get

$$\begin{aligned} \dot{s}(t) &= \lambda(s_0(t)+, t)\dot{s}_0(t) + v(s_0(t)+, t) \;=\; \lambda(s_0(t)-, t)\dot{s}_0(t) + v(s_0(t)-, t) = \\ &= \lambda^+\dot{s}_0 + v^+ \;=\; \lambda^-\dot{s}_0 + v^-. \end{aligned} \tag{6.6}$$

The two representations of $\dot{s}(t)$ in (6.6) can be shown to be equivalent in view of (6.5).

We next turn to the balance laws. We can treat all of the balance laws simultaneously by writing them in the generic form

$$\int_{\mathcal{D}_0} \beta(\mathbf{x}, t)\, dV_x \;+\; \int_{\partial\mathcal{D}_0} \zeta(\mathbf{x}, t)\, dA_x = \frac{d}{dt}\int_{\mathcal{D}_0} \omega(\mathbf{x}, t)\, \mathrm{d}V_x. \tag{6.7}$$

This is the referential form of the generic spatial balance law (4.2).

In the present one dimensional setting the body occupies the region $\mathcal{R}_0 = \{x : 0 \le x \le L\}$ and an arbitrary part of the body occupies a region $\mathcal{D}_0 = \{x : x_1 < x < x_2\} \subset \mathcal{R}_0$; here x_1 and x_2 are arbitrary except for the requirement that $0 < x_1 < x_2 < L$; see Figure 6.2. The generic balance law in one-dimension takes the form

$$\int_{x_1}^{x_2} \beta(x, t)\, dx \;+\; \zeta(x_2, t) - \zeta(x_1, t) = \frac{d}{dt}\int_{x_1}^{x_2} \omega(x, t)\, dx \tag{6.8}$$

where we have used the fact that the boundary of $\mathcal{D}_0$ consists of the two points $x = x_1$ and $x = x_2$. Equation (6.8) must hold for all $\mathcal{D}_0$, i.e. for all x_1 and $x_2 (> x_1)$ in $[0, L]$.

First consider any interval $[x_1, x_2]$ which does *not* contain $x = s_0(t)$. Thus the fields are smooth on $[x_1, x_2]$ and so we can write (6.8) as

$$\int_{x_1}^{x_2} \beta\, dx \;+\; \int_{x_1}^{x_2} \frac{\partial\zeta}{\partial x} dx = \int_{x_1}^{x_2} \frac{\partial\omega}{\partial t}\, dx. \tag{6.9}$$

This must hold for all intervals $[x_1, x_2]$ on either side of $x = s_0$, and so by localization we conclude that

$$\beta \;+\; \frac{\partial\zeta}{\partial x} \;=\; \frac{\partial\omega}{\partial t}. \tag{6.10}$$

This field equation must hold at each $x \in [0, L]$ with the exception of $x = s_0(t)$.

Next consider an interval $[x_1, x_2]$ that contains the singular point $x = s_0(t)$ in its interior. It is natural in this case to consider the segment $[x_1, x_2]$ as the union of the two segments

$[x_1, s_0]$ and $[s_0, x_2]$. First note that

$$\begin{aligned}
\zeta(x_2,t)-\zeta(x_1,t) &= \zeta(x_2,t)-\zeta(s_0+,t)+\zeta(s_0-,t)-\zeta(x_1,t)+\zeta(s_0+,t)-\zeta(s_0-,t) = \\
&= \int_{s_0}^{x_2} \frac{\partial \zeta}{\partial x}\,dx + \int_{x_1}^{s_0} \frac{\partial \zeta}{\partial x}\,dx + [\![\ \zeta\ ]\!] = \\
&= \int_{x_1}^{x_2} \frac{\partial \zeta}{\partial x}\,dx + [\![\ \zeta\ ]\!],
\end{aligned} \tag{6.11}$$

where in the first step we have added and subtracted the quantities $\zeta(s_0\pm, t)$. Similarly we have

$$\begin{aligned}
\frac{d}{dt}\int_{x_1}^{x_2} \omega(x,t)\,dx &= \frac{d}{dt}\int_{x_1}^{s_0(t)} \omega(x,t)\,dx + \frac{d}{dt}\int_{s_0(t)}^{x_2} \omega(x,t)\,dx = \\
&= \int_{x_1}^{s_0} \frac{\partial \omega}{\partial t}\,dx + \omega(s_0(t)-,t)\dot{s}_0 \ + \int_{s_0}^{x_2} \frac{\partial \omega}{\partial t}\,dx - \omega(s_0(t)+,t)\dot{s}_0(t) = \\
&= \int_{x_1}^{x_2} \frac{\partial \omega}{\partial t}\,dx - [\![\omega]\!]\ \dot{s}_0.
\end{aligned} \tag{6.12}$$

Therefore the balance law (6.8) takes the form

$$\int_{x_1}^{x_2} \left(\beta \ + \ \frac{\partial \zeta}{\partial x} \ - \frac{\partial \omega}{\partial t}\right) dx \ + \ [\![\ \zeta\ ]\!] + [\![\omega]\!]\ \dot{s}_0 \ = \ 0. \tag{6.13}$$

In view of the field equation (6.10) the integrand vanishes almost everywhere and so we obtain the *jump condition*

$$[\![\ \zeta\ ]\!] + [\![\omega]\!]\ \dot{s}_0 = 0 \tag{6.14}$$

that must hold at $x = s_0(t)$.

We now apply this to the various balance laws of continuum mechanics. First consider linear momentum balance (cf. (4.67)). Here we take $\beta = \rho_0 b$ for the body force, $\zeta = \sigma$ for the stress, and $\omega = \rho_0 v$ for the linear momentum in (6.8); note that ρ_0 is the mass density in the reference configuration. The jump condition associated with linear momentum balance then follows from (6.14):

$$[\![\ \sigma\ ]\!] + [\![\rho_0 v]\!]\ \dot{s}_0 = 0. \tag{6.15}$$

Next consider the first law of thermodynamics. In equation (6.8) we now take $\beta = \rho_0 bv + \rho_0 r$ for the rate at which the body forces do work and the heat supply, $\zeta = \sigma v + q_0$ for the rate at which the stress does work and the heat flux, and $\omega = \rho_0\varepsilon + \rho_0 v^2/2$ for the internal energy density and the kinetic energy density; cf. (5.21). The jump condition associated with the first law of thermodynamics then follows from (6.14):

$$[\![\ \sigma v + q_0\ ]\!] + [\![\ \rho_0\varepsilon + \rho_0 v^2/2\ ]\!]\ \dot{s}_0 = 0. \tag{6.16}$$

Finally consider the second law of thermodynamics (cf. (5.22)). In this case the equality in (6.8) must be replaced by the inequality $\leq$; and we must take $\beta = \rho_0 r/\theta$ for the entropy source, $\zeta = q_0/\theta$ for the entropy flux, and $\omega = \rho_0\eta$ for the specfic entropy in (6.8). The jump inequality condition associated with the second law of thermodynamics then follows from (6.14) with the inequality replaced by $\leq$:

$$[\![\ q_0/\theta\ ]\!] + [\![\ \rho_0\eta\ ]\!]\ \dot{s}_0 \leq 0. \tag{6.17}$$

In summary, we have the following jump conditions in the one dimensional theory:

$$\left.\begin{aligned}
&[\![\ v\ ]\!] + [\![\ \lambda\ ]\!]\dot{s}_0 = 0,\\
&[\![\ \sigma\ ]\!] + [\![\rho_0 v]\!]\ \dot{s}_0 = 0,\\
&[\![\ \sigma v + q_0\ ]\!] + [\![\ \rho_0\varepsilon + \rho_0 v^2/2\ ]\!]\ \dot{s}_0 = 0,\\
&[\![\ q_0/\theta\ ]\!] + [\![\ \rho_0\eta\ ]\!]\ \dot{s}_0 \leq 0,
\end{aligned}\right. \tag{6.18}$$

which arise from the continuity of the motion, balance of linear momentum, and the first and second laws of thermodynamics respectively. They must hold at the singular point $x = s_0(t)$.

There is no one-dimensional counterpart of the angular momentum balance law. In three dimensions it turns out that the jump condition stemming from angular momentum balance is automatically implied by the other jump conditions, and it therefore does not impose any additional restrictions on a singular surface.

6.3 Worked Examples and Exercises.

Problem 6.1. *Generalized transport theorem.* For a smooth field $\beta(y,t)$, the one dimensional counterpart of the three dimensional transport equation $(3.89)_1$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{y_1}^{y_2} \beta\ \mathrm{d}y = \int_{y_1}^{y_2} \beta'\mathrm{d}y + \beta_2 v_2 - \beta_1 v_1 \tag{a}$$

where $y_\alpha = y(x_\alpha, t)$ are the current locations of two particles x_1 and x_2, $v(y,t)$ is the particle velocity field, $\beta' = \partial\beta(y,t)/\partial t$ and we have set $\beta_\alpha = \beta(y_\alpha, t)$ and $v_\alpha = v(y_\alpha, t)$, $\alpha = 1, 2$.

When the motion involves a singular point at $y = s(t) \in (y_1(t), y_2(t))$, show that (a) generalizes to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{y_1}^{y_2} \beta \,\mathrm{d}y = \int_{y_1}^{y_2} \beta' \mathrm{d}y + \beta_2 v_2 - \beta_1 v_1 - [\![\beta]\!]\dot{s}.$$

Solution: During the motion, the particles x_1 and x_2 are located at

$$y_1(t) = y(x_1, t), \qquad y_2(t) = y(x_2, t),$$

and have speeds

$$v_1 = \dot{y}_1(t), \qquad v_2 = \dot{y}_2(t).$$

Consider the part of the body that occupies $\mathcal{D}_0 = [x_1, x_2]$ in the reference configuration. At time t, this part occupies the segment $\mathcal{D}_t = [y_1(t), y_2(t)] = [y(x_1,t), y(x_2,t)]$; see Figure 6.2.

Recall the standard formula for differentiating an integral when the interval of integration is variable:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} f(y,t)\,\mathrm{d}y = \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(y,t)\,\mathrm{d}y + f(b(t),t)\,\dot{b}(t) - f(a(t),t)\,\dot{a}(t).$$

On using this formula we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{y_1}^{y_2} \beta\,\mathrm{d}y \quad &= \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{y_1}^{s} \beta\,\mathrm{d}y \;+\; \int_{s}^{y_2} \beta\,\mathrm{d}y\right) \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\int_{y_1(t)}^{s(t)} \beta\,\mathrm{d}y \;+\; \frac{\mathrm{d}}{\mathrm{d}t}\int_{s(t)}^{y_2(t)} \beta\,\mathrm{d}y \\
&= \int_{y_1}^{s} \beta'\,dy \;+\beta^-\dot{s} - \beta_1\,v_1 \;+\; \int_{s}^{y_2} \beta'\,\mathrm{d}y \;+\beta_2\,v_2 - \beta^+\dot{s} \\
&= \int_{y_1}^{y_2} \beta'\,\mathrm{d}y + \beta_2 v_2 - \beta_1 v_1 - [\![\, \beta \,]\!]\,\dot{s}
\end{aligned}$$

which is what we set out to prove.

Problem 6.2. *Jump conditions in Eulerian form.* In a one dimensional setting, the global balance laws for mass, linear momentum, energy and the entropy inequality in *Eulerian* form are

$$\left.\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{y_1}^{y_2} \rho\,\mathrm{d}y \quad &= \quad 0\,, \\
T(y_2,t) - T(y_1,t) + \int_{y_1}^{y_2} \rho b\,\mathrm{d}y \quad &= \quad \frac{\mathrm{d}}{\mathrm{d}t}\int_{y_1}^{y_2} \rho v\,\mathrm{d}y\,, \\
T(y_2,t)\,v(y_2,t) - T(y_1,t)\,v(y_1,t) + \int_{y_1}^{y_2} \rho b v\,\mathrm{d}y & \\
\quad &= \quad \frac{\mathrm{d}}{\mathrm{d}t}\int_{y_1}^{y_2} \rho\varepsilon\,\mathrm{d}y \;+\; \frac{\mathrm{d}}{\mathrm{d}t}\int_{y_1}^{y_2} \frac{1}{2}\rho v^2\,\mathrm{d}y\,, \\
\frac{q(y_2,t)}{\theta(y_2,t)} - \frac{q(y_1,t)}{\theta(y_1,t)} + \int_{y_1}^{y_2} \frac{\rho r}{\theta}\,\mathrm{d}y \quad &= \quad \frac{\mathrm{d}}{\mathrm{d}t}\int_{y_1}^{y_2} \rho\eta\,\mathrm{d}y\,.
\end{aligned}\right\}$$

Here we have denoted the particle velocity, mass density, specific body force, true stress, specific internal energy, specific entropy, specific heat supply, temperature and heat flux by $v(y,t), \rho(y,t)$, $b(y,t)$, $T(y,t)$, $\varepsilon(y,t)$, $\eta(y,t)$, $r(y,t)$, $\theta(y,t)$, $q(y,t)$ respectively.

Show from these that the respective *jump conditions in Eulerian form* associated with the balance laws for mass, linear momentum, energy and the entropy inequality are

$$\left.\begin{array}{lcl} [\![\rho(\dot{s}-v)]\!] & = & 0\ , \\ [\![T+\rho v(\dot{s}-v)]\!] & = & 0\ , \\ \left[\!\left[Tv+q+\rho(\dot{s}-v)\left(\varepsilon+\frac{1}{2}v^2\right)\right]\!\right] & = & 0\ , \\ [\![q/\theta+\rho(\dot{s}-v)\eta]\!] & \leq & 0, \end{array}\right\}$$

which are to hold at the singular point $y = s(t)$. Observe that the shock speed enters through the terms $\dot{s} - v^{\pm}$, and note that $\dot{s} - v^{\pm}$ is the speed of the singular point relative to the particle at $y = s\pm$.

6.4 Kinematic Jump Conditions in 3-D.

We now turn to the three dimensional Lagrangian framework for studying a singular surface $\mathcal{S}_0(t)$ as presented in Section 6.1 and illustrated in Figure 6.1(a). In particular, $V_0(\mathbf{x}, t)$ denotes the propagation speed of a point of this surface in the direction $\mathbf{n}_0$ normal to the surface.

The singular surface $\mathcal{S}_0(t)$ can be described parametrically by

$$\mathbf{x} = \tilde{\mathbf{x}}(\xi_1, \xi_2, t) \tag{6.19}$$

where the parameters (ξ_1, ξ_2) belong to some fixed domain Π. The pair (ξ_1, ξ_2) identifies a particular *point of the surface*, while $\mathbf{x}$ given by (6.19) is the spatial location of this point at time t. We recall that a given surface can be characterized by different parameterizations. Of particular interest are intrinsic features of the surface, i.e. features that do not depend on the particular parameterization.

For simplicity, we shall not keep repeating the phrase "at the point (ξ_1, ξ_2) of the surface at time t" in what follows though it applies to many sentences (starting with the next one).

One can choose the parameterization such that the vectors $\boldsymbol{\ell}_1$ and $\boldsymbol{\ell}_2$, defined by

$$\boldsymbol{\ell}_\alpha = \frac{\partial \tilde{\mathbf{x}}}{\partial \xi_\alpha}, \tag{6.20}$$

are (linearly independent) unit vectors that are tangent to $\mathcal{S}_0(t)$. The velocity of propagation of point (ξ_1, ξ_2) of the surface $\mathcal{S}_0(t)$ represented by $\partial\tilde{\mathbf{x}}/\partial t$ is dependent on the parameterization and is therefore not an intrinsic property of the surface. On the other hand, one can show that the velocity component normal to the surface,

$$V_0 = \frac{\partial \tilde{\mathbf{x}}}{\partial t} \cdot \mathbf{n}_0, \tag{6.21}$$

is independent of the parameterization and is therefore an intrinsic property of the surface. Note the distinction between the velocity of *a point of the surface* $\mathcal{S}_0$ and the velocity of *a particle of the body*, even if their locations happen to coincide at a given time.

Let $\mathcal{S}_t$ be the image of $\mathcal{S}_0(t)$ in the current configuration. Then $\mathcal{S}_t$ is described by

$$\mathbf{y} = \tilde{\mathbf{y}}(\xi_1, \xi_2, t) = \mathbf{y}(\tilde{\mathbf{x}}(\xi_1, \xi_2, t), t), \qquad (\xi_1, \xi_2) \in \Pi, \tag{6.22}$$

where $\mathbf{y}(\mathbf{x}, t)$ is the motion of the body. Since the function $\mathbf{y}$ is continuous across $\mathcal{S}_0$ by assumption, the function $\tilde{\mathbf{y}}(\xi_1, \xi_2, t)$ is independent of whether one calculates it by approaching the surface from the $+$ or $-$ side. In particular, this tells us that the left hand sides of (6.23) and (6.24) below are equal, as are the left hand sides of the two equations in (6.29).

First consider the limit from the plus side. Differentiating (6.22) with respect to ξ_α and using (6.20) yields

$$\frac{\partial \tilde{\mathbf{y}}}{\partial \xi_\alpha} = \mathbf{F}^+ \frac{\partial \tilde{\mathbf{x}}}{\partial \xi_\alpha} = \mathbf{F}^+ \boldsymbol{\ell}_\alpha. \tag{6.23}$$

If we do the same from the minus side, we get

$$\frac{\partial \tilde{\mathbf{y}}}{\partial \xi_\alpha} = \mathbf{F}^- \frac{\partial \tilde{\mathbf{x}}}{\partial \xi_\alpha} = \mathbf{F}^- \boldsymbol{\ell}_\alpha. \tag{6.24}$$

Subtracting the former from the latter gives

$$\mathbf{F}^+ \boldsymbol{\ell}_\alpha - \mathbf{F}^- \boldsymbol{\ell}_\alpha = [\![\mathbf{F}]\!] \boldsymbol{\ell}_\alpha = \mathbf{o}.$$

Since the pair of vectors $\boldsymbol{\ell}_1$ and $\boldsymbol{\ell}_2$ are linearly independent, it follows that the *jump condition*

$$[\![\mathbf{F}]\!] \boldsymbol{\ell} = \mathbf{o} \tag{6.25}$$

must hold for all unit vectors $\boldsymbol{\ell}$ that are tangent to $\mathcal{S}_0$.

Since $(\mathbf{F}^+ - \mathbf{F}^-)\boldsymbol{\ell} = \mathbf{0}$ for all unit vectors $\boldsymbol{\ell}$ that are tangent to $\mathcal{S}_0$, it follows that this set of tangent vectors is the null space of the tensor $\mathbf{F}^+ - \mathbf{F}^-$. Since the null space is 2-dimensional, while the vector space itself is 3-dimensional, it follows that the rank of the tensor $\mathbf{F}^+ - \mathbf{F}^-$ must be one $(= 3 - 2)$. Thus from Chapter 2 of Volume I it follows that there necessarily exist two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ in terms of which

$$[\![\mathbf{F}]\!] = \mathbf{F}^+ - \mathbf{F}^- = \mathbf{a} \otimes \mathbf{b}; \tag{6.26}$$

without loss of generality we can take $\mathbf{b}$ to be a unit vector. Since $(\mathbf{F}^+ - \mathbf{F}^-)\boldsymbol{\ell} = (\mathbf{a}\otimes\mathbf{b})\boldsymbol{\ell} = (\mathbf{b}\cdot\boldsymbol{\ell})\mathbf{a} = \mathbf{0}$ it follows that $\mathbf{b}\cdot\boldsymbol{\ell} = 0$ for all vectors $\boldsymbol{\ell}$ tangent to $\mathcal{S}_0$. Thus the unit vector $\mathbf{b}$ is normal to the plane $\mathcal{S}_0$ and so $\mathbf{b} = \mathbf{n}_0$. Observe from (6.26) that $\mathbf{a} = [\![\mathbf{F}]\!]\mathbf{b}$. Thus we conclude that (6.25) requires

$$[\![\mathbf{F}]\!] = \mathbf{F}^+ - \mathbf{F}^- = \mathbf{a}\otimes\mathbf{n}_0, \qquad \mathbf{a} = [\![\mathbf{F}]\!]\mathbf{n}_0. \tag{6.27}$$

Conversely, one can readily verify that if $\mathbf{F}^+$ and $\mathbf{F}^-$ satisfy (6.27) for any non-zero vector $\mathbf{a}$ with $\mathbf{n}_0$ being normal to the plane $\mathcal{S}_0$, then the jump condition (6.25) holds. An application of the jump condition (6.27)) is explored in Problems NNN and NNN.

The physical significance of $(6.27)_1$ can be seen by writing it in the illuminating form

$$\mathbf{F}^- = \Big(\mathbf{I} - \beta\,\mathbf{n}\otimes\mathbf{n}\Big)\Big(\mathbf{I} - \gamma\,\mathbf{m}\otimes\mathbf{n}\Big)\,\mathbf{F}^+. \tag{6.28}$$

In this representation $\beta = \mathbf{b}\cdot\mathbf{n}$, $\mathbf{m} = \mathbf{b}_*/|\mathbf{b}_*|$ and $\gamma = |\mathbf{b}_*|$ where $\mathbf{b} = |(\mathbf{F}^+)^{-\mathrm{T}}\mathbf{n}_0|\mathbf{a}$ and $\mathbf{b}_* = \mathbf{b} - (\mathbf{b}\cdot\mathbf{n})\mathbf{n}$. The unit vector $\mathbf{n}$ here is normal to the image $\mathcal{S}_t$ of the singular surface in the current configuration and is given by the familiar relation (6.33) below. One can readily verify that the unit vector $\mathbf{m}$ is perpendicular to $\mathbf{n}$. It follows from (6.28) that at each instant t, locally, near each point $\mathbf{x}\in\mathcal{S}_0$, the deformation on $\mathcal{R}_0^-$ differs from the deformation on $\mathcal{R}_0^+$ by a simple shear $\mathbf{I} - \gamma\,\mathbf{m}\otimes\mathbf{n}$ parallel to the plane $\mathcal{S}_t$ followed by a uniaxial stretch $\mathbf{I} - \beta\,\mathbf{n}\otimes\mathbf{n}$ in the direction normal to $\mathcal{S}_t$.

Next differentiate (6.22) with respect to t. Considering the limits from the plus and minus sides lead to the respective equations

$$\frac{\partial\tilde{\mathbf{y}}}{\partial t} = \mathbf{F}^+\frac{\partial\tilde{\mathbf{x}}}{\partial t} + \mathbf{v}^+, \qquad \frac{\partial\tilde{\mathbf{y}}}{\partial t} = \mathbf{F}^-\frac{\partial\tilde{\mathbf{x}}}{\partial t} + \mathbf{v}^-. \tag{6.29}$$

Subtracting the former from the latter yields

$$(\mathbf{F}^+ - \mathbf{F}^-)\frac{\partial\tilde{\mathbf{x}}}{\partial t} + (\mathbf{v}^+ - \mathbf{v}^-) = \mathbf{o}.$$

Note from (6.25) that when $\mathbf{F}^+ - \mathbf{F}^-$ operates on the tangential component of any vector, the result is the null vector. Therefore the action of $\mathbf{F}^+ - \mathbf{F}^-$ on any vector equals its action on the component of that vector in the normal direction. Thus we can replace $\partial\tilde{\mathbf{x}}/\partial t$ in the preceding equation by its normal component and so get

$$(\mathbf{F}^+ - \mathbf{F}^-)\left(\frac{\partial\tilde{\mathbf{x}}}{\partial t}\cdot\mathbf{n}_0\right)\mathbf{n}_0 + (\mathbf{v}^+ - \mathbf{v}^-) = \mathbf{o}.$$

Finally, using (6.21) in this leads to the *jump condition*

$$V_0[\![\mathbf{F}]\!]\mathbf{n}_0 + [\![\mathbf{v}]\!] = \mathbf{o}. \tag{6.30}$$

This jump condition relates the limiting values of the deformation gradient tensor and the particle velocity and involves the speed V_0 of the (Lagrangian) surface in the normal direction $\mathbf{n}_0$. It must hold at each $\mathbf{x} \in \mathcal{S}_0(t)$.

The image of $\mathcal{S}_0(t)$ in the current configuration is $\mathcal{S}_t$; see Figure 6.1(b). Its propagation speed

$$V = \frac{\partial \tilde{\mathbf{y}}}{\partial t} \cdot \mathbf{n} \tag{6.31}$$

in the normal direction $\mathbf{n}$ can be related to the speed V_0. On using (6.22) in (6.31) we get

$$V = \frac{\partial \tilde{\mathbf{y}}}{\partial t} \cdot \mathbf{n} = \left(\mathbf{F}^{\pm}\frac{\partial \tilde{\mathbf{x}}}{\partial t}\right) \cdot \mathbf{n} + \mathbf{v}^{\pm} \cdot \mathbf{n} \tag{6.32}$$

Recall the relation

$$\mathbf{n} = \frac{\mathbf{F}_{\pm}^{-T}\mathbf{n}_0}{|\mathbf{F}_{\pm}^{-T}\mathbf{n}_0|} \tag{6.33}$$

between the unit normals $\mathbf{n}$ and $\mathbf{n}_0$; see (2.36). Using this in the preceding equation and simplifying the result leads to

$$V = \frac{V_0}{|\mathbf{F}_{+}^{-T}\mathbf{n}_0|} + \mathbf{v}^{+} \cdot \mathbf{n} = \frac{V_0}{|\mathbf{F}_{-}^{-T}\mathbf{n}_0|} + \mathbf{v}^{-} \cdot \mathbf{n}. \tag{6.34}$$

The two representations of V here are equivalent by (6.30).

6.5 Momentum, Energy and Entropy Jump Conditions in 3-D.

6.5.1 Linear Momentum Balance Jump Condition.

We now derive the jump condition associated with linear momentum balance (in referential form), starting from the global balance law (4.67):

$$\int_{\partial\mathcal{D}_0} \mathbf{S}\mathbf{n}_0 \, dA_x \; + \int_{\mathcal{D}_0} \rho_0 \mathbf{b} \, dV_x \; = \frac{d}{dt}\int_{\mathcal{D}_0} \rho_0 \mathbf{v} \, dV_x \tag{6.35}$$

that holds for all subregions $\mathcal{D}_0 \subset \mathcal{R}_0$. By limiting attention to all subregions that do *not* intersect the singular surface $\mathcal{S}_0(t)$ we derive, in the usual way, the field equation,

$$\text{Div } \mathbf{S} + \rho_0 \mathbf{b} = \rho_0 \dot{\mathbf{v}}; \tag{6.36}$$

this must hold at all $\mathbf{x}$ in $\mathcal{R}_0$ that do *not* lie on $\mathcal{S}_0$.

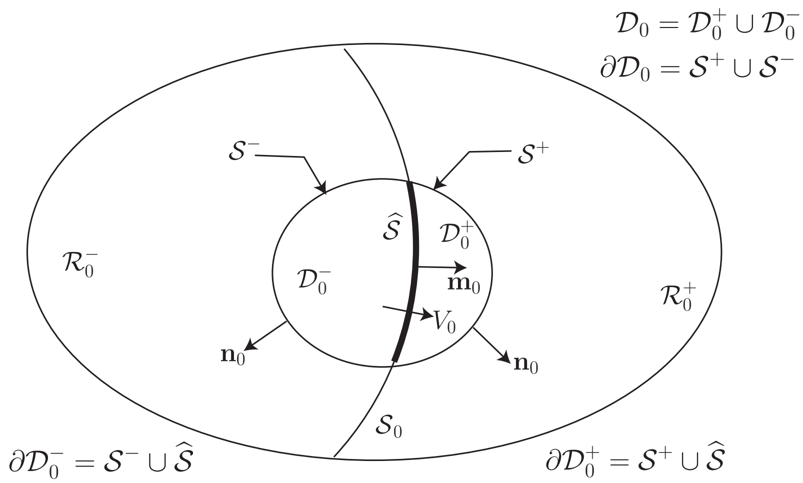

Figure 6.3: An arbitrary subregion $\mathcal{D}_0$ that intersects the singular surface $\mathcal{S}_0(t)$: $\mathcal{D}_0 = \mathcal{D}_0^+(t) \cup \mathcal{D}_0^-(t)$. The segment of $\mathcal{S}_0$ that lies within $\mathcal{D}_0$ is denoted by $\widehat{\mathcal{S}}(t)$. Note that $\partial\mathcal{D}_0 = \mathcal{S}^+(t) \cup \mathcal{S}^-(t)$, $\partial\mathcal{D}_0^+(t) = \mathcal{S}^+(t) \cup \widehat{\mathcal{S}}(t)$ and $\partial\mathcal{D}_0^-(t) = \mathcal{S}^-(t) \cup \widehat{\mathcal{S}}(t)$.

Now consider an arbitrary subregion $\mathcal{D}_0$ that *does intersect* $\mathcal{S}_0(t)$ as shown in Figure 6.3: $\mathcal{S}_0(t)$ separates $\mathcal{D}_0$ into two regions $\mathcal{D}_0^+(t)$ and $\mathcal{D}_0^-(t)$, the unit vector $\mathbf{m}_0$ is normal to the singular surface $\mathcal{S}_0(t)$ and points into $\mathcal{D}_0^+(t)$, and the unit vector $\mathbf{n}_0$ is normal to the boundary $\partial\mathcal{D}_0(t)$ of $\mathcal{D}_0$ and points outwards from $\mathcal{D}_0(t)$. Futhermore we denote the segment of $\mathcal{S}_0$ that lies within $\mathcal{D}_0$ by $\widehat{\mathcal{S}}(t)$, and the surfaces $\mathcal{S}^+(t)$ and $\mathcal{S}^-(t)$ are as shown in Figure 6.3 such that the boundary of $\mathcal{D}_0$ is $\partial\mathcal{D}_0(t) = \mathcal{S}^+(t) \cup \mathcal{S}^-(t)$, the boundary of $\mathcal{D}_0^+$ is $\partial\mathcal{D}_0^+(t) = \mathcal{S}^+(t) \cup \widehat{\mathcal{S}}(t)$ and the boundary of $\mathcal{D}_0^-$ is $\partial\mathcal{D}_0^-(t) = \mathcal{S}^-(t) \cup \widehat{\mathcal{S}}(t)$. A point on $\mathcal{S}_0(t)$ propagates with speed V_0 in the direction $\mathbf{m}_0$.

We first simplify the traction term in (6.35) as follows by considering $\mathcal{D}_0$ to be the union of $\mathcal{D}_0^+$ and $\mathcal{D}_0^-$:

$$
\begin{aligned}
\int_{\partial\mathcal{D}_0} \mathbf{S}\mathbf{n}_0 \, dA_x &= \int_{\mathcal{S}^+} \mathbf{S}\mathbf{n}_0 \, dA_x + \int_{\mathcal{S}^-} \mathbf{S}\mathbf{n}_0 \, dA_x \\
&= \int_{\mathcal{S}^+} \mathbf{S}\mathbf{n}_0 \, dA_x + \int_{\widehat{\mathcal{S}}} \mathbf{S}^+(-\mathbf{m}_0) \, dA_x + \int_{\widehat{\mathcal{S}}} \mathbf{S}^+\mathbf{m}_0 \, dA_x + \\
&+ \int_{\mathcal{S}^-} \mathbf{S}\mathbf{n}_0 \, dA_x + \int_{\widehat{\mathcal{S}}} \mathbf{S}^-\mathbf{m}_0 \, dA_x - \int_{\widehat{\mathcal{S}}} \mathbf{S}^-\mathbf{m}_0 \, dA_x \\
&= \int_{\partial\mathcal{D}_0^+} \mathbf{S}\mathbf{n}_0 \, dA_x + \int_{\partial\mathcal{D}_0^-} \mathbf{S}\mathbf{n}_0 \, dA_x + \int_{\widehat{\mathcal{S}}} \Big(\mathbf{S}^+ - \mathbf{S}^-\Big)\mathbf{m}_0 \, dA_x \\
&= \int_{\mathcal{D}_0^+} \operatorname{Div}\mathbf{S} \, dV_x + \int_{\mathcal{D}_0^-} \operatorname{Div}\mathbf{S} \, dV_x + \int_{\widehat{\mathcal{S}}} [\![\mathbf{S}]\!]\mathbf{m}_0 \, dA_x \\
&= \int_{\mathcal{D}_0} \operatorname{Div}\mathbf{S} \, dV_x + \int_{\widehat{\mathcal{S}}} [\![\mathbf{S}]\!]\mathbf{m}_0 \, dA_x,
\end{aligned}
\tag{6.37}
$$

where $\mathbf{S}^{\pm}$ are the limiting values of the first Piola-Kirchhoff stress tensor at a point on $\mathcal{S}_0$ as it is approached from the plus and minus sides of the surface. In the second step we have added and subtracted the same quantity; in the third step we have observed that $\partial\mathcal{D}_0^+(t) = \mathcal{S}^+(t) \cup \widehat{\mathcal{S}}(t)$ and $\partial\mathcal{D}_0^-(t) = \mathcal{S}^-(t) \cup \widehat{\mathcal{S}}(t)$, and used the symbol $\mathbf{n}_0$ to denote the unit outward normal to the regions $\partial\mathcal{D}_0^{\pm}$; and we have used the divergence theorem in the third step.

Next we simplify the rate of change of linear momentum term in (6.35). On using the analog of the transport equation (3.89) in the present setting we have

$$
\frac{d}{dt}\int_{\mathcal{D}_0^+} \rho_0 \mathbf{v} \, dV_x = \int_{\mathcal{D}_0^+} \rho_0 \dot{\mathbf{v}} \, dV_x + \int_{\widehat{\mathcal{S}}} \rho_0 \mathbf{v}^+(-V_0) \, dA_x,
$$

$$
\frac{d}{dt}\int_{\mathcal{D}_0^-} \rho_0 \mathbf{v} \, dV_x = \int_{\mathcal{D}_0^-} \rho_0 \dot{\mathbf{v}} \, dV_x + \int_{\widehat{\mathcal{S}}} \rho_0 \mathbf{v}^-(V_0) \, dA_x,
$$

which when added together yields

$$
\frac{d}{dt}\int_{\mathcal{D}_0} \rho_0 \mathbf{v} \, dV_x = \int_{\mathcal{D}_0} \rho_0 \dot{\mathbf{v}} \, dV_x - \int_{\widehat{\mathcal{S}}} \rho_0 [\![\mathbf{v}]\!] V_0 \, dA_x. \tag{6.38}
$$

Substituting (6.37) and (6.38) into (6.35) leads to

$$
\int_{\mathcal{D}_0} (\operatorname{Div}\,\mathbf{S} + \rho_0\mathbf{b} - \rho_0\dot{\mathbf{v}}) \; dV_x + \int_{\widehat{\mathcal{S}}} ([\![\mathbf{S}]\!]\mathbf{m}_o + \rho_o[\![\mathbf{v}]\!]V_0) \; dA_x = 0. \tag{6.39}
$$

The first term here vanishes in view of the field equation (6.36) and so we are led to

$$
\int_{\widehat{\mathcal{S}}} ([\![\mathbf{S}]\!]\mathbf{m}_o + \rho_o[\![\mathbf{v}]\!]V_0) \; dA_x = 0. \tag{6.40}
$$

Since this must hold for all $\mathcal{D}_0$, and therefore all $\widehat{\mathcal{S}}\,(\subset \mathcal{S}_0)$, we conclude by localization that

$$[\![\mathbf{S}]\!]\mathbf{m}_o + \rho_o[\![\mathbf{v}]\!]V_0 = \mathbf{0}. \tag{6.41}$$

This is the jump condition associated with linear momentum balance. It must hold at all $\mathbf{x} \in \mathcal{S}_0$.

Remark: It is useful to consider the origin of each term of the jump condition (6.41) in light of each term of the global balance law (6.35). Keeping Figure 6.3 in mind, consider a limiting process in which $\mathcal{S}^+$ is collapsed onto $\widehat{\mathcal{S}}$. The traction on $\mathcal{S}^+$ then reduces to the traction $\mathbf{S}^+\mathbf{m}_0$ on $\widehat{\mathcal{S}}$ since $\mathbf{m}_0 = \mathbf{n}_0$. On the other hand a limiting process in which $\mathcal{S}^-$ is collapsed onto $\widehat{\mathcal{S}}$ results in the traction on $\mathcal{S}^-$ reducing to the traction $\mathbf{S}^-(-\mathbf{m}_0)$ on $\widehat{\mathcal{S}}$ since $\mathbf{m}_0 = -\mathbf{n}_0$ in this case. This results in the traction term $\mathbf{S}^+\mathbf{m}_0 - \mathbf{S}^-\mathbf{m}_0 = [\![\mathbf{S}]\!]\mathbf{m}_0$ on $\widehat{\mathcal{S}}$ which led to the first term in (6.41). The body force term in (6.35) disappears in the aforementioned limit. The rate of change of linear momentum of $\mathcal{D}_0$ consists of two parts: the change of linear momentum in the bulk of $\mathcal{D}_0$ which also disappears in this limit, and a second contribution due to the motion of the singular surface. We see from Figure 6.3 that as the singular surface advances with speed V_0, particles with momentum density $\rho_0\mathbf{v}^+$ are "replaced" by particles with momentum density $\rho_0\mathbf{v}^-$. This leads to the rate of change of linear momentum term $(-\rho_0\mathbf{v}^+ + \rho\mathbf{v}^-)V_0 = -[\![\rho_0\mathbf{v}]\!]V_0$ over the surface $\widehat{\mathcal{S}}$, which led to the second term in (6.41).

This remark tells us how, in many cases, we can write down the jump condition associated with a balance law directly by inspection. See Problem 6.5.

6.5.2 Summary: Jump Conditions in Lagrangian Formulation.

The jump conditions associated with angular momentum balance and the first and second laws of thermodynamics can be derived as above. This leads to the following complete set of jump conditions which are to hold at all points $\mathbf{x} \in \mathcal{S}_0(t)$:

$$\left.\begin{aligned}
&[\![\mathbf{v}]\!] \;+\; [\![\mathbf{F}\mathbf{n}_0]\!]\,V_0 = \mathbf{0}, \qquad [\![\,\mathbf{F}]\!] = \mathbf{a}\otimes\mathbf{n}_0 \;\text{ where } \mathbf{a} = [\![\mathbf{F}]\!]\mathbf{n}_0,\\
&[\![\,\mathbf{S}\mathbf{n}_0\,]\!] \;+\; [\![\,\rho_0\mathbf{v}\,]\!]V_0 = \mathbf{0},\\
&[\![\mathbf{S}\mathbf{n}_0\cdot\mathbf{v}]\!] + [\![\rho_0\,(\varepsilon + \mathbf{v}\cdot\mathbf{v}/2)]\!]\;V_0 + [\![\mathbf{q}_0\cdot\mathbf{n}_0]\!] = 0,\\
&[\![\rho_0\eta]\!]\;V_0 + [\![\mathbf{q}_0\cdot\mathbf{n}_0/\theta]\!] \le 0.
\end{aligned}\right\} \tag{6.42}$$

The third line results from the first law of thermodynamics and the inequality in the final line is associated with the second law of thermodynamics. The jump condition stemming from angular momentum balance can be shown to be implied by the other jump conditions and so does not impose an additional condition at the singular surface.

Remark: In the special case of *mechanical equilibrium*, the particle velocity $\mathbf{v}$ and the propagation speed V_0 of the surface vanish at all times. In this case the general jump conditions (6.42) reduce to

$$[\![\, \mathbf{F}\boldsymbol{\ell}]\!] = \mathbf{0}, \qquad [\![\mathbf{S}\mathbf{n}_0]\!] = 0 \ , \qquad [\![\mathbf{q}_0 \cdot \mathbf{n}_0]\!] = 0 \ . \tag{6.43}$$

Thus momentum balance requires the traction to be continuous while the first law of thermodynamics requires the heat flux to be continuous. The jump condition associated with the second law of thermodynamics holds automatically in view of $(6.43)_3$, assuming the temperature to be continuous.

Remark: A different special case corresponds to the setting where the singular surface is a *material surface*, i.e. it is attached to the same set of material particles (even though the body might be undergoing a dynamic process) e.g. the interface between two perfectly bonded materials in a composite material. In this case the (Lagrangian) speed of the surface $V_0 = 0$ and the general jump conditions (6.42) specialize to

$$[\![\, \mathbf{F}\boldsymbol{\ell}]\!] = \mathbf{0}, \qquad [\![\mathbf{v}]\!] \ = \mathbf{0}, \quad [\![\, \mathbf{S}\mathbf{n}_0 \,]\!] = \mathbf{0}, \quad [\![\mathbf{S}\mathbf{n}_0 \cdot \mathbf{v}]\!] + +[\![\mathbf{q}_0 \cdot \mathbf{n}_0]\!] = 0, \quad [\![\mathbf{q}_0 \cdot \mathbf{n}_0]\!] \leq 0. \tag{6.44}$$

6.5.3 Jump Conditions in Eulerian Formulation.

Consider the Eulerian framework for studying a singular surface; see Figure 6.1(b). Let $\mathcal{S}_t$ denote the image of the Lagrangian singular surface $\mathcal{S}_0(t)$ in the current configuration. The unit vector $\mathbf{n}(\mathbf{y}, t)$ is normal to $\mathcal{S}_t$, and the surface propagates at a speed $V(\mathbf{y}, t)$ in the direction $\mathbf{n}$. The speeds V and V_0 are related by (6.34) while the unit normals $\mathbf{n}$ and $\mathbf{n}_0$ are related by (6.33).

One can show that mass balance, linear momentum balance, the first law of thermodynamics and the second law of thermodynamics require that at each instant t the following jump conditions hold at all points $\mathbf{y} \in \mathcal{S}_t$:

$$\left.\begin{aligned} [\![\rho(V - \mathbf{v}\cdot\mathbf{n})]\!] &= 0 \ , \\ [\![\rho\mathbf{v}(V - \mathbf{v}\cdot\mathbf{n}) + \mathbf{T}\mathbf{n}]\!] &= 0 \ , \\ [\![\mathbf{T}\mathbf{n}\cdot\mathbf{v} + \mathbf{q}\cdot\mathbf{n} + \rho(\varepsilon + \mathbf{v}\cdot\mathbf{v}/2)(V - \mathbf{v}\cdot\mathbf{n})]\!] &= 0 \ , \\ [\![\mathbf{q}\cdot\mathbf{n}/\theta + \rho\eta(V - \mathbf{v}\cdot\mathbf{n})]\!] &\leq 0. \end{aligned}\right\} \tag{6.45}$$

Here $\mathbf{T}$ and $\mathbf{q}$ are the true stress and true heat flux vector. Note that ρ here is the mass density in the current configuration and, in general, is discontinuous at the singular surface. Observe that it is the velocity of the moving surface relative to the underlying particle velocity, $V - \mathbf{v}^{\pm}\cdot\mathbf{n}$, that enters all of the above expressions. These jump conditions are established in Problem 6.8.

6.6 Worked Examples and Exercises.

Problem 6.3. Consider the motion of a singular surface characterized by $\mathbf{x} = \tilde{\mathbf{x}}(\xi_1, \xi_2, t)$ as in Section 6.4. Does the velocity vector of the moving surface $\mathbf{V}_0 = \partial\tilde{\mathbf{x}}/\partial t$ depend on the parameterization? Show that the normal component of velocity $\mathbf{V}_0\cdot\mathbf{n}_0$ is an intrinsic property of the surface in that it does *not* depend on the parameterization. Note that it is only $\mathbf{V}\cdot\mathbf{n}_0$ that appeared in the jump conditions (6.42).

Problem 6.4. *Motivation:* Consider a *smooth* motion $\mathbf{y} = \mathbf{y}(\mathbf{x}, t)$ of a body that occupies a region $\mathcal{R}_0$ in a reference configuration. The particle velocity is given by

$$\mathbf{v} = \frac{\partial\mathbf{y}}{\partial t}.$$

Taking the referential gradient of this equation, changing the order of differentiation on the right hand side, and using $\mathbf{F} = \operatorname{Grad}\mathbf{y}$ leads to

$$\frac{\partial\mathbf{v}}{\partial\mathbf{x}} = \frac{\partial\mathbf{F}}{\partial t}. \tag{a}$$

Since (a) holds at each particle $\mathbf{x} \in \mathcal{R}_0$ we can integrate this equation over an arbitrary subregion $\mathcal{D}_0$ of $\mathcal{R}_0$ to get

$$\int_{\mathcal{D}_0} \frac{\partial\mathbf{v}}{\partial\mathbf{x}}\, dV_x \ = \int_{\mathcal{D}_0} \frac{\partial\mathbf{F}}{\partial t}\, dV_x.$$

Since $\mathcal{D}_0$ does not depend on time we can take the time derivative outside the integral in the right hand term. On the left hand side we can use the divergence theorem[2] to convert the volume integral over $\mathcal{D}_0$ into

[2]See Section 5.2 of Volume 1.

a surface integral over its boundary $\partial\mathcal{D}_0$ leading to

$$\int_{\partial\mathcal{D}_0} \mathbf{v}\otimes\mathbf{n}\,dA_x \;=\frac{d}{dt}\int_{\mathcal{D}_0}\mathbf{F}\,dV_x. \tag{b}$$

This must hold for all subregions $\mathcal{D}_0 \subset \mathcal{R}_0$. Conversely, if we require (b) to hold for all smooth motions and all subregions $\mathcal{D}_0$, one can readily reverse the preceding steps and derive (a).

Since (b) is of integral form, each term is mathematically meaningful for classes of motions that are *not as smooth as assumed above*. This motivates the following question:

Problem Statement: Suppose that the conservation law (b) is required to hold for motions of the class considered in this chapter. Derive the field equation and jump condition associated with it.

Solution: First consider subregions $\mathcal{D}_0$ that do not intersect the singular surface $\mathcal{S}_0$. One can then reverse the steps that led from (a) to (b) to conclude that

$$\frac{\partial\mathbf{v}}{\partial\mathbf{x}}=\frac{\partial\mathbf{F}}{\partial t}\qquad\text{at each } \mathbf{x}\in\mathcal{R}_0,\ \mathbf{x}\notin\mathcal{S}_0(t). \tag{c}$$

Next consider a region $\mathcal{D}_0$ that intersects $\mathcal{S}_0(t)$. The jump condition associated with the conservation law (b) can be written down by inspection, for example by using the approach outlined in the Remark at the end of Section 6.5.1. The jump condition associated with (b) is

$$(\mathbf{v}^+-\mathbf{v}^-)\otimes\mathbf{n}_0\;-\;(\mathbf{F}^- -\mathbf{F}^+)V_0=\mathbf{O}\qquad\text{at each } \mathbf{x}\in\mathcal{S}_0. \tag{d}$$

The unit vector $\mathbf{n}_0$ is normal to the singular surface $\mathcal{S}_0$ and points into the "positive" side.

Thus in summary the conservation law (b) leads to the field equation (c) and the jump condition (d).

To examine (d) more closely, operate both sides of (d) on the unit normal $\mathbf{n}_0$. This gives

$$(\mathbf{v}^+-\mathbf{v}^-)\;-\;(\mathbf{F}^- -\mathbf{F}^+)\mathbf{n}_0V_0=\mathbf{o}\qquad\text{for all } \mathbf{x}\in\mathcal{S}_0. \tag{e}$$

Similarly operating both sides of (d) on any unit vector $\boldsymbol{\ell}$ that is tangent to $\mathcal{S}_0$ leads to

$$V_0(\mathbf{F}^- -\mathbf{F}^+)\boldsymbol{\ell}=\mathbf{o}\qquad\text{at each } \mathbf{x}\in\mathcal{S}_0. \tag{f}$$

The pair of (vector) jump conditions (e) and (f) are equivalent to the (tensor) jump condition (d).

Observe that when $V_0\neq 0$, the two jump conditions (e) and (f) coincide with the kinematic jump conditions $(6.42)_1$. However they differ when $V_0=0$.

Remark: Consider the time-integral version of (b) obtained by integrating it with respect to time from some initial instant t_0 to time t. Derive the jump conditions associated with this time integral statement. Is the case $V_0=0$ exceptional?

Problem 6.5. Consider the generic balance law (in Lagrangian form)

$$\int_{\partial\mathcal{D}_0}\zeta(\mathbf{x},t,\mathbf{n}_0)\,dA_x\;+\int_{\mathcal{D}_0}\beta(\mathbf{x},t)\,dV_x\;=\;\frac{d}{dt}\int_{\mathcal{D}_0}\omega(\mathbf{x},t)\,\mathrm{d}V_x \tag{6.46}$$

where $\mathbf{n}_0$ is the unit outward normal vector to $\partial\mathcal{D}_0$. Here $\zeta(\mathbf{x}, t, \mathbf{n}_0)$, $\beta(\mathbf{x}, t)$ and $\omega(\mathbf{x}, t)$ are generic fields. The dependence of $\zeta(\mathbf{x}, t, \mathbf{n}_0)$ on the unit vector $\mathbf{n}_0$ is assumed to be linear so that, in particular,

$$\zeta(\mathbf{x}, t, -\mathbf{n}_0) = -\zeta(\mathbf{x}, t, \mathbf{n}_0). \tag{a}$$

Show that the jump condition associated with the balance law (6.46) is

$$\zeta^+(\mathbf{x}, t, \mathbf{m}_0) - \zeta^-(\mathbf{x}, t, \mathbf{m}_0) + \Big(\omega^+(\mathbf{x}, t) - \omega^-(\mathbf{x}, t)\Big) V_0 = 0. \tag{6.47}$$

The unit vector $\mathbf{m}_0$ here is normal to the singular surface and points into the "plus" side.

Solution We shall follow the approach outlined in the Remark at the end of Section 6.5.1. In all limiting processes below, it is assumed that the various fields are such that the limits exist as needed.

Keep Figure 6.3 in mind and consider the first term of (6.46). Note first that $\partial\mathcal{D}_0 = \mathcal{S}^+ \cup \mathcal{S}^-$. Consider a limiting process in which $\mathcal{S}^+$ and $\mathcal{S}^-$ collapse onto $\widehat{\mathcal{S}}$ from the "plus" and "minus" sides respectively. In this limit the integral of ζ on $\mathcal{S}^+$ reduces to the integral of $\zeta^+(\mathbf{x}, t, \mathbf{m}_0)$ on $\widehat{\mathcal{S}}$ since, in the limit, $\mathbf{n}_0 = \mathbf{m}_0$; see Figure 6.3. Similarly the integral of ζ on $\mathcal{S}^-$ reduces to the integral of $\zeta^-(\mathbf{x}, t, -\mathbf{m}_0)$ on $\widehat{\mathcal{S}}$ since $\mathbf{n}_0 = -\mathbf{m}_0$ in this case. Thus the integral of ζ over $\partial\mathcal{D}_0$ reduces to the integral of $\zeta^+(\mathbf{x}, t, \mathbf{m}_0) - \zeta^-(\mathbf{x}, t, \mathbf{m}_0)$ on $\widehat{\mathcal{S}}$ where we have used the linearity condition (a) to take the negative sign out of the argument of ζ^-:

$$\int_{\partial\mathcal{D}_0} \zeta(\mathbf{x}, t, \mathbf{n}_0)\, dA_x \;\to\; \int_{\widehat{\mathcal{S}}} \Big(\zeta^+(\mathbf{x}, t, \mathbf{m}_0) \;-\; \zeta^-(\mathbf{x}, t, \mathbf{m}_0)\Big)\, dA_x.$$

The volume integral of β in (6.46) disappears in the aforementioned limit since the volume of $\mathcal{D}_0$ tends to zero:

$$\int_{\mathcal{D}_0} \beta(\mathbf{x}, t)\, dV_x \;\to\; 0.$$

Next consider the right hand side of (6.46). The rate of change of the integral of ω on $\mathcal{D}_0$ consists of two parts: the change in the bulk, i.e. the integral of $\partial\omega/\partial t$ over $\mathcal{D}_0$, which also disappears in this limit since the volume of $\mathcal{D}_0 \to 0$, and a second contribution due to the motion of the singular surface. We see from Figure 6.3 that as the singular surface advances with speed V_0, particles on the "plus" side of the singular surface with $\omega = \omega^+$ are "replaced" by particles on the "minus" side with $\omega = \omega^-$. Thus in this limit, the rate of change of the integral of ω on $\mathcal{D}_0$ is given by the surface integral of $-\omega^+ V_0 + \omega^- V_0$ over the surface $\widehat{\mathcal{S}}$:

$$\frac{d}{dt}\int_{\mathcal{D}_0} \omega(\mathbf{x}, t)\, \mathrm{d}V_x \;\to\; \int_{\widehat{\mathcal{S}}} \Big(-\,\omega^+(\mathbf{x}, t)V_0 + \omega^-(\mathbf{x}, t)V_0\Big) dA_x.$$

Collecting the preceding results shows that in this limit, equation (6.46) leads to

$$\int_{\widehat{\mathcal{S}}} \Big(\zeta^+(\mathbf{x}, t, \mathbf{m}_0) - \zeta^-(\mathbf{x}, t, \mathbf{m}_0) + \omega^+(\mathbf{x}, t)V_0 - \omega^-(\mathbf{x}, t)V_0\Big)\, dA_x = 0.$$

Since this must hold for all $\widehat{\mathcal{S}}$ it follows by localization that the jump condition (6.47) must hold.

Problem 6.6. Derive the jump condition associated with balance of *angular momentum* (in the Lagrangian formulation). Show that the resulting jump condition is implied by the jump conditions (6.42) and is therefore not an additional restriction on the limiting values of the fields.

Solution: Consider a body occupying a region $\mathcal{R}_0$ in a reference configuration and undergoing a motion $\mathbf{y} = \widehat{\mathbf{y}}(\mathbf{x}, t)$. The balance of angular momentum requires that

$$\int_{\partial \mathcal{D}_0} \widehat{\mathbf{y}} \times \mathbf{S}\mathbf{n}_0 \, dA_x + \int_{\mathcal{D}_0} \widehat{\mathbf{y}} \times \rho_0 \mathbf{b} \, dV_x \;=\; \frac{d}{dt} \int_{\mathcal{D}_0} \widehat{\mathbf{y}} \times \rho_0 \mathbf{v} \, dV_x, \tag{a}$$

at each instant t and for every subregion $\mathcal{D}_0 \subset \mathcal{R}_0$. Suppose that a singular surface $\mathcal{S}_0(t)$ intersects the region $\mathcal{D}_0$ as in Figure 6.3. One can write down the jump condition by specializing the generic jump condition given in Problem 6.5. Accordingly, the jump condition associated with (a) is

$$\widehat{\mathbf{y}} \times (\mathbf{S}^+\mathbf{n}_0 - \mathbf{S}^-\mathbf{n}_0) = \widehat{\mathbf{y}} \times (-\rho_0 \mathbf{v}^+ + \rho_0 \mathbf{v}^-) V_0. \tag{b}$$

However by $(6.42)_2$, linear momentum balance ensures that $[\![\mathbf{S}]\!]\mathbf{n}_0 + [\![\rho_0 \mathbf{v}]\!] V_0 = \mathbf{o}$. In light of this, the jump condition (b) is automatic in that it is implied by linear momentum balance.

Problem 6.7. Generalize the *transport equations* (3.89) to a setting where the region occupied by the body at time t involves a singular surface.

Solution: Suppose first that the motion is smooth and consider an arbitrary subregion $\mathcal{D}_t \subset \mathcal{R}_t$. The unit vector $\mathbf{n}$ is normal to $\mathcal{D}_t$ and points in the outward direction. From (3.89), for a smooth scalar field $\beta(\mathbf{y}, t)$ and a smooth vector field $\mathbf{b}(\mathbf{y}, t)$ we have the transport equations

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{D_t} \beta \, \mathrm{d}V_y = \int_{D_t} \beta' \, \mathrm{d}V_y + \int_{\partial D_t} \beta \, (\mathbf{v} \cdot \mathbf{n}) \, \mathrm{d}A_y, \tag{a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{D_t} \mathbf{b} \, \mathrm{d}V_y = \int_{D_t} \mathbf{b}' \, \mathrm{d}V_y + \int_{\partial D_t} \mathbf{b} \, (\mathbf{v} \cdot \mathbf{n}) \, \mathrm{d}A_y, \tag{b}$$

where as usual, $\beta' = \partial \beta(\mathbf{y}, t)/\partial t$ and $\mathbf{b}' = \partial \mathbf{b}(\mathbf{y}, t)/\partial t$. Note that the scalar $\mathbf{v} \cdot \mathbf{n}$ is the propagation speed of the closed surface $\partial \mathcal{D}_t$ in the outward normal direction.

The key observation for our purposes is that (a) and (b) *hold for nonmaterial regions* $\mathcal{D}_t$ *as well* with $\mathbf{v} \cdot \mathbf{n}$ replaced by the propagation speed of the boundary in the outward normal direction[3].

Now consider a motion that involves a singular surface $\mathcal{S}_t$. Let $\mathcal{D}_t$ be an arbitrary subregion that intersects $\mathcal{S}_t$ and suppose that $\mathcal{S}_t$ separates $\mathcal{D}_t$ into two regions $\mathcal{D}_t^+$ and $\mathcal{D}_t^-$. The surfaces $\mathcal{S}_t^+$, $\mathcal{S}_t^-$ are as shown in Figure 6.4, and $\widehat{\mathcal{S}}_t = \mathcal{S}_t \cap \mathcal{D}_t$ is the portion of the singular surface that lies within $\mathcal{D}_t$.

We first apply (a) to $\mathcal{D}_t^-$. Note that its boundary $\partial \mathcal{D}_t^-$ is comprised of the union of the surfaces $\mathcal{S}_t^-$ and $\widehat{\mathcal{S}}_t$ and that the outward normal speed with which these surfaces propagate are $\mathbf{v} \cdot \mathbf{n}$ and V respectively. Applying (a) to the region $\mathcal{D}_t^-$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{D_t^-} \beta \, \mathrm{d}V_y = \int_{D_t^-} \beta' \, \mathrm{d}V_y + \int_{S_t^-} \beta \, \mathbf{v} \cdot \mathbf{n} + \int_{\widehat{S}_t} \beta^- \, V \, \mathrm{d}A_y,$$

where β^- is the limiting value of β at a point on $\mathcal{S}_t$ is approached from the minus side. Next we apply (a) to $\mathcal{D}_t^+$. Its boundary of $\partial \mathcal{D}_t^+$ is comprised of the union of the surfaces $\mathcal{S}_t^+$ and $\widehat{\mathcal{S}}_t$ and the outward normal

[3]See for example *Vectorial Mechanics*, by E.A. Milne, Interscience, NY, 1948.

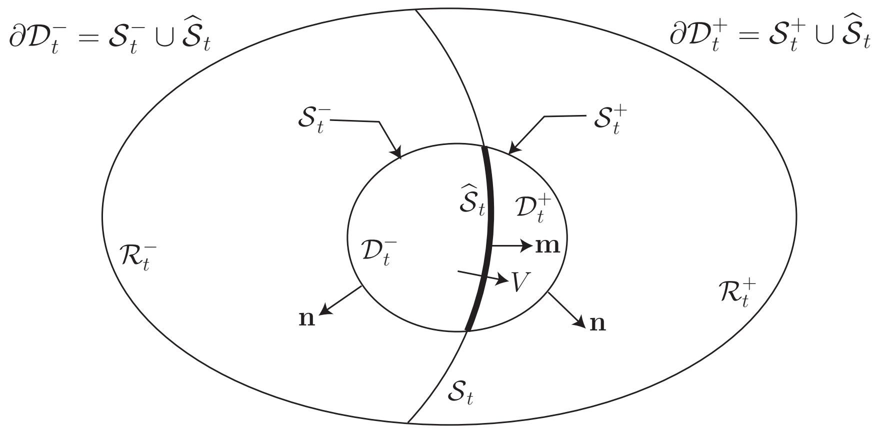

Figure 6.4: An arbitrary subregion $\mathcal{D}_t$ that intersects the singular surface $\mathcal{S}_t$: $\mathcal{D}_t = \mathcal{D}_t^+ \cup \mathcal{D}_t^-$. The segment of $\mathcal{S}_t$ that lies within $\mathcal{D}_t$ is denoted by $\widehat{\mathcal{S}}_t$. Note that $\partial\mathcal{D}_t^+ = \mathcal{S}_t^+ \cup \widehat{\mathcal{S}}_t$ and $\partial\mathcal{D}_t^- = \mathcal{S}_t^- \cup \widehat{\mathcal{S}}_t$. The unit vector $\mathbf{m}$ is normal to $\mathcal{S}_t$ and points into $\mathcal{R}_t^+$ The propagation speed of the surface is V in the direction $\mathbf{m}$.

speed with which these surfaces propagate are $\mathbf{v}\cdot\mathbf{n}$ and $-V$ respectively. Applying (a) to the region $\mathcal{D}_t^+$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{D_t^+} \beta \, \mathrm{d}V_y = \int_{D_t^+} \beta' \, \mathrm{d}V_y + \int_{S_t^+} \beta \, \mathbf{v}\cdot\mathbf{n} - \int_{\widehat{S}_t} \beta^+ \, V \, \mathrm{d}A_y.$$

On combining the preceding equations we obtain the desired result

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{D_t} \beta \, \mathrm{d}V_y = \int_{D_t} \beta' \, \mathrm{d}V_y + \int_{\partial D_t} \beta \, \mathbf{v}\cdot\mathbf{n} \, \mathrm{d}A_y - \int_{\widehat{S}_t} \left(\beta^+ - \beta^-\right) \, V \, \mathrm{d}A_y$$

Similarly for a vector field $\mathbf{b}(\mathbf{y}, t)$ we use (b) and show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{D_t} \mathbf{b} \, \mathrm{d}V_y = \int_{D_t} \mathbf{b}' \, \mathrm{d}V_y + \int_{\partial D_t} \mathbf{b} \, (\mathbf{v}\cdot\mathbf{n}) \, \mathrm{d}A_y - \int_{\widehat{S}_t} \left(\mathbf{b}^+ - \mathbf{b}^-\right) \, V \, \mathrm{d}A_y$$

Problem 6.8. Derive the *Eulerian jump conditions* (6.45).

<u>*Solution*</u>: We shall derive the jump condition associated with linear momentum balance; the other jump conditions can be obtained is an entirely analogous manner. The global balance of linear momentum requires that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{D_t} \rho\mathbf{v} \, \mathrm{d}V_y = \int_{D_t} \rho\mathbf{b} \, \mathrm{d}V_y + \int_{\partial D_t} \mathbf{T}\mathbf{n} \, \mathrm{d}A_y \tag{a}$$

for all subregions $\mathcal{D}_t \subset \mathcal{R}_t$. By applying the generalized transport equation derived in Problem 6.7 to the first term in (a) we get

$$\int_{D_t} (\rho\mathbf{v})' \, \mathrm{d}V_y + \int_{\partial D_t} \rho\mathbf{v} \, (\mathbf{v}\cdot\mathbf{n}) \, \mathrm{d}A_y - \int_{\widehat{S}_t} \left((\rho\mathbf{v})^+ - (\rho\mathbf{v})^-\right) \, V \, \mathrm{d}A_y = \int_{D_t} \rho\mathbf{b} \, \mathrm{d}V_y + \int_{\partial D_t} \mathbf{T}\mathbf{n} \, \mathrm{d}A_y, \tag{b}$$

where $\widehat{\mathcal{S}}_t = \mathcal{D}_t \cap \mathcal{S}_t$ is the portion of the singular surface that is in the interior of $\mathcal{D}_t$; see Figure 6.4.

We now take the limit of (b) as $\mathcal{D}_t$ is shrunk so as to collapse onto $\widehat{\mathcal{S}}_t$, i.e. we keep $\widehat{\mathcal{S}}_t$ fixed as $\mathcal{D}_t$ shrinks. Observe from Figure 6.4 that $\partial\mathcal{D}_t = \mathcal{S}_t^+ \cup \mathcal{S}_t^-$, and $\mathbf{m}$ denotes the unit vector normal to $\mathcal{S}_t$ that points into $\mathcal{R}_t^+$. It can be seen from Figure 6.4 that in the limit we are concerned with, $\mathcal{S}_t^+ \to \widehat{\mathcal{S}}_t$ and the unit outward normal to $\mathcal{S}_t^+$ approaches $+\mathbf{m}$; similarly $\mathcal{S}_t^- \to \widehat{\mathcal{S}}_t$ and the unit outward normal to $\mathcal{S}_t^-$ approaches $-\mathbf{m}$. In this limit

$$\begin{aligned}
\int_{\partial D_t} \mathbf{T}\mathbf{n}\, \mathrm{d}A_y &= \int_{\mathcal{S}_t^+} \mathbf{T}\mathbf{n}\, \mathrm{d}A_y + \int_{\mathcal{S}_t^-} \mathbf{T}\mathbf{n}\, \mathrm{d}A_y \to \int_{\widehat{\mathcal{S}}_t} \mathbf{T}^+\mathbf{m}\, \mathrm{d}A_y + \int_{\widehat{\mathcal{S}}_t} \mathbf{T}^-(-\mathbf{m})\, \mathrm{d}A_y,\\
\int_{\partial D_t} \rho\mathbf{v}(\mathbf{v}\cdot\mathbf{n})\, \mathrm{d}A_y &= \int_{\mathcal{S}_t^+} \rho\mathbf{v}(\mathbf{v}\cdot\mathbf{n})\, \mathrm{d}A_y + \int_{\mathcal{S}_t^-} \rho\mathbf{v}(\mathbf{v}\cdot\mathbf{n})\, \mathrm{d}A_y \to\\
&\to \int_{\widehat{\mathcal{S}}_t} (\rho\mathbf{v})^+(\mathbf{v}^+\cdot\mathbf{m})\, \mathrm{d}A_y + \int_{\widehat{\mathcal{S}}_t} (\rho\mathbf{v})^-(\mathbf{v}^-\cdot(-\mathbf{m}))\, \mathrm{d}A_y,
\end{aligned}$$

and the volume integrals are taken to vanish in this limit. Thus in this limit (b) yields

$$\begin{aligned}
\int_{\widehat{\mathcal{S}}_t} (\rho\mathbf{v})^+(\mathbf{v}^+\cdot\mathbf{m})\, \mathrm{d}A_y \;&+\; \int_{\widehat{\mathcal{S}}_t} (\rho\mathbf{v})^-\big[\mathbf{v}^-\cdot(-\mathbf{m})\big]\, \mathrm{d}A_y - \int_{\widehat{\mathcal{S}}_t} \big[(\rho\mathbf{v})^+ - (\rho\mathbf{v})^-\big]\, V\, \mathrm{d}A_y =\\
&= \int_{\widehat{\mathcal{S}}_t} \mathbf{T}^+\mathbf{m}\, \mathrm{d}A_y - \int_{\widehat{\mathcal{S}}_t} \mathbf{T}^-\mathbf{m}\, \mathrm{d}A_y,
\end{aligned}$$

which can be written as

$$\int_{\widehat{\mathcal{S}}_t} \big\{(\mathbf{T}^+\mathbf{m} - \mathbf{T}^-\mathbf{m}) + (\rho\mathbf{v})^+\big(V - \mathbf{v}^+\cdot\mathbf{m}\big) - (\rho\mathbf{v})^-\big(V - \mathbf{v}^-\cdot\mathbf{m}\big)\big\}\, \mathrm{d}A_y$$

or

$$\int_{\widehat{\mathcal{S}}_t} \{[\![\mathbf{T}\mathbf{m}]\!] + [\![(\rho\mathbf{v})(V - \mathbf{v}\cdot\mathbf{m})]\!]\}\, \mathrm{d}A_y = 0.$$

Since this must hold for all choices of $\widehat{\mathcal{S}}_t$ it follows that the integrand must vanish at each point on $\mathcal{S}_t$. Thus we have the jump condition associated with linear momentum balance as

$$[\![\mathbf{T}\mathbf{m}]\!] + [\![(\rho\mathbf{v})(V - \mathbf{v}\cdot\mathbf{m})]\!] = 0 \qquad (c)$$

which must hold at all points of the singular surface $\mathcal{S}_t$.

Problem 6.9. Derive again the *Eulerian jump conditions* (6.45) but now by substituting $\mathbf{S} = J\mathbf{T}\mathbf{F}^{-T}$ and $\mathbf{q}_0 = J\mathbf{F}^{-1}\mathbf{q}$ into the Lagrangian jump conditions (6.42).

Problem 6.10. Show that the jump condition $(6.42)_3$ associated with the first law of thermodynamics can be written alternately as

$$\Big([\![\rho_0\varepsilon]\!] - \frac{1}{2}(\mathbf{S}^+ + \mathbf{S}^-)\cdot[\![\mathbf{F}]\!]\Big)V_0 + [\![\mathbf{q}_0\cdot\mathbf{n}_0]\!] = 0. \qquad (6.48)$$

Solution: To establish the desired result we simplify the terms $[\![\mathbf{S}\mathbf{n}_0 \cdot \mathbf{v}]\!] + [\![\rho_0 \mathbf{v}\cdot\mathbf{v}/2]\!]V_0$ that appear in $(6.42)_3$ as follows:

$$\begin{aligned}
\mathbf{S}^+\mathbf{n}_0\cdot\mathbf{v}^+ \;&-\; \mathbf{S}^-\mathbf{n}_0\cdot\mathbf{v}^- + \tfrac{1}{2}\rho_0(\mathbf{v}^+\cdot\mathbf{v}^+ - \mathbf{v}^-\cdot\mathbf{v}^-)V_0 = \\
&= \mathbf{S}^+\mathbf{n}_0\cdot\mathbf{v}^+ - \mathbf{S}^-\mathbf{n}_0\cdot\mathbf{v}^- + \tfrac{1}{2}\rho_0(\mathbf{v}^+ - \mathbf{v}^-)\cdot(\mathbf{v}^+ + \mathbf{v}^-)V_0 = \\
&= \mathbf{S}^+\mathbf{n}_0\cdot\mathbf{v}^+ - \mathbf{S}^-\mathbf{n}_0\cdot\mathbf{v}^- - \tfrac{1}{2}(\mathbf{S}^+\mathbf{n}_0 - \mathbf{S}^-\mathbf{n}_0)\cdot(\mathbf{v}^+ + \mathbf{v}^-) = \\
&= \tfrac{1}{2}(\mathbf{S}^+ + \mathbf{S}^-)\mathbf{n}_0\cdot(\mathbf{v}^+ - \mathbf{v}^-) = \\
&= -\tfrac{1}{2}(\mathbf{S}^+ + \mathbf{S}^-)\mathbf{n}_0\cdot(\mathbf{F}^+ - \mathbf{F}^-)\mathbf{n}_0 V_0 = \\
&= -\tfrac{1}{2}(\mathbf{S}^+ + \mathbf{S}^-)\mathbf{n}_0\cdot\mathbf{a}V_0 = \\
&= -\tfrac{1}{2}(\mathbf{S}^+ + \mathbf{S}^-)\cdot(\mathbf{a}\otimes\mathbf{n}_0)V_0 = \\
&= -\tfrac{1}{2}(\mathbf{S}^+ + \mathbf{S}^-)\cdot(\mathbf{F}^+ - \mathbf{F}^-)V_0.
\end{aligned}$$

In this sequence of calculations we have used the linear momentum jump condition $(6.42)_2$ in the second step; the kinematic jump condition (6.30) in the fourth step; and the kinematic jump condition (6.27) in the fifth and seventh steps. We have also used the algebraic result $(\mathbf{a}\otimes\mathbf{n}_0)\mathbf{n}_0 = (\mathbf{n}_0\cdot\mathbf{n}_0)\mathbf{a} = \mathbf{a}$ and the identity $\mathbf{A}\mathbf{x}\cdot\mathbf{y} = \mathbf{A}\cdot(\mathbf{y}\otimes\mathbf{x})$.

We now replace the terms $[\![\mathbf{S}\mathbf{n}_0 \cdot \mathbf{v}]\!] + [\![\rho_0 \mathbf{v}\cdot\mathbf{v}/2]\!]V_0$ in the jump condition $(6.42)_3$ by the preceding representation which leads to (6.48).

Problem 6.11. Under modest assumptions on the temperature and heat flux, show that the rate of entropy production at a singular surface can be written as

$$\Gamma^{\text{jump}} = \int_{\mathcal{S}_0(t)\cap\mathcal{D}_0} \frac{f}{<\theta>} V_0 \, dA_x \tag{6.49}$$

where the *driving force* $f(\mathbf{x},t)$ on the singular surface is given by

$$f = [\![\rho_0\psi]\!] - <\mathbf{S}>\cdot[\![\mathbf{F}]\!] + <\rho_0\eta>[\![\theta]\!]. \tag{6.50}$$

Here $\psi = \varepsilon - \theta\eta$ is the Helmholtz free energy function. We use the notation $<\alpha>$ to denote the average value of a generic field $\alpha(\mathbf{x},t)$ at the singular surface:

$$<\alpha> = \frac{1}{2}(\alpha^+ + \alpha^-). \tag{6.51}$$

Solution: The rate of entropy production $\Gamma(t)$ associated with a subregion $\mathcal{D}_0$ of $\mathcal{R}_0$ is defined to be the excess of the rate of increase of the total entropy in $\mathcal{D}_0$ over the rate that entropy is supplied to $\mathcal{D}_0$ through the heat flux $\mathbf{q}_0$ and the heat supply r:

$$\Gamma(t) = \frac{d}{dt}\int_{\mathcal{D}_0} \rho_0\eta \, dV_x - \int_{\partial\mathcal{D}_0} \frac{\mathbf{q}_0\cdot\mathbf{n}_0}{\theta} \, dA_x - \int_{\mathcal{D}_0} \frac{\rho_0 r}{\theta} \, dV_x. \tag{a}$$

Suppose that the singular surface $\mathcal{S}_0(t)$ intersects $\mathcal{D}_0$ for some time interval of interest. One can decompose the entropy production rate into a term $\Gamma^{\rm bulk}$ characterizing the contribution from the particles in the bulk of $\mathcal{D}_0$ plus a term $\Gamma^{\rm jump}$ characterizing the contribution from the particles on that portion of the singular surface within $\mathcal{D}_0$, viz. $\mathcal{S}_0(t) \cap \mathcal{D}_0$. Such an alternate representation for Γ may be obtained from (a) with the help of a calculation analogous to the one that led from (6.35) to (6.39). The result may be written in the form

$$\Gamma \;=\; \Gamma^{\rm bulk} + \Gamma^{\rm jump}, \tag{b}$$

where

$$\Gamma^{\rm bulk} \;=\; \int_{\mathcal{D}_0} \{\rho_0 \dot{\eta} - {\rm Div}(\mathbf{q}_0/\theta) - \rho_0 r/\theta\} \; dV_x, \tag{c}$$

and

$$\Gamma^{\rm jump} \;=\; -\int_{\mathcal{S}_0(t)\cap\mathcal{D}_0} \{\rho_0 [\![\eta]\!] \; V_0 + [\![\mathbf{q}_0 \cdot \mathbf{n}_0/\theta]\!]\} \; dA_x. \tag{d}$$

Observe that the non-negativity of the integrands in (c) and (d) correspond to the field and jump inequalities, $(\mathbf{??})_2$ and $(6.42)_4$, associated with the second law of thermodynamics.

By using the alternative form (6.48) of the jump condition associated with the first law, one can rewrite part of the integrand in (d) as follows:

$$\begin{aligned} [\![\mathbf{q}_0 \cdot \mathbf{n}_0/\theta]\!] \;&=\; \left(-\frac{\rho_0 [\![\varepsilon]\!] - <\mathbf{S}> \cdot \; [\![\mathbf{F}]\!]}{<\theta>} \right) V_0 + \\ &\quad + (<1/\theta> - 1/<\theta>) \, [\![\mathbf{q}_0 \cdot \mathbf{n}_0]\!] + [\![1/\theta]\!] < \mathbf{q}_0 \cdot \mathbf{n}_0 > . \end{aligned} \tag{e}$$

Suppose that the thermomechanical processes considered are either adiabatic, in which case $\mathbf{q}_0 = \mathbf{0}$, $r = 0$; or involve heat conduction and the temperature is continuous. For adiabatic processes the last two terms on the right in (e) vanish by virtue of the vanishing of $\mathbf{q}_0$. For processes involving heat conduction, these same terms vanish because of the continuity of the temperature. Thus in all thermomechanical processes of the aforementioned type, (e) reduces to

$$[\![\mathbf{q}_0 \cdot \mathbf{n}_0/\theta]\!] \;=\; -\frac{1}{<\theta>} \left(\rho_0 [\![\varepsilon]\!] - \mathbf{S} \cdot \; [\![\mathbf{F}]\!] \right) \; V_0.$$

The rate of entropy production at the singular surface, $\Gamma^{\rm jump}$ given by (d), may now be rewritten as

$$\Gamma^{\rm jump} \;=\; \int_{\mathcal{S}_0(t)\cap\mathcal{D}} \frac{[\![\rho_0 \varepsilon]\!] - <\mathbf{S}> \cdot \; [\![\mathbf{F}]\!] - <\theta> [\![\rho_0 \eta]\!]}{<\theta>} \; V_0 \; dA_x.$$

In terms of the Helmholtz free energy $\psi = \varepsilon - \theta\eta$, this reduces to the desired result (6.49), (6.50).

Note that the second law of thermodynamics requires that

$$f V_0 \geq 0 \tag{6.52}$$

at each $\mathbf{x} \in \mathcal{S}_0$

Problem 6.12. In this problem we will derive the classical jump conditions (*Rankine-Hugoniot conditions*) of *gas dynamics* at a "normal shock" in a uniaxial flow of an inviscid fluid. In an inviscid fluid the Cauchy stress tensor necessarily has the form $\mathbf{T} = -p\mathbf{I}$ where $p(\mathbf{y}, t)$ is the pressure. Consider a one dimensional flow of such a fluid: $\mathbf{v} = v\mathbf{e}_1$. Suppose that the flow involves a shock (a singular surface) perpendicular to the flow and that the Eulerian shock velocity is $V\mathbf{e}_1$.

Show that the general three-dimensional jump conditions (in Eulerian form) can be simplified and combined to give

$$[\![\rho U]\!] = 0, \qquad [\![p + \rho U^2]\!] = 0, \qquad [\![\rho U(\varepsilon + p/\rho + U^2/2)] + q]\!] = 0, \tag{a}$$

where $U^\pm = V - v^\pm$ are the relative velocities of the shock relative to the flow. The entropy inequality must also hold of course.

<u>*Solution*</u>: Setting $\mathbf{n} = \mathbf{e}_1, \mathbf{v}^\pm = v^\pm \mathbf{e}_1, \mathbf{q} = q\,\mathbf{e}_1$, $U^\pm = V - v^\pm, \mathbf{T}^\pm\mathbf{n} = -p^\pm \mathbf{e}_1, \mathbf{T}^\pm\mathbf{n}\cdot\mathbf{v}^\pm = -p^\pm v^\pm$ and $\mathbf{v}^\pm\cdot\mathbf{v}^\pm = (v^\pm)^2$ in the general jump conditions $(6.45)_{1,2,3}$ leads to

$$[\![\rho U]\!] = 0, \qquad [\![\rho v U - p]\!] = 0, \qquad [\![-pv + q + \rho(\varepsilon + v^2/2)U]\!] = 0 \tag{b}$$

Equation $(b)_1$ establishes $(a)_1$.

Next, $(b)_1$ tells us that

$$\rho^+ U^+ = \rho^- U^-, \tag{c}$$

and so we can simplify $(b)_2$ as follows:

$$[\![p]\!] = [\![\rho v U]\!] = \rho^+U^+v^+ - \rho^-U^-v^- = \rho^+U^+(v^+ - v^-) = -\rho^+U^+(U^+ - U^-) = -\rho^+U_+^2 + \rho_- U_-^2 = -[\![\rho U^2]\!]$$

where we have used (c) in the third and fifth steps and used $v^\pm = V - U^\pm$ in the fourth step. Thus

$$[\![p + \rho U^2]\!] = 0. \tag{d}$$

Equation (d) establishes $(a)_2$.

We now turn to $(b)_3$ and work on each term separately:

$$[\![\rho U\varepsilon]\!] = \rho^+U^+[\![\varepsilon]\!], \tag{i}$$

$$\begin{aligned}[\![\rho U v^2]\!] &= \rho^+U^+[\![v^2]\!] = \rho^+U^+((V-U^+)^2 - (V-U^-)^2) = \rho^+U^+(-2VU^+ + U_+^2 + 2VU^- - U_-^2) \\ &= \rho^+U^+[\![U^2 - 2VU]\!],\end{aligned} \tag{ii}$$

$$\begin{aligned}[\![pv]\!] &= p^+(V-U^+) - p^-(V-U^-) = [\![p]\!]V - p^+U^+ + p^-U^- \\ &= -[\![\rho U^2]\!]V - (p^+\rho^-/\rho^+)U^- + (p^-\rho^+/\rho^-)U^+ \\ &= -[\![\rho U^2]\!]V - [\![p/\rho]\!]\rho^+U^+ = -(\rho^+U_+^2 - \rho^-U_-^2)V - [\![p/\rho]\!]\rho^+U^+ \\ &= -\rho^+U^+(U^+ - U^-)V - [\![p/\rho]\!]\rho^+U^+ = -\rho^+U^+\Big([\![U]\!]V + [\![p/\rho]\!]\Big) = -\rho^+U^+[\![UV + p/\rho]\!].\end{aligned} \tag{iii}$$

In the preceding calculations we have used $(c), (d)$ and $v^{\pm} = V - U^{\pm}$ at several steps. Therefore from (ii) and (iii) we get

$$[\![\rho U v^2/2]\!] - [\![pv]\!] = \frac{1}{2}\rho^+ U^+ [\![U^2 - 2VU]\!] + \rho^+ U^+ \Big([\![UV + p/\rho]\!]\Big) = \rho^+ U^+ [\![U^2/2 + p/\rho]\!]. \qquad (iv)$$

Finally on combining (i) and (iv) we have

$$[\![\rho U\varepsilon + \rho U v^2/2 - pv]\!] = \rho^+ U^+ [\![\varepsilon]\!] + \rho^+ U^+ [\![U^2/2 + p/\rho]\!] = \rho^+ U^+ [\![\varepsilon + U^2/2 + p/\rho]\!] = [\![\rho U(\varepsilon + U^2/2 + p/\rho)]\!]$$

which in view of $(b)_3$ establishes $(a)_3$.

Remark: If the flow is occurring at high speed, the dynamic processes will occur much faster than the heat transfer processes and so one frequently assumes the flow to be adiabatic, i.e. $\mathbf{q} = \mathbf{0}$, $r = 0$.

Problem 6.13. In this problem you are asked to explicitly solve the kinematic jump condition $(6.27)_1$ in a particular case.

Suppose that the limiting values of the deformation gradient tensor at a point on a singular surface are $\mathbf{F}^+ = \mathbf{Q}\mathbf{U}_2$ and $\mathbf{F}^- = \mathbf{U}_1$. Here $\mathbf{U}_1$ and $\mathbf{U}_2$ are symmetric positive definite tensors such that

$$\mathbf{U}_2 = \mathbf{R}^T \mathbf{U}_1 \mathbf{R} \qquad (a)$$

where

$$\mathbf{R} = -1 + 2\mathbf{e} \otimes \mathbf{e} \qquad (b)$$

for some unit vector $\mathbf{e}$. (Note that the tensor $\mathbf{R}$ represents a 180-degree rotation about the unit vector $\mathbf{e}$. Note also that $\det \mathbf{U}_1 = \det \mathbf{U}_2$.).

Consider the kinematic jump condition

$$\mathbf{Q}\mathbf{U}_2 - \mathbf{U}_1 = \mathbf{a} \otimes \mathbf{n}_0 \qquad (c)$$

for some orthogonal tensor $\mathbf{Q}$, non-zero vector $\mathbf{a}$ and unit vector $\mathbf{n}_0$.

(i) Given $\mathbf{U}_1$ and $\mathbf{U}_2$ that obey $(a), (b)$, verify that one solution $\{\mathbf{n}_0, \mathbf{a}, \mathbf{Q}\}$ of (c) is

$$\text{Solution I}: \qquad \mathbf{n}_0 = \mathbf{e}, \qquad \mathbf{a} = 2\left(\frac{\mathbf{U}_1^{-1}\mathbf{e}}{|\mathbf{U}_1^{-1}\mathbf{e}|^2} - \mathbf{U}_1\mathbf{e}\right),$$

and that a second solution is

$$\text{Solution II}: \qquad \mathbf{n}_0 = 2\alpha\left(\mathbf{e} - \frac{\mathbf{U}_1^2\mathbf{e}}{|\mathbf{U}_1\mathbf{e}|^2}\right), \qquad \mathbf{a} = \frac{1}{\alpha}\,\mathbf{U}_1\mathbf{e}.$$

Here the scalar α simply makes $\mathbf{n}_0$ a unit vector. The orthogonal tensors $\mathbf{Q}$ associated with these two solutions have not been displayed. (One can show that there are no other solutions of (c).)

(ii) Consider tensors $\mathbf{U}_1$ and $\mathbf{U}_2$ whose components with respect to some orthonormal basis are

$$[\mathbf{U}_1] = \begin{pmatrix} \eta_2 & 0 & 0 \\ 0 & \eta_1 & 0 \\ 0 & 0 & \eta_1 \end{pmatrix}, \quad [\mathbf{U}_2] = \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_2 & 0 \\ 0 & 0 & \eta_1 \end{pmatrix},$$

where $\eta_1 > 0$ and $\eta_2 > 0$ are constants. Show that this pair of tensors obey (a) and (b) for some $\mathbf{e}$.

Remark: See K. Bhattacharya, *Microstructure of Martensite*, Oxford, 2003, for a discussion of how this problem relates to a particular problem in the crystallography of martensite.

Problem 6.14. In this problem you are to examine an example where the kinematic jump condition $(6.27)_1$ does *not* have a solution and therefore, where the proposed tensors $\mathbf{F}^+$ and $\mathbf{F}^-$ cannot be the limiting deformation gradient tensors at a point on a singular surface with continuous deformation.

Consider the symmetric positive definite tensors $\mathbf{U}_1, \mathbf{U}_2$ and $\mathbf{U}_3$ whose components with respect to an orthonormal basis are

$$[\mathbf{U}_1] = \begin{pmatrix} \eta_2 & 0 & 0 \\ 0 & \eta_1 & 0 \\ 0 & 0 & \eta_1 \end{pmatrix}, \quad [\mathbf{U}_2] = \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_2 & 0 \\ 0 & 0 & \eta_1 \end{pmatrix}, \quad [\mathbf{U}_3] = \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_1 & 0 \\ 0 & 0 & \eta_2 \end{pmatrix}.$$

Here $\eta_1 > 0$ and $\eta_2 > 0$ are constants. Show that there do *not* exist a proper orthogonal tensor $\mathbf{Q}$, a non-zero vector $\mathbf{a}$ and unit vector $\mathbf{n}_0$ for which

$$\mathbf{F}^+ - \mathbf{F}^- = \mathbf{a} \otimes \mathbf{n}_0$$

when $\mathbf{F}^+ = \mathbf{Q}\mathbf{U}_i$, $i = 1, 2, 3$, and $\mathbf{F}^- = \mathbf{I}$.

Remark: See K. Bhattacharya, *Microstructure of Martensite*, Oxford, 2003, for a discussion of how this problem relates to a particular problem in the crystallography of austenite/martensite.

Problem 6.15. Consider a singular surface as discussed in the present chapter but now suppose that the motion $\mathbf{y}(\mathbf{x}, t)$ is once continuously differentiable everywhere so that in particular the particle velocity field $\dot{\mathbf{y}}$ and deformation gradient tensor field $\text{Grad}\,\mathbf{y}$ are both continuous across the singular surface. However the second derivatives of $\mathbf{y}(\mathbf{x}, t)$ are discontinuous across this singular surface. Such a surface is called an *acceleration wave*. Specialize the general (Eulerian) jump conditions (6.45) to this case.

References:

1. K. Bhattacharya, *Microstructure of Martensite*, Oxford, 2003.

2. P. Chadwick, *Continuum Mechanics: Concise Theory and Problems*, Chapter 3, Dover, 1999.

3. M.E. Gurtin, E. Fried and L. Anand, *The Mechanics and Thermodynamics of Continua*, Chapters 32 and 33, Cambridge University Press, 2010.

Chapter 7

Constitutive Principles

In this brief chapter we make some general remarks about the constitutive response of a material. In the subsequent chapters we explore these ideas in detail, in the context of certain specific classes of materials.

The basic fields of the continuum theory, i.e. the velocity vector $\mathbf{v}(\mathbf{y},t)$, mass density $\rho(\mathbf{y},t)$, Cauchy stress tensor $\mathbf{T}(\mathbf{y},t)$, heat flux vector $\mathbf{q}(\mathbf{y},t)$, specific internal energy $\varepsilon(\mathbf{y},t)$, specific entropy $\eta(\mathbf{y},t)$ and temperature $\theta(\mathbf{y},t)$, must satisfy the field equations/inequality

$$\left.\begin{array}{rcll} \dot{\rho} \ + \ \rho \,\mathrm{div}\, \mathbf{v} & = & 0, & \text{Mass balance,} \\[2ex] \mathrm{div}\, \mathbf{T} \ + \ \rho \mathbf{b} & = & \rho \dot{\mathbf{v}}, & \text{Linear momentum balance,} \\[2ex] \mathbf{T} & = & \mathbf{T}^T, & \text{Angular momentum balance,} \\[2ex] \mathbf{T}\cdot\mathbf{D} \ + \ \mathrm{div}\, \mathbf{q} + \rho r & = & \rho\dot{\varepsilon}, & \text{Energy balance,} \\[2ex] \rho\dot{\eta} & \geq & \mathrm{div}\left(\dfrac{\mathbf{q}}{\theta}\right) + \dfrac{\rho r}{\theta}, & \text{Entropy inequality.} \end{array}\right\} \tag{7.1}$$

The body force density $\mathbf{b}(\mathbf{y},t)$ and heat supply density $r(\mathbf{y},t)$ are applied on the body by agents external to the body. They are viewed as prescribed.

These field equations hold for all materials so long as the body can be treated as a continuum[1]. While our daily experience tells us that different materials respond differently to the same stimulus, so far there is nothing that brings this characteristic into the theory.

Moreover, in terms of components, there are 16 scalar-valued fields in (7.1) above (taking

[1]and it does not involve additional ingredients such as contact and body torques.

the Cauchy stress to be symmetric) and only 5 field equations (the entropy condition being an inequality). The additional 11 scalar equations that are needed describe the material behavior. They are provided by a set of *constitutive relations*.

A material is characterized by a set of *constitutive response functions*[2] $\widehat{\mathbf{T}}$, $\widehat{\mathbf{q}}$, $\widehat{\varepsilon}$ and $\widehat{\eta}$: given the motion $\mathbf{y}(\mathbf{x},t)$ and the temperature field $\theta(\mathbf{x},t)$ at all particles of the body for all times up to and including the present, the role of the constitutive response functions is to provide the current values of stress, heat flux, specific internal energy and specific entropy at each particle in the body. A simple nontrivial example of a set of constitutive relations would be

$$\mathbf{T} = \widehat{\mathbf{T}}(\mathbf{F},\dot{\mathbf{F}},\theta), \qquad \mathbf{q} = \widehat{\mathbf{q}}(\mathbf{F},\theta,\mathrm{Grad}\;\theta), \qquad \varepsilon = \widehat{\varepsilon}(\mathbf{F},\theta), \qquad \eta = \widehat{\eta}(\mathbf{F},\theta), \tag{7.2}$$

where $\mathbf{F} = \mathrm{Grad}\,\mathbf{y}$ and $\dot{\mathbf{F}}$ is its material time derivative. In terms of components, there are 11 scalar-valued equations here (the function $\widehat{\mathbf{T}}$ being assumed to be symmetric tensor-valued).

In order to determine the specific constitutive response functions that describe a given material, one might, say, use first principles atomistic calculations, or use physical experiments, or a combination thereof[3]. However it is natural to *ask whether the constitutive response functions are completely arbitrary as far as continuum theory is concerned or whether the basic theory we have laid out thus far places any restrictions on them* (which might, for example, reduce the number of independent constitutive response functions that must ultimately be determined by other means). We address this question in the remainder of this chapter.

1. *Causality.* We note first that we would never allow the current value of any of the fields to depend on a future value of that or any other field.

2. *Field Equations.* As noted already, angular momentum balance places a clear restriction on the constitutive law for stress: the function $\widehat{\mathbf{T}}$ must be *symmetric* tensor-valued. We assume that the body force $\mathbf{b}$ and heat supply r – applied by agents external to the body – can in *principle* be prescribed arbitrarily. Thus given any motion and temperature field, and any set of constitutive response functions generating stress, heat flux, etc., one can always find a body force $\mathbf{b}$ and heat supply r *a posteriori* such that the equation of motion $(7.1)_2$ and the energy equation $(7.1)_4$ hold. Thus these equations do not place any restrictions on the constitutive response functions. This leaves the entropy inequality.

[2]Though we use the term "functions" they might in fact sometimes be functionals.

[3]We note that a particular set of constitutive response functions, no matter how carefully determined, provides a *model* of the material. It would not describe the material's behavior exactly.

Once r has been chosen to satisfy the energy equation, the entropy inequality becomes a restriction since it has *no arbitrarily prescribable entities.* This restriction can be made explicit by eliminating r between the energy equation $(7.1)_4$ and the entropy inequality $(7.1)_5$ leading to

$$\rho\theta\dot{\eta} \ \geq \ \rho\dot{\varepsilon} - \mathbf{T}\cdot\mathbf{D} - \frac{\mathbf{q}\cdot \operatorname{grad}\theta}{\theta}$$

in terms of the specific internal energy; or equivalently as either

$$\rho\dot{\psi} - \mathbf{T}\cdot\mathbf{D} + \rho\eta\dot{\theta} \ - \frac{\mathbf{q}\cdot \operatorname{grad}\theta}{\theta} \ \leq 0 \tag{7.3}$$

or

$$\rho_0\dot{\psi} \ - \ \mathbf{S}\cdot\dot{\mathbf{F}} \ + \ \rho_0\eta\dot{\theta} \ - \ \mathbf{q}_o\cdot\frac{\operatorname{Grad}\theta}{\theta} \ \leq \ 0,$$

in terms of the specific Helmholtz free energy $\psi = \varepsilon - \eta\theta$. Thus the entropy inequality will impose certain restrictions on the constitutive response functions. We shall illustrate this in Section 7.2, and explore it in detail for several classes of constitutive relations in the subsequent chapters.

3. *Material Frame Indifference.* Next, we know from the discussions on objectivity in Sections 3.8 and 5.5 how the various physical fields are related in two different motions that differ by a rigid body motion. For example we know that $\varepsilon^* = \varepsilon, \mathbf{F}^* = \mathbf{QF}$ and $\theta^* = \theta$ where the starred and unstarred quantities refer to the two motions and $\mathbf{Q}$ is the proper orthogonal tensor that relates the two motions. Thus if, for example, the constitutive response function for internal energy $\widehat{\varepsilon}$ depends on $\mathbf{F}$ and θ, the constitutive relation gives $\varepsilon = \widehat{\varepsilon}(\mathbf{F},\theta)$ and $\varepsilon^* = \widehat{\varepsilon}(\mathbf{F}^*,\theta^*) = \widehat{\varepsilon}(\mathbf{QF},\theta)$ in the two motions. objectivity requires that $\varepsilon = \varepsilon^*$ and therefore it is necessary that

$$\widehat{\varepsilon}(\mathbf{F},\theta) = \widehat{\varepsilon}(\mathbf{QF},\theta)$$

for all proper orthogonal $\mathbf{Q}$. This restricts the allowable form of $\widehat{\varepsilon}$ which is a consequence of requiring the constitutive response functions to be consistent with *material frame indifference*. It reflects the fact that physical laws should be independent of the observer. We shall illustrate such a restriction and its consequences in Section 7.2, and explore this issue in detail for several classes of constitutive relations in the subsequent chapters.

4. *Material Symmetry.* Next, if we have knowledge of some microstructural symmetry characteristics of the material – for example the material might be a single crystal of a crystalline material with a cubic lattice – then the symmetry at the microstrutural level translates into a symmetry at the continuum level. Requiring the constitutive response

functions to be consistent with this *material symmetry* (if any) will place further restrictions on them.

Remark: Internal Constraints. The response of a specific material might always display some particular kinematic characteristic. For example one might observe that the volume change in all motions in some material is very small compared with other geometric changes. Or in some other material, one might observe that the extension or contraction in a particular direction is almost zero (perhaps because there are stiff fibers inside the material in that direction). One might then *idealize* these materials by saying that the former is strictly incompressible, and the latter strictly inextensible in some direction. *Internal constraints* of this sort restrict the set of motions that the material can undergo and this in turn has consequences on the constitutive response functions.

Remark: Equipresence. For internal consistency, if we allow one of the constitutive response functions in the set $\widehat{\mathbf{T}}$, $\widehat{\mathbf{q}}$, $\widehat{\varepsilon}, \widehat{\eta}$ to depend on a particular quantity, say temperature gradient, then we have no *a priori* reason not to allow all of them to depend on this quantity. Note that the example given in (7.2) is not consistent with this convention since only $\widehat{\mathbf{T}}$, for example, has been allowed to depend on $\dot{\mathbf{F}}$.

7.1 Different Functional Forms of Constitutive Response Functions. Some Examples.

The particular way in which a specific constitutive response function depends on the motion and temperature depends on the characteristics of the material being modeled. For example, consider the stress and how it may depend on the motion. The examples below describe cases where the material has (a) no knowledge of its past (no memory), (b) short term memory and (c) long term memory. Likewise the examples describe materials where the stress at a particle depends (d) only on what goes on in an infinitesimal neighborhood of that particle, (e) the neighborhood of influence is slightly larger than in the preceding case, and (f) the neighborhood of influence is large. What follow are merely *examples* of stress response functions with these properties.

(i) [No memory. Short range forces.] Suppose that the stress at a particle p at a time t depends only on the motion of the particles in an infinitesimal neighborhood of p at that time t. In this case we might expect $\widehat{\mathbf{T}}$ to be given by

$$\widehat{\mathbf{T}}(\mathbf{F}(\mathbf{x},t)).$$

(ii) [Short memory. Short range forces.] Suppose that the stress at a particle p at a time t depends only on the motion of the particles in an infinitesimal neighborhood of p at times up to *and close to* times preceding time t. Such a material has some slight memory of the past. In this case we might expect $\widehat{\mathbf{T}}$ to depend on $\mathbf{F}(\mathbf{x},t)$ as well some number of time derivatives of $\mathbf{F}$:

$$\widehat{\mathbf{T}}(\mathbf{F}(\mathbf{x},t), \dot{\mathbf{F}}(\mathbf{x},t), \ddot{\mathbf{F}}(\mathbf{x},t), \ldots, \overset{(n)}{\mathbf{F}}(\mathbf{x},t)).$$

(iii) [Long memory. Short range forces.] Suppose that the stress at a particle p at a time t depends only on the motion of the particles in an infinitesimal neighborhood of p at *all times up to* and preceding time t. Such a material has memory of the past. In this case we might expect $\widehat{\mathbf{T}}$ to involve terms such as

$$\int_{-\infty}^{t} \mathrm{e}^{-(t-\tau)/\lambda} \mathbf{f}\big(\mathbf{F}(\mathbf{x},\tau)\big)\, d\tau, \qquad \lambda > 0.$$

(iv) [No memory. Medium range forces.] Suppose that the stress at a particle p at a time t depends only on the motion of the particles in a *small but not infinitesimal* neighborhood of p at that time t. Such a material is said to be nonlocal. In this case we might expect $\widehat{\mathbf{T}}$ to depend on $\mathbf{F}(\mathbf{x},t)$ as well as some spatial derivatives of $\mathbf{F}$:

$$\widehat{\mathbf{T}}\Big(\mathbf{F},\ \mathrm{Grad}\,\mathbf{F},\ \mathrm{Grad}\big(\mathrm{Grad}\,\mathbf{F}\big)\Big).$$

(v) [No memory. Long range forces.] Suppose that the stress at a particle p at a time t depends only on the motion of *all of the the particles* of the body at that time t. In this case we might expect $\widehat{\mathbf{T}}$ to involve terms such as

$$\int_{\mathcal{R}_t} \mathrm{e}^{-\alpha|\mathbf{x}-\xi|}\, \mathbf{f}\big(\mathbf{F}(\boldsymbol{\xi}-\mathbf{x},t)\big) d\boldsymbol{\xi}.$$

The preceding are meant only to be illustrative and not exhaustive in any sense. There are other constitutive relations whose primitive form is quite different to the preceding, for example,

$$\alpha \overset{\triangle}{\mathbf{T}} + \mathbf{T} = \mathbf{f}(\mathbf{F})$$

where the first term is the convected time derivative of stress.

7.2 Illustration.

In this section we use a simple example to illustrate how the restrictions placed on a set of constitutive relations by the requirements of objectivity and the entropy inequality can be used to infer certain characteristics of the constitutive response functions. In the example below, we will find that there is a considerable simplification in the form of the constitutive relations.

Kinematics and field equations: For simplicity, we consider the so-called purely mechanical theory of a continuum which involves the motion $\mathbf{y}(\mathbf{x}, t)$, velocity $\mathbf{v}(\mathbf{y}, t)$, mass density $\rho(\mathbf{y}, t)$, Cauchy stress $\mathbf{T}(\mathbf{y}, t)$ and free energy $\psi(\mathbf{y}, t)$ but no thermodynamic quantities. These fields must obey the kinematic equations

$$\mathbf{F} = \operatorname{Grad}\mathbf{y}, \qquad J = \det\mathbf{F}, \qquad \mathbf{v} = \dot{\mathbf{y}}, \qquad \mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T), \qquad \mathbf{L} = \operatorname{grad}\mathbf{v}, \tag{7.4}$$

and the field equations

$$\left.\begin{array}{lcll} \rho_0 & = & \rho J, & \\ \operatorname{div}\mathbf{T} + \rho\mathbf{b} & = & \rho\dot{\mathbf{v}}, & \mathbf{T} = \mathbf{T}^T, \\ \mathbf{T}\cdot\mathbf{D} & \geq & \rho\dot{\psi}, & \end{array}\right\} \tag{7.5}$$

associated with mass balance, linear and angular momentum balance, and the entropy inequality. The inequality $(7.5)_3$ is the purely mechanical version of (7.3).

Constitutive equations: The 11 scalar fields corresponding to ρ, ψ and the components of $\mathbf{y}$ and $\mathbf{T}$ ($\mathbf{T}$ being taken to be symmetric) that appear in (7.5) are to obey the 4 scalar field equations $(7.5)_{1,2}$. The requisite 7 additional scalar equations are given through appropriate constitutive relations. As an example suppose that the stress and energy depend on the motion through constitutive response functions $\overline{\mathbf{T}}$ and $\overline{\psi}$ that are such that

$$\mathbf{T} = \overline{\mathbf{T}}(J), \qquad \psi = \overline{\psi}(J) \qquad \text{where } J = \det\mathbf{F}.$$

This material has no memory and only short range forces, and specifically, the stress and energy depend on the motion only through the determinant of the deformation gradient tensor. Alternatively, since mass balance relates the mass density ρ to J through $(7.5)_1$, we may consider an equivalent set of constitutive relations

$$\mathbf{T} = \widehat{\mathbf{T}}(\rho), \qquad \psi = \widehat{\psi}(\rho). \tag{7.6}$$

The aim of this section is to illustrate how objectivity and the entropy inequality impose certain restrictions on the constitutive response functions $\widehat{\mathbf{T}}$ and $\widehat{\psi}$. We shall start with the set of constitutive relations (7.6) that involve *seven* independent scalar constitutive functions $\widehat{\mathrm{T}}_{ij}$ and $\widehat{\psi}$; after ensuring that (7.6) is consistent with the requirements of objectivity and the entropy inequality, we will find that, in fact, they only involve *one* independent scalar-valued constitutive function.

Material frame indifference: We first consider the implications of objectivity. Here we examine the physical quantities associated with two motions $\mathbf{y}(\mathbf{x}, t)$ and $\mathbf{y}^*(\mathbf{x}, t) = \mathbf{Q}(t)\mathbf{y}(\mathbf{x}, t)$ where $\mathbf{Q}(t)$ is a rigid rotation at each t. The Jacobians in the two motions are related by $J = J^*$ since $J = \det \mathbf{F}$ and $J^* = \det \mathbf{F}^* = \det(\mathbf{QF}) = (\det \mathbf{Q})(\det \mathbf{F}) = \det \mathbf{F}$. Mass balance $(7.5)_1$ now implies that the mass densities associated with the two motions are related by $\rho^* = \rho$. The constitutive relations (7.6) give the energies and stresses associated with these two motions to be

$$\psi = \widehat{\psi}(\rho), \quad \psi^* = \widehat{\psi}(\rho^*); \qquad \mathbf{T} = \widehat{\mathbf{T}}(\rho), \quad \mathbf{T}^* = \widehat{\mathbf{T}}(\rho^*). \tag{7.7}$$

The relationship between the energies ψ, ψ^* and the stresses $\mathbf{T}, \mathbf{T}^*$ associated with the two motions must be postulated based on physical grounds. As discussed in Section 5.5 we require that

$$\psi^* = \psi, \qquad \mathbf{T}^* = \mathbf{QTQ}^T, \tag{7.8}$$

for all orthogonal tensors $\mathbf{Q}$.

Restrictions due to objectivity: We now determine the restrictions that objectivity (7.8) places on the two constitutive response functions $\widehat{\psi}$ and $\widehat{\mathbf{T}}$. First, it follows from $(7.8)_1$ and $(7.7)_{1,2}$ that the energy response function must obey the requirement $\widehat{\psi}(\rho^*) = \widehat{\psi}(\rho)$; but this is automatic since $\rho^* = \rho$. Next, it follows from $(7.8)_2$, $(7.7)_{3,4}$ and $\rho_* = \rho$ that we must have

$$\widehat{\mathbf{T}}(\rho) = \mathbf{Q}\widehat{\mathbf{T}}(\rho)\mathbf{Q}^T. \tag{7.9}$$

This must hold for all $\rho > 0$ and all proper orthogonal tensors $\mathbf{Q}$. It is a restriction on the stress response function $\widehat{\mathbf{T}}$.

Constitutive relations consistent with objectivity: We now explore the implications of the restriction (7.9). We showed in one of the problems in Chapter 3 of Volume I, that a second order symmetric tensor $\mathbf{A}$ obeys $\mathbf{A} = \mathbf{QAQ}^T$ for all rotations $\mathbf{Q}$ if and only if $\mathbf{A}$ is a scalar multiple of the identity tensor. Thus in the present context objectivity implies that $\widehat{\mathbf{T}}(\rho) = -\widehat{p}(\rho)\mathbf{I}$ for some scalar-valued function $\widehat{p}(\rho)$. In summary, the constitutive relations

(7.6) conform to the requirement of objectivity, if and only if they take the simpler form

$$\widehat{\mathbf{T}}(\rho) = -\widehat{p}(\rho)\mathbf{I}, \qquad \psi = \widehat{\psi}(\rho), \tag{7.10}$$

where $\widehat{p}$ is an arbitrary scalar-valued constitutive function, there being no restrictions on $\widehat{\psi}$. Observe that the 7 scalar constitutive functions $\widehat{\mathrm{T}}_{ij}$ and $\widehat{\psi}$ have been reduced by the requirement of objectivity to 2 independent scalar-valued constitutive functions $\widehat{p}$ and $\widehat{\psi}$.

Restrictions due to the entropy inequality: We now turn to the entropy inequality $(7.5)_3$. First note that

$$\mathbf{T} \cdot \mathbf{D} = -\widehat{p}(\rho)\ \mathbf{I} \cdot \mathbf{D} = -\widehat{p}(\rho)\ \mathrm{tr}\,\mathbf{D} = -\widehat{p}(\rho)\ \mathrm{div}\,\mathbf{v} = \widehat{p}(\rho)\ \frac{\dot{\rho}}{\rho}$$

where we have used (3.23), (3.18) in the third step and the alternative form $\dot{\rho} + \rho\,\mathrm{div}\,\mathbf{v} = 0$ of mass balance in the last step. Thus the entropy inequality $(7.5)_3$ requires that

$$\widehat{p}(\rho)\ \frac{\dot{\rho}}{\rho}\ \geq\ \rho\,\widehat{\psi}'(\rho)\,\dot{\rho}. \tag{7.11}$$

This must hold in all thermodynamic processes and therefore places a restriction on the constitutive functions $\widehat{p}$ and $\widehat{\psi}$.

Constitutive relations consistent with the entropy inequality: We now explore the implications of (7.11). We first rewrite (7.11) as

$$\Big(-\widehat{p}(\rho)\ +\ \rho^2\,\widehat{\psi}'(\rho)\Big)\ \dot{\rho}\ \leq\ 0 \tag{7.12}$$

and then note that, since this must hold in all thermodynamic processes, it must hold for all $\rho > 0$ and all $\dot{\rho}$. (See an important note at the end of this section.) It follows that the term in the parenthesis must vanish, since if it was either positive or negative, we could chose $\dot{\rho}$ to be negative or positive respectively, thus violating the inequality (7.12). Consequently we find that the two constitutive functions $\widehat{p}$ and $\widehat{\psi}$ are not independent but are in fact related by

$$\widehat{p}(\rho)\ =\ \rho^2\,\widehat{\psi}'(\rho). \tag{7.13}$$

Summary: If a material is described by a set of constitutive relations of the form (7.6), in view of objectivity and the entropy inequality they must in fact be of the form[4]

$$\mathbf{T} =\ -\rho^2\,\widehat{\psi}'(\rho)\,\mathbf{I}, \qquad \psi = \widehat{\psi}(\rho). \tag{7.14}$$

[4]The reader may recognize that these constitutive relations describe a compressible inviscid fluid in circumstances where thermal effects are not important.

Observe that the 7 scalar constitutive functions $\widehat{\mathrm{T}}_{ij}$ and $\widehat{p}$ that we started with have been reduced by the requirements of objectivity and the entropy inequality to 1 independent scalar constitutive function $\widehat{\psi}$.

Note: We stated above, that (7.12) must hold for all $\rho > 0$ and all $\dot{\rho}$. We thus claimed that at any particular spatial and temporal point $(\mathbf{y}_\dagger, t_\dagger)$ the values of ρ and $\dot{\rho}$ can be specified independently and arbitrarily. Consider the uniform velocity field $\mathbf{v}(\mathbf{y}, t) = \mathbf{L}\mathbf{y}$ where $\mathbf{L}$ is an arbitrary constant tensor. Set $\beta = \operatorname{tr}\mathbf{L} = \operatorname{div}\mathbf{v}$; β is also constant. Now consider the spatially uniform mass density field

$$\rho(t) = \alpha \mathrm{e}^{-\beta(t-t_\dagger)}$$

where $\alpha > 0$ is an arbitrary constant. Observe that this mass density satisfies the requirement of mass balance, $\dot{\rho} + \rho \operatorname{div}\mathbf{v} = 0$, and that $\rho(t)$ is positive for all t. Note that $\rho(t_\dagger) = \alpha$ and $\dot{\rho}(t_\dagger) = -\alpha\beta$. Since the constants $\alpha > 0$ and β can be chosen independently and arbitrarily, this shows that the values of $\rho(t_\dagger)$ and $\dot{\rho}(t_\dagger)$ can be chosen independently and arbitrarily as was claimed.

References:

1. P. Chadwick, *Continuum Mechanics: Concise Theory and Problems*, Chapter 4 §§1-3, Dover, 1999.

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3. C. Truesdell, *A First Course in Rational Continuum Mechanics*, Chapter 4, Academic Press, New York 1977.

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Chapter 8

Thermoelastic Materials

In this chapter we introduce and study the constitutive relations for a thermoelastic material, starting in Section 8.1 from their most primitive form. The entropy inequality is used to reduce them in Section 8.2 and material frame indifference is then used in Section 8.3 to reduce them even further. In Section 8.4 we discuss various consequences of the reduced constitutive relations, and in particular specialize the field equations, introduce the specific heat at constant strain, and provide an alternative characterization of the internal energy potential as a function of deformation gradient and entropy. The notion of material symmetry is introduced in Section 8.5 and we specialize the constitutive relations to the isotropic and transversely isotropic cases. In Section 8.6 the constitutive theory is modified to the case of materials that are kinematically constrained (e.g. incompressible materials). Various explicit special constitutive relations are given in Section 8.7, and the linearized theory of thermoelasticity is derived by making the appropriate approximations of smallness.

8.1 Constitutive Characterization in Primitive Form.

An elastic material has no memory of its past. Moreover, the response at a particle $\mathbf{x}$ depends only on the motion of the particles in an infinitesimal neighborhood of it. Thus as far as dependency on the motion goes, the constitutive response functions depend on the motion solely through the current value of the deformation gradient tensor at $\mathbf{x}$: $\mathbf{F}(\mathbf{x}, t)$. For a thermoelastic material we assume an additional dependency on the temperature of the same form: $\theta(\mathbf{x}, t)$. However, we know from simple models of heat transfer, e.g. Fourier's law, that the heat flux depends on the temperature *gradient*. Thus the heat flux response function $\widehat{\mathbf{q}}$ (or $\widehat{\mathbf{q}}_0$) must depend on $\mathrm{Grad}\,\theta$. By the notion of equipresence, we allow all of the constitutive response functions to depend on $\mathrm{Grad}\,\theta$.

Thus a thermoelastic material is characterized by a set of constitutive relations

$$\left.\begin{array}{rcl} \mathbf{T} &=& \widehat{\mathbf{T}}(\mathbf{F},\ \theta,\ \mathrm{Grad}\,\theta), \\ \varepsilon &=& \widehat{\varepsilon}(\mathbf{F},\ \theta,\ \mathrm{Grad}\,\theta), \\ \mathbf{q} &=& \widehat{\mathbf{q}}(\mathbf{F},\ \theta,\ \mathrm{Grad}\,\theta), \\ \eta &=& \widehat{\eta}(\mathbf{F},\ \theta,\ \mathrm{Grad}\,\theta), \end{array}\right\} \quad \text{or equivalently} \quad \left.\begin{array}{rcl} \mathbf{S} &=& \widehat{\mathbf{S}}(\mathbf{F},\ \theta,\ \mathrm{Grad}\,\theta), \\ \psi &=& \widehat{\psi}(\mathbf{F},\ \theta,\ \mathrm{Grad}\,\theta), \\ \mathbf{q}_0 &=& \widehat{\mathbf{q}}_0(\mathbf{F},\ \theta,\ \mathrm{Grad}\,\theta), \\ \eta &=& \widehat{\eta}(\mathbf{F},\ \theta,\ \mathrm{Grad}\,\theta), \end{array}\right\}$$

for the Cauchy stress $\mathbf{T}$, the specific internal energy ε, the true heat flux vector $\mathbf{q}$ and the specific entropy η; or equivalently for the first Piola-Kirchhoff stress $\mathbf{S}$, the specific Helmholtz free-energy ψ, the nominal heat flux vector $\mathbf{q}_0$ and the specific entropy η.

The constitutive response functions for stress, $\widehat{\mathbf{T}}$ and $\widehat{\mathbf{S}}$, are assumed to satisfy the (angular momentum) requirements

$$\widehat{\mathbf{T}}(\mathbf{F},\theta,\mathbf{g}) = \widehat{\mathbf{T}}^T(\mathbf{F},\theta,\mathbf{g}), \qquad \widehat{\mathbf{S}}(\mathbf{F},\theta,\mathbf{g})\mathbf{F}^T \ = \ \mathbf{F}\widehat{\mathbf{S}}^T(\mathbf{F},\theta,\mathbf{g})$$

for all nonsingular $\mathbf{F}$, all $\theta > 0$, and all vectors $\mathbf{g}$.

8.2 Implications of the Entropy Inequality.

It is convenient for our present purposes to consider the referential constitutive characterization

$$\left.\begin{array}{rcl} \mathbf{S} &=& \widehat{\mathbf{S}}(\mathbf{F},\ \theta,\ \mathrm{Grad}\,\theta), \\ \psi &=& \widehat{\psi}(\mathbf{F},\ \theta,\ \mathrm{Grad}\,\theta), \\ \mathbf{q}_0 &=& \widehat{\mathbf{q}}_0(\mathbf{F},\ \theta,\ \mathrm{Grad}\,\theta), \\ \eta &=& \widehat{\eta}(\mathbf{F},\ \theta,\ \mathrm{Grad}\,\theta). \end{array}\right\} \tag{8.1}$$

Consider the entropy inequality written in the form

$$\rho_0\dot{\psi} \ - \ \mathbf{S}\cdot\dot{\mathbf{F}} \ + \ \rho_0\eta\dot{\theta} \ - \ \mathbf{q}_o\cdot\frac{\mathrm{Grad}\,\theta}{\theta} \ \le \ 0. \tag{8.2}$$

Substituting (8.1) into (8.2) and rearranging terms leads to

$$\begin{aligned}
&\Big\{\rho_0\widehat{\psi}_{\mathbf{F}}(\mathbf{F},\theta,\mathbf{g}) - \widehat{\mathbf{S}}(\mathbf{F},\theta,\mathbf{g})\Big\}\cdot\dot{\mathbf{F}} \\
&\quad + \rho_0\left\{\widehat{\psi}_\theta(\mathbf{F},\theta,\mathbf{g}) + \widehat{\eta}(\mathbf{F},\theta,\mathbf{g})\right\}\dot{\theta} \\
&\quad + \rho_0\left\{\widehat{\psi}_{\mathbf{g}}(\mathbf{F},\theta,\mathbf{g})\right\}\cdot\dot{\mathbf{g}} \\
&\quad - \widehat{\mathbf{q}}_0(\mathbf{F},\theta,\mathbf{g})\cdot\mathbf{g}/\theta \qquad\qquad \leq \ 0,
\end{aligned} \tag{8.3}$$

where the subscripts such as $\mathbf{F},\theta$ and $\mathbf{g}$ denote partial differentiation with respect to these quantities. Equation (8.3) must hold in all thermomechanical processes.

The fact that (8.3) must hold in all thermomechanical processes implies that this inequality must hold for *all arbitrarily chosen*[1] tensors $\dot{\mathbf{F}}$, real numbers $\dot{\theta}$ and vectors $\dot{\mathbf{g}}$. *Notice that*

[1]We must show that the values of $\mathbf{F}(\mathbf{x}_0,t_0),\ \dot{\mathbf{F}}(\mathbf{x}_0,t_0),\ \theta(\mathbf{x}_0,t_0),\ \dot{\theta}(\mathbf{x}_0,t_0),\ \mathbf{g}(\mathbf{x}_0,t_0),\ \dot{\mathbf{g}}(\mathbf{x}_0,t_0)$ can be specified arbitrarily at any particle $\mathbf{x}_0$ and instant t_0.

Following Gurtin, Fried and Anand, choose two arbitrary constant tensors $\mathbf{L}_0$ and $\mathbf{F}_0$ with $\det\mathbf{F}_0 > 0$, and consider the motion

$$\mathbf{y}(\mathbf{x},t) = \mathbf{x}_0 + e^{(t-t_0)\mathbf{L}_0}\,\mathbf{F}_0(\mathbf{x}-\mathbf{x}_0)\ .$$

Observe that the deformation gradient tensor (is independent of position $\mathbf{x}$) and is given by

$$\mathbf{F}(t) = e^{(t-t_0)\mathbf{L}_0}\,\mathbf{F}_0;$$

one can show that $\det\mathbf{F}(t) > 0$. The time derivative of $\mathbf{F}$ is

$$\dot{\mathbf{F}}(t) = \mathbf{L}_0\,e^{(t-t_0)\mathbf{L}_0}\,\mathbf{F}_0.$$

Thus

$$\mathbf{F}(\mathbf{x}_0,t_0) = \mathbf{F}_0, \qquad \dot{\mathbf{F}}(\mathbf{x}_0,t_0) = \mathbf{L}_0\mathbf{F}_0.$$

Since $\mathbf{L}_0$ and $\mathbf{F}_0$ were prescribed arbitrarily it follows that $\mathbf{F}(\mathbf{x}_0,t_0),\ \dot{\mathbf{F}}(\mathbf{x}_0,t_0)$ can be specified arbitrarily.

Next choose two arbitrary (scalar) constants $\theta_0,\overline{\theta}_0$ with $\theta_0 > 0$, and two arbitrary constant vectors $\mathbf{g}_0$, $\overline{\mathbf{g}}_0$. Define the scalar-valued function

$$\phi(\mathbf{x},t) = (t-t_0)\,\overline{\theta}_0 + \mathbf{g}_0\cdot(\mathbf{x}-\mathbf{x}_0) + (t-t_0)\,\overline{\mathbf{g}}_0\cdot(\mathbf{x}-\mathbf{x}_0)$$

and consider the temperature field

$$\theta(\mathbf{x},t) = \theta_0\,e^{\phi(\mathbf{x},t)/\theta_0}.$$

Note that $\theta(\mathbf{x},t) > 0$. Then

$$\theta(\mathbf{x}_0,t_0) = \theta_0, \quad \dot{\theta}(\mathbf{x}_0,t_0) = \overline{\theta}_0, \quad \mathbf{g}(\mathbf{x}_0,t_0) = \mathrm{Grad}\,\theta(\mathbf{x}_0,t_0) = \mathbf{g}_0, \quad \dot{\mathbf{g}}(\mathbf{x}_0,t_0) = \mathrm{Grad}\,\dot{\theta}(\mathbf{x}_0,t_0) = \overline{\mathbf{g}}_0.$$

Since θ_0, $\overline{\theta}_0$, $\mathbf{g}_0$ and $\overline{\mathbf{g}}_0$ were prescribed arbitrarily, it follows that $\theta(\mathbf{x}_0,t_0),\ \dot{\theta}(\mathbf{x}_0,t_0),\ \mathbf{g}(\mathbf{x}_0,t_0),\ \dot{\mathbf{g}}(\mathbf{x}_0,t_0)$ can be specified arbitrarily.

the terms within each pair of braces in (8.3), do not involve the quantity immediately outside it. We now exploit this factor.

Pick and fix all of the quantities $\mathbf{F}, \theta, \mathbf{g}, \dot{\theta}$ and $\dot{\mathbf{g}}$, i.e. all of the quantities in (8.3) except for $\dot{\mathbf{F}}$. Then (8.3) has the form

$$\mathbf{A} \cdot \dot{\mathbf{F}} + \beta \leq 0 \text{ for all tensors } \dot{\mathbf{F}}$$

where $\mathbf{A}$ and β are independent of $\dot{\mathbf{F}}$. If we now pick $\dot{\mathbf{F}} = \alpha\mathbf{A}$, this implies that $\alpha(\mathbf{A}\cdot\mathbf{A})+\beta \leq 0$ for all real numbers α which implies that $\mathbf{A} \cdot \mathbf{A} = 0$ and therefore that $\mathbf{A} = \mathbf{0}$, i.e. that

$$\rho_0 \widehat{\psi}_{\mathbf{F}}(\mathbf{F}, \theta, \mathbf{g}) - \widehat{\mathbf{S}}(\mathbf{F}, \theta, \mathbf{g}) = \mathbf{0}.$$

Proceeding in this manner, we conclude from (8.3) that

$$\left.\begin{array}{rcl} \rho_0 \widehat{\psi}_{\mathbf{F}}(\mathbf{F}, \theta, \mathbf{g}) - \widehat{\mathbf{S}}(\mathbf{F}, \theta, \mathbf{g}) & = & \mathbf{0}, \\ \widehat{\psi}_{\theta}(\mathbf{F}, \theta, \mathbf{g}) + \widehat{\eta}(\mathbf{F}, \theta, \mathbf{g}) & = & 0, \\ \widehat{\psi}_{\mathbf{g}}(\mathbf{F}, \theta, \mathbf{g}) & = & \mathbf{0}, \\ -\widehat{\mathbf{q}}_0(\mathbf{F}, \theta, \mathbf{g}) \cdot \mathbf{g} & \leq & 0, \end{array}\right\} \tag{8.4}$$

where the inequality $(8.4)_4$ is what is leftover of (8.3) after one has concluded that $(8.4)_{1,2,3}$ hold. It simply states that the direction of the heat flux vector cannot oppose that of the temperature gradient vector, i.e. heat flows in the direction of high temperature to low temperature.

The third of (8.4) states that the Helmholtz free-energy potential $\widehat{\psi}$ is independent of the temperature gradient $\mathbf{g}$ and therefore from the first and second of (8.4) it follows that the stress and entropy response functions $\widehat{\mathbf{S}}$ and $\widehat{\eta}$ are also independent of $\mathbf{g}$. Thus (8.4) can be further simplified to read

$$\left.\begin{array}{rcl} \psi & = & \widehat{\psi}(\mathbf{F}, \theta), \\ \widehat{\mathbf{S}}(\mathbf{F}, \theta) & = & \rho_0 \widehat{\psi}_{\mathbf{F}}(\mathbf{F}, \theta), \\ \widehat{\eta}(\mathbf{F}, \theta) & = & -\widehat{\psi}_{\theta}(\mathbf{F}, \theta), \\ \widehat{\mathbf{q}}_0(\mathbf{F}, \theta, \mathbf{g}) \cdot \mathbf{g} & \geq & 0. \end{array}\right\} \tag{8.5}$$

Conversely if (8.5) holds then so does (8.3). Thus (8.5) describes the most general thermoelastic material which is consistent with the entropy inequality.

Observe that in order to characterize a thermoelastic material it is only necessary to specify the *two* constitutive response functions $\widehat{\psi}(\mathbf{F},\theta)$ and $\widehat{\mathbf{q}}_0(\mathbf{F},\ \theta,\ \mathrm{Grad}\,\theta)$.

8.3 Implications of Material Frame Indifference.

We now explore the implications of material frame indifference on the constitutive response functions $\widehat{\psi}(\mathbf{F},\theta)$ and $\widehat{\mathbf{q}}_0(\mathbf{F},\ \theta,\ \mathrm{Grad}\,\theta)$.

Throughout this discussion we will be concerned with two processes $\mathbf{y}^*,\theta^*$ and $\mathbf{y},\theta$, which are related to each other by

$$\mathbf{y}^*(\mathbf{x},t) = \mathbf{Q}(t)\mathbf{y}(\mathbf{x},t), \qquad \theta^*(\mathbf{x},t) = \theta(\mathbf{x},t),$$

where $\mathbf{Q}(t)$ is a rotation tensor at each time. We focus attention on a particle p of the body. Let $\mathbf{F}^*,\psi^*,\ \mathbf{q}_0^*$ and $\mathbf{F},\psi,\mathbf{q}_0$ be the deformation gradient tensors, Helmholtz free-energies and heat flux vectors at this one particle p at time t in these two processes. Then the constitutive relation, when applied to these two processes, yields

$$\left.\begin{array}{rcl} \psi &=& \widehat{\psi}(\mathbf{F},\ \theta), \\[2ex] \mathbf{q}_0 &=& \widehat{\mathbf{q}}_0(\mathbf{F},\ \theta,\ \mathrm{Grad}\,\theta), \end{array}\right\} \qquad \left.\begin{array}{rcl} \psi^* &=& \widehat{\psi}(\mathbf{F}^*,\ \theta), \\[2ex] \mathbf{q}_0^* &=& \widehat{\mathbf{q}}_0(\mathbf{F}^*,\ \theta,\ \mathrm{Grad}\,\theta). \end{array}\right\} \tag{8.6}$$

Note that since $\mathrm{Grad}\,\theta$ is the referential spatial gradient of temperature, it is the same in the two motions; $\mathrm{grad}\,\theta$ on the other hand would be different.

From our previous discussion on objectivity, see Sections 3.8 and 5.5, we know that

$$\mathbf{F}^* = \mathbf{Q}\mathbf{F}, \qquad \psi^* = \psi, \quad \text{and} \quad \mathbf{q}_0^* = \mathbf{q}_0. \tag{8.7}$$

Constitutive laws should be independent of the observer. Thus, on using (8.6) and (8.7) we see that the constitutive response functions $\widehat{\psi},\ \widehat{\mathbf{q}}_0$ must be such that

$$\widehat{\psi}(\mathbf{Q}\mathbf{F},\theta) = \widehat{\psi}(\mathbf{F},\theta), \qquad \widehat{\mathbf{q}}_0(\mathbf{Q}\mathbf{F},\theta,\mathbf{g}) = \widehat{\mathbf{q}}_0(\mathbf{F},\theta,\mathbf{g}) \tag{8.8}$$

for all tensors $\mathbf{F}$ with positive determinant, all proper orthogonal tensors $\mathbf{Q}$, all positive real numbers θ and all vectors $\mathbf{g}$.

Equation (8.8) places restrictions of the constitutive response functions $\widehat{\psi},\widehat{\mathbf{q}}_0$. For example the function $\widehat{\psi}(\mathbf{F},\theta) = \mathrm{tr}\,(\mathbf{F}^T\mathbf{F})$ does satisfy $(8.8)_1$ but $\widehat{\psi}(\mathbf{F},\theta,\mathbf{g}) = \mathrm{tr}\,\mathbf{F}$ does not.

Next we turn our attention to determining the most general constitutive response functions $\widehat{\psi}$, $\widehat{\mathbf{q}}_0$ which are consistent with (8.8). Since the temperature and temperature gradient are the same on the two sides of (8.8), they play no role in the present discussion, and so for convenience we shall suppress them in what follows.

First consider the Helmholtz free energy response function $\widehat{\psi}$ which is required to obey

$$\widehat{\psi}(\mathbf{F}) \;=\; \widehat{\psi}(\mathbf{QF}). \tag{8.9}$$

We begin by deriving a necessary condition implied by (8.9). Since (8.9) is to hold for all rotations $\mathbf{Q}$, it must necessarily hold for the particular choice $\mathbf{Q} = \mathbf{R}^T$ where $\mathbf{R}$ is the rotational part in the polar decomposition $\mathbf{F} = \mathbf{RU}$. Then (8.9) implies

$$\widehat{\psi}(\mathbf{F}) = \widehat{\psi}(\mathbf{U}) \tag{8.10}$$

where $\mathbf{U} = \sqrt{(\mathbf{F}^T\mathbf{F})}$. Conversely, let $\widehat{\psi}(\cdot)$ be any real-valued function defined for tensors with positive determinant which obeys (8.10). Then, since $\mathbf{QF} = \mathbf{Q}(\mathbf{RU}) = (\mathbf{QR})\mathbf{U}$, the rotational and right stretch factors in the (unique) polar decomposition of $\mathbf{QF}$ are $\mathbf{QR}$ and $\mathbf{U}$ respectively. Thus when (8.10) holds it follows that

$$\widehat{\psi}(\mathbf{QF}) = \widehat{\psi}(\mathbf{U}) \tag{8.11}$$

for all $\mathbf{F}$ with positive determinant and all rotations $\mathbf{Q}$ where $\mathbf{U} = \sqrt{(\mathbf{F}^T\mathbf{F})}$. Combining (8.11) with (8.10) shows that (8.9) holds. Thus, it follows that (8.10) is necessary and sufficient for (8.9) to hold.

Remark: Intuitively, we would have expected the free energy to not depend on the rotation, and therefore that it should depend on $\mathbf{F}$ only through the stretch. However it may not have been immediately obvious as to which of the two stretches $\mathbf{U}$ and $\mathbf{V}$ would be appropriate. The above analysis shows that it is $\mathbf{U}$. Intuitively, we could have guessed this too, since frame indifference states that a post-rotation doesn't affect the energy; it does not refer to a pre-rotation (which as we shall discuss later is related to the symmetry of the material, and which would, in general, affect the energy).

We turn next to the referential heat flux response function $\widehat{\mathbf{q}}_0$ which according to (8.8) must be such that

$$\widehat{\mathbf{q}}_0(\mathbf{F}) \;=\; \widehat{\mathbf{q}}_0(\mathbf{QF}) \tag{8.12}$$

for all $\mathbf{F}$ with positive determinant and all rotations $\mathbf{Q}$. An analysis entirely analogous to the preceding shows that a necessary and sufficient condition for (8.12) to hold is that

$$\widehat{\mathbf{q}}_0(\mathbf{F}) = \widehat{\mathbf{q}}_0(\mathbf{U}) \tag{8.13}$$

where $\mathbf{U} = \sqrt{(\mathbf{F}^T\mathbf{F})}$.

Therefore *the most general set of constitutive response functions for an elastic material that is both frame-indifferent and consistent with the second law of thermodynamics* is as follows:

$$\left.\begin{aligned} \psi &= \widehat{\psi}(\mathbf{U},\, \theta), \\ \mathbf{q}_0 &= \widehat{\mathbf{q}}_0(\mathbf{U},\, \theta,\, \mathrm{Grad}\,\theta). \end{aligned}\right\}$$

Since $\mathbf{C} = \mathbf{U}^2$, we can define a function $\tilde{\psi}$ by $\tilde{\psi}(\mathbf{C}, \theta) = \widehat{\psi}(\sqrt{\mathbf{C}}, \theta)$. Since $\mathbf{C} = \mathbf{F}^T\mathbf{F}$, it follows that

$$\frac{\partial C_{pq}}{\partial F_{ij}} = F_{iq}\delta_{pj} + F_{ip}\delta_{qj}$$

so that by the chain rule and $\mathbf{S} = \rho_0 \partial\widehat{\psi}/\partial\mathbf{F}$, we get

$$\mathbf{S} = 2\rho_0\, \mathbf{F}\tilde{\psi}_{\mathbf{C}}(\mathbf{C}, \theta).$$

Thus in summary, a thermoelastic material is characterized by the following set of constitutive relations:

$$\left.\begin{aligned} \psi &= \tilde{\psi}(\mathbf{C}, \theta), \\ \mathbf{S} &= 2\rho_0\, \mathbf{F}\, \tilde{\psi}_{\mathbf{C}}(\mathbf{C}, \theta), \\ \eta &= -\tilde{\psi}_{\theta}(\mathbf{C}, \theta), \\ \mathbf{q}_0 &= \tilde{\mathbf{q}}_0(\mathbf{C}, \theta, \mathrm{Grad}\,\theta), \end{aligned}\right\} \tag{8.14}$$

where $\tilde{\mathbf{q}}_0$ is subject to the inequality

$$\tilde{\mathbf{q}}_0(\mathbf{C}, \theta, \mathbf{g}) \cdot \mathbf{g} \geq 0. \tag{8.15}$$

8.4 Discussion.

Remark 1: The Helmholtz free-energy function $\tilde{\psi}(\mathbf{C}, \theta)$ completely characterizes the response of a thermoelastic material (with the exception of its heat transfer characteristics). Once $\tilde{\psi}(\mathbf{C}, \theta)$ has been determined for a particular material, the 1^{st} Piola-Kirchhoff stress $\mathbf{S}$ and the entropy η can be calculated from (8.14). The Cauchy stress $\mathbf{T} = J^{-1}\mathbf{S}\mathbf{F}^T$ and the internal energy $\varepsilon = \psi + \eta\theta$ are given by

$$\tilde{\mathbf{T}}(\mathbf{C}, \theta) = 2\rho\, \mathbf{F}\, \tilde{\psi}_{\mathbf{C}}(\mathbf{C}, \theta)\, \mathbf{F}^T, \qquad \tilde{\varepsilon}(\mathbf{C}, \theta) = \tilde{\psi}(\mathbf{C}, \theta) - \theta\, \tilde{\psi}_{\theta}(\mathbf{C}, \theta). \tag{8.16}$$

It can be verified that if, instead of the free energy $\tilde{\psi}(\mathbf{C},\theta)$, the internal energy $\tilde{\varepsilon}(\mathbf{C},\theta)$ (alone) is known, it is *not* possible to determine the stress and entropy from it without additional information.

Remark 2: The balance of *angular momentum* requires that $\mathbf{S}\mathbf{F}^T = \mathbf{F}\mathbf{S}^T$. In principle, this imposes a restriction on the constitutive response function $\widehat{\mathbf{S}}$. However since $\tilde{\psi}_{\mathbf{C}}$ is a symmetric tensor, the stress that results from the constitutive equation $\mathbf{S} = 2\rho_0\,\mathbf{F}\,\tilde{\psi}_{\mathbf{C}}(\mathbf{C},\theta)$ automatically satisfies the angular momentum requirement. Thus, for a thermoelastic material the two laws of thermodynamics and the principle of material frame indifference imply the angular momentum principle!

Remark 3: We now return to the field equations. The material time derivative of the internal energy can be written as follows:

$$\begin{aligned}\rho_o\dot{\varepsilon} &= \rho_o(\psi+\eta\theta)^{\cdot} = \rho_o\left(\frac{\partial\psi}{\partial\mathbf{F}}\cdot\dot{\mathbf{F}} + \frac{\partial\psi}{\partial\theta}\,\dot{\theta} + \dot{\eta}\theta + \eta\dot{\theta}\right)\\ &= \mathbf{S}\cdot\dot{\mathbf{F}} + \rho_o\left(-\eta\dot{\theta} + \dot{\eta}\theta + \eta\dot{\theta}\right) = \mathbf{S}\cdot\dot{\mathbf{F}} + \rho_o\dot{\eta}\theta\end{aligned}$$

where we have made use of (8.5). Therefore the energy equation $\mathbf{S}\cdot\dot{\mathbf{F}} + \mathrm{Div}\,\mathbf{q}_0 + \rho_0 r = \rho_0\dot{\varepsilon}$ reduces to

$$\mathrm{Div}\,\mathbf{q}_0 + \rho_0 r = \rho_0\theta\dot{\eta}. \tag{8.17}$$

This together with the equation of motion

$$\mathrm{Div}\,\mathbf{S} + \rho_0\mathbf{b} = \rho_0\dot{\mathbf{v}} \tag{8.18}$$

form the *basic field equations* which are to be satisfied in any thermomechanical process for a thermoelastic material. Even though these equations appear to have decoupled into a thermal one and a mechanical one, the motion $\mathbf{y}(\mathbf{x},t)$ and the temperature field $\theta(\mathbf{x},t)$ are coupled through the constitutive relations (8.14).

Observe from the energy equation (8.17) that if the process happens to be *adiabatic*, i.e. if $\mathbf{q}_0 = \mathbf{0}$ and $r = 0$ at all $\mathbf{x}$ and t, (which may, for instance, be a good approximation for dynamic processes where the inertial time scale is much smaller than the thermal time scale) then the energy equation simplifies further to $\dot{\eta} = 0$ and so η is independent of time at each particle. An adiabatic process in a thermoelastic material is therefore necessarily *isentropic*.

Remark 4: A particular example of the heat conduction law $\mathbf{q}_o = \tilde{\mathbf{q}}_o(\mathbf{C},\theta,\mathbf{g})$ is given by the generalized Fourier law

$$\tilde{\mathbf{q}}_0(\mathbf{C},\theta,\mathbf{g}) = \mathbf{K}(\mathbf{C},\theta)\mathbf{g}$$

where the second order tensor $\mathbf{K}$ is known as the conductivity tensor. The entropy inequality (8.15) requires that $\mathbf{K}$ be positive semi-definite.

<u>*Remark 5*</u>: Let Γ denote the rate of entropy production at time t in a (material) subregion $\mathcal{D}_0 \subset \mathcal{R}_0$:

$$\Gamma = \frac{d}{dt}\int_{D_0} \rho_o \eta dV_x - \int_{D_0} \frac{\rho_o r}{\theta} dV_x - \int_{\partial D_o} \frac{\mathbf{q}_o \cdot \mathbf{n}_o}{\theta} dA_x.$$

By using (8.17) this can be simplified to

$$\Gamma = \int_{D_0} \rho_o \gamma \; dV_x \qquad \text{where} \qquad \gamma = \frac{\mathbf{q}_0 \cdot \text{Grad}\,\theta}{\rho_0^2\theta^2} \;\geq\; 0; \tag{8.19}$$

here γ represents the (local) entropy production rate per unit mass and the inequality in (8.19) arises due to the second law. Thus the production of entropy is due solely to the flux of heat.

<u>*Remark 6*</u>: When Grad $\theta = 0$ at a particle one says that that particle is in *"thermal equilibrium"*. The entropy production γ at a particle vanishes if it is in thermal equilibrium.

In order to further examine the entropy inequality, for each $\mathbf{C}$ and θ, define a function

$$f(\mathbf{g}) = \tilde{\mathbf{q}}_0(\mathbf{C}, \theta, \mathbf{g}) \cdot \mathbf{g} \tag{8.20}$$

for all vectors $\mathbf{g}$. Observe that

$$\left.\begin{array}{rcll} f(\mathbf{o}) & = & 0 & \text{and} \\ f(\mathbf{g}) & \geq & 0 & \text{for all vectors } \mathbf{g} \end{array}\right\} \tag{8.21}$$

where the inequality here follows from the entropy inequality (8.15). It follows from (8.21) that f has a minimum at $\mathbf{g} = \mathbf{o}$ and therefore

$$\left.\frac{\partial f}{\partial \mathbf{g}}\right|_{\mathbf{g}=\mathbf{o}} = \mathbf{o}.$$

By evaluating $\partial f/\partial \mathbf{g}$ using (8.20) and then setting $\mathbf{g} = \mathbf{o}$ gives

$$\mathbf{q}_0(\mathbf{C}, \theta, \mathbf{0}) = \mathbf{o}. \tag{8.22}$$

Equation (8.22) says that when the temperature gradient vanishes, so does the heat flux vector. Thus when a particle is in thermal equilibrium, the heat flux vector at it necessarily vanishes.

<u>*Remark 7*</u>: Since $\varepsilon = \psi + \eta\theta$, the internal energy can be expressed as a function of $\mathbf{C}$ and θ by

$$\tilde{\varepsilon}(\mathbf{C}, \theta) = \tilde{\psi}(\mathbf{C}, \theta) + \tilde{\eta}(\mathbf{C}, \theta)\,\theta. \tag{8.23}$$

Define the quantity $c(\mathbf{C}, \theta)$ by

$$c(\mathbf{C}, \theta) = \frac{\partial \tilde{\varepsilon}(\mathbf{C}, \theta)}{\partial \theta} \tag{8.24}$$

which represents the change of specific internal energy with respect to change of temperature at constant strain; we see below that this is a specific heat of the material. By using (8.23) and $(8.14)_3$ in (8.24), we can express c alternatively as

$$c = \theta \frac{\partial \tilde{\eta}(\mathbf{C}, \theta)}{\partial \theta} = -\theta \frac{\partial^2 \tilde{\psi}(\mathbf{C}, \theta)}{\partial \theta^2}. \tag{8.25}$$

By using (8.14) and (8.25), the energy equation (8.17) can now be written as

$$\text{Div}\,\mathbf{q}_0 + \rho_0 r = \rho_o c\dot{\theta} - \rho_0 \theta\, \frac{\partial^2 \tilde{\psi}(\mathbf{C}, \theta)}{\partial \theta \partial \mathbf{C}} \cdot \dot{\mathbf{C}}.$$

Observe that in the case of a special process in which the stretching of the body remains constant, i.e. $\dot{\mathbf{C}} = \mathbf{O}$, this reduces to the classical "heat equation"

$$\text{Div}\,\mathbf{q}_0 + \rho_0 r = \rho_o c\dot{\theta}.$$

This shows that c is the *specific heat per unit mass at constant strain.*

<u>*Remark 8*</u>: Assume that the specific heat at constant strain is positive: $c(\mathbf{C}, \theta) > 0$ for all symmetric positive definite tensors $\mathbf{C}$ and all positive numbers θ. Then it follows from $c = \partial\tilde{\varepsilon}/\partial\theta = \theta\partial\tilde{\eta}/\partial\theta$ that

$$\frac{\partial \tilde{\eta}(\mathbf{C}, \theta)}{\partial \theta} > 0.$$

Thus, the relation $\eta = \tilde{\eta}(\mathbf{C}, \theta)$ is invertible at each fixed $\mathbf{C}$ and leads to the inverse relation $\theta = \bar{\theta}(\mathbf{C}, \eta)$. We can now use this to swap θ for η in $\tilde{\varepsilon}(\mathbf{C}, \theta)$ thereby obtaining another internal energy potential $\bar{\varepsilon}(\mathbf{C}, \eta)$ by

$$\bar{\varepsilon}(\mathbf{C}, \eta) = \tilde{\varepsilon}\left(\mathbf{C}, \bar{\theta}(\mathbf{C}, \eta)\right) = \tilde{\psi}(\mathbf{C}, \bar{\theta}(\mathbf{C}, \eta)) + \eta\bar{\theta}(\mathbf{C}, \eta).$$

Differentiating this with respect to $\mathbf{C}$ and η shows that

$$\frac{\partial \bar{\varepsilon}(\mathbf{C}, \eta)}{\partial \mathbf{C}} = \frac{\partial \tilde{\psi}(\mathbf{C}, \theta)}{\partial \mathbf{C}},$$

$$\frac{\partial \bar{\varepsilon}(\mathbf{C}, \eta)}{\partial \eta} = \theta,$$

with $\theta = \bar{\theta}(\mathbf{C}, \eta)$. Thus the constitutive relations for a thermoelastic material can be written alternatively, and equivalently, as

$$\mathbf{S} \;=\; 2\rho_0\,\mathbf{F}\,\frac{\partial \bar{\varepsilon}(\mathbf{C},\eta)}{\partial \mathbf{C}} \;=\; \rho_0\,\frac{\partial \bar{\varepsilon}(\mathbf{C},\eta)}{\partial \mathbf{F}}, \qquad \theta = \frac{\partial \bar{\varepsilon}(\mathbf{C},\eta)}{\partial \eta}. \tag{8.26}$$

While both forms of the constitutive relationships are always valid, (8.14) is particularly convenient if the process happens to be isothermal, while (8.26) is more convenient to use if the process is isentropic.

It is worth noting that once $\bar{\varepsilon}(\mathbf{C}, \eta)$ has been determined for a particular material, all other thermoelastic characteristics of that material (e.g. $\bar{\mathbf{T}}$, $\bar{\mathbf{S}}$, $\bar{\theta}$ and $\bar{\psi}$) can be calculated. If instead the Helmholtz free-energy function $\bar{\psi}(\mathbf{C}, \eta)$ was known, it is *not* possible to determine all of the other thermomechanical characteristics of the material without additional information. Thus $\mathbf{C}, \eta$ are the "natural variables" for the internal energy potential while $\mathbf{C}, \theta$ are the natural variables for the Helmholtz free energy potential.

8.5 Material Symmetry.

The Cauchy stress in some current configuration does not depend on the choice of reference configuration. In fact, recall from Section 4 that our discussion of the (true) Cauchy stress was carried out without any mention of a reference configuration.

However the deformation gradient tensor $\mathbf{F}$ enters into the constitutive relation[2] $\mathbf{T} = \widehat{\mathbf{T}}(\mathbf{F})$, and the deformation gradient tensor does depend on the choice of reference configuration. Since $\mathbf{F}$ depends on the choice of reference configuration while $\mathbf{T}$ does not, it is necessary that the stress response function $\widehat{\mathbf{T}}$ must also depend on the reference configuration; and the way in which $\mathbf{F}$ and $\widehat{\mathbf{T}}$ depend on the reference configuration must be such that $\widehat{\mathbf{T}}(\mathbf{F})$ is independent of reference configuration.

To explore this further, let χ_1 and χ_2 be two reference configurations and let $\mathbf{F}_1$ and $\mathbf{F}_2$ be the deformation gradient tensors in the current configuration relative to these two reference configurations (at some particle p). Let $\widehat{\mathbf{T}}_1$ and $\widehat{\mathbf{T}}_2$ be the two stress response functions associated with these two reference configurations. Since the Cauchy stress in the current configuration is given by both $\mathbf{T} = \widehat{\mathbf{T}}_1(\mathbf{F}_1)$ and $\mathbf{T} = \widehat{\mathbf{T}}_2(\mathbf{F}_2)$, we must have

$$\widehat{\mathbf{T}}_2(\mathbf{F}_2) = \widehat{\mathbf{T}}_1(\mathbf{F}_1).$$

[2]Since the temperature θ plays no role in the present discussion we suppress it.

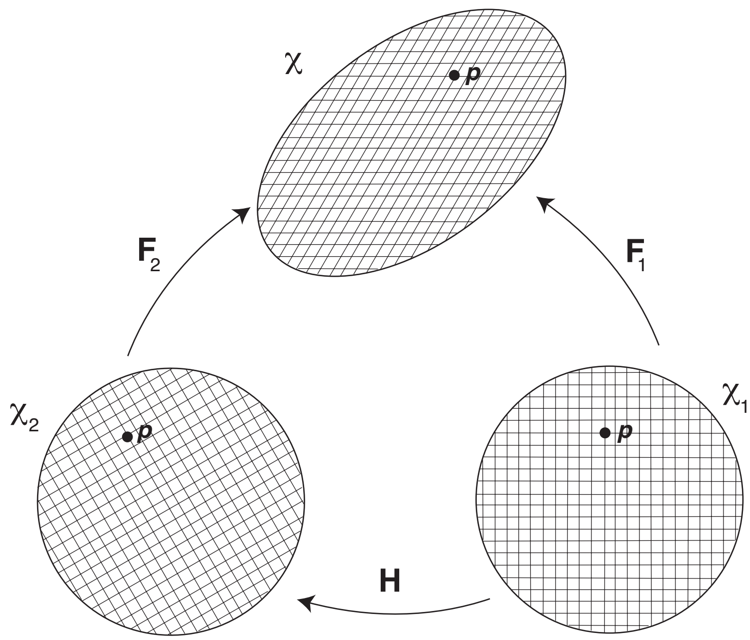

Figure 8.1: A sketch of the regions occupied by a body in a (current) configuration χ and two reference configurations χ_1 and χ_2. The deformation gradient tensors from $\chi_1 \to \chi$ and $\chi_2 \to \chi$ are $\mathbf{F}_1$ and $\mathbf{F}_2$. The stress at the particle p in the deformed configuration is $\mathbf{T}$ and is independent of the choice of reference configuration.

Let $\mathbf{H}$ be the gradient of the mapping (at p) from the first reference configuration to the second. Then $\mathbf{F}_1 = \mathbf{F}_2\mathbf{H}$ and so we must have

$$\widehat{\mathbf{T}}_2(\mathbf{F}) = \widehat{\mathbf{T}}_1(\mathbf{FH}) \qquad \text{for all nonsingular } \mathbf{F}.$$

Thus if we know the stress response function $\widehat{\mathbf{T}}_1$ associated with one reference configuration, and the mapping $\mathbf{H}$ from it to another reference configuration, the stress response function in the second reference configuration can be found from the preceding equation.

If the two reference configurations happen to be such that

$$\widehat{\mathbf{T}}_1(\mathbf{F}) = \widehat{\mathbf{T}}_2(\mathbf{F}) \qquad \text{for all nonsingular } \mathbf{F},$$

then these two reference configurations have the same stress response functions; see Figure 8.2. Consider the set of all tensors $\mathbf{H}$ that take χ_1 to a configuration with the identical stress response function. This set is

$$\{\mathbf{H} : \ \det \mathbf{H} \neq 0, \ \widehat{\mathbf{T}}(\mathbf{F}) = \widehat{\mathbf{T}}(\mathbf{FH}) \text{ for all nonsingular } \mathbf{F}\}.$$

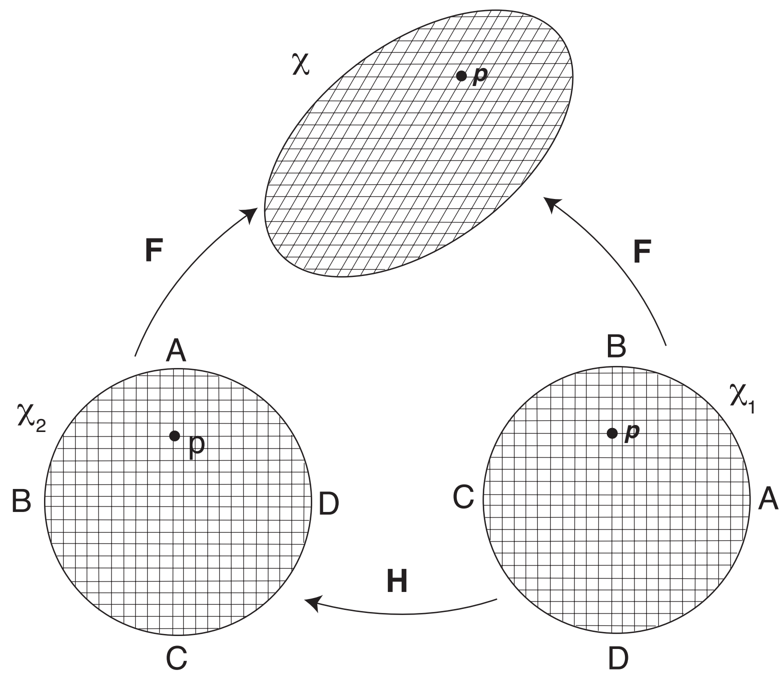

Figure 8.2: A sketch of the regions occupied by a body in a (deformed) configuration χ and two reference configurations χ_1 and χ_2. The locations of 4 material points A, B, C, D in the two reference configuration are shown. Note the symmetry between the reference configurations χ_1 and χ_2 even though they are distinct. The transformation $\mathbf{H}$ from $\chi_1 \to \chi_2$ preserves the symmetry of the material.

This is the set of transformations of the reference configuration that leave the stress response function unchanged.

Next recall that the mass densities, ρ_1 and ρ_2 in the two reference configurations are related by

$$\rho_2 = \rho_1 |\det \mathbf{H}|.$$

The mass densities in the two reference configurations are identical if $|\det \mathbf{H}| = 1$.

We speak of two configurations as being mechanically indistinguishable from each other if their mass densities and stress response functions coincide. Thus in order to study mechanically indistinguishable configurations we must restrict attention to tensors $\mathbf{H}$ whose determinant is ± 1. (If $\det \mathbf{H} = -1$ such a tensor could not be the deformation gradient of an actual deformation since $\det \mathbf{H} \not> 0$. However we admit such tensors in our discussion of symmetry since we wish to include reflection tensors in the set of permissible symmetry transformations.) A tensor with determinant equal to ± 1 is said to be *unimodular*. In view of the two preceding observations we consider the set of tensors

$$\mathcal{G} = \{\mathbf{H} : \ \det \mathbf{H} = \pm 1, \ \widehat{\mathbf{T}}(\mathbf{F}) = \widehat{\mathbf{T}}(\mathbf{F}\mathbf{H}) \text{ for all nonsingular } \mathbf{F}\}. \tag{8.27}$$

The set $\mathcal{G}$ characterizes the set of all configurations that are mechanically indistinguishable from χ_1. It is called the *material symmetry group* of the configuration χ_1.

Note that *symmetry is a property of a configuration.* In general, the same body will have different symmetries in different configurations. Symmetry transformations are the particular transformations that leave the "material microstructure" invariant.

One can show from (8.27) that *(a)* if $\mathbf{H}_1 \in \mathcal{G}$ and $\mathbf{H}_2 \in \mathcal{G}$ then $\mathbf{H}_1\mathbf{H}_2 \in \mathcal{G}$; and *(b)* if $\mathbf{H} \in \mathcal{G}$ then $\mathbf{H}^{-1} \in \mathcal{G}$. Therefore $\mathcal{G}$ is a group[3].

We could alternatively have carried out the discussion of symmetry based on the free energy potential $\widehat{\psi}(\mathbf{F})$, in which case the set of tensors

$$\mathcal{G} = \{\mathbf{H} : \ \det \mathbf{H} = \pm 1, \ \widehat{\psi}(\mathbf{F}) = \widehat{\psi}(\mathbf{F}\mathbf{H}) \text{ for all nonsingular } \mathbf{F}\} \tag{8.28}$$

would be said to describe the symmetry of a configuration. We leave it as an exercise to the reader to explore the relation between these two definitions, keeping in mind the relation $\widehat{\mathbf{T}}(\mathbf{F}) = \rho_0 J^{-1} \widehat{\psi}_{\mathbf{F}}(\mathbf{F})\, \mathbf{F}^{\mathrm{T}}$ between the two constitutive response functions $\widehat{\mathbf{T}}$ and $\widehat{\psi}$.

The set of all tensors whose determinant is ± 1 is a group $\mathcal{U}$ referred to as the *unimodular group*. It follows from (8.27) that the material symmetry group $\mathcal{G}$ is a subset of the unimodular group:

$$\mathcal{G} \subset \mathcal{U}. \tag{8.29}$$

In principle, the material symmetry group could coincide with any subgroup of the unimodular group. Two particular subgroups of $\mathcal{U}$ are the set of orthogonal tensors $\mathcal{O}$, and the set of proper orthogonal tensors $\mathcal{O}^+$; see Chapter 4 of Volume I. The set of all rotations about a fixed axis is a subgroup of $\mathcal{O}^+$ and is therefore another potential material symmetry group. Frequently, the material symmetry group is composed of certain rotations and/or reflections. One can show using material symmetry, together with material frame indifference, that a necessary and sufficient condition for an orthogonal tensor $\mathbf{Q}$ to be in $\mathcal{G}$ is for

$$\widehat{\mathbf{T}}(\mathbf{Q}\mathbf{F}\mathbf{Q}^T) \ = \ \mathbf{Q}\widehat{\mathbf{T}}(\mathbf{F})\mathbf{Q}^T \qquad \text{for all nonsingular tensors } \mathbf{F}; \tag{8.30}$$

or alternatively in terms of the free energy

$$\widehat{\psi}(\mathbf{Q}\mathbf{F}\mathbf{Q}^T) \ = \ \widehat{\psi}(\mathbf{F}) \qquad \text{for all nonsingular tensors } \mathbf{F}. \tag{8.31}$$

[3] See Chapter 4 of Volume I.

Remark: As emphasized above, symmetry is a property of a configuration of a body. However when there is no chance for confusion, we shall use the (imprecise but convenient and common) terminology that attributes symmetry to the material, e.g. we shall speak of an "isotropic material" when we mean that we consider a configuration of a body in which it is isotropic, and so on.

8.5.1 Some Examples of Material Symmetry Groups.

Example: The largest possible material symmetry group is the unimodular group itself: $\mathcal{G} = \mathcal{U}$. If the material symmetry group of a certain configuration of a body is the unimodular group, one can show that

$$\widehat{\mathbf{T}}(\mathbf{F}) = f(\det \mathbf{F})\mathbf{I};$$

i.e. the Cauchy stress is hydrostatic and depends on the deformation only through the Jacobian $\det \mathbf{F}$. Thus the body responds as a *fluid*. As shown in one of the exercises, if the material symmetry group in one configuration is $\mathcal{U}$, then it is $\mathcal{U}$ in every configuration. Thus if the body behaves as a fluid in one configuration it behaves as a fluid in all configurations.

Example: Let $\mathbf{R}_{\mathbf{e}}^{\phi}$ denote the rotation through an angle ϕ about an axis $\mathbf{e}$. The set of all rotations about a fixed axis $\mathbf{e}$, $\mathbf{R}_{\mathbf{e}}^{\phi}$ with $0 \leq \phi \leq 2\pi$, is a group. If the material symmetry group coincides with this group we say that the body is *transversely isotropic* about the axis $\mathbf{e}$ in this reference configuration.

Example: If the material symmetry group is generated[4] by $\pi/2$ rotations $\mathbf{R}_{\mathbf{i}}^{\pi/2}, \mathbf{R}_{\mathbf{j}}^{\pi/2}, \mathbf{R}_{\mathbf{k}}^{\pi/2}$ about the three orthonormal vector $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, the material is said to be *cubic*[5] in this reference configuration.

Example: If $\mathcal{G} = \mathcal{O}$ we say that the body is *isotropic* in this reference configuration.

[4] A set of tensors $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_N$ is said to generate a group if products of their powers exhaust the group.

[5] The converse is not true. If a material is cubic it does not have to be generated by the rotations listed here.

8.5.2 Imposing Symmetry Requirements on Constitutive Response Functions.

Thus far we have simply learned how to characterize the symmetry of a material. We now examine how this information helps further reduce the constitutive relations. We shall address this issue in two cases. Green and Adkins have studied this question in great detail; see also Spencer.

8.5.2.1 An isotropic material. If a configuration is *isotropic* then $\widehat{\psi}(\mathbf{QFQ}^T) = \widehat{\psi}(\mathbf{F})$ for all orthogonal tensors $\mathbf{Q}$. In this event one can show that[6] $\widehat{\psi}$ has the representation

$$\widehat{\psi}(\mathbf{F}) = \psi\big(I_1(\mathbf{C}), I_2(\mathbf{C}), I_3(\mathbf{C})\big) \tag{8.32}$$

where

$$I_1(\mathbf{C}) = \operatorname{tr}\mathbf{C}, \qquad I_2(\mathbf{C}) = \frac{1}{2}\big[(\operatorname{tr}\mathbf{C})^2 - \operatorname{tr}(\mathbf{C}^2)\big], \qquad I_3(\mathbf{C}) = \det\,\mathbf{C} = J^2, \tag{8.33}$$

are the principal scalar invariants of the right Cauchy Green tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2$. (Recall that the eigenvalues and principal scalar invariants of $\mathbf{C}$ and $\mathbf{B} = \mathbf{F}\mathbf{F}^T = \mathbf{V}^2$ coincide.) On differentiating (8.33) with respect to $\mathbf{C}$ we find we find

$$\frac{\partial I_1}{\partial \mathbf{C}} = \mathbf{I}, \qquad \frac{\partial I_2}{\partial \mathbf{C}} = I_1\,\mathbf{I} - \mathbf{C} \qquad \frac{\partial I_3}{\partial \mathbf{C}} = J^2\mathbf{C}^{-1}. \tag{8.34}$$

In order to determine the stress tensor $\mathbf{S}$, we may use $\mathbf{S} = 2\rho_0\mathbf{F}\widehat{\psi}_{\mathbf{C}}$; and since $\mathbf{T} = J^{-1}\mathbf{S}\mathbf{F}^T$ it follows that $\mathbf{T} = 2\rho_0 J^{-1}\mathbf{F}\widehat{\psi}_{\mathbf{C}}\mathbf{F}^T$ which can be used to find $\mathbf{T}$. Using these, together with the chain rule and (8.34), lead to the following constitutive relations for an elastic material with respect to an isotropic reference configuration:

$$\left.\begin{aligned} \rho_0^{-1}\mathbf{T} &= 2J\frac{\partial\psi}{\partial I_3}\mathbf{I} \;+\; \frac{2}{J}\left[\frac{\partial\psi}{\partial I_1} + I_1\frac{\partial\psi}{\partial I_2}\right]\mathbf{B} \;-\; \frac{2}{J}\frac{\partial\psi}{\partial I_2}\mathbf{B}^2, \\ \rho_0^{-1}\mathbf{S} &= 2I_3\frac{\partial\psi}{\partial I_3}\mathbf{F}^{-T} \;+\; 2\left[\frac{\partial\psi}{\partial I_1} + I_1\frac{\partial\psi}{\partial I_2}\right]\mathbf{F} \;-\; 2\frac{\partial\psi}{\partial I_2}\mathbf{B}\mathbf{F}. \end{aligned}\right\} \tag{8.35}$$

Note that for an isotropic material the principal directions of $\mathbf{T}$ and $\mathbf{B}$ coincide.

<u>*Remark*</u>: For an elastic material, we frequently refer to its *strain energy function* $W(\mathbf{F})$. This is related to the Helmholtz free energy function by

$$W(\mathbf{F}) = \rho_0\widehat{\psi}(\mathbf{F}) \tag{8.36}$$

[6] See for example Chapter 4 of Volume I.

so that W represents the free energy per unit referential volume.

Remark: In terms of the principal stretches $\lambda_1, \lambda_2, \lambda_3$ one can write the principal scalar invariants of $\mathbf{C}$ (or $\mathbf{B}$) as

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \qquad I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2, \qquad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{8.37}$$

For an isotropic material, since the strain energy function can be written as $W(I_1, I_2, I_3)$, it follows that it can equivalently be written in the form

$$W = \widehat{W}(\lambda_1, \lambda_2, \lambda_3) \tag{8.38}$$

where the constitutive response function $\widehat{W}$ remains invariant if any two of its arguments are switched:

$$\widehat{W}(\lambda_1, \lambda_2, \lambda_3) = \widehat{W}(\lambda_2, \lambda_1, \lambda_3) = \widehat{W}(\lambda_1, \lambda_3, \lambda_2) = \ldots. \tag{8.39}$$

Remark: Let λ_i^2 and $\mathbf{r}_i$ be an eigenvalue and corresponding eigenvector of $\mathbf{C}$. One can show by differentiating $\mathbf{C}\mathbf{r}_i = \lambda_i^2\mathbf{r}_i$ (no sum on i) and using the fact that $\mathbf{r}_i$ is a unit vector that

$$\frac{\partial \lambda_i}{\partial \mathbf{C}} = \frac{1}{2\lambda_i}\mathbf{r}_i \otimes \mathbf{r}_i \qquad \text{(no sum on } i). \tag{8.40}$$

The constitutive relation for the Cauchy stress can now be rewritten by changing $\widetilde{W}(\mathbf{C})$ to $\widehat{W}(\lambda_1, \lambda_2, \lambda_3)$ in $(8.16)_1$, using the chain rule, (8.40) and $\mathbf{F}\mathbf{r}_i = \lambda_i \boldsymbol{\ell}_i$. This leads to

$$\mathbf{T} = \sum_{i=1}^{3} \frac{\lambda_i}{J}\frac{\partial \widehat{W}}{\partial \lambda_i}\, \boldsymbol{\ell}_i \otimes \boldsymbol{\ell}_i, \tag{8.41}$$

showing that the principal components of the Cauchy stress are

$$T_{11} = \frac{\lambda_1}{\lambda_1\lambda_2\lambda_3}\frac{\partial \widehat{W}}{\partial \lambda_1}, \qquad T_{22} = \frac{\lambda_2}{\lambda_1\lambda_2\lambda_3}\frac{\partial \widehat{W}}{\partial \lambda_2}, \qquad T_{33} = \frac{\lambda_3}{\lambda_1\lambda_2\lambda_3}\frac{\partial \widehat{W}}{\partial \lambda_3}. \tag{8.42}$$

Thus given $\mathbf{F}$, one can find the principal values of $\mathbf{T}$ from (8.42) and the principal directions of $\mathbf{T}$ by finding the principal directions of $\mathbf{B} = \mathbf{F}\mathbf{F}^T$:

Remark: Since the 1st Piola-Kirchhoff stress tensor $\mathbf{S}$ is not symmetric in general, it may not have principal values. However a calculation just like the one above but now applied to $(8.14)_2$ allows us to write the constitutive relation for $\mathbf{S}$ as

$$\mathbf{S} = \sum_{k=1}^{3} \frac{\partial \widehat{W}}{\partial \lambda_k}\, \boldsymbol{\ell}_k \otimes \mathbf{r}_k. \tag{8.43}$$

where $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ and $\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3$ are the eigenvectors of the right and left stretch tensors $\mathbf{U}$ and $\mathbf{V}$ respectively.

Remark: If the deformation is such that $\mathbf{F} = \lambda_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_3 \mathbf{e}_3 \otimes \mathbf{e}_3$, then the bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ and $\{\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3\}$ all coincide and

$$\mathbf{S} = S_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + S_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + S_3 \mathbf{e}_3 \otimes \mathbf{e}_3 \qquad \text{where } S_i = \frac{\partial \widehat{W}}{\partial \lambda_i}.$$

If one considers spherically symmetric problems for isotropic materials, one finds that the bases $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_Z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ coincide, that they are the principal bases for $\mathbf{B}$ and $\mathbf{T}$, and that $[F]$ and $[S]$ are diagonal in these bases.

Remark: Perhaps it is worth mentioning that if one is to conduct laboratory experiments to find $\widehat{W}$, it is necessary to carry out experiments that probe various parts of $\lambda_1, \lambda_2, \lambda_3$-space and not simply a hundred uniaxial tension tests which would only probe a single path (many times) in this space.

8.5.2.2 A transversely isotropic material. If a configuration is transversely isotropic about an axis $\mathbf{e}$ then $\widehat{\psi}(\mathbf{QFQ}^T) = \widehat{\psi}(\mathbf{F})$ for all rotation tensors $\mathbf{Q}$ about the axis $\mathbf{e}$. In this event one can show that[7] $\widehat{\psi}$ has the representation

$$\widehat{\psi}(\mathbf{F}) = \psi(I_1(\mathbf{C}),\ I_2(\mathbf{C}),\ I_3(\mathbf{C}),\ I_4(\mathbf{C}, \mathbf{e}),\ I_5(\mathbf{C}, \mathbf{e})) \tag{8.44}$$

where $I_1(\mathbf{C}), I_2(\mathbf{C})$ and $I_3(\mathbf{C})$ continue to be given by (8.33) and

$$I_4(\mathbf{C}, \mathbf{e}) = \operatorname{tr}\Big[\mathbf{C}^2(\mathbf{e} \otimes \mathbf{e})\Big], \qquad\qquad I_5(\mathbf{C}, \mathbf{e}) = \operatorname{tr}\Big[\mathbf{C}(\mathbf{e} \otimes \mathbf{e})\Big]. \tag{8.45}$$

By taking components with respect to an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ where $\mathbf{e}_3 = \mathbf{e}$, this can be written as

$$\widehat{\psi}(\mathbf{F}) = \psi(I_1(\mathbf{C}), I_2(\mathbf{C}), I_3(\mathbf{C}), C_{33}, C_{31}^2 + C_{32}^2 + C_{33}^2). \tag{8.46}$$

The presence of the C_{3i} components here is because the $\mathbf{e}_3$-direction is special.

One can use $\mathbf{S} = 2\rho_0 \mathbf{F} \widehat{\psi}_{\mathbf{C}}$ and $\mathbf{T} = 2\rho_0 J^{-1} \mathbf{F} \widehat{\psi}_{\mathbf{C}} \mathbf{F}^T$ to derive expressions for the stresses by differentiating (8.33) and (8.45) with respect to $\mathbf{C}$ and using the chain rule.

[7] See for example Chapter 4 of Volume I.

8.6 Materials with Internal Constraints.

Thus far we have assumed that the body under consideration can undergo *any* motion at all by subjecting it to suitable body forces and surface tractions. In certain circumstances it is possible, and indeed convenient, to idealize the body such that it can *only* undergo motions of a certain restricted class. For example a *rigid body* can only undergo rigid motions, i.e. motions in which $\mathbf{F}(\mathbf{x}, t)$ is independent of $\mathbf{x}$ and is proper orthogonal at all t; an *incompressible body* can only undergo volume-preserving motions, i.e. motions in which $\det \mathbf{F}(\mathbf{x}, t) = 1$; a body which is inextensible in a certain (referential) direction $\mathbf{e}$ can only undergo motions in which $|\mathbf{F}(\mathbf{x}, t)\mathbf{e}| = 1$ at every particle and instant.

A material is said to be subjected to a "simple internal constraint" if it can only undergo motions in which

$$\widehat{\phi}(\mathbf{F}(\mathbf{x}, t)) = 0 \qquad \text{for all } \mathbf{x} \in \mathcal{R},\ t \in [t_o, t_1] \tag{8.47}$$

where the function $\widehat{\phi}$ describes the constraint. The constraints of rigidity, incompressibility and inextensibility are described by $\widehat{\phi}(\mathbf{F}) = \mathbf{F}^T\mathbf{F} - \mathbf{I}$, $\widehat{\phi}(\mathbf{F}) = \det \mathbf{F} - 1$ and $\widehat{\phi}(\mathbf{F}) = \mathbf{F}\mathbf{e}\cdot\mathbf{F}\mathbf{e} - 1$ respectively.

It is natural to require the constraint (8.47) to be material frame indifferent, i.e. if a deformation gradient tensor $\mathbf{F}$ obeys the constraint (8.47), a subsequent rigid rotation should not lead to a violation of the constraint, i.e. if $\mathbf{F}$ obeys (8.47) then $\mathbf{QF}$ must also satisfy (8.47) for every rotation $\mathbf{Q}$. Accordingly we require that $\widehat{\phi}(\mathbf{F}) = \widehat{\phi}(\mathbf{QF})$ for all nonsingular tensors $\mathbf{F}$ and all rotations $\mathbf{Q}$. The discussion on objectivity earlier in this chapter can be readily adapted to be present context to show that (8.47) is objective if and only if the constraint can be expressed in the form

$$\phi(\mathbf{C}(\mathbf{x}, t)) = 0 \qquad \text{for all } \mathbf{x} \in \mathcal{R},\ t \in [t_o, t_1], \tag{8.48}$$

where $\mathbf{C} = \mathbf{F}^T\mathbf{F}$. Observe that the three examples given earlier can be expressed in this form: rigidity as $\mathbf{C} = \mathbf{I}$, incompressibility as $\det \mathbf{C} = 1$, and inextensibility as $\mathbf{Ce} \cdot \mathbf{e} = 1$.

We now consider the *stresses* in a constrained body. In order to explain the basic idea, consider as an example, a sphere composed of an incompressible isotropic material that is subjected to a uniform applied pressure q on its boundary. Since the geometry, the material and the loading are all symmetric, let us restrict attention to spherically symmetric deformations. Thus the sphere can only become a smaller/larger sphere when q is applied. However due to incompressibility the sphere cannot remain spherical and change its size.

Thus, irrespective of the value of q, incompressibility implies that any symmetric deformation of the sphere must be the trivial one: $\widehat{\mathbf{y}}(\mathbf{x}, t) = \mathbf{x}$ and therefore that necessarily $\mathbf{F}(\mathbf{x}, t) = \mathbf{I}$ no matter what the value of q. The stress on the other hand would certainly depend on the value of the applied pressure q. Thus the stress $\mathbf{T}$ is not completely determined by the deformation gradient $\mathbf{F}$, or equivalently, *different stress fields can correspond to the same deformation*. This contradicts our earlier notion of determinism, according to which the stress is completely determined by the history of the motion until time t. We must therefore modify this notion when considering a constrained body.

It is customary to suppose that the stress $\mathbf{T}$ can be additively decomposed into two parts, one that is determined by the history of the deformation gradient tensor $\mathbf{F}$, and the other, say $\mathbf{N}$, that is not. The stress $\mathbf{N}$ is assumed to do no work in any motion consistent with the constraint. Thus, for example, one might have

$$\mathbf{T} = \tilde{\mathbf{T}}(\mathbf{F}, \dot{\mathbf{F}}) \; + \; \mathbf{N} \tag{8.49}$$

where

$$\mathbf{N} \cdot \mathbf{D} = 0 \tag{8.50}$$

for all stretching tensors $\mathbf{D}$ consistent with the constraint (8.48). The stress field $\mathbf{N}(\mathbf{x}, t)$ arises in "reaction" to the constraint and is often referred to as the *reaction stress*. One sometimes refers to $\tilde{\mathbf{T}}$ and $\mathbf{N}$ as the *active* and *reactive* parts of the stress.

Note that (8.50) does not hold for all symmetric $\mathbf{D}$ but only for those that are consistent with the constraint (8.48). Thus we now determine the restriction that the constraint places on the stretching tensor. Differentiating (8.48) with respect to t shows that

$$\phi_{\mathbf{C}}(\mathbf{C}) \cdot \dot{\mathbf{C}} = 0. \tag{8.51}$$

On making use of the kinematic identity $\dot{\mathbf{C}} = 2\mathbf{F}^T\mathbf{D}\mathbf{F}$, (8.51) provides the following restriction on the possible values of $\mathbf{D}$:

$$\mathbf{F}\phi_{\mathbf{C}}(\mathbf{C})\mathbf{F}^T \cdot \mathbf{D} = 0. \tag{8.52}$$

Equation (8.52) states that $\mathbf{D}$ is orthogonal to $\mathbf{F}\phi_{\mathbf{C}}(\mathbf{C})\mathbf{F}^T$. Equations (8.50) and (8.52) state that the tensor $\mathbf{N}$ is orthogonal to every tensor $\mathbf{D}$ that is orthogonal to $\mathbf{F}\phi_{\mathbf{C}}(\mathbf{C})\mathbf{F}^T$. Thus $\mathbf{N}$ must be parallel to $\mathbf{F}\phi_{\mathbf{C}}(\mathbf{C})\mathbf{F}^T$, and so there exists a scalar field $q(\mathbf{x}, t)$ such that

$$\mathbf{N} = q\,\mathbf{F}\phi_{\mathbf{C}}(\mathbf{C})\mathbf{F}^T. \tag{8.53}$$

Thus in *summary*, the reaction stress in a body subjected to a simple constraint (8.48) is given by (8.53).

As an example, consider an *incompressible body* in which case $\phi(\mathbf{C}) = \det \mathbf{C} - 1$. Thus $\phi_{\mathbf{C}}(\mathbf{C}) = (\det \mathbf{C})\mathbf{C}^{-T} = \mathbf{C}^{-1}$, so that (8.53) yields the well-known result

$$\mathbf{N} = q\, \mathbf{I} \tag{8.54}$$

which says that the reaction stress is a pressure. Similarly, for a body that is inextensible in the referential direction $\mathbf{e}$, one finds that

$$\mathbf{N} = q\, \mathbf{Fe} \otimes \mathbf{Fe}; \tag{8.55}$$

here, $\mathbf{N}$ is a uniaxial stress in the fiber direction $\mathbf{e}$.

The constitutive theory for a constrained material now proceeds as for an unconstrained material. Thus in particular, $\tilde{\mathbf{T}}$ is required to be objective and a discussion of material symmetry is carried out in terms of $\tilde{\mathbf{T}}$. In particular, for an incompressible elastic body that is isotropic in the reference configuration, one finds that the Helmholtz free-energy $\psi(\mathbf{F})$ has the form $\psi(I_1(\mathbf{C}), I_2(\mathbf{C}))$, and that the stress tensors $\mathbf{T}$ and $\mathbf{S}$ are related to the deformation through

$$\mathbf{T} = -p\,\mathbf{I} \; + \; 2\left[\frac{\partial W}{\partial I_1} + I_1\frac{\partial W}{\partial I_2}\right]\mathbf{B} \; - \; 2\frac{\partial W}{\partial I_2}\mathbf{B}^2, \tag{8.56}$$

$$\mathbf{S} = -p\,\mathbf{F}^{-T} \; + \; 2\left[\frac{\partial W}{\partial I_1} + I_1\frac{\partial W}{\partial I_2}\right]\mathbf{F} \; - \; 2\frac{\partial W}{\partial I_2}\mathbf{BF}. \tag{8.57}$$

Note that $I_3(\mathbf{C}) = \det\, \mathbf{C} = (\det\, \mathbf{F})^2 = 1$ due to incompressibility.

If the strain energy function is expressed in terms of the principal stretches

$$W = \widehat{W}(\lambda_1, \lambda_2, \lambda_3) \tag{8.58}$$

with $\widehat{W}$ being invariant if any two of its arguments are switched,

$$\widehat{W}(\lambda_1, \lambda_2, \lambda_3) = \widehat{W}(\lambda_2, \lambda_1, \lambda_3) = \widehat{W}(\lambda_1, \lambda_3, \lambda_2) = \ldots. \tag{8.59}$$

then the principal Cauchy stress components can be written at

$$T_{11} = \lambda_1 \frac{\partial \widehat{W}}{\partial \lambda_1} - p, \qquad T_{22} = \lambda_2 \frac{\partial \widehat{W}}{\partial \lambda_2} - p, \qquad T_{33} = \lambda_3 \frac{\partial \widehat{W}}{\partial \lambda_3} - p. \tag{8.60}$$

8.7 Some Models of Elastic Materials.

In order to describe the response of some particular elastic material we need to know the strain energy function $W(I_1(\mathbf{C}), I_2(\mathbf{C}), I_3(\mathbf{C}))$ (or the free energy function $\psi = W/\rho_0$) that characterizes the material. In this section we give some examples of strain energy functions W from the literature. The list is by no means complete.

It is convenient to record again the constitutive relations. For an (unconstrained) isotropic elastic material we have

$$\left.\begin{aligned}
\mathbf{T} &= 2J\frac{\partial W}{\partial I_3}\mathbf{I} + \frac{2}{J}\left[\frac{\partial W}{\partial I_1} + I_1\frac{\partial W}{\partial I_2}\right]\mathbf{B} - \frac{2}{J}\frac{\partial W}{\partial I_2}\mathbf{B}^2,\\
\mathbf{S} &= 2I_3\frac{\partial W}{\partial I_3}\mathbf{F}^{-T} + 2\left[\frac{\partial W}{\partial I_1} + I_1\frac{\partial W}{\partial I_2}\right]\mathbf{F} - 2\frac{\partial W}{\partial I_2}\mathbf{BF},
\end{aligned}\right\} \tag{8.61}$$

and for an incompressible isotropic elastic material we have

$$\begin{aligned}
\mathbf{T} &= -p\,\mathbf{I} + 2\left[\frac{\partial W}{\partial I_1} + I_1\frac{\partial W}{\partial I_2}\right]\mathbf{B} - 2\frac{\partial W}{\partial I_2}\mathbf{B}^2,\\
\mathbf{S} &= -p\,\mathbf{F}^{-T} + 2\left[\frac{\partial W}{\partial I_1} + I_1\frac{\partial W}{\partial I_2}\right]\mathbf{F} - 2\frac{\partial W}{\partial I_2}\mathbf{BF}.
\end{aligned} \tag{8.62}$$

8.7.1 A Compressible Fluid.

Consider a material characterized by a strain energy function $W(I_1, I_2, I_3)$ that depends only on the third invariant:

$$W = W(I_3).$$

It follows immediately from $(8.61)_1$ that the constitutive relation for stress is

$$\mathbf{T} = 2J\,W'(I_3)\,\mathbf{I}. \tag{8.63}$$

Note that the Cauchy stress is hydrostatic and depends on the deformation only through the Jacobian $\det\mathbf{F}$; therefore the material under discussion is a perfect fluid, i.e. an inviscid, compressible fluid.

We can write the preceding constitutive relation in a more familiar form as follows. Recall that the Helmholtz free energy function is given by $\psi = W(I_3)/\rho_0$, and that the mass density in the curent configuration is given by $\rho_0 = \rho J,\ J = \sqrt{I_3}$. Define the function $\overline{\psi}(\rho)$ by

$$\overline{\psi}(\rho) = \frac{1}{\rho_0}W\Big((\rho_0/\rho)^2\Big).$$

Then it is readily shown from (8.63) that

$$\mathbf{T} = -\rho^2\, \overline{\psi}\,'(\rho)\, \mathbf{I}.$$

We can write this as

$$\mathbf{T} = -p\, \mathbf{I}$$

with the *pressure* p given by

$$p = \rho^2\, \overline{\psi}\,'(\rho). \tag{8.64}$$

This is the familiar form for stress in an inviscid, compressible fluid.

Note that we studied this material previously in Section 7.2.

8.7.2 Neo-Hookean Model.

The neo-Hookean strain energy density function is perhaps the simplest model for an *incompressible isotropic* rubber-like material. It is characterized by

$$W(I_1, I_2) = \frac{\mu}{2}(I_1 - 3), \tag{8.65}$$

where $\mu > 0$ is a material constant. Substituting (8.65) into $(8.62)_1$ leads to the constitutive relation

$$\mathbf{T} = -p\,\mathbf{I} \ + \ \mu\mathbf{B} \tag{8.66}$$

for a neo-Hookean material. We now consider the response of this material in two deformations:

<u>*Uniaxial Stress*</u>: Consider a state of uniaxial stress in the $\mathbf{e}_1$-direction and the corresponding pure homogeneous deformation. The Cauchy stress tensor and deformation gradient tensor have components in the form

$$[T] = \begin{pmatrix} T_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad [F] = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}; \tag{8.67}$$

here we have set $\lambda_1 = \lambda$ for the stretch in the direction of stressing and set $\lambda_2 = \lambda_3$ for the transverse stretch. Since the material is incompressible we must have $\det \mathbf{F} = \lambda\lambda_2^2 = 1$ and so

$$\lambda_2 = \lambda^{-1/2}; \tag{8.68}$$

this describes how the *transverse stretch is related to the longitudinal stretch in uniaxial stress*. Using $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ and substituting (8.67) into (8.66) leads to

$$\begin{pmatrix} T_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -p+\mu\lambda^2 & 0 & 0 \\ 0 & -p+\mu\lambda_2^2 & 0 \\ 0 & 0 & -p+\mu\lambda_2^2 \end{pmatrix}. \tag{8.69}$$

Therefore from $T_{22} = T_{33} = 0$ we get $p = \mu\lambda_2^2 = \mu/\lambda$ where we have used (8.68). Substituting this into T_{11} gives

$$T_{11} = \mu(\lambda^2 - \lambda^{-1}),$$

which is the *stress-stretch response in uniaxial stress*. We can calculate the components of the first Piola-Kirchhoff stress tensor by using the formula $\mathbf{S} = J\mathbf{T}\mathbf{F}^{-T}$. Alternative by physical considerations, if the cross-section normal to the axis of stress has dimensions 1×1 in the reference configuration, in the current configuration its dimensions are $\lambda_2 \times \lambda_2$. Thus the axial force is $S_{11} \times 1 = T_{11} \times \lambda_2^2$. Thus

$$S_{11} = T_{11}\lambda_2^2 = \mu(\lambda - \lambda^{-2}),$$

with all the other stress components S_{ij} being zero. Figure 8.3 shows plots of the stresses T_{11} and S_{11} versus the stretch λ in uniaxial stress for a neo-Hookean material.

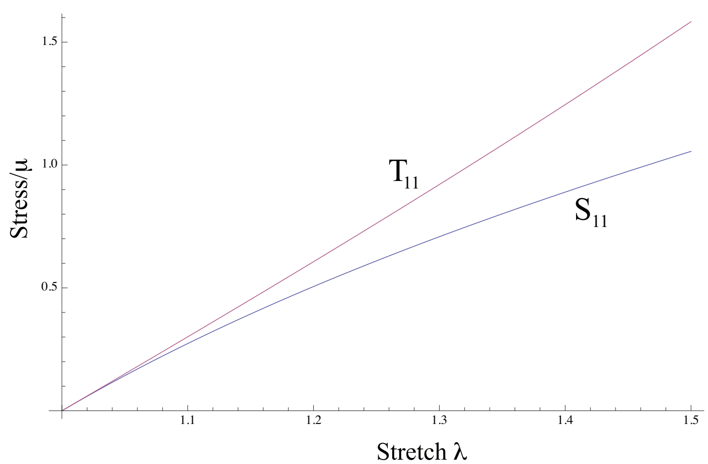

Figure 8.3: Stress-Stretch curves for the neo-Hookean material in uniaxial tension. The upper and lower curves correspond respectively to the Cauchy stress and first Piola-Kirchhoff stress.

<u>*Simple Shear*</u>: We now consider a simple shear which is characterized by a deformation

gradient tensor with components

$$[F] = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad [B] = [F][F]^T = \begin{pmatrix} 1+k^2 & k & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{8.70}$$

Note that $\det \mathbf{F} = 1$ automatically. Substituting (8.70) into (8.66) gives

$$[T] = \begin{pmatrix} \mu(1+k^2) - p & \mu k & 0 \\ \mu k & \mu - p & 0 \\ 0 & 0 & \mu - p \end{pmatrix}. \tag{8.71}$$

This is all that one can say without some further knowledge. In particular, note that the reaction pressure p is not determined.

Suppose that the stress normal to the plane of shearing is zero: $T_{33} = 0$. Then $p = \mu$ and

$$T_{12} = \mu k, \qquad T_{11} = \mu k^2.$$

Observe that in this material, the shear stress T_{12} depends linearly on the amount of shear k for all values of k. Observe also the presence of the *nonzero normal stress* $T_{11} = \mu k^2$. This is in contrast to the linearized theory where the shear stress T_{12} is the only nonzero stress. Note that for infinitesimal amounts of shear $T_{11} = O(k^2)$ and so can be neglected.

8.7.3 Blatz-Ko Model.

The Blatz-Ko strain energy function provides a model for a class of (isotropic compressible) foam rubber materials:

$$W(I_1, I_2, I_3) = \frac{\mu}{2}\left(\frac{I_2}{I_3} + 2\sqrt{I_3} - 5\right) \tag{8.72}$$

where $\mu > 0$ is a material constant. Substituting (8.72) into $(8.61)_1$ gives

$$\mathbf{T} = \frac{\mu}{J^3}\Big[(J^3 - I_2)\mathbf{I} + I_1\mathbf{B} - \mathbf{B}^2\Big]. \tag{8.73}$$

Simple Shear: When $[F]$ and $[B]$ are given by (8.70) we have

$$[B^2] = \begin{pmatrix} k^4+3k^2+1 & k(k^2+2) & 0 \\ k(k^2+2) & 1+k^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{8.74}$$

Therefore the principal scalar invariants are

$$\left.\begin{aligned} I_1(\mathbf{B}) &= \operatorname{tr}\mathbf{B} = 3+k^2, \\ I_2(\mathbf{B}) &= \tfrac{1}{2}\left[(\operatorname{tr}\mathbf{B})^2 - \operatorname{tr}(\mathbf{B}^2)\right] = 3+k^2, \\ I_3(\mathbf{B}) &= \det\mathbf{B} = 1, \qquad J = \sqrt{I_3} = 1. \end{aligned}\right\} \tag{8.75}$$

Substituting (8.75), (8.74) and $(8.70)_2$ into (8.73) leads to

$$T_{12} = \mu k, \qquad T_{22} = -\mu k^2. \tag{8.76}$$

Again, the shear stress happens to depend linearly on the amount of shear and there is a nonzero normal stress in the body.

Uniaxial Stress: When $[F]$ is given by $(8.67)_2$, we have

$$[B] = [F][F]^T = \begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_2^2 \end{pmatrix}, \qquad [B^2] = \begin{pmatrix} \lambda^4 & 0 & 0 \\ 0 & \lambda_2^4 & 0 \\ 0 & 0 & \lambda_2^4 \end{pmatrix}, \tag{8.77}$$

and so

$$\left.\begin{aligned} I_1(\mathbf{B}) &= \operatorname{tr}\mathbf{B} = \lambda^2 + 2\lambda_2^2, \\ I_2(\mathbf{B}) &= \tfrac{1}{2}\left[(\operatorname{tr}\mathbf{B})^2 - \operatorname{tr}(\mathbf{B}^2)\right] = \lambda_2^2(2\lambda^2+\lambda_2^2), \\ I_3(\mathbf{B}) &= \det\mathbf{B} = \lambda^2\lambda_2^4, \qquad J = \sqrt{I_3} = \lambda\lambda_2^2. \end{aligned}\right\} \tag{8.78}$$

Therefore substituting (8.78) and (8.77) into (8.73) yields $T_{22} = \mu J^{-1}(\lambda\lambda_2^4 - 1)\lambda^2\lambda_2^2$. But this stress component must vanish because the state of uniaxial stress is characterized by $(8.67)_1$. Therefore we set $T_{22} = 0$ and derive the following relation between the transverse stretch and the longitudinal stretch:

$$\lambda_2 = \lambda^{-1/4}. \tag{8.79}$$

Substituting (8.77), (8.78) and (8.79) into (8.73) now yields the stress-stretch relation

$$T_{11} = \mu(1 - \lambda^{-5/2}).$$

The corresponding formula for the first Piola-Kirchhoff can be readily obtained from $\mathbf{S} = J\mathbf{T}\mathbf{F}^{-T}$:

$$S_{11} = T_{11}\lambda_2^2 = \mu(\lambda^{-1/2} - \lambda^{-3}). \tag{8.80}$$

Figure 8.4 shows plots of the stresses T_{11} and S_{11} versus the stretch λ in uniaxial stress for a Blatz-Ko material.

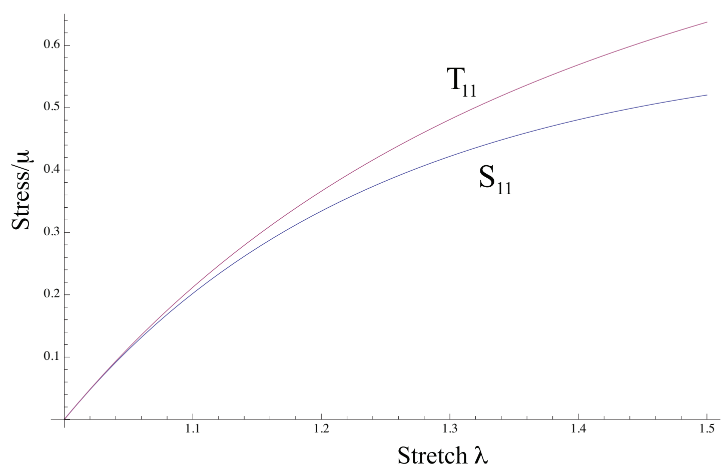

Figure 8.4: Stress-Stretch curves for the Blatz-Ko material in uniaxial tension. The upper and lower curves correspond respectively to the Cauchy stress and first Piola-Kirchhoff stress.

8.7.4 Gent Model. Limited Extensibility.

The Gent model for an incompressible isotropic elastic material is characterized by the strain energy function

$$W = W(I_1) = -\frac{\mu}{2}\, J_m \,\ln\left(1 - \frac{I_1 - 3}{J_m}\right) \tag{8.81}$$

where μ and J_m are positive material constants. Since the argument of the logarithm must be positive, the principal invariant I_1 cannot exceed $3 + J_m$:

$$I_1 < 3 + J_m. \tag{8.82}$$

One can show by linearization that μ is the shear modulus at infinitesimal deformations. One can also show that in the limit $J_m \to \infty$, the Gent model reduces to the neo-Hookean model (8.65).

Substituting the particular form (8.81) into the general constitutive equation (8.62) leads to

$$\mathbf{T} = -p\mathbf{I} + \frac{\mu J_m}{3 + J_m - I_1}\,\mathbf{B}. \tag{8.83}$$

<u>*Uniaxial Stress in the* $\mathbf{e}_1$*-direction*</u>: Set $\lambda_1 = \lambda$ and take the transverse stretches to be equal $\lambda_2 = \lambda_3$. Then incompressibility gives $\lambda\lambda_2^2 = 1$ and so the transverse stretch is given in terms of λ by

$$\lambda_2 = \lambda^{-1/2}. \tag{8.84}$$

The principal invariant I_1 is now found to be

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \lambda^2 + 2\lambda^{-1}. \tag{8.85}$$

Since I_1 cannot exceed $3 + J_m$ we must have $\lambda^2 + 2\lambda^{-1} < 3 + J_m$ which limits the maximum possible stretch to some value λ_m.

The stress components T_{11} and $T_{22} = T_{33}$ can now be found from (8.83) to be

$$T_{11} = -p + \frac{\mu J_m\,\lambda^2}{3 + J_m - \lambda^2 - 2\lambda^{-1}}, \qquad T_{22} = -p + \frac{\mu J_m}{\lambda(3 + J_m - \lambda^2 - 2\lambda^{-1})}. \tag{8.86}$$

Since $T_{22} = T_{33} = 0$ in a state of uniaxial stress in the $\mathbf{e}_1$-direction, the second equation above can be solved for p. When this is substituted into the first equation we find the stress-stretch relation in uniaxial stress:

$$T_{11} = \left(\lambda^2 - \lambda^{-1}\right)\left(\frac{\mu J_m}{3 + J_m - \lambda^2 - 2\lambda^{-1}}\right). \tag{8.87}$$

Note that $T_{11} \to \infty$ as $\lambda \to \lambda_m$.

8.7.5 Fung Model for Soft Biological Tissue.

Reference: J. D. Humphrey, Continuum biomechanics of soft biological tissues, *Proceeding of the Royal Society: Series A*, Vol. 459, 2003, pp. 3 - 46.

The mechanical response of soft biological tissue under quasi-static loading is dominated by its fibrous constituents: collagen and elastin. At small strains, the collagen fibers are unstretched and the mechanical response is almost entirely due to the soft, isotropic elastin. As the load increases, the collagen fibers straighten-out and align with the direction of loading. This leads to a rapid increase in the stiffness as well as anisotropic material behavior due to the preferred direction of collagen orientation.

One of the earliest constitutive models for soft tissue was proposed by Fung using a strain energy function in the form

$$W = W(\mathbf{C}) = \alpha\Big(\mathrm{e}^{Q(\mathbf{C})} - 1\Big),$$

where α is a material constant and Q is a function of the left Cauchy Green tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$:

$$Q = Q(\mathbf{C}).$$

Different functional forms of Q have been considered with a general quadratic form the most common. By suitably choosing the form of $Q(\mathbf{C})$ the material anisotropy can be built in. The rapid stiffening is modeled by the exponential.

8.8 An Elastic Body with One Preferred Direction.

Reference: A.J.M. Spencer *Deformation of Fiber-Reinforced Materials*, Clarendon Press, Oxford, 1972.

In this section we consider an elastic material that has a "preferred direction" $\mathbf{a}$ in the reference configuration. The material might, for example, be reinforced by a family of fibers oriented in direction $\mathbf{a}$, (though our analysis models a uniform material, not one with two constituents corresponding to a matrix and fibers). Because of the preferred direction the material is not isotropic. In fact, since arbitrary rotations about $\mathbf{a}$ preserves material symmetry, the material is transversely isotropic about $\mathbf{a}$. While the models we construct will be identical to those obtained in Section 8.5.2 our approach will be different[8].

We shall consider two material models. The first involves no constraints. The second treats the material as incompressible and inextensible in the direction $\mathbf{a}$.

We start by stating without proof a theorem from algebra which will be used in what follows:

[8]In Section 8.5.2 we considered the energy to have the form $W(\mathbf{C})$ and required it be invariant under all rotations about the axis $\mathbf{a}$. Here we consider the energy to have the form $W(\mathbf{C}, \mathbf{a})$ and required it be invariant under all rotations.

Theorem: Let $\phi(\mathbf{A}_1, \mathbf{A}_2)$ be a scalar-valued polynomial of two symmetric tensors $\mathbf{A}_1$ and $\mathbf{A}_2$. Moreover, suppose that ϕ is invariant under rotations in the sense that $\phi(\mathbf{A}_1, \mathbf{A}_2) = \phi(\mathbf{Q}\mathbf{A}_1\mathbf{Q}^T, \mathbf{Q}\mathbf{A}_2\mathbf{Q}^T)$ for all orthogonal $\mathbf{Q}$. Then ϕ can be expressed as a polynomial function of the following basic invariants:

$$\begin{gathered}\operatorname{tr}\mathbf{A}_1, \quad \operatorname{tr}\mathbf{A}_2, \quad \operatorname{tr}\mathbf{A}_1^2, \quad \operatorname{tr}\mathbf{A}_2^2, \quad \operatorname{tr}\mathbf{A}_1^3, \quad \operatorname{tr}\mathbf{A}_2^3, \\ \operatorname{tr}\left(\mathbf{A}_1\mathbf{A}_2\right), \quad \operatorname{tr}\left(\mathbf{A}_1\mathbf{A}_2^2\right), \quad \operatorname{tr}\left(\mathbf{A}_1^2\mathbf{A}_2\right), \quad \operatorname{tr}\left(\mathbf{A}_1^2\mathbf{A}_2^2\right).\end{gathered}$$

Proof: R.S. Rivlin, Journal of Rational Mechanics and Analysis, **4**(1955), pp. 681.

Special Case: Suppose that $\mathbf{A}_2 = \mathbf{a} \otimes \mathbf{a}$ where $\mathbf{a}$ is a unit vector. In this case note that $\mathbf{A}_2 = \mathbf{A}_2^2 = \mathbf{A}_2^3$ and $\operatorname{tr}\mathbf{A}_2 = 1$. In this case the list of basic invariants reduces to

$$\operatorname{tr}\mathbf{A}_1, \quad \operatorname{tr}\mathbf{A}_1^2, \quad \operatorname{tr}\mathbf{A}_1^3, \quad \operatorname{tr}\left(\mathbf{A}_1\mathbf{A}_2\right), \quad \operatorname{tr}\left(\mathbf{A}_1^2\mathbf{A}_2\right) \ . \tag{8.88}$$

<u>*Material 1*</u>: Consider an elastic solid with a single preferred orientation represented by the unit vector $\mathbf{a}$ in the reference configuration. The strain energy function can be expressed as a function of the deformation gradient tensor $\mathbf{F}$ and the unit vector $\mathbf{a}$, i.e.

$$W = W(\mathbf{F}, \mathbf{a}). \tag{8.89}$$

Material frame indifference requires that $W(\mathbf{QF}, \mathbf{a}) = W(\mathbf{F}, \mathbf{a})$ for all rotations $\mathbf{Q}$ which holds if and only if one can express W in the form

$$W = W(\mathbf{C}, \mathbf{a}) \tag{8.90}$$

where $\mathbf{C} = \mathbf{F}^T\mathbf{F}$; recall that $\mathbf{a}$ is a direction in the *reference* configuration and so is unaffected by $\mathbf{Q}$.) Since the sense of the vector $\mathbf{a}$ is immaterial, changing $\mathbf{a} \to -\mathbf{a}$ does not change the energy. Therefore we can take

$$W = W(\mathbf{C}, \mathbf{a} \otimes \mathbf{a}). \tag{8.91}$$

Requiring W to be invariant under all orthogonal changes of the reference configuration (and assuming W to be a polynomial of arbitrary order of its arguments) implies that W can be expressed as a function of the isotropic polynomials listed in the Theorem above. Therefore setting $\mathbf{A}_1 = \mathbf{C}$ and $\mathbf{A}_2 = \mathbf{a} \otimes \mathbf{a}$ leads to

$$W = W(J_1, J_2, J_3, J_4, J_5) \tag{8.92}$$

where from (8.88),

$$J_1 = \operatorname{tr}\mathbf{C}, \quad J_2 = \operatorname{tr}\mathbf{C}^2, \quad J_3 = \operatorname{tr}\mathbf{C}^3, \quad J_4 = \operatorname{tr}\left[\mathbf{C}(\mathbf{a} \otimes \mathbf{a})\right], \quad J_5 = \operatorname{tr}\left[\mathbf{C}^2\left(\mathbf{a} \otimes \mathbf{a}\right)\right]. \tag{8.93}$$

These are the same five invariants that Green and Adkins use in their discussion of a transversely isotropic material about the axis $\mathbf{a}$. There are several equivalent but alternative ways in which to write the invariants J_i given in (8.93). One that is convenient for our purposes is

$$\begin{gathered} I_1 = \operatorname{tr}\mathbf{C}, \qquad I_2 = 1/2\big[(\operatorname{tr}\mathbf{C})^2 - \operatorname{tr}\mathbf{C}^2\big], \qquad I_3 = \det\mathbf{C}, \\ I_4 = \operatorname{tr}\left[\mathbf{C}^2(\mathbf{a}\otimes\mathbf{a})\right], \qquad I_5 = \operatorname{tr}\left[\mathbf{C}(\mathbf{a}\otimes\mathbf{a})\right]. \end{gathered} \tag{8.94}$$

Observe that these are the same invariants we found in Section 8.45 for a transversely isotropic material. One readily finds from (8.94) that

$$\left.\begin{aligned} \frac{\partial I_1}{\partial\mathbf{C}} &= \mathbf{I}; \\ \frac{\partial I_2}{\partial\mathbf{C}} &= I_1\,\mathbf{I} - \mathbf{C}; \\ \frac{\partial I_3}{\partial\mathbf{C}} &= J^2\mathbf{C}^{-1}; \\ \frac{\partial I_4}{\partial\mathbf{C}} &= \mathbf{a}\otimes\mathbf{Ca} + \mathbf{Ca}\otimes\mathbf{a}; \\ \frac{\partial I_5}{\partial\mathbf{C}} &= \mathbf{a}\otimes\mathbf{a}. \end{aligned}\right\} \tag{8.95}$$

The Cauchy stress can now be calculated using $\mathbf{T} = 2J^{-1}\mathbf{F}W_{\mathbf{C}}\mathbf{F}^T$, $W = W(I_1, I_2, I_3, I_4, I_5)$, the chain rule and (8.95) whence

$$\mathbf{T} = \frac{2}{J}\Big(W_1\mathbf{B} + W_2\big[I_1\,\mathbf{B} - \mathbf{B}^2\big] + J^2W_3\mathbf{I} + W_4\big[\mathbf{b}\otimes\mathbf{Bb} + \mathbf{Bb}\otimes\mathbf{b}\big] + W_5\mathbf{b}\otimes\mathbf{b}\Big)$$

where $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ is the right Cauchy Green tensor; $\mathbf{b} = \mathbf{Fa}$ is the image of $\mathbf{a}$ in the deformed configuration; and we have set $W_i = \partial W/\partial I_i$.

<u>*Material 2*</u>: Suppose that the material at hand coincides with the material above except that now it is, in addition, both incompressible and inextensible in the direction $\mathbf{a}$. These two kinematic constraints are described by $\det\mathbf{F} = 1$ and $|\mathbf{a}| = |\mathbf{Fa}|$, and they can be written equivalently as

$$\det\mathbf{C} = 1, \qquad \operatorname{tr}\big[\mathbf{C}(\mathbf{a}\otimes\mathbf{a})\big] = 1, \tag{8.96}$$

where $\mathbf{C} = \mathbf{F}^T\mathbf{F}$.

As noted previously in Section 8.6, the Cauchy stress in a kinematically constrained material is additively decomposed into a part $\tilde{\mathbf{T}}$ determined by the deformation and a part $\mathbf{N}$, called the reaction stress, which is not determined by the deformation:

$$\mathbf{T} = \tilde{\mathbf{T}} + \mathbf{N}.$$

From (8.54) and (8.55), the reaction stress $\mathbf{N}$ arising from the constraints of incompressibility and inextensibility is

$$\mathbf{N} = -p\mathbf{I} + \tau\mathbf{C}\mathbf{a}\otimes\mathbf{a} \ = \ -p\mathbf{I} + \tau\mathbf{b}\otimes\mathbf{b},$$

where $\mathbf{b} = \mathbf{F}\mathbf{a}$ is the image of $\mathbf{a}$ in the deformed configuration. Here p and τ are constitutively indeterminate.

Observe that because of the kinematic constraints (8.96), two of the invariants in the list (8.94) must equal unity:

$$I_3 = I_5 = 1, \tag{8.97}$$

and so the strain energy function has the form $W = W(I_1, I_2, I_4)$.

The kinematically determined portion of the stress is calculated from the constitutive relation $\tilde{\mathbf{T}} = 2J^{-1}\mathbf{F}W_{\mathbf{C}}\mathbf{F}^T$. A straightforward calculation using the chain rule and $(8.94)_{1,2,3}$ leads to

$$\tilde{\mathbf{T}} = 2(W_1 + I_1W_2)\mathbf{B} - 2W_2\mathbf{B}^2 + 2W_4(\mathbf{b}\otimes\mathbf{B}\mathbf{b} + \mathbf{B}\mathbf{b}\otimes\mathbf{b}) \tag{8.98}$$

where $\mathbf{b} = \mathbf{F}\mathbf{a}$, $W_i = \partial W/\partial I_i$ and $\mathbf{B} = \mathbf{F}\mathbf{F}^T$. Therefore the Cauchy stress $\mathbf{T} = \tilde{\mathbf{T}} + \mathbf{N}$ is given by

$$\mathbf{T} = 2(W_1 + I_1W_2)\mathbf{B} - 2W_2\mathbf{B}^2 + 2W_4(\mathbf{b}\otimes\mathbf{B}\mathbf{b} + \mathbf{B}\mathbf{b}\otimes\mathbf{b}) - p\mathbf{I} + \tau\mathbf{b}\otimes\mathbf{b}. \tag{8.99}$$

8.9 Linearized Thermoelasticity.

Reference: D.E. Carlson, Linear Thermoelasticity, in *Mechanics of Solids*, Volume II, (Edited by C. Truesdell), Springer, 1984, pp. 299–321.

On various occasions in the preceding chapters on kinematics and stress, we specialized the general theory to the case where the displacement gradient tensor $\mathbf{H} = \mathrm{Grad}\,\mathbf{u} = \mathbf{F} - \mathbf{I}$ was infinitesimal: $|\mathbf{H}| << 1$. In particular the *infinitesimal strain tensor* $\boldsymbol{\gamma}$ was defined by

$$\boldsymbol{\gamma} = \frac{1}{2}(\mathbf{H} + \mathbf{H}^T), \qquad \gamma_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{8.100}$$

and we found that *all* strain measures, $\mathbf{E}(\mathbf{U})$ and $\mathbf{E}(\mathbf{V})$, equaled the infinitesimal strain tensor $\boldsymbol{\gamma}$ to leading order:

$$E_{ij} \;=\; \gamma_{ij} + O(|\mathbf{H}|^2).$$

Note also that $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ with $\mathbf{F} = \mathbf{I} + \mathbf{H}$ and (8.100) tells us that

$$\mathbf{C} = \mathbf{I} + 2\boldsymbol{\gamma} + O(|\mathbf{H}|^2).$$

In the case of stress, we found that the Cauchy stress and the first Piola-Kirchhoff stress agreed to leading order and we called this common *stress* $\boldsymbol{\sigma}$:

$$S_{ij}, T_{ij} \to \sigma_{ij} \qquad \text{where} \quad \sigma_{ij} = \sigma_{ji}.$$

We now approximate the general constitutive relationships under the assumption that the displacement gradient tensor is infinitesimal, that the temperature θ is close to some uniform reference temperature θ_0 and the temperature gradient $|\mathbf{g}| = |\mathrm{Grad}\,\theta|$ is infinitesimal.

First consider the *constitutive relation for stress*: it is convenient to start with the expression $(8.14)_2$ for the first Piola-Kirchhoff stress:

$$\mathbf{S} = 2\rho_0 \mathbf{F}\frac{\partial \psi}{\partial \mathbf{C}}(\mathbf{C}, \theta). \tag{8.101}$$

Setting $(\mathbf{F}, \mathbf{C}, \theta) = (\mathbf{I}, \mathbf{I}, \theta_0)$ on the right hand side of (8.101) gives the stress in the reference configuration which we denote by $\overset{o}{\boldsymbol{\sigma}}$:

$$\overset{o}{\boldsymbol{\sigma}} = 2\rho_0 \left.\frac{\partial \psi}{\partial \mathbf{C}}\right|_{\substack{\mathbf{C}=\mathbf{I}\\ \theta=\theta_0}}. \tag{8.102}$$

This is frequently called the *residual stress.*

We now approximate (8.101) by carrying out a Taylor expansion of its right hand side and using $\mathbf{F} = \mathbf{I} + \mathbf{H}$, $\mathbf{C} = \mathbf{I} + 2\boldsymbol{\gamma}$:

$$\begin{aligned}
\sigma_{ij} \;&=\; 2\rho_0\left(\delta_{ik} + \frac{\partial u_i}{\partial x_k}\right)\frac{\partial \psi}{\partial C_{kj}}(\mathbf{C}, \theta), \\
&=\; 2\rho_0\left(\delta_{ik} + \frac{\partial u_i}{\partial x_k}\right)\left[\left.\frac{\partial \psi}{\partial C_{kj}}\right|_{\substack{\mathbf{C}=\mathbf{I}\\ \theta=\theta_0}} + \left.\frac{\partial^2 \psi}{\partial C_{kj}\partial C_{pq}}\right|_{\substack{\mathbf{C}=\mathbf{I}\\ \theta=\theta_0}} 2\gamma_{pq} + \left.\frac{\partial^2 \psi}{\partial C_{kj}\partial \theta}\right|_{\substack{\mathbf{C}=\mathbf{I}\\ \theta=\theta_0}} (\theta - \theta_0) + O(\epsilon^2)\right], \\
&=\; \left(\delta_{ik} + \frac{\partial u_i}{\partial x_k}\right)\left[\overset{o}{\sigma}_{kj} + \mathbb{C}_{kjpq}\gamma_{pq} \;+ \mathsf{M}_{kj}(\theta - \theta_0) + O(\epsilon^2)\right], \\
&=\; \overset{o}{\sigma}_{ij} + \overset{o}{\sigma}_{kj}\frac{\partial u_i}{\partial x_k} + \mathbb{C}_{ijpq}\gamma_{pq} \;+\; \mathsf{M}_{ij}(\theta - \theta_0) + O(\epsilon^2),
\end{aligned}$$

where we used $(8.102)_2$ and set

$$\mathbb{C}_{ijk\ell} = 4\rho_0 \frac{\partial^2\psi}{\partial C_{ij} C_{k\ell}}\bigg|_{\substack{\mathbf{C}=\mathbf{I}\\ \theta=\theta_0}} \tag{8.103}$$

and

$$\mathsf{M}_{ij} = 2\rho_0 \frac{\partial^2\psi}{\partial C_{ij}\partial\theta}\bigg|_{\substack{\mathbf{C}=\mathbf{I}\\ \theta=\theta_0}} . \tag{8.104}$$

The components of the fourth-order *elasticity tensor* $\mathbb{C}$ and the second-order *stress-temperature tensor* M are material constants. In particular, the components of the elasticity tensor represent the various elastic moduli of the material. Observe from (8.103) that

$$\mathbb{C}_{ijk\ell} = \mathbb{C}_{k\ell ij}, \qquad \mathbb{C}_{ijk\ell} = \mathbb{C}_{jik\ell}, \qquad \mathbb{C}_{ijk\ell} = \mathbb{C}_{ij\ell k}, \tag{8.105}$$

and therefore that the most general (anisotropic) elastic material has 21 elastic constants. Since the stress-temperature tensor is symmetric, it involves 6 material constants:

$$\mathsf{M}_{ij} = \mathsf{M}_{ji}. \tag{8.106}$$

The constitutive relation for stress in the linearized theory is thus

$$\sigma_{ij} = \overset{o}{\sigma}_{ij} + \overset{o}{\sigma}_{kj} \frac{\partial u_i}{\partial x_k} + \mathbb{C}_{ijk\ell}\gamma_{k\ell} + \mathsf{M}_{ij}(\theta - \theta_0).$$

From hereon we take the residual stress to be zero in the reference configuration, $\overset{o}{\boldsymbol{\sigma}} = \mathbf{0}$, in which event the preceding equation reduces to

$$\sigma_{ij} = \mathbb{C}_{ijk\ell}\gamma_{k\ell} + \mathsf{M}_{ij}(\theta - \theta_0). \tag{8.107}$$

If the elasticity tensor is invertible, one can solve (8.107) for the strain $\boldsymbol{\gamma}$ in terms of stress and temperature to get an equation of the form

$$\gamma_{ij} = \mathbb{K}_{ijk\ell}\sigma_{k\ell} + \mathsf{A}_{ij}\,(\theta - \theta_0).$$

The components of the compliance tensor $\mathbb{K}$ and the *thermal expansion tensor* A are material constants.

We next turn to the *constitutive relation for the specific entropy.* Recalling that $\eta = -\partial\psi(\mathbf{C},\theta)/\partial\theta$ and carrying out a Taylor expansion of the right hand side about $(\mathbf{C},\theta) = (\mathbf{I},\theta_0)$ and using $\mathbf{C} = \mathbf{I} + 2\boldsymbol{\gamma}$ yields

$$\begin{aligned} \eta &= -\frac{\partial\psi}{\partial\theta}\bigg|_{\substack{\mathbf{C}=\mathbf{I}\\ \theta=\theta_0}} - \frac{\partial^2\psi}{\partial C_{ij}\partial\theta}\bigg|_{\substack{\mathbf{C}=\mathbf{I}\\ \theta=\theta_0}} 2\gamma_{ij} - \frac{\partial^2\psi}{\partial\theta^2}\bigg|_{\substack{\mathbf{C}=\mathbf{I}\\ \theta=\theta_0}} (\theta-\theta_0) \\ &= -\rho_0^{-1}\mathsf{M}_{ij}\gamma_{ij} + c\,(\theta/\theta_0 - 1)\,, \end{aligned} \tag{8.108}$$

where we have used (8.104) and set

$$c = -\theta_0 \frac{\partial^2 \psi}{\partial \theta^2}\Big|_{\substack{\mathbf{C}=\mathbf{I} \\ \theta=\theta_0}}$$

for the specific *specific heat* in the reference configuration at the reference temperature. We have taken the entropy in the reference configuration at the reference temperature, i.e. $\partial\psi/\partial\theta$ at $(\mathbf{C}, \theta) = (\mathbf{I}, \theta_0)$, to be zero which can be done by choosing the datum for the entropy appropriately. The constitutive relation for specific entropy in the linearized theory is

$$\eta = -\rho_0^{-1} \mathsf{M}_{ij}\gamma_{ij} \; + c\,(\theta/\theta_0 \, - \, 1)\,. \tag{8.109}$$

For completeness, we note the appropriate approximation for the *Helmholtz free-energy* ψ. A Taylor expansion of $\psi(\mathbf{C}, \theta)$ about $(\mathbf{C}, \theta) = (\mathbf{I}, \theta_0)$ is readily shown to lead to

$$\rho_0\psi = \frac{1}{2}\mathbb{C}_{ijk\ell}\gamma_{ij}\gamma_{kl} + \mathbb{M}_{ij}\gamma_{ij}(\theta - \theta_0) - \frac{\rho_0 c}{2\theta_0}(\theta - \theta_0)^2.$$

Next we turn to *the constitutive relation for the heat flux:* $\mathbf{q}_0 = \widehat{\mathbf{q}}_0(\mathbf{C}, \theta, \mathbf{g}),\ \ \mathbf{g} = \text{Grad}\,\theta$. It is useful to first recall from (8.22) that the entropy inequality requires that

$$\widehat{\mathbf{q}}_0(\mathbf{C}, \theta, \mathbf{0}) = \mathbf{0} \tag{8.110}$$

for all symmetric positive definite tensors $\mathbf{C}$ and all positive numbers θ. Differentiating this shows that we also must have

$$\frac{\partial \widehat{\mathbf{q}}_0}{\partial \mathbf{C}}(\mathbf{C}, \theta, \mathbf{0}) = \mathbf{0}, \qquad \frac{\partial \widehat{\mathbf{q}}_0}{\partial \theta}(\mathbf{C}, \theta, \mathbf{0}) = \mathbf{0}. \tag{8.111}$$

We now take the constitutive law for the referential heat flux vector, $\mathbf{q}_0 = \widehat{\mathbf{q}}_0(\mathbf{C}, \theta, \mathbf{g})$ and carry out a Taylor expansion about $(\mathbf{C}, \theta, \mathbf{g}) = (\mathbf{I}, \theta_0, \mathbf{0})$. This gives

$$q_i^0 = \frac{\partial \widehat{q}_i^0}{\partial g_j}\Big|_{\mathbf{C}=\mathbf{I},\,\theta=\theta_0,\,\mathbf{g}=\mathbf{0}} \; \frac{\partial \theta}{\partial x_j}$$

where we have used (8.110) and (8.111) and set $\mathbf{g} = \text{Grad}\,\theta$. On setting

$$K_{ij} = \frac{\partial \widehat{q}_i^0}{\partial g_j}\Big|_{\mathbf{C}=\mathbf{I},\,\theta=\theta_0,\,\mathbf{g}=\mathbf{0}}$$

for the *heat conduction tensor* (in the reference configuration at the reference temperature in thermal equilibrium) we have are led to the (Fourier) heat conduction relation

$$\mathbf{q}_0 = \mathbf{K}\,\text{Grad}\,\theta. \tag{8.112}$$

Finally we linearize the *energy equation* for a thermoelastic material which, by (8.17), is

$$\text{Div}\ \mathbf{q}_0 + \rho_0 r = \rho_0 \theta \dot{\eta}.$$

On substituting the linearized heat conduction relation (8.112) and the constitutive relation (8.109) for the entropy into this we are led to the linearized energy equation

$$\frac{\partial}{\partial x_i}\left(K_{ij}\frac{\partial \theta}{\partial x_j}\right) + \theta_0 \mathsf{M}_{ij}\dot{\gamma}_{ij} + \rho_0 r = \rho_0 c\,\dot{\theta}.$$

In **summary**, the linearized theory of thermoelasticity is characterized by the system of equations

$$\left.\begin{aligned}
\frac{\partial \sigma_{ij}}{\partial x_j} + \rho_0 b_i &= \rho_0 \ddot{u}_i,\\
\frac{\partial q_i}{\partial x_i} + \theta_0 \mathsf{M}_{ij}\dot{\gamma}_{ij} + \rho_0 r &= c\,\dot{\theta},\\
\sigma_{ij} &= \mathbb{C}_{ijk\ell}\,\gamma_{k\ell} + \mathsf{M}_{ij}(\theta - \theta_0),\\
q_i &= \mathrm{K}_{ij}\frac{\partial \theta}{\partial x_j},\\
\gamma_{ij} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),
\end{aligned}\right\} \tag{8.113}$$

where the material is characterized by the elastic moduli $\mathbb{C}_{ijk\ell}$, the heat conductivity coefficients K_{ij} and the stress-temperature coefficients M_{ij} (which are related to the coefficients of thermal expansion). Equation (8.109) can be used to calculate the specific entropy a posteriori.

This system of equations (8.113) can be reduced to a system of 4 (scalar) equations for the displacement field $\mathbf{u}(\mathbf{x}, t)$ and the temperature field $\theta(\mathbf{x}, t)$ by substituting $(8.113)_{3,4}$ into $(8.113)_{1,2}$ to eliminate the stress and heat flux, and then using $(8.113)_5$ in the resulting pair of equations to eliminate the strain. This leads to

$$\left.\begin{aligned}
\frac{\partial}{\partial x_j}\left(\mathbb{C}_{ijk\ell}\frac{\partial u_k}{\partial x_\ell}\right) + \mathsf{M}_{ij}\frac{\partial \theta}{\partial x_j} + \rho_0 b_i &= \rho_0 \frac{\partial^2 u_i}{\partial t^2},\\
\frac{\partial}{\partial x_i}\left(K_{ij}\frac{\partial \theta}{\partial x_j}\right) + \theta_0 \mathsf{M}_{ij}\frac{\partial^2 u_i}{\partial t\,\partial x_j} + \rho_0 r &= \frac{c\,\rho_0}{\theta_0}\frac{\partial \theta}{\partial t},
\end{aligned}\right\}$$

where we have also made use of the symmetries (11.9) and (11.10) of $\mathbb{C}$ and M. Observe that the mechanical effects (characterized by the terms involving the displacement $\mathbf{u}$) are coupled to the thermal effects (characterized by the terms involving the temperature θ) by the stress-temperature tensor M. If $\mathsf{M} = \mathbf{0}$ the first equation becomes the wave equation while the second becomes the heat equation.

8.9.1 Linearized Isotropic Thermoelastic Material.

For an isotropic material we know that $\psi = \psi(I_1(\mathbf{C}), I_2(\mathbf{C}), I_3(\mathbf{C}), \theta)$ and so a lengthy calculation can be used to evaluated the second derivative to ψ with respect to $\mathbf{C}$ which eventually leads to the specific expression:

$$\mathbb{C}_{ijkl} = \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) + \lambda\delta_{ij}\delta_{k\ell}$$

where (with $W = \rho_0\psi$)

$$\mu = 2\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\right)\Big|_{I_1=I_2=3, I_3=1, \theta=\theta_0},$$

$$\begin{aligned}\lambda = 4\Bigg(\frac{\partial W}{\partial I_2} + \frac{\partial W}{\partial I_3} \quad &+ \quad \frac{\partial^2 W}{\partial I_1^2} + 4\frac{\partial^2 W}{\partial I_2^2} + \frac{\partial^2 W}{\partial I_3^2} + 4\frac{\partial^2 W}{\partial I_1 \partial I_2} + \\ &+ \quad 2\frac{\partial^2 W}{\partial I_1 \partial I_3} + 4\frac{\partial^2 W}{\partial I_2 \partial I_3}\Bigg)\Big|_{I_1=I_2=3, I_3=1, \theta=\theta_0}.\end{aligned}$$

In addition, for an isotropic material we also have

$$\mathsf{M}_{ij} = m\delta_{ij}, \qquad K_{ij} = k\delta_{ij}.$$

so that the material has a single coefficient of thermal expansion and a single coefficient of heat conduction.

8.10 Worked Examples and Exercises.

Problem 8.1. Suppose that the stress-stretch relation (for the Cauchy stress) in uniaxial tension of a particular isotropic incompressible elastic material is given by

$$T = c_1(\lambda - 1) + c_2(\lambda - 1)^n + c_3 \ln \lambda, \qquad \text{where } n > 1 \text{ is a positive integer;} \quad c_1 > 0,\ c_3 > 0.$$

Note that $T \to -\infty$ when $\lambda \to 0$ and $T \to \infty$ when $\lambda \to \infty$.

Determine *two* strain energy functions $W(I_1, I_2)$ that yield this same stress-stretch relation. Calculate the shear stress - amount of shear relations corresponding to these two $W's$.

Problem 8.2. Consider an isotropic incompressible elastic material. In general, such a material is characterized by a strain energy function $W(I_1, I_2)$ where I_1 and I_2 are the first two principal invariants of the Cauchy Green tensor $\mathbf{C}$ (or $\mathbf{B}$). Many examples of material models mentioned in the notes, e.g. the neo-Hookean model, the Gent model, the Arruda-Boyce model, do not depend on the second invariant I_2 and so have the special form $W = W(I_1)$. Devise an experiment that can determine if the strain energy function of a given isotropic incompressible elastic material depends on I_2 or not.

Problem 8.3. Let $\widehat{\mathbf{T}}(\mathbf{F})$ be the Cauchy stress response function. It is symmetruc tensor-valued and defined for all tensors with positive determinant.

(i) Show that $\widehat{\mathbf{T}}(\mathbf{F})$ is material frame indifferent if and only if

$$\widehat{\mathbf{T}}(\mathbf{F}) = \mathbf{R}\widehat{\mathbf{T}}(\mathbf{U})\mathbf{R}^T$$

where $\mathbf{R}$ and $\mathbf{U}$ are the rotation and stretch tensors in the polar decomposition of $\mathbf{F}$.

(ii) Show that this can be written equivalently as

$$\widehat{\mathbf{T}}(\mathbf{F}) = \mathbf{F}\overline{\mathbf{T}}(\mathbf{C})\mathbf{F}^T$$

where $\mathbf{C} = \mathbf{F}^T\mathbf{F}$.

(iii) Show from this that the second Piola-Kirchhoff stress tensor $\mathbf{S}^{(2)}$ can be expressed as a function of the Lagrangian Cauchy-Green tensor $\mathbf{C}$ only.

(iv) Suppose that $\overline{\mathbf{T}}(\mathbf{C})$ is a polynomial of arbitrary (finite) order:

$$\overline{\mathbf{T}}(\mathbf{C}) = \sum_{n=0}^{N} \alpha_n \, \mathbf{C}^n$$

Show that

$$\widehat{\mathbf{T}}(\mathbf{F}) = \sum_{n=0}^{N} \alpha_n \, \mathbf{B}^{n+1}$$

where $\mathbf{B} = \mathbf{F}\mathbf{F}^T$.

<u>Solution</u>: Given two motions $\mathbf{y}$ and $\mathbf{y}^* = \mathbf{Q}\mathbf{y}$ the associated deformation gradient tensors and Cauchy stress tensors are given by $\mathbf{F}, \mathbf{T}$ and $\mathbf{F}^*, \mathbf{T}^*$ respectively. Therefore from the constitutive relation we have

$$\mathbf{T} = \widehat{\mathbf{T}}(\mathbf{F}), \qquad \mathbf{T}^* = \widehat{\mathbf{T}}(\mathbf{F}^*).$$

However we know that $\mathbf{F}^* = \mathbf{Q}\mathbf{F}$ and $\mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^T$. Therefore we have

$$\mathbf{T} = \widehat{\mathbf{T}}(\mathbf{F}), \qquad \mathbf{Q}\mathbf{T}\mathbf{Q}^T = \widehat{\mathbf{T}}(\mathbf{Q}\mathbf{F}),$$

and on combining these we get

$$\widehat{\mathbf{T}}(\mathbf{F}) = \mathbf{Q}^T\widehat{\mathbf{T}}(\mathbf{Q}\mathbf{F})\mathbf{Q}. \qquad (a)$$

Equation (a) must hold for all nonsingular tensors $\mathbf{F}$ and all rotations $\mathbf{Q}$. This is what material frame indifference requires. We are asked to determine the most most general form of the constitutive relation that satisfies this requirement.

(i) Since (a) must hold for all rotations $\mathbf{Q}$ it must necessarily hold for $\mathbf{Q} = \mathbf{R}^T$ where $\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}$ with $\mathbf{U} = (\mathbf{F}^T\mathbf{F})^{1/2}$. Therefore necessarily

$$\widehat{\mathbf{T}}(\mathbf{F}) = \mathbf{R}\widehat{\mathbf{T}}(\mathbf{U})\mathbf{R}^T \qquad \text{where} \;\; \mathbf{U} = (\mathbf{F}^T\mathbf{F})^{1/2}, \; \mathbf{R} = \mathbf{F}\mathbf{U}^{-1}. \tag{b}$$

Conversely suppose that (b) hold for all nonsingular tensors $\mathbf{F}$. Then it must also hold for $\mathbf{QF}$ where $\mathbf{Q}$ is any rotation. The associated stretch and rotation tensors are

$$((\mathbf{QF})^T(\mathbf{QF}))^{1/2} = (\mathbf{F}^T\mathbf{F})^{1/2} = \mathbf{U}, \qquad (\mathbf{QF})\mathbf{U}^{-1} = \mathbf{QR}.$$

and so (b) when applied to $\mathbf{QF}$ gives

$$\widehat{\mathbf{T}}(\mathbf{QF}) = \mathbf{QR}\widehat{\mathbf{T}}(\mathbf{U})\mathbf{R}^T\mathbf{Q}^T \tag{c}$$

Combining (b) and (c) yields (a). Thus (b) is necessary and sufficient for the constitutive relation to be consistent with material frame indifference.

(ii) Next, let $\tilde{\mathbf{T}}$ be the function defined by

$$\tilde{\mathbf{T}}(\mathbf{U}) = \mathbf{U}^{-1}\widehat{\mathbf{T}}(\mathbf{U})\mathbf{U}^{-1}$$

Substituting this into (b) gives

$$\widehat{\mathbf{T}}(\mathbf{F}) = \mathbf{F}\tilde{\mathbf{T}}(\mathbf{U})\mathbf{F}^T$$

and so if we introduce the function $\overline{\mathbf{T}}(\mathbf{C}) = \tilde{\mathbf{T}}(\sqrt{\mathbf{C}})$ we get

$$\widehat{\mathbf{T}}(\mathbf{F}) = \mathbf{F}\overline{\mathbf{T}}(\mathbf{C})\mathbf{F}^T. \tag{d}$$

(iii) The second Piola-Kirchhoff tensor $\mathbf{S}^{(2)}$ is defined by $\mathbf{S}^{(2)} = J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T}$ and so substituting (d) into this yields

$$\mathbf{S}^{(2)} = J\;\overline{\mathbf{T}}(\mathbf{C}) = (\det \mathbf{F})\;\overline{\mathbf{T}}(\mathbf{C}) = (\det \mathbf{U})\;\overline{\mathbf{T}}(\mathbf{C}) = (\det \sqrt{\mathbf{C}})\;\overline{\mathbf{T}}(\mathbf{C}),$$

and so we can write

$$\mathbf{S}^{(2)} = \widehat{\overline{\mathbf{T}}}(\mathbf{C}).$$

(iv) From $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ and $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ it is readily shown that

$$\mathbf{F}\mathbf{C}^n\mathbf{F}^T = \mathbf{B}^{n+1} \qquad n = 0, 1, \ldots \tag{e}$$

For example by induction, if (e) holds for $n = p$, then because $\mathbf{B}^{p+2} = \mathbf{B}^{p+1}\mathbf{B} = \mathbf{F}\mathbf{C}^p\mathbf{F}^T\mathbf{B} = \mathbf{F}\mathbf{C}^p\mathbf{F}^T\mathbf{F}\mathbf{F}^T = \mathbf{F}\mathbf{C}^p\mathbf{C}\mathbf{F}^T = \mathbf{F}\mathbf{C}^{p+1}\mathbf{F}^T$, it will hold for $n = p + 1$. It is trivial to verify that the given identity holds for $n = 0$. This establishes (e). The result we are asked to prove follows immediately since $\widehat{\mathbf{T}}(\mathbf{F}) = \mathbf{F}\overline{\mathbf{T}}(\mathbf{C})\mathbf{F}^T$.

Problem 8.4. We know that symmetry is a property of a configuration, not a body. A body which is isotropic in one configuration will not, in general, be isotropic in another configuration. More generally, let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the material symmetry groups associated with two reference configurations χ_1 and χ_2 respectively. Then in general $\mathcal{G}_1 \neq \mathcal{G}_2$. However, knowing $\mathcal{G}_1$ allows one to calculate $\mathcal{G}_2$ provided that one knows the deformation gradient tensor of the mapping from $\chi_1 \to \chi_2$.

If $\mathcal{G}_1$ and $\mathcal{G}_2$ are the material symmetry groups of two configurations show that

$$\mathcal{G}_2 = \left\{ \mathbf{P} \mid \mathbf{H}^{-1}\mathbf{P}\mathbf{H} \in \mathcal{G}_1 \right\}. \tag{8.114}$$

where $\mathbf{H}$ is the gradient of the mapping from $\chi_1 \to \chi_2$. Thus for any $\mathbf{A} \in \mathcal{G}_1$ we have that $\mathbf{H}\mathbf{A}\mathbf{H}^{-1} \in \mathcal{G}_2$ and conversely for any $\mathbf{B} \in \mathcal{G}_2$ we have that $\mathbf{H}^{-1}\mathbf{B}\mathbf{H} \in \mathcal{G}_1$.

Solution: Consider a particle p and let $\mathbf{H}$ be the gradient of the mapping from configuration χ_1 to configuration χ_2 at p. It is not necessary for the mass densities at p in these two configurations to be the same, and so det $\mathbf{H}$ need not be ± 1. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the material symmetry groups at p in these two configurations. The stress response functions $\hat{\mathbf{T}}_1$ and $\hat{\mathbf{T}}_2$ in these two configurations are related by

$$\hat{\mathbf{T}}_1(\mathbf{F}) = \hat{\mathbf{T}}_2(\mathbf{F}\mathbf{H}^{-1}) \qquad \text{for all nonsingular tensors } \mathbf{F}.$$

Recall from (8.27) that a tensor $\mathbf{P}$ belongs to symmetry group $\mathcal{G}_1$ if and only if

$$\hat{\mathbf{T}}_1(\mathbf{F}\mathbf{P}) = \hat{\mathbf{T}}_1(\mathbf{F}) \qquad \text{for all nonsingular tensors } \mathbf{F},$$

or, by making use of (a), if and only if

$$\hat{\mathbf{T}}_2(\mathbf{F}\mathbf{P}\mathbf{H}^{-1}) = \hat{\mathbf{T}}_2(\mathbf{F}\mathbf{H}^{-1}) \qquad \text{for all nonsingular tensors } \mathbf{F},$$

which in turn holds if and only if

$$\hat{\mathbf{T}}_2(\mathbf{F}\mathbf{H}\mathbf{P}\mathbf{H}^{-1}) = \hat{\mathbf{T}}_2(\mathbf{F}) \qquad \text{for all nonsingular tensors } \mathbf{F}.$$

Thus $\mathbf{P}$ belongs to $\mathcal{G}_1$ if and only if $\mathbf{H}\mathbf{P}\mathbf{H}^{-1}$ belongs to $\mathcal{G}_2$ and so we have shown that

$$\mathcal{G}_2 = \left\{ \mathbf{P} \mid \mathbf{H}^{-1}\mathbf{P}\mathbf{H} \in \mathcal{G}_1 \right\}.$$

The material symmetry group $\mathcal{G}_2$ therefore consists of all elements of $\mathcal{G}_1$, premultiplied by $\mathbf{H}$ and postmultiplied by $\mathbf{H}^{-1}$. One often writes $\mathcal{G}_2 = \mathbf{H}\mathcal{G}\mathbf{H}^{-1}$.

Problem 8.5. Let $\widehat{\mathbf{T}}(\mathbf{F})$ be the objective Cauchy stress response function of an elastic material.

(i) If the material symmetry group of the reference configuration is the full unimodular group $\mathcal{U}$ show that there is a scalar-valued function $\widehat{p}$ such that

$$\widehat{\mathbf{T}}(\mathbf{F}) = -\widehat{p}(\det \mathbf{F})\mathbf{I}.$$

Truesdell and Noll call such a material an elastic (i.e. compressible, inviscid fluid).

(ii) Next, if the material symmetry group of one reference configuration is the full unimodular group $\mathcal{U}$, then show that the material symmetry group of *every* reference configuration is also $\mathcal{U}$. (Thus, if the material is an elastic fluid in one reference configuration then it is an elastic fluid in every reference configuration).

Solution:

(i) An elastic material here is characterized by the constitutive relation $\mathbf{T} = \widehat{\mathbf{T}}(\mathbf{F})$ (rather than in terms of the gradient of the free energy as in the preceding notes). The deformation gradient tensor $\mathbf{F}$ and the Cauchy stress response function $\widehat{\mathbf{T}}$ depend on the choice of reference configuration. By definition, a tensor $\mathbf{H}$ belongs to the material symmetry group of the reference configuration if it is unimodular ($\det \mathbf{H} = \pm 1$) and

$$\widehat{\mathbf{T}}(\mathbf{F}) = \widehat{\mathbf{T}}(\mathbf{F}\mathbf{H}) \tag{a}$$

for all nonsingular $\mathbf{F}$. The tensor $\mathbf{H}$ represents a transformation of the reference configuration in which the stress response function $\widehat{\mathbf{T}}$ and the mass density remain invariant.

Now consider the particular class of elastic materials described in this problem. For it, equation (a) holds for all nonsingular $\mathbf{F}$ and *all* unimodular $\mathbf{H}$. Since this holds for all unimodular $\mathbf{H}$ it necessarily holds for the particular choice

$$\mathbf{H} = J^{1/3}\mathbf{F}^{-1}, \qquad J = \det \mathbf{F}.$$

Substituting this into (a) tells us that

$$\widehat{\mathbf{T}}(\mathbf{F}) = \widehat{\mathbf{T}}(J^{1/3}\mathbf{I}), \qquad J = \det \mathbf{F}, \tag{b}$$

for all nonsingular $\mathbf{F}$. Material frame indifference requires

$$\widehat{\mathbf{T}}(\mathbf{Q}\mathbf{F}) = \mathbf{Q}\widehat{\mathbf{T}}(\mathbf{F})\mathbf{Q}^T \tag{c}$$

for all orthogonal $\mathbf{Q}$ and all nonsingular $\mathbf{F}$. Thus by (b) and (c) we must have

$$\widehat{\mathbf{T}}(J^{1/3}\mathbf{I}) = \mathbf{Q}\widehat{\mathbf{T}}(J^{1/3}\mathbf{I})\mathbf{Q}^T \tag{d}$$

for all orthogonal $\mathbf{Q}$ where we have used $\det \mathbf{Q}\mathbf{F} = \det \mathbf{F} = J$. Recall that if a symmetric tensor $\mathbf{A}$ is such that $\mathbf{A} = \mathbf{Q}\mathbf{A}\mathbf{Q}^T$ for all orthogonal $\mathbf{Q}$ then $\mathbf{A}$ must be a scalar multiple of the identity tensor. Thus in our setting (d) implies that

$$\widehat{\mathbf{T}}(J^{1/3}\mathbf{I}) = \alpha\mathbf{I} \qquad \text{where} \quad \alpha = \alpha(J);$$

or by (b)

$$\widehat{\mathbf{T}}(\mathbf{F}) = \alpha(J)\mathbf{I}, \qquad J = \det \mathbf{F}, \tag{e}$$

for all nonsingular $\mathbf{F}$.

Conversely if (e) holds it can be readily verified that (a) holds for all unimodular $\mathbf{H}$ and (c) hold for all orthogonal $\mathbf{Q}$.

(ii) Next, consider a body whose material symmetry group in some configuration χ_1 is the unimodular group: $\mathcal{G}_1 = \mathcal{U}$. Let χ_2 be any other configuration and let $\mathbf{H}$ be the gradient of the mapping from χ_1 to

χ_2. If $\mathbf{P}$ is any element of $\mathcal{U}$ then so is $\mathbf{H}^{-1}\mathbf{P}\mathbf{H}$, and thus $\mathbf{H}^{-1}\mathbf{P}\mathbf{H} \in \mathcal{G}_1$. Therefore by (8.114), $\mathbf{P} \in \mathcal{G}_2$. Thus we have shown that any element $\mathbf{P}$ of $\mathcal{U}$ is necessarily an element of $\mathcal{G}_2$, i.e. $\mathcal{U} \subset \mathcal{G}_2$. However, all material symmetry groups are subsets of the unimodular group and therefore $\mathcal{G}_2 \subset \mathcal{U}$. By combining these, $\mathcal{U} \subset \mathcal{G}_2 \subset \mathcal{U}$, and so $\mathcal{G}_2 = \mathcal{U}$. This means that *if the material symmetry group coincides with the unimodular group in any one configuration, then it coincides with it in every configuration.*

Problem 8.6. If the material symmetry group $\mathcal{G}$ of some material is a subgroup of the orthogonal group $\mathcal{O}$ in some configuration, does this imply that the material symmetry group in every configuration is a subgroup of the orthogonal group? Truesdell and Noll call such a material an elastic solid. (Compare this with a similar result for the unimodular group $\mathcal{U}$ in Problem 8.5.)

Problem 8.7. Let $\widehat{\mathbf{T}}(\mathbf{F})$ be the objective stress response function of an elastic material. If the body is isotropic in the reference configuration, so that the material symmetry group contains the orthogonal group, show that $\widehat{\mathbf{T}}(\mathbf{F})$ admits the representation

$$\hat{\mathbf{T}}(\mathbf{F}) = \overline{\mathbf{T}}(\mathbf{B})$$

where $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ is the Eulerian Cauchy Green tensor, and where the function $\overline{\mathbf{T}}$ obeys

$$\overline{\mathbf{T}}(\mathbf{Q}\mathbf{B}\mathbf{Q}^T) = \mathbf{Q}\overline{\mathbf{T}}(\mathbf{B})\mathbf{Q}^T$$

for all orthogonal tensors $\mathbf{Q}$ and all symmetric positive definite tensors $\mathbf{B}$.

It follows from this that $\overline{\mathbf{T}}(\mathbf{B})$ is an isotropic function and therefore admits the representation[9]

$$\overline{\mathbf{T}}(\mathbf{B}) = \tau_o\mathbf{I} + \tau_1\mathbf{B} + \tau_2\mathbf{B}^2 \qquad \text{where} \quad \tau_i = \tau_i\big(I_1(\mathbf{B}), I_2(\mathbf{B}), I_3(\mathbf{B})\big),$$

and

$$I_1(\mathbf{B}) = \mathrm{tr}\,\mathbf{B}, \qquad I_2(\mathbf{B}) - \frac{1}{2}\big[(\mathrm{tr}\,\mathbf{B})^2 - \mathrm{tr}\,(\mathbf{B}^2)\big], \qquad I_3(\mathbf{B}) = \det\,\mathbf{B},$$

are the principal scalar invariants of $\mathbf{B}$.

Problem 8.8. Suppose that one has carried out a series of experiments on a particular elastic (inviscid compressible) gas and found that its specific heat at constant volume, c_v, and its specific heat at constant pressure, c_p, are both *constants.* Determine a complete characterization of the constitutive response function for the Helmholtz free energy of this material in terms of a set of parameters (unknown constants) only.

Solution: Since the rate of heat supply to a point in the body, per unit reference volume, is given by $\mathrm{Div}\mathbf{q}_o + \rho_o r$, it follows from the definition of specific heat that

$$\mathrm{Div}\mathbf{q}_o + \rho_o r = \begin{cases} \rho_o c_v \dot{\theta} & \text{in processes in which } \dot{\rho} = 0, \\[2ex] \rho_o c_p \dot{\theta} & \text{in processes in which } \dot{p} = 0. \end{cases}$$

[9] See for example Chapter 4 of Volume I.

It follows from this and the energy equation (8.17) that

$$\dot{\eta} = \begin{cases} c_v\dot{\theta}/\theta & \text{in processes in which } \dot{\rho} = 0, \\ \\ c_p\dot{\theta}/\theta & \text{in processes in which } \dot{p} = 0. \end{cases} \tag{a}$$

First consider processes in which $\dot{\rho} = 0$. Differentiating $\eta = \eta(\rho, \theta)$ with respect to time, and using $(a)_1$, gives

$$\dot{\eta} = \frac{\partial \eta}{\partial \rho}\dot{\rho} + \frac{\partial \eta}{\partial \theta}\dot{\theta} = \frac{\partial \eta}{\partial \theta}\dot{\theta} = c_v\frac{\dot{\theta}}{\theta} \qquad \Rightarrow \qquad \left(\frac{\partial \eta}{\partial \theta} - \frac{c_v}{\theta}\right)\dot{\theta} = 0.$$

Since this must hold for all proccess of this type, it must necessarily hold for arbitrary $\dot{\theta}$ and so

$$\frac{\partial \eta}{\partial \theta} = \frac{c_v}{\theta}.$$

Integrating this yields

$$\eta = c_v \log\left(\theta/\theta_o\right) + f(\rho), \tag{b}$$

where the constant θ_o and the function $f(\rho)$ arise from the integration. Since $\eta = -\partial\psi/\partial\theta$ we can integrate this once more to obtain

$$\psi = -c_v\theta\left[\log\left(\theta/\theta_o\right) - 1\right] - \theta f(\rho) - g(\rho), \tag{c}$$

where $g(\rho)$ is a function arising from the integration.

Next consider processes in which $\dot{p} = 0$. By calculating $\dot{\eta}$ from (b) and using the result in $(a)_2$ gives

$$f'(\rho)\dot{\rho} + c_v\frac{\dot{\theta}}{\theta} = c_p\frac{\dot{\theta}}{\theta} \tag{d}$$

which must hold in all processes in which $\dot{p} = 0$. This does *not* mean that (d) must hold for all $\dot{\rho}$ and $\dot{\theta}$ because the requirement $\dot{p} = 0$ relates $\dot{\rho}$ to $\dot{\theta}$. In order to proceed further, we need to first determine this relationship and then eliminate either $\dot{\rho}$ or $\dot{\theta}$ from (d). Substituting (c) into $(8.64)_1$ gives

$$p = \rho^2\frac{\partial \psi}{\partial \rho} = -\rho^2\theta f'(\rho) - \rho^2 g'(\rho), \tag{e}$$

and then setting $\dot{p} = 0$ leads to

$$\left[\theta(\rho^2 f'' + 2\rho f') + \rho^2 g'' + 2\rho g'\right]\dot{\rho} = -\rho^2 f'\dot{\theta}$$

which is the relationship between $\dot{\rho}$ and $\dot{\theta}$. Using this to eliminate $\dot{\rho}$ from (d), and recognizing that the result must hold in all processes with $\dot{p} = 0$, i.e. for arbitrary $\dot{\theta}$, leads one to

$$\theta\left[\rho^2 f'' + 2\rho f' + \frac{\rho^2(f')^2}{R}\right] + \left[\rho^2 g'' + 2\rho g'\right] = 0 \tag{f}$$

where we have set

$$R = c_p - c_v.$$

Since (f) must hold for all $\rho > 0$ and $\theta > 0$ it follows that both of the following must hold:

$$\rho^2 f'' + 2\rho f' + \frac{\rho^2(f')^2}{R} = 0, \tag{g}$$

$$\rho^2 g'' + 2\rho g' = 0. \tag{h}$$

Solving the differential equations (g) and (h) leads to

$$g(\rho) = d_2 - \frac{d_1}{\rho}, \qquad f(\rho) = -R\log\rho + d_3, \tag{i}$$

where d_1, d_2 and d_3 are constants.

In summary, combining (b), (c), (e) and (i) provide the following constitutive characterization of this gas:

$$\left.\begin{array}{rcl}
\psi & = & R\theta\log\rho - c_v\theta\left[\log\left(\dfrac{\theta}{\theta_0}\right) - 1\right] - d_3\theta + \dfrac{d_1}{\rho} - d_2 \\
p & = & R\rho\theta - d_1, \\
\eta & = & -R\log\rho + c_v\log\left(\dfrac{\theta}{\theta_0}\right) + d_3, \\
\varepsilon & = & c_v\theta + \dfrac{d_1}{\rho} - d_2.
\end{array}\right\}$$

The constant $-d_1$ is seen to represent the pressure in the gas at absolute zero temperature. But on physical ground this should be zero. Also, the constants d_2 and d_3 simply set the datum values of energy and entropy and can therefore be chosen arbitrarily. Taking them also to be zero leads to the classical equations for an ideal polytropic gas:

$$\left.\begin{array}{rcl}
\psi & = & R\theta\log\rho - c_v\theta\left[\log\left(\dfrac{\theta}{\theta_0}\right) - 1\right] \\
p & = & R\rho\theta, \\
\eta & = & -R\log\rho + c_v\log\left(\dfrac{\theta}{\theta_0}\right), \\
\varepsilon & = & c_v\theta.
\end{array}\right\} \tag{j}$$

Eliminating θ between $(j)_2$ and $(j)_3$ leads to

$$p = R\ \theta_0\ \rho^\gamma\ \exp[\eta/c_v - 1] \qquad \text{where } \gamma = R/c_v + 1. \tag{k}$$

The preceding equations are valid in all processes. Note that $(j)_2$ is the "stress-strain-temperature relation" and is particularly suited for use in isothermal processess, while (k) is the "stress-strain-entropy relation" and its form is particularly convenient for isentropic processes.

Problem 8.9. Consider a so-called anti-plane deformation. described with respect to a fixed orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ by

$$y_1 = x_1, \qquad y_2 = x_2, \qquad y_3 = x_3 + u(x_1, x_2).$$

Consider an arbitrary incompressible isotropic elastic material and determine all of the field equations that u must satisfy.

What happens in the special case when $W = W(I_1)$?

Solution:

Reference: J. K. Knowles, On finite anti-plane shear for incompressible elastic materials, *Journal of the Australian Mathematical Society*, Series B, **19**(1976), pp. 400-415.

We are given the deformation field

$$y_1 = x_1, \quad y_2 = x_2, \quad y_3 = x_3 + u(x_1, x_2). \tag{a}$$

Using $F_{ij} = \partial y_i / \partial x_j$ we can calculate the matrix of components of the deformation gradient tensor:

$$[F] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k_1 & k_1 & 1 \end{pmatrix}$$

where for simplicity we use the notation

$$k_1 = \frac{\partial u}{\partial x_1}, \qquad k_2 = \frac{\partial u}{\partial x_2}, \qquad k = |\nabla u| = \sqrt{k_1^2 + k_2^2}.$$

Observe that

$$\det[F] = 1$$

automatically. We can now calculate the components of the Cauchy-Green tensor $[B] = [F][F]^T$ and its square $[B]^2$:

$$[B] = [F][F]^T = \begin{pmatrix} 1 & 0 & k_1 \\ 0 & 1 & k_2 \\ k_1 & k_2 & 1+k^2 \end{pmatrix}, \qquad [B]^2 = \begin{pmatrix} 1+k_1^2 & k_1k_2 & (2+k^2)k_1 \\ k_1k_2 & 1+k_2^2 & (2+k^2)k_2 \\ (2+k^2)k_1 & (2+k^2)k_2 & 1+3k^2+k^4 \end{pmatrix}. \tag{b}$$

The principal invariants are

$$I_1(\mathbf{B}) = \mathrm{tr}\,[B] = 3 + k^2, \qquad I_2(\mathbf{B}) = \frac{1}{2}[(\mathrm{tr}[B])^2 - \mathrm{tr}\,([B]^2)] = 3 + k^2. \tag{c}$$

We now substitute (b) and (c) into the constitutive relation for the Cauchy stress for an isotropic incompressible material

$$\mathbf{T} = -p\mathbf{I} + 2(W_1 + I_1 W_2)\mathbf{B} - 2W_2\mathbf{B}^2$$

where $W_\alpha = \partial W / \partial I_\alpha$, $\alpha = 1, 2$. After some simplification we find

$$\left.\begin{aligned} T_{11} &= -p + 2W_1 + 2(2+k_2^2)W_2, \\ T_{22} &= -p + 2W_1 + 2(2+k_1^2)W_2, \\ T_{33} &= -p + 2(1+k^2)W_1 + 2(2+k^2)W_2, \\ T_{12} &= T_{21} = -2W_2k_1k_2, \\ T_{13} &= T_{31} = 2k_1(W_1+W_2), \\ T_{23} &= T_{32} = 2k_2(W_1+W_2), \end{aligned}\right\} \tag{d}$$

where it is understood that

$$W_\alpha = \left.\frac{\partial W}{\partial I_\alpha}\right|_{I_1=I_2=3+k^2}, \qquad \alpha = 1, 2.$$

If we substitute the stresses (d) into the equilibrium equations without body forces

$$\frac{\partial T_{ij}}{\partial y_j} = 0 \ ,$$

we obtain 3 partial differential equations for 2 unknown fields $u(x_1, x_2)$, $p(x_1, x_2, x_3)$. And in general these equations are not self-consistent and therefore have no solution. For an analysis that proves this rigorously see the reference above. This means that *an anti-plane deformation* (a) *cannot be sustained by an arbitrary isotropic incompressible material (in the absence of body forces).*

Now consider the special material $W = W(I_1)$. The expressions (d) for stress now simplify to

$$T_{11} = -p + 2W' \qquad T_{22} = -p + 2W' \qquad T_{33} = -p + 2(1 + k^2)W'$$

$$T_{12} = T_{21} = 0 \tag{e}$$

$$T_{13} = T_{31} = 2k_1 W' \qquad T_{23} = T_{32} = 2k_2 W'$$

where $W' = dW/dI_1$. For simplicity let us assume that p is also independent of x_3 so that $p = p(x_1, x_2)$. It is not necessary to make this assumption. Note that all the stress components now depend on only x_1, x_2 at most.

We now substitute (e) into the equilibrium equations. The third equation $\partial T_{3j}/\partial y_j = 0$ becomes

$$\frac{\partial}{\partial x_1}\left(W'(3 + |\nabla u|^2)\,\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(W'(3 + |\nabla u|^2)\,\frac{\partial u}{\partial x_2}\right) = 0 \tag{f}$$

where we have made use of the fact that $y_1 = x_1, y_2 = x_2$. This is a single partial differential equation for $u(x_1, x_2)$. The first equilibrium equation $\partial T_{1j}/\partial y_j = 0$ gives

$$\frac{\partial}{\partial x_1}\ (-p + 2W') = 0, \tag{g}$$

while the second equilibrium equation $\partial T_{2j}/\partial y_j = 0$ gives

$$\frac{\partial}{\partial x_2}\ (-p + 2W') = 0. \tag{h}$$

It follows from (g) and (h) that

$$p(x_1, x_2) = 2W'(3 + |\nabla u|^2).$$

Remark: Observe that formally, when we wrote out the equilibrium equations in terms of u and p we had the 3 equations (f), (g) and (h) for the two unknown fields. But in this case equations (g) and (h) are self-consistent in that if one differentiates the former by x_2 one gets the same result as when differentiating the latter by x_1.

Problem 8.10. Here we consider a body subjected to a certain kinematic constraint. Let the unit vector $\mathbf{m}$ denote a direction in the reference configuration, and suppose that the area of any plane normal to $\mathbf{m}$ cannot change. (Though the body is treated as a homogeneous continuum, it might, for example, be a solid that has a family of stiff parallel planes aligned normal to the direction $\mathbf{m}$.) Determine the corresponding reaction stress.

Problem 8.11. Ericksen has suggested that certain elastic crystals obey the kinematic constraint

$$\operatorname{tr}\mathbf{C} = 3.$$

Determine the associated reaction stress.

Reference: J. L. Ericksen, Constitutive theory for some constrained elastic crystals, *International Journal of Solids and Structures*, Vol. 22, 1986, pp. 951-964.

Problem 8.12. Show that the symmetry of a configuration is unchanged by a pure dilatation.

Solution:

If the configurations χ_1 and χ_2 happen to be related by a dilatation, so that $\mathbf{H} = h\mathbf{I}$, it follows from (8.114) that $\mathcal{G}_2 = \mathcal{G}_1$. Thus the *symmetry of a configuration is unchanged by a pure dilatation.*

Problem 8.13. Let $\mathcal{G}(\mathbf{e})$ be the set of all volume preserving tensors that leaves a given unit vector $\mathbf{e}$ invariant to within sign, i.e.

$$\mathcal{G}(\mathbf{e}) = \Big\{\mathbf{H} \ : \ \det\mathbf{H} = \pm 1, \ \mathbf{He} = \pm\mathbf{e}\Big\}.$$

(i) Show that $\mathcal{G}(\mathbf{e})$ is a group.

(ii) Show that $\mathcal{G}(\mathbf{e})$ is *not* the unimodular group.

(iii) Show that $\mathcal{G}(\mathbf{e})$ is *not* a subgroup of the orthogonal group.

(iv) Suppose that $\mathcal{G}(\mathbf{e})$ is the material symmetry group of a body in a certain reference configuration. Show that the symmetry group of this body in *every* reference configuration is (*a*) *not* the unimodular group, and (*b*) *not* a subgroup of the orthogonal group. (Thus in the terminology of Truesdell and Noll, this simple material is neither a fluid nor a solid.)

Reference B. D. Coleman, Simple Liquid Crystals, *Archive for Rational Mechanics and Analysis*, **20**(1965), pp. 41-58.

Solution:

(*i*) In order to show that $\mathcal{G}(\mathbf{e})$ is a group we must show that if any two tensors belong to $\mathcal{G}(\mathbf{e})$ then so does their product; and that if any tensor belongs to $\mathcal{G}(\mathbf{e})$ so does its inverse.

Suppose that $\mathbf{H}_1$ and $\mathbf{H}_2$ both belong to $\mathcal{G}(\mathbf{e})$. Then

$$\det \mathbf{H}_1 = \pm 1, \qquad \mathbf{H}_1\mathbf{e} = \pm\mathbf{e}, \qquad \det \mathbf{H}_2 = \pm 1, \qquad \mathbf{H}_2\mathbf{e} = \pm\mathbf{e}. \tag{a}$$

First, it follows from $(a)_{2,4}$ that $\mathbf{H}_1\mathbf{H}_2\mathbf{e} = \mathbf{H}_1(\mathbf{H}_2\mathbf{e}) = \mathbf{H}_1(\pm\mathbf{e}) = \pm\mathbf{e}$; and from $(a)_{1,3}$ that $\det(\mathbf{H}_1\mathbf{H}_2) = \det \mathbf{H}_1 \det \mathbf{H}_2 = \pm 1$. Therefore $\mathbf{H}_1\mathbf{H}_2 \in \mathcal{G}(\mathbf{e})$. Second, it follows from $(a)_{1,2}$ that $\det \mathbf{H}_1^{-1} = \pm 1$ and that $\mathbf{H}_1^{-1}\mathbf{e} = \pm\mathbf{e}$. Therefore $\mathbf{H}_1^{-1} \in \mathcal{G}(\mathbf{e})$. In view of these two properties of $\mathcal{G}(\mathbf{e})$ is a group.

(*ii*) If $\mathcal{G}(\mathbf{e})$ coincides with the unimodular group $\mathcal{U}$ then necessarily every member of $\mathcal{U}$ must also be a member of $\mathcal{G}(\mathbf{e})$. Thus if we can find any one tensor that is in $\mathcal{U}$ but not in $\mathcal{G}(\mathbf{e})$ this would establish the desired result $\mathcal{G}(\mathbf{e}) \neq \mathcal{U}$.

Consider an orthonormal basis $\{\mathbf{e}, \mathbf{e}_2, \mathbf{e}_3\}$ and let $\mathbf{P}$ be the tensor whose components in this basis are

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \alpha \neq \pm 1.$$

Since $\det \mathbf{P} = 1$ it follows that $\mathbf{P}$ is a member of the unimodular group $\mathcal{U}$. However since the components of $\mathbf{e}$ in this basis are $\{1, 0, 0\}$ we see that $\mathbf{P}\mathbf{e} = \alpha\mathbf{e} \neq \pm\mathbf{e}$ and so $\mathbf{P}$ is *not* a member of $\mathcal{G}(\mathbf{e})$. Therefore $\mathcal{G}(\mathbf{e}) \neq \mathcal{U}$.

(*iii*) If $\mathcal{G}(\mathbf{e})$ is a subgroup of the orthogonal group $\mathcal{O}$ then every member of $\mathcal{G}(\mathbf{e})$ must also be a member of $\mathcal{O}$. Thus if we can find any one tensor that is in $\mathcal{G}(\mathbf{e})$ but not in $\mathcal{O}$ this would establish the desired result $\mathcal{G}(\mathbf{e}) \not\subset \mathcal{U}$.

Consider again the orthonormal basis $\{\mathbf{e}, \mathbf{e}_2, \mathbf{e}_3\}$ and now let $\mathbf{P}$ be the tensor whose components in this basis are

$$\begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \alpha \neq 0.$$

Since $\det \mathbf{P} = 1$ and $\mathbf{P}\mathbf{e} = \mathbf{e}$ it follows that $\mathbf{P} \in \mathcal{G}(\mathbf{e})$. However $\mathbf{P}\mathbf{P}^T \neq \mathbf{I}$ and so $\mathbf{P}$ is *not* a member of the orthogonal group $\mathcal{O}$. Therefore $\mathcal{G}(\mathbf{e}) \not\subset \mathcal{U}$.

(*iv*) It was shown in Problem 8.5 that if the material symmetry group in any one reference configuration is the unimodular group then the material symmetry group in every reference configuration will necessarily be the unimodular group. Therefore if the material symmetry group in some reference configuration is not the unimodular group it follows that there is no reference configuration in which the material symmetry group will be the unimodular group. The desired result of the first part of (*iv*) now follows from this and the result of part (*ii*).

Turning to the second part, let $\mathcal{G}_2$ be the material symmetry group in an arbitrary second reference configuration that is related to the first by the deformation gradient tensor $\mathbf{F}$. Then from Problem 8.4,

$\mathcal{G}_2 = \mathbf{F}\mathcal{G}(\mathbf{e})\mathbf{F}^{-1}$, i.e. for every $\mathbf{P} \in \mathcal{G}(\mathbf{e})$ we have $\mathbf{FPF}^{-1} \in \mathcal{G}_2$. Since $\mathbf{P} \in \mathcal{G}(\mathbf{e})$ it follows that $\mathbf{Pe} = \pm\mathbf{e}$ which we can write as $\mathbf{PF}^{-1}\mathbf{Fe} = \pm\mathbf{e}$ whence $\mathbf{FPF}^{-1}(\mathbf{Fe}) = \pm(\mathbf{Fe})$. Therefore $\mathcal{G}_2 = \mathcal{G}(\mathbf{Fe})$. We can now repeat the argument of part (iii) to show that $\mathcal{G}_2 = \mathcal{G}(\mathbf{Fe}) \not\subset \mathcal{O}$.

Thus we have shown that there is no reference configuration in which the material symmetry group coincides with the unimodular group, and that there is no reference configuration in which the material symmetry group is a subset of the orthogonal group.

Problem 8.14. Show that the constitutive relation of an elastic material can be written in terms of the *second Piola-Kirchhoff stress tensor* $\mathbf{P}$ as

$$\mathbf{P} = \frac{\partial \overline{W}}{\partial \mathbf{E}} \tag{8.115}$$

where $\overline{W}(\mathbf{E})$ is the strain energy function expressed in terms of the Green strain tensor $\mathbf{E}$:

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}). \tag{a}$$

<u>Solution</u>: Recall that we introduced the second Piola-Kirchhoff stress tensor $\mathbf{S}^{(2)}$ in Section 4.9. For notational simplicity, in this section we shall denote it by $\mathbf{P}$. Recall that $\mathbf{P}$ is conjugate to the time rate of change of the Green strain in the sense that the

$$\text{Stress power per unit reference volume } = \mathbf{S} \cdot \dot{\mathbf{F}} = \mathbf{P} \cdot \dot{\mathbf{E}}.$$

Also, it is related to the Cauchy stress tensor $\mathbf{T}$ and the first Piola-Kirchhoff stress tensor $\mathbf{S}$ by

$$\mathbf{P} = J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T}, \qquad \mathbf{P} = \mathbf{F}^{-1}\mathbf{S}.$$

Finally, recall that it is symmetric due to angular momentum balance:

$$\mathbf{P} = \mathbf{P}^T.$$

Now consider an elastic material. By material frame indifference the strain energy function W depends on $\mathbf{F}$ only through the right Cauchy Green tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$, or in view of (a), only through $\mathbf{E}$:

$$W(\mathbf{F}) = \overline{W}(\mathbf{E}).$$

The components of the first Piola-Kirchoff stress tensor can be calculated as follows:

$$S_{ij} = \frac{\partial W}{\partial F_{ij}} = \frac{\partial \overline{W}}{\partial E_{k\ell}}\frac{\partial E_{k\ell}}{\partial F_{ij}} = \frac{1}{2}\frac{\partial \overline{W}}{\partial E_{k\ell}}(\delta_{kj}F_{i\ell} + F_{ik}\delta_{\ell j}) = F_{ik}\frac{\partial \overline{W}}{\partial E_{kj}},$$

where we have used the fact that

$$\frac{\partial E_{k\ell}}{\partial F_{ij}} = \frac{1}{2}(\delta_{kj}F_{i\ell} + F_{ik}\delta_{\ell j})$$

which follows by differentiating (a) with respect to $\mathbf{F}$. Finally, since $\mathbf{S} = \mathbf{FP}$, this leads to (8.115). Observe that equation (8.115) is a further reflection of the fact that $\mathbf{P}$ is conjugate to $\mathbf{E}$.

Problem 8.15. A Van Der Waals Gas is an inviscid compressible fluid characterized by the Helmholtz free energy response function

$$\psi(v,\theta) = -R\theta\log(v-b) - \frac{a}{v} - c_v\theta\log\frac{\theta}{\theta_o},$$

where $v = 1/\rho$ is the specific volume. Determine explicit expressions for $\varepsilon(v,\theta), p(v,\theta)$ and $\eta(v,\theta)$.

Solution:

$$\varepsilon = \psi - \theta\frac{\partial\psi}{\partial\theta} = c_v\theta + a\left(1 - \frac{1}{v}\right) + R\theta_o\log(1-b),$$

$$p = -\frac{\partial\psi}{\partial v} = \rho_o\frac{R\theta}{v-b} - \rho_o\frac{a}{v^2},$$

Remark: The ideal gas law can be derived from molecular scale arguments based on the kinetic theory of gases. Among the various assumptions underlying this derivation is one that treats the gas as a collection of point masses (which occupy no volume) and another that neglects any forces between pairs of molecules. An intuitive way in which to correct for the former is to replace v by $v-b$ where the emperical constant $b > 0$ accounts for the volume occupied by the gas molecules. With regard to the second assumption, the presence of interatomic forces will tend to slow down a molecule as it approaches a wall and therefore to reduce the pressure in proportion to the number of interacting pairs of molecules suggests that p should be reduced by a term proportional to $1/v^2$. Based on these heuristic arguments Van der Waals replaced the ideal gas law $p = R\theta/v$ by

$$p = \frac{R\theta}{v-b} - \frac{a}{v^2}.$$

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Chapter 9

Elastic Materials: Micromechanical Models

Continuum theory says that an elastic material is characterized by a free energy function $\psi(\mathbf{C}, \theta)$. If additional information on material symmetry is available, this can be reduced further, for example to the form $\psi(I_1(\mathbf{C}), I_2(\mathbf{C}), I_3(\mathbf{C}), \theta)$ for an isotropic material. However, that is as far as the theory goes. The examples of explicit functions ψ given in Section 8.7 (corresponding, for example, to the Blatz-Ko or Fung models) are "phenomenological models" of particular elastic materials, i.e. the functional form of ψ is laid out at the outset at the continuum level, and subsequent laboratory experiments are used to refine it.

On the other hand the macroscopic (or continuum) behavior of a material reflects its underlying microscopic behavior. If one could describe the processes at the microscopic scale, and knew how to homogenize them across scales, one could then infer the response at the macroscopic scale. When this is possible, the continuum model so developed captures the microscopic physics.

In this chapter we shall start at the respective microscopic scales and develop two explicit forms for the free energy function $\psi(\mathbf{C}, \theta)$ describing two specific elastic materials, viz. a rubber-like material (in Section 9.1) and a crystalline solid (in Section 9.2). The microscale models we use are the simplest conceivable ones, and our purpose is *merely to illustrate* how one might develop continuum models from microscale models.

9.1 Example: Rubber Elasticity.

The reader is referred to Weiner and references therein, and Treloar, for an in-depth discussion of the material to follow. The discussion in Sections 9.1.1 - 9.1.4 below concern a single long chain molecule. In Section 9.1.5 a network of molecules built-up from a unit cell will be considered, and from this (microscopic) unit cell model we shall derive explicit (macroscopic) strain energy functions of the generalized Hookean type, i.e. $\psi = \psi(I_1)$.

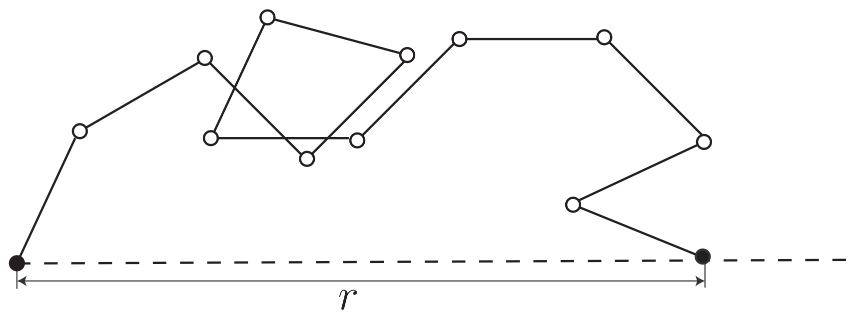

Figure 9.1: A long chain of N identical rigid links, each of length ℓ, each attached to its adjacent links by ball joints which allow free rotation. The end-to-end distance between the two ends of the chain is r.

Rubber consists of long, flexible, chain-like organic molecules. The molecule is chain-like in the sense that it is composed of a chain of links along the molecule's backbone. Since the $C - C$ bond of each link is very strong, each link is almost inextensible. However the links can rotate easily relative to each other, and the deformation of the molecule is related to the easy rotation of the links (one C-C bond relative to the next) rather than to the extensibility of the bond. See Figure 9.1. The molecule takes up a random conformation (configuration) in a stress-free state and progressively straightens out and gets oriented if a tensile force is applied. Once the molecule is perfectly straight, it cannot extend appreciably and the chain locks-up. A block of rubber consists of a large number of such molecules which are connected to each other (cross-linked) at certain points along their lengths.

In the simplest models, the distance r between the two ends of a molecule is the primary kinematic variable. Thus the Helmholtz free energy would have the form $\psi(r, \theta)$. As discussed in Section 5.2 of Weiner, experimental observations suggest that the dependency of the internal energy on r is negligible, and moreover that entropic effects dominate the stress-stretch-temperature behavior. Thus for the free energy we have

$$\psi(r, \theta) = \varepsilon(\theta) - \theta\eta(r, \theta)$$

so that the force-displacement-temperature relation is

$$f = \frac{\partial}{\partial r}\psi(r,\theta) = -\theta\frac{\partial}{\partial r}\eta(r,\theta).$$

Let $p(r)$ denote the probability density function. Then, the probability that the end-to-end distance of the molecule has a value between r and $r+dr$ is $p(r)\,dr$ in one dimension, and $4\pi r^2\,p(r)\,dr$ in three dimensions. While thermal fluctuations can affect p we neglect the effect of temperature on p here. From statistical mechanics, the entropy is then given by

$$\eta = k\;\ln p$$

where k is Boltzmann's constant, e.g. see Weiner. Thus for the free energy we can take

$$\psi(r,\theta) = \varepsilon(\theta) - k\theta\,\ln p(r). \tag{9.1}$$

and the relation between the force f, the end-to-end distance r, and the temperature θ is

$$f = -k\theta\,\frac{\partial}{\partial r}\Big(\ln p(r)\Big). \tag{9.2}$$

This relation is completely determined once the probability density function $p(r)$ is known.

9.1.1 A Single Long Chain Molecule: A One-Dimensional Toy Model.

Consider a chain consisting of N rigid links, each of length ℓ, so that its total (contour) length is $L = N\ell$. This is the length of the chain if it is completely stretched out. The links are connected by ball-joints at its ends. In the present one-dimensional toy model, every link must lie on the $x-$axis.

In some arbitrary configuration, one end of the chain is at the origin and the other is at $x = r$ where $r = j\ell$ and j is some integer in the interval $-N \le j \le N$. If the chain is viewed as an oriented curve, then in this configuration some of the links are oriented in the $+x$-direction while the rest are oriented in the $-x$-direction; see illustration in Figure 9.2. Let N_+ and N_- denote the number of links that are oriented in the $+x$ and $-x$ directions. Then necessarily

$$\left.\begin{aligned} N_+ + N_- &= N, \\ \ell N_+ - \ell N_- &= r = j\ell, \end{aligned}\right\}$$

which can be solved to give

$$\left.\begin{aligned} N_+ &= \frac{N+j}{2}, \\ N_- &= \frac{N-j}{2}. \end{aligned}\right\} \tag{9.3}$$

Thus, given the total number of links N, the end-to-end length of the chain r and the link length ℓ, equation (9.3) gives the number of links which are oriented forwards and the number oriented backwards.

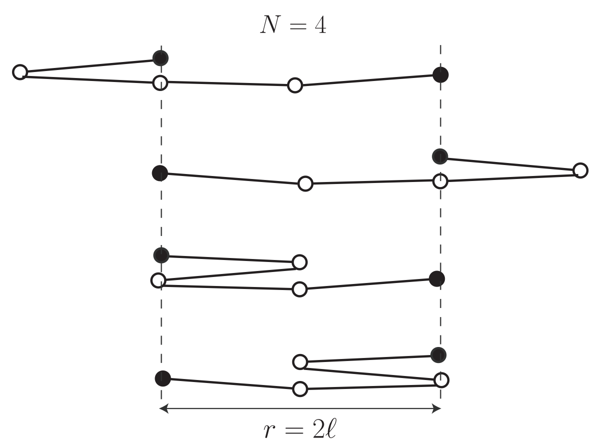

Figure 9.2: A chain consisting of $N = 4$ identical rigid links, each of length ℓ. One end of the chain is at the origin. In the configurations shown, the end-to-end distance between the two ends is $r = 2\ell$. There are $4 = 4!/(3!\,1!)$ ways in which the links can be arranged consistent with this information.

Since N_+ and N_- must be integers, it follows from $(9.3)_2$ that $N - j$ must be an even integer (in which case $N + j$ is automatically an even integer). In essence, this states that the end-to-end length of the chain r *cannot* take all values in the interval $-L \le r \le L$. The only locations at which the movable end of the chain can lie correspond to the values of r, or equivalently to the integers j, which are such that $N - j$ is an even integer. Observe that if j satisfies this condition then so do $j \pm 2$ but not $j \pm 1$. Thus the free end of the chain can only be located at every other node. (This restrictive feature is due to the extreme simplicity of the present one-dimensional model; if, for example, the two ends of the chain were located on the $x-$axis but the individual links were allowed to be anywhere in three-dimensional space, then the movable end can be located anywhere in $[-L, L]$.) In the present model, the movable end of the chain can be located at $x = r = j\ell$ if and only if $j \in \mathbb{Z}$ where

$$\mathbb{Z} = \{ i \mid i = \text{integer},\ -N \le i \le N,\ i = N - \text{even integer} \}. \tag{9.4}$$

Now pick and fix a $j \in \mathbb{Z}$. This fixes both ends of the chain. We want to know the number of different arrangements (configurations) of the chain that are consistent with these end locations. Since we know N and j, this determines the number N_+ of links oriented one way and the number N_- oriented the other way. This does *not* however completely determine the configuration of the chain[1] because the individual links can be arranged in several different ways, all of them with the same values of N_+ and N_- as illustrated in Figure 9.2 for the case $N = 4, j = 2$. The number of different ways in which the links can be arranged is

$$W(j) \ = \ \frac{N!}{N_+!\ N_-!} \ = \ \frac{N!}{\left(\dfrac{N+j}{2}\right)!\ \left(\dfrac{N-j}{2}\right)!}, \qquad j \in \mathbb{Z}. \tag{9.5}$$

It is customary to use the symbol W for this and it should not be confused with the strain energy function.

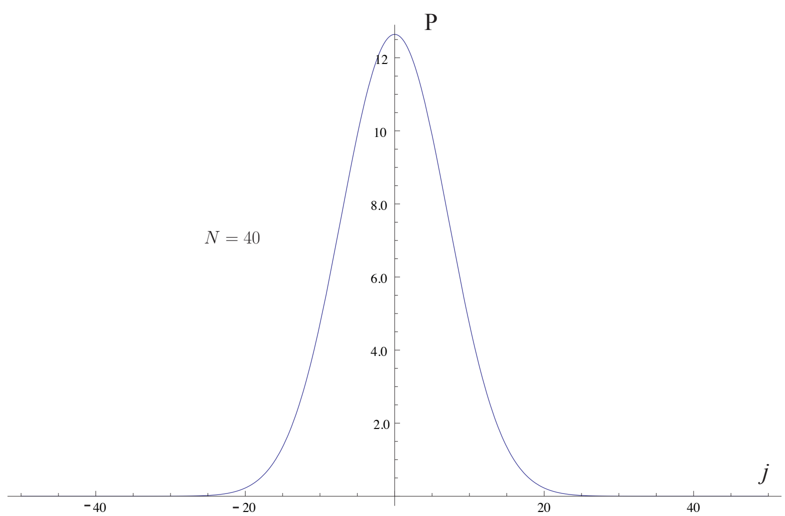

Figure 9.3: The binomial probability $P(j)$ versus j where P is given by (9.5), (9.7). It characterizes the probability of finding the free end of the chain at a certain location j. This graph should only show the set of discrete points corresponding to $j \in \mathbb{Z}$ rather than a curve.

Next, we know that the location of the movable end of the chain, characterized by j, can take any value $j \in \mathbb{Z}$, and that for each such j, the number of possible configurations is

[1]This is analogous to a one-dimensional random walk: if a person walks along the $x-$axis and takes a total of N steps, each of length ℓ, and ends up at the position $r = j\ell$, then the number of steps taken forwards N_+ and the number taken backwards N_- are fixed but the order in which these steps are taken is not fixed.

$W(j)$. Thus the *sum total* number of possible configurations of the chain is

$$\sum_{j\in\mathbb{Z}} W(j). \tag{9.6}$$

If each of these configurations can occur with equal likelihood, the *probability* of the chain having length $r = j\ell$ is

$$P(j) = \frac{W(j)}{\displaystyle\sum_{j\in\mathbb{Z}} W(j)}. \tag{9.7}$$

This describes a "binomial distribution" of probability. Figure 9.3 shows a plot of $P(j)$ versus the free-end location of the chain j for a particular value of N. Note that the most likely location of the free end of the chain is at the origin. The mean square of the end-to-end length, say $\overline{r^2}$, is proportional to $N\,\ell^2$.

Thus by (9.2), the relationship between the force f, length r and temperature θ for a one-dimensional chain with N links, each of length ℓ is given[2] explicitly by

$$f = f(r,\theta) = -k\theta\,\frac{\partial}{\partial r}\ln P\big(r/\ell\big), \tag{9.8}$$

where the probablity function $P(j)$ is given by (9.5), (9.7) with $j = r/\ell = Nr/L$. A graph of force[3] f versus end-to-end distance r is shown in Figure 9.4. Note the characteristic upward turning of the force-length curve. Since the chain cannot have a length greater than L, the chain "locks-up" when $r \to L$.

9.1.2 A Special Case of the Preceding One-Dimensional Long Chain Molecule.

We now approximate the exact solution of the preceding problem to the special case where (i) the chain has many links, and (ii) we limit attention to configurations in which the chain is far from being fully stretched:

$$N >> 1, \qquad |r| << L.$$

[2]Since r takes this discrete values in this model the derivative $\partial/\partial r$ is meaningful only for $N >> 1$.

[3]The calculation proceeds as follows: given the total number of links N, we construct the set $\mathbb{Z}$ from (9.4); the number of configurations at a fixed j, $W(j)$, from (9.5); the total number of configurations from (9.6); and the probability of a particular configuration from (9.7). The force is then given by (9.8).

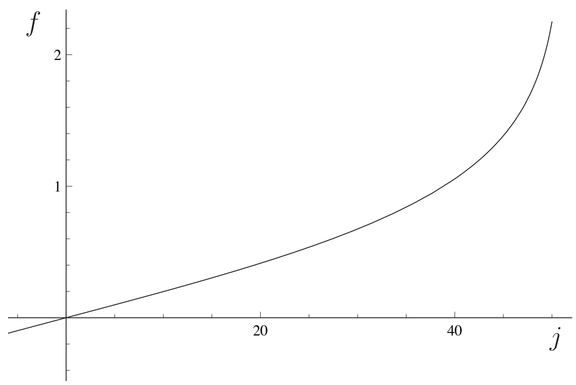

Figure 9.4: The force f versus end-to-end distance r of the chain as given by (9.4) - (9.8). This graph should only show the set of discrete points corresponding to $j \in \mathbb{Z}$ rather than a curve.

It is convenient to set

$$\xi = \frac{r}{L} = \frac{j}{N}. \tag{9.9}$$

Note that $|\xi| << 1$. For large values of the integer n, the Stirling approximation for $n!$ gives

$$n! \approx n \ln n - n + \frac{1}{2} \ln n + \frac{1}{2} \ln(2\pi).$$

It can therefore be readily shown that for fixed ξ and large N,

$$\ln \left[\frac{N!}{\left(\dfrac{N+j}{2}\right)! \left(\dfrac{N-j}{2}\right)!} \right] \approx -\frac{N}{2} \left[-2 \ln 2 + (1+\xi) \ln(1+\xi) + (1-\xi) \ln(1-\xi) \right],$$

to leading order. Next, this can be approximated for small ξ to get

$$\ln W = \ln \left[\frac{N!}{\left(\dfrac{N+j}{2}\right)! \left(\dfrac{N-j}{2}\right)!} \right] \approx -\frac{N}{2} \left[-2 \ln 2 + \xi^2 \right], \tag{9.10}$$

or

$$W = 2^N \, \mathrm{e}^{-b^2 r^2} \qquad \text{where} \quad b^2 = \frac{1}{2N\ell^2}.$$

Therefore the expression (9.7) leads to

$$p(r) = \frac{2^N \, \mathrm{e}^{-b^2 r^2}}{\int_{-\infty}^{\infty} 2^N \, \mathrm{e}^{-b^2 r^2} \, dr} = \frac{b}{\sqrt{\pi}} \, \mathrm{e}^{-b^2 r^2}. \tag{9.11}$$

In summary, the *probability density function* $p(r)$ is given by

$$p(r) = \frac{b}{\sqrt{\pi}}\, \mathrm{e}^{-b^2 r^2} \qquad \text{where} \quad b = \sqrt{\frac{1}{2N\ell^2}}\ . \tag{9.12}$$

The probability of finding the free end of the chain lying between r and $r + dr$ is therefore $p(r)\, dr$. Note that, since $|r/L| = |r/(N\ell)| << 1$ and $N >> 1$ we are now treating r as a continuous variable in the interval $(-\infty, \infty)$. Observe that the probability density function $(9.12)_1$ is proportional to $\exp(-b^2r^2)$ and so we have been led to "Gaussian statistics" in this limit.

Since $p(r)$ attains its maximum value at $r = 0$, this is the most probable end-to-end distance of the chain. The root mean square length of the chain – given by the square root of the integral of $r^2 \times p(r)$ with respect to r from negative infinity to infinity – is readily calculated to be

$$\left(\overline{r^2}\right)^{1/2} = \frac{1}{2b} = \ell\sqrt{N}\,\sqrt{2}.$$

Substituting the probability density function (9.12) into the general constitutive law (9.2) leads to the following relationship between the force, end-to-end distance and temperature:

$$f = \frac{k\theta}{\ell}\ \frac{r}{L}. \tag{9.13}$$

Observe the linear relationship between f and r, and that the modulus $k\theta/\ell$ of the material is completely determined in terms of micromechanical parameters. This approximation is, of course, only valid for small stretches $|r/L| << 1$; in particular, it does not therefore display the phenomenon of "locking-up" of the chain which was captured by the exact solution (9.5) - (9.8).

Remark: Note that by (9.7), P is the ratio of two numbers and is dimensionless. On the other hand due to the integral in the denominator of $(9.11)_1$ the dimension of p is length^{-1}. In the three dimensional model considered next, p will have dimension length^{-3}. When calculating the entropy we take the logarithm of the probability (density), and in order to be able to take the logarithm the probability (density) should be made non-dimensional using an appropriate length scale. However this factor will become an additive constant in the entropy and so we disregard it.

9.1.3 A Single Long Chain Molecule in Three Dimensions.

Suppose that the rigid links of the chain are *not restricted* to lie on the x-axis but can lie anywhere in 3-dimensional space (provided the links remain connected), with the connection between any two adjacent links being a ball joint. Each link is free to take any configuration subject only to the restriction that it be joined to its two neighboring links.

We limit attention to the special case where the number of links is large, $N >> 1$, and the chain is not close to being fully extended, i.e. $|r| << L$. (This is the three-dimensional version of the special case we considered before.)

Suppose that one end of the chain is held fixed at the origin, while the other end is to lie within the infinitesimal box $\{(x, y, z) \ : \ y_1 < x < dy_1,\ y_2 < y < y_2 + dy_2,\ y_3 < z < y_3 + dy_3\}$. The probability that the free end of the chain lies in this box is given by $p(y_1, y_2, y_3)dy_1dy_2dy_3$ where p is the *probability density function*. Recall that Gaussian probability distributions are ubiquitous in a wide range of physical problems; in particular we encountered it in the preceding problem. Assume that in the current circumstances p is given by Gaussian statistics:

$$
\begin{aligned}
p(y_1, y_2, y_3) &= \frac{\exp\left(-b^2y_1^2\right)}{\int_{-\infty}^{\infty} \exp\left(-b^2y_1^2\right)\, dy_1} \cdot \frac{\exp\left(-b^2y_2^2\right)}{\int_{-\infty}^{\infty} \exp\left(-b^2y_2^2\right)\, dy_2} \cdot \frac{\exp\left(-b^2y_3^2\right)}{\int_{-\infty}^{\infty} \exp\left(-b^2y_3^2\right)\, dy_3} \\
&= \frac{\exp\left[-b^2(y_1^2 + y_2^2 + y_3^2)\right]}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[-b^2(y_1^2 + y_2^2 + y_3^2)\right]\, dy_1dy_2dy_3} \\
&= \left(\frac{b}{\sqrt{\pi}}\right)^3 \exp\left[-b^2(y_1^2 + y_2^2 + y_3^2)\right].
\end{aligned}
\tag{9.14}
$$

One can show that in the present three dimensional model

$$
b = \sqrt{\frac{3}{2N\ell^2}}; \tag{9.15}
$$

see, for example, Section 5.5 of Weiner.

Observe the spherical symmetry of the probability density function (9.14) in that it only depends on $y_1^2 + y_2^2 + y_3^2$. Let $r = (y_1^2 + y_2^2 + y_3^2)^{1/2}$ denote the linear distance between the two ends of the chain. Then we can write (9.14) as

$$
p(r) = \frac{b^3}{\pi^{3/2}} \exp\left[-b^2r^2\right] \tag{9.16}
$$

where

$$
b = \sqrt{\frac{3}{2N\ell^2}}\ . \tag{9.17}
$$

Observe the similarity between (9.12) and (9.16), (9.17), the two factors of 3 in the latter being due to the 3-dimensional character of the present setting. Now, with one end of the chain fixed, the probability that the end-to-end distance of the chain lies between r and $r + dr$ (irrespective of direction in three-dimensional space) is $p(r)\, 4\pi r^2\, dr$ where $4\pi r^2\, dr$ is the volume of a spherically symmetric differential element.

By differentiating $4\pi r^2 p(r)$ we find that it attains its maximum value at

$$r = \ell\sqrt{N}\,\sqrt{2/3}\,;$$

this is therefore the most probable value of r, i.e. the most probable end-to-end length of the chain. The root mean square length of the chain – given by the square root of the integral of $r^2 \times [4\pi r^2 p(r)]$ with respect to r from zero[4] to infinity – is

$$\left(\overline{r^2}\right)^{1/2} = \ell\sqrt{N}. \tag{9.18}$$

Substituting the probability density function (9.16), (9.17) into the general constitutive law (9.2) leads to the following relationship between the force f, end-to-end distance r and temperature θ:

$$f = 3\frac{k\theta}{\ell}\,\frac{r}{L}\ . \tag{9.19}$$

Compare this with the result (9.13) of the one-dimensional model.

The case of limited rotation: In many polymeric molecules, e.g. paraffin, the angle between two adjacent bonds is difficult to change and can be treated as fixed. Thus if one link is held fixed, the adjacent link can only lie on the cone whose vertex is at the joint connecting the links and whose angle is the given fixed one. If this angle is $\alpha\,(< \pi/2)$ then the mean square end-to-end distance can be shown to be

$$\overline{r^2} = N\ell^2\,\left(\frac{1+\cos\alpha}{1-\cos\alpha}\right)\,;$$

see Chapter 3.1 of Treloar. Observe that the Gaussian probability density function (9.16) involves a single parameter b; in the preceding example the value of b was given by (9.17). Suppose here that we continue to adopt the Gaussian probability distribution function but leave b arbitrary. The mean square end-to-end distance associated with this probability density function is

$$\overline{r^2} = \int_0^\infty r^2\, p(r)\, 4\pi r^2\, dr = \int_0^\infty r^2\Big(\frac{b^3}{\pi^{3/2}}\exp\left[-b^2r^2\right]\Big)\, 4\pi r^2\, dr = \frac{3}{2}b^{-2}.$$

[4]In the one dimensional model r took values in $(-\infty,\infty)$, whereas here, r is a radial distance and so takes values in $(0,\infty)$.

If we set the two preceding expressions for $\overline{r^2}$ equal to each other, we can solve for b and find

$$b = \sqrt{\frac{3}{2N\ell^2}\,\left(\frac{1-\cos\alpha}{1+\cos\alpha}\right)}. \tag{9.20}$$

Therefore the probability density function for such a constrained long chain molecule can be taken to be given by (9.16) with b given by (9.20). This in turn can now be used to calculate the force-distance-temperature relation using (9.2).

9.1.4 A Single Long Chain Molecule: Langevin Statistics.

As noted previously, the Gaussian distribution used in the preceding two examples is only valid for $|r/L| << 1$, i.e. for small values of stretch. In order to develop a model that is appropriate at large stretches one must use a distribution function that is more accurate than the Gaussian distribution. This is provided by the Langevin distribution defined below; see Chapter 6.2 of Treloar.

In Langevin statistics, the *probability density function* is given by

$$p(r) = c\exp\left[-N\left(\beta\,\mathcal{L}(\beta) + \ln\frac{\beta}{\sinh\beta}\right)\right] \qquad \text{where} \quad \beta = \mathcal{L}^{-1}\left(\frac{r}{L}\right), \tag{9.21}$$

where c is a constant that can be fixed by normalization and $\mathcal{L}^{-1}$ denotes the inverse of the Langevin function

$$\mathcal{L}(x) = \coth x \;-\; \frac{1}{x}.$$

As before, r is the linear end-to-end distance between the two ends of the chain, $L = N\ell$ is the (contour) length of the entire chain, and each link is free to take any configuration subject only to the restriction that it be joined to its two neighboring links. The inessential constant c can be determined by requiring the integral of $p(r)4\pi r^2$ with respect to r from zero to infinity to be unity. Since $\mathcal{L}^{-1}(x) = 3x + 9x^3/5 + ...$ for small x, it can be readily seen using a Taylor expansion that for small r/L the Langevin probability density function (9.21) reduces to the Gaussian probability density function (9.16).

Again, with one end of the chain held fixed, the probability that the other end is at a radial distance between r and $r + dr$ (irrespective of direction) is $p(r)\,4\pi r^2\,dr$ where $4\pi r^2\,dr$ is the volume of a spherically symmetric differential element.

The relation between the force f, end-to-end distance r and temperature θ is given by

$$f = -k\theta\,\frac{\partial}{\partial r}\big(\ln\,p(r)\big) \tag{9.22}$$

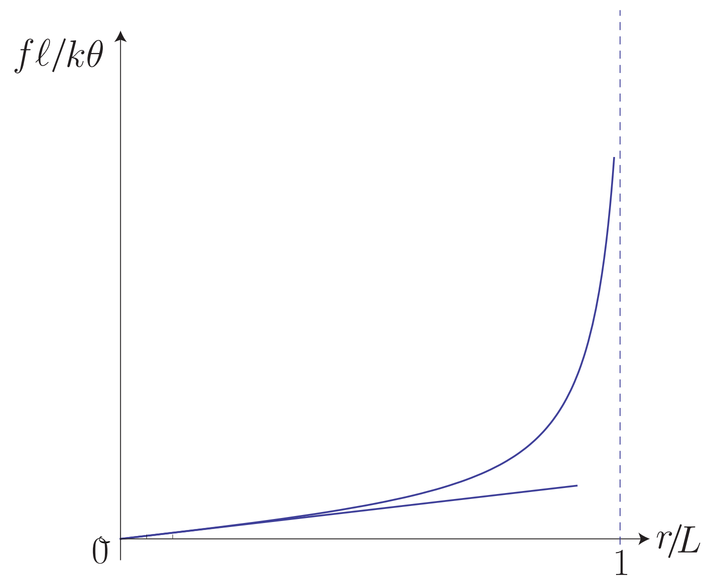

Figure 9.5: The force f versus end-to-end distance of the chain r. The curve corresponds to Langevin statistics and is characterized by (9.22), (9.21). Note the rapid rise as $r \to L$ corresponding to locking-up of the chain. The straight line corresponds to Gaussian statistics and is characterized by (9.19).

with $p(r)$ is given by (9.21).

Figure 9.5 shows plots of the force f versus the distance r according to the Gaussian relation (9.19) and the Langevin relation (9.22), (9.21). Observe that the Langevin model shows the upward trend of the f, r-curve associated with chain locking.

9.1.5 A Molecular Model for a Generalized Neo-Hookean Material.

Network models of rubber account for the presence of several molecules. A class of particularly useful network models is based on building up from a unit cell. For example in a 8-chain cubic unit cell, one end of each of eight molecules are linked together in the interior of a cube, their eight other ends being located at the eight vertices of the cube. The unit cell is assumed to deform with the macroscopic deformation, for example, it maybe subject to stretches $\lambda_1, \lambda_2, \lambda_3$ in the cubic directions.

Suppose that in an unstressed configuration the cube has dimensions $a_0 \times a_0 \times a_0$. The distance r_0 from a vertex to the centre of the cube is $r_0 = a_0\sqrt{3}/2$. Since r_0 is the end-to-end

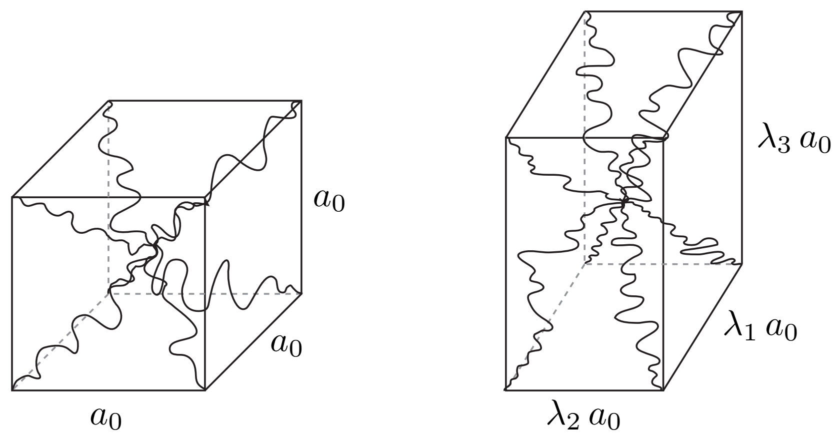

Figure 9.6: A unit cell with 8 molecular chains. In the reference configuration the unit cell is a cube. The molecules are linked together in the interior of the cube at one of their ends, their eight other ends being attached to the vertices of the cube. The deformation stretches the cube along its edges.

distance of a molecule in the unstressed state, we take r_0 to be the root mean square length of a chain[5] In the deformed configuration the unit cell is a tetrahedron, $\lambda_1 a_0 \times \lambda_2 a_0 \times \lambda_3 a_0$, and the distance r from a vertex to the centre is

$$r = \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}\,\frac{a_0}{2} \;=\; \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}\,\frac{r_0}{\sqrt{3}}. \tag{9.23}$$

This is the end-to-end distance of a chain in the deformed configuration.

Since the first invariant of the Cauchy Green tensor $\mathbf{C}$ can be expressed in terms of the principal stretches by

$$I_1(\mathbf{C}) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

it follows that we can write

$$r = \frac{r_0}{\sqrt{3}}\sqrt{I_1} \tag{9.24}$$

where we are now assuming that the principal stretches at the macroscopic scale coincide with those of the unit cell.

To calculate the *macroscopic energy* we multiply the free of one molecule given by (9.1) by n, the molecular density per unit volume to get

$$\psi = n\varepsilon(\theta) \;-\; nk\theta\;\ln p\Big(r_0\sqrt{I_1/3}\Big) \tag{9.25}$$

[5]In, for example, the Gaussian model $r_0 = \ell\sqrt{N}$; see (9.18).

which is of the form

$$\psi = \psi(I_1).$$

An isotropic incompressible elastic material whose response is independent of the second invariant I_2 is often referred to as a *generalized neo-Hookean material*. Thus the 8-chain network model leads to a family of generalized neo-Hookean materials, each one corresponding to a particular probability distribution function $p(r)$ and its root mean square chain length r_0.

If we use the Gaussian distribution (9.16), (9.17) for p, this specializes to

$$\psi = -nk\theta(-b^2r^2) = \frac{nk\theta}{2}(\lambda_1^2+\lambda_2^2+\lambda_3^2) = \frac{nk\theta}{2}I_1$$

where we have omitted all terms that are independent of the kinematic variable r. This is precisely the macroscopic free energy of the *neo-Hookean model* (8.7.2) (except for an additive constant).

Use of the Langevin probability density function in (9.25) leads to a far more accurate model for rubber elasticity based on the eight-chain unit cell; see Arruda and Boyce.

References:

E. Arruda and M.C. Boyce, A three-dimensional constitutive model for the large deformation stretch behavior of rubber elastic materials, *Journal of the Mechanics and Physics of Solids*, **41**(1993), pp. 389-412.

L.R. G. Treloar, Chapters 2, 3 and 6, *The Physics of Rubber Elasticity*, Clarendon Press, Oxford, 1975.

J.H. Weiner, Chapter 5, *Statistical Mechanics of Elasticity*, Wiley, 1983.

9.2 Example: Lattice Theory of Elasticity.

The notes in this section closely follow the unpublished lecture notes of Professor Kaushik Bhattacharya of Caltech. I am most grateful to him for sharing them with me.

The aim of this section is to illustrate how a simple atomistic model of a crystalline solid can be used to *derive* explicit continuum scale constitutive response functions $\widehat{\mathbf{T}}$ and $\widehat{W}$ for the Cauchy stress and the strain energy function in terms of the deformation gradient tensor. We will see that the expressions to be derived *automatically satisfy the requirements of material frame indifference, material symmetry and the entropy inequality.* Moreover we find that the traction - stress relation $\mathbf{t} = \mathbf{T}\mathbf{n}$ and the symmetry condition $\mathbf{T} = \mathbf{T}^T$ hold automatically. The expressions for $\widehat{\mathbf{T}}$ and $\widehat{W}$ that we derive are explicit in terms of the lattice geometry and the interatomic force potential; see (9.36) and (9.41).

9.2.1 A Bravais Lattice. Pair Potential.

A Bravais lattice $\mathcal{L}$ is an infinite set of points in $\mathbb{R}^3$ generated by translating a point $\mathbf{y}_o$ through three linearly independent vectors $\{\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3\}$: i.e.,

$$\mathcal{L}(\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3) = \left\{ \mathbf{y} : \ \mathbf{y} \in \mathbb{R}^3, \ \mathbf{y} = \mathbf{y}_o + \nu_i \boldsymbol{\ell}_i \quad \text{for all integers } \nu_1, \nu_2, \nu_3 \ \right\} . \tag{9.26}$$

The *lattice vectors* $\{\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3\}$ define a *unit cell.* Note the distinction between the lattice $\mathcal{L}$, which is an infinite set of periodically arranged points in space, and the lattice vectors. In particular, it is generally possible to generate the same lattice $\mathcal{L}$ from more than one set of lattice vectors, i.e., a given set of lattice vectors generates a unique lattice, but the converse is not necessarily true. More on this later. We shall take the orientation of the lattice vectors to be such that the volume of the unit cell is

$$\text{vol (unit cell)} = (\boldsymbol{\ell}_1 \times \boldsymbol{\ell}_2) \cdot \boldsymbol{\ell}_3 \ > \ 0. \tag{9.27}$$

The neighborhood of any lattice point, say $\mathbf{y}_A$, is identical to that of any other lattice point, say $\mathbf{y}_C$. To see this we simply note that if $\mathbf{y}_B$ is any third lattice point, then there necessarily is a fourth lattice point $\mathbf{y}_D$ such that $\mathbf{y}_D - \mathbf{y}_C = \mathbf{y}_B - \mathbf{y}_A$. Thus the position of $\mathbf{y}_B$ relative to $\mathbf{y}_A$ is the same as the position of $\mathbf{y}_D$ relative to $\mathbf{y}_C$. Any two lattice points $\mathbf{y}_A$ and $\mathbf{y}_C$ of a Bravais lattice are therefore *geometrically equivalent.* Bravais lattices can represent only monoatomic lattices; in particular, no alloy is a Bravais lattice[6].

We will ignore lattice vibrations and assume that the atoms are located at the lattice points. Therefore the calculations we carry out are valid at zero degrees Kelvin.

[6]Even some monoatomic lattices – e.g. a hexagonal close-packed lattice – cannot be represented as a Bravais lattice.

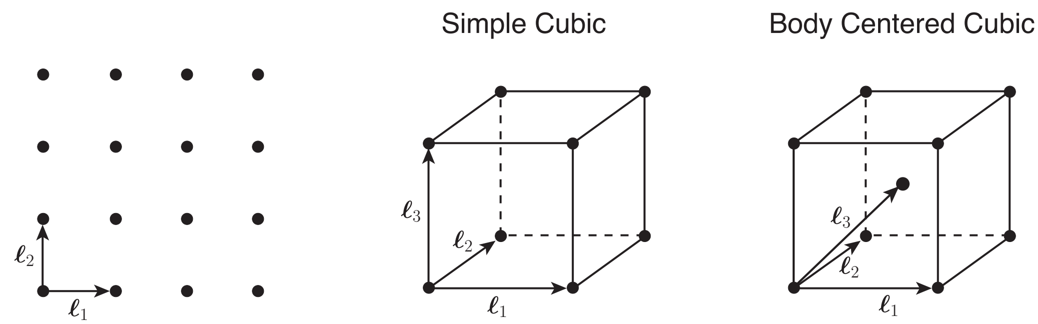

Figure 9.7: Examples of lattices in $\mathbb{R}^2$ and $\mathbb{R}^3$.

In the simplest model of interatomic interactions one assumes the existence of a *pair potential* $\phi(\rho)$ such that the force exerted by atom A on atom B, say $\mathbf{f}_{A,B}$, is the gradient of this potential:

$$\mathbf{f}_{A,B} = -\boldsymbol{\nabla}_y \phi(|\mathbf{y}|)\Big|_{\mathbf{y}=\mathbf{y}_B-\mathbf{y}_A} = -\phi'(|\mathbf{y}_B - \mathbf{y}_A|)\ \frac{\mathbf{y}_B - \mathbf{y}_A}{|\mathbf{y}_B - \mathbf{y}_A|}\ . \tag{9.28}$$

In this model the force exerted by one atom on the other depends solely on the relative positions of *those* two atoms and is independent of the positions of the surrounding atoms. Note that the force (9.28) is a *central force* in that it acts along the line joining those two atoms. Also observe that if the distance ρ between the atoms is such that $\phi'(\rho) < 0$, then the force between them is repulsive; if $\phi'(\rho) > 0$ it is attractive. Finally, observe from (9.28) that $\mathbf{f}_{A,B} = -\mathbf{f}_{B,A}$ so that the force exterted by atom A on atom B is equal in magnitude and opposite in direction to the force applied by atom B on atom A.

Figure 9.8 shows a graph of a typical pair-potential $\phi(\rho)$ versus the distance ρ between the pair of atoms. Note that the associated force is repulsive at short distances ($< \rho_o$) and attractive at large distances ($> \rho_o$).

Several of the calculations to follow will involve infinite sums over all lattice points of terms involving ϕ, ϕ' and ϕ''; it is necessary to ensure that these sums converge to finite values. This requires that $\phi(\rho) \to 0$ fast enough as $\rho \to \infty$. We assume that ϕ possesses the requisite[7] decay rate.

[7] To determine the required decay rate, one can consider a sphere of radius, say $\overline{\rho}$, and separate the infinite sum over the entire lattice into a finite sum over the finite number of lattice points in the interior of the sphere plus a sum over the infinite number of lattice points in the exterior of the sphere. An upper bound for the second term can then be written by replacing the sum by an integral (over the entire three dimensional region exterior to the sphere). Convergence of the integral guarantees convergence of the sum. For example

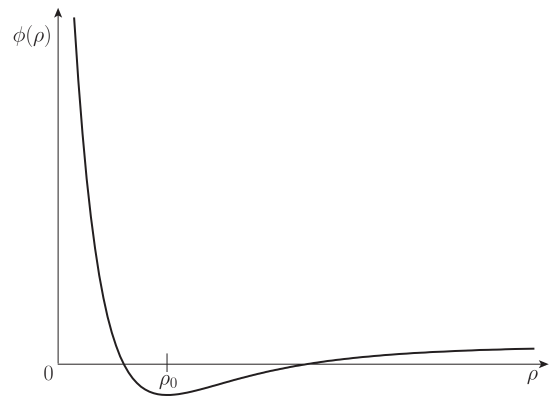

Figure 9.8: Typical graph of the pair-potential ϕ.

Because of the periodicity and symmetry of a Bravais lattice, if $\mathbf{y}_A$ and $\mathbf{y}_B$ are any two lattice points, there necessarily is a third lattice point $\mathbf{y}_C$ which is such that $\mathbf{y}_A - \mathbf{y}_B = -(\mathbf{y}_A - \mathbf{y}_C)$. Therefore according to the force law (9.28), the forces exerted on atom A by atoms B and C are equal in magnitude and opposite in direction. Consequently for each atom B that exerts a force on atom A, there is another atom C that exerts an equal and opposite force on A. Thus a Bravais lattice is *always in equilibrium.*

9.2.2 Homogenous Deformation of a Bravais Lattice.

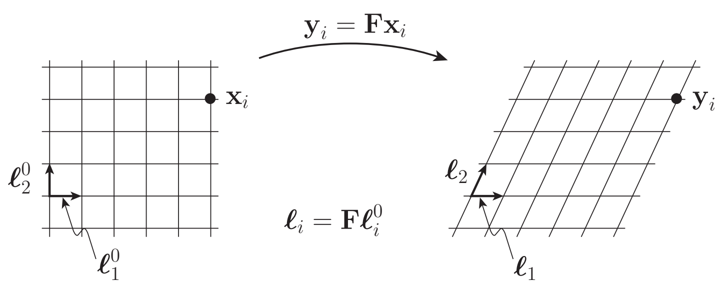

Figure 9.9: Homogeneous deformation of a lattice. The lattice vectors $\{\boldsymbol{\ell}_1^o, \boldsymbol{\ell}_2^o\}$ of the reference lattice are mapped by $\mathbf{F}$ into the lattice vectors $\{\boldsymbol{\ell}_1, \boldsymbol{\ell}_2\}$ of the deformed lattice.

Most, but not all, of the discussion to follow will be carried out entirely in the current (deformed) lattice. There will however be a few occasions when we wish to consider a

the energy (9.41) will converge if the integral of $\rho^2\phi(\rho)$ over the interval $[\overline{\rho}, \infty)$ converges, which would be true if $\phi \to 0$ faster than ρ^{-3} as $\rho \to \infty$.

reference lattice. In this event we will consider a second Bravais lattice $\mathcal{L}_0$:

$$\mathcal{L}(\boldsymbol{\ell}_1^o, \boldsymbol{\ell}_2^o, \boldsymbol{\ell}_3^o) = \left\{ \mathbf{x} : \ \mathbf{x} \in \mathbb{R}^3, \ \mathbf{x} = \mathbf{x}_o + \nu_i \boldsymbol{\ell}_i^o \quad \text{for all integers } \nu_1, \nu_2, \nu_3 \ \right\}$$

where the lattice vectors $\{\boldsymbol{\ell}_1^o, \boldsymbol{\ell}_2^o, \boldsymbol{\ell}_3^o\}$ define a unit cell of the reference lattice. Since each set of lattice vectors is linearly independent, it follows that there is a nonsingular tensor $\mathbf{F}$ that maps $\{\boldsymbol{\ell}_1^o, \boldsymbol{\ell}_2^o, \boldsymbol{\ell}_3^o\} \to \{\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3\}$:

$$\boldsymbol{\ell}_i = \mathbf{F}\boldsymbol{\ell}_i^o, \quad i = 1, 2, 3. \tag{9.29}$$

This is illustrated in Figure 9.9.

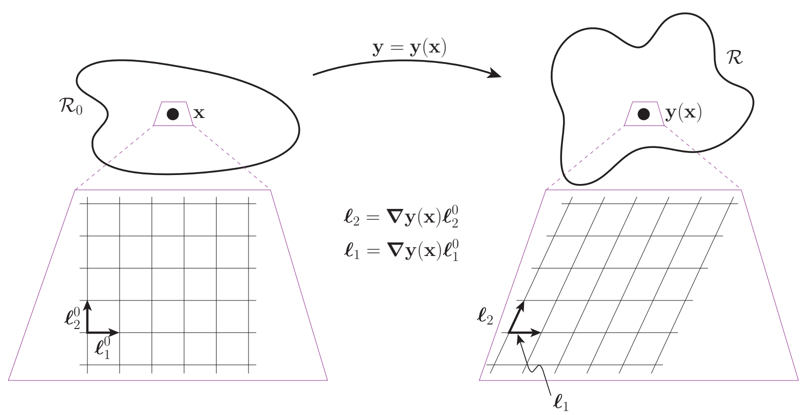

Figure 9.10: The deformation $\mathbf{y}(\mathbf{x})$ carries the three dimensional region $\mathcal{R}_0$ into $\mathcal{R}$. The figure shows blown-up views of infinitesimal neighborhoods of $\mathbf{x}$ and $\mathbf{y}(\mathbf{x})$. The mapping of the lattice vectors is assumed to be described by Grad $\mathbf{y}(\mathbf{x})$ ($= \boldsymbol{\nabla}\mathbf{y}(\mathbf{x})$) as depicted in the figure.

Suppose that we associate a (continuum) body with the lattice. The lattices $\mathcal{L}_0$ and $\mathcal{L}$ are associated with two configurations of the body. Let $\mathbf{y}(\mathbf{x})$ be the deformation of the continuum that maps $\mathcal{R}_0$ into $\mathcal{R}$. The deformation gradient tensor is Grad $\mathbf{y}(\mathbf{x})$. As discussed in Section 2.2, Grad $\mathbf{y}(\mathbf{x})$ maps material fibers of the continuum from the reference to the deformed configurations. The tensor $\mathbf{F}$ introduced above maps the reference lattice vectors to the deformed lattice vectors through (9.29). The Cauchy-Born hypothesis states that the "continuum deforms with the lattice" in the sense that Grad $\mathbf{y}(\mathbf{x}) = \mathbf{F}$. This is illustrated in Figure 9.10.

9.2.3 Traction and Stress.

We now establish a notion of traction and then derive an explicit expression for it in terms of the interatomic forces. Let $\mathcal{P}$ be an arbitrary plane through the lattice and let $\mathbf{n}$ denote a unit vector normal to $\mathcal{P}$. Let $\mathcal{L}^+$ and $\mathcal{L}^-$ denote the two subsets of the lattice $\mathcal{L}$ which are on, respectively, the side into which and the side away from which $\mathbf{n}$ points; see Figure 9.11. Let $\mathcal{A}$ be a subregion of the plane $\mathcal{P}$. Consider two lattice points $\mathbf{y}_+ \in \mathcal{L}^+$ and $\mathbf{y}_- \in \mathcal{L}^-$ such that the line joining them intersects the subregion $\mathcal{A}$; see Figure 9.11. By summing the forces between all such pairs of atoms, we can associate a force with the region $\mathcal{A}$. The traction $\mathbf{t}$ can then be defined as the normalization of this force by the area of $\mathcal{A}$:

$$\mathbf{t}(\mathcal{A}) = \frac{1}{\text{area }(\mathcal{A})} \sum \mathbf{f}_{i,j} \ = \ \frac{1}{\text{area }(\mathcal{A})} \sum -\phi'(|\mathbf{y}_- - \mathbf{y}_+|)\ \frac{\mathbf{y}_- - \mathbf{y}_+}{|\mathbf{y}_- - \mathbf{y}_+|}, \tag{9.30}$$

where the summation is carried out over all $\mathbf{y}_+ \in \mathcal{L}^+$ and $\mathbf{y}_- \in \mathcal{L}^-$ which are such that the line joining $\mathbf{y}_+$ to $\mathbf{y}_-$ intersects $\mathcal{A}$.

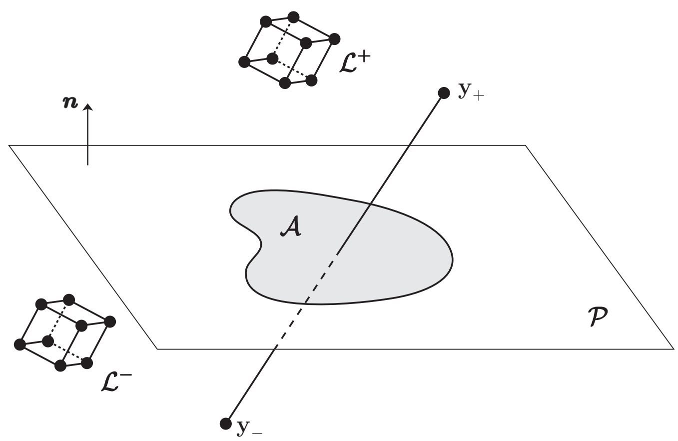

Figure 9.11: A plane $\mathcal{P}$ separating the lattice into two parts $\mathcal{L}^+$ and $\mathcal{L}^-$. The two lattice points $\mathbf{y}_+ \in \mathcal{L}^+$ and $\mathbf{y}_- \in \mathcal{L}^-$ are such that the line joining them intersects the subregion $\mathcal{A} \subset \mathcal{P}$.

If (9.30) is to be useful, we need to characterize the range of summation in a simpler form. First, since $\mathbf{y}_-$ and $\mathbf{y}_+$ are lattice points, it follows that there are integers $\{\nu_1, \nu_2, \nu_3\}$ for which $\mathbf{y}_- - \mathbf{y}_+ = \nu_i \boldsymbol{\ell}_i$. Conversely, given any three integers $\{\nu_1, \nu_2, \nu_3\}$ which are such that $(\nu_i \boldsymbol{\ell}_i) \cdot \mathbf{n} < 0$ (which simply means that the vector $\nu_i \boldsymbol{\ell}_i$ points in the $-\mathbf{n}$ direction), there exist pairs (note plural) of lattice points $\mathbf{y}_+ \in \mathcal{L}^+$ and $\mathbf{y}_- \in \mathcal{L}^-$ such that $\mathbf{y}_- - \mathbf{y}_+ = \nu_i \boldsymbol{\ell}_i$; of these, the number of pairs whose line of connection intersects $\mathcal{A}$ can be estimated to be

$$\begin{aligned} N \ &= \ \frac{\text{volume of the (non-prismatic) cylinder with base } \mathcal{A} \text{ and generator } \nu_i \boldsymbol{\ell}_i}{\text{volume of the unit cell}} \\ &= \ \frac{\text{area }(\mathcal{A})\ |(\nu_i \boldsymbol{\ell}_i) \cdot \mathbf{n}|}{(\boldsymbol{\ell}_1 \times \boldsymbol{\ell}_2) \cdot \boldsymbol{\ell}_3} = -\text{area }(\mathcal{A}) \frac{(\nu_i \boldsymbol{\ell}_i) \cdot \mathbf{n}}{(\boldsymbol{\ell}_1 \times \boldsymbol{\ell}_2) \cdot \boldsymbol{\ell}_3} \end{aligned} \tag{9.31}$$

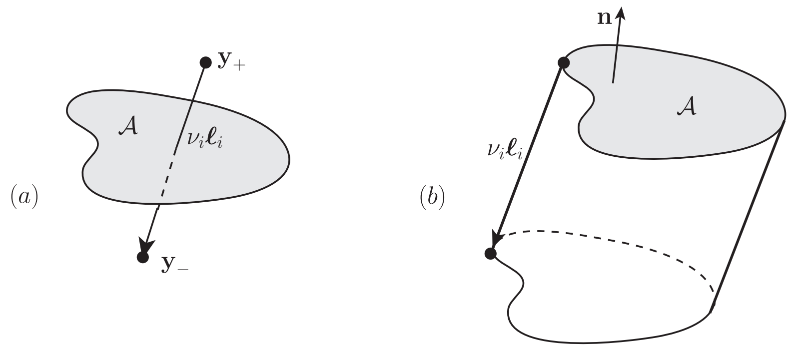

Figure 9.12: (a) Two lattice points $\mathbf{y}_+ \in \mathcal{L}^+$ and $\mathbf{y}_- \in \mathcal{L}^-$: $\mathbf{y}_- - \mathbf{y}_+ = \nu_i\boldsymbol{\ell}_i$ for some integers ν_1, ν_2, ν_3. (b) Non-prismatic cylinder whose base is $\mathcal{A}$ and generator is $\nu_i\boldsymbol{\ell}_i$.

when the area of $\mathcal{A}$ is sufficiently large[8]; see Figure 9.12. In the last step we have used the fact that $(\nu_i\boldsymbol{\ell}_i) \cdot \mathbf{n} < 0$. Given the triplet of integers $\{\nu_1, \nu_2, \nu_3\}$, equation (9.31) gives the corresponding number of pairs of points whose line of connection intersects $\mathcal{A}$.

We can now evaluate the summation in (9.30) in two steps: first, for given $\{\nu_1, \nu_2, \nu_3\}$ with $(\nu_i\boldsymbol{\ell}_i)\cdot\mathbf{n} < 0$, we sum over all pairs of lattice points $\mathbf{y}_-$ and $\mathbf{y}_+$ which have $\mathbf{y}_- - \mathbf{y}_+ = \nu_i\boldsymbol{\ell}_i$ and where the line connecting them intersects $\mathcal{A}$. Then, we sum over all triplets of integers $\{\nu_1, \nu_2, \nu_3\}$ obeying $(\nu_i\boldsymbol{\ell}_i) \cdot \mathbf{n} < 0$. This leads to

$$\mathbf{t}(\mathcal{A}) = \frac{1}{\text{area}\,(\mathcal{A})} \sum_{\substack{\{\nu_1,\nu_2,\nu_3\}\ \ni \\ (\nu_i\boldsymbol{\ell}_i)\cdot\mathbf{n}<0}} -\phi'(|\nu_p\boldsymbol{\ell}_p|)\, \frac{\nu_i\boldsymbol{\ell}_i}{|\nu_k\boldsymbol{\ell}_k|}\, N\ . \tag{9.32}$$

Substituting (9.31) into this yields

$$\mathbf{t}(\mathcal{A}) = \frac{1}{(\boldsymbol{\ell}_1 \times \boldsymbol{\ell}_2)\cdot\boldsymbol{\ell}_3} \sum_{\substack{\{\nu_1,\nu_2,\nu_3\}\ \ni \\ (\nu_i\boldsymbol{\ell}_i)\cdot\mathbf{n}<0}} \phi'(|\nu_p\boldsymbol{\ell}_p|)\, \frac{\nu_i\boldsymbol{\ell}_i}{|\nu_k\boldsymbol{\ell}_k|}\, (\nu_j\boldsymbol{\ell}_j)\cdot\mathbf{n}\ . \tag{9.33}$$

Finally, observe that if we change $\{\nu_1, \nu_2, \nu_3\} \to \{-\nu_1, -\nu_2, -\nu_3\}$, the term within the summation sign remains unchanged though $(\nu_i\boldsymbol{\ell}_i) \cdot \mathbf{n}$ changes sign. Therefore, the sum above with the restriction $(\nu_i\boldsymbol{\ell}_i) \cdot \mathbf{n} < 0$ equals one-half the sum without this restriction. Therefore

[8]In a homogeneously deformed continuum, the traction on the plane $\mathcal{P}$ would be uniform, i.e. it would be the same at all point on $\mathcal{P}$. The lattice at hand has a uniform geometry and we want (9.30) to be related to the continuum notion of traction. This requires that the right-hand side of (9.30) be independent of the size of $\mathcal{A}$. This in turn requires that the subregion $\mathcal{A}$ be sufficiently large.

we obtain the following expression for the *traction* on the plane $\mathcal{P}$:

$$\mathbf{t}(\mathcal{A}) = \left[\frac{1}{2(\boldsymbol{\ell}_1 \times \boldsymbol{\ell}_2)\cdot\boldsymbol{\ell}_3} \sum_{\{\nu_1,\nu_2,\nu_3\}} \phi'(|\nu_p \boldsymbol{\ell}_p|) \, \frac{(\nu_i \boldsymbol{\ell}_i)\otimes(\nu_j \boldsymbol{\ell}_j)}{|\nu_k \boldsymbol{\ell}_k|} \right] \mathbf{n} \tag{9.34}$$

where the summation is taken over all triplets of integers $\{\nu_1, \nu_2, \nu_3\}$.

Observe that the traction given by (9.34) depends linearly on the unit normal vector $\mathbf{n}$. This suggests that we define the *Cauchy stress tensor* $\mathbf{T}$ by

$$\mathbf{T} = \frac{1}{2(\boldsymbol{\ell}_1 \times \boldsymbol{\ell}_2)\cdot\boldsymbol{\ell}_3} \sum_{\{\nu_1,\nu_2,\nu_3\}} \phi'(|\nu_p \boldsymbol{\ell}_p|) \, \frac{(\nu_i \boldsymbol{\ell}_i)\otimes(\nu_j \boldsymbol{\ell}_j)}{|\nu_k \boldsymbol{\ell}_k|} \ . \tag{9.35}$$

Note that $\mathbf{t} = \mathbf{T}\mathbf{n}$. Moreover $\mathbf{T} = \mathbf{T}^T$ as required by the balance of angular momentum. Given a Bravais lattice and a pair potential, equation (9.35) provides *an explicit formula for the stress*. It involves the (current) geometry of the lattice and the pair-potential.

Finally we provide a representation for $\mathbf{T}$ in terms of a referential lattice by replacing the deformed lattice vectors $\{\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3\}$ in (9.35) by reference lattice vectors. To this end, consider a reference lattice defined by lattice vectors $\{\boldsymbol{\ell}_1^o, \boldsymbol{\ell}_2^o, \boldsymbol{\ell}_3^o\}$. The lattice vectors $\{\boldsymbol{\ell}_1^o, \boldsymbol{\ell}_2^o, \boldsymbol{\ell}_3^o\}$ of the reference lattice are related to the lattice vectors $\{\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3\}$ of the deformed lattice through the nonsingular tensor $\mathbf{F}$ where $\boldsymbol{\ell}_i = \mathbf{F}\boldsymbol{\ell}_i^o$. The stress (in the deformed lattice) given by (9.35) can now be written in terms of the referential lattice vectors and $\mathbf{F}$ as

$$\mathbf{T} = \widehat{\mathbf{T}}(\mathbf{F}) = \frac{1}{2(\mathbf{F}\boldsymbol{\ell}_1^o \times \mathbf{F}\boldsymbol{\ell}_2^o)\cdot\mathbf{F}\boldsymbol{\ell}_3^o} \sum_{\{\nu_1,\nu_2,\nu_3\}} \phi'(|\nu_p \, F\boldsymbol{\ell}_p^o|) \frac{(\nu_i \, \mathbf{F}\boldsymbol{\ell}_i^o)\otimes(\nu_j \, \mathbf{F}\boldsymbol{\ell}_j^o)}{|\nu_k \, \mathbf{F}\boldsymbol{\ell}_k^o|} \ . \tag{9.36}$$

This provides an explicit formula for the *stress response function* $\widehat{\mathbf{T}}$ in terms of the referential lattice, the tensor $\mathbf{F}$, and the pair potential. If we associate a continuum with this lattice and invoke the Cauchy-Born hypothesis, $\mathbf{F}$ would be the deformation gradient tensor.

9.2.4 Energy.

We begin by calculating the energy of a single atom located at a lattice point $\mathbf{y}$. The energy associated with the pair of atoms located at $\mathbf{y}$ and $\boldsymbol{\xi}$ is $\phi(|\mathbf{y}-\boldsymbol{\xi}|)$. Assume that this energy is equally shared by the two atoms. Then, the energy of the atom located at $\mathbf{y}$ due to its interaction with all other atoms of the lattice is

$$\frac{1}{2} \sum_{\substack{\boldsymbol{\xi}\in\mathcal{L} \\ \boldsymbol{\xi}\neq\mathbf{y}}} \phi(|\mathbf{y}-\boldsymbol{\xi}|) \ = \ \frac{1}{2} \sum_{\{\nu_1,\nu_2,\nu_3\}} \phi(|\nu_i \, \boldsymbol{\ell}_i|). \tag{9.37}$$

Observe that this energy does not depend on $\mathbf{y}$, reflecting the fact that the lattice is uniform and the energy of each atom is the same. Now consider the energy associated with some region $\mathcal{R}$ of three dimensional space. If $\mathcal{R}$ is sufficiently large, the number of lattice points in $\mathcal{R}$ is

$$\frac{\text{vol }(\mathcal{R})}{(\boldsymbol{\ell}_1 \times \boldsymbol{\ell}_2) \cdot \boldsymbol{\ell}_3} \tag{9.38}$$

where the denominator denotes the volume of the unit cell. Therefore the energy associated with the region $\mathcal{R}$ is given by the product of two preceding expressions:

$$\frac{\text{vol }(\mathcal{R})}{(\boldsymbol{\ell}_1 \times \boldsymbol{\ell}_2) \cdot \boldsymbol{\ell}_3} \quad \frac{1}{2} \sum_{\{\nu_1,\nu_2,\nu_3\}} \phi(|\nu_i\, \boldsymbol{\ell}_i|). \tag{9.39}$$

On dividing by vol $(\mathcal{R})$, we get the energy per unit current volume. Thus, given a Bravais lattice and a pair potential, equation (9.39) provides *an explicit formula for the energy per unit current volume.* It involves the (current) geometry of the lattice and the pair-potential.

Finally we express this in terms of a referential lattice. Consider the lattice defined by lattice vectors $\{\boldsymbol{\ell}_1^o, \boldsymbol{\ell}_2^o, \boldsymbol{\ell}_3^o\}$ that are related to the current lattice vectors by $\boldsymbol{\ell}_i = \mathbf{F}\boldsymbol{\ell}_i^o$. Substituting $\boldsymbol{\ell}_i = \mathbf{F}\boldsymbol{\ell}_i^o$ and using the fact that the volumes of $\mathcal{R}$ and its pre-image $\mathcal{R}_o$ in the reference configuration are related by $\text{vol}(\mathcal{R}) = \det \mathbf{F}\, \text{vol}(\mathcal{R}_o)$ allows us to write (9.39) as

$$\frac{\text{vol }(\mathcal{R}_o)\ \det \mathbf{F}}{(\mathbf{F}\boldsymbol{\ell}_1^o \times \mathbf{F}\boldsymbol{\ell}_2^o) \cdot \mathbf{F}\boldsymbol{\ell}_3^o} \quad \frac{1}{2} \sum_{\{\nu_1,\nu_2,\nu_3\}} \phi(|\nu_i\, \mathbf{F}\boldsymbol{\ell}_i^o|). \tag{9.40}$$

Finally, on using the identity $(\mathbf{Aa} \times \mathbf{Ab}) \cdot \mathbf{Ac} = \det \mathbf{A}\ (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ and dividing by $\text{vol}(\mathcal{R}_o)$ gives the following expression for the *energy per unit referential volume*:

$$\widehat{W}(\mathbf{F}) \ = \ \frac{1}{(\boldsymbol{\ell}_1^o \times \boldsymbol{\ell}_2^o) \cdot \boldsymbol{\ell}_3^o} \quad \frac{1}{2} \sum_{\{\nu_1,\nu_2,\nu_3\}} \phi(|\nu_i\, \mathbf{F}\boldsymbol{\ell}_i^o|). \tag{9.41}$$

This provides an explicit formula for the *strain energy response function* $\widehat{W}$ in terms of the referential lattice, the tensor $\mathbf{F}$, and the pair potential. If we associate a continuum with this lattice and invoke the Cauchy-Born hypothesis, $\mathbf{F}$ would be the deformation gradient tensor.

Note from (9.41) and (9.29) that the function $\widehat{W}$ and the tensor $\mathbf{F}$ both depend on the choice of reference lattice vectors. However, the energy of the deformed lattice does not depend on the choice of reference lattice vectors. Therefore the way in which $\mathbf{F}$ and $\widehat{W}$ depend on the reference lattice vectors must balance each other out such that the value of $\widehat{W}$ is independent of the choice of reference lattice vectors.

It is shown in Problem 9.2 that the stress response function (9.36) derived previously and the energy response function (9.41) are related *automatically* through the relation

$$\widehat{\mathbf{T}}(\mathbf{F}) = \frac{1}{\det \mathbf{F}} \frac{\partial \widehat{W}}{\partial \mathbf{F}}(\mathbf{F})\, \mathbf{F}^T \tag{9.42}$$

which is precisely what the entropy inequality would require of the continuum theory.

9.2.5 Material Frame Indifference.

It is shown in Problem 9.1 that the constitutive response function $\widehat{\mathbf{T}}(\mathbf{F})$ defined by (9.36) *automatically* obeys the relation

$$\widehat{\mathbf{T}}(\mathbf{QF}) = \mathbf{Q}\widehat{\mathbf{T}}(\mathbf{F})\mathbf{Q}^T$$

for all proper orthogonal tensors $\mathbf{Q}$ as would be required by material frame indifference in the continuum theory.

It can similarly be verified that the energy response function (9.41) has the property that

$$\widehat{W}(\mathbf{F}) = \widehat{W}(\mathbf{QF})$$

for all proper orthogonal tensors $\mathbf{Q}$. This shows that $\widehat{W}$ is automatically consistent with material frame indifference.

9.2.6 Linearized Elastic Moduli. Cauchy Relations.

In Problem 9.3 we shall linearize the constitutive relation (9.41), (9.42) to the special case of infinitesimal deformations. This leads to the constitutive relation of linear elasticity with the material characterized by an elasticity tensor $\mathbb{C}$. In fact, Problem 9.3 provides an explicit formula for the components $\mathbb{C}_{ijk\ell}$ of the elasticity tensor in terms of the referential lattice and the pair potential.

The elastic moduli obtained in this way exhibit the symmetries

$$\mathbb{C}_{ijk\ell} = \mathbb{C}_{k\ell ij} = \mathbb{C}_{jik\ell} = \mathbb{C}_{ij\ell k}, \tag{9.43}$$

just as required by the continuum theory; see Section (8.9). However *in addition*, $\mathbb{C}_{ijk\ell}$ here is found to also posses the symmetry

$$\mathbb{C}_{ijk\ell} = \mathbb{C}_{i\ell kj} \tag{9.44}$$

which is not required by the continuum theory. The symmetries (9.44) obtained from the present lattice model are known as the *Cauchy relations*. The Cauchy relations are known to be not obeyed by most elastic materials[9] and this is therefore a limitation of the lattice theory formulated here. This limitation is directly related to the use of a pair-potential to model interatomic interactions. More realistic interatomic interaction models remove this limitation.

9.2.7 Lattice and Continuum Symmetry.

Since the stress and strain energy response functions $\widehat{\mathbf{T}}$ and $\widehat{W}$ given by (9.36) and (9.41) were derived from lattice considerations, they inherit the appropriate invariance characteristics associated with the symmetry of the underlying lattice. In this section we address three issues:

1. We examine the geometric invariance characteristics of a Bravais lattice and construct its "lattice symmetry group".

2. We show that the lattice symmetry group plays the role of the material symmetry group for the response functions $\widehat{\mathbf{T}}$ and $\widehat{W}$ derived above.

3. We remark on the suitability of using the lattice symmetry group to characterize the symmetry of a continuum.

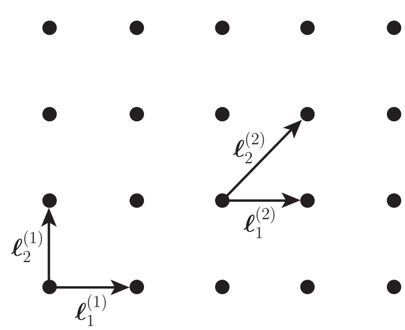

Figure 9.13: Two sets of lattice vectors that describe the same lattice.

[9]For example, for an isotropic material, the Cauchy relations imply that the Poisson ratio must always be 0.25.

Characterizing the symmetry of a Bravais lattice: First observe that because of its inherent symmetry, more than one set of lattice vectors may generate the same lattice. For example, the two-dimensional lattice shown in Figure 9.13 is generated by both $\{\boldsymbol{\ell}_1^{(1)}, \boldsymbol{\ell}_2^{(1)}\}$ and $\{\boldsymbol{\ell}_1^{(2)}, \boldsymbol{\ell}_2^{(2)}\}$. Observe that

$$\begin{aligned} \boldsymbol{\ell}_1^{(2)} &= \boldsymbol{\ell}_1^{(1)}, \\ \boldsymbol{\ell}_2^{(2)} &= \boldsymbol{\ell}_1^{(1)} + \boldsymbol{\ell}_2^{(1)}, \end{aligned} \tag{9.45}$$

so that the 2×2 matrix $[\mu]$, whose elements relate the two sets of lattice vectors through $\boldsymbol{\ell}_i^{(2)} = \mu_{ij}\boldsymbol{\ell}_j^{(1)}$, is

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{9.46}$$

Note that the elements of $[\mu]$ are integers and that $\det[\mu] = 1$.

In general, let $\mathcal{L}(\boldsymbol{\ell}_1^{(1)}, \boldsymbol{\ell}_2^{(1)}, \boldsymbol{\ell}_3^{(1)})$ be the lattice generated by a given set of lattice vectors $\{\boldsymbol{\ell}_1^{(1)}, \boldsymbol{\ell}_2^{(1)}, \boldsymbol{\ell}_3^{(1)}\}$. Suppose that $\{\boldsymbol{\ell}_1^{(2)}, \boldsymbol{\ell}_2^{(2)}, \boldsymbol{\ell}_3^{(2)}\}$ is a second[10] set of lattice vectors that generates *this same lattice*, i.e.

$$\mathcal{L}(\boldsymbol{\ell}_1^{(1)}, \boldsymbol{\ell}_2^{(1)}, \boldsymbol{\ell}_3^{(1)}) = \mathcal{L}(\boldsymbol{\ell}_1^{(2)}, \boldsymbol{\ell}_2^{(2)}, \boldsymbol{\ell}_3^{(2)}).$$

One can show that two sets of lattice vectors generate the same lattice if and only if the matrix $[\mu]$, whose elements relate the two sets of lattice vectors through

$$\boldsymbol{\ell}_i^{(2)} = \mu_{ij}\boldsymbol{\ell}_j^{(1)}, \tag{9.47}$$

has elements that are integers and whose determinant is 1.

An alternative more useful way in which to characterize symmetry is as follows: given a set of lattice vectors $\{\boldsymbol{\ell}_1^{(1)}, \boldsymbol{\ell}_2^{(1)}, \boldsymbol{\ell}_3^{(1)}\}$ and the associated Bravis lattice $\mathcal{L} = \mathcal{L}(\boldsymbol{\ell}_1^{(1)}, \boldsymbol{\ell}_2^{(1)}, \boldsymbol{\ell}_3^{(1)})$, let $\mathcal{G}(\mathcal{L})$ denote the set of all nonsingular tensors $\mathbf{H}$ that map $\{\boldsymbol{\ell}_1^{(1)}, \boldsymbol{\ell}_2^{(1)}, \boldsymbol{\ell}_3^{(1)}\}$ into a set of vectors $\{\mathbf{H}\boldsymbol{\ell}_1^{(1)}, \mathbf{H}\boldsymbol{\ell}_2^{(1)}, \mathbf{H}\boldsymbol{\ell}_3^{(1)}\}$ that generate the same lattice:

$$\mathcal{L}(\boldsymbol{\ell}_1^{(1)}, \boldsymbol{\ell}_2^{(1)}, \boldsymbol{\ell}_3^{(1)}) = \mathcal{L}(\mathbf{H}\boldsymbol{\ell}_1^{(1)}, \mathbf{H}\boldsymbol{\ell}_2^{(1)}, \mathbf{H}\boldsymbol{\ell}_3^{(1)}) \qquad \text{for all } \mathbf{H} \in \mathcal{G}(\mathcal{L}).$$

It follows from (9.47) that $\mathcal{G}(\mathcal{L})$ admits the representation

$$\mathcal{G}(\mathcal{L}) = \Big\{ \mathbf{H}\colon\ \mathbf{H}\boldsymbol{\ell}_i^{(1)} = \mu_{ij}\, \boldsymbol{\ell}_j^{(1)} \text{ for all } \mu_{ij} \text{ that are integers with } \det[\mu] = 1 \Big\}\ . \tag{9.48}$$

Two sets of lattice vectors generate the same lattice if and only if

$$\boldsymbol{\ell}_i^{(2)} = \mathbf{H}\boldsymbol{\ell}_i^{(1)}, \quad i = 1, 2, 3, \tag{9.49}$$

[10] We shall only consider lattice vector sets that have the same orientation.

where $\mathbf{H} \in \mathcal{G}(\mathcal{L})$. This is equivalent to (9.47). Despite the presence of the lattice vectors on the right hand side of (9.48), by its definition, $\mathcal{G}(\mathcal{L})$ depends on the lattice but not on the particular set of lattice vectors used to represent it. The set $\mathcal{G}(\mathcal{L})$ can be shown to be a group. It characterizes the symmetry of the lattice $\mathcal{L}$ and may be referred to as the "lattice symmetry group".

It is shown in Problem 9.5 that

$$\det \mathbf{H} = 1 \qquad \text{for all } \mathbf{H} \in \mathcal{G}(\mathcal{L}). \tag{9.50}$$

As a consequence, note that the volumes of the unit cells formed by lattice vectors $\{\boldsymbol{\ell}_1^{(1)}, \boldsymbol{\ell}_2^{(1)}, \boldsymbol{\ell}_3^{(1)}\}$ and $\{\boldsymbol{\ell}_1^{(2)}, \boldsymbol{\ell}_2^{(2)}, \boldsymbol{\ell}_3^{(2)}\}$ are equal if the lattice vectors are related through a symmetry transformation:

$$(\boldsymbol{\ell}_1^{(2)} \times \boldsymbol{\ell}_2^{(2)}) \cdot \boldsymbol{\ell}_3^{(2)} = (\boldsymbol{\ell}_1^{(1)} \times \boldsymbol{\ell}_2^{(1)}) \cdot \boldsymbol{\ell}_3^{(1)} \tag{9.51}$$

provided

$$\boldsymbol{\ell}_i^{(2)} = \mathbf{H}\boldsymbol{\ell}_i^{(1)}, \qquad \mathbf{H} \in \mathcal{G}(\mathcal{L}). \tag{9.52}$$

Symmetry of the response functions $\widehat{\mathbf{T}}$ and $\widehat{W}$: As noted at the beginning of this subsection, since the stress and strain energy response functions $\widehat{\mathbf{T}}$ and $\widehat{W}$ given by (9.36) and (9.41) were derived from lattice considerations, they inherit the appropriate invariance characteristics associated with the symmetry of the underlying lattice. We shall now verify this claim and show, for example, that

$$\widehat{W}(\mathbf{F}) = \widehat{W}(\mathbf{FH}) \qquad \text{for all } \mathbf{H} \in \mathcal{G}(\mathcal{L}_o) \tag{9.53}$$

where $\mathcal{G}(\mathcal{L}_o)$ is the lattice symmetry group (9.48) of the reference lattice $\mathcal{L}_o$ and $\widehat{W}$ is the strain energy response function (9.41).

Recall from Section 8.5 that when examining symmetry in the continuum theory, we considered a deformed configuration χ and two reference configurations χ_1 and χ_2. We were interested in the special case when a symmetry transformation took $\chi_1 \to \chi_2$. In the lattice theory we analogously consider a deformed lattice $\mathcal{L}$ that is generated by lattice vectors $\{\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3\}$ and two reference lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ that are generated by lattice vectors $\{\boldsymbol{\ell}_1^{(1)}, \boldsymbol{\ell}_2^{(1)}, \boldsymbol{\ell}_3^{(1)}\}$ and $\{\boldsymbol{\ell}_1^{(2)}, \boldsymbol{\ell}_2^{(2)}, \boldsymbol{\ell}_3^{(2)}\}$. We are interested in the special case when a symmetry transformation takes $\{\boldsymbol{\ell}_1^{(1)}, \boldsymbol{\ell}_2^{(1)}, \boldsymbol{\ell}_3^{(1)}\}$ to $\{\boldsymbol{\ell}_1^{(2)}, \boldsymbol{\ell}_2^{(2)}, \boldsymbol{\ell}_3^{(2)}\}$ in which case the reference lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ are identical: $\mathcal{L}_1 = \mathcal{L}_2$.

Let $\{\boldsymbol{\ell}_1^{(1)}, \boldsymbol{\ell}_2^{(1)}, \boldsymbol{\ell}_3^{(1)}\}$ be a set of lattice vectors characterizing a reference lattice $\mathcal{L}_1$, and let

$\widehat{W}_1$ be the stored energy response function with respect to this reference lattice:

$$\widehat{W}_1(\mathbf{F}) \;=\; \frac{1}{(\boldsymbol{\ell}_1^{(1)} \times \boldsymbol{\ell}_2^{(1)}) \cdot \boldsymbol{\ell}_3^{(1)}} \quad \frac{1}{2} \sum_{\{\nu_1,\nu_2,\nu_3\}} \phi(|\nu_i\, \mathbf{F}\boldsymbol{\ell}_i^{(1)}|). \tag{9.54}$$

Let $\{\boldsymbol{\ell}_1^{(2)}, \boldsymbol{\ell}_2^{(2)}, \boldsymbol{\ell}_3^{(2)}\}$ be another set of lattice vectors characterizing a (possibly different) reference lattice $\mathcal{L}_2$, and let $\widehat{W}_2$ be the stored energy response function with respect to this reference lattice:

$$\widehat{W}_2(\mathbf{F}) \;=\; \frac{1}{(\boldsymbol{\ell}_1^{(2)} \times \boldsymbol{\ell}_2^{(2)}) \cdot \boldsymbol{\ell}_3^{(2)}} \quad \frac{1}{2} \sum_{\{\nu_1,\nu_2,\nu_3\}} \phi(|\nu_i\, \mathbf{F}\boldsymbol{\ell}_i^{(2)}|). \tag{9.55}$$

If the two sets of reference lattice vectors are related by (9.47), or equivalently by (9.49), then they generate the same reference lattice ($\mathcal{L}_1 = \mathcal{L}_2$) in which case

$$\widehat{W}_1(\mathbf{F}) = \widehat{W}_2(\mathbf{F}). \tag{9.56}$$

It then follows from (9.54) and (9.49) that

$$\begin{aligned}
\widehat{W}_1(\mathbf{FH}) \;&=\; \frac{1}{(\boldsymbol{\ell}_1^{(1)} \times \boldsymbol{\ell}_2^{(1)}) \cdot \boldsymbol{\ell}_3^{(1)}} \quad \frac{1}{2} \sum_{\{\nu_1,\nu_2,\nu_3\}} \phi(|\nu_i\, \mathbf{FH}\boldsymbol{\ell}_i^{(1)}|) \\
&=\; \frac{1}{(\boldsymbol{\ell}_1^{(1)} \times \boldsymbol{\ell}_2^{(1)}) \cdot \boldsymbol{\ell}_3^{(1)}} \quad \frac{1}{2} \sum_{\{\nu_1,\nu_2,\nu_3\}} \phi(|\nu_i\, \mathbf{F}\boldsymbol{\ell}_i^{(2)}|) \\
&=\; \frac{1}{(\boldsymbol{\ell}_1^{(2)} \times \boldsymbol{\ell}_2^{(2)}) \cdot \boldsymbol{\ell}_3^{(2)}} \quad \frac{1}{2} \sum_{\{\nu_1,\nu_2,\nu_3\}} \phi(|\nu_i\, \mathbf{F}\boldsymbol{\ell}_i^{(2)}|) \\
&=\; \widehat{W}_2(\mathbf{F})
\end{aligned}$$

where in the penultimate step we have used (9.51) and in the ultimate step we have used (9.55). It follows from this and (9.56) that

$$\widehat{W}_1(\mathbf{FH}) = \widehat{W}_1(\mathbf{F}) \qquad \text{for all nonsingular } \mathbf{F} \text{ and all } \mathbf{H} \in \mathcal{G}(\mathcal{L}_1).$$

Similarly one can show that

$$\widehat{\mathbf{T}}_1(\mathbf{F}) = \widehat{\mathbf{T}}_1(\mathbf{FH}) \qquad \text{for all } \mathbf{H} \in \mathcal{G}(\mathcal{L}_1)\ . \tag{9.57}$$

Thus the stress response function $\widehat{\mathbf{T}}$ and the energy response function $\widehat{W}$ derived from the present lattice theory, i.e. (9.36) and (9.41), are invariant under the group of transformations $\mathcal{G}(\mathcal{L}_o)$ that map the reference lattice back onto itself.

The lattice symmetry group and the symmetry of a continuum. Suppose that the lattice underlying the reference configuration of some elastic solid is a known Bravais lattice

$\mathcal{L}_0$. However, suppose that one does not adopt the elementary pair potential model for interatomic interactions but arrives at a form for the strain energy function $\widehat{W}(\mathbf{F})$ by some other method, i.e. consider a strain energy response function $\widehat{W}(\mathbf{F})$ for the lattice that is *not* given by (9.41).

Even though the pair potential model for interatomic interactions was not used, the underlying lattice is (by assumption) a known Bravais lattice. Thus in particular the symmetry of the lattice is characterized by a known lattice symmetry group $\mathcal{G}(\mathcal{L}_o)$. Should one require that the continuum model exhibit all of the symmetries of the lattice? i.e. should we require

$$\widehat{W}(\mathbf{FH}) = \widehat{W}(\mathbf{F}) \quad \text{for all } \mathbf{H} \in \mathcal{G}(\mathcal{L}_o)? \tag{9.58}$$

The generally accepted answer is “no”: the material symmetry group of the continuum should be a suitable subgroup of $\mathcal{G}(\mathcal{L}_o)$. This is based on the fact that in addition to rotations and reflections, the lattice symmetry group $\mathcal{G}(\mathcal{L}_o)$ contains finite shears as well; e.g. see the example (9.45), (9.46). Such shears cause large distortions of the lattice and are usually associated with lattice slip and plasticity. It is natural therefore to exclude these large shears when modeling thermoelastic materials.

Based on the work of Ericksen & Pitteri (see Bhattacharya) the appropriate material symmetry group of the continuum should be the subgroup of rotations in $\mathcal{G}(\mathcal{L}_o)$:

$$\mathsf{P}(\mathcal{L}_o) = \{\mathbf{R} : \ \mathbf{R} \in SO(3), \mathbf{R} \in \mathcal{G}(\mathcal{L}_o)\} \ . \tag{9.59}$$

Thus we would require $\widehat{W}(\mathbf{FR}) = \widehat{W}(\mathbf{F})$ for all $\mathbf{R} \in \mathsf{P}(\mathcal{L}_o)$ instead of the more stringent requirement (9.58). $\mathsf{P}(\mathcal{L}_o)$ is called the “point group” or “ Laue group” of the lattice. It is the group of rotations which map the lattice[11] back into itself. The point group associated with any Bravais lattice is a finite group.

9.2.8 Worked Examples and Exercises.

Problem 9.1. Show that the stress response function $\widehat{\mathbf{T}}(\mathbf{F})$ given explicitly in (9.36) *automatically* satisfies the condition $\widehat{\mathbf{T}}(\mathbf{QF}) = \mathbf{Q}\widehat{\mathbf{T}}(\mathbf{F})\mathbf{Q}^T$ for all proper orthogonal tensors $\mathbf{Q}$. (Therefore this $\widehat{\mathbf{T}}(\mathbf{F})$ is automatically material frame indifferent.)

[11] For example, the point group of a simple cubic lattice consists of the 24 rotations that map the unit cube back into itself.

<u>*Solution*</u>: From (9.36),

$$\mathbf{T}(\mathbf{QF}) = \frac{1}{2(\mathbf{QF}\boldsymbol{\ell}_1^o \times \mathbf{QF}\boldsymbol{\ell}_2^o)\cdot\mathbf{QF}\boldsymbol{\ell}_3^o} \sum_{\{\nu_1,\nu_2,\nu_3\}} \left[\phi'(|\nu_p\,\mathbf{QF}\boldsymbol{\ell}_p^o|)\;.\;\frac{(\nu_i\,\mathbf{QF}\boldsymbol{\ell}_i^o)\otimes(\nu_j\,\mathbf{QF}\boldsymbol{\ell}_j^o)}{|\nu_k\,\mathbf{QF}\boldsymbol{\ell}_k^o|}\right]\;. \qquad (a)$$

By using the vector identity $(\mathbf{Aa}\times\mathbf{Ab})\cdot\mathbf{Ac} = \det\mathbf{A}\;(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ and the fact that $\det\mathbf{Q}=1$ we can write

$$(\mathbf{QF}\boldsymbol{\ell}_1^o \times \mathbf{QF}\boldsymbol{\ell}_2^o)\cdot\mathbf{QF}\boldsymbol{\ell}_3^o \;=\; (\mathbf{F}\boldsymbol{\ell}_1^o \times \mathbf{F}\boldsymbol{\ell}_2^o)\cdot\mathbf{F}\boldsymbol{\ell}_3^o\;. \qquad (b)$$

Next, since $\mathbf{Q}$ is orthogonal, it preserves length, i.e. $|\mathbf{Qy}| = |\mathbf{y}|$ for all vectors $\mathbf{y}$, and consequently

$$|\nu_i\,\mathbf{QF}\boldsymbol{\ell}_i^o| = \;|\nu_i\,\mathbf{F}\boldsymbol{\ell}_i^o|\;. \qquad (c)$$

Finally, in view of the vector identity $(\mathbf{Aa})\otimes(\mathbf{Bb}) = \mathbf{A}(\mathbf{a}\otimes\mathbf{b})\mathbf{B}^T$ can write

$$(\nu_i\mathbf{QF}\boldsymbol{\ell}_i^o)\otimes(\nu_i\mathbf{QF}\boldsymbol{\ell}_i^o) = \mathbf{Q}\left((\nu_i\,\mathbf{F}\boldsymbol{\ell}_i^o)\otimes(\nu_j\,\mathbf{F}\boldsymbol{\ell}_j^o)\right)\mathbf{Q}^T. \qquad (d)$$

Therefore we can simplify (a) by using $(b), (c)$ and (d) to get

$$\begin{aligned}\mathbf{T}(\mathbf{QF}) \quad &= \quad \frac{1}{2(\mathbf{F}\boldsymbol{\ell}_1^o\times\mathbf{F}\boldsymbol{\ell}_2^o)\cdot\mathbf{F}\boldsymbol{\ell}_3^o}\,\mathbf{Q}\left(\sum_{\{\nu_1,\nu_2,\nu_3\}}\left[\;\phi'(|\nu_p\,F\boldsymbol{\ell}_p^o|)\frac{(\nu_i\,\mathbf{F}\boldsymbol{\ell}_i^o)\otimes(\nu_j\,\mathbf{F}\boldsymbol{\ell}_j^o)}{|\nu_k\,\mathbf{F}\boldsymbol{\ell}_k^o|}\right]\right)\mathbf{Q}^T \\ &= \quad \mathbf{QT}(\mathbf{F})\mathbf{Q}^T\;.\end{aligned}$$

Problem 9.2. Show that the Cauchy stress response function $\widehat{\mathbf{T}}(\mathbf{F})$ given explicitly by (9.36) and the strain energy response function $\widehat{W}(\mathbf{F})$ given explicitly by (9.41) are *automatically* related by

$$\widehat{\mathbf{T}}(\mathbf{F}) = \frac{1}{\det\mathbf{F}}\,\frac{\partial\widehat{W}}{\partial\mathbf{F}}(\mathbf{F})\,\mathbf{F}^T.$$

(Therefore the stress and strain energy response functions provided by the lattice theory automatically satisfy the relation imposed by the entropy inequality.)

<u>*Solution*</u>: Differentiating (9.41) with respect to $\mathbf{F}$ gives

$$2(\boldsymbol{\ell}_1^o\times\boldsymbol{\ell}_2^o)\cdot\boldsymbol{\ell}_3^o\,\left(\frac{\partial\widehat{W}}{\partial\mathbf{F}}\right) = \sum_{\{\nu_1,\nu_2,\nu_3\}}\left[\phi'(|\nu_p\,\mathbf{F}\boldsymbol{\ell}_p^o|)\,\left(\frac{\partial}{\partial\mathbf{F}}(|\nu_i\,\mathbf{F}\boldsymbol{\ell}_i^o|)\right)\right]\;. \qquad (a)$$

The following identity can be readily verified for an arbitrary vector $\mathbf{y}$:

$$\frac{\partial}{\partial\mathbf{F}}(|\mathbf{Fy}|) = \frac{1}{2|\mathbf{Fy}|}\,\frac{\partial}{\partial\mathbf{F}}(|\mathbf{Fy}|^2)\;=\;\frac{1}{2|\mathbf{Fy}|}\,\frac{\partial}{\partial\mathbf{F}}\Big(\mathbf{Fy}\cdot\mathbf{Fy}\Big)\;=\;\frac{1}{2|\mathbf{Fy}|}\,\Big(2\mathbf{Fy}\otimes\mathbf{y}\Big),$$

from which it follows that

$$\frac{\partial}{\partial\mathbf{F}}(|\nu_p\,\mathbf{F}\boldsymbol{\ell}_p^o|) = \frac{1}{|\nu_k\,\mathbf{F}\boldsymbol{\ell}_k^o|}\left[(\nu_i\,\mathbf{F}\boldsymbol{\ell}_i^o)\otimes\nu_j\,\boldsymbol{\ell}_j^o\right]\;. \qquad (b)$$

Substituting (b) into (a) yields

$$2(\boldsymbol{\ell}_1^o\times\boldsymbol{\ell}_2^o)\cdot\boldsymbol{\ell}_3^o\,\left(\frac{\partial\widehat{W}}{\partial\mathbf{F}}\right) = \sum_{\{\nu_1,\nu_2,\nu_3\}}\left[\phi'(|\nu_p\,\mathbf{F}\boldsymbol{\ell}_p^o|)\frac{(\nu_i\,\mathbf{F}\boldsymbol{\ell}_i^o)\otimes\nu_j\,\boldsymbol{\ell}_j^o}{|\nu_k\,\mathbf{F}\boldsymbol{\ell}_k^o|}\right],$$

from which it follows that

$$2(\boldsymbol{\ell}_1^o \times \boldsymbol{\ell}_2^o) \cdot \boldsymbol{\ell}_3^o \left(\frac{\partial \widehat{W}}{\partial \mathbf{F}} \right) \mathbf{F}^T = \sum_{\{\nu_1,\nu_2,\nu_3\}} \left[\phi'(|\nu_p\, \mathbf{F}\boldsymbol{\ell}_p^o|) \frac{(\nu_i\, \mathbf{F}\boldsymbol{\ell}_i^o) \otimes \nu_j\, \mathbf{F}\boldsymbol{\ell}_j^o}{|\nu_k\, \mathbf{F}\boldsymbol{\ell}_k^o|} \right] . \tag{c}$$

Finally, because of the identity $(\mathbf{Aa} \times \mathbf{Ab}) \cdot \mathbf{Ac} = \det \mathbf{A}\ (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$, we see from (c) and (9.36) that the relation (9.42) between $\widehat{\mathbf{T}}$ and $\widehat{W}$ holds.

Problem 9.3. Derive an explicit expression for the elasticity tensor $\mathbb{C}$ of linear elasticity by linearization of the results of this chapter. Show that the resulting components $\mathbb{C}_{ijk\ell}$ posses the usual symmetries

$$\mathbb{C}_{ijk\ell} = \mathbb{C}_{k\ell ij} = \mathbb{C}_{jik\ell} = \mathbb{C}_{ij\ell k}, \tag{a}$$

as well as the additional symmetry

$$\mathbb{C}_{ijk\ell} = \mathbb{C}_{i\ell kj} \tag{b}$$

known as the Cauchy relations.

<u>Solution</u>: We first show that the energy response function $\widehat{W}$ given by (9.41) depends on $\mathbf{F}$ only through the Cauchy-Green tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$; thereafter we determine the components of the elasticity tensor $\mathbb{C}$ by recalling that

$$\mathbb{C}_{ijk\ell} = \left. \frac{\partial^2 W(\mathbf{C})}{\partial C_{ij}\, \partial C_{kl}} \right|_{\mathbf{C}=\mathbf{I}} .$$

The fact that $\widehat{W}$ depends on $\mathbf{F}$ only through $\mathbf{C}$ follows from

$$|\nu_i\, \mathbf{F}\boldsymbol{\ell}_i^0| \;=\; \Big((\nu_i\, \mathbf{F}\boldsymbol{\ell}_i^0) \cdot (\nu_i\, \mathbf{F}\boldsymbol{\ell}_i^0) \Big)^{1/2} \;=\; \Big(\mathbf{F}^T\mathbf{F}(\nu_i\, \boldsymbol{\ell}_i^0) \cdot \nu_i\, \boldsymbol{\ell}_i^0 \Big)^{1/2} \;=\; \Big(\mathbf{C}(\nu_i\, \boldsymbol{\ell}_i^0) \cdot \nu_i\, \boldsymbol{\ell}_i^0 \Big)^{1/2}$$

whence we can write (9.41) as

$$W(\mathbf{C}) = \frac{1}{2(\boldsymbol{\ell}_1^0 \times \boldsymbol{\ell}_2^0) \cdot \boldsymbol{\ell}_3^0} \sum_{\{\nu_1,\nu_2,\nu_3\}} \phi\Big((\mathbf{C}(\nu_i\, \boldsymbol{\ell}_i^0) \cdot \nu_i\, \boldsymbol{\ell}_i^0)^{1/2} \Big) .$$

In order to calculate the elasticity tensor we must calculate the second derivative of W with respect to $\mathbf{C}$ and then evaluate it in the reference configuration where $\mathbf{C} = \mathbf{I}$. In order to simplify the writing it is convenient to introduce the notation

$$\alpha = 2(\boldsymbol{\ell}_1^0 \times \boldsymbol{\ell}_2^0) \cdot \boldsymbol{\ell}_3^0, \qquad \mathbf{y} = \nu_i\, \boldsymbol{\ell}_i^0, \qquad \rho(\mathbf{C}) = (\mathbf{Cy} \cdot \mathbf{y})^{1/2} ,$$

so that

$$W(\mathbf{C}) = \frac{1}{\alpha} \sum_{\{\nu_1,\nu_2,\nu_3\}} \phi(\rho(\mathbf{C})).$$

It is straightforward to show that

$$\frac{\partial \rho(\mathbf{C})}{\partial C_{k\ell}} \;=\; \frac{y_k\, y_\ell}{2\rho(\mathbf{C})}.$$

Therefore the first derivative of W is

$$\frac{\partial W}{\partial C_{k\ell}}(\mathbf{C}) = \sum_{\{\nu_1,\nu_2,\nu_3\}} \frac{1}{\alpha}\,\phi'(\rho(\mathbf{C}))\,\frac{\partial \rho(\mathbf{C})}{\partial C_{k\ell}} \;=\; \sum_{\{\nu_1,\nu_2,\nu_3\}} \frac{1}{2\alpha\rho(\mathbf{C})}\,\phi'(\rho(\mathbf{C}))\,y_k\,y_\ell\ .$$

The second derivative can be calculated similarly by differentiating this once more, which leads after some calculation to

$$\frac{\partial^2 W(\mathbf{C})}{\partial C_{ij}\,\partial C_{k\ell}} = \sum_{\{\nu_1,\nu_2,\nu_3\}} \frac{1}{4\alpha\rho^2(\mathbf{C})}\left(\phi''(\rho(\mathbf{C})) - \frac{1}{\rho(\mathbf{C})}\phi'(\rho(\mathbf{C}))\right) y_i\,y_j\,y_k\,y_\ell\ .$$

In order to calculate the components of the elasticity tensor we set $\mathbf{C} = \mathbf{I}$, $\rho(\mathbf{C}) = \rho(\mathbf{I}) = |\mathbf{y}|$ in the preceding expression to obtain

$$\mathbb{C}_{ijk\ell} = \left.\frac{\partial^2 W(\mathbf{I})}{\partial C_{ij}\,\partial C_{k\ell}}\right|_{\mathbf{C}=\mathbf{I}} = \frac{1}{2(\boldsymbol{\ell}_1^0 \times \boldsymbol{\ell}_2^0)\cdot\boldsymbol{\ell}_3^0} \sum_{\{\nu_1,\nu_2,\nu_3\}} \frac{1}{4|\mathbf{y}|^2}\left(\phi''(|\mathbf{y}|) - \frac{1}{|\mathbf{y}|}\phi'(|\mathbf{y}|)\right) y_i\,y_j\,y_k\,y_\ell$$

where the vector $\mathbf{y} = \nu_i\boldsymbol{\ell}_i$. The right-hand side of this is invariant with respect to the change of any pair of subscripts, and therefore so is the left-hand side. This establishes the symmetries (a) and (b).

Problem 9.4. Show that two sets of lattice vectors $\{\boldsymbol{\ell}_1^{(1)}, \boldsymbol{\ell}_2^{(1)}, \boldsymbol{\ell}_3^{(1)}\}$ and $\{\boldsymbol{\ell}_1^{(2)}, \boldsymbol{\ell}_2^{(2)}, \boldsymbol{\ell}_3^{(2)}\}$ generate the same lattice if and only if the matrix $[\mu]$, whose elements relate the lattice vectors through

$$\boldsymbol{\ell}_i^{(2)} = \mu_{ij}\boldsymbol{\ell}_j^{(1)},$$

has elements that are integers and has determinant 1.

Problem 9.5. Let $\mathbf{H}$ be any member of the lattice symmetry group $\mathcal{G}(\mathcal{L})$ defined in (9.48). Show that $\det\mathbf{H} = 1$.

<u>*Solution*</u>: Substitute

$$\mathbf{H}\boldsymbol{\ell}_i^{(1)} = \mu_{ij}\boldsymbol{\ell}_j^{(1)}$$

into the vector identity

$$(\mathbf{H}\boldsymbol{\ell}_1^{(1)} \times \mathbf{H}\boldsymbol{\ell}_2^{(1)})\cdot\mathbf{H}\boldsymbol{\ell}_3^{(1)} = \det\mathbf{H}\ (\boldsymbol{\ell}_1^{(1)} \times \boldsymbol{\ell}_2^{(1)})\cdot\boldsymbol{\ell}_3^{(1)}$$

and expand out the result. This leads to

$$\det\,[\mu] = \det\mathbf{H}$$

after making use of the fact that

$$(\boldsymbol{\ell}_1^{(1)} \times \boldsymbol{\ell}_2^{(1)})\cdot\boldsymbol{\ell}_3^{(1)} = (\boldsymbol{\ell}_2^{(1)} \times \boldsymbol{\ell}_3^{(1)})\cdot\boldsymbol{\ell}_1^{(1)} = (\boldsymbol{\ell}_3^{(1)} \times \boldsymbol{\ell}_1^{(1)})\cdot\boldsymbol{\ell}_2^{(1)}$$

where each of these expressions represents the volume of the unit cell. Finally, since $\det\,[\mu] = 1$ it follows that $\det\mathbf{H} = 1$.

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Chapter 10

Some Nonlinear Effects: Illustrative Examples

Nonlinearity can lead to phenomena that are absent from the linearized theory, and in some cases, phenomena that may be totally unexpected; and even counterintuitive. In this chapter we illustrate some examples of these.

10.1 Example (1): Simple Shear.

Consider an isotropic elastic body that occupies a unit cube in its reference configuration. This cube is subjected to the simple shear deformation

$$y_1 = x_1 + kx_2, \qquad y_2 = x_2, \qquad y_3 = x_3, \tag{10.1}$$

where, as usual, (x_1, x_2, x_3) and (y_1, y_2, y_3) are the coordinates of a particle in the reference and deformed configurations respectively. The amount of shear $k > 0$ is given. All components are taken with respect to a fixed orthonormal basis. We wish to calculate the components of the Cauchy stress tensor $\mathbf{T}$ associated with this deformation.

The components $F_{ij} = \partial y_i/\partial x_j$ of the deformation gradient tensor associated (10.1) are

$$[F] = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and so the tensors $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ and $\mathbf{B}^2$ have components

$$[B] \;=\; \begin{pmatrix} 1+k^2 & k & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad [B]^2 \;=\; \begin{pmatrix} 1+3k^2+k^4 & k(2+k^2) & 0 \\ k(2+k^2) & 1+k^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{10.2}$$

The corresponding principal scalar invariants therefore are given by

$$I_1 = \mathrm{tr}\,[B] = 3+k^2, \qquad I_2 = \frac{1}{2}\left[\mathrm{tr}\,([B]^2) - (\mathrm{tr}\,[B])^2\right] = 3+k^2, \qquad I_3 = \det[B] = 1. \tag{10.3}$$

For an isotropic elastic material $\mathbf{T}$ is related to the deformation through the constitutive relation

$$\mathbf{T} = 2\sqrt{I_3}W_3\mathbf{I} \;+\; \frac{2}{\sqrt{I_3}}(W_1 + I_1 W_2)\mathbf{B} \;-\; \frac{2}{\sqrt{I_3}}W_2\mathbf{B}^2, \qquad \mathbf{B} = \mathbf{F}\mathbf{F}^T, \tag{10.4}$$

where $W(I_1, I_2, I_3)$ is the strain-energy function with respect to the reference configuration and we have written $W_i = \partial W/\partial I_i$.

On substituting (10.2) and (10.3) into (10.4) we find that the components of Cauchy stress in simple shear are

$$\left.\begin{aligned} T_{12} &= 2k(W_1 + W_2), \\ T_{23} &= T_{31} = 0, \\ T_{11} &= (2W_1 + 4W_2 + 2W_3) \;+\; 2k^2(W_1 + W_2), \\ T_{22} &= (2W_1 + 4W_2 + 2W_3), \\ T_{33} &= (2W_1 + 4W_2 + 2W_3) \;+\; 2k^2 W_2, \end{aligned}\right\} \tag{10.5}$$

where the functions $W_i(I_1, I_2, I_3)$ are evaluated at (I_1, I_2, I_3)= $(3+k^2, 3+k^2, 1)$.

<u>Remarks</u>:

1. Observe that in contrast to the classical linearized theory, the normal stress components T_{11}, T_{22}, T_{33} do *not* vanish in general. This is sometimes called the Poynting effect. Thus in order to maintain a simple shear deformation, one must apply the appropriate shear stress *as well as* suitable normal stresses.

2. If one linearizes the expressions for stress in (10.5) for small amounts of shear, $|k| << 1$, one obtains

$$\begin{aligned} T_{12} &= 2k(W_1 + W_2)\Big|_{I_1=3,\ I_2=3,\ I_3=1} \ +\ O(k^3), \\ T_{11} &= (2W_1 + 4W_2 + 2W_3)\Big|_{I_1=3,\ I_2=3,\ I_3=1} \ +\ O(k^2), \end{aligned} \tag{10.6}$$

as $k \to 0$.

It follows from the first of these that the *shear modulus* at infinitesimal strains, $\mu = T_{12}/k$, is given by

$$\mu = 2(W_1 + W_2)\Big|_{I_1=3,\ I_2=3,\ I_3=1} . \tag{10.7}$$

Note that $T_{12} = \mu\, k$.

On the other hand note that the leading order term in the expression for T_{11} in (10.6) does not depend on k. Therefore it is the value of T_{11} when $k = 0$, i.e. in the reference configuration. If the reference configuration is stress-free, this term vanishes and then we see from $(10.5)_3$ that $T_{11} = O(k^2)$ as $k \to 0$. Thus T_{11} is quadratic in k which is why it is absent from the linearized theory.

3. Consider the restriction of the strain energy function W to simple shear deformations, i.e. specialize the strain energy function to simple shear by making use of the fact that $I_1 = 3 + k^2, I_2 = 3 + k^2, I_3 = 1$ and so define a strain energy that is a function only of k:

$$\overline{W}(k) \stackrel{\text{def}}{=} W(3 + k^2, 3 + k^2, 1).$$

Differentiating this with respect to k gives $\overline{W}'(k) = 2k(W_1 + W_2)$ where W_1 and W_2 are evaluated at $I_1 = 3 + k^2, I_2 = 3 + k^2, I_3 = 1$. Therefore from (10.5) we see that

$$T_{12} = \overline{W}\,'(k).$$

Thus the shear stress T_{12} is the gradient of the energy with respect to the amount of shear k. Note that, knowing the function $\overline{W}(k)$ alone allows us to calculate the shear stress T_{12}, but *not* any of the other stress components.

4. Observe from (10.5) that $T_{11} - T_{22} = kT_{12}$. This is independent of the strain energy function W that characterizes the material and so holds for *all* elastic materials. It is sometimes called a “universal relation”.

5. Instead of the simple shearing of a cube, consider instead a circular shaft that is subjected to a torsional deformation, i.e. a twisting about its axis. Locally, at each particle, a torsional deformation is just a simple shear. It follows from the present discussion that we would have to apply *both* a torque and an axial force in order to maintain such a deformation. This is in contrast in the linearized theory where only a torque is required.

6. Similarly, consider a large thin sheet which contains a small planar crack in its interior. If far from the crack the sheet is subjected to a simple shear deformation in the plane of the sheet, with the direction of shearing being parallel to the crack – a so-called Mode II loading –, it follows from the preceding discussion that the crack faces will either move apart and so the crack with open up, or the crack faces will press together and be in contact; which of these occurs depends on whether the normal stress in the direction perpendicular to the crack faces would be tensile or compressive in the absence of the crack. In contrast, in the linearized theory, the crack faces simply slide parallel to each other.

7. Analogous to here, normal stresses are involved in the shear flow of a non-Newtonian fluid. If such a fluid is placed between two coaxial circular cylindrical tubes, one of which is rotating about its axis, the fluid will climb up along the tubes. This is because in order to maintain a shear flow, a suitable normal stress must be applied. But there is nothing at the free surface to apply such a stress.

10.2 Example (2): Deformation of an Incompressible Cube Under Prescribed Tensile Forces.

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2. R.S. Rivlin, Stability of pure homogeneous deformations of an elastic cube under dead loading, *Quarterly of Applied Mathematics*, **32**(1974), pp. 265-271

Equilibrium configurations of a cube: Consider an incompressible isotropic elastic body which occupies a unit cube in a reference configuration. The cube is composed of a neo-Hookean material. The strain-energy function characterizing the material is therefore given by

$$W = \frac{\mu}{2}(I_1 - 3)$$

where $\mu > 0$ is a constant. The general constitutive relation for the Cauchy stress in an incompressible isotropic elastic body, $(8.62)_1$, specializes for a neo-Hookean material to

$$\mathbf{T} = \mu\mathbf{B} - p\mathbf{I}, \qquad \mathbf{B} = \mathbf{F}\mathbf{F}^T. \tag{10.8}$$

Here p is the pressure field that arises due to the incompressibility constraint.

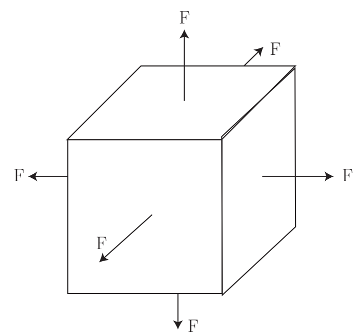

Figure 10.1: A unit cube in the reference configuration. All six of its faces are subjected to uniformly distributed normal tractions whose resultant force, on each face, is F. The figure only shows the resultant force and not the distributed traction.

Each of the six faces of the cube is subjected to a tensile force $F(> 0)$ (which are in fact the resultants of uniformly distributed normal tractions that are applied on each face). This is illustrated in Figure 10.1 where the uniform distribution of normal traction is not shown, and only the resultant forces are shown. We wish to determine the resulting pure homogeneous deformation of the body.

It should be noted that in the problem considered here it is the force F that is prescribed (or equivalently the first Piola-Kirchhoff traction that is prescribed). The associated Cauchy (true) tractions on the faces of the cube will depend on the areas of the faces in the deformed configuration. One could consider the problem in which the Cauchy tractions are prescribed on each face. This is a different problem to the one we study here.

Because of the symmetry of the body, the loading and the material, it is natural to assume that the deformation will also be symmetric. However we wish to look at not-necessarily symmetric pure homogeneous deformations, and so we shall not assume a priori that the cube deforms symmetrically. If it does, then we will find this out. Thus, suppose that the cube undergoes a pure homogeneous deformation

$$y_1 = \lambda_1 x_1, \qquad y_2 = \lambda_2 x_2, \qquad y_3 = \lambda_3 x_3. \tag{10.9}$$

Incompressibility of the material requires that

$$\lambda_1 \lambda_2 \lambda_3 = 1. \tag{10.10}$$

The deformed faces of the body have areas $\lambda_2\lambda_3$, $\lambda_3\lambda_1$ and $\lambda_1\lambda_2$. Thus the prescribed boundary conditions require that the Cauchy stress components be given by

$$T_{11} = \frac{F}{\lambda_2\lambda_3}, \qquad T_{22} = \frac{F}{\lambda_3\lambda_1}, \qquad T_{33} = \frac{F}{\lambda_1\lambda_2}. \tag{10.11}$$

The problem at hand is to find the principal stretches λ_i, given F.

Since the deformation is homogeneous, and assuming that the pressure field is constant, the stress field will also be homogeneous throughout the body. Therefore (ignoring body forces) the equilibrium equations are satisfied automatically. The boundary conditions have already been enforced above. All that remains is to enforce the constitutive law. To this end we first note that the deformation gradient tensor $\mathbf{F}$ and the Cauchy-Green tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ associated with the deformation (10.9) have components

$$[F] = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \qquad [B] = \begin{pmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{pmatrix}. \tag{10.12}$$

Therefore the constitutive relation (10.8) for a neo-Hookean material together with (10.12) gives

$$T_{11} = \mu\lambda_1^2 - p, \qquad T_{22} = \mu\lambda_2^2 - p, \qquad T_{33} = \mu\lambda_3^2 - p. \tag{10.13}$$

Combining (10.13) with (10.11) and using (10.10) leads to

$$F\lambda_1 = \mu\lambda_1^2 - p, \qquad F\lambda_2 = \mu\lambda_2^2 - p, \qquad F\lambda_3 = \mu\lambda_3^2 - p. \tag{10.14}$$

Equations (10.14) and (10.10) provide four scalar algebraic equations involving $\lambda_1, \lambda_2, \lambda_3$ and p.

In order to solve these equations systematically it is convenient to first eliminate p. Thus, subtracting the second of (10.14) from the first, and similarly the third of (10.14) from the second leads to

$$\left.\begin{aligned} [F-\mu(\lambda_1+\lambda_2)](\lambda_1-\lambda_2) &= 0, \\ [F-\mu(\lambda_2+\lambda_3)](\lambda_2-\lambda_3) &= 0. \end{aligned}\right\} \tag{10.15}$$

Equations (10.10) and (10.15) are to be solved for the principal stretches. There are now three cases to consider.

Case (1): Suppose first that all of the λ's are distinct: $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$. Then (10.15) yields

$$F = \mu(\lambda_1+\lambda_2), \qquad F = \mu(\lambda_2+\lambda_3),$$

which implies that

$$\lambda_1 = \lambda_3;$$

this contradicts the assumption that the $\lambda's$ are all distinct. Thus there is no solution in which all of the λ's are distinct.

Case (2): Suppose next that all of the λ's are equal: $\lambda_1 = \lambda_2 = \lambda_3$. In this case equations (10.15) are automatically satisfied while (10.10) requires that

$$\lambda_1 = \lambda_2 = \lambda_3 = 1. \tag{10.16}$$

Thus one solution of the problem, for every value of the applied force F, is given by (10.9), (10.16). This corresponds to a configuration of the body in which, geometrically, it remains a unit cube, but one that is under stress.

Case (3): Finally consider the remaining possibility that two λ's are equal and different to the third:

$$\lambda_2 = \lambda_3 = \lambda \text{ (say)}, \qquad \lambda_1 \neq \lambda. \tag{10.17}$$

Incompressibility (10.10) requires that

$$\lambda_1 = \lambda^{-2}$$

while equations (10.15) reduce to $F = \mu(\lambda_1+\lambda_2) = \mu(\lambda^{-2}+\lambda)$, i.e.

$$\lambda + \lambda^{-2} = F/\mu. \tag{10.18}$$

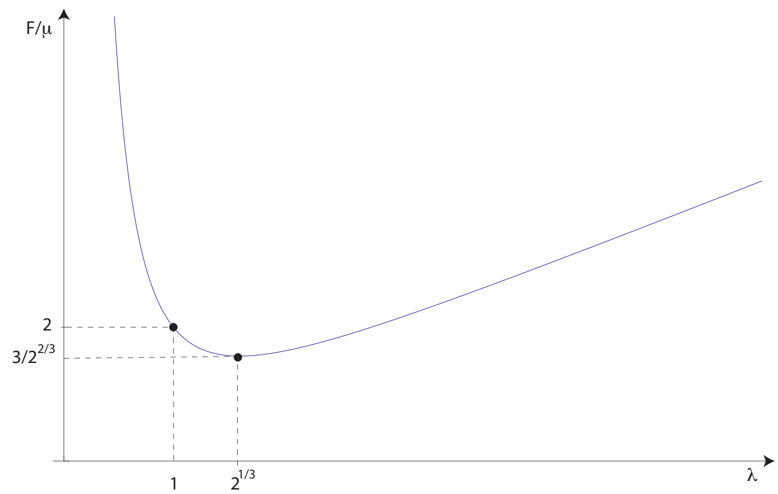

Figure 10.2: Graph of F versus λ as given by (10.18).

Given F/μ, if (10.18) can be solved for one or more roots $\lambda > 0$, then (10.17), (10.9), (10.18) provides the corresponding solution to the problem[1]. Whether (10.18) can be solved or not depends on the value of F/μ. To examine this consider the graph of F/μ versus λ shown in Figure 10.2. From it, we see that

if $F/\mu < 3/2^{2/3}$ then equation (10.18) has no roots,

if $F/\mu = 3/2^{2/3}$ then equation (10.18) has one root $\lambda = 2^{1/3}$, and

if $F/\mu > 3/2^{2/3}$ then equation (10.18) has two roots.

Note that solutions with $\lambda > 1$ describe configurations in which the deformed body has two relatively long equal edges and one relatively short unequal edge ($\lambda_2 = \lambda_3 > 1, \lambda_1 < 1$), i.e. the block has a flattened shape. On the other hand $\lambda > 1$ describes configurations in which the deformed body has two relatively short equal edges and one relatively long unequal edge ($\lambda_2 = \lambda_3 < 1, \lambda_1 > 1$), i.e. the block has a pillar-like shape.

Thus in *summary*, there are two types of configurations which the body can adopt. In one, the body remains a unit cube in the deformed configuration and this is possible for all values of the applied force F. The other is possible only if $F/\mu \geq 3/2^{2/3}$ and here the deformed body is no longer a cube; rather, it has two sides equal and the third side different; there are two possibilities of this form corresponding to the two roots of (10.18).

[1]There are of course additional configurations corresponding to permutations of the λ's, e.g. $\lambda_3 = \lambda_1 = \lambda,\ \lambda_2 = \lambda^{-2}$.

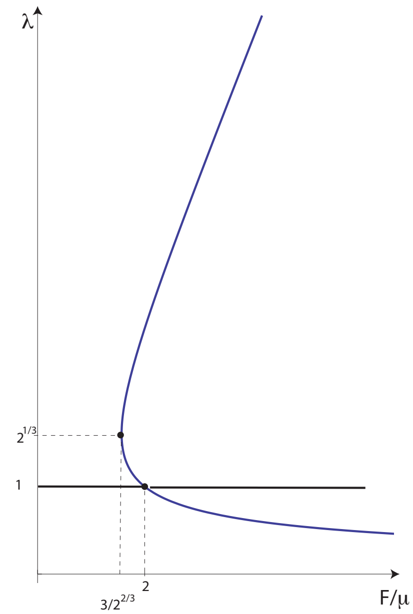

Figure 10.3: Equilibrium configurations of the cube: the symmetric configuration corresponds to the line $\lambda = 1$. The curve corresponds to the non-symmetric configurations given by (10.18).

Both types of solutions are depicted in Figure 10.3. The symmetric solution corresponds to the horizontal line $\lambda = 1$ which extends indefinitely to the right. The unsymmetric solution corresponds to the curve (which is the same curve as in Figure 10.2 but with the axes switched). The figure shows that

> if $F/\mu < 3/2^{2/3}$ the body must be in the cubic configuration,
>
> if $F/\mu > 3/2^{2/3}$ the body can be in either the cubic configuration or a configuration in which one stretch is different to the other two; in fact there are two configurations of this type.

Thus the solution to the problem is *non-unique*. This lack of uniqueness implies that one should examine the stability of the various equilibrium configurations.

Stability of the cube: Rivlin has shown that

> The symmetric configurations of the body are stable for $F/\mu < 2$. They are unstable otherwise.
>
> The unsymmetric configurations of the body with $\lambda > 2^{1/3}$ are stable. The others are unstable.

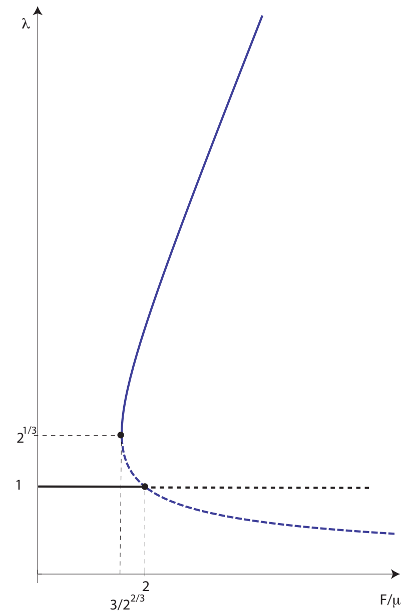

Figure 10.4: The stable and unstable solutions are depicted by the solid and dashed curves respectively.

Figure 10.4 depicts the solutions again, now with the solid line/curve corresponding to the stable solutions and the dashed line/curve the unstable ones.

Remarks on studying the stability of the equilibrium configurations: As mentioned in Chapter 7 of Volume I, an alternative approach for studying equilibrium configurations of an elastic solid/structure is via the minimization of the potential energy. One considers *all geometrically possible deformation fields* $\mathbf{z}(\mathbf{x})$, and minimizes the potential energy over this class of functions. If the potential energy has an extremum at say $\mathbf{z}(\mathbf{x}) = \mathbf{y}(\mathbf{x})$ then $\mathbf{y}(\mathbf{x})$ describes an equilibrium configuration of the body. Moreover, if this extremum corresponds to a minimum of the potential energy, then this configuration is stable.

Suppose that an elastic body occupies a region $\mathcal{R}_0$ in a reference configuration and that the deformation field is prescribed on a portion $\partial\mathcal{R}_{\text{def}}$ of its boundary, and that the first Piola-Kirchhoff traction ("dead load") is prescribed on the remaining portion $\partial\mathcal{R}_{\text{loads}}$ of the boundary; here $\partial\mathcal{R}_0 = \partial\mathcal{R}_{\text{def}} \cup \partial\mathcal{R}_{\text{loads}}$. A kinematically possible deformation field is any smooth enough vector field $\mathbf{z}(\mathbf{x})$ defined on $\mathcal{R}_0$ that obeys all geometric constraints. One requirement would be that its value coincide with the prescribed deformation on $\partial\mathcal{R}_{\text{def}}$. If there are internal kinematic constraints such as incompressibility, then these too must be satisfied. The potential energy associated with a geometrically possible deformation field

$\mathbf{z}(\mathbf{x})$ is

$$\Phi = \int_{\mathcal{R}_0} W(\text{Grad}\,\mathbf{z})\; dV_x \;-\; \int_{\partial\mathcal{R}_{\text{loads}}} \mathbf{Sn}\cdot\mathbf{z}\; dA_x. \tag{10.19}$$

The first term on the right hand side describes the elastic energy stored in the body while the second term corresponds to the potential energy of the applied dead loading. One seeks to minimize this functional over the set of all geometrically possible deformation fields.

Exercise: Specialize this to the problem of the triaxially loaded cube under discussion and show that the potential energy

$$\Phi = \int_{\mathcal{R}_0} W(\text{Grad}\,\mathbf{z})\; dV_x \;-\; \int_{\partial\mathcal{R}_0} \mathbf{Sn}\cdot\mathbf{z}\; dA_x$$

is to be minimized over the set of all smooth enough vector field $\mathbf{z}(\mathbf{x})$ subject to the incompressibility requirement $\det\big(\text{Grad}\,\mathbf{z}\big) = 1$.

Exercise: Rather than minimizing this over the set of all geometrically possible kinematic fields suppose that we minimize over the smaller class of all geometrically possible *homogeneous* deformation fields: $\mathbf{z}(\mathbf{x}) = \mathbf{Fx}$ where $\mathbf{F}$ is a constant tensor with unit determinant. Show that in this case

$$\Phi = W(\mathbf{F}) - \mathbf{S}\cdot\mathbf{F} \tag{10.20}$$

plus an inessential additive constant.

Exercise: Suppose that we further limit attention to geometrically possible deformation fields of the even more restricted form

$$y_1 = \lambda_1 x_1, \quad y_2 = \lambda_2 x_2, \quad y_3 = \lambda_3 x_3, \qquad \lambda_1\lambda_2\lambda_3 = 1. \tag{10.21}$$

Show that the potential energy (10.20) (for the neo-Hookean material) now takes the explicit form

$$\Phi(\lambda_1,\lambda_2,\lambda_3) = \frac{\mu}{2}(\lambda_1^2+\lambda_2^2+\lambda_3^2) - F(\lambda_1+\lambda_2+\lambda_3).$$

We are to minimize this over all $(\lambda_1,\lambda_2,\lambda_3)$ subject to the constraint $\lambda_1\lambda_2\lambda_3 = 1$. We can simplify this by eliminating λ_3 using the incompressibility constraint. This gives

$$\Phi(\lambda_1,\lambda_2) = \frac{\mu}{2}(\lambda_1^2+\lambda_2^2+\frac{1}{\lambda_1^2\lambda_2^2}) - F(\lambda_1+\lambda_2+\frac{1}{\lambda_1\lambda_2}) \tag{10.22}$$

which we must minimize over all (λ_1,λ_2).

Finding the extrema of (10.22) leads to the equilibrium configurations found previously (depicted in Figure 10.3). Examining whether or not they are local minima provides necessary information about their stability. Necessary, but not sufficient, because we are minimizing over a subset of all geometrically possible deformations. If the current analysis shows an equilibrium state to be a local minimizer, it may or may not be a minimizer (stable) under the wider class of all deformations. However if the current analysis shows an equilibrium state to be not a local minimizer, then it is not a minimizer even under the wider class of all deformations (and so is unstable).

Exercise Rather than minimizing over the class of deformations (10.21) consider the subset of such deformations of the form

$$y_1 = \lambda^{-2} x_1, \qquad y_2 = \lambda x_2, \qquad y_3 = \lambda x_3, \quad \lambda > 0.$$

Show that the potential energy specializes to

$$\Phi(\lambda) = \frac{\mu}{2}(\lambda^{-4} + 2\lambda^2) - F(\lambda^{-2} + 2\lambda) \tag{10.23}$$

which we are to minimize over all $\lambda > 0$.

Show that the equilibrium configurations found by setting $\Phi'(\lambda) = 0$ are

$$\text{either } \lambda = 1 \quad \text{or} \quad \lambda + \lambda^{-2} = F/\mu$$

which are the two equilibrium configurations that we found earlier. To examine their stability show that

$$\Phi''(\lambda)|_{\lambda=1} \;=\; 6(2\mu - F)$$

and conclude that the symmetric configuration is unstable when

$$F > 2\mu.$$

Similarly show that

$$\Phi''(\lambda)|_{\lambda+\lambda^{-2}=F/\mu} \;=\; 2\mu\,(\lambda^3 - 1)\lambda^{-6}(\lambda^3 - 2).$$

and conclude that the non-symmetric solutions are unstable when

$$1 \;<\; \lambda \;<\; 2^{1/3}.$$

As noted previously, a configuration found to be unstable by this calculation will be unstable irregardless of the set of possible deformation fields. On the other hand a solution that is found to be stable by this analysis may not be stable when the full set of admissible deformations are considered. In fact as noted previously, Rivlin has shown that the unsymmetric configurations are stable only if $\lambda > 2^{1/3}$. Thus the unsymmetric solutions corresponding to $\lambda < 1$ that are minimizers of (10.23) are not minimizers in the more general setting.

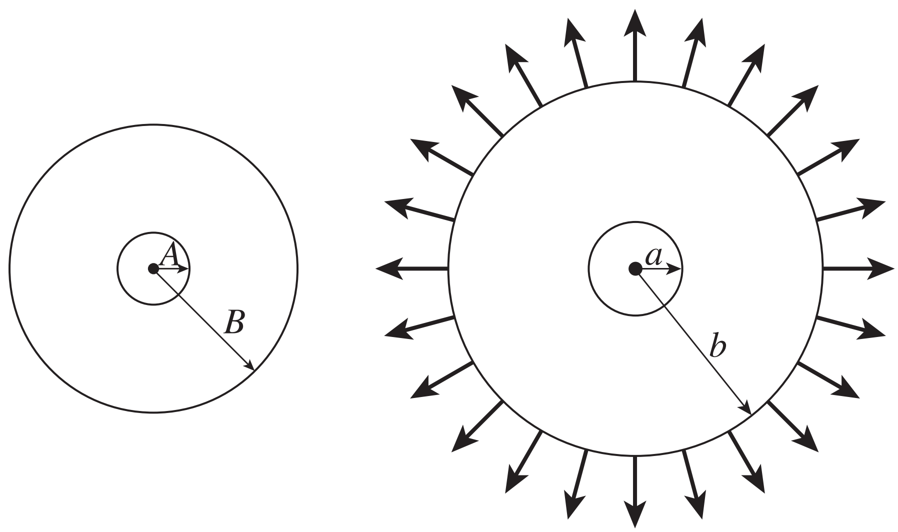

Figure 10.5: A hollow sphere in the reference configuration (left) and in the deformed configuration (right). The uniform radial dead load is applied on the outer surface of the sphere.

10.3 Example (3): Growth of a Cavity.

References:

1. A.N. Gent and P.B. Lindley, Internal rupture of bonded rubber cylinders in tension, *Proceedings of the Royal Society (London)*, **A249**, (1958), 195-205.
2. J.M. Ball, Discontinuous equilibrium solutions and cavitation in nonlinear elasticity, *Philosophical Transactions of the Royal Society (London)*, **A306**, (1982), 557-611.

Suppose that the region occupied by a body in a reference configuration is a hollow sphere of inner radius A and outer radius B, and suppose that this body is composed of a neo-Hookean material:

$$\mathbf{T} = \mu\mathbf{B} - p\,\mathbf{I}. \tag{10.24}$$

A uniformly distributed radial tensile (Piola-Kirchhoff) traction S is applied on the outer surface of the body while the inner surface remains traction-free. We wish to determine the radius a of the deformed cavity as a function of the applied stress S.

We assume that the resulting deformation is spherically symmetric so that it can be described by

$$r = f(R), \quad \theta = \Theta, \quad \phi = \Phi, \tag{10.25}$$

where (R, Θ, Φ) and (r, θ, ϕ) are the spherical coordinates of a particle in the reference and deformed configurations respectively. The principal stretches associated with this deformation are[2]

$$\lambda_R = f'(R), \quad \lambda_\Theta = \lambda_\Phi = f(R)/R. \tag{10.26}$$

Since the material is incompressible, the deformation (10.25) must be volume preserving so that

$$\det \mathbf{F} \;=\; \lambda_R \lambda_\Theta \lambda_\Phi = f^2(R) f'(R)/R^2 = 1.$$

Solving the first order differential equation $f^2 f' = R^2$ leads to

$$f(R) = (R^3 + a^3 - A^3)^{1/3}, \qquad A \le R \le B, \tag{10.27}$$

where a is a constant of integration. Note from (10.25) and (10.27) that $r = a$ when $R = A$, and so the constant a represents the (unknown) *radius of the deformed cavity.*

From (10.24), the normal components of Cauchy stress are given by

$$T_{rr} = \mu \lambda_R^2 - p, \quad T_{\theta\theta} = T_{\phi\phi} = \mu \lambda_\Theta^2 - p, \tag{10.28}$$

where $p = p(r)$ is the reaction pressure arising due to the incompressibility constraint, and the principal stretches λ_R and λ_Θ are given by (10.26), (10.27); the shear components of stress vanish. The equilibrium equation $\operatorname{div} \mathbf{T} = \mathbf{0}$ when expressed in spherical components and then specialized to the present case where $T_{rr} = T_{rr}(r), T_{\theta\theta}(r) = T_{\phi\phi}(r)$ reduces to

$$\frac{dT_{rr}}{dr} + \frac{2}{r}(T_{rr} - T_{\theta\theta}) = 0. \tag{10.29}$$

Calculating $T_{rr} - T_{\theta\theta}$ from (10.28), then expressing the result explicitly in terms of r by using (10.25), (10.26) and (10.27), and finally substituting the result into (10.29), leads to the following differential equation for T_{rr}:

$$\frac{dT_{rr}}{dr} + \frac{2\mu}{r}\left[\frac{(r^3 - a^3 + A^3)^{4/3}}{r^4} - \frac{r^2}{(r^3 - a^3 + A^3)^{2/3}}\right] = 0. \tag{10.30}$$

The given loading on the boundaries implies that $S_{RR} = 0$ at the inner cavity wall and $S_{RR} = S$ at the outer wall. By using the relation $\mathbf{T} = \mathbf{S}\mathbf{F}^T$ between the Cauchy and first

[2] These expressions can be calculated either by finding the eigenvalues of $\mathbf{B}$, or more easily by calculating the ratio between the deformed and undeformed lengths of two infinitesimal material fibers, one in the radial direction and the other in the circumferential direction.

Piola-Kirchhoff stresses, these boundary conditions can be written in terms of the Cauchy stress as

$$T_{rr} = 0 \quad \text{at } r = a, \qquad T_{rr} = \frac{B^2}{(B^3 - A^3 + a^3)^{2/3}} S \quad \text{at } r = (B^3 - A^3 + a^3)^{1/3}. \tag{10.31}$$

We wish to integrate (10.30) from the inner radius $r = a$ to the outer radius $r = (B^3 - A^3 + a^3)^{1/3}$ and enforce the boundary conditions (10.31).

It order to carry out this calculation it is convenient to make the substitution

$$\lambda = \frac{r}{(r^3 + A^3 - a^3)^{1/3}}$$

(which is nothing more that the stretch r/R in the hoop direction). Then some calculation shows that the differential equation (10.30) takes the form

$$\frac{\mathrm{d}T_{rr}}{\mathrm{d}\lambda} \;=\; -2\mu(\lambda^{-5} + \lambda^{-2}).$$

This can be readily integrated. After integrating this and enforcing the boundary conditions, one finds the following relation between the deformed cavity radius a and the applied stress S:

$$\frac{S}{2\mu} = \left(\frac{a^3}{B^3} - \frac{A^3}{B^3} + 1\right)^{1/3} + \frac{1}{4}\left(\frac{a^3}{B^3} - \frac{A^3}{B^3} + 1\right)^{-2/3} - \left(\frac{A}{a} + \frac{1}{4}\frac{A^4}{a^4}\right)\left(\frac{a^3}{B^3} - \frac{A^3}{B^3} + 1\right)^{2/3}. \tag{10.32}$$

We are given the radii A and B of the undeformed body, the shear modulus μ and the applied stress S, and are to determine the radius a of the deformed cavity from this equation.

Equation (10.32) is of the form $S/2\mu = h(a)$. One can show that $h(a)$ increases monotonically with a; moreover $h(A) = 0$ and $h(a) \to \infty$ as $a \to \infty$. Thus, for each given value of applied stress $S > 0$, equation (10.32) can be solved for a unique root a. The graph in Figure 10.6 shows the variation of the cavity radius a/B with the applied stress $S/2\mu$ according to (10.32) for a cavity of initial radius $A/B = 0.3$.

Thus far this problem in the nonlinear theory has not been qualitatively different to the linear problem. However, if we plot graphs of a versus S for progressively decreasing initial cavity radii A, the family of curves obtained shows an interesting trend as seen in Figure 10.7. As $A/B \to 0$, the graph of a/B versus $S/2\mu$ approaches the curve $\mathcal{C}$. Note that $\mathcal{C}$ is composed of two portions: the straight line segment $a = 0$ for $0 < S/2\mu \leq 1.25$ and the curved portion for $S/2\mu \geq 1.25$.

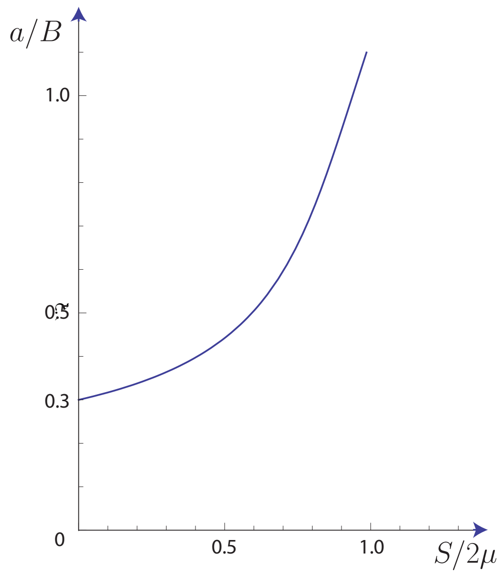

Figure 10.6: Variation of the deformed cavity radius a with applied stress S. The figure has been drawn for the case $A/B = 0.3$.

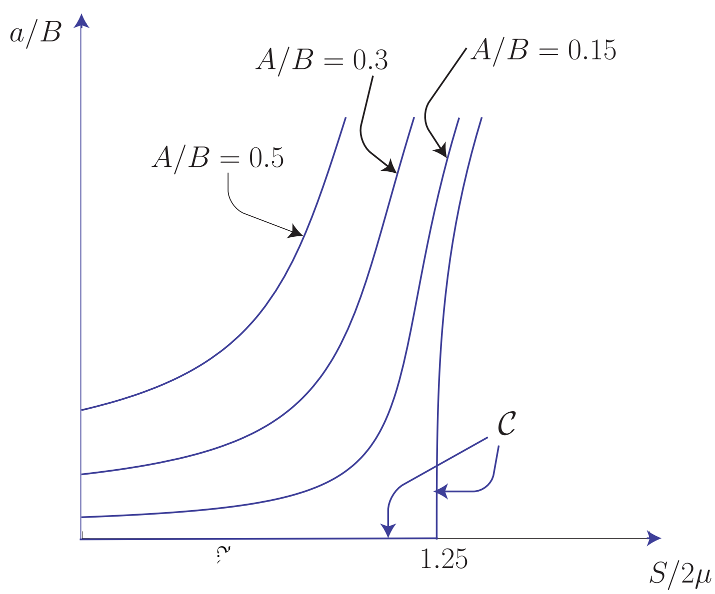

Figure 10.7: Variation of the deformed cavity radius a with applied stress S. The different curves correspond to different values of the undeformed cavity radius A. Observe that as $A/B \to 0$ these curves approach the curve $\mathcal{C}$.

To examine this analytically, one takes the limit $A/B \to 0$ at fixed a/B of the right-hand side of (10.32). The third term on the right hand side vanishes and the other two terms combine to give

$$\frac{S}{2\mu} = \frac{a^3/B^3 + 5/4}{(a^3/B^3+1)^{2/3}}.$$

This is the equation of the curved portion of $\mathcal{C}$. Note that $S/2\mu \to 5/4$ as $a/B \to 0$ in this equation. This is reflected in Figure 10.7 as well.

We therefore conclude that a cavity which is infinitesimally small in the undeformed configuration remains infinitesimally small as the applied stress S increases until it reaches the critical value $S_{cr}/2\mu = 5/4$; when S exceeds S_{cr}, the cavity grows (i.e. $a > 0$) in the manner described by the curved portion of $\mathcal{C}$. This describes the phenomenon of *cavitation*.

An entirely similar calculation shows that, if the body is composed of a *general* isotropic incompressible material with associated strain energy function $W(\lambda_1, \lambda_2, \lambda_3)$, the value of the critical stress for cavitation is formally given by

$$S_{cr} = \int_1^\infty \frac{1}{\lambda^3 - 1} \overline{W}\,'(\lambda) d\lambda \tag{10.33}$$

where $\overline{W}(\lambda) \stackrel{\text{def}}{=} W(\lambda^{-2}, \lambda, \lambda)$. Observe that the integrand has a potential singularity at $\lambda = 1$ unless $\overline{W}\,'(1)$ behaves suitably; moreover, since the range of this integral is infinite, its convergence depends on the behavior of $\overline{W}(\lambda)$ as $\lambda \to \infty$. Thus for certain elastic materials, i.e. certain functions $\overline{W}$, the integral in (10.33) will not converge (i.e. $S_{cr} = \infty$) and so an infinitesimally small void remains infinitesimally small for all values of applied stress. For other materials (for which the integral does converge) cavitation will occur and the infinitesimal cavity will begin to growth when S exceeds the critical value given by (10.33).

10.4 Example (4): Inflation of a Thin-Walled Tube.

References:

1. S. Kyriakides and Y-C. Chang, On the initiation and propagation of a localized instability in an inflated elastic tube, *International Journal of Solids and Structures*, **27**, (1991), 1085-1111.

2. J.L. Ericksen, *Introduction to the Thermodynamics of Solids*, Chapman & Hall, 1991, Chapters 3 and 5.

Consider a long *thin-walled* circular cylindrical tube whose mean radius and wall thickness in a stress-free reference configuration are R and D respectively. The tube is composed of an incompressible isotropic elastic material, and it is subjected to an internal pressure p (per unit current area). In the deformed state, the tube has mean radius r and wall thickness d. We assume a state of plane strain so that there is no change in the axial dimension.

We shall exploit the fact that the tube is thin-walled, $D/R << 1$, and use it to carry out an *approximate analysis*. The results can be justified by carrying out an exact analysis of a thick-walled internally pressurized tube and then taking the limit $D/R \to 0$ of the results.

Since the deformation must be volume preserving, and there is no change in the axial dimension, one must have $2\pi RD = 2\pi rd$. Thus

$$d = \frac{RD}{r}. \tag{10.34}$$

The stretch ratio in the circumferential direction is $\lambda_\Theta = 2\pi r/2\pi R$; in the axial direction $\lambda_Z = 1$; and in the radial direction $\lambda_R = d/D$ which because of (10.34) can be written as $\lambda_R = R/r$. Thus

$$\lambda_R = \frac{R}{r}, \qquad \lambda_\Theta = \frac{r}{R}, \qquad \lambda_Z = 1. \tag{10.35}$$

The principal Cauchy stresses T_i are related to the strain energy function $W = W(\lambda_1, \lambda_2, \lambda_3)$ by $T_i = \lambda_i W_i - q$ where W_i denotes $\partial W/\partial\lambda_i$. (We use the symbol q for the reaction pressure due to the incompressibility constraint since the applied internal pressure has been denoted by p.) Thus on taking the 1- and 2- directions to coincide with the radial and circumferential directions respectively we find from the constitutive relation that

$$T_{rr} = \lambda_R W_1 - q, \qquad T_{\theta\theta} = \lambda_\Theta W_2 - q.$$

The reaction pressure q can be eliminated by subtracting the second equation from the first. This leads to

$$T_{\theta\theta} - T_{rr} = \lambda_\Theta W_2 - \lambda_R W_1. \tag{10.36}$$

Consider next the equilibrium of a longitudinal section of the tube in the deformed configuration; see Figure 10.8. One readily finds by setting the resultant force on this segment of the tube equal to zero that $p \times 2r = 2(T_{\theta\theta} \times d)$, where $T_{\theta\theta}$ is the mean (true) Cauchy hoop stress. Thus

$$T_{\theta\theta} = \frac{pr}{d} = \frac{pr^2}{RD} \tag{10.37}$$

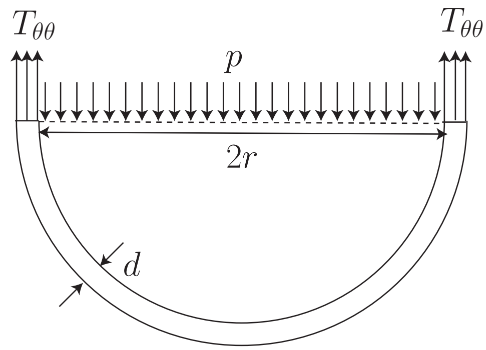

Figure 10.8: Free body diagram of a longitudinal section of the tube including the fluid it contains (in the deformed configuration). The two segments of the tube itself each have thickness d and a stress $T_{\theta\theta}$ acts on them. The relevant portion of fluid has length $2r$ and pressure p. The force $2 \times (T_{\theta\theta} d)$ must balance the force $p(2r)$.

where we used (10.34) in the last step. Since the tube is thin-walled ($R/D >> 1$ and so presumably $r/d >> 1$), the circumferential stress $T_{\theta\theta} = p \times (r/d) >> p$, i.e. $T_{\theta\theta}$ is considerably greater than the applied pressure p. On the other hand the radial stress T_{rr} has the value $-p$ at the inner-wall of the tube and vanishes at the outer-wall of the tube and so is presumably of the order of p through the entire thickness. Thus $T_{\theta\theta} >> T_{rr}$ and so in (10.36) we may neglect the radial stress in comparison with the circumferential stress to get

$$T_{\theta\theta} \approx \lambda_\Theta W_2 - \lambda_R W_1. \tag{10.38}$$

Finally we combine (10.38) and (10.37) to obtain

$$p = \frac{RD}{r^2} \left(\lambda_\Theta W_2 - \lambda_R W_1 \right). \tag{10.39}$$

In this equation the principal stretches λ_R and λ_Θ on the right hand side are given in terms of r by (10.35), and the functions W_i are evaluated at $(R/r, r/R, 1)$. Thus this equation relates the applied pressure p to the deformed radius r of the tube. Given the deformed radius r, it gives the corresponding pressure p. Or conversely given the pressure p, it is to be solved for the radius r.

Since "volume" is the natural conjugate kinematic variable to pressure, the results simplify if we work with the volume (enclosed by a unit length of the tube) instead of the radius r. Accordingly let

$$v = \pi r^2$$

denote the volume enclosed by a unit length of the tube in the deformed configuration. Then the principal stretches (10.35) can be expressed as

$$\lambda_R = \sqrt{\frac{\pi R^2}{v}}, \qquad \lambda_\Theta = \sqrt{\frac{v}{\pi R^2}}, \qquad \lambda_Z = 1. \tag{10.40}$$

Next let $\overline{W}(v)$ be the strain energy function (per unit reference length of the tube) expressed as a function of volume v, i.e. evaluate $W(\lambda_1, \lambda_2, \lambda_3)$ at $(\lambda_1, \lambda_2, \lambda_3) = (\lambda_R, \lambda_\Theta, 1)$ where λ_R and λ_Θ are given by (10.40):

$$\overline{W}(v) \stackrel{\text{def}}{=} 2\pi RD\, W\left(\sqrt{\frac{\pi R^2}{v}},\ \sqrt{\frac{v}{\pi R^2}},\ 1\right); \tag{10.41}$$

the factor $2\pi RD$ here converts "per unit reference volume" to "per unit reference length". Differentiating (10.41) with respect to v leads to

$$\overline{W}\,'(v) = \frac{\pi RD}{v}\left\{\sqrt{\frac{v}{\pi R^2}}\, W_2 - \sqrt{\frac{\pi R^2}{v}}\, W_1\right\} = \frac{RD}{r^2}\Big\{\lambda_\Theta\, W_2 - \lambda_R\, W_1\Big\} \tag{10.42}$$

and therefore from (10.42) and (10.39) we have the following relation between the applied pressure p and the enclosed volume v:

$$p = \overline{W}\,'(v). \tag{10.43}$$

In summary, given the strain energy function $W(\lambda_1, \lambda_2, \lambda_3)$ that characterizes the material, we calculate $\overline{W}(v)$ from (10.41). Then, given the volume v, equation (10.43) yields the corresponding pressure p. Or conversely given the pressure p, equation (10.43) is to be solved for the corresponding volume v.

Given a specific material model (strain energy function) one can work out the details above and obtain an explicit expression for the pressure-volume relation $p = \overline{W}\,'(v)$. For example for the neo-Hookean strain energy function

$$W(\lambda_1, \lambda_2, \lambda_3) = \frac{\mu}{2}\,(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)$$

(10.43) takes the explicit form

$$\frac{p}{\mu} = \frac{D}{R}\left\{1 - \left(\frac{\pi R^2}{v}\right)^2\right\}. \tag{10.44}$$

For certain strain energy functions such as the neo-Hookean and Gent models, the relationship $p = \overline{W}\,'(v)$ between pressure and volume is a monotonically increasing one. Consequently given the pressure p there is a unique corresponding value of volume v (and vice versa of course).

However for certain other strain energy functions this relationship is non-monotonic. This is the case for example for the following strain energy function that models the latex rubber used by Kyriakides and Chang in certain experiments:

$$W = \sum_{n=1}^{3} \frac{\mu_n}{\alpha_n}((\lambda_1^{\alpha_n} + \lambda_2^{\alpha_n} + \lambda_3^{\alpha_n})$$

where

$$\mu_1 = 617kPa, \quad \mu_2 = 1.86kPa, \quad \mu_3 = -9.79kPa, \quad \alpha_1 = 1.30, \quad \alpha_2 = 5.08, \quad \alpha_3 = -2.00.$$

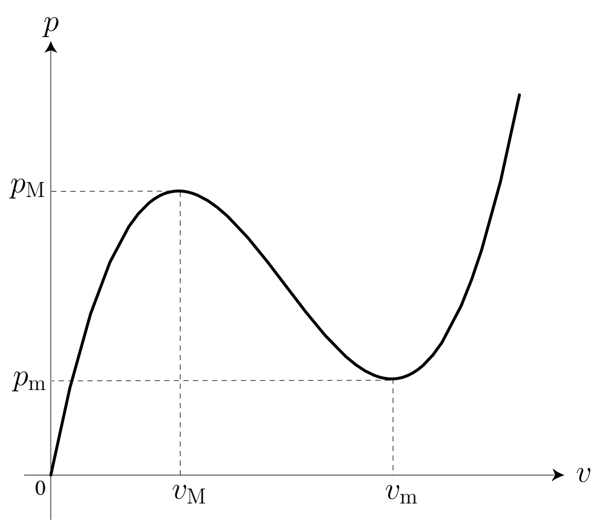

Figure 10.9: Schematic graph of p versus v as given by (10.43) for a certain class of materials (strain energy functions). The pressure reaches a (local) maximum value p_M at $v = v_M$ and a (local) minimum value p_m at $v = v_m$.

For this material, as the volume v increases, the pressure p first rises until it reaches a maximum value, it then decreases until it reaches a minimum value, and finally increases again. Figure 10.9 depicts such a case *schematically* where the (local) maximum value of pressure is attained at $v = v_M$ and this value is $p = p_M$, and the (local) minimum value of pressure is attained at $v = v_m$ and this value is $p = p_m$.

We shall now discuss the consequences of having a rising-falling-rising pressure-volume curve. In order to describe the behavior of the tube under various conditions we will state without proof several results. Proofs can be found in Ericksen.

Case of prescribed pressure: If the given value of pressure lies in the range $p_m < p < p_M$, there are three values of v, say v_1, v_2 and v_3, corresponding to the three branches of the pressure-volume curve as depicted in Figure 10.10. Thus the solution to the equilibrium problem is non-unique. Additional considerations must be taken into account if this non-uniqueness is to be resolved.

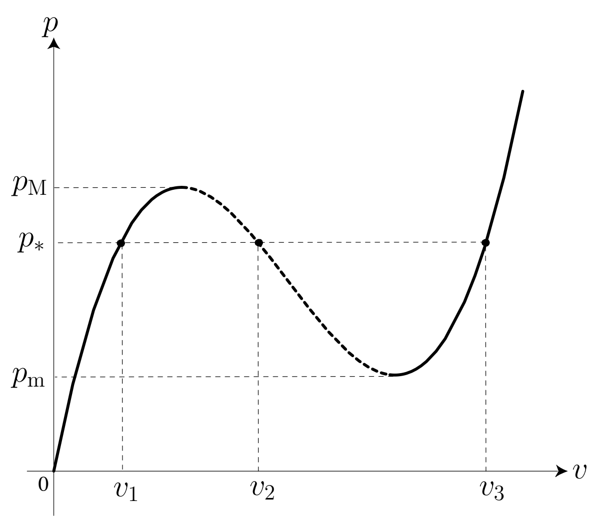

Figure 10.10: Three values of the volume, v_1, v_2 and v_3, correspond to the given value of pressure p_*. The first rising branch is associated with relatively small values of volume v (and therefore tube radius r) while the second rising branch is associated with relatively large values of volume v (and therefore tube radius r). Thus the tube has a relatively small radius in the configuration associated with v_1 and a large radius in the configuration associated with v_3. The configuration associated with v_2 is unstable.

Equilibrium solutions that are observed in the laboratory must be stable. Thus it is natural to look at the stability of these multiple equilibrium states. In order to look at this, one must describe more carefully the manner in which the loading is controlled. Suppose that the pressure is controlled – often called loading by a “soft device”. This can be achieved, for example, by inflating the tube with an incompressible fluid using a piston carrying a weight.

Changing the magnitude of the weight changes the pressure.

If an equilibrium configuration is required to be stable against small disturbances, then the solution $v = v_2$ is found to be unstable while the solutions v_1 and v_3 are both stable. Thus there are multiple stable solutions to the equilibrium problem.

Thus in this case we have *non-uniqueness* of stable equilibrium solutions[3] (for certain values of the prescribed pressure).

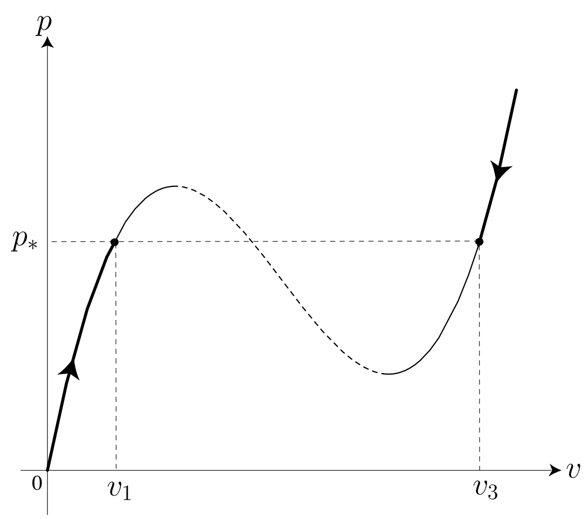

Figure 10.11: A process during which the pressure increases from 0 to p_*; and a second process during which the pressure decreases from some large value to p_*. The first process necessarily starts on the first rising branch of the p, v-curve, while the second process necessarily starts on the second rising branch of the p, v-curve.

One approach for examining this further would be to consider the *process* by which the tube is pressurized instead of considering just a pure equilibrium problem. The observed value of volume would now depend on the process by which the pressure p_* is reached: one might surmise from Figure 10.11 that if the pressure had increased monotonically from 0 to p_* the associated volume would be v_1; on the other hand if the pressure had decreased monotonically from some large value to p_*, the associated volume would be v_3.

An alternative approach would be to require that an equilibrium configuration be stable against *all* disturbances. In this event one finds that the solution $v = v_1$ is stable if the

[3]As can be seen from Figure 10.4, this also occurs in Example (2) for applied force values in the range $3/2^{2/3} < F/\mu < 2$.

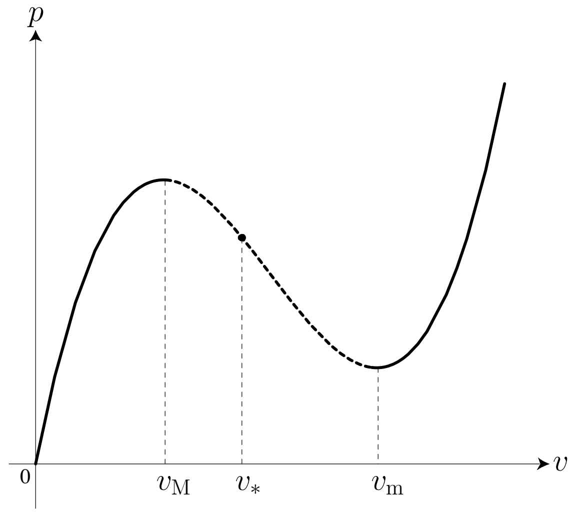

Figure 10.12: There is only one equilibrium configuration of the type discussed here corresponding to the prescribed volume v_* and it is unstable.

pressure is less than a certain critical pressure (called the Maxwell pressure) and unstable for larger values of pressure. On the other hand the solution $v = v_3$ is found to be stable when the pressure is greater than the Maxwell pressure and unstable for smaller values of pressure. Thus the unique solution corresponding to a prescribed value of pressure smaller than the Maxwell pressure is v_1, while that corresponding to a value of pressure exceeding the Maxwell pressure is v_3. This only leaves unresolved the uniqueness of solution at the Maxwell stress itself.

Case of prescribed volume: As seen from Figure 10.12, there is a unique value of pressure corresponding to any value of the prescribed volume. However not all of these configurations are stable. Again, one must describe the manner in which the loading is controlled before stability can be discussed. In this case suppose that the volume v is controlled – often called loading by a "hard device". This can be achieved, for example, by inflating the tube with an incompressible fluid using a screw: moving the screw in or out would increase or decrease the prescribed volume v.

The solution corresponding to a value of v in the ranges $v < v_M$ or $v > v_m$, i.e. the two rising branches of the pressure-volume curve, is found to be stable against small disturbances, while a solution corresponding to a value of v in the intermediate range $v_M < v < v_m$, i.e. the falling branch of the pressure-volume curve, is unstable. Thus if the given value of volume lies in the range $v_M < v < v_m$ there is no stable solution to the problem

Thus in this case we have *non-existence* of a stable equilibrium solution (for certain values of the prescribed volume).

What configuration does the tube take if the value of v, which we control and can therefore choose as we please, takes the value v_* shown in Figure 10.12? Since this value of v_* is associated with the falling branch of the pressure-volume curve we know there is no stable equilibrium state *of the form we have been considering*, i.e. where the tube remains in a homogeneously deformed cylindrical configuration. Thus necessarily the tube must take on a configuration which is not of this form.

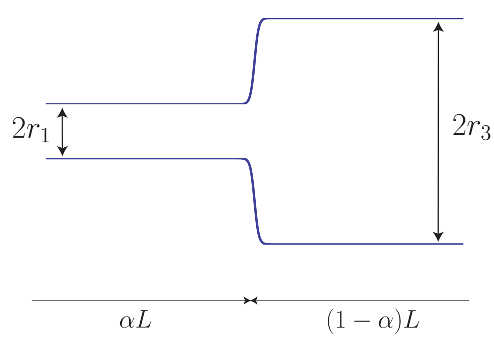

Figure 10.13: A configuration of the tube in which a length αL of the tube has a radius r_1 (where $v_1 = \pi r_1^2$ is associated with the first rising branch of the p, v-curve); and the remaining length $(1-\alpha)L$ of the tube has a radius r_3 (where $v_3 = \pi r_3^2$ is associated with the second rising branch of the p, v-curve);

Consider a length L of the tube. It turns out that when $v = v_*$, some fraction of this length of tube, say $\alpha L\,(0 < \alpha < 1)$, has a circular cylindrical shape with a relatively small radius corresponding to a value of v on the first rising branch of the pressure-volume curve; and the remaining length $(1-\alpha)L$ has a circular cylindrical shape with a relatively large radius corresponding to a value of v on the second rising branch of the pressure-volume curve. These two values of v average out to give the value v_*. Thus the equilibrium configuration of the tube involves lengths of two different radii (with a transition zone joining them); see Figure 10.13. As the value of v_* increases from v_M to v_m, the length that has the small radius gets monotonically shorter (and the length having the large radius gets longer), i.e. α decreases from 1 to 0. See Ericksen for the theory behind this and Kyriakides and Chang for experiments that exhibit this behavior.

10.5 Example (5): Nonlinear Wave Propagation.

An elastic bar occupies the interval $[0, L]$ in an unstressed reference configuration. At the initial instant the bar is unstretched and at rest. The left hand end of the bar is suddenly given a velocity V at the initial instant and this velocity is maintained thereafter. Using the one-dimensional purely mechanical theory, calculate the stretch field in the bar for $t > 0$. Consider only sufficiently small times prior to the reflection of waves from the right hand end $x = L$, and therefore take $L = \infty$ with no loss of generality. Consider both cases $V > 0$ (sudden "impact" on the bar) and $V < 0$ (sudden extension of the bar).

Formulation: In the one-dimensional theory, the particle located at x in the reference configuration is taken to $y = y(x, t)$ at time t during a motion. The stretch $\lambda(x, t)$ and particle velocity $v(x, t)$ are

$$\lambda = \frac{\partial y}{\partial x}, \qquad v = \frac{\partial y}{\partial t}, \tag{10.45}$$

from which it follows that λ and v must obey the compatibility equation

$$\frac{\partial \lambda}{\partial t} = \frac{\partial v}{\partial x}. \tag{10.46}$$

Since the mapping from $x \to y$ is to be one-to-one we have $\partial y/\partial x > 0$, i.e.

$$\lambda > 0. \tag{10.47}$$

Next, let $\sigma(x, t)$ denote the stress. The equation of motion is

$$\frac{\partial \sigma}{\partial x} = \rho \frac{\partial v}{\partial t} \tag{10.48}$$

where the positive constant ρ is the mass density in the reference configuration.

For an elastic material, the stress response function $\widehat{\sigma}$ gives the stress as a function of stretch:

$$\sigma = \widehat{\sigma}(\lambda), \qquad \widehat{\sigma}(1) = 0, \tag{10.49}$$

where $(10.49)_2$ indicates that we have taken the unstretched configuration to be stress free. Suppose that

$$\widehat{\sigma}'(\lambda) > 0 \quad \text{and} \quad \widehat{\sigma}''(\lambda) < 0 \qquad \text{for all } \lambda > 0, \tag{10.50}$$

so that $\widehat{\sigma}$ is monotonically *increasing* and *concave*.

It is convenient to introduce the function

$$c(\lambda) = \sqrt{\widehat{\sigma}'(\lambda)/\rho} \; > 0. \tag{10.51}$$

Observe, in view of $(10.50)_2$, that

$$c'(\lambda) < 0 \qquad \text{for all } \lambda > 0. \tag{10.52}$$

Since c' does not change sign, c has a unique inverse c^{-1} defined on the interval $(c(\infty), c(0))$.

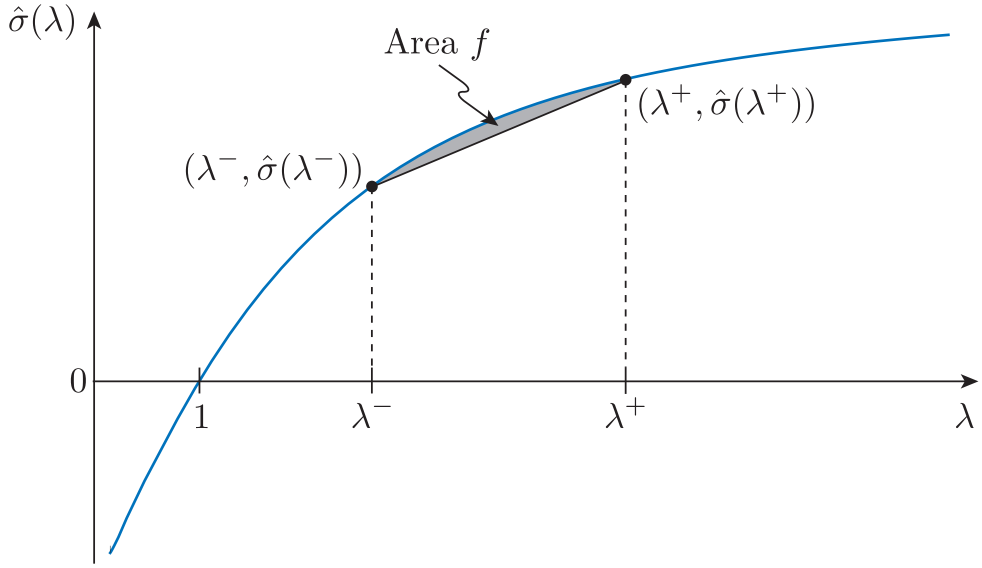

Figure 10.14: A graph of stress σ versus stretch λ for a concave stress response function $\widehat{\sigma}(\lambda)$. The chord joining $(\lambda^-, \widehat{\sigma}(\lambda^-))$ and $(\lambda^+, \widehat{\sigma}(\lambda^+))$ lies below the curve.

Substituting $\sigma = \widehat{\sigma}(\lambda)$ into (10.48) and using (10.51) allows the equation of motion to be written equivalently as

$$c^2(\lambda)\frac{\partial \lambda}{\partial x} = \frac{\partial v}{\partial t}. \tag{10.53}$$

Remark: If the pair of partial differential equations (10.46) and (10.53) are linearized about a configuration of constant stretch λ_0 we get the linear system

$$c^2(\lambda_0)\frac{\partial \lambda}{\partial x} = \frac{\partial v}{\partial t}, \qquad \frac{\partial \lambda}{\partial t} = \frac{\partial v}{\partial x}. \tag{10.54}$$

In terms of y, this pair of linearized equations reduce to the linear wave equation

$$c^2(\lambda_0)\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}. \tag{10.55}$$

This shows that $c(\lambda_0)$ is the propagation speed of infinitesimal waves superposed on a uniformly stretched state of the bar at a stretch λ_0, i.e. it is the speed of sound at the stretch λ_0.

Problem Statement: Find the stretch $\lambda(x,t)$ and velocity $v(x,t)$ that obey the field equations

$$c^2(\lambda)\frac{\partial \lambda}{\partial x} = \frac{\partial v}{\partial t}, \qquad \frac{\partial \lambda}{\partial t} = \frac{\partial v}{\partial x}, \qquad x > 0,\ t > 0, \tag{10.56}$$

except possibly at shocks (at which the appropriate jump conditions must hold), together with the initial conditions

$$\lambda(x,0) = 1, \qquad v(x,0) = 0, \qquad x > 0, \tag{10.57}$$

and the boundary condition

$$v(0,t) = V, \qquad t > 0. \tag{10.58}$$

The stress response function $\widehat{\sigma}$ is given and obeys the monotonicity and concavity conditions (10.49). The speed of sound $c(\lambda)$ is defined in terms of $\widehat{\sigma}$ by (10.51). The velocity V at the boundary is given.

Observe that the scaling $x \to kx$ and $t \to kt$ leaves this boundary-initial value problem invariant. It is natural therefore to seek a solution that also displays this same scale invariance, i.e. to look for a solution of the form

$$\lambda = \overline{\lambda}(\xi), \quad v = \overline{v}(\xi) \qquad \text{where} \quad \xi = x/t\ > 0. \tag{10.59}$$

We start by seeking a continuous solution of this form. Specifically, in view of the boundary and initial conditions at hand, we seek a continuous solution of the boundary-initial value problem that has the form

$$\lambda(x,t), v(x,t) = \begin{cases} \lambda^-, v^-, & 0 < x < \xi^- t, \\ \widehat{\lambda}(x/t),\ \widehat{v}(x/t), & \xi^- t < x < \xi^+ t, \\ \lambda^+, v^+, & \xi^+ t < x, \end{cases} \tag{10.60}$$

where the constants $\lambda^\pm, v^\pm, \xi^\pm$ and the functions $\widehat{\lambda}(x/t), \widehat{v}(x/t)$ are to be determined. Figure 10.15 shows a schematic plot of $\lambda(x,t)$ versus x at a fixed time t as described by this form of solution.

Note that the solution involves two wave fronts: a front running wave front at $x = \xi^+ t$ which is followed by a wave front at $x = \xi^- t$. The states ahead of the first wave front and behind

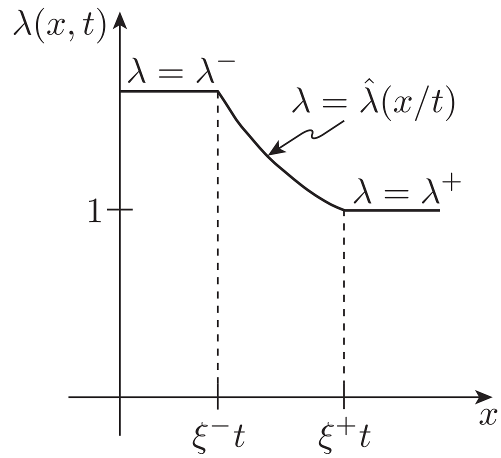

Figure 10.15: Graph of the stretch $\lambda(x,t)$ versus x at fixed t for a continuous solution of the form (10.60).

the second wave front are constant states. The solution varies smoothly between the two wavefronts from one constant state to the other. In view of (10.47) and the inequalities in (10.60), the unknown quantities $\lambda^\pm, v^\pm, \xi^\pm$ must obey

$$\lambda^\pm > 0, \qquad 0 < \xi^- < \xi^+. \tag{10.61}$$

The initial conditions (10.57) and boundary condition (10.58) tell us that

$$\lambda^+ = 1, \qquad v^+ = 0, \qquad v^- = V. \tag{10.62}$$

The continuity of the solution (10.60) requires

$$\widehat{\lambda}(\xi^+) = \lambda^+, \quad \widehat{v}(\xi^+) = v^+, \quad \widehat{\lambda}(\xi^-) = \lambda^-, \quad \widehat{v}(\xi^-) = v^-, \tag{10.63}$$

which upon using (10.62) simplifies to

$$\widehat{\lambda}(\xi^+) = 1, \quad \widehat{v}(\xi^+) = 0, \quad \widehat{\lambda}(\xi^-) = \lambda^-, \quad \widehat{v}(\xi^-) = V. \tag{10.64}$$

Substituting (10.60) into (10.46) and (10.53) leads to the pair of equations

$$c(\widehat{\lambda}(\xi)) = \xi, \qquad \widehat{v}'(\xi) = -\xi\widehat{\lambda}'(\xi), \qquad \xi^- < \xi < \xi^+, \tag{10.65}$$

where we have assumed that $\widehat{\lambda}'(\xi)$ does not vanish identically on the entire range $\xi^- < \xi < \xi^+$. We now solve (10.65) subject to (10.64) to determine $\widehat{\lambda}(\xi), \widehat{v}(\xi)$, λ^- and $\xi^\pm$.

As noted below (10.52), the function c is invertible and so $(10.65)_1$ leads to

$$\widehat{\lambda}(\xi) = c^{-1}(\xi), \qquad \xi^- < \xi < \xi^+. \tag{10.66}$$

Integrating $(10.65)_2$ from ξ^- to ξ and enforcing $(10.64)_4$ gives

$$\widehat{v}(\xi) = V - \int_{\xi^-}^{\xi} \zeta \widehat{\lambda}'(\zeta)\, d\zeta, \qquad \xi^- < \xi < \xi^+. \tag{10.67}$$

Changing variables in this integration allows us to write $\widehat{v}(\xi)$ as

$$\widehat{v}(\xi) = V - \int_{\lambda^-}^{\widehat{\lambda}(\xi)} c(\zeta)\, d\zeta, \qquad \xi^- < \xi < \xi^+. \tag{10.68}$$

Enforcing the remaining conditions $(10.64)_{1,2,3}$ leads to the following three equations which are to be solved for ξ^-, ξ^+ and λ^-:

$$c(1) = \xi^+, \qquad c(\lambda^-) = \xi^-, \qquad -V = \int_1^{\lambda^-} c(\lambda)\, d\lambda. \tag{10.69}$$

If equations $(10.69)_{1,2,3}$ can be solved for $\xi^\pm$ and λ^- in the ranges $\lambda^- > 0$, $0 < \xi^- < \xi^+$, then equation (10.60) with (10.66) and (10.68) is a solution to the boundary-initial value problem at hand.

The value of ξ^+ is given by equation $(10.69)_1$ and it is > 0 since $c > 0$. Likewise, if we know the value of λ^-, the value of ξ^- is given by equation $(10.69)_2$ and it too is > 0. Moreover, since c is a monotonically decreasing function these values will obey $\xi^- < \xi^+$ if (and only if) $\lambda^- > 1$, i.e. if we know the value of λ^- and it is > 1, then $(10.69)_{1,2}$ will provide values of $\xi^\pm$ in the proper ranges. Thus it remains to solve equation $(10.69)_3$ for a root $\lambda^- > 1$ corresponding to the given V. Since $c > 0$, the function $g(\lambda)$ defined by

$$g(\lambda) = \int_1^{\lambda} c(\zeta)\, d\zeta, \qquad \lambda > 0, \tag{10.70}$$

and associated with the right hand side of $(10.69)_3$, is monotonically increasing. Moreover $g(\lambda) < 0$ for $\lambda < 1$ and $g(\lambda) > 0$ for $\lambda > 1$. Therefore equation $(10.69)_3$ has a unique root $\lambda^- > 1$ provided $-V$ lies in the appropriate range of the function g, i.e. provided $-g(\infty) < V < 0$. Note that since $V < 0$ the loading on the left hand boundary is extensional.

In summary, for each value of the boundary velocity in the range $-g(\infty) < V < 0$, equation $(10.69)_3$ has a unique root $\lambda^- > 1$. Equations $(10.69)_{1,2}$ then give the values of $\xi^\pm$. These values are guaranteed to lie in the required ranges. With these choices for $\xi^\pm, \lambda^-$ (and $\lambda^+ = 1$), equations (10.60) , (10.66), (10.68) provide a solution to the boundary-initial value problem at hand.

Observe that if $V > 0$, i.e. for "impact loading", there is no solution of the form (10.60).

Case $V > 0$. Since the boundary-initial value problem has no continuous solution of the form (10.60) when $V > 0$, we now admit the possibility of solutions with discontinuities. Accordingly we seek a solution involving a single shock wave propagating into the bar at an (unknown) speed $\dot{s}$. Such a solution has the form

$$\lambda(x,t), v(x,t) = \begin{cases} \lambda^-, v^-, & 0 < x < \dot{s}t, \\ \lambda^+, v^+, & x > \dot{s}t, \end{cases} \tag{10.71}$$

where $\dot{s}, \lambda^\pm, v^\pm$ are to be determined with $\dot{s} > 0, \lambda^\pm > 1$. Figure 10.16 shows a schematic plot of $\lambda(x,t)$ versus x at a fixed time t as described by this form of solution.

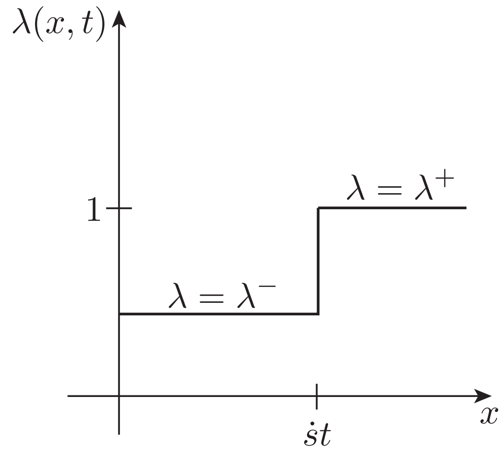

Figure 10.16: Graph of the stretch $\lambda(x,t)$ versus x at fixed t for a solution of the form (10.71) involving a shock.

The shock at $x = \dot{s}t$ separates the upstream stretch and particle velocity λ^+, v^+ from the downstream stretch and particle velocity λ^-, v^- . On enforcing the initial conditions (10.57) and boundary condition (10.58) we find

$$\lambda^+ = 1, \qquad v^+ = 0, \qquad v^- = V. \tag{10.72}$$

It remains to determine λ^- and $\dot{s}$, and for this we have the kinematic jump condition $(v^+ - v^-) + \dot{s}(\lambda^+ - \lambda^-) = 0$ and the linear momentum jump condition $(\sigma^+ - \sigma^-) + \rho\dot{s}(v^+ - v^-) = 0$. The former gives

$$V + \dot{s}(\lambda^- - 1) = 0. \tag{10.73}$$

As one would expect, this shows that the bar is compressed ($\lambda^- < 1$) behind the shock since $\dot{s} > 0, V > 0$. The momentum jump condition, combined with (10.73), gives

$$\rho V^2 = (\lambda^- - 1)\widehat{\sigma}(\lambda^-) \tag{10.74}$$

since the unstretched configuration has been assumed to be stress-free. It remains for us to solve (10.74) for a root λ^- in the range $0 < \lambda^- < 1$.

For $0 < \lambda < 1$ we have $\widehat{\sigma}(\lambda) < 0$ in view of $(10.49)_2$ and $(10.50)_1$. This, together with $\widehat{\sigma}'(\lambda) > 0$, shows that the function $\widehat{\sigma}(\lambda)(\lambda - 1)$ associated with the right hand side of (10.74) is a monotonically decreasing function on $(0, 1]$ that takes values in the interval $[0, -\widehat{\sigma}(0+))$. Thus for each value of the prescribed velocity in the range $0 < V < (-\widehat{\sigma}(0+)/\rho)^{1/2}$, equation (10.74) has a unique root λ^- in the range $0 < \lambda^- < 1$; (10.73) then gives the associated value of $\dot{s}$. Equation (10.71) then provides (tentatively) a solution to the boundary-initial value problem at hand with these values of $v^\pm, \lambda^\pm, \dot{s}$.

We know from Section 6.5 that there is an entropy inequality that must hold at a shock, which in the present purely mechanical theory takes the form of a dissipation inequality. Before declaring (10.71) to be a solution we must verify that this inequality holds.

By analogy with the thermomechanical entropy inequality, see equations (6.49) and (6.50) in Problem 6.11, the dissipation inequality in the purely mechanical theory is the requirement that

$$f\,\dot{s} \geq 0 \qquad \text{at the shock } x = \dot{s}t, \tag{10.75}$$

where the driving force f is defined as

$$f(\lambda^+, \lambda^-) = \int_{\lambda^-}^{\lambda^+} \widehat{\sigma}(\lambda)\,d\lambda - \frac{\widehat{\sigma}(\lambda^+) + \widehat{\sigma}(\lambda^-)}{2}\,(\lambda^+ - \lambda^-). \tag{10.76}$$

Observe the geometric interpretation of f as the area shown shaded in Figure 10.14. Since $\dot{s} > 0$, the dissipation inequality requires the driving force to be nonnegative:

$$\int_{\lambda^-}^{\lambda^+} \widehat{\sigma}(\lambda)\,d\lambda - \frac{\widehat{\sigma}(\lambda^+) + \widehat{\sigma}(\lambda^-)}{2}\,(\lambda^+ - \lambda^-) \quad \geq 0. \tag{10.77}$$

Geometrically, this requires that the area under the stress response function between λ^- and λ^+ must not be smaller than the area below the chord joining the two points $(\lambda^-, \widehat{\sigma}(\lambda^-))$ and $(\lambda^+, \widehat{\sigma}(\lambda^+))$. Since the stress response function is concave, we know that for any λ^- and λ^+ the chord joining $(\lambda^-, \widehat{\sigma}(\lambda^-))$ and $(\lambda^+, \widehat{\sigma}(\lambda^+))$ lies below the segment of the curve $\sigma = \widehat{\sigma}(\lambda)$ on that same domain. Therefore the area under the curve is not smaller than the area under the chord, i.e. $f \geq 0$. Thus (10.77) holds.

The dissipation inequality is therefore satisfied at the shock and so, finally, we conclude that equation (10.71) does indeed provide a solution to the boundary-initial value problem at hand with the previously mentioned values for $v^\pm, \lambda^\pm, \dot{s}$.

In **summary**, suppose that the stress response function is *concave* as assumed above. Then if the applied boundary velocity V is *extensional*, i.e. $V < 0$, (more precisely $-g(\infty) < V < 0$), the boundary-initial value problem has a *continuous* solution of the form (10.60). If V is *compressive*, i.e. $V > 0$, (more precisely $0 < V < (-\widehat{\sigma}(0+)/\rho)^{1/2}$), the problem has no continuous solution but has a *discontinuous* solution of the form (10.71) that satisfies the dissipation inequality (an "admissible" discontinuous solution).

Remark 1: One can verify that in the case $V < 0$, the boundary-initial value problem also has a discontinuous solution involving a shock but that it *violates the dissipation inequality.* Thus in this case the problem has a continuous solution and no admissible discontinuous solution.

Remark 2: Suppose that the stress response function is *convex.* By analyses similar those above, one can show that the situation is now reversed: if the applied boundary velocity V is *compressive*, i.e. $V > 0$, the problem has a *continuous* solution of the form (10.60) but no admissible discontinuous solution; if V is *extensional*, i.e. $V < 0$, the problem has an admissible *discontinuous* solution of the form (10.71) (that obeys the dissipation inequality) but no continuous solution.

10.6 Worked Examples and Exercises.

Problem 10.1. Consider a body that occupies a circular cylindrical region of space, length L and radius R, in a reference configuration. We consider a coordinate system in which the x_3-axis coincides with the axis of the shaft and its cross-section is parallel to the x_1, x_2-plane. A torsional deformation of the shaft about its axis can be described as follows: let the coordinates (x_1, x_2, x_3) of a particle in the reference configuration be written in cylindrical polar coordinates as $(r\cos\theta, r\sin\theta, z)$. When subjected to a torsional deformation, the coordinates (y_1, y_2, y_3) of this particle in the deformed configuration are given by

$$y_1 = r\cos(\theta+\phi), \qquad y_2 = r\sin(\theta+\phi), \qquad y_3 = z,$$

where $\phi = \gamma z$ is the angle through which the cross-section at $x_3 = z$ rotates about its axis. The constant γ represents the angle of rotation (twist) per unit shaft length. The lateral surface of the shaft is traction-free. Show that, locally, at each particle, this deformation is effectively a simple shear. If the shaft is composed of a neo-Hookean material, calculate the torque and axial force that must be applied on the shaft in order to maintain a torsional deformation.

Problem 10.2. Consider a thin rubber sheet that occupies a region $\ell \times \ell \times h$, $h << \ell$, in a reference configuration. The cube is composed of a Mooney-Rivlin material, i.e. an incompressible material characterized by the strain energy function

$$W(I_1, I_2) = c_1(I_1 - 3) \; + \; c_2(I_2 - 3)$$

where c_1, c_2 are material constants. Each of the four edges of the sheet is subjected to a tensile force F (which are in fact the resultants of uniformly distributed in-plane bi-axial normal tractions applied on the four edges.) Determine all possible homogeneous equilibrium configurations of the sheet and examine their stability. [Experiments of this nature have been carried out by Treloar as described in Ericksen.]

Problem 10.3. Derive the formula (10.33) for the cavitation stress for a general incompressible isotropic elastic material. Give an explicit example of a material (i.e. a strain energy function) which does not exhibit the phenomenon of cavitation (i.e. the stress required for cavitation is infinite). A material that does exhibit cavitation is, of course, the neo-Hookean material.

Problem 10.4. Calculate the Maxwell pressure referred to in the pressurized tube problem of Section 10.4. [You may wish to find the global minimum of the potential energy function

$$\Phi(v) = \overline{W}(v) - p_* v,$$

i.e. minimize Φ over all $0 < v < \infty$, p_* being the prescribed pressure.]

Problem 10.5. Consider the pressurized tube problem of Section 10.4. In the case of prescribed volume, calculate the pressure in the tube corresponding to a given value of v_* in the range $v_M < v_* < v_m$. Also, calculate the length αL of the tube that is associated with the first rising branch of the p, v-curve. [You may wish to minimize the potential energy function

$$\Phi(v_1, v_2, \alpha) = \alpha\, \overline{W}(v_1) + (1 - \alpha)\, \overline{W}(v_3)$$

over all $0 < v_1 < \infty, 0 < v_3 < \infty, 0 < \alpha < 1$, where v_1, v_3 and α are subject to the constraint

$$v_* = \alpha v_1 + (1 - \alpha) v_3,$$

v_* being the prescribed volume.]

Problem 10.6. In the respective cases of prescribed pressure and prescribed volume in the pressurized tube problem of Section 10.4, we referred to a "soft loading device" and a "hard loading device". When stating the form of the potential energy in the cube under tension in Section 10.2 we referred to portions $\partial\mathcal{R}_{\text{load}}$ and $\partial\mathcal{R}_{\text{def}}$ of the boundary of the body on which, respectively, the first Piola Kirchhoff traction and the deformation were prescribed. How do these concepts relate to each other?

Chapter 11

Linearized (Thermo)Elasticity

Given the important role that the linearized theory of elasticity plays in many practical applications, here we shall collect (from Sections 2.8, 4.10, and 8.9) many of the basic equations pertinent to this theory into one section. Some limited additional material is included, especially in the section of Worked Examples and Exercises at the end. Volume III in this series of lecture notes will explore the linear theory in detail.

11.1 Linearized Thermoelasticity

A body occupies the region $\mathcal{R}$ in a stress-free reference configuration[1]. A particle is identified by its position $\mathbf{x}$ and the displacement field is denoted by $\mathbf{u}(\mathbf{x}, t)$. When components are used they will always be with respect to a fixed orthonormal basis. The particle velocity is

$$\mathbf{v} = \dot{\mathbf{u}}, \qquad v_i = \dot{u}_i; \tag{11.1}$$

and the (infinitesimal) strain tensor is

$$\boldsymbol{\varepsilon} = \frac{1}{2}\big(\mathrm{Grad}\,\mathbf{u} + (\mathrm{Grad}\,\mathbf{u})^T\big), \qquad \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{11.2}$$

where a subscript comma followed by an index denotes partial differentiation with respect to the x coordinate associated with that index, e.g. $u_{i,j} = \partial u_i / \partial x_j$.

In the linearized theory, spatial derivatives (gradient, divergence etc.) will always be taken with respect to $\mathbf{x}$; the difference between it and derivatives with respect to $\mathbf{y}$ are of

[1] Therefore by assumption the residual stress vanishes.

the order of magnitude of the terms neglected in the linearized theory. Similarly all fields are defined on the reference regions and all calculations are carried out there. The distinction between the mass densities in the reference and current configurations can be neglected and we simply use the symbol ρ.

The stress tensor $\boldsymbol{\sigma}$ is symmetric

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T, \qquad \sigma_{ij} = \sigma_{ji}, \tag{11.3}$$

and obeys the equation of motion

$$\mathrm{Div}\,\boldsymbol{\sigma} + \rho\mathbf{b} = \rho\ddot{\mathbf{u}}, \qquad \sigma_{ij,j} + \rho b_i = \rho\ddot{u}_i \tag{11.4}$$

where ρ is the mass density and $\mathbf{b}$ the body force per unit mass. The traction $\mathbf{t}$ on a surface with unit normal $\mathbf{n}$ is related to the stress by

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}, \qquad t_i = \sigma_{ij}n_j. \tag{11.5}$$

At all times and all particles, the temperature field in the body $\theta(\mathbf{x}, t)$ is assumed to be close to some reference temperature θ_0: $|\theta - \theta_0| << 1$ The linearized energy equation reads

$$\mathrm{Div}\ \mathbf{q} + \rho r = \rho\theta_0\dot{\eta}. \tag{11.6}$$

The material is characterized by the strain energy function $W(= \rho\psi)$:

$$W = \frac{1}{2}\mathbb{C}_{ijk\ell}\varepsilon_{i,j}\varepsilon_{k,l} + \mathsf{M}_{ij}\varepsilon_{ij}(\theta - \theta_0) - \frac{\rho c}{2\theta_0}(\theta - \theta_0)^2; \tag{11.7}$$

it involves material constants $\mathbb{C}_{ijk\ell}$, M_{ij} and c. The associated stress-strain-temperature relation is

$$\boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \theta) = \mathbb{C}\ \boldsymbol{\varepsilon} +\ (\theta - \theta_0)\mathsf{M}, \qquad \sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}}(\boldsymbol{\varepsilon}, \theta) = \mathbb{C}_{ijpq}\varepsilon_{pq}\ +\ (\theta - \theta_0)\mathsf{M}_{ij}. \tag{11.8}$$

Thus the components of the fourth-order *elasticity tensor* $\mathbb{C}$ represent the elastic moduli of the material. It has the symmetries

$$\mathbb{C}_{ijk\ell} = \mathbb{C}_{k\ell ij}, \qquad \mathbb{C}_{ijk\ell} = \mathbb{C}_{jik\ell}, \qquad \mathbb{C}_{ijk\ell} = \mathbb{C}_{ij\ell k}, \tag{11.9}$$

and therefore the most general (anisotropic) elastic material has 21 elastic constants. The components of the second-order *stress-temperature tensor* M are material constants that couple the mechanical and thermal fields. It has the symmetries

$$\mathsf{M}_{ij} = \mathsf{M}_{ji} \tag{11.10}$$

and therefore M involves 6 material constants. Finally, the constitutive relation for the specific entropy in the linearized theory is (recalling that $W = \rho\psi$)

$$\eta = -\rho^{-1}\frac{\partial W}{\partial \theta} = -\rho^{-1}\mathsf{M}\cdot\boldsymbol{\varepsilon} + c\,(\theta/\theta_0 - 1)\,, \qquad \eta = -\rho^{-1}\frac{\partial W}{\partial \theta} = -\rho^{-1}\mathsf{M}_{ij}\varepsilon_{ij} + c\,(\theta/\theta_0 - 1)\,. \tag{11.11}$$

where the material constant c denotes the *specific heat at constant strain.*

Substituting the constitutive relation for the entropy (11.11) into the energy equation (11.6) allows us to write the energy equation in the form

$$\mathrm{Div}\,\mathbf{q} + \theta_0\,\mathsf{M}\cdot\dot{\boldsymbol{\varepsilon}} + \rho r = \rho c\dot{\theta}, \qquad \frac{\partial q_i}{\partial x_i} + \theta_0 \mathsf{M}_{ij}\dot{\varepsilon}_{ij} + \rho r = \rho c\,\dot{\theta}. \tag{11.12}$$

Next we turn to *the constitutive relation for the heat flux* $\mathbf{q}$ which in the linearized theory is the classical Fourier heat conduction relation

$$\mathbf{q} = \mathbf{K}\,\mathrm{Grad}\,\theta, \qquad q_i = K_{ij}\,\theta_{,j} \tag{11.13}$$

where $\mathbf{K}$ is the *heat conduction tensor*. The 9 components of $\mathbf{K}$ are material constants.

From Section 8.2, the *second law of thermodynamics* reduces in thermoelasticity to the requirement $\mathbf{q}(\mathbf{g})\cdot\mathbf{g} \geq 0$ for all vectors $\mathbf{g}$. This requires that $\mathbf{K}$ be positive semi-definite:

$$(\mathbf{K}\mathbf{g})\cdot\mathbf{g} = K_{ij}g_i g_j \geq 0 \qquad \text{for all vectors } \mathbf{g}. \tag{11.14}$$

If the material is inhomogeneous, then $\rho = \rho(\mathbf{x})$, $\mathbb{C} = \mathbb{C}(\mathbf{x})$, $\mathsf{M} = \mathsf{M}(\mathbf{x})$ and $\mathbf{K} = \mathbf{K}(\mathbf{x})$.

If the elasticity tensor is invertible, one can solve the stress-strain-temperature relation for the strain $\boldsymbol{\varepsilon}$ in terms of stress and temperature to get an equation of the form

$$\boldsymbol{\varepsilon} = \mathbb{K}\sigma + \mathsf{A}\,(\theta - \theta_0), \qquad \boldsymbol{\varepsilon}_{ij} = \mathbb{K}_{ijk\ell}\sigma_{k\ell} + \mathsf{A}_{ij}\,(\theta - \theta_0). \tag{11.15}$$

The 21 components of the *compliance tensor* $\mathbb{K} = \mathbb{C}^{-1}$ and the 6 components of the symmetric *thermal expansion tensor* $\mathsf{A} = \mathbb{C}^{-1}\mathsf{M}$ are material constants. Since $\mathbb{K}$ is the inverse of $\mathbb{C}$ we have

$$\mathbb{C}_{ijk\ell}\mathbb{K}_{k\ell mn} = \frac{1}{2}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}).$$

In **summary**, the linearized theory of thermoelasticity is characterized by the system of equations

$$\left.\begin{aligned}
\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i &= \rho \ddot{u}_i, \\
\frac{\partial q_i}{\partial x_i} + \theta_0 \mathsf{M}_{ij}\dot{\varepsilon}_{ij} + \rho r &= \rho c\, \dot{\theta}, \\
\sigma_{ij} &= \mathbb{C}_{ijk\ell}\, \varepsilon_{k\ell} \; + \mathsf{M}_{ij}(\theta - \theta_0), \\
q_i &= \mathrm{K}_{ij}\, \frac{\partial \theta}{\partial x_j}, \\
\varepsilon_{ij} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),
\end{aligned}\right\} \tag{11.16}$$

where the material is characterized by the elastic moduli $\mathbb{C}_{ijk\ell}$, the specific heat at constant strain c, the heat conductivity coefficients K_{ij} and the stress-temperature coefficients M_{ij} (which are related to the coefficients of thermal expansion). We have not included the constitutive relation for entropy in the list above; the specific entropy can be calculated a posteriori from (11.13).

The system of equations (11.16) can be reduced to a system of 4 (scalar) equations for the displacement field $\mathbf{u}(\mathbf{x}, t)$ and the temperature field $\theta(\mathbf{x}, t)$ by substituting $(11.16)_{3,4}$ into $(11.16)_{1,2}$ to eliminate the stress and heat flux, and then using $(11.16)_5$ in the resulting pair of equations to eliminate the strain. This leads to

$$\left.\begin{aligned}
\frac{\partial}{\partial x_j}\left(\mathbb{C}_{ijk\ell}\frac{\partial u_k}{\partial x_\ell}\right) + \mathsf{M}_{ij}\frac{\partial \theta}{\partial x_j} + \rho b_i &= \rho \ddot{u}_i, \\
\frac{\partial}{\partial x_i}\left(K_{ij}\frac{\partial \theta}{\partial x_j}\right) + \theta_0 \mathsf{M}_{ij}\frac{\partial \dot{u}_i}{\partial x_j} + \rho r &= \rho c\, \dot{\theta},
\end{aligned}\right\} \tag{11.17}$$

where we have also made use of the symmetries (11.9) and (11.10) of $\mathbb{C}$ and M; see for example Problem 11.3. Observe that the mechanical effects (characterized by the terms involving the displacement $\mathbf{u}$) are coupled to the thermal effects (characterized by the terms involving the temperature θ) by the stress-temperature tensor M. If $\mathsf{M} = \mathbf{0}$ the first equation becomes the wave equation (assuming $\mathbb{C}$ to be elliptic) while the second becomes the heat equation (assuming $\mathbf{K}$ to be positive definite).

If an orthogonal tensor $\mathbf{Q}$ is a symmetry transformation of the reference configuration then the material properties must obey the relations

$$\mathbb{C}_{ijk\ell} = \mathbb{C}_{pqrs} Q_{pi} Q_{qj} Q_{rk} Q_{s\ell}, \qquad \mathsf{M}_{ij} = Q_{ip} Q_{jq} \mathsf{M}_{pq}, \qquad K_{ij} = Q_{ip} Q_{jq} K_{pq}. \tag{11.18}$$

The material is *isotropic* if these equations hold for all orthogonal $\mathbf{Q}$ in which case

$$\left.\begin{aligned} \mathbb{C}_{ijkl} &= \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) + \lambda\delta_{ij}\delta_{k\ell} \\ \mathsf{M}_{ij} &= m\delta_{ij}, \\ K_{ij} &= k\delta_{ij}. \end{aligned}\right\} \tag{11.19}$$

Thus an isotropic material has two independent elastic moduli denoted by λ and μ and called the Lamé moduli, a single coefficient of thermal expansion α (related to m as shown below) and a single coefficient of heat conduction k. The stress-strain-temperature relation for an isotropic material takes the explicit form

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda(\operatorname{tr}\boldsymbol{\varepsilon})\mathbf{I} + m(\theta - \theta_0)\mathbf{I}, \qquad \sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda(\varepsilon_{kk})\delta_{ij} + m(\theta - \theta_0)\delta_{ij}. \tag{11.20}$$

If

$$\mu \neq 0, \quad 2\mu + 3\lambda \neq 0 \tag{11.21}$$

the isotropic elasticity tensor is invertible and the stress-strain-temperature relation can be inverted to give the strain in terms of stress and temperature by

$$\boldsymbol{\varepsilon} = \frac{1}{2\mu}\,\boldsymbol{\sigma} - \frac{\lambda}{2\mu(2\mu + 3\lambda)}(\operatorname{tr}\boldsymbol{\sigma})\mathbf{I} + \alpha(\theta - \theta_0)\mathbf{I}, \quad \varepsilon_{ij} = \frac{1}{2\mu}\,\sigma_{ij} - \frac{\lambda}{2\mu(2\mu + 3\lambda)}\sigma_{kk}\delta_{ij} + \alpha(\theta - \theta_0)\delta_{ij} \tag{11.22}$$

where

$$\alpha = -\,\frac{m}{3\lambda + 2\mu} \tag{11.23}$$

is the *coefficient of thermal expansion.*

A frequently used approximation of the exact theory above is the "partially coupled quasistatic theory". Here one neglects the inertial term in the equation of motion $(11.17)_1$ as well as the coupling between mechanical and thermal effects in the energy equation $(11.17)_2$; coupling in the equation of (quasi-static) motion is retained.

11.1.1 Worked Examples and Exercises

Problem 11.1. For an isotropic material, show that the coefficient of thermal expansion α is related to the Lamé moduli λ, μ and the stress-temperature coefficient m by

$$\alpha = -\,\frac{m}{3\lambda + 2\mu}.$$

Solution: Essentially, we need to invert the stress-strain-temperature relation

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij} + m(\theta - \theta_0)\delta_{ij}. \tag{11.24}$$

To do this we first we set $i = j = k$ in (11.24) to get

$$\sigma_{kk} = 2\mu\varepsilon_{kk} + 3\lambda\varepsilon_{kk} + 3m(\theta - \theta_0),$$

which yields

$$\varepsilon_{kk} = \frac{1}{2\mu + 3\lambda}\,\sigma_{kk} \; - \frac{3m}{2\mu + 3\lambda}\,(\theta - \theta_0)$$

provided $2\mu + 3\lambda \neq 0$. Solving (11.24) for ε_{ij} gives

$$\varepsilon_{ij} = \frac{1}{2\mu}\,\sigma_{ij} - \frac{\lambda}{2\mu}\varepsilon_{kk}\delta_{ij} - \frac{m}{2\mu}(\theta - \theta_0)\delta_{ij}$$

where we have further assumed that $\mu \neq 0$. We now substitute the previous expression for ε_{kk} into the preceding equation which eliminates the strain from its righthand side and leads to the strain-stress-temperature relation

$$\varepsilon_{ij} = \frac{1}{2\mu}\,\sigma_{ij} - \frac{\lambda}{2\mu(2\mu + 3\lambda)}\sigma_{kk}\delta_{ij} - \frac{m}{3\lambda + 2\mu}(\theta - \theta_0)\delta_{ij}.$$

By definition, the coefficient of $(\theta - \theta_0)\delta_{ij}$ is the coefficient of thermal expansion α and so we get the desired result

$$\alpha = -\,\frac{m}{3\lambda + 2\mu}.$$

Note that $\varepsilon_{ij} = \alpha(\theta - \theta_0)\delta_{ij}$ when the stress is zero.

Problem 11.2. The propagation of mechanical waves and the diffusion of heat occur on two different time scales that are related to the wave speed $\sqrt{E/\rho}$ and the diffusivity $k/(\rho c)$ respectively. Thus dynamical processes driven by inertia occur on a much faster time scale than thermal diffusion. Such circumstances are often modeled as being *adiabatic*: i.e. one neglects the heat flux and heat supply by taking $\mathbf{q} = \mathbf{0}, r = 0$. Derive the displacement equations of motion associated with an adiabatic process and identify the so-called *adiabatic elasticity tensor*.

Solution: Assume for simplicity that the material is homogeneous. Then we can write $(11.17)_1$ as

$$\mathbb{C}_{ijk\ell}u_{k,\ell j} \; + \mathsf{M}_{ij}\theta_{,j} \; + \rho b_i = \rho\ddot{u}_i.$$

After setting $K_{ij} = 0, r = 0$ in $(11.17)_2$, we have

$$\theta_0 \mathsf{M}_{k\ell} \dot{u}_{k,\ell} \ = \rho c \, \dot{\theta}.$$

Differentiating the first equation with respect to time and the second with respect to x_j yields

$$\mathbb{C}_{ijk\ell} \dot{u}_{k,\ell j} \ + \ \mathsf{M}_{ij} \dot{\theta}_{,j} = \rho \, \dddot{u}_i$$

and

$$\theta_0 \mathsf{M}_{k\ell} \dot{u}_{k,\ell j} \ = \rho c \, \dot{\theta}_{,j}.$$

We can eliminate the temperature θ between these two to get

$$\Big(\mathbb{C}_{ijk\ell} + \big(\theta_0/\rho c\big) \mathsf{M}_{ij} \mathsf{M}_{k\ell} \Big) \dot{u}_{k,\ell j} = \rho \, \dddot{u}_i.$$

Integrating this with respect to time leads to

$$\Big(\mathbb{C}_{ijk\ell} + \big(\theta_0/\rho c\big) \mathsf{M}_{ij} \mathsf{M}_{k\ell} \Big) u_{k,\ell j} + \rho b_i = \rho \, \ddot{u}_i \tag{11.25}$$

where we have taken the additive function of $\mathbf{x}$ that results from time integration to be the constant $\rho \mathbf{b}$. Equation (11.25) indicates that the 4-tensor

$$\mathbb{A}_{ijk\ell} = \mathbb{C}_{ijk\ell} + \big(\theta_0/\rho c\big) \mathsf{M}_{ij} \mathsf{M}_{k\ell}$$

plays the role of a modified elasticity tensor in the adiabatic theory in the sense that $(\mathbb{A}_{ijk\ell} u_{k,\ell})_{,j} + \rho b_i = \rho \ddot{u}_i$ has the form of an effective equation of motion. The 4-tensor $\mathbb{A}$ is known as the adiabatic elasticity tensor. For an isotropic material

$$\mathbb{A}_{ijk\ell} = \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) + \big(\lambda + \theta_0 m^2/\rho c\big) \delta_{ij} \delta_{k\ell}.$$

11.2 Linearized Elasticity: The Purely Mechanical Theory

Here one simply drops all the thermal terms above[2] . The strain-displacement equations and the equations of motion are

$$\left.\begin{aligned} \varepsilon_{ij} &= \frac{1}{2}\left(u_{i,j}+u_{j,i}\right), \\ \sigma_{ij,j}+\rho b_i &= \rho\ddot{u}_i, \end{aligned}\right\} \tag{11.26}$$

respectively. The traction $\mathbf{t}$ on a surface with unit normal $\mathbf{n}$ is given by

$$t_i = \sigma_{ij}n_j. \tag{11.27}$$

The material is characterized by the strain energy function

$$W(\boldsymbol{\varepsilon}) = \frac{1}{2}\left(\mathbb{C}\boldsymbol{\varepsilon}\right)\cdot\boldsymbol{\varepsilon} \;=\; \frac{1}{2}\mathbb{C}_{ijk\ell}\varepsilon_{ij}\varepsilon_{k\ell}. \tag{11.28}$$

In fact it is merely the elasticity tensor $\mathbb{C}$ that characterizes the material. The stress is related to the strain by

$$\sigma_{ij} = \frac{\partial W(\boldsymbol{\varepsilon})}{\partial\varepsilon_{ij}} \;=\; \mathbb{C}_{ijk\ell}\,\varepsilon_{k\ell}. \tag{11.29}$$

The elasticity tensor $\mathbb{C}$ has the symmetries

$$\mathbb{C}_{ijk\ell} = \mathbb{C}_{k\ell ij} = \mathbb{C}_{jik\ell} = \mathbb{C}_{ij\ell k}, \tag{11.30}$$

and a general anisotropic elastic material has 21 independent elastic moduli $\mathbb{C}_{ijk\ell}$. The elasticity tensor is positive definite if

$$\mathbb{C}_{ijk\ell}\varepsilon_{ij}\varepsilon_{k\ell} > 0 \quad \text{for all symmetric tensors } \boldsymbol{\varepsilon} \neq \mathbf{0}. \tag{11.31}$$

Problem 11.13 establishes one consequence of positive definiteness.

On substituting (11.29) and $(11.26)_1$ into $(11.26)_2$ we get the displacement equations of motion:

$$\left(\mathbb{C}_{ijk\ell}u_{k,\ell}\right)_{,j} + \rho b_i = \rho\ddot{u}_i \tag{11.32}$$

[2]One could view this as a theory on its own standing. Alternatively, if the process is *isothermal* so that the temperature is constant everywhere and at all times, $\theta(\mathbf{x},t)=\theta_0$, then the equations of thermoelasticity reduce to the equations presented here with the energy equation $(11.16)_2$ giving the heat supply r needed to maintain such an isothermal process.

where we have made use of the result established in Problem 11.3. This is a set of three partial differential equations involving the three components of displacement $u_i(\mathbf{x}, t)$. They must hold at all $\mathbf{x} \in \mathcal{R}$ and all times during the motion.

Consider again the strain-displacement relation $(11.26)_1$. Given the displacement field $\mathbf{u}(\mathbf{x})$, it is a set of formulae for calculating the corresponding strain field $\boldsymbol{\varepsilon}(\mathbf{x})$. However if one is given the strain field $\boldsymbol{\varepsilon}(\mathbf{x})$, the strain-displacement relations written in the form

$$u_{i,j} + u_{j,i} = 2\varepsilon_{ij} \tag{11.33}$$

is a set of six partial differential equations for calculating the three displacement components. Thus it is (in general) an overdetermined set of equations: i.e. given an arbitrary strain field it does not generally have a solution. If this set of equations are to be solvable, the strain field must satisfy the *compatibility equations*

$$\varepsilon_{ij,k\ell} + \varepsilon_{k\ell,ij} = \varepsilon_{ik,j\ell} + \varepsilon_{j\ell,ik}. \tag{11.34}$$

One can show that there are only six independent equations contained in (11.34).

For any subregion $\mathcal{D}$ of $\mathcal{R}$ one has the following balance between the rate of working and the rate of change of energy:

$$\int_{\partial\mathcal{D}} \mathbf{t}\cdot\mathbf{v}\; dA + \int_{\mathcal{D}} \rho\mathbf{b}\cdot\mathbf{v}\; dV = \frac{d}{dt}\int_{\mathcal{D}} \Big(W(\boldsymbol{\varepsilon}) + \frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v}\Big)\; dV. \tag{11.35}$$

If the material is isotropic, then

$$\mathbb{C}_{ijk\ell} = \lambda\delta_{ij}\delta_{k\ell} + \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) \tag{11.36}$$

and so the stress-strain relation specializes to

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda(\varepsilon_{kk})\delta_{ij}.$$

An isotropic material is characterized by two independent material constants that, here, have been taken to be the Lamé moduli λ and μ. Alternative commonly used elastic constants are the Young's modulus E, Poisson's ratio ν and bulk modulus κ which are related to λ and μ by

$$E = \frac{\mu(2\mu+3\lambda)}{\mu+\lambda}, \qquad \nu = \frac{\lambda}{2(\lambda+\mu)}, \qquad \kappa = \frac{3\lambda+2\mu}{3}.$$

The Lamé constant μ is in fact the shear modulus.

If

$$\mu \neq 0, \quad 2\mu + 3\lambda \neq 0 \tag{11.37}$$

the isotropic elasticity tensor is invertible and the strain in terms of stress is given by

$$\varepsilon_{ij} = \frac{1}{2\mu}\,\sigma_{ij} + \frac{\lambda}{2\mu(2\mu+\lambda)}\sigma_{kk}\delta_{ij}. \tag{11.38}$$

In terms of E and ν the strain-stress relation can be written as

$$\varepsilon_{ij} = \frac{1}{E}\,\Big[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\Big].$$

In order to solve the equations of elasticity for a specific circumstance, one must specify initial conditions

$$\mathbf{u}(\mathbf{x}, t_0) = \mathbf{u}_0(\mathbf{x}), \qquad \mathbf{v}(\mathbf{x}, t_0) = \mathbf{v}_0(\mathbf{x}), \tag{11.39}$$

at all points in $\mathcal{R}$ at the initial instant t_0. In addition one must specify boundary conditions at all points on $\partial\mathcal{R}$ for all times. For example the boundary $\partial\mathcal{R}$ may be composed of two distinct portions $\partial\mathcal{R}_{\text{disp}}$ and $\partial\mathcal{R}_{\text{trac}}$ on which the displacement and traction respectively are specified during the motion:

$$\mathbf{u}(\mathbf{x}, t) = \overline{\mathbf{u}}(\mathbf{x}, t), \quad \mathbf{x} \in \partial\mathcal{R}_{\text{disp}}, \qquad \boldsymbol{\sigma}(\mathbf{x}, t)\mathbf{n}(\mathbf{x}) = \overline{\mathbf{t}}(\mathbf{x}, t), \quad \mathbf{x} \in \partial\mathcal{R}_{\text{trac}}. \tag{11.40}$$

The functions $\mathbf{u}_0(\mathbf{x})$, $\mathbf{v}_0(\mathbf{x})$, $\overline{\mathbf{u}}(\mathbf{x}, t)$ and $\overline{\mathbf{t}}(\mathbf{x}, t)$ are given (on their respective domains of definition).

11.2.1 Worked Examples and Exercises

Problem 11.3. Show that $\mathbb{C}\,\boldsymbol{\varepsilon} = \mathbb{C}\boldsymbol{\nabla}\mathbf{u}$.

Solution: Let $\mathbf{A}$ be any 2- tensor. Then

$$\mathbb{C}_{ijk\ell}A_{k\ell} = \frac{1}{2}\mathbb{C}_{ijk\ell}A_{k\ell} + \frac{1}{2}\mathbb{C}_{ijk\ell}A_{k\ell} = \frac{1}{2}\mathbb{C}_{ijk\ell}A_{k\ell} + \frac{1}{2}\mathbb{C}_{ij\ell k}A_{\ell k} = \frac{1}{2}\mathbb{C}_{ijk\ell}A_{k\ell} + \frac{1}{2}\mathbb{C}_{ijk\ell}A_{\ell k}.$$

In the second step we have simply changed the dummy (repeated) indices, while in the fourth step we have used the symmetry $\mathbb{C}_{ijk\ell} = \mathbb{C}_{ij\ell k}$. Therefore

$$\mathbb{C}_{ijk\ell}A_{k\ell} = \mathbb{C}_{ijk\ell}\Big(\frac{1}{2}(A_{k\ell} + A_{\ell k})\Big).$$

The desired result follows for the special choice $A_{ij} = u_{i,j}$.

Problem 11.4. Show that the stress-strain relation for an isotropic material,

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda(\varepsilon_{kk})\delta_{ij},$$

can be inverted to give the strain in terms of stress by

$$\varepsilon_{ij} = \frac{1}{2\mu}\,\sigma_{ij} + \frac{\lambda}{2\mu(2\mu+\lambda)}\sigma_{kk}\delta_{ij} \tag{11.41}$$

provided that

$$\mu \neq 0, \quad 2\mu + 3\lambda \neq 0. \tag{11.42}$$

Problem 11.5. The elasticity tensor is said to be *positive definite* if

$$\mathbb{C}_{ijk\ell}S_{ij}S_{k\ell} > 0$$

for all symmetric tensors $\mathbf{S} \neq \mathbf{0}$. For an isotropic material show that positive definiteness is equivalent to the inequalities

$$\mu > 0, \qquad 2\mu + 3\lambda > 0. \tag{11.43}$$

Problem 11.6. For any subregion $\mathcal{D}$ of $\mathcal{R}$, and any smooth motion, establish the following balance between the rate of working and the rate of change of energy:

$$\int_{\partial\mathcal{D}} \mathbf{t}\cdot\mathbf{v}\; dA + \int_{\mathcal{D}} \rho\mathbf{b}\cdot\mathbf{v}\; dV = \frac{d}{dt}\int_{\mathcal{D}} \Big(W(\boldsymbol{\varepsilon}) + \frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v}\Big)\; dV. \tag{11.44}$$

Solution: Starting with the left hand side of (11.44) we get

$$\begin{aligned}
\int_{\partial\mathcal{D}} t_i\, v_i\; dA + \int_{\mathcal{D}} \rho b_i\, v_i\; dV &= \int_{\partial\mathcal{D}} \sigma_{ij}n_j\, v_i\; dA + \int_{\mathcal{D}} \rho b_i\, v_i\; dV \\
&= \int_{\mathcal{D}} \left((\sigma_{ij}\, v_i)_{,j} \;+\; \rho b_i\, v_i\right) dV \\
&= \int_{\mathcal{D}} \left(\sigma_{ij,j}\, v_i \;+\; \rho b_i\, v_i + \sigma_{ij}\, v_{i,j}\right) dV \\
&= \int_{\mathcal{D}} \left(\rho\dot{v}_i v_i \;+ \sigma_{ij}\,\dot{\varepsilon}_{ij}\right) dV \\
&= \int_{\mathcal{D}} \left(\rho\dot{v}_i v_i \;+ \frac{\partial W}{\partial\varepsilon_{ij}}\,\dot{\varepsilon}_{ij}\right) dV \\
&= \int_{\mathcal{D}} \Big(\frac{\partial}{\partial t}(\frac{1}{2}\rho v_i v_i) \;+ \frac{\partial}{\partial t}W(\boldsymbol{\varepsilon})\Big)\; dV.
\end{aligned}$$

In the first step we have used the relation $t_i = \sigma_{ij} n_j$ between the traction and stress; in the second step we have used the divergence theorem; in the fourth step we have used the equations of motion $(11.26)_2$, the strain displacement relation $(11.26)_1$ and the fact that $v_i = \dot{u}_i$; and in the fifth step we have made use of (11.29).

Problem 11.7. In the finite deformation theory we know that a rigid deformation is characterized by

$$\mathbf{y}(\mathbf{x}) = \mathbf{Q}\mathbf{x} + \mathbf{c} \tag{11.45}$$

where $\mathbf{Q}$ is a constant proper orthogonal tensor and $\mathbf{c}$ is a constant vector. In such a deformation, both Cauchy Green tensors $\mathbf{B}$ and $\mathbf{C}$ equal the identity and so any measure of finite strain vanishes.

We now consider the analogous issue within the linear theory. Suppose that the infinitesimal strain tensor $\boldsymbol{\varepsilon}(\mathbf{x})$ vanishes everywhere on $\mathcal{R}$. What is the most general form of displacement field for which this is true?

Solution: We first consider the second derivatives of the displacement field, i.e. $u_{i,jk}$, and show that they vanish on $\mathcal{R}$. To see this we carry out the following sequence of manipulations:

$$\begin{aligned} u_{i,jk} &= u_{i,kj} = (u_{i,k})_{,j} = -(u_{k,i})_{,j} = \\ &= -u_{k,ij} = -u_{k,ji} = -(u_{k,j})_{,i} = (u_{j,k})_{,i} = \\ &= u_{j,ki} = u_{j,ik} = (u_{j,i})_{,k} = -(u_{i,j})_{,k} = \\ &= -u_{i,jk}. \end{aligned}$$

In each line of calculations above, in the last step we have used the fact that $u_{a,b} = -u_{b,a}$ since the strain $\varepsilon_{ab} = 0$. In the other steps we have either changed the order of partial differentiation or done some rearranging of terms.

Thus we conclude that $u_{i,jk}(\mathbf{x}) = 0$ at all $\mathbf{x} \in \mathcal{R}$. Integrating this once gives

$$u_{i,j}(\mathbf{x}) = W_{ij}$$

where $\mathbf{W}$ is a constant tensor. Since $u_{i,j} = -u_{j,i}$ the tensor $\mathbf{W}$ is skew-symmetric. Integrating this again gives

$$\mathbf{u}(\mathbf{x}) = \mathbf{W}\mathbf{x} + \mathbf{c}$$

where $\mathbf{c}$ is a constant vector. This is the most general form of displacement field corresponding to a vanishing infinitesimal strain field.

We can write this in an alternative form by recalling the algebraic result that, corresponding to any skew symmetric tensor $\mathbf{W}$ there is vector $\mathbf{w}$ such that $\mathbf{W}\mathbf{x} = \mathbf{w} \times \mathbf{x}$ for all vectors $\mathbf{x}$. Thus we can write the preceding equation as

$$\mathbf{u}(\mathbf{x}) = \mathbf{w} \times \mathbf{x} + \mathbf{c} \tag{11.46}$$

where $\mathbf{w}$ and $\mathbf{c}$ are constant vectors. One sometimes finds (11.46) referred to as an " infinitesimal rigid displacement field" just as (11.45) is a (finite) rigid deformation.

Problem 11.8. If the strain-displacement equations $u_{i,j} + u_{j,i} = 2\varepsilon_{ij}$ hold, show that the components of strain must necessarily satisfy the *compatibility equations*

$$\varepsilon_{ij,k\ell} + \varepsilon_{k\ell,ij} = \varepsilon_{ik,j\ell} + \varepsilon_{j\ell,ik}. \tag{11.47}$$

Solution: Differentiating $\varepsilon_{ij} = (1/2)(u_{i,j} + u_{j,i})$ gives

$$2\varepsilon_{ij,k\ell} = u_{i,jk\ell} + u_{j,ik\ell}$$

and

$$2\varepsilon_{k\ell,ij} = u_{k,\ell ij} + u_{\ell,kij}.$$

Adding these two equations and changing the order of partial differentiation gives

$$\begin{aligned} 2(\varepsilon_{ij,k\ell} + \varepsilon_{k\ell,ij}) &= u_{i,kj\ell} + u_{j,\ell ik} + u_{k,ij\ell} + u_{\ell,jik} \\ &= u_{i,kj\ell} + u_{k,ij\ell} + u_{j,\ell ik} + u_{\ell,jik} \\ &= 2(\varepsilon_{ik,j\ell} + \varepsilon_{j\ell,ik}) \end{aligned}$$

which is the desired result.

Problem 11.9. The elasticity tensor is said to be *strongly elliptic* if

$$\mathbb{C}_{ijk\ell} a_i b_j a_k b_\ell > 0$$

for all unit vectors $\mathbf{a}, \mathbf{b}$. For an isotropic material show that strong ellipticity is equivalent to the inequalities

$$\mu > 0, \qquad 2\mu + \lambda > 0. \tag{11.48}$$

The next problem provides a physical interpretation of strong ellipticity.

Problem 11.10. Consider a homogeneous elasticity body with no body forces that undergoes a motion of the form

$$\mathbf{u}(\mathbf{x}, t) = \varphi(\mathbf{x} \cdot \mathbf{n} - ct)\, \mathbf{m} \tag{11.49}$$

where $\mathbf{m}$ and $\mathbf{n}$ are constant unit vectors and the scalar c is a constant. This describes a *wave that propagates* in the the direction $\mathbf{n}$ with propagation speed c, the particle motion being in the direction $\mathbf{m}$. Under what conditions can this motion be sustained in a homogeneous elastic solid with no body forces? Specialize your results to an isotropic material.

Solution: Let $\mathbf{A}(\mathbf{n})$ be the symmetric tensor whose components are

$$A_{ik}(\mathbf{n}) = \rho^{-1}\, \mathbb{C}_{ijk\ell} n_j n_\ell. \tag{11.50}$$

It is called the *acoustic tensor*. One can readily see by differentiation that for this motion,

$$\rho\ddot{\mathbf{u}} = \rho c^2 \varphi''\, \mathbf{m}, \qquad \mathrm{Div}\,(\mathbb{C}\boldsymbol{\varepsilon}) = \rho\varphi'' \mathbf{A}(\mathbf{n})\mathbf{m}. \tag{11.51}$$

Thus the equations of motion become

$$\mathbf{A}(\mathbf{n})\mathbf{m} = c^2\mathbf{m} \tag{11.52}$$

where we have assumed that $\varphi'' \neq 0$. Thus for a wave of the assumed form to propagate in the direction $\mathbf{n}$, the particle velocity direction $\mathbf{m}$ must be an eigenvector of the acoustic tensor $\mathbf{A}(\mathbf{n})$.

If the symmetric tensor $\mathbf{A}(\mathbf{n})$ is positive definite for each $\mathbf{n}$, then it has three positive eigenvalues and therefore three (real) speeds of propagation corresponding to each $\mathbf{n}$. The three associated mutually orthogonal eigenvectors are the directions of particle motion. The acoustic tensor $\mathbf{A}(\mathbf{a})$ is positive definite if $\mathbf{A}_{ij}(\mathbf{a})b_i b_j > 0$ for all unit vectors $\mathbf{b}$ or

$$\mathbb{C}_{ijk\ell} a_i b_j a_k b_\ell > 0 \qquad \text{for all unit vectors } \mathbf{a}, \mathbf{b}. \tag{11.53}$$

An elasticity tensor that satisfies this condition is said to be strongly elliptic.

For an isotropic material one finds from (11.36) and the definition of $\mathbf{A}$ that

$$\rho\,\mathbf{A}(\mathbf{n}) = \frac{\mu+\lambda}{\rho}\,\mathbf{n}\otimes\mathbf{n} \;+\; \frac{\mu}{\rho}\,\mathbf{I}. \tag{11.54}$$

The three eigenvalues of this tensor are c_L^2, c_S^2 and c_S^2 where

$$c_L^2 = \frac{2\mu+\lambda}{\rho}, \qquad c_S^2 = \frac{\mu}{\rho}. \tag{11.55}$$

For $\mathbf{A}(\mathbf{n})$ to be positive definite (or equivalently for $\mathbb{C}$ to be strongly elliptic) we need these eigenvalues to be positive, i.e. we need

$$2\mu+\lambda > 0, \qquad \mu > 0,$$

in which case there are two real wave speeds

$$c_L = \sqrt{\frac{2\mu+\lambda}{\rho}}, \qquad c_S = \sqrt{\frac{\mu}{\rho}}. \tag{11.56}$$

Observe that (because of the isotropy of the material) the wave speeds and strong ellipticity conditions are in fact independent of the propagation direction $\mathbf{n}$.

It can be readily verified from (11.54) that

$$\mathbf{A}(\mathbf{n})\mathbf{n} = \frac{\lambda+2\mu}{\rho}\mathbf{n}, \qquad \mathbf{A}(\mathbf{n})\mathbf{n}_\perp = \frac{\mu}{\rho}\mathbf{n}_\perp.$$

where $\mathbf{n}_\perp$ is any vector perpendicular to $\mathbf{n}$. It follows that the eigenvectors corresponding to the eigenvalues c_L^2 and c_S^2 are $\mathbf{n}$ and $\mathbf{n}_\perp$ respectively. Thus the particle motion direction (which equals the eigenvector direction) corresponding to the wave speed c_L is $\mathbf{n}$ and therefore this wave propagates in the same direction as that of the particle motion. Thus this is a *longitudinal wave.* The particle motion direction associated with waves with speed c_S is perpendicular to $\mathbf{n}$ and therefore the wave propagates in a direction normal to the particle motion. Thus this is a *shear wave.*

Problem 11.11. Show that the Young's modulus E, Poisson ratio ν and bulk modulus κ are given in terms of the Lamé constants as follows:

$$E = \frac{\mu(2\mu+3\lambda)}{\mu+\lambda}, \qquad \nu = \frac{\lambda}{2(\lambda+\mu)}, \qquad \kappa = \frac{3\lambda+2\mu}{3}$$

Solution To examine the Young's modulus and Poisson ratio we must consider a state of uniaxial stress (say in the x_1 direction):

$$[\sigma] = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Substituting this into the right hand side of the strain-stress relation

$$\varepsilon_{ij} = \frac{1}{2\mu}\sigma_{ij} + \frac{\lambda}{2\mu(2\mu+\lambda)}\sigma_{kk}\delta_{ij}$$

gives the corresponding strain tensor components to be

$$[\varepsilon] = \begin{pmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{22} \end{pmatrix}, \qquad \text{where} \quad \varepsilon_{11} = \frac{\mu+\lambda}{\mu(2\mu+\lambda)}\sigma, \qquad \varepsilon_{22} = \frac{\lambda}{2\mu(2\mu+\lambda)}\sigma.$$

The Young's modulus is defined as the ratio between the stress and strain, $\sigma_{11}/\varepsilon_{11}$, in a state of uniaxial stress and therefore

$$E = \frac{\mu(2\mu+3\lambda)}{\mu+\lambda},$$

The Poisson's ratio is defined as the ratio between the transverse strain and longitudinal strain, $\varepsilon_{22}/\varepsilon_{11}$, in a state of uniaxial stress and therefore

$$\nu = \frac{\lambda}{2(\lambda+\mu)},$$

To examine the bulk modulus we consider a state of hydrostatic strain

$$[\varepsilon] = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}$$

Substituting this into the right hand side of the stress-strain relation

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda(\varepsilon_{kk})\delta_{ij}.$$

gives the associated stress:

$$[\sigma] = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix} \qquad \text{where} \quad p = -3(\lambda + 2\mu/3)\varepsilon.$$

The bulk modulus is defined as the ratio between the mean hydrostatic stress $p/3$ and the volumetric strain ε in a state of hydrostatic strain and so

$$\kappa = \frac{3\lambda + 2\mu}{3}$$

Problem 11.12. *Variational formulation of elasticity.* Consider an elastic body in equilibrium that occupies a region $\mathcal{R}$. The material is characterized by the elasticity tensor $\mathbb{C}$. The boundary $\partial\mathcal{R}$ is composed of two portions $\partial\mathcal{R}_{\text{disp}}$ and $\partial\mathcal{R}_{\text{trac}}$ on which displacement and traction boundary conditions are specified

$$u_i(\mathbf{x}) = \overline{u}_i(\mathbf{x}), \quad \mathbf{x} \in \partial\mathcal{R}_{\text{disp}}; \qquad \mathbb{C}_{ijk\ell}u_{k,\ell}(\mathbf{x})n_j(\mathbf{x}) = \overline{t}_i(\mathbf{x}), \quad \mathbf{x} \in \partial\mathcal{R}_{\text{trac}}. \tag{11.57}$$

Here $\partial\mathcal{R} = \partial\mathcal{R}_{\text{disp}} \cup \partial\mathcal{R}_{\text{trac}}$.

Any vector field that is defined and sufficiently smooth on $\mathcal{R}$ and obeys the displacement boundary condition $\mathbf{w} = \overline{\mathbf{u}}$ on $\partial\mathcal{R}_{\text{disp}}$ is said to be *geometrically admissible*. Consider the potential energy functional $\Phi\{\mathbf{w}\}$ *defined on the set of all geometrically admissible displacement fields* $\mathbf{w}(\mathbf{x})$ by

$$\Phi\{\mathbf{w}\} = \int_{\mathcal{R}} \frac{1}{2}\mathbb{C}_{ijk\ell}w_{i,j}w_{k,\ell}\, dV \ - \int_{\mathcal{R}} \rho b_i w_i\, dV \ - \int_{\partial\mathcal{R}_{\text{trac}}} \overline{t}_i w_i\, dA$$

Show that a vector field $\mathbf{w}(\mathbf{x})$ that extremizes the potential energy necessarily satisfies the conditions

$$(\mathbb{C}_{ijk\ell}w_{k,\ell})_{,j} + \rho b_i = 0 \quad \mathbf{x} \in \mathcal{R}, \qquad \mathbb{C}_{ijk\ell}w_{k,\ell}n_j - \overline{t}_i = 0, \quad \mathbf{x} \in \partial\mathcal{R}_{\text{trac}}$$

which are seen to be the equilibrium equations and traction boundary condition if we set $\sigma_{ij} = \mathbb{C}_{ijk\ell}w_{k,\ell}$.

Solution: The calculus of variations used here is reviewed in Chapter 7 of Volume I.

Suppose that $\mathbf{w}(\mathbf{x})$ extremizes the potential energy functional and let $\delta\mathbf{w}(\mathbf{x})$ be an admissible variation, i.e. any sufficiently smooth function defined on $\mathcal{R}$ with $\delta\mathbf{w}(\mathbf{x}) = 0$ on $\partial\mathcal{R}_{\text{disp}}$. To simplify writing let Φ_1 denote the first term on the righthand side of Φ. Its first variation is

$$\begin{aligned}
\delta\Phi_1 &= \int_{\mathcal{R}} \frac{1}{2}\mathbb{C}_{ijk\ell}\delta w_{i,j}w_{k,\ell}\ dV\ + \int_{\mathcal{R}} \frac{1}{2}\mathbb{C}_{ijk\ell}w_{i,j}\delta w_{k,\ell}\ dV \\
&= \int_{\mathcal{R}} \mathbb{C}_{ijk\ell}w_{k,\ell}\delta w_{i,j}\ dV \\
&= \int_{\mathcal{R}} (\mathbb{C}_{ijk\ell}w_{k,\ell}\delta w_i)_{,j}\ dV\ - \int_{\mathcal{R}} (\mathbb{C}_{ijk\ell}w_{k,\ell})_{,j}\delta w_i\ dV \\
&= \int_{\partial\mathcal{R}} (\mathbb{C}_{ijk\ell}w_{k,\ell})n_j\delta w_i\ dA\ - \int_{\mathcal{R}} (\mathbb{C}_{ijk\ell}w_{k,\ell})_{,j}\delta w_i\ dV \\
&= \int_{\partial\mathcal{R}_{\text{trac}}} (\mathbb{C}_{ijk\ell}w_{k,\ell})n_j\delta w_i\ dA\ - \int_{\mathcal{R}} (\mathbb{C}_{ijk\ell}w_{k,\ell})_{,j}\delta w_i\ dV
\end{aligned}$$

where is the second step we have used the symmetries of $\mathbb{C}_{ijk\ell}$ and the result established in Problem 11.3; in the fourth step we have used the divergence theorem; and in the final step we have used the facts that $\partial\mathcal{R} = \partial\mathcal{R}_{\text{disp}} \cup \partial\mathcal{R}_{\text{trac}}$ and $\delta w_i = 0$ on $\partial\mathcal{R}_{\text{disp}}$. Let Φ_2 denote the second and third terms on the righthand side of Φ. Its first variation is

$$\delta\Phi_2 = -\int_{\mathcal{R}} \rho b_i \delta w_i\ dV\ - \int_{\partial\mathcal{R}_{\text{trac}}} \bar{t}_i\delta w_i\ dA.$$

The first variation of Φ vanishes at an extremum; $\delta\Phi = \delta\Phi_1 + \delta\Phi_2 = 0$. Therefore we have

$$\int_{\partial\mathcal{R}_{\text{trac}}} \Big((\mathbb{C}_{ijk\ell}w_{k,\ell})n_j - \bar{t}_i\Big)\delta w_i\ dA\ - \int_{\mathcal{R}} \Big((\mathbb{C}_{ijk\ell}w_{k,\ell})_{,j} + \rho b_i\Big)\delta w_i\ dV = 0$$

which must hold for all variations $\delta\mathbf{w}(\mathbf{x})$. It follows (see Chapter 7 of Volume I) that

$$\mathbb{C}_{ijk\ell}w_{k,\ell}n_j - \bar{t}_i = 0, \qquad \mathbf{x} \in \partial\mathcal{R}_{\text{trac}},$$

$$(\mathbb{C}_{ijk\ell}w_{k,\ell})_{,j} + \rho b_i = 0 \qquad \mathbf{x} \in \mathcal{R}.$$

Problem 11.13. Consider again the framework of the preceding problem. If the elasticity tensor is positive definite, show that the solution of the boundary value problem *minimizes* the potential energy functional over the set of all admissible displacement fields. (In the preceding problem we showed only that the solution of the boundary value problem was an *extremum* of the potential energy. Note that we did not assume the elasticity tensor to be positive definite there.)

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Chapter 12

Compressible Fluids. Viscous Fluids.

In this chapter we shall develop and discuss some of the most common constitutive models used to describe many fluids. We will consider (*i*) compressible fluids with no viscous effects, (*ii*) (not-necessarily Newtonian) viscous fluids with no compressibility effects, and (iii) fluids with neither compressibility nor viscous effects. The remaining case, a compressible fluid with viscous effects, is sketched in an exercise. Both mechanical and thermodynamic effects are considered in the first model, but for reasons of simplicity, thermodynamic effects are left out thereafter. Various exercises in this chapter illustrate certain phenomena, generalizations and/or additional results.

A roadmap of this chapter is as follows: in Section 12.1 we consider compressible inviscid (elastic) fluids. We begin from a set of constitutive relations in primitive form and then determine the most general set of such constitutive relationships that is material frame indifferent and conforms to the entropy inequality. Next we consider the class of adiabatic flows of such a fluid (which is relevant, for example, in gas dynamics). Several examples involving shock waves are given. In Section 12.2 we turn to incompressible viscous fluids. Again we begin by determining the most general set of constitutive relations within a certain class that is material frame indifferent and conforms to the dissipation inequality. Newtonian and non-Newtonian models are presented (the latter being limited to the generalized Newtonian (including power-law) model and the Reiner Rivlin model). In the third and final section, Section 12.3, we consider incompressible inviscid fluids.

In subsequent chapters we shall consider viscoelastic fluids and liquid crystals. These materials involve microstructural effects that have to be suitably accounted for in the continuum theory. Viscoelastic fluids typically consist of a base fluid containing particles (e.g. a slurry) or long molecules (e.g. oil paints). The molecules in a liquid crystal have orientational order though they have no positional order.

12.1 Compressible Inviscid Fluids (Elastic Fluids).

An elastic fluid is a special case of a general elastic material. For the class of fluids we consider here the material symmetry group of a reference configuration is the *largest possible symmetry group*, i.e. the set of all transformations whose determinant is ± 1, the unimodular group[1] . This implies, as was shown in Problem 8.5 (i), that the manner in which the constitutive response functions depend on the deformation gradient tensor $\mathbf{F}$ is solely through its determinant $\det \mathbf{F}$. Moreover, it was shown in Problem 8.5 (ii) that if the material symmetry group of one reference configuration is the unimodular group, then the material symmetry group of every reference configuration is also the unimodular group.

Next, since mass balance tells us that the mass densities in the reference and current configurations, ρ_0 and ρ respectively, are related by $\rho_0 = \rho \det \mathbf{F}$, it follows that dependence on $\det \mathbf{F}$ is equivalent to dependence on the mass density ρ.

Thus an elastic (or inviscid) fluid is characterized by the set of constitutive relations

$$\left.\begin{aligned} \mathbf{T} &= \widehat{\mathbf{T}}(\rho,\, \theta,\, \operatorname{grad}\theta), \\ \psi &= \widehat{\psi}(\rho,\, \theta,\, \operatorname{grad}\theta), \\ \eta &= \widehat{\eta}(\rho,\, \theta,\, \operatorname{grad}\theta), \\ \mathbf{q} &= \widehat{\mathbf{q}}(\rho,\, \theta,\, \operatorname{grad}\theta), \end{aligned}\right\} \tag{12.1}$$

for the Cauchy stress $\mathbf{T}$, the specific Helmholtz free energy ψ, the specific entropy η and the true heat flux vector $\mathbf{q}$; $\widehat{\mathbf{T}}$ is assumed to be symmetric tensor-valued so that angular momentum balance is then automatic.

Implications of the Entropy Inequality: Consider the entropy inequality written in the form

$$\rho\dot{\psi} \;-\; \mathbf{T}\cdot\mathbf{D} \;+\; \rho\eta\dot{\theta} \;-\; \mathbf{q}\cdot\frac{\operatorname{grad}\theta}{\theta} \;\le\; 0; \tag{12.2}$$

[1]Thus a fluid, by this definition, is necessarily isotropic. A liquid crystal on the other hand has a preferred microstructural orientation and at the same time flows. Materials such as this are sometimes referred to as anisotropic fluids.

see (5.27). Substituting (12.1) into (12.2) and rearranging terms leads to

$$\begin{aligned} &\Big\{-\rho^2\widehat{\psi}_\rho(\rho,\theta,\mathbf{g})\,\mathbf{I} \;-\; \widehat{\mathbf{T}}(\rho,\theta,\mathbf{g})\Big\}\cdot\mathbf{D} \\ &\quad+\rho\Big\{\widehat{\psi}_\theta(\rho,\theta,\mathbf{g})+\widehat{\eta}(\rho,\theta,\mathbf{g})\Big\}\,\dot{\theta} \\ &\quad+\rho\Big\{\widehat{\psi}_{\mathbf{g}}(\rho,\theta,\mathbf{g})\Big\}\cdot\dot{\mathbf{g}} \;-\; \widehat{\mathbf{q}}(\rho,\theta,\mathbf{g})\cdot\mathbf{g}/\theta \;\le\; 0, \end{aligned} \tag{12.3}$$

where the subscripts $\rho, \theta, \mathbf{g}$ denote partial differentiation with respect to these quantities and we have set

$$\mathbf{g} = \operatorname{grad}\theta.$$

In writing (12.3) we have made use of the fact that $\mathbf{I}\cdot\mathbf{D} = -\dot{\rho}/\rho$ which follows from mass balance $\dot{\rho} + \rho\,\operatorname{div}\mathbf{v} = 0$ and $\operatorname{div}\mathbf{v} = \operatorname{tr}\mathbf{D} = \mathbf{I}\cdot\mathbf{D}$.

By adapting to the current context arguments similar to those used previously in Sections 7.2 and 8.2, the inequality (12.3) must hold for all arbitrarily chosen symmetric tensors $\mathbf{D}$, real numbers $\dot{\theta}$ and vectors $\dot{\mathbf{g}}$ from which we conclude that

$$\left.\begin{aligned} -\rho^2\widehat{\psi}_\rho(\rho,\theta,\mathbf{g})\mathbf{I} - \widehat{\mathbf{T}}(\rho,\theta,\mathbf{g}) &= \mathbf{0}, \\ \widehat{\psi}_\theta(\rho,\theta,\mathbf{g}) + \widehat{\eta}(\rho,\theta,\mathbf{g}) &= 0, \\ \widehat{\psi}_{\mathbf{g}}(\rho,\theta,\mathbf{g}) &= \mathbf{o}, \\ -\widehat{\mathbf{q}}(\rho,\theta,\mathbf{g})\cdot\mathbf{g} &\le 0, \end{aligned}\right\} \tag{12.4}$$

where the inequality $(12.4)_4$ is what is leftover from (12.3) after one has concluded that $(12.4)_{1,2,3}$ hold.

The third of (12.4) states that the Helmholtz free-energy potential $\widehat{\psi}$ is independent of the temperature gradient $\mathbf{g}$. Therefore from the first and second of (12.4) it follows that the stress and entropy response functions $\widehat{\mathbf{T}}$ and $\widehat{\eta}$ are also independent of $\mathbf{g}$. Thus (12.4) can be further simplified to read

$$\left.\begin{aligned} \psi &= \widehat{\psi}(\rho,\theta), \\ \widehat{\mathbf{T}}(\rho,\theta) &= -\rho^2\widehat{\psi}_\rho(\rho,\theta)\mathbf{I}, \\ \widehat{\eta}(\rho,\theta) &= -\widehat{\psi}_\theta(\rho,\theta), \\ \widehat{\mathbf{q}}(\rho,\theta,\mathbf{g})\cdot\mathbf{g} &\ge 0. \end{aligned}\right\} \tag{12.5}$$

Conversely if (12.5) holds then so does (12.3). Thus (12.5) describes the most general elastic fluid which is consistent with the entropy inequality. Note that the Cauchy stress is necessarily hydrostatic.

Implications of Material Frame Indifference: We now explore the implications of material frame indifference on the constitutive response functions $\widehat{\psi}(\rho, \theta)$ and $\widehat{\mathbf{q}}(\rho,\, \theta,\, \mathrm{grad}\, \theta)$.

Consider two processes $\mathbf{y}, \theta$ and $\mathbf{y}^*, \theta^*$ that are related by $\mathbf{y}^*(\mathbf{x}, t) = \mathbf{Q}(t)\mathbf{y}(\mathbf{x}, t)$, $\theta^*(\mathbf{x}, t) = \theta(\mathbf{x}, t)$, where $\mathbf{Q}(t)$ is a rotation tensor at each instant. Since the mass density ρ and the free energy ψ are objective, see (5.31), and the temperature is the same in the two processes, material frame indifference does not impose any restrictions on $\widehat{\psi}$. Thus we only need consider $\widehat{\mathbf{q}}$.

Note from the chain rule that $\mathbf{g} = \mathrm{grad}\, \theta$ and $\mathbf{g}^* = \mathrm{grad}^*\, \theta$ are related by $\mathbf{g}^* = \mathbf{Q}\mathbf{g}$. A constitutive law must be independent of the observer and so the response function $\widehat{\mathbf{q}}$ is the same for both observers. Thus the constitutive relation, when applied to these two processes, tells us that

$$\mathbf{q} = \widehat{\mathbf{q}}(\rho,\ \theta,\ \mathbf{g}), \qquad \mathbf{q}^* = \widehat{\mathbf{q}}(\rho,\ \theta,\ \mathbf{g}^*), \tag{12.6}$$

where $\mathbf{q}$ and $\mathbf{q}^*$ denote the heat flux vectors at a particle p at a time t in these two processes.

Recall our previous discussion on objectivity where we discussed why we should require

$$\mathbf{q}^* = \mathbf{Q}\mathbf{q}; \tag{12.7}$$

see (5.36). On combining (12.6) with (12.7) we find that the constitutive response function $\widehat{\mathbf{q}}$ must be such that

$$\widehat{\mathbf{q}}(\mathbf{Q}\mathbf{g}) = \mathbf{Q}\widehat{\mathbf{q}}(\mathbf{g}) \tag{12.8}$$

for all proper orthogonal tensors $\mathbf{Q}$ and all vectors $\mathbf{g}$. (The mass density and temperature play no role in the present discussion and so we have suppressed them.)

Equation (12.8) tells us that $\widehat{\mathbf{q}}$ must be an isotropic vector-valued function and therefore we can write

$$\widehat{\mathbf{q}}(\mathbf{g}) = k(|\mathbf{g}|)\, \mathbf{g}, \tag{12.9}$$

where k is an arbitrary scalar-valued function; see Section 11 of Truesdell and Noll.

Therefore the most general set of constitutive response functions for an elastic fluid that is both frame-indifferent and consistent with the entropy inequality is as follows:

$$\left.\begin{aligned}\psi &= \psi(\rho,\theta),\\ \mathbf{T}(\rho,\theta) &= -p(\rho,\theta)\mathbf{I},\\ \eta(\rho,\theta) &= -\psi_\theta(\rho,\theta),\\ \mathbf{q}(\rho,\theta,\mathbf{g}) &= k(\rho,\theta,|\mathbf{g}|)\,\mathbf{g}\end{aligned}\right\} \tag{12.10}$$

where p denotes the pressure,

$$p(\rho,\theta) = \rho^2\psi_\rho(\rho,\theta), \tag{12.11}$$

$\mathbf{g} = \operatorname{grad}\theta$ and the entropy inequality requires that

$$k(\rho,\theta,|\mathbf{g}|) \geq 0. \tag{12.12}$$

Example: As deduced above, a compressible fluid is completely characterized (other than for its heat conduction properties) by the single constitutive response function $\psi(\rho,\theta)$. Given $\psi(\rho,\theta)$, one can calculate the various other fields such as the pressure, entropy, internal energy etc. using appropriate constitutive relations.

In the literature on compressible fluids it is often customary to characterize the fluid by specifying the *two* constitutive response functions $p(\rho,\theta)$ and $\varepsilon(\rho,\theta)$ instead. Consider for example the following commonly used characterization of a "perfect gas":

$$p(\rho,\theta) = R\rho\theta, \qquad \varepsilon(\rho,\theta) = c\theta \tag{12.13}$$

where R and c are material constants. From (12.11) and $(12.13)_1$ we get $\rho^2\psi_\rho(\rho,\theta) = R\rho\theta$ which can be integrated to yield

$$\psi(\rho,\theta) = R\theta\,\ln\,\rho + f(\theta). \tag{12.14}$$

Here $f(\theta)$ arises from integration with respect to ρ. From this and $(12.10)_3$ we get

$$-\eta(\rho,\theta) = \psi_\theta(\rho,\theta) = R\ln\rho + f'(\theta). \tag{12.15}$$

The internal energy $\varepsilon = \psi + \eta\theta$ can now be calculated from (12.14) and (12.15), and compared with $(12.13)_2$:

$$\varepsilon(\rho,\theta) = f(\theta) - \theta f'(\theta) = c\theta.$$

Solving this first order differential equation gives $f(\theta)$ and substituting the result back into (12.14) gives the Helmholtz free energy function

$$\psi(\rho,\theta) = R\theta\ln\rho - c\theta\ln(\theta/\theta_0) + c\theta \tag{12.16}$$

that characterizes the material completely. In particular, the specific entropy $\eta = -\psi_\theta(\rho,\theta)$ is

$$\eta = -R\ln\rho + c\ln(\theta/\theta_0). \tag{12.17}$$

We emphasize that specifying only one of the two constitutive response functions $p(\rho,\theta)$ or $\varepsilon(\rho,\theta)$ does *not* fully characterize the fluid. One must specify *both* as in the present example. In contrast, specifying the *single* constitutive response function $\psi(\rho,\theta)$ does completely characterize the fluid.

Thermomechanical processes of a compressible fluid are governed by the constitutive relations (12.10) - (12.12) together with the field equations. Since $\mathbf{T} = -p\mathbf{I}$ we have $\operatorname{div}\mathbf{T} = -\operatorname{grad} p$ and so the equations of motion specialize to

$$-\operatorname{grad} p + \rho\mathbf{b} = \rho\dot{\mathbf{v}}. \tag{12.18}$$

Mass balance requires

$$\dot{\rho} + \rho \operatorname{div}\mathbf{v} = 0, \tag{12.19}$$

and, just as for a general elastic material, the energy equation specializes to

$$\operatorname{div}\mathbf{q} + \rho r = \rho\theta\dot{\eta} \tag{12.20}$$

which is the Eulerian version of (8.17). The entropy inequality reduces to (12.12).

In summary, the fields $\mathbf{v}(\mathbf{y},t), p(\mathbf{y},t), \rho(\mathbf{y},t), \mathbf{q}(\mathbf{y},t), \eta(\mathbf{y},t), \psi(\mathbf{y},t)$ and $\theta(\mathbf{y},t)$ associated with the flow of an elastic fluid are governed by the system of equations (12.10) - (12.12), (12.18)-(12.20). In component form there are 11 scalar fields to be determined by the same number of equations.

12.1.1 Worked Examples and Exercises.

Problem 12.1. *Specific heat at constant mass density.* Since $\varepsilon = \psi + \eta\theta$ we can define a constitutive response function for the specific internal energy by

$$\widehat{\varepsilon}(\rho,\theta) \;=\; \widehat{\psi}(\rho,\theta) - \theta\,\widehat{\psi}_\theta(\rho,\theta). \tag{12.21}$$

Define the quantity $c(\rho,\theta)$ by

$$c(\rho,\theta) = \frac{\partial\widehat{\varepsilon}(\rho,\theta)}{\partial\theta} \tag{12.22}$$

which represents the change of specific internal energy with respect to change of temperature at constant density; we see below that this is a specific heat of the material. By using (12.21) we can express c alternatively as

$$c = -\theta\frac{\partial^2\widehat{\psi}(\rho,\theta)}{\partial\theta^2}. \tag{12.23}$$

Differentiating $\eta = -\psi_\theta$ with respect to time and using (12.23) allows us to write the energy equation (12.20) as

$$\operatorname{div}\mathbf{q} + \rho r = \rho c\dot{\theta} - \rho\theta\,\frac{\partial^2\widehat{\psi}(\rho,\theta)}{\partial\theta\partial\rho}\;\;\dot{\rho}.$$

If the mass density at each particle remains constant during some process, i.e. $\dot{\rho} = 0$, this reduces to the "classical energy equation"

$$\text{div}\,\mathbf{q} + \rho r = \rho c \dot{\theta}.$$

This shows that c is the *specific heat per unit mass at constant strain, i.e. at constant mass density or constant specific volume* $1/\rho$.

Problem 12.2. *Alternative constitutive description of an elastic fluid.* In what has preceded this, we characterized the material through the single constitutive function $\psi(\rho, \theta)$. One can verify that no other function of ρ and θ, e.g. $\varepsilon(\rho, \theta)$, by itself characterizes the material completely.

In certain circumstances (such as the setting considered in the next section), it is more convenient to work with functions of ρ and η rather than with functions of ρ and θ. One can verify that the single constitutive function $\overline{\psi}(\rho, \eta)$ by itself does *not* completely characterize the material. However the internal energy function $\overline{\varepsilon}(\rho, \eta)$ *does*, as we shall now show.

Essentially we want to swap θ for η and in order to do this we must invert the function $\eta = \eta(\rho, \theta)$ to get $\theta = \overline{\theta}(\rho, \eta)$. We can invert $\eta(\rho, \theta)$ uniquely provided that η is a monotonic function of θ (at fixed ρ). Since the specific heat at constant density $c(\rho, \theta) = \theta \partial \widehat{\eta} / \partial \theta$ it follows that if we assume

$$c(\rho, \theta) > 0$$

then the relation $\eta = \widehat{\eta}(\rho, \theta)$ is invertible at each fixed ρ and leads to the inverse relation $\theta = \overline{\theta}(\rho, \eta)$. We can now use this to swap θ for η.

In particular suppose we swap θ for η in $\varepsilon(\rho, \theta)$ to obtain another internal energy potential $\overline{\varepsilon}(\rho, \eta)$:

$$\overline{\varepsilon}(\rho, \eta) = \varepsilon\left(\rho, \overline{\theta}(\rho, \eta)\right) = \psi(\rho, \overline{\theta}(\rho, \eta)) + \eta \overline{\theta}(\rho, \eta). \tag{12.24}$$

We now show that the derivatives of $\overline{\varepsilon}$ with respect to ρ and η give certain other physical quantities. Differentiating (12.24) with respect to ρ at fixed η, and with respect to η at fixed ρ, give

$$\frac{\partial \overline{\varepsilon}}{\partial \rho}(\rho, \eta) = \frac{\partial \widehat{\psi}}{\partial \rho}(\rho, \theta), \qquad \frac{\partial \overline{\varepsilon}}{\partial \eta}(\rho, \eta) = \theta,$$

where $\theta = \overline{\theta}(\rho, \eta)$. Thus the pressure and temperature are given by the constitutive relations

$$p = \rho^2 \frac{\partial \overline{\varepsilon}}{\partial \rho}(\rho, \eta), \qquad \theta = \frac{\partial \overline{\varepsilon}}{\partial \eta}(\rho, \eta). \tag{12.25}$$

Other physical quantities such as the Helmholtz free-energy can now be calculated.

Thus is summary, an elastic fluid can be completely characterized by either of the potentials $\psi(\rho, \theta)$ and $\overline{\varepsilon}(\rho, \eta)$. While both are always valid, working with ρ and θ is particularly convenient if the process happens to be isothermal, while working with ρ and η is more convenient if the process is isentropic.

Finally it is worth reiterating that $\psi(\rho, \theta)$ characterizes the fluid completely but $\overline{\psi}(\rho, \eta)$ does not. Likewise $\overline{\varepsilon}(\rho, \eta)$ characterizes the fluid completely but $\varepsilon(\rho, \theta)$ does not. Thus ρ, η are the "natural variables" for the internal energy potential while ρ, θ are the natural variables for the Helmholtz free energy potential.

Problem 12.3. *Bernoulli's theorem.* See Problem 12.17.

12.1.2 Adiabatic Flows.

Suppose that the dynamical processes occur on a time scale that is much faster than the thermal processes. In this event one can neglect the heat flux and heat supply and set $\mathbf{q} = \mathbf{0}, r = 0$ in the energy equation. The energy equation (12.20) then reduces to $\dot{\eta} = 0$ which says that the flow is isentropic, i.e. the entropy of each particle remains constant during the flow. If all particles have the same entropy at the initial instant, then the entropy field remains constant in both space and time throughout the flow.

The field equations of momentum and mass balance are

$$-\mathrm{grad}\, p + \rho \mathbf{b} = \rho \dot{\mathbf{v}}, \tag{12.26}$$

and

$$\dot{\rho} + \rho \,\mathrm{div}\, \mathbf{v} = 0, \tag{12.27}$$

while the first law reduces (as noted above) to

$$\dot{\eta} = 0. \tag{12.28}$$

When studying adiabatic flows it is more convenient to have the constitutive equation for pressure in the form $p = \overline{p}(\rho, \eta)$ rather than in the form $p = p(\rho, \theta)$ because the entropy is constant at each particle during the flow. This can be obtained as described in Problem 12.2:

$$p = \overline{p}(\rho, \eta) = p(\rho, \overline{\theta}(\rho, \eta)) \tag{12.29}$$

where $\theta = \overline{\theta}(\rho, \eta)$ is the inverse of $\eta = -\psi_\theta(\rho, \theta)$.

In an adiabatic flow, the fields $\mathbf{v}(\mathbf{y}, t), p(\mathbf{y}, t)$ and $\rho(\mathbf{y}, t)$ are governed by the system of equations (12.26) - (12.29).

Example: Reconsider the perfect gas characterized by $p = R\rho\theta$ and $\varepsilon = c\theta$ that we examined in a previous example. Recall that there, we found in particular that

$$\eta = \eta(\rho, \theta) = -R \ln \rho + c \ln(\theta/\theta_0).$$

Solving this for the temperature gives

$$\theta/\theta_0 = \rho^{R/c}\, e^{\eta/c},$$

which when substituted into $p = R\rho\theta$ leads to

$$p = \overline{p}(\rho, \eta) = R\theta_0 \rho^{\gamma}\, e^{\eta/c}, \qquad \text{where } \gamma = 1 + R/c.$$

Thus in an isentropic flow of a perfect gas, $p \sim \rho^{\gamma}$.

Note that the internal energy of a perfect gas can be written in the useful form

$$\varepsilon = c\theta = \frac{c}{R}\frac{p}{\rho} = \frac{1}{\gamma - 1}\frac{p}{\rho}$$

where we have used $p = R\rho\theta$ and $\gamma = 1 + R/c$.

12.1.3 Worked Examples and Exercises.

Problem 12.4. *Acoustic waves in an adiabatic flow.* Consider an equilibrium state

$$\mathbf{v}(\mathbf{y}, t) = \mathbf{o}, \qquad p(\mathbf{y}, t) = p_0, \qquad \rho(\mathbf{y}, t) = \rho_0,$$

where p_0 and ρ_0 are constants with corresponding specific entropy η_0. Now consider a small disturbance of this state

$$\mathbf{v}(\mathbf{y}, t) = \tilde{\mathbf{v}}(\mathbf{y}, t), \qquad p(\mathbf{y}, t) = p_0 + \tilde{p}(\mathbf{y}, t), \qquad \rho(\mathbf{y}, t) = \rho_0 + \tilde{\rho}(\mathbf{y}, t) \tag{12.30}$$

where $\tilde{\mathbf{v}}, \tilde{p}$ and $\tilde{\rho}$ are small in some suitable sense. Substituting (12.30) into (12.26) and (12.27), and linearizing leads to

$$-\mathrm{grad}\, \tilde{p} = \rho_0 \frac{\partial \tilde{\mathbf{v}}}{\partial t}, \qquad \frac{\partial \tilde{\rho}}{\partial t} + \rho_0 \,\mathrm{div}\, \tilde{\mathbf{v}} = 0. \tag{12.31}$$

The constitutive relation $p = \overline{p}(\rho, \eta)$ when linearized leads to $p_0 = \overline{p}(\rho_0, \eta_0)$ and to next order, to

$$\tilde{p} = \overline{p}_{\rho}(\rho_0, \eta_0)\tilde{\rho}.$$

Substituting this into $(12.31)_1$ gives

$$-a^2\, \mathrm{grad}\, \tilde{\rho} = \rho_0 \frac{\partial \tilde{\mathbf{v}}}{\partial t} \tag{12.32}$$

where we have set

$$a = \sqrt{\frac{\partial \overline{p}}{\partial \rho}(\rho_0, \eta_0)}$$

provided the term within the square root is non-negative. Thus the pair of linear partial differential equations governing $\tilde{\rho}(\mathbf{y}, t)$ and $\tilde{\mathbf{v}}(\mathbf{y}, t)$ are

$$-a^2\, \mathrm{grad}\, \tilde{\rho} = \rho_0 \frac{\partial \tilde{\mathbf{v}}}{\partial t}, \qquad \frac{\partial \tilde{\rho}}{\partial t} + \rho_0 \,\mathrm{div}\, \tilde{\mathbf{v}} = 0.$$

Taking the gradient of the first equation and the time derivative of the second equation allows us to eliminate $\tilde{\mathbf{v}}$. This leads to the single equation

$$\frac{\partial^2 \tilde{\rho}}{\partial t^2} = a^2 \nabla^2 \tilde{\rho}$$

for the mass density $\tilde{\rho}(\mathbf{y}, t)$; ∇^2 is the Laplacian operator. This is a linear wave equation with wave speed a which represents the speed of acoustic waves (the speed of sound).

For the perfect gas discussed in a previous example, we had $p = \overline{p}(\rho, \eta) = R\theta_0 \rho^\gamma e^{\eta/c}$. Differentiating this particular function $\overline{p}$ gives the acoustic speed in a perfect gas to be

$$a = \sqrt{\gamma p/\rho}.$$

Problem 12.5. *Shock waves in an adiabatic flow.* Consider an adiabatic flow of some elastic fluid that involves a shock wave across which the pressure p, the mass density ρ, specific entropy η and the particle velocity $\mathbf{v}$ (as well as other related fields such as the temperature θ) suffer jump discontinuities. Referring back to how we concluded that the entropy of a particle remains constant, we see that we implicitly assumed that the various fields varied smoothly. Thus when there is a shock wave, the entropy η of a particle can change discontinuously as the particle crosses the shock wave, though its entropy will remain constant on either side of the shock wave.

Recall the jump conditions at a shock wave $\mathcal{S}_t$ established in (6.45). For an adiabatic flow we drop the terms $\mathbf{q}\cdot\mathbf{n}$ from the energy and entropy conditions there and find that, at each instant t, the following must hold on $\mathcal{S}_t$:

$$\left.\begin{aligned} [\![\rho(V - \mathbf{v}\cdot\mathbf{n})]\!] &= 0\ , \\ [\![\rho\mathbf{v}(V - \mathbf{v}\cdot\mathbf{n}) + \mathbf{T}\mathbf{n}]\!] &= 0\ , \\ [\![\mathbf{T}\mathbf{n}\cdot\mathbf{v} + \rho(\varepsilon + \mathbf{v}\cdot\mathbf{v}/2)(V - \mathbf{v}\cdot\mathbf{n})]\!] &= 0\ , \\ [\![\rho\eta(V - \mathbf{v}\cdot\mathbf{n})]\!] &\leq 0. \end{aligned}\right\} \tag{12.33}$$

As usual, for any field $g(\mathbf{y}, t)$, we write $[\![g]\!] = g^+ - g^-$ for the difference between its limiting values $g^\pm$ from the positive and negative side of the shock surface; the positive side is the side into which the unit normal $\mathbf{n}$ points and V is the propagation speed of the shock in the direction $\mathbf{n}$. These jump conditions are known as the Rankine-Hugoniot conditions.

The shock is called a *normal shock* if the particle velocities $\mathbf{v}^\pm$ are normal to the shock surface: $\mathbf{v}^\pm = v^\pm\,\mathbf{n}$. If we let $U^\pm = V - v^\pm$ denote the speed of the shock relative to the flow, and use the fact that $\mathbf{T} = -p\mathbf{I}$ for the class of fluids under consideration, one can rewrite the jump conditions (12.33) after some algebra as

$$[\![\rho U]\!] = 0, \qquad [\![p + \rho U^2]\!] = 0, \qquad [\![\rho U(\varepsilon + p/\rho + U^2/2)]\!] = 0, \qquad [\![\rho\eta U]\!] \leq 0. \tag{12.34}$$

The calculation leading from (12.33) to (12.34) were carried out in Problem 6.12.

If $\rho^{\pm}U^{\pm} \neq 0$, we can use $(12.34)_1$ to drop the ρU terms from $(12.34)_{3,4}$; in the latter case the sign of $\rho^{\pm}U^{\pm}$ is also important. This allows us to write (12.34) as

$$[\![\rho U]\!] = 0, \qquad [\![p + \rho U^2]\!] = 0, \qquad [\![\varepsilon + p/\rho + U^2/2]\!] = 0, \qquad [\![\eta]\!] \leq 0. \tag{12.35}$$

where we have assumed $\rho^{\pm}U^{\pm} > 0$.

Problem 12.6. *A Shock Tube.* [Chadwick] A piston can slide freely (at one end) in a semi-infinite cylinder of gas. Initially the piston and gas are at rest, and the gas is at a pressure p_0 and mass density ρ_0. The piston is instantaneously given a speed V_{piston} which is held constant thereafter. The direction of motion of the piston is into the gas. Assume that all motion is adiabatic and one dimensional in the direction of the pipe axis, that a shock wave forms and moves ahead of the piston into the gas, and that the states ahead of and behind the shock are each uniform (pressure, density etc.) Take the fluid to be a perfect gas so that in particular the specific internal energy is $\varepsilon = (\gamma - 1)^{-1}\, p/\rho$.

Show that the propagation speed of the shock wave V_{shock} is

$$V_{\text{shock}} = \frac{1}{4}(\gamma+1)V_{\text{piston}} + \left[\frac{1}{16}(\gamma+1)^2 V_{\text{piston}}^2 + a_0^2\right]^{1/2}$$

where a_0 is the speed of sound in the state ahead of the shock:

$$a_0 = \sqrt{\gamma p_0/\rho_0}.$$

What happens if the piston is moved in the opposite direction (away from the gas)?

Problem 12.7. *Conservation laws and shock waves.* Show that the pair of equations (12.26) and (12.27) that govern a general adiabatic flow can be written equivalently as

$$\frac{\partial}{\partial t}(\rho) + \frac{\partial}{\partial y_i}(\rho v_i) = 0, \qquad \frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial y_j}(p\delta_{ij} + \rho v_i v_j) = 0. \tag{12.36}$$

Observe that these equations are in conservation law form

$$\frac{\partial P}{\partial t} + \frac{\partial Q_i}{\partial y_i} = 0, \qquad \frac{\partial P_i}{\partial t} + \frac{\partial Q_{ij}}{\partial y_j} = 0.$$

Show that the generic form of the jump conditions at a shock associated with such conservation laws are

$$-[\![P]\!]\,U \;+\; [\![Q_i n_i]\!] \;= 0, \qquad -[\![P_i]\!]\,U \;+\; [\![Q_{ij} n_j]\!] \;= 0,$$

respectively.

12.2 Incompressible Viscous Fluids.

In the simplest (Newtonian) model of a viscous fluid in one dimension, the shear stress τ and the rate of shearing (strain rate) $\dot{\gamma}$ are related linearly by $\tau = \eta\dot{\gamma}$ where the material constant η is the coefficient of viscosity. Thus if a constant stress is applied on the fluid, the corresponding strain rate will be non zero and the strain will evolve with time implying that the fluid will flow. This is in contrast to a linearly elastic solid where one can apply a constant stress τ and the body can be in equilibrium at the strain level $\gamma = \tau/k$ where k is the elastic modulus.

If we generalize the elementary constitutive relation $\tau = \eta\dot{\gamma}$ to a nonlinear setting we might write $\tau = \widehat{\tau}(\dot{\gamma})$ where $\widehat{\tau}$ is the constitutive response function for shear stress. If further generalized to a three-dimensional setting we might write $\mathbf{T} = \widehat{\mathbf{T}}(\mathbf{L})$ where $\mathbf{T}$ is the Cauchy stress and $\mathbf{L} = \operatorname{grad}\mathbf{v}$ is the velocity gradient. Accordingly in this section we take the constitutive relation for stress $\mathbf{T}$ to depend on the *deformation rate* as measured by the velocity gradient tensor $\mathbf{L}$: $\mathbf{T} = \widehat{\mathbf{T}}(\mathbf{L})$. This is complementary to the class of constitutive relations for an elastic material where the stress depends on the deformation as measured by the deformation gradient tensor $\mathbf{F}$: $\mathbf{T} = \widehat{\mathbf{T}}(\mathbf{F})$.

We shall assume that the fluid is incompressible. Therefore it can only undergo motions that are isochoric (locally volume preserving):

$$\operatorname{div}\mathbf{v} = \operatorname{tr}\mathbf{L} = \operatorname{tr}\mathbf{D} = 0.$$

Recall that the velocity gradient tensor $\mathbf{L}$ and stretching tensor (rate of deformation tensor) $\mathbf{D}$ are defined by

$$\mathbf{L} = \operatorname{grad}\mathbf{v}, \qquad \mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T). \tag{12.37}$$

For simplicity, we shall ignore thermodynamic effects and consider the so-called purely mechanical theory of a continuum which only involves the following fields: particle velocity $\mathbf{v}(\mathbf{y}, t)$, Cauchy stress $\mathbf{T}(\mathbf{y}, t)$ and specific free energy $\psi(\mathbf{y}, t)$. These fields must obey the field equations/inequality

$$\left.\begin{array}{lcll} \operatorname{div}\mathbf{v} & = & 0, & \\ \operatorname{div}\mathbf{T} + \rho\mathbf{b} & = & \rho\dot{\mathbf{v}}, & \mathbf{T} = \mathbf{T}^T, \\ \mathbf{T}\cdot\mathbf{D} & \geq & \rho\dot{\psi}. & \end{array}\right\} \tag{12.38}$$

Since the material is taken to be incompressible the mass density $\rho(\mathbf{y}, t)$ remains constant at each particle.

In its primitive form, the constitutive relations for the class of viscous fluids we consider are taken to have the form

$$\left.\begin{aligned} \mathbf{T} &= -p\mathbf{I} + \widehat{\boldsymbol{\tau}}(\mathbf{L}), \\ \psi &= \widehat{\psi}(\mathbf{L}), \end{aligned}\right\} \tag{12.39}$$

where the function $\widehat{\boldsymbol{\tau}}$ is required to be symmetric tensor-valued. The term $-p\mathbf{I}$ in the stress arises in reaction to the incompressibility constraint.

We recall that no reference configuration is needed to develop the Eulerian formulation of the field equations, nor is a reference configuration involved in the notions of Cauchy stress $\mathbf{T}$ and velocity gradient $\mathbf{L}$. Thus there is no reference configuration underlying the basic system of equations (12.38) and (12.39). Consequently, provided the initial and boundary conditions are given appropriately, the Eulerian fields $p(\mathbf{v}, t), \mathbf{v}(\mathbf{y}, t)$ can be determined independently of a reference configuration. This is not uncommon even for fluids characterized by other constitutive relations. Of course *a reference configuration is needed if we wish to label particles*, and without one, we can only describe the variation of the various fields at points in space but not at fluid particles. Given a velocity field $\mathbf{v}(\mathbf{y}, t)$ one finds the motion $\mathbf{y}(\mathbf{x}, t)$ by solving the initial value problem

$$\left.\begin{aligned} \dot{\mathbf{y}}(\mathbf{x}, t) &= \mathbf{v}(\mathbf{y}(\mathbf{x}, t), t), \\ \mathbf{y}(\mathbf{x}, t_0) &= \mathbf{y}_0(\mathbf{x}), \end{aligned}\right\} \tag{12.40}$$

on the region $\mathcal{R}_0$ that the body occupies in the reference configuration (with respect to which $\mathbf{y}(\mathbf{x}, t)$ is defined).

Implications of the Dissipation Inequality We begin by determining the restrictions that the dissipation inequality $(12.38)_3$ places on the constitutive response functions. Substituting (12.39) into $(12.38)_3$ leads to

$$\widehat{\boldsymbol{\tau}}(\mathbf{L}) \cdot \mathbf{D} \;-\; \rho\, \widehat{\psi}_{\mathbf{L}}(\mathbf{L}) \cdot \dot{\mathbf{L}} \qquad \geq 0 \tag{12.41}$$

where we have used the fact that $-p\,\mathbf{I} \cdot \mathbf{D} = -p\,\mathrm{tr}\mathbf{D} = -p\,\mathrm{div}\,\mathbf{v} = 0$ in isochoric motions.

Equation (12.41) must hold in all isochoric processes. In particular it must hold for all $\dot{\mathbf{L}}$ and so we conclude in the usual way that

$$\left.\begin{aligned} \widehat{\psi}_{\mathbf{L}} &= 0, \\ \widehat{\boldsymbol{\tau}}(\mathbf{L}) \cdot \mathbf{D} &\geq 0, \end{aligned}\right\} \tag{12.42}$$

where $(12.42)_2$ is just the dissipation inequality (12.41) simplified in light of $(12.42)_1$. We conclude that $\widehat{\psi}(\mathbf{L})$ does not depend on $\mathbf{L}$ and so it is constant.

Remark: If we set $g(\mathbf{L}) = \widehat{\boldsymbol{\tau}}(\mathbf{L}) \cdot \mathbf{L}$, then $g(\mathbf{0}) = 0$ while the dissipation inequality requires $g(\mathbf{L}) \geq 0$. Thus $g(\mathbf{L})$ has a minimum at $\mathbf{L} = \mathbf{0}$ and consequently we must have $\partial g/\partial \mathbf{L} = \mathbf{0}$ at $\mathbf{L} = \mathbf{0}$. This shows that $\widehat{\boldsymbol{\tau}}(\mathbf{0}) = \mathbf{0}$. Thus the stress $\boldsymbol{\tau}$ must vanish in equilibrium.

Implications of Material Frame Indifference Recall the definitions of the stretch tensor $\mathbf{D} = (\mathbf{L} + \mathbf{L}^T)/2$ and the spin tensor $\mathbf{W} = (\mathbf{L} - \mathbf{L}^T)/2$, and observe that there is a one-to-one relationship between the tensor $\mathbf{L}$ and the pair of tensors $\{\mathbf{D}, \mathbf{W}\}$. Thus, instead of $\widehat{\boldsymbol{\tau}}(\mathbf{L})$ we may equivalently take

$$\boldsymbol{\tau} = \check{\boldsymbol{\tau}}(\mathbf{D}, \mathbf{W}). \tag{12.43}$$

Turning to material frame indifference, we consider as usual two motions that are related by a rigid rotation $\mathbf{Q}$. The tensors $\mathbf{D}, \mathbf{W}$ and their counterparts $\mathbf{D}^*, \mathbf{W}^*$ in the two motions are related by

$$\mathbf{D}^* = \mathbf{Q}\mathbf{D}\mathbf{Q}^T, \qquad \mathbf{W}^* = \mathbf{Q}\mathbf{W}\mathbf{Q}^T + \boldsymbol{\Omega}, \tag{12.44}$$

where $\boldsymbol{\Omega}$ is a skew symmetric tensor; see Section 3.8. As discussed in Section 5.5 the Cauchy stress $\mathbf{T}$ must be objective and therefore so must $\boldsymbol{\tau}$. Material frame indifference therefore requires that

$$\check{\boldsymbol{\tau}}(\mathbf{D}^*, \mathbf{W}^*) = \mathbf{Q}\, \check{\boldsymbol{\tau}}(\mathbf{D}, \mathbf{W})\, \mathbf{Q}^T \tag{12.45}$$

which we can write as

$$\check{\boldsymbol{\tau}}(\mathbf{Q}\mathbf{D}\mathbf{Q}^T,\ \mathbf{Q}\mathbf{W}\mathbf{Q}^T + \boldsymbol{\Omega}) = \mathbf{Q}\check{\boldsymbol{\tau}}(\mathbf{D}, \mathbf{W})\mathbf{Q}^T. \tag{12.46}$$

Equation (12.46) must hold for all rotations $\mathbf{Q}$ and all skew symmetric tensors $\boldsymbol{\Omega}$. Thus it must hold in particular for $\mathbf{Q} = \mathbf{I}$ and $\boldsymbol{\Omega} = -\mathbf{W}$. Then (12.46) reduces to

$$\check{\boldsymbol{\tau}}(\mathbf{D},\ \mathbf{0}) = \check{\boldsymbol{\tau}}(\mathbf{D}, \mathbf{W}). \tag{12.47}$$

Since this must hold for all skew symmetric $\mathbf{W}$ it follows that $\boldsymbol{\tau}$ much be independent of the spin tensor $\mathbf{W}$. Thus from hereon we may simply write $\boldsymbol{\tau}(\mathbf{D})$ and the material frame indifference restriction (12.46) becomes

$$\widehat{\boldsymbol{\tau}}(\mathbf{Q}\mathbf{D}\mathbf{Q}^T) = \mathbf{Q}\widehat{\boldsymbol{\tau}}(\mathbf{D})\mathbf{Q}^T. \tag{12.48}$$

Since this must hold for all rotations $\mathbf{Q}$ it follows that the function $\boldsymbol{\tau}(\cdot)$ is an isotropic function of $\mathbf{D}$; see Chapter 4 of Volume I. Therefore $\widehat{\boldsymbol{\tau}}$ has the representation

$$\widehat{\boldsymbol{\tau}}(\mathbf{D}) = \tau_o\mathbf{I} + \tau_1\mathbf{D} + \tau_2\mathbf{D}^2 \tag{12.49}$$

where the coefficients τ_k, $k = 0, 1, 2$ are functions of the three principal scalar invariants of $\mathbf{D}$, viz. $I_1(\mathbf{D}), I_2(\mathbf{D}), I_3(\mathbf{D})$:

$$\tau_k = \tau_k(I_2(\mathbf{D}), I_3(\mathbf{D})), \qquad k = 0, 1, 2, \tag{12.50}$$

where

$$I_1(\mathbf{D}) = \operatorname{tr}\mathbf{D} = 0, \qquad I_2(\mathbf{D}) = \frac{1}{2}\big[(\operatorname{tr}\mathbf{D})^2 - \operatorname{tr}\mathbf{D}^2\big], \qquad I_3(\mathbf{D}) = \det\,\mathbf{D}.$$

Note that $I_1(\mathbf{D}) = 0$ because of incompressibility. Moreover the term $\tau_0\mathbf{I}$ in (12.49) can be absorbed into the term $-p\mathbf{I}$ in the stress and so can be omitted. The $\tau_k' s$ are restricted by the dissipation inequality $\boldsymbol{\tau}\cdot\mathbf{D} \geq 0$, i.e.

$$\tau_1 \operatorname{tr}(\mathbf{D}^2) + \tau_2 \operatorname{tr}(\mathbf{D}^3) \geq 0. \tag{12.51}$$

Thus **in summary** the Cauchy stress is given by

$$\mathbf{T} = -p\mathbf{I} + \tau_1\mathbf{D} + \tau_2\mathbf{D}^2, \tag{12.52}$$

together with (12.50), (12.51). A fluid characterized by this constitutive relation is known as a *Reiner-Rivlin fluid.*

It maybe worth pointing out the similarity between (12.52) and the constitutive relation

$$\mathbf{T} = -p\mathbf{I} + \varphi_1\mathbf{B} + \varphi_2\mathbf{B}^2, \tag{12.53}$$

of an incompressible elastic solid; here $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ and the φ_k's are functions of the principal scalar invariants of $\mathbf{B}$. Perhaps this is not surprising since the mathematical analyses used in the two cases are similar. Though the effects of material frame indifference and isotropy are intimately coupled in each representation one might, very roughly, say that (12.53) is a result of isotropy while (12.52) is a result of material frame indifference.

Example: Consider a *steady simple shearing flow* in the $1,2$-plane with the particle velocity being in the 1-direction. The associated velocity field is

$$v_1(\mathbf{y}, t) = \dot{\gamma} y_2, \quad v_2(\mathbf{y}, t) = 0, \quad v_3(\mathbf{y}, t) = 0, \qquad \mathbf{y} \in \mathbb{R}^3,\ -\infty < t < \infty; \tag{12.54}$$

here the constant $\dot{\gamma}$ is the rate of shearing. In this motion, planes normal to 2-direction slide in the 1-direction.

The components of the associated velocity gradient tensor $\mathbf{L} = \operatorname{grad}\mathbf{v}$, the stretching tensor $\mathbf{D} = (\mathbf{L} + \mathbf{L}^T)/2$ and its square $\mathbf{D}^2$ can be readily calculated to be

$$[L] = \begin{pmatrix} 0 & \dot{\gamma} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad [D] = \begin{pmatrix} 0 & \dot{\gamma}/2 & 0 \\ \dot{\gamma}/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad [D^2] = \begin{pmatrix} \dot{\gamma}^2/4 & 0 & 0 \\ 0 & \dot{\gamma}^2/4 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and from this we find that the principal scalar invariants of $\mathbf{D}$ take the values

$$I_1(\mathbf{D}) = \mathrm{tr}\,\mathbf{D} = 0, \qquad I_2(\mathbf{D}) = \frac{1}{2}\big[(\mathrm{tr}\,\mathbf{D})^2 - \mathrm{tr}\,(\mathbf{D}^2)\big] = -\dot{\gamma}^2/4, \qquad I_3(\mathbf{D}) = \det\,\mathbf{D} = 0. \tag{12.55}$$

The constitutive relation

$$\mathbf{T} = -p\mathbf{I} + \tau_1\mathbf{D} + \tau_2\mathbf{D}^2, \tag{12.56}$$

now yields the Cauchy stress

$$[T] = \begin{pmatrix} -p + \tau_2\dot{\gamma}^2/4 & \tau_1\dot{\gamma}/2 & 0 \\ \tau_1\dot{\gamma}/2 & -p + \tau_2\dot{\gamma}^2/4 & 0 \\ 0 & 0 & -p \end{pmatrix},$$

where the constitutive functions $\tau_k(I_2(\mathbf{D}), I_3(\mathbf{D}))$, $k = 1, 2$, are evaluated at $(I_2(\mathbf{D}), I_3(\mathbf{D})) = (-\dot{\gamma}^2/4, 0)$.

Remark: As one would expect by symmetry, the shear stress components $T_{23} = T_{31} = 0$; moreover the shear stress T_{12} that drives the flow however is non-zero. We note however that the normal stress components T_{11}, T_{22}, T_{33} do *not* vanish in general even though this is a shear flow. The presence of non-zero normal stresses here is the analog of the Poynting effect that we encountered previously for a nonlinear elastic solid; see Section 10.1.

Note that if $\tau_2 = 0$, so that the $\mathbf{D}^2$ term is absent from the constitutive relation, then the normal stresses may vanish if the boundary conditions are such that $T_{33} = -p = 0$. Observe that $\tau_2 = 0$ does *not* mean the constitutive relation is linear since τ_1 may depend nonlinearly on the invariants of $\mathbf{D}$; see for example the power law fluid below.

Note the "universal relation" $T_{11} - T_{22} = 0$ which holds for all Reiner-Rivlin fluids. Compare this with the universal relation $T_{11} - T_{22} = kT_{12}$ in an elastic solid.

12.2.1 Example: A Newtonian Fluid.

As a specific example, consider the case where $\boldsymbol{\tau}$ is linear in $\mathbf{D}$. It follows from the representation (12.52) and the linearity in $\mathbf{D}$ that we must have $\tau_2 = 0$ and that τ_1 must be independent of $\mathbf{D}$. Thus

$$\widehat{\boldsymbol{\tau}}(\mathbf{D}) = 2\eta\,\mathbf{D} \tag{12.57}$$

where we have set $\tau_1 = 2\eta$ which is a material constant. So in a Newtonian fluid the Cauchy stress is given by the constitutive relation

$$\mathbf{T} = -p\,\mathbf{I} + 2\eta\,\mathbf{D}. \tag{12.58}$$

In a steady simple shearing flow we find that

$$T_{12} = \eta\dot{\gamma}$$

and so the material constant η represents the *shear viscosity.*

Returning to a general flow, the dissipation inequality requires that $\boldsymbol{\tau} \cdot \mathbf{D} \geq 0$, i.e.

$$2\eta \ \mathrm{tr}\,(\mathbf{D}^2) \geq 0 \tag{12.59}$$

and so the *shear viscosity* η must be nonnegative:

$$\eta \geq 0.$$

Substituting (12.58) into the equation of motion $\mathrm{div}\,\mathbf{T}+\rho\mathbf{b} = \rho\dot{\mathbf{v}}$ (where incompressibility implies that the mass density ρ is constant) leads to the classical *Navier-Stokes equation*

$$\dot{\mathbf{v}} = -\frac{1}{\rho}\mathrm{grad}\,p + \nu\nabla^2\mathbf{v} + \mathbf{b}, \qquad \nu = \frac{\eta}{\rho}, \qquad \mathrm{div}\,\mathbf{v} = 0. \tag{12.60}$$

12.2.2 Example: A Generalized Newtonian Fluid.

In the simplest one-dimensional setting of a Newtonian fluid we have the relation $\tau = \eta\dot{\gamma}$ between the shear stress and the strain rate. A natural generalization would be to allow the viscosity to depend on the strain rate and consider a constitutive relation $\tau = \eta(\dot{\gamma})\,\dot{\gamma}$. In the general theory of a Newtonian fluid we have $\widehat{\boldsymbol{\tau}}(\mathbf{D}) = 2\eta\,\mathbf{D}$ with η being a constant. This can be generalized according to (12.49) by allowing η to depend on the two invariants $I_2(\mathbf{D})$ and $I_3(\mathbf{D})$. However recall from the example above that in a steady simple shearing flow $I_2 = \dot{\gamma}^2/4, I_3 = 0$ so that the shear rate $\dot{\gamma}$ and I_2 can be used interchangeably in such a flow. This suggests one natural generalization of the Newtonian model to be

$$\mathbf{T} = -p\mathbf{I} + 2\eta(\dot{\gamma})\,\mathbf{D}, \qquad \dot{\gamma} = \sqrt{2\mathbf{D}\cdot\mathbf{D}} \geq 0.$$

The function $\eta(\dot{\gamma})$ represents a "generalized viscosity" of the fluid. The dissipation inequality requires that $\eta\,\mathbf{D}\cdot\mathbf{D} \geq 0$ whence $\eta \geq 0$. We call this model a *generalized Newtonian fluid.*

A special case of this is obtained by taking the generalized viscosity to be a *power law*:

$$\eta(\dot{\gamma}) = \eta_0\,\dot{\gamma}^{(n-1)}$$

which leads to the power law constitutive relation

$$\mathbf{T} = -p\mathbf{I} + 2\eta_0\,\dot{\gamma}^{(n-1)}\,\mathbf{D}, \qquad \dot{\gamma} = \sqrt{2\mathbf{D}\cdot\mathbf{D}} \geq 0.$$

Here η_0 and n are material constants. A power-law fluid with $n = 1$ is Newtonian; if $n < 1$ it is said to be *shear thinning* since the viscosity decreases as the rate of shearing increases; and if $n > 1$ it is said to be *shear thickening* since the viscosity increases as the rate of shearing increases. Examples of shear thinning fluids are slurries, greases, toothpaste and oil paints; shear thickening fluids are less common.

12.2.3 Worked Examples and Exercises.

Problem 12.8. Consider the class of *compressible viscous fluids* characterized by the constitutive response functions

$$\left.\begin{aligned} \psi &= \widehat{\psi}(\rho, \theta, \mathbf{L}), \\ \eta &= \widehat{\eta}(\rho, \theta, \mathbf{L}), \\ \mathbf{T} &= \widehat{\mathbf{T}}(\rho, \theta, \mathbf{L}), \\ \mathbf{q} &= \widehat{\mathbf{q}}(\rho, \theta, \mathbf{g}). \end{aligned}\right\} \tag{12.61}$$

Work within the thermomechanical theory and develop the most general set of constitutive relationships of this form that is consistent with the dissipation inequality and material frame indifference.

Solution: An outline of the steps one might go through are as follows:

- Use the entropy inequality to show that both $\widehat{\psi}$ and $\widehat{\eta}$ are independent of $\mathbf{L}$; that $\widehat{\eta} = -\widehat{\psi}_\theta$ and that the entropy inequality reduces to

$$(\rho^2 \widehat{\psi}_\rho(\rho, \theta)\, \mathbf{I} + \widehat{\mathbf{T}}) \cdot \mathbf{D} + \widehat{\mathbf{q}} \cdot \mathbf{g}/\theta \geq 0.$$

- Define

$$\widehat{\boldsymbol{\tau}}(\rho, \theta, \mathbf{L}) = \widehat{\mathbf{T}}(\rho, \theta, \mathbf{L}) + \rho^2 \widehat{\psi}_\rho(\rho, \theta)\, \mathbf{I}$$

so that the stress can be additively decomposed as

$$\widehat{\mathbf{T}}(\rho, \theta, \mathbf{L}) = -\rho^2 \widehat{\psi}_\rho(\rho, \theta)\, \mathbf{I} + \widehat{\boldsymbol{\tau}}(\rho, \theta, \mathbf{L})$$

and the entropy inequality can be written as

$$\widehat{\boldsymbol{\tau}}(\rho, \theta, \mathbf{L}) \cdot \mathbf{D} + \widehat{\mathbf{q}}(\rho, \theta, \mathbf{g}) \cdot \mathbf{g}/\theta \geq 0.$$

- Show from material frame indifference that $\widehat{\boldsymbol{\tau}}$ depends on $\mathbf{L}$ only through its symmetric part $\mathbf{D}$ so that one can write $\widehat{\boldsymbol{\tau}}(\rho, \theta, \mathbf{D})$.

- Show that the entropy inequality further reduces to the two separate inequalities

$$\widehat{\boldsymbol{\tau}}(\rho, \theta, \mathbf{D}) \cdot \mathbf{D} \geq 0, \qquad \widehat{\mathbf{q}}(\rho, \theta, \mathbf{g}) \cdot \mathbf{g} \geq 0$$

– Show from the entropy inequality that

$$\widehat{\boldsymbol{\tau}}(\rho, \theta, \mathbf{0}) = \mathbf{0}.$$

Therefore when the fluid is at rest, the preceding decomposition shows that $\widehat{\mathbf{T}}(\rho, \theta, \mathbf{L}) = -\rho^2 \widehat{\psi}_\rho(\rho, \theta)\, \mathbf{I}$. Thus the hydrostatic stress $-\rho^2 \widehat{\psi}_\rho(\rho, \theta)\, \mathbf{I}$ represents the equilibrium stress and the component $\boldsymbol{\tau}$ is due to the motion.

– Show that material frame indifference further requires that $\widehat{\boldsymbol{\tau}}(\mathbf{D})$ and $\widehat{\mathbf{q}}(\mathbf{g})$ obey

$$\widehat{\boldsymbol{\tau}}(\mathbf{Q}\mathbf{D}\mathbf{Q}^T) = \mathbf{Q}\widehat{\boldsymbol{\tau}}(\mathbf{D})\mathbf{Q}^T, \qquad \widehat{\mathbf{q}}(\mathbf{Q}\mathbf{g}) = \mathbf{Q}\widehat{\mathbf{q}}(\mathbf{g}).$$

(where ρ and θ have been suppressed). Thus $\widehat{\boldsymbol{\tau}}(\mathbf{D})$ and $\widehat{\mathbf{q}}(\mathbf{g})$ must be isotropic and infer from this that

$$\widehat{\boldsymbol{\tau}}(\mathbf{D}) = \tau_0 \mathbf{I} + \tau_1 \mathbf{D} + \tau_2 \mathbf{D}^2, \qquad \tau_p = \tau_p(\rho, \theta, I_1(\mathbf{D}), I_2(\mathbf{D}), I_3(\mathbf{D}))$$

where $I_p(\mathbf{D})$, $p = 1, 2, 3$ are the three principal invariants of $\mathbf{D}$; and that

$$\widehat{\mathbf{q}}(\rho, \theta, \mathbf{g}) = k(\rho, \theta, |\mathbf{g}|)\mathbf{g}.$$

Here $\tau_p, p = 0, 1, 2$ and k are scalar-valued constitutive functions.

– Show that the entropy inequality reduces to

$$\tau_0 \operatorname{tr} \mathbf{D} \;+\; \tau_1 \operatorname{tr} \mathbf{D}^2 \;+ \tau_2 \operatorname{tr} \mathbf{D}^3 \geq 0, \qquad k \geq 0.$$

– Thus in summary one concludes that the most general set of constitutive equations of the form considered that is consistent with the entropy inequality and material frame indifference is:

$$\left.\begin{aligned}
&\psi = \widehat{\psi}(\rho, \theta) \\
&\eta = -\widehat{\psi}_\theta(\rho, \theta) \\
&\widehat{\mathbf{T}}(\rho, \theta, \mathbf{L}) = -\rho^2 \widehat{\psi}_\rho(\rho, \theta)\, \mathbf{I} + \boldsymbol{\tau} = -\rho^2 \widehat{\psi}_\rho(\rho, \theta)\, \mathbf{I} + \tau_0 \mathbf{I} + \tau_1 \mathbf{D} + \tau_2 \mathbf{D}^2, \\
&\widehat{\mathbf{q}}(\rho, \theta, \mathbf{g}) = k(\rho, \theta, |\mathbf{g}|)\mathbf{g}.
\end{aligned}\right\}$$

where $\tau_p = \tau_p(\rho, \theta, I_1(\mathbf{D}), I_2(\mathbf{D}), I_3(\mathbf{D}))$ and $k = k(\rho, \theta, |\mathbf{g}|)$ are subject to the inequalities above.

– *Remark:* In the special case where the stress depends linearly on $\mathbf{D}$ (*a compressible Newtonian fluid*) show that

$$\widehat{\boldsymbol{\tau}}(\rho, \theta, \mathbf{D}) = 2\eta(\rho, \theta)\mathbf{D} + \lambda(\rho, \theta) \operatorname{tr} \mathbf{D}\, \mathbf{I}$$

where η and λ are material constants. Show that the entropy inequality gives

$$\eta \geq 0, \qquad \lambda + \frac{2}{3}\eta \geq 0.$$

Show that η is the shear viscosity, and $\lambda + 2\eta/3$ is the bulk viscosity.

Problem 12.9. *Poiseuille Flow of a Generalized Newtonian Fluid.* Consider a pipe of length L and circular cross-section of radius a. An incompressible generalized Newtonian fluid flows steadily through the pipe. Assume that there is no slip between the fluid and the pipe wall. Suppose that the drop in fluid pressure between the two ends of the pipe is $\Delta p\,(>0)$, the pressure at the inlet being higher than the pressure at the outlet. Calculate the volumetric flow rate of the fluid. When specialized to a power law fluid show that the volumetric flow rate is

$$\frac{\pi a^3}{3+1/n}\left(\frac{a\Delta p}{2L\eta_0}\right)^{1/n}.$$

Solution: Consider circular cylindrical coordinates (r,θ,z) with the z-axis coinciding with the axis of the pipe. Because of symmetry, we take the steady uniform flow to have the form

$$\mathbf{v}(\mathbf{y},t)=v(r)\,\mathbf{e}_z,$$

i.e. the particle velocity only has a component in the axial direction and it depends solely on the radial coordinate. The corresponding particle acceleration is readily found to vanish: $\dot{\mathbf{v}}(\mathbf{y},t)=\mathbf{o}$. In cylindrical coordinates

$$\operatorname{div}\mathbf{v}=\frac{\partial v_r}{\partial r}+\frac{1}{r}v_r+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial v_z}{\partial z}$$

which vanishes automatically for a flow of the assumed form, whence it is automatically isochoric.

Next, we find the velocity gradient tensor

$$\mathbf{L}=\operatorname{grad}\mathbf{v}=v'(r)\,\mathbf{e}_z\otimes\mathbf{e}_r$$

and its symmetric part

$$\mathbf{D}=\frac{1}{2}v'(r)(\mathbf{e}_z\otimes\mathbf{e}_r+\mathbf{e}_r\otimes\mathbf{e}_z).$$

One can readily verify that $\dot\gamma$ defined by $\dot\gamma=\sqrt{2\mathbf{D}\cdot\mathbf{D}}$ in this flow is

$$\dot\gamma=|v'(r)|.$$

The no slip boundary condition at the wall tells us that $v(a)=0$ while the pressure gradient will drive the flow in the $+z$-direction and so we expect $v(r)\geq 0$. Thus we shall tentatively assume that $v'(r)\leq 0$ for $0\leq r\leq a$ and thus take

$$\dot\gamma=-v'(r).$$

The constitutive relation is

$$\mathbf{T}=-p\mathbf{I}+2\eta(\dot\gamma)\mathbf{D}$$

and so

$$\mathbf{T}=-p\mathbf{I}-\eta(\dot\gamma)\dot\gamma(\mathbf{e}_z\otimes\mathbf{e}_r+\mathbf{e}_r\otimes\mathbf{e}_z).$$

Thus

$$T_{rr}=T_{\theta\theta}=T_{zz}=-p,\quad T_{rz}=-\dot\gamma\eta(\dot\gamma),\quad T_{r\theta}=T_{\theta z}=0.$$

At this point we allow the pressure to depend on all three coordinates: $p=p(r,\theta,z)$.

We now turn to the equations of motion in the absence of body force. The equations in the radial and azimuthal directions are

$$\frac{\partial T_{rr}}{\partial r} = 0, \qquad \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} = 0,$$

respectively. On using the constitutive relations in these two equations we get $\partial p/\partial r = 0$ and $\partial p/\partial \theta = 0$ and so the pressure p is at most a function of z alone: $p = p(z)$. The remaining equation of motion now reads

$$\frac{dT_{rz}}{dr} + \frac{dT_{zz}}{dz} + \frac{T_{rz}}{r} = 0$$

or

$$\frac{1}{r}\frac{d}{dr}(rT_{rz}) - p'(z) = 0.$$

Since the first term is a function of r only and the second a function of z only they must each be constant, say, c:

$$\frac{1}{r}\frac{d}{dr}(rT_{rz}) = c, \qquad p'(z) = c.$$

Therefore

$$p(z) = cz = -\frac{\Delta p}{L}z$$

where we have dropped the inessential additive constant, and

$$\frac{1}{r}\frac{d}{dr}(rT_{rz}) = -\frac{\Delta p}{L}$$

whence

$$T_{rz} = -\frac{\Delta p}{L}\frac{r}{2};$$

the constant arising in this integration must be zero since otherwise there would be a $1/r$ term in the stress.

Now using the constitutive relation $T_{rz} = -\dot\gamma\,\eta(\dot\gamma)$ gives is a first order nonlinear ordinary differential equation for $v(r)$:

$$-v'(r)\eta(-v'(r)) = \frac{\Delta p}{L}\frac{r}{2}.$$

Define the function

$$\tau(\dot\gamma) = \dot\gamma\,\eta(\dot\gamma) \qquad \text{for } \dot\gamma \geq 0,$$

and assume it to be invertible. [In a steady simple shearing flow (12.54) one can show that $T_{12} = \tau(\dot\gamma)$.] Then we get

$$-v'(r) = \tau^{-1}\Big(\frac{\Delta p}{L}\frac{r}{2}\Big)$$

Integrating this from some arbitrary radius r to the pipe wall $r = a$, and using the no slip boundary condition $v(a) = 0$ gives the axial component of velocity to be

$$v(r) = \int_r^a \tau^{-1}\Big(\frac{\Delta p}{L}\frac{\xi}{2}\Big)\,d\xi$$

where ξ is just a dummy variable of integration. Finally the volumetric flow rate through the pipe is given by

$$\int_0^a 2\pi r v(r)dr = \int_0^a 2\pi r\left[\int_r^a \tau^{-1}\Big(\frac{\Delta p}{L}\frac{\xi}{2}\Big)\,d\xi\right]\,dr.$$

To illustrate this consider a power law fluid $\eta(\dot\gamma) = \eta_0 \dot\gamma^{n-1}$. Then $\tau(\dot\gamma) = \eta_0\dot\gamma^n$ and so

$$\tau^{-1}(x) = (x/\eta_0)^{1/n}.$$

Therefore by evaluating the above integrals we find the axial velocity to be given by

$$v(r) = \left(\frac{\Delta p}{2L\eta_0}\right)^{1/n} \frac{a^{1+1/n} - r^{1+1/n}}{1+1/n}$$

and the volumetric flow rate to be

$$\frac{\pi a^3}{3+1/n}\left(\frac{a\Delta p}{2L\eta_0}\right)^{1/n}.$$

When $n = 1$ we recover the classical result for a Poiseuille flow of a Newtonian fluid where, in particular, the flow rate depends linearly on the pressure gradient. If the fluid is shear thinning ($n < 1$) the volume flux increases more rapidly with pressure gradient $\Delta p/L$ than for a Newtonian fluid.

Note that $v(r)$ decreases monotonically and so is consistent with the assumption $v'(r) \le 0$ made earlier.

Problem 12.10. *Rod climbing in a spinning flow.* Consider circular cylindrical coordinates with the z-axis vertically upwards. Consider a rigid circular cylindrical rod of radius a and infinite length. The rod rotates about its axis at a constant angular speed Ω. The axis of the rod is vertical and coincides with the z-axis. The rod is immersed in a bath of an incompressible viscous fluid which occupies the region $r > a, -\infty < z < h(r), 0 \le \theta < 2\pi$ where $z = h(r)$ describes the (unknown) free-surface of the fluid. We are asked to determine $h(r)$.

Solution: Due to the viscosity, the fluid will also undergo a rotational motion. We assume that this motion has the form

$$\mathbf{v} = v(r)\mathbf{e}_\theta. \qquad (i)$$

The acceleration of a particle is given, according to (1.29), by $\dot{\mathbf{v}} = \mathbf{v}' + (\operatorname{grad}\mathbf{v})\mathbf{v}$ which in the current setting yields

$$\dot{\mathbf{v}} = -\frac{v^2(r)}{r}\mathbf{e}_r. \qquad (ii)$$

One can verify that $\operatorname{div}\mathbf{v} = 0$ automatically for this flow and therefore it is isochoric.

Set

$$\dot\gamma = v'(r) - \frac{v(r)}{r} = r\frac{d}{dr}\left(\frac{v(r)}{r}\right). \qquad (iii)$$

The velocity gradient corresponding to the flow (i) can be readily calculated and from it we find that the stretching tensor is

$$\mathbf{D} = \frac{1}{2}\dot\gamma(\mathbf{e}_r\otimes\mathbf{e}_\theta + \mathbf{e}_\theta\otimes\mathbf{e}_r), \qquad (iv)$$

and its square is

$$\mathbf{D}^2 = \frac{1}{4}\dot\gamma^2(\mathbf{e}_r\otimes\mathbf{e}_r + \mathbf{e}_\theta\otimes\mathbf{e}_\theta). \qquad (v)$$

One can verify that $\dot\gamma = (2\mathbf{D}\cdot\mathbf{D})^{1/2}$.

Suppose that the constitutive relation of the fluid is

$$\mathbf{T} = -p\mathbf{I} + 2\alpha_1\mathbf{D} + \alpha_2\mathbf{D}^2 \tag{vi}$$

where α_1, α_2 are constants. Substituting (iv) and (v) into (vi) leads to

$$\mathbf{T} = T_{rr}\,\mathbf{e}_r\otimes\mathbf{e}_r + T_{\theta\theta}\,\mathbf{e}_\theta\otimes\mathbf{e}_\theta + T_{zz}\,\mathbf{e}_z\otimes\mathbf{e}_z + T_{r\theta}\,(\mathbf{e}_r\otimes\mathbf{e}_\theta + \mathbf{e}_\theta\otimes\mathbf{e}_r), \tag{vii}$$

where

$$T_{rr} = -p + \frac{\alpha_2}{4}\dot{\gamma}^2, \qquad T_{\theta\theta} = T_{rr}, \qquad T_{zz} = -p, \qquad T_{r\theta} = \alpha_1\dot{\gamma}. \tag{viii}$$

There is a gravitational body force in this problem and the body force per unit mass is $\mathbf{b} = -g\mathbf{e}_z$. The particle acceleration $\dot{\mathbf{v}}$ was given above. Therefore the equations of motion in the present setting reduce to

$$\left.\begin{aligned} \frac{\partial}{\partial r}T_{rr} &= -\rho\frac{v^2}{r}, \\ \frac{d}{dr}T_{r\theta} + \frac{2}{r}T_{r\theta} &= 0, \\ \frac{\partial}{\partial z}T_{zz} - \rho g &= 0. \end{aligned}\right\} \tag{ix}$$

Note from the constitutive relation for $T_{r\theta}$ that this stress component depends solely on r which is why we have used the ordinary derivative d/dr in $(ix)_2$. On the other hand the pressure (and therefore the normal stress components) may depend on both r and z. This is why we used partial derivatives in the other two equations of motion.

The equation of motion $(ix)_2$ can be written as

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2T_{r\theta}\right) = 0$$

which can be integrated to give

$$T_{r\theta} = \frac{c}{r^2}$$

where c is a constant. Combining this with the constitutive relation $T_{r\theta} = \alpha_1\dot{\gamma}$ and using the expression (iii) for $\dot{\gamma}$ gives

$$\alpha_1 r\frac{d}{dr}\left(\frac{v}{r}\right) = \frac{c}{r^2}.$$

Integrating this gives

$$v(r) = -\frac{c/\alpha_1}{2r} \tag{x}$$

where the constant of integration that arises from this second integration must vanish since otherwise $v \to \infty$ as $r \to \infty$.

At the surface of the rod, the fluid and rod move together and therefore $v(a) = a\Omega$. Using this in (x) yields the constant c to be

$$c = -2a^2\Omega\alpha_1.$$

This and (x) determine the velocity field completely:

$$v(r) = \frac{a^2\Omega}{r}. \tag{xi}$$

Equations (xi) and (iii) give

$$\dot{\gamma} = -\frac{2a^2\Omega}{r^2}. \qquad (xii)$$

Next consider the radial equation of motion $(ix)_3$ which on using (xi) reads

$$\frac{\partial}{\partial r} T_{rr} = -\rho\frac{v^2}{r} = -\rho\frac{a^4\Omega^2}{r^3}.$$

Integrating this yields

$$T_{rr} = \rho\frac{a^4\Omega^2}{2r^2} + \xi(z) \qquad (xiii)$$

where the function $\xi(z)$ arises since the integration is with respect to r. From $(xiii)_1$ and $(xiii)$ we have

$$p = \alpha_2\frac{a^4\Omega^2}{r^4} - \rho\frac{a^4\Omega^2}{2r^2} - \xi(z). \qquad (xiv)$$

Finally consider the axial equation of motion $(ix)_3$:

$$\frac{\partial}{\partial z} T_{zz} - \rho g = 0.$$

Integrating this gives

$$T_{zz} = \rho g z + \zeta(r)$$

and on using the constitutive relation $(xiii)_3$ this leads to

$$p = -\rho g z - \zeta(r). \qquad (xv)$$

The two equations (xiv) and (xv) for the pressure must be identical. Matching them determines $\xi(z)$ and $\zeta(r)$, and this in turn gives the pressure field $p(r,z)$:

$$p = \alpha_2\frac{a^4\Omega^2}{r^4} - \rho\frac{a^4\Omega^2}{2r^2} - \rho g z. \qquad (xvi)$$

The pressure at the free surface must equal the atmospheric pressure, which we take to be zero; thus we have $p = 0$ at $z = h(r)$. Using this in (xvi) gives the profile of the free surface to be

$$h(r) = -\frac{a^4\Omega^2}{2gr^2} + \frac{\alpha_2}{\rho g}\frac{a^4\Omega^2}{r^4}. \qquad (xvii)$$

Observe that the first term does not involve the material constants α_1 and α_2. It characterizes a purely inertial effect. As one might expect (since inertia tends to move the particles outwards), this term describes a depression of the surface. Next note that the material parameter α_1 does not enter either term in $(xvii)$ and so has no effect on the free surface profile. Thus in particular if $\alpha_2 = 0$, in which case the fluid is Newtonian, there is no constitutive effect on the free surface profile. The second term however is constitutive and is related to the normal stress effect in shearing flows discussed in the example below (12.52). If α_2 is positive and sufficiently large, the second term will dominate the first and the net effect will be that the free surface will rise, i.e. "the fluid will climb up the rod".

Problem 12.11. Viscous fluids are dissipative and it is of some interest to determine how fast the energy of a fluid that is initially in motion decays to zero (assuming there is no source of energy input into the fluid). Accordingly consider an incompressible Newtonian fluid that occupies a region $\mathcal{R}$ and has its boundary held fixed: $\mathbf{v} = \mathbf{o}$ on $\partial\mathcal{R}$ for $t \geq 0$. Though the boundary is fixed, the fluid in the interior is in motion. Show that the energy dissipation rate, which for an incompressible fluid equals the rate of decrease of kinetic energy,

$$= 2\eta \int_{\mathcal{R}} |\mathbf{D}|^2 \, dV = \eta \int_{\mathcal{R}} |\boldsymbol{\omega}|^2 \, dV.$$

Solution:

$$\begin{aligned}
\frac{d}{dt}\int_{\mathcal{R}} \frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v}\; dV &= \frac{d}{dt}\int_{\mathcal{R}} \frac{1}{2}\rho v_i v_i dV \\
&= \int_{\mathcal{R}} \rho v_i \dot{v}_i dV \\
&= \int_{\mathcal{R}} v_i T_{ij,j} dV \\
&= \int_{\mathcal{R}} v_i\big(-p\delta_{ij} + \eta(v_{i,j} + v_{j,i})\big)_{,j} dV \\
&= \int_{\mathcal{R}} \big(-p_{,i} v_i + \eta v_i (v_{i,jj} + v_{j,ij})\big) dV \\
&= \int_{\mathcal{R}} \big(-(pv_i)_{,i} + p v_{i,i} + \eta v_i v_{i,jj}\big) dV \\
&= \int_{\mathcal{R}} \big(-(pv_i)_{,i} + \eta(v_i v_{i,j})_{,j} - \eta v_{i,j} v_{i,j}\big) dV \\
&= \int_{\partial\mathcal{R}} \big(-p v_i n_i + \eta v_i v_{i,j} n_j\big) dA + \int_{\mathcal{R}} -2\eta D_{ij} D_{ij} dV \\
&= -2\eta \int_{\mathcal{R}} \mathbf{D}\cdot\mathbf{D} dV
\end{aligned}$$

In the sequence of calculations above at various points we have used the equations of motion $T_{ij,j} = \rho\dot{v}_i$, the constitutive equation $T_{ij} = -p\delta_{ij} + 2\eta D_{ij}$, the definition $D_{ij} = (v_{i,j} + v_{j,i})/2$, incompressibility $v_{i,i} = 0$, the divergence theorem, and the boundary condition $\mathbf{v} = \mathbf{o}$ on $\partial\mathcal{R}$.

To establish the alternative representation, note first that

$$\boldsymbol{\omega}\cdot\boldsymbol{\omega} = \omega_i\omega_i = e_{ijk} v_{j,k} e_{ipq} v_{p,q} = (\delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}) v_{j,k} v_{p,q} = v_{p,q} v_{p,q} - v_{q,p} v_{p,q}$$

where we have used the formula $e_{ijk}e_{ipq} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}$ relating the alternator to the Kronecker delta. Observe also that

$$\int_{\mathcal{R}} v_{q,p} v_{p,q} dV = \int_{\mathcal{R}} \Big((v_{q,p} v_p)_{,q} - v_{q,pq} v_p \Big) dV = \int_{\partial\mathcal{R}} v_{q,p} v_p n_q \, dA = 0$$

where we have used incompressibility $v_{i,i} = 0$, the divergence theorem, and the boundary condition $\mathbf{v} = \mathbf{o}$ on $\partial\mathcal{R}$. Therefore

$$\int_{\mathcal{R}} \boldsymbol{\omega}\cdot\boldsymbol{\omega}\, dV = \int_{\mathcal{R}} v_{p,q} v_{p,q}\, dV = \int_{\mathcal{R}} 2D_{pq} D_{pq}\, dV = \int_{\mathcal{R}} 2\mathbf{D}\cdot\mathbf{D}\, dV$$

and so the rate of change of kinetic energy is

$$\frac{d}{dt}\int_{\mathcal{R}} \frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v} \;= -\eta \int_{\mathcal{R}} \boldsymbol{\omega}\cdot\boldsymbol{\omega}\, dV$$

12.2.4 An Important Remark:

Our starting point $\mathbf{T} = \mathbf{T}(\mathbf{L})$ is a natural generalization of the elementary constitutive relation $\tau = \eta\dot{\gamma}$, and through a systematic analysis this led to the reduced form (12.52). Unfortunately (12.52) is a poor model for non-Newtonian fluids. The special Newtonian case is of course very useful in describing many fluids. Thus the models used to describe most non-Newtonian fluids are based on a different starting point and lead to different forms of constitutive relations. We shall say more in the subsequent chapter on viscoelastic fluids.

12.3 Incompressible Inviscid Fluids.

Since viscous effects are absent in this case we simply set $\boldsymbol{\tau} = \mathbf{0}$ in the basic equations of Section 12.2. The stress is therefore purely a pressure, but in contrast to a compressible fluid as considered in Section 12.1, is not determined constitutively. The pressure is a reaction to the incompressibility constraint.

The system of equations governing the fields $\mathbf{v}(\mathbf{y}, t)$ and $p(\mathbf{y}, t)$ for a flow of an incompressible inviscid fluid is

$$\left.\begin{aligned} \operatorname{div}\mathbf{v} &= 0, \\ -\operatorname{grad} p + \rho\mathbf{b} &= \rho\dot{\mathbf{v}}, \end{aligned}\right\} \tag{12.62}$$

where we have substituted

$$\mathbf{T} = -p\,\mathbf{I} \tag{12.63}$$

into the equation of motion.

When the body force is conservative and its associated potential is β, then

$$\mathbf{b} = -\operatorname{grad}\beta$$

and the equation of motion (12.62) can be written as

$$\dot{\mathbf{v}} = -\operatorname{grad}\left(\frac{p}{\rho} + \beta\right). \tag{12.64}$$

The conservation of mass tells us that $\dot{\rho} = 0$ whence the density of each particle does not change with time. For simplicity we shall assume in most of the examples to follow that the density is spatially uniform as well so that the density everywhere and at all times is the same. We note that there are interesting and important problems where the density varies spatially and where in fact the density gradient is critical to some phenomenon. An example of this is a fluid where the density varies with depth (such as in the deep ocean) where the density gradient leads[2] to what are called "internal waves".

Recall that the vorticity is defined by

$$\boldsymbol{\omega} = \mathrm{curl}\,\mathbf{v}. \tag{12.65}$$

The vorticity is a measure of the angular velocity of the fluid and $\boldsymbol{\omega}/2$ is the axial vector corresponding to the spin tensor $\mathbf{W}$. The simple form of the pair of governing equations $(12.62)_1$ and (12.64) leads the vorticity to have certain important characteristics for an inviscid incompressible fluid; this is illustrated in Problems 12.13 - 12.16 below.

12.3.1 Worked Examples and Exercises.

Problem 12.12. *Bernoulli's theorem* If an incompressible inviscid fluid undergoes a steady irrotational motion in the presence of a conservative body force show that

$$\varphi \stackrel{\text{def}}{=} \frac{p}{\rho} + \frac{1}{2}|\mathbf{v}|^2 + \beta$$

is constant everywhere and for all time. This is known as *Bernoulli's theorem.*

Solution: We will first show that the material time derivative $\dot{\varphi} = 0$ indicating that φ is constant at each particle. We will then show that $\mathrm{grad}\,\varphi = \mathbf{o}$ indicating that φ is constant spatially as well.

First, taking the material time derivative of φ and keeping in mind that ρ is constant for an incompressible fluid gives

$$\dot{\varphi} \;=\; \frac{\dot{p}}{\rho} + \mathbf{v}\cdot\dot{\mathbf{v}} + \dot{\beta}.$$

Recall the relation (1.28) between the referential and spatial time derivatives. On using this on the pressure p and the potential β we get

$$\dot{p} = p' + (\mathrm{grad}\,p)\cdot\mathbf{v} = (\mathrm{grad}\,p)\cdot\mathbf{v} \qquad \text{and} \qquad \dot{\beta} = \beta' + (\mathrm{grad}\,\beta)\cdot\mathbf{v} = (\mathrm{grad}\,\beta)\cdot\mathbf{v}$$

[2]Perhaps it is worth noting that fluid compressibility allowed for the propagation of waves in a compressible fluid as observed previously. This is absent in an incompressible fluid. Waves in an incompressible fluid are associated with some other source such as a free-surface or density gradient.

where in the respective second steps we have used the fact that $p' = 0$ and $\beta' = 0$ in a steady flow. Thus we can write the preceding equation as

$$\dot{\varphi} = \left[\operatorname{grad}\left(\frac{p}{\rho} + \beta\right) + \dot{\mathbf{v}}\right] \cdot \mathbf{v}.$$

Thus we have $\dot{\varphi} = 0$ because of the equation of motion (12.64). Therefore φ is constant at each particle. Note that the irrotationality of the flow has not been used in establishing this result.

Second, we are to show that $\operatorname{grad}\varphi = \mathbf{o}$. Consider the equation of motion written in the form given in equation (i) of Problem 12.13, Since the flow is steady, $\mathbf{v}' = \mathbf{o}$. Since the flow is irrotational $\boldsymbol{\omega} = \mathbf{o}$. Thus the left hand side of (i) vanishes from which the desired result follows immediately.

Problem 12.13. *Vorticity transport.* Consider an incompressible inviscid fluid with conservative body forces. Show that the vorticity obeys the equation

$$\dot{\boldsymbol{\omega}} = \mathbf{L}\boldsymbol{\omega} \tag{12.66}$$

where $\mathbf{L} = \operatorname{grad}\mathbf{v}$ is the velocity gradient tensor.

Solution: In view of the relation

$$\dot{\mathbf{v}} = \mathbf{v}' + \mathbf{L}\mathbf{v} \qquad \text{where } \mathbf{L} = \operatorname{grad}\mathbf{v}$$

between the material and spatial time derivatives of $\mathbf{v}$, we can write the equation of motion (12.64) as

$$\mathbf{v}' + \mathbf{L}\mathbf{v} = -\operatorname{grad}\left(\frac{p}{\rho} + \beta\right).$$

Next, one can readily establish the vector calculus identity

$$\mathbf{L}\mathbf{v} = (\operatorname{curl}\mathbf{v}) \times \mathbf{v} + \operatorname{grad}\left(\frac{1}{2}|\mathbf{v}|^2\right)$$

which can be used to write the equation of motion as

$$\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{v} = -\operatorname{grad}\left(\frac{p}{\rho} + \beta + \frac{1}{2}|\mathbf{v}|^2\right) \tag{i}$$

where $\boldsymbol{\omega} = \operatorname{curl}\mathbf{v}$ is the vorticity.

We now take the curl of both sides of this equation and recall that the curl of a gradient vanishes. This leads to

$$\boldsymbol{\omega}' + \operatorname{curl}(\boldsymbol{\omega} \times \mathbf{v}) = \mathbf{o}.$$

Evaluating the term $\operatorname{curl}(\boldsymbol{\omega}\times\mathbf{v})$ and making use of incompressibility shows that $\operatorname{curl}(\boldsymbol{\omega}\times\mathbf{v}) = (\operatorname{grad}\boldsymbol{\omega})\mathbf{v} - \mathbf{L}\boldsymbol{\omega}$ and so finally we

$$\boldsymbol{\omega}' + (\operatorname{grad}\boldsymbol{\omega})\mathbf{v} = \mathbf{L}\boldsymbol{\omega},$$

which, since the material and spatial time derivatives of $\boldsymbol{\omega}$ are related by $\dot{\boldsymbol{\omega}} = \boldsymbol{\omega}' + (\operatorname{grad}\boldsymbol{\omega})\mathbf{v}$, leads to the desired result

$$\dot{\boldsymbol{\omega}} = \mathbf{L}\boldsymbol{\omega}.$$

Problem 12.14. *Circulation and Kelvin's theorem.* Show that the circulation associated with an arbitrary closed material curve in an incompressible inviscid fluid is time invariant.

Solution: Consider a one-parameter family of closed material curves $\mathcal{C}_t$ where the parameter is t. Since $\mathcal{C}_t$ is a material curve the same particles are associated with it at all times. We can define $\mathcal{C}_t$ parametrically by

$$\mathcal{C}_t: \qquad \mathbf{y} = \overline{\mathbf{y}}(s,t), \qquad 0 \le s \le \ell(t),$$

where s is the arc length along the curve $\mathcal{C}_t$ and $\ell(t)$ is its total length at time t. The unit tangent vector to the curve $\mathcal{C}_t$ is

$$\mathbf{s}(s,t) = \frac{\partial \overline{\mathbf{y}}}{\partial s}.$$

Let $\mathbf{v}(\mathbf{y},t)$ be the velocity field. Then at any instant t one calls the integral of $\mathbf{v}\cdot\mathbf{s}$ around $\mathcal{C}_t$ the *circulation* associated with $\mathcal{C}_t$:

$$\int_{\mathcal{C}_t} \mathbf{v}\big(\overline{\mathbf{y}}(s,t),t\big)\cdot\mathbf{s}(s,t)\ ds.$$

Note that the circulation is solely a function of time (given $\mathcal{C}_t$ and $\mathbf{v}$).

On using the transport theorem $(3.86)_2$ – and this is where we make use of the fact that $\mathcal{C}_t$ is a material curve – we can write the rate of change of circulation as

$$\frac{d}{dt}\int_{\mathcal{C}_t} \mathbf{v}\cdot\mathbf{s}\ ds = \int_{\mathcal{C}_t}\Big(\dot{\mathbf{v}} + \mathbf{L}^T\mathbf{v}\Big)\cdot\mathbf{s}\ ds.$$

However the second term on the right hand side vanishes:

$$\int_{\mathcal{C}_t} \mathbf{L}^T\mathbf{v}\cdot\mathbf{s}\ ds = \int_{\mathcal{C}_t}\big(\operatorname{grad}|\mathbf{v}|^2/2\big)\cdot\mathbf{s}\ ds = \int_{\mathcal{C}_t}\frac{\partial}{\partial s}(|\mathbf{v}|^2/2)\,ds = 0$$

because (a) $\mathbf{L}^T\mathbf{v} = \operatorname{grad}|\mathbf{v}|^2/2$, (b) $(\operatorname{grad}\chi)\cdot\mathbf{s} = \partial\chi/\partial s$ for any scalar-valued field χ, and (c) the integral over the entire closed path $\mathcal{C}_t$ of $\partial\chi/\partial s$ vanishes provided χ is well behaved. Thus the rate of change of circulation can be written as

$$\frac{d}{dt}\int_{\mathcal{C}_t} \mathbf{v}\cdot\mathbf{s}\ ds = \int_{\mathcal{C}_t}\dot{\mathbf{v}}\cdot\mathbf{s}\ ds. \tag{i}$$

From (12.64), the equation of motion for an inviscid incompressible fluid under conservative body forces is

$$\dot{\mathbf{v}} = -\operatorname{grad}\left(\frac{p}{\rho}+\beta\right) = -\operatorname{grad}\chi \tag{ii}$$

where we have set $\chi = p/\rho + \beta$. Substituting (ii) into (i) leads to

$$\frac{d}{dt}\int_{\mathcal{C}_t} \mathbf{v}\cdot\mathbf{s}\ ds = -\int_{\mathcal{C}_t}(\operatorname{grad}\chi)\cdot\mathbf{s}\ ds = -\int_{\mathcal{C}_t}\frac{\partial\chi}{\partial s}\ ds \ = \ 0 \tag{iii}$$

where we have again used the facts that $(\operatorname{grad}\chi)\cdot\mathbf{s} = \partial\chi/\partial s$ for any well-behaved scalar-valued field χ and the integral over the entire closed path $\mathcal{C}_t$ of $\partial\chi/\partial s$ vanishes. Thus the rate of change of circulation vanishes and so the circulation is constant.

Problem 12.15. *Irrotational flows.* If the vorticity at every particle of the fluid vanishes at some instant of time the flow is said to be *irrotational* at that instant. Show that if a flow is irrotational at one instant t_0 then it is irrotational for all instants t assuming the fluid to be incompressible and inviscid.

Solution: One approach is via the vorticity transport equation (12.66):

$$\dot{\boldsymbol{\omega}} = \mathbf{L}\boldsymbol{\omega}.$$

Since the time derivative in this equation is at fixed $\mathbf{x}$, if we are to integrate it the velocity gradient tensor and vorticity vector must be expressed in Lagrangian form: $\mathbf{L}(\mathbf{x}, t), \boldsymbol{\omega}(\mathbf{x}, t)$. Then the preceding first-order ordinary differential equation, together with the fact that

$$\boldsymbol{\omega}(\mathbf{x}, t_0) = \mathbf{o}$$

constitute an initial-value problem. (The particle $\mathbf{x}$ is held fixed.) Assume that this problem has a unique solution. Since one can verify by direct substitution that $\boldsymbol{\omega}(\mathbf{x}, t) = \mathbf{o}$ is one solution for all t, it then follows that

$$\boldsymbol{\omega}(\mathbf{x}, t) = \mathbf{o}$$

is the solution[3] for all t. One can now use the inverse motion $\mathbf{x} = \mathbf{x}(\mathbf{y}, t)$ to conclude that this is true in Eulerian form as well: $\boldsymbol{\omega}(\mathbf{y}, t) = \mathbf{o}$.

A second approach is via Kelvin circulation theorem. By Stokes' theorem (see Section 5.2 of Volume I) we can write the circulation of $\mathcal{C}_t$ as

$$\int_{\mathcal{C}_t} \mathbf{v} \cdot \mathbf{s}\, ds = \int_{\mathcal{S}_t} \operatorname{curl} \mathbf{v} \cdot \mathbf{n}\, dA_y = \int_{\mathcal{S}_t} \boldsymbol{\omega} \cdot \mathbf{n}\, dA_y$$

where $\mathcal{S}_t$ is any smooth surface in $\mathcal{R}_t$ whose boundary is $\mathcal{C}_t$ and $\mathbf{n}$ is a unit normal on $\mathcal{S}_t$; the direction of $\mathbf{n}$ is selected using the right hand rule with respect to the sense of integration. Since $\boldsymbol{\omega}$ vanishes at time t_0, the rightmost expression for the circulation shows that the circulation must also vanish at time t_0. But Kelvin's theorem says that the circulation is time independent. Therefore the circulation must vanish at all times:

$$\int_{\mathcal{S}_t} \boldsymbol{\omega} \cdot \mathbf{n}\, dA_y = 0.$$

Since this is to hold for all choices of $\mathcal{C}_t$ we can localize it to get the desire result $\boldsymbol{\omega}(\mathbf{y}, t) = \mathbf{o}$.

Problem 12.16. *Velocity potential.* Consider an irrotational flow of an inviscid incompressible fluid. Explore the simplifications resulting from the preceding information.

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[3]One can verify by direct substitution that if the initial condition had been $\boldsymbol{\omega}(\mathbf{x}, t_0) = \boldsymbol{\omega}_0(\mathbf{x})$, a solution of the initial-value problem is given by $\boldsymbol{\omega}(\mathbf{x}, t) = \mathbf{F}(\mathbf{x}, t)\boldsymbol{\omega}_0(\mathbf{x})$; see also Problem 3.16 specialized to isochoric motions.

Solution: We are given that

$$\operatorname{curl}\mathbf{v}(\mathbf{y},t)=\mathbf{o} \qquad \text{for all} \quad \mathbf{y}\in\mathcal{R}_t,\ t\in[t_0,t_1].$$

A well-known theorem in calculus tells us that if the curl of a vector field vanishes, then that vector field is the gradient of a scalar field. Thus in the present context there must exist a scalar-valued function $\phi(\mathbf{y},t)$ such that

$$\mathbf{v}=\operatorname{grad}\phi;$$

ϕ is called the *velocity potential.* Since the fluid is incompressible the flow must obey $\operatorname{div}\mathbf{v}=0$. Substituting $\mathbf{v}=\operatorname{grad}\phi$ into $\operatorname{div}\mathbf{v}=0$ shows that ϕ satisfies Laplace's equation

$$\nabla^2\phi=0.$$

Combining this with the result of Problem 12.15 leads to the very useful result that if the flow of an incompressible inviscid fluid is irrotational at one instant of time, then the velocity field for all times can be expressed as the gradient of a solution to Laplace's equation.

Problem 12.17. *Bernoulli's theorem for a compressible inviscid fluid.* Consider the purely mechanical theory for an inviscid **compressible** fluid. Given the free energy function $\widehat{\psi}(\rho)$, the system of equations governing the pressure $p(\mathbf{y},t)$, density $\rho(\mathbf{y},t)$ and velocity $\mathbf{v}(\mathbf{y},t)$ are

$$\left.\begin{aligned}-\operatorname{grad}p+\rho\mathbf{b} &= \rho\dot{\mathbf{v}},\\ \dot{\rho}+\rho\operatorname{div}\mathbf{v} &= 0,\\ p=\widehat{p}(\rho) &= \rho^2\,\widehat{\psi}'(\rho).\end{aligned}\right\}$$

Suppose that the body force is conservative with potential β: $\mathbf{b}=-\operatorname{grad}\beta$. Define

$$\varphi=\psi+\frac{p}{\rho}+\frac{1}{2}|\mathbf{v}|^2+\beta$$

and show that φ is constant for all time and everywhere for a steady irrotational flow.

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Chapter 13

Liquid Crystals

13.1 Introduction.

In a crystalline solid the molecules that form the material are located in an ordered manner on a lattice; moreover the molecular orientation is also ordered. If the solid is heated until it melts into its liquid phase the molecules lose both their positional and orientational order: they are free to move about and tumble around. Certain materials however when heated, first change from the solid phase to a *liquid crystalline phase*, and only upon further heating do they change to the liquid phase. In the liquid crystal phase the molecules have no positional order and can move about freely. However the molecules retain their orientational order. This is depicted schematically in Figure 13.1. In the simplest case the molecules orient themselves in one preferred direction and this direction is identified by a vector called the *director* which we denote by $\mathbf{d}$.

More precisely, the molecular orientation in the liquid crystal phase is *not* constrained to lie *precisely* in the director direction (as it would in a solid). Rather, the molecular orientation varies stochastically about $\mathbf{d}$ with the molecules spending significantly more time in this preferred direction than in any other. An order parameter (or order tensor) can be used to measure the degree to which the molecular orientation coincides with $\mathbf{d}$. The order parameter varies from the value unity (at perfect ordering) to zero (at completely random ordering). As the temperature (and therefore the degree of disorder) increases, the value of the order parameter will decrease as the liquid crystal "moves" from being closer to a solid towards being closer to a liquid. We shall not discuss this aspect of liquid crystals in these

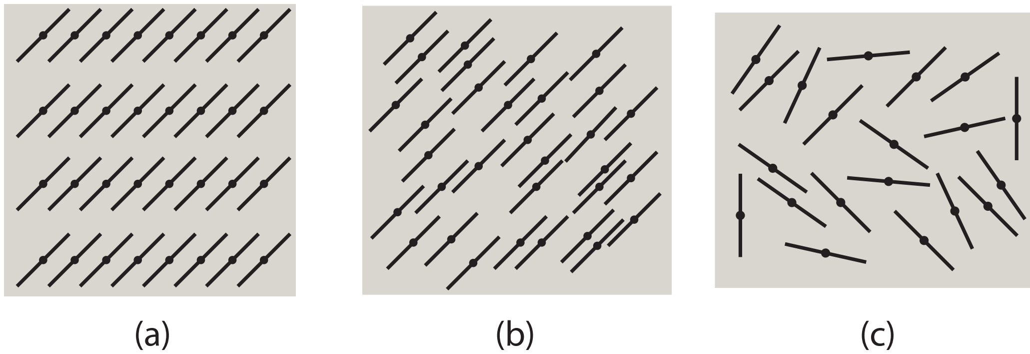

Figure 13.1: Schematic diagram illustrating order: (a) Both positional and orientational order; (b) orientational order but no positional order; (c) neither positional nor orientational order.

notes. The interested reader may consult the book by de Gennes and Prost.

The preceding description characterizes a *nematic liquid crystal*. There are other types of liquid crystals. For example in a chiral nematic liquid crystal, the molecules prefer to orient themselves at a small angle to each other. A string of molecules can then arrange themselves with the director rotating in a spiral as one moves along the string (at the microscopic scale), returning to the starting orientation after a certain distance. Liquid crystals are classified broadly into three types: nematic, cholesteric and smectic; a qualitative description of these can be found in Collings.

Most materials do not exist in the liquid crystal phase though several do. Collings describes certain common characteristics of the molecules of a material that can exist in a liquid crystal phase. He also describes the role of these materials in liquid crystal displays (LCDs) as well as in molecular and cellular biology.

Liquid crystals respond to electric and magnetic fields. For example when placed in an electric field, the positive and negative charges of a liquid crystal molecule get displaced slightly producing an electric dipole. The external electric field now applies a pair of forces on the charged ends of the molecule that are equal in magnitude and opposite in direction. Thus there is no resultant force on the molecule, but if the pair of forces is non-collinear, there will be a resultant *couple*. This couple will tend to rotate the molecule until it is aligned with the field. Thus the orientation of the director can be controlled by the electric field. In the theory developed here, the effect of the external electric or magnetic field enters solely through a body couple field (analogous to a body force field) that is determined by the electric/magnetic field.

Boundary conditions can play an important role in the behavior of a liquid crystal. For example if the forces between the liquid crystal and the container wall (in which the liquid crystal is held) are very strong, the container will hold the molecules in some specific direction along the wall. In this case one says that there is *strong anchoring* at the boundary. The anchoring direction, i.e. the director orientation at the wall, can be controlled by carefully rubbing the surface of the container in the desired direction. To illustrate the role of anchoring, consider for example a liquid crystal that is contained between two parallel plates. If the molecules are anchored in the same direction on both plates, then (in the absence of external forcing) the director field throughout the body will coincide with this direction. However if the direction of the director at one plate differs from that at the other plate, then the director field will vary through the body; this is illustrated in Problem 13.3. Or, returning to the case where the anchoring direction is the same at the two plates, suppose there is an electric field applied in some other direction. In this event there will be a competition between molecules wanting to orient in the direction preferred due to the boundary condition and wanting to orient in the direction preferred due to the field; Problem 13.4 illustrates this.

In this chapter we limit attention to nematic liquid crystals. The theory that we present is often referred to as the *Leslie-Ericksen theory*. Note that the director is a field $\mathbf{d}(\mathbf{y}, t)$ so that if a liquid crystal is subjected to a deformation, the preferred direction at different locations may be different depending on the local conditions. In contrast to the classical continuum theory developed in previous Chapters, the Leslie-Ericksen theory involves *two* fundamental kinematic fields: the usual velocity field $\mathbf{v}(\mathbf{y}, t)$ and the director field $\mathbf{d}(\mathbf{y}, t)$ (or a field related to the director field). This second characterizes the microstructural effects of the material at the continuum scale.

An outline of the material in this chapter is as follows: as alluded to above the continuum theory of a nematic liquid crystal involves certain new ingredients, viz. body couples and contact couples (analogous to body forces and contact forces), a conjugate kinematic field representing the rotation rate of a director (as distinct from the rotation rate of the continuum), and the related notion of rotational inertia. These concepts are introduce in Section 13.2. The basic balance laws of continuum mechanics for linear momentum, angular momentum and the dissipation inequality are restated in light of these new concepts, and the associated field equations/inequality are derived. The section ends with a statement of the constitutive relations in primitive form. In Section 13.3 the constitutive equations are made to conform to material frame indifference and the dissipation inequality. A specific example of the free energy function and the extra stress are given in Section 13.4. The boundary condition to be imposed on the director field is discussed in Section 13.5. The final Section 13.6 contains several worked examples and exercises which illustrate some basic phenomena and fill gaps in our presentation. In particular two examples look at the derivation of the aforementioned example free energy function and extra stress by linearizing the general constitutive relations. Another exercise asks for a physics-based derivation of this free energy function. A variational formulation of the theory and a problem that looks at stability are also included.

13.2 Formulation of basic concepts.

KINEMATICS: Consider a fluid undergoing a motion characterized by a velocity field $\mathbf{v}(\mathbf{y},t)$. The velocity gradient tensor $\mathbf{L}$, the stretching tensor $\mathbf{D}$ and the spin tensor $\mathbf{W}$ are given by

$$\mathbf{L} = \operatorname{grad}\mathbf{v}, \qquad \mathbf{D} = \frac{1}{2}(\mathbf{L}+\mathbf{L}^T), \qquad \mathbf{W} = \frac{1}{2}(\mathbf{L}-\mathbf{L}). \tag{13.1}$$

The spin tensor $\mathbf{W}$ is a measure of the angular velocity of the fluid. The axial vector associated with the skew symmetric tensor $\mathbf{W}$ is

$$\frac{1}{2}\operatorname{curl}\mathbf{v},$$

and this represents the local angular velocity of the fluid; see Section 3.3. If the fluid is incompressible as we shall assume here it can only undergo locally volume preserving motions:

$$\operatorname{div}\mathbf{v} = \operatorname{tr}\mathbf{D} = 0. \tag{13.2}$$

To characterize the microstructure underlying a liquid crystal we assign to each particle p a *director* characterized by a unit vector $\mathbf{d}(\mathbf{y},t)$. A motion of the liquid crystal is characterized by *two independent kinematic fields* $\mathbf{v}(\mathbf{y},t)$ and $\mathbf{w}(\mathbf{y},t)$, where the former represents the velocity of a particle and the latter characterizes the angular velocity of a director[1]. Since $\mathbf{w}$ is the angular velocity of the director $\mathbf{d}$ we must have

$$\dot{\mathbf{d}} = \mathbf{w}\times\mathbf{d}, \qquad \dot{d}_i = e_{ijk}w_jd_k \tag{13.3}$$

[1]As we shall see shortly, having introduced a new independent kinematic field (director angular velocity) we must introduce conjugate force fields (body and surface couples).

where e_{ijk} is the alternator. Observe from this that the angular velocity field $\mathbf{w}(\mathbf{y}, t)$ and the director field $\mathbf{d}(\mathbf{y}, t)$ are *not independent*.

The difference $\boldsymbol{\omega}$ between the angular velocity of the director field and the angular velocity of the fluid,

$$\boldsymbol{\omega} = \mathbf{w} - \frac{1}{2}\mathrm{curl}\,\mathbf{v},$$

is the angular velocity with which the director spins relative to the fluid.

Since the director is a unit vector[2] we have $|\mathbf{d}| = 1$. Taking the material time derivative of $\mathbf{d} \cdot \mathbf{d} = 1$ gives

$$\dot{\mathbf{d}} \cdot \mathbf{d} = 0 \tag{13.4}$$

which shows that $\dot{\mathbf{d}}$ is perpendicular to $\mathbf{d}$.

Let $\overset{o}{\mathbf{d}}$ be the co-rotational rate of change of the director (see Section 3.9):

$$\overset{o}{\mathbf{d}} = \dot{\mathbf{d}} - \mathbf{W}\mathbf{d}. \tag{13.5}$$

One can verify that

$$\overset{o}{\mathbf{d}} = \boldsymbol{\omega} \times \mathbf{d} \tag{13.6}$$

and so $\overset{o}{\mathbf{d}}$ represents the rate of change of the director as viewed by an observer spinning with the fluid. It follows from (13.6) that

$$\overset{o}{\mathbf{d}} \cdot \mathbf{d} = 0.$$

Thus the co-rotational rate $\overset{o}{\mathbf{d}}$ is also perpendicular to $\mathbf{d}$.

We assume the director to be frame indifferent. Thus if $\mathbf{d}$ and $\mathbf{d}^*$ describe the director as seen by two observers related by $\mathbf{Q}$, we postulate that

$$\mathbf{d}^* = \mathbf{Q}\mathbf{d}.$$

As noted in Section 3.9, when any vector $\mathbf{d}$ is objective, its material time derivative $\dot{\mathbf{d}}$ is not objective in general but its co-rotational derivative $\overset{o}{\mathbf{d}}$ is.

Finally it is convenient for later purposes to note that

$$\dot{\overline{\mathrm{grad}\,\mathbf{d}}} = \mathrm{grad}\,(\dot{\mathbf{d}}) - (\mathrm{grad}\,\mathbf{d})\mathbf{L}, \qquad \dot{\overline{d_{i,j}}} = (\dot{d}_i)_{,j} - d_{i,k}v_{k,j}. \tag{13.7}$$

[2] By requiring the director to be a unit vector at all times, we are constraining it to be inextensible. This constraint can be relaxed.

This can be readily established by using the following expressions for the material time derivatives of the respective quantities G_{ij} and d_i,

$$\dot{G}_{ij} = G'_{ij} + G_{ij,k}v_k, \qquad \dot{d}_i = d'_i + d_{i,k}v_k$$

in the notation of Section 1.6. Here we have set $G_{ij} = d_{i,j}$. In this chapter we use the notation $(\cdot)_{,k} = \partial(\cdot)/\partial y_k$.

FORCES and TORQUES: In addition to the usual body forces and contact forces (tractions) of continuum mechanics, when modeling a liquid crystal one must also account for the torques that act on the directors. Therefore one must introduce *body couple* and *contact couple* fields into the theory.

Thus, in addition to a body force $\mathbf{b}(\mathbf{y}, t)$ there is also a *body couple* $\mathbf{c}(\mathbf{y}, t)$ per unit mass acting at each particle of the body. Likewise, at any point $\mathbf{y}$ on a surface $\mathcal{S}_t$, in addition to the contact force $\mathbf{t}(\mathbf{y}, t, \mathbf{n})$ there is a *contact couple* $\mathbf{m}(\mathbf{y}, t, \mathbf{n})$; here $\mathbf{n}$ is a unit normal vector at a point on a surface in the body and $\mathbf{m}$ is the couple applied by the material on the positive side of $\mathcal{S}_t$ on the material on the negative side. (The "positive side" of $\mathcal{S}_t$ is the side into which $\mathbf{n}$ points.)

In order to define these notions precisely we must specify how each contributes to the resultant force, resultant torque and total rate of working on an arbitrary part $\mathcal{P}$ of the body. Consider a part $\mathcal{P}$ that occupies a region $\mathcal{D}_t$ at time t. Then we postulate that the resultant force, the resultant torque and the total rate of working are

$$\left.\begin{aligned}
&\int_{\partial\mathcal{D}_t} \mathbf{t}\, \mathrm{d}A_y + \int_{\mathcal{D}_t} \rho\mathbf{b}\, \mathrm{d}V_y,\\[2ex]
&\int_{\partial\mathcal{D}_t} \mathbf{y}\times\mathbf{t}\, \mathrm{d}A_y + \int_{\mathcal{D}_t} \mathbf{y}\times\rho\mathbf{b}\, \mathrm{d}V_y + \int_{\partial\mathcal{D}_t} \mathbf{m}\, \mathrm{d}A_y + \int_{\mathcal{D}_t} \rho\mathbf{c}\, \mathrm{d}V_y,\\[2ex]
&\int_{\partial\mathcal{D}_t} \mathbf{t}\cdot\mathbf{v}\, \mathrm{d}A_y + \int_{\mathcal{D}_t} \rho\mathbf{b}\cdot\mathbf{v}\, \mathrm{d}V_y + \int_{\partial\mathcal{D}_t} \mathbf{m}\cdot\mathbf{w}\, \mathrm{d}A_y + \int_{\mathcal{D}_t} \rho\mathbf{c}\cdot\mathbf{w}\, \mathrm{d}V_y.
\end{aligned}\right\} \tag{13.8}$$

Note from the expression for the rate of working that the kinematic field conjugate to the body couples and contact couples is the director angular velocity field[3] $\mathbf{w}(\mathbf{y}, t)$.

The source of the body couple is often an electric (or magnetic) field which polarizes each liquid crystal molecule and thereby applies a pair of forces on each molecule that are equal

[3]In Problem 4.7 we considered the effect of body couples and contact couples in a continuum, but there, we did not introduce an independent kinematic field such as the angular velocity field $\mathbf{w}$. Therefore we can appropriate some but *not all* of the results from that problem to the current discussion.

in magnitude and opposite in direction. If these forces are non-collinear, they result in a couple. In the continuum theory this is represented by the body couple field $\mathbf{c}(\mathbf{y}, t)$. In the presence of an electric field $\mathbf{E}$, the resulting body couple is given by

$$\rho\mathbf{c} = \mathbf{d} \times \mathbf{g}, \tag{13.9}$$

where

$$\mathbf{g} = \epsilon_0 \chi \left(\mathbf{d} \cdot \mathbf{E}\right) \mathbf{E}; \tag{13.10}$$

here the constant parameters ϵ_0 and χ are the permittivity of free space and the dielectric anisotropy respectively. Observe that the body couple vanishes if the director $\mathbf{d}$ is parallel to the electric field $\mathbf{E}$. A detailed discussion of the effect of electric and magnetic fields on liquid crystals can be found, for example, in de Gennes and Prost.

BALANCE LAWS AND FIELD EQUATIONS: The global balance laws for linear momentum and angular momentum are postulated to be

$$\left.\begin{aligned} \frac{d}{dt}\int_{\mathcal{D}_t} \rho\mathbf{v}\, \mathrm{d}V_y &= \int_{\partial\mathcal{D}_t} \mathbf{t}\, \mathrm{d}A_y + \int_{\mathcal{D}_t} \rho\mathbf{b}\, \mathrm{d}V_y, \\ \frac{d}{dt}\int_{\mathcal{D}_t} \left(\mathbf{y} \times \rho\dot{\mathbf{y}} + \mathbf{d} \times \sigma\dot{\mathbf{d}}\right) \mathrm{d}V_y &= \int_{\partial\mathcal{D}_t} \mathbf{y} \times \mathbf{t}\, \mathrm{d}A_y + \int_{\mathcal{D}_t} \mathbf{y} \times \rho\mathbf{b}\, \mathrm{d}V_y + \\ &\quad + \int_{\partial\mathcal{D}_t} \mathbf{m}\, \mathrm{d}A_y + \int_{\mathcal{D}_t} \rho\mathbf{c}\, \mathrm{d}V_y. \end{aligned}\right\} \tag{13.11}$$

The first of these describes a balance between the resultant force and the rate of change of linear momentum. The second likewise balances the resultant torque and the rate of change of angular momentum.

Since in the preceding discussion we introduced additional kinematic and torque fields, here we have introduced a corresponding measure of rotational inertia represented by the term $\mathbf{d} \times \sigma\dot{\mathbf{d}}$ in the left hand side of $(13.11)_2$. The constant σ is a measure of the rotational inertia (just as ρ is a measure of the translational inertia). In order to motivate this term, consider the moment of inertia tensor $\mathbf{J}$ of the director about its centre of mass. If the director is treated as a slender rigid rod, its moment of inertia about the director axis is zero while its moment of inertia about any axis perpendicular to the director is $\sigma = m\ell^2/12$ where the mass and length of the director are m and ℓ respectively. Thus $\mathbf{J} = \sigma(\mathbf{I} - \mathbf{d} \otimes \mathbf{d})$. The angular momentum of the director is then $\mathbf{Jw}$ where $\mathbf{w}$ is the angular velocity of the director. On using (13.3) we find that $\mathbf{Jw} = \mathbf{d} \times \sigma\dot{\mathbf{d}}$ which is the term introduced into the left hand side of $(13.11)_2$. In what follows we shall ignore this inertial effect and take

$$\sigma = 0;$$

the case $\sigma \neq 0$ is considered in Problem 13.7.

Since the material is taken to be incompressible, the mass density ρ is constant and the balance of mass is automatic.

Keeping in mind that $\mathbf{t} = \mathbf{t}(\mathbf{y}, t; \mathbf{n})$, the linear momentum balance law leads first to the usual traction-stress relation, and thereafter to the usual equation of motion:

$$\mathbf{t} = \mathbf{T}\mathbf{n}, \qquad t_i = T_{ij}n_j, \tag{13.12}$$

$$\operatorname{div}\mathbf{T} + \rho\mathbf{b} = \rho\dot{\mathbf{v}}, \qquad T_{ij,j} + \rho b_i = \rho\dot{v}_i. \tag{13.13}$$

Turning to the angular momentum balance law, first using $\mathbf{t} = \mathbf{T}\mathbf{n}$ and then the divergence theorem allows us to convert the first surface integral on the right hand side into a volume integral. On applying the angular momentum balance principle in this form to a tetrahedral region and shrinking the region to a point, the volume integrals approach zero faster than the surface integral, and so in the limit, the only remaining contribution is the limit of the surface integral of $\mathbf{m}(\mathbf{y}, t; \mathbf{n})$ over the boundary. Then mimicking the steps one uses to show the existence of the stress tensor $\mathbf{T}$ allows one to conclude that there exists a *couple stress tensor* $\mathbf{Z}(\mathbf{y}, t)$ that is independent of $\mathbf{n}$ such that

$$\mathbf{m}(\mathbf{y}, t, \mathbf{n}) = \mathbf{Z}(\mathbf{y}, t)\mathbf{n}; \tag{13.14}$$

see Problem 4.7. One can now return to $(13.11)_2$ and substitute $\mathbf{t} = \mathbf{T}\mathbf{n}, \mathbf{m} = \mathbf{Z}\mathbf{n}$ and $\sigma = 0$ to get

$$\int_{\partial\mathcal{D}_t} \mathbf{y} \times \mathbf{T}\mathbf{n}\,\mathrm{d}A_y + \int_{\mathcal{D}_t} \mathbf{y} \times \rho\mathbf{b}\,\mathrm{d}V_y + \int_{\partial\mathcal{D}_t} \mathbf{Z}\mathbf{n}\,\mathrm{d}A_y + \int_{\mathcal{D}_t} \rho\mathbf{c}\,\mathrm{d}V_y = \int_{\mathcal{D}_t} \rho\mathbf{y} \times \dot{\mathbf{v}}\,\mathrm{d}V_y$$

or in terms of components

$$\int_{\partial\mathcal{D}_t} e_{ijk}y_jT_{kp}n_p\,\mathrm{d}A_y + \int_{\mathcal{D}_t} e_{ijk}y_j\rho b_k\,\mathrm{d}V_y + \int_{\partial\mathcal{D}_t} Z_{ip}n_p\,\mathrm{d}A_y + \int_{\mathcal{D}_t} \rho c_i\,\mathrm{d}V_y = \int \rho e_{ijk}y_j\dot{v}_k\,\mathrm{d}V_y.$$

Using the divergence theorem to convert the surface integrals to volume integrals and then localizing the result in the familiar way leads to

$$e_{ijk}\delta_{jp}T_{kp} + e_{ijk}y_jT_{kp,p} + e_{ijk}y_j\rho b_k + Z_{ip,p} + \rho c_i - e_{ijk}y_j\rho\dot{v}_k = 0,$$

which simplifies on using the equations of motion (13.13) to the following field equation associated with the angular momentum balance law:

$$e_{ijk}T_{kj} + Z_{ip,p} + \rho c_i = 0. \tag{13.15}$$

This set of equations can be written in an illuminating alternative form by first multiplying it by e_{ipq} and then using the familiar identity $e_{ipq}e_{ijk} = \delta_{pj}\delta_{qk} - \delta_{pk}\delta_{qj}$. This leads to

$$T_{qp} - T_{pq} = -e_{ipq}Z_{ij,j} - \rho e_{ipq}c_i$$

which is an expression for the anti-symmetric part of stress $\mathbf{T} - \mathbf{T}^T$ in terms of the couple stress and the body couple. Note from this that the Cauchy stress is *not symmetric* in general.

DISSIPATION INEQUALITY We now turn to the counterpart of the entropy inequality in the present purely mechanical context. We postulate that the rate of external work on any part $\mathcal{P}$ of the body cannot be less than the rate of increase of kinetic energy and free energy ψ. Accordingly let the dissipation rate per unit mass be denoted by D so that by definition

$$\begin{aligned}\int_{\mathcal{D}_t} \rho \mathsf{D}\, dV_y \;=\; & \int_{\partial\mathcal{D}_t} \mathbf{t}\cdot\mathbf{v}\, \mathrm{d}A_y + \int_{\mathcal{D}_t} \rho\mathbf{b}\cdot\mathbf{v}\, \mathrm{d}V_y + \int_{\partial\mathcal{D}_t} \mathbf{m}\cdot\mathbf{w}\, \mathrm{d}A_y + \int_{\mathcal{D}_t} \rho\mathbf{c}\cdot\mathbf{w}\, \mathrm{d}V_y - \\ & - \frac{d}{dt}\int_{\mathcal{D}_t} \rho\psi\, dV_y - \frac{d}{dt}\int_{\mathcal{D}_t} \frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v}\, dV_y - \frac{d}{dt}\int_{\mathcal{D}_t} \frac{1}{2}\sigma\dot{\mathbf{d}}\cdot\dot{\mathbf{d}}\, dV_y\end{aligned} \tag{13.16}$$

where ψ is the free energy per unit mass. The dissipation inequality postulates that each side of (13.16) must be non-negative.

The last term in (13.16) represents the kinetic energy associated with the spinning of the directors. To see this recall that the director kinetic energy is given by $\mathbf{Jw}\cdot\mathbf{w}/2$ where $\mathbf{J}$ is the inertia tensor of the director at its centre of mass and $\mathbf{w}$ is the angular velocity of the director. By using $\mathbf{Jw} = \mathbf{d}\times\sigma\dot{\mathbf{d}}$ (which was established previously) and (13.3) this can be written in the alternate form $\sigma\dot{\mathbf{d}}\cdot\dot{\mathbf{d}}/2$. As mentioned previously, in our present analysis we neglect the director rotational inertia and take $\sigma = 0$; the case $\sigma \neq 0$ is considered in Problem 13.7.

In order to localize (13.16), we first substitute for the traction $\mathbf{t}$ and contact couple $\mathbf{m}$ in terms of stress $\mathbf{T}$ and couple stress $\mathbf{Z}$ using (13.12) and (13.14) respectively. The divergence theorem is then used to convert the integrals over the boundary $\partial\mathcal{D}_t$ into volume integrals. Finally the equations of motion (13.13) and (13.15) are used to eliminate the terms $\partial T_{ij}/\partial y_j$ and $\partial Z_{ij}/\partial y_j$. This leads to

$$\int_{\mathcal{D}_t} \rho\mathsf{D}\, dV_y = \int_{\mathcal{D}_t} \left(T_{ij}v_{i,j} + Z_{ij}w_{i,j} - w_i e_{ijk}T_{kj} - \rho\dot{\psi}\right) dV_y \geq 0.$$

Assuming adequate smoothness, we can localize this using the usual argument to write the dissipation rate per unit volume in the form

$$\rho \mathsf{D} = T_{ij} v_{i,j} + Z_{ij} w_{i,j} - w_i e_{ijk} T_{kj} - \rho \dot{\psi} \geq 0. \tag{13.17}$$

Note that since the stress tensor $\mathbf{T}$ is not necessarily symmetric, we *cannot* simply replace $T_{ij} v_{i,j}$ by $T_{ij} D_{ij}$ where $\mathbf{D}$ is the stretching tensor as in previous chapters.

CONSTITUTIVE RESPONSE The field equations (13.2), (13.13) and (13.15) provide 7 scalar equations that involve the fields $\mathbf{T}(\mathbf{y}, t), \mathbf{Z}(\mathbf{y}, t)$ and $\mathbf{v}(\mathbf{y}, t)$ which have 21 scalar components. In order to complete the mathematical model we must have an appropriate set of constitutive equations which we assume to have the primitive form

$$\left.\begin{array}{rcl} \psi & = & \widehat{\psi}(\mathbf{d}, \mathbf{G}, \dot{\mathbf{d}}, \mathbf{L}), \\ \mathbf{T} & = & -p\mathbf{I} + \widehat{\mathbf{T}}(\mathbf{d}, \mathbf{G}, \dot{\mathbf{d}}, \mathbf{L}), \\ \mathbf{Z} & = & \widehat{\mathbf{Z}}(\mathbf{d}, \mathbf{G}, \dot{\mathbf{d}}, \mathbf{L}), \end{array}\right\} \tag{13.18}$$

where we have set

$$\mathbf{G} = \operatorname{grad} \mathbf{d}. \tag{13.19}$$

Note that the pressure $p(\mathbf{y}, t)$, the director field $\mathbf{d}(\mathbf{y}, t)$ and the free energy $\psi(\mathbf{y}, t)$ that appear in these constitutive relationships did not enter into the field equations (13.2), (13.13) and (13.15). The pressure has been included because of the incompressibility constraint. In terms of components, there are 19 scalar constitutive equations here and we have introduced 5 additional scalar quantities.

If the director has no polarity as we assume here, the vectors $\mathbf{d}$ and $-\mathbf{d}$ are physically indistinguishable. In this event the transformation

$$\mathbf{d} \to -\mathbf{d} \tag{13.20}$$

must leave the constitutive response functions $\widehat{\psi}, \widehat{\mathbf{T}}$ and $\widehat{\mathbf{Z}}$ unchanged.

13.3 Reduced Constitutive Relations.

13.3.1 Restrictions due to dissipation inequality.

In order to determine the restrictions placed on this set of constitutive relations by the dissipation inequality, we wish to substitute (13.18) into (13.17). We therefore begin by calculating

$$\dot{\psi} = \frac{\partial \psi}{\partial d_i}\dot{d}_i + \frac{\partial \psi}{\partial G_{ij}}\dot{G}_{ij} + \frac{\partial \psi}{\partial \dot{d}_i}\ddot{d}_i + \frac{\partial \psi}{\partial L_{ij}}\dot{L}_{ij}.$$

Observe that the preceding equation involves $\mathbf{d}$ and its time and spatial derivatives. Note that some other terms in the dissipation inequality (13.17) involve $\mathbf{w}$ and its spatial derivative. But recall that $\mathbf{w}(\mathbf{y}, t)$ and $\mathbf{d}(\mathbf{y}, t)$ are not independent since they are related by (13.3). Thus if we are to use the dissipation inequality in the usual manner to draw some useful conclusions we need to express it in a form where certain terms can be arbitrarily chosen (whence their coefficients must vanish). Thus we now establish the necessary relationships between $\mathbf{d}, \mathbf{w}$ and their derivatives.

Since $\dot{\mathbf{d}} = \mathbf{w} \times \mathbf{d}$, or in component form $\dot{d}_i = e_{ijk} w_j d_k$, we have

$$\dot{d}_{i,j} = e_{i\ell k} w_{\ell,j} d_k + e_{i\ell k} w_\ell d_{k,j}.$$

Recall from (13.7) that $\dot{\overline{d_{i,j}}} = \dot{d}_{i,j} - d_{i,k} v_{k,j}$. On combining these we have

$$\dot{G}_{ij} = \dot{\overline{d_{i,j}}} = \dot{d}_{i,j} - d_{i,k} v_{k,j} = e_{i\ell k} w_{\ell,j} d_k + e_{i\ell k} w_\ell d_{k,j} - d_{i,k} v_{k,j}.$$

Therefore the third term in the expression for $\dot{\psi}$ above can be written as

$$\frac{\partial \psi}{\partial G_{ij}}\dot{G}_{ij} = \Big(\frac{\partial \psi}{\partial G_{ij}} e_{i\ell k} d_k\Big) w_{\ell,j} + \Big(\frac{\partial \psi}{\partial G_{ij}} e_{i\ell k} G_{kj}\Big) w_\ell - \Big(\frac{\partial \psi}{\partial G_{ij}} G_{ik}\Big) v_{k,j},$$

and so $\dot{\psi}$ itself can be written as

$$\begin{aligned} \dot{\psi} \quad &= \quad \Big(\frac{\partial \psi}{\partial d_i} e_{ijk} d_k\Big) w_j + \Big(\frac{\partial \psi}{\partial G_{ij}} e_{i\ell k} d_k\Big) w_{\ell,j} + \Big(\frac{\partial \psi}{\partial G_{ij}} e_{i\ell k} G_{kj}\Big) w_\ell \\ &- \quad \Big(\frac{\partial \psi}{\partial G_{ij}} G_{ik}\Big) v_{k,j} + \frac{\partial \psi}{\partial \dot{d}_i}\ddot{d}_i + \frac{\partial \psi}{\partial L_{ij}}\dot{L}_{ij}. \end{aligned}$$

The dissipation inequality can now be written as

$$\begin{aligned}\rho\mathsf{D} &= \left[\widehat{T}_{ij} + \rho\frac{\partial\psi}{\partial G_{kj}}G_{ki}\right]v_{i,j}\\ &+ \left[\widehat{Z}_{ij} - \rho\, e_{ik\ell}\, d_k\frac{\partial\psi}{\partial G_{\ell j}}\right]w_{i,j}\\ &- e_{ijk}\, w_i\left[\widehat{T}_{kj} - \rho d_k\frac{\partial\psi}{\partial d_j} - \rho G_{kp}\frac{\partial\psi}{\partial G_{jp}}\right]\\ &- \left[\rho\frac{\partial\psi}{\partial\dot{d}_i}\right]\ddot{d}_i - \left[\rho\frac{\partial\psi}{\partial L_{ij}}\right]\dot{L}_{ij} \geq 0.\end{aligned}$$

Since this must hold for arbitrary $\dot{\mathbf{L}}$ and $\ddot{\mathbf{d}}$, we conclude that ψ must be independent of $\mathbf{L}$ and $\dot{\mathbf{d}}$ whence we can write

$$\psi = \psi(\mathbf{d}, \mathbf{G}).$$

The dissipation inequality is now reduced to

$$\begin{aligned}\rho\mathsf{D} &= \left[\widehat{T}_{ij} + \rho\frac{\partial\psi}{\partial G_{kj}}G_{ki}\right]v_{i,j}\\ &+ \left[\widehat{Z}_{ij} - \rho\, e_{ik\ell}\, d_k\frac{\partial\psi}{\partial G_{\ell j}}\right]w_{i,j}\\ &- e_{ijk}\, w_i\left[\widehat{T}_{kj} - \rho d_k\frac{\partial\psi}{\partial d_j} - \rho G_{kp}\frac{\partial\psi}{\partial G_{jp}}\right] \geq 0.\end{aligned}$$

Let $\boldsymbol{\tau}$ and $\boldsymbol{\zeta}$ denote the respective portions of stress and couple stress that are not determined by the free energy:

$$\tau_{ij} = \widehat{T}_{ij} + \rho\frac{\partial\psi}{\partial G_{pj}}G_{pi}, \qquad \zeta_{ij} = \widehat{Z}_{ij} - \rho\, e_{ik\ell}\, d_k\frac{\partial\psi}{\partial G_{\ell j}},$$

so that the Cauchy stress can be written as

$$T_{ij} = -p\delta_{ij} - \rho\frac{\partial\psi}{\partial G_{pj}}G_{pi} + \tau_{ij}\ ,$$

and the couple stress as

$$Z_{ij} = \rho\, e_{ik\ell}\, d_k\frac{\partial\psi}{\partial G_{\ell j}} + \zeta_{ij}\ .$$

The dissipation inequality now reads

$$\rho \mathsf{D} = \tau_{ij} v_{i,j} + \zeta_{ij} w_{i,j} - w_i e_{ijk} \tau_{kj} \geq 0. \tag{13.21}$$

In obtaining this inequality we have used the identity

$$e_{ijk} \left(d_q \frac{\partial \psi}{\partial d_p} + G_{q\ell} \frac{\partial \psi}{\partial G_{p\ell}} + G_{\ell q} \frac{\partial \psi}{\partial G_{\ell p}} \right) = 0 \tag{13.22}$$

which is a consequence of material frame indifference: since ψ must be frame indifferent we shall show shortly that $\psi(\mathbf{d}, \mathbf{G}) = \psi(\mathbf{Qd}, \mathbf{QGQ}^T)$ for all orthogonal $\mathbf{Q}(t)$. Differentiate this with respect to time, choose $\mathbf{Q} = \mathbf{I}$, set $\dot{\mathbf{Q}} = \boldsymbol{\Omega}$ where $\boldsymbol{\Omega}$ must be skew symmetric since $\mathbf{Q}\dot{\mathbf{Q}}^T$ is skew symmetric, and finally use the representation for $\boldsymbol{\Omega}$ in terms of its axial vector. Requiring the result to be true for all axial vectors leads to the identity above.

We now assume that the "extra stresses" $\boldsymbol{\tau}$ and $\boldsymbol{\zeta}$ do not depend on $\mathbf{G}$, the gradient of $\mathbf{d}$. Thus the associated constitutive response functions have the form

$$\boldsymbol{\tau} = \boldsymbol{\tau}(\mathbf{d}, \dot{\mathbf{d}}, \mathbf{L}), \qquad \boldsymbol{\zeta} = \boldsymbol{\zeta}(\mathbf{d}, \dot{\mathbf{d}}, \mathbf{L}) \ . \tag{13.23}$$

Returning to the dissipation inequality (13.21) and noting that it must hold for all $w_{i,j}$, and that because of (13.23) the only occurrence of $w_{i,j}$ is the explicit one in that inequality, the coefficient of $w_{i,j}$ must vanish: $\boldsymbol{\zeta} = \mathbf{0}$. Thus we have the following *constitutive relation for couple stress*:

$$Z_{ij} = \rho\, e_{ik\ell}\, d_k \frac{\partial \psi}{\partial G_{\ell j}}, \tag{13.24}$$

and the dissipation inequality now reads

$$\rho \mathsf{D} = \tau_{ij} v_{i,j} - e_{ijk} w_i \tau_{kj} \geq 0. \tag{13.25}$$

Note that $\boldsymbol{\tau}$ is not generally symmetric and so we cannot replace $\tau_{ij} v_{i,j}$ in the dissipation inequality by $\tau_{ij} D_{ij}$ as in previous chapters. However we can move towards this as follows: As noted earlier, $\mathbf{w} - \boldsymbol{\omega} = \mathrm{curl}\, \mathbf{v}/2$ is the axial vector associated with the spin tensor $\mathbf{W}$. Thus first replacing $\mathrm{grad}\, \mathbf{v}$ in (13.25) using $\mathrm{grad}\, \mathbf{v} = \mathbf{L} = \mathbf{D} + \mathbf{W}$ and then using the representation of $\mathbf{W}$ in terms of its axial vector allows us to rewrite the dissipation inequality in the form

$$\rho \mathsf{D} = \tau_{ij} D_{ij} - e_{ijk} \omega_j \tau_{ik} \geq 0. \tag{13.26}$$

In summary, we have reduced the constitutive relations to

$$\left.\begin{aligned}
\psi &= \psi(\mathbf{d},\mathbf{G}),\\
T_{ij} &= -p\delta_{ij} - \rho\frac{\partial\psi}{\partial G_{kj}}G_{ki} + \tau_{ij}(\mathbf{d},\dot{\mathbf{d}},\mathbf{L}),\\
Z_{ij} &= \rho\, e_{ik\ell}\, d_k\frac{\partial\psi}{\partial G_{\ell j}},
\end{aligned}\right\} \tag{13.27}$$

and the dissipation inequality to

$$\tau_{ij}D_{ij} - e_{ijk}\omega_j\tau_{ik} \geq 0.$$

Remark In Problem 13.8 we look into whether the dissipation inequality implies that the extra stress $\boldsymbol{\tau}$ vanishes in equilibrium, i.e. when $\dot{\mathbf{d}} = \mathbf{o}, \mathbf{D} = \mathbf{0}$.

13.3.2 Restrictions due to material frame indifference.

First consider the free energy

$$\psi = \psi(\mathbf{d},\mathbf{G})$$

where $\mathbf{G} = \operatorname{grad}\mathbf{d}$. We assume the director to be frame indifferent. Thus if $\mathbf{d}$ and $\mathbf{d}^*$ describe the director as seen by two observers related by $\mathbf{Q}$, we postulate that

$$\mathbf{d}^* = \mathbf{Q}\mathbf{d}.$$

Recall from Section 3.9, that the co-rotational time derivative of any objective vector is objective, and so here, $\overset{o}{\mathbf{d}}$ is necessarily objective.

We begin by examining whether $\mathbf{G}$ is objective or not. Let $\mathbf{x} = \overline{\mathbf{x}}(\mathbf{y},t)$ be the inverse of the motion $\mathbf{y} = \mathbf{y}(\mathbf{x},t)$. The spatial gradient $\partial\overline{\mathbf{x}}/\partial\mathbf{y}$ is $\mathbf{F}^{-1}$ where $\mathbf{F}$ is the deformation gradient tensor. If $\overline{\mathbf{d}}(\mathbf{x},t)$ is defined by $\overline{\mathbf{d}}(\mathbf{x},t) = \mathbf{d}(\mathbf{y}(\mathbf{x},t),t)$ then $\mathbf{d}(\mathbf{y},t) = \overline{\mathbf{d}}(\overline{\mathbf{x}}(\mathbf{y},t),t)$. Let

$$G_{ij} = \frac{\partial d_i}{\partial y_j} \quad \text{and} \quad H_{ij} = \frac{\partial \overline{d}_i}{\partial x_j}.$$

Differentiating $\mathbf{d}(\mathbf{y},t) = \overline{\mathbf{d}}(\overline{\mathbf{x}}(\mathbf{y},t),t)$ and using the chain rule gives

$$G_{ij} = \frac{\partial d_i}{\partial y_j} = \frac{\partial \overline{d}_i}{\partial x_p}\frac{\partial \overline{x}_p}{\partial y_j}$$

and so

$$\mathbf{G} = \mathbf{H}\mathbf{F}^{-1}.$$

Since $\mathbf{d}^* = \mathbf{Q}\mathbf{d}$ and $\mathbf{x}^* = \mathbf{x}$ it follows that $\mathbf{H}^* = \mathbf{Q}\mathbf{H}$. We know from Section 3.8 that $\mathbf{F}^* = \mathbf{Q}\mathbf{F}$. Therefore

$$\mathbf{G}^* = \mathbf{H}^*\mathbf{F}_*^{-1} = (\mathbf{Q}\mathbf{H})(\mathbf{Q}\mathbf{F})^{-1} = \mathbf{Q}\mathbf{H}\mathbf{F}^{-1}\mathbf{Q}^T = \mathbf{Q}\mathbf{G}\mathbf{Q}^T$$

and therefore $\mathbf{G}$ is objective.

Since ψ has to be frame indifferent the function $\psi(\mathbf{d}, \mathbf{G})$ must have the property

$$\psi(\mathbf{d}, \mathbf{G}) = \psi(\mathbf{Q}\mathbf{d}, \mathbf{Q}\mathbf{G}\mathbf{Q}^T) \tag{13.28}$$

for all rotations $\mathbf{Q}$, i.e. thus ψ must be an isotropic function of all its arguments. Note that the tensor $\mathbf{G}$ is not necessarily symmetric and therefore the classical representation theorems for isotropic functions cannot be used.

Consider now the frame indifference of the constitutive relation for the extra stress:

$$\boldsymbol{\tau} = \boldsymbol{\tau}(\mathbf{d}, \dot{\mathbf{d}}, \mathbf{L}).$$

Since $\dot{\mathbf{d}} = \overset{o}{\mathbf{d}} + \mathbf{W}\mathbf{d}$, $\mathbf{D} = (\mathbf{L} + \mathbf{L}^T)/2$ and $\mathbf{W} = (\mathbf{L} - \mathbf{L}^T)/2$ it follows that there is a one-to-one relation between the sets $\{\mathbf{d}, \dot{\mathbf{d}}, \mathbf{L}\}$ and $\{\mathbf{d}, \overset{o}{\mathbf{d}}, \mathbf{D}, \mathbf{W}\}$. Thus $\boldsymbol{\tau}$ admits the alternative (equivalent) representation

$$\boldsymbol{\tau} = \boldsymbol{\tau}(\mathbf{d}, \overset{o}{\mathbf{d}}, \mathbf{D}, \mathbf{W}).$$

Turning to material frame indifference we use the usual notation that a starred and unstarred symbol refer to a quantity as seen by two observers related to each other by the rotation $\mathbf{Q}$. The quantities $\mathbf{d}, \overset{o}{\mathbf{d}}, \mathbf{D}$ and $\mathbf{G}$ are objective and so

$$\mathbf{d}^* = \mathbf{Q}\mathbf{d}, \quad (\overset{o}{\mathbf{d}})^* = \mathbf{Q}\,\overset{o}{\mathbf{d}}, \quad \mathbf{D}^* = \mathbf{Q}\mathbf{D}\mathbf{Q}^T, \quad \mathbf{G}^* = \mathbf{Q}\mathbf{G}\mathbf{Q}^T.$$

Since the Cauchy stress is objective, the extra stress must also be objective:

$$\boldsymbol{\tau}^* = \mathbf{Q}\boldsymbol{\tau}\mathbf{Q}^T.$$

Finally considering the spin tensor we recall from Section 3.8 that it is not objective but rather obeys the relation

$$\mathbf{W}^* = \mathbf{Q}\mathbf{W}\mathbf{Q}^T + \boldsymbol{\Omega}$$

where $\mathbf{\Omega} = \dot{\mathbf{Q}}\mathbf{Q}^T$. As has been noted before, since $\mathbf{Q}(t)\mathbf{Q}^T(t) = \mathbf{I}$ it follows by differentiation that $\mathbf{\Omega}$ must be skew-symmetric.

Material frame indifference therefore requires that the extra stress response function obey

$$\boldsymbol{\tau}(\mathbf{Qd}, \mathbf{Q}\overset{o}{\mathbf{d}}, \mathbf{QDQ}^T, \mathbf{QWQ}^T + \mathbf{\Omega}) = \mathbf{Q}\boldsymbol{\tau}(\mathbf{d}, \overset{o}{\mathbf{d}}, \mathbf{D}, \mathbf{W})\mathbf{Q}^T$$

for all rotations $\mathbf{Q}$ and skew symmetric $\mathbf{\Omega}$. Picking $\mathbf{Q} = \mathbf{I}$ and $\mathbf{\Omega} = -\mathbf{W}$ leads to

$$\boldsymbol{\tau}(\mathbf{d}, \overset{o}{\mathbf{d}}, \mathbf{D}, \mathbf{0}) = \boldsymbol{\tau}(\mathbf{d}, \overset{o}{\mathbf{d}}, \mathbf{D}, \mathbf{W})$$

which must hold for all skew symmetric $\mathbf{W}$. Thus $\boldsymbol{\tau}$ must be independent of $\mathbf{W}$:

$$\boldsymbol{\tau} = \boldsymbol{\tau}(\mathbf{d}, \overset{o}{\mathbf{d}}, \mathbf{D})$$

and objectivity of the extra stress $\boldsymbol{\tau}$ thus requires

$$\boldsymbol{\tau}(\mathbf{Qd}, \mathbf{Q}\overset{o}{\mathbf{d}}, \mathbf{QDQ}^T) = \mathbf{Q}\boldsymbol{\tau}(\mathbf{d}, \overset{o}{\mathbf{d}}, \mathbf{D})\mathbf{Q}^T, \tag{13.29}$$

i.e. $\boldsymbol{\tau}$ must be an isotropic function of all its arguments.

Remark: Recall the two different discussions of a transversely isotropic elastic material in Sections 8.5.2 and 8.8. In one, we took the energy to be of the form $\psi(\mathbf{F})$ and required that it exhibit the symmetries associated with transversely isotropic transformations about the preferred direction $\mathbf{d}$; and in the other, we took the energy to be of the form $\psi(\mathbf{F}, \mathbf{d})$ and required it to be isotropic. These were equivalent treatments. The situation here is similar to the latter treatment.

13.3.3 Summary.

In summary, we have reduced the constitutive relations to

$$\left.\begin{aligned} \psi &= \psi(\mathbf{d}, \mathbf{G}), \\ T_{ij} &= -p\delta_{ij} - \rho\frac{\partial\psi}{\partial G_{pj}}G_{pi} + \tau_{ij}(\mathbf{d}, \overset{o}{\mathbf{d}}, \mathbf{D}), \\ Z_{ij} &= \rho\, e_{ik\ell}\, d_k \frac{\partial\psi}{\partial G_{\ell j}}, \end{aligned}\right\} \tag{13.30}$$

where $\mathbf{G} = \operatorname{grad}\mathbf{d}$, and $\psi(\mathbf{d}, \mathbf{G})$ and $\boldsymbol{\tau}(\mathbf{d}, \overset{o}{\mathbf{d}}, \mathbf{D})$ are isotropic functions of their respective arguments. The dissipation inequality reads

$$\tau_{ij}D_{ij} - e_{ijk}\omega_j\tau_{ik} \geq 0. \tag{13.31}$$

The field equations of the Leslie-Ericksen theory are the equations of linear and angular momentum, incompressibility and the constraint $|\mathbf{d}| = 1$:

$$\left.\begin{aligned} T_{ij,j} + \rho b_i &= \rho \dot{v}_i, \\ e_{ijk}T_{kj} + Z_{ij,j} + \rho c_i &= 0, \\ v_{i,i} &= 0, \\ d_i d_i &= 1. \end{aligned}\right\} \tag{13.32}$$

In the special case of equilibrium, assuming that the extra stress $\boldsymbol{\tau}$ vanishes in equilibrium, the problem decouples as follows: substituting (13.30) into $(13.32)_2$ leads to set of 3 scalar partial differential equations for $d_i(\mathbf{y})$; see Problem 13.1 for a more careful discussion of this.

13.4 A Particular Constitutive Model.

In the Leslie-Ericksen theory, a nematic liquid crystal is characterized by two constitutive functions: the free energy $\psi(\mathbf{d}, \operatorname{grad}\mathbf{d})$ and the extra stress $\boldsymbol{\tau}(\mathbf{d}, \overset{o}{\mathbf{d}}, \mathbf{D})$. In this section we lay down, with no motivation, a particular free energy and a particular extra stress. The particular forms presented here are motivated in Problems 13.9, 13.10 and 13.12.

13.4.1 A Free Energy Function ψ: the Frank Energy

As a specific example of a constitutive relation for the free energy we consider

$$\rho\psi(\mathbf{d}, \operatorname{grad}\mathbf{d}) = \frac{1}{2}K_1(\operatorname{div}\mathbf{d})^2 + \frac{1}{2}K_2(\mathbf{d}\cdot\operatorname{curl}\mathbf{d})^2 + \frac{1}{2}K_3|\mathbf{d}\times\operatorname{curl}\mathbf{d}|^2 \tag{13.33}$$

which is referred to as the Frank free energy, and the material constants K_1, K_2 and K_3 are called the Frank elastic constants. As discussed in Problem 13.10 the moduli K_1, K_2 and K_3 are associated with the three basic modes of liquid crystal deformation illustrated in Figure 13.3. One can show that $\psi \geq 0$ if and only if

$$K_1 \geq 0, \quad K_2 \geq 0, \quad K_3 \geq 0.$$

Warner and Terentjev (Section 2.5) use an expansion based on the order parameter (mentioned previously in Section 13.1 that measures the degree of orientational order) that to leading quadratic order gives $K_2 = K_3$. In this case one has the two-parameter Frank free energy

$$\rho\psi = \frac{1}{2}K_1(\mathrm{div}\,\mathbf{d})^2 + \frac{1}{2}K|\mathrm{curl}\,\mathbf{d}|^2 \tag{13.34}$$

where we have set $K = K_2 = K_3$. Observe that this is quadratic in $\mathbf{d}$. The one-parameter Frank free energy,

$$\rho\psi = \frac{1}{2}K|\mathrm{grad}\,\mathbf{d}|^2 \ + \ \text{a null lagrangian} \tag{13.35}$$

is obtained by setting $K = K_1 = K_2 = K_3$. The null lagrangian is a term that does not affect the Euler-Lagrange equations associated with ψ (and therefore has no effect on the equations of motion).

13.4.2 An Extra Stress $\boldsymbol{\tau}$.

As a specific example of a constitutive relation for the extra stress we take

$$\begin{aligned} \boldsymbol{\tau}(\mathbf{d}, \overset{o}{\mathbf{d}}, \mathbf{D}) \ = \ & \alpha_1\,(\mathbf{d}\cdot\mathbf{D}\mathbf{d})\mathbf{d}\otimes\mathbf{d} \ + \ \alpha_2\ \overset{o}{\mathbf{d}}\otimes\mathbf{d} \ + \ \alpha_3\,\mathbf{d}\otimes\overset{o}{\mathbf{d}} \ + \\ & + \alpha_4\,\mathbf{D} \ + \ \alpha_5\,\mathbf{D}\mathbf{d}\otimes\mathbf{d} \ + \ \alpha_6\,\mathbf{d}\otimes\mathbf{D}\mathbf{d}. \end{aligned} \tag{13.36}$$

Observe that the extra stress vanishes in equilibrium: $\boldsymbol{\tau}(\mathbf{d}, \mathbf{o}, \mathbf{0}) = \mathbf{0}$. It can be readily verified that this constitutive relation has the property $\boldsymbol{\tau}(\mathbf{Q}\mathbf{d}, \mathbf{Q}\overset{o}{\mathbf{d}}, \mathbf{Q}\mathbf{D}\mathbf{Q}^T) = \mathbf{Q}\boldsymbol{\tau}(\mathbf{d}, \overset{o}{\mathbf{d}}, \mathbf{D})\mathbf{Q}^T$ for all rotations $\mathbf{Q}$ as is required by material frame indifference.

The material constants $\alpha_1, \alpha_2, \ldots, \alpha_6$ are known as the Leslie viscosity coefficients. One can show that the dissipation inequality (13.31) holds if and only if (see Stewart)

$$\begin{aligned} \alpha_3 - \alpha_2 \ &\geq \ 0, \\ \alpha_4 \ &\geq \ 0, \\ 2\alpha_4 + \alpha_5 + \alpha_6 \ &\geq \ 0, \\ 2\alpha_1 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 \ &\geq \ 0, \\ 4(\alpha_3 - \alpha_2)(2\alpha_4 + \alpha_5 + \alpha_6) \ &\geq \ (\alpha_2 + \alpha_3 + \alpha_6 - \alpha_5)^2. \end{aligned} \tag{13.37}$$

13.5 Boundary Conditions: Anchoring.

In order to solve an initial-boundary-value problem we must supplement the system of partial differential equations arising from the field equations with appropriate boundary conditions.

If the forces between the container and the liquid crystal are very strong, the director will be oriented in some specific direction on its boundary. This can be achieved physically by carefully rubbing the surface of the container. In this case one says that there is *strong anchoring* at the boundary. Thus in the case of strong anchoring the director direction is prescribed on the boundary: $\mathbf{d}(\mathbf{y},t) = \mathbf{d}_0(\mathbf{y},t)$ for $\mathbf{y} \in \partial\mathcal{R}$ where $\mathbf{d}_0$ is given. For example $\mathbf{d}_0$ may be tangential to the boundary $\partial\mathcal{R}$; or perhaps normal to it.

In the case of *no anchoring* the boundary surface imposes no restrictions on the director direction. Mathematically, the appropriate boundary condition is then the natural boundary condition arising from a variational formulation; see Chapter 7 of Volume I, and also Problem 13.2 here.

Other possible boundary conditions include *weak anchoring* and *conical anchoring* as discussed, for example, by de Gennes and Prost.

13.6 Worked Examples and Exercises.

Problem 13.1. Specialize and discuss the angular momentum field equation $(13.32)_2$ when the extra stress $\boldsymbol{\tau}(\mathbf{d}, \overset{o}{\mathbf{d}}, \mathbf{D})$ and free energy $\psi(\mathbf{d}, \operatorname{grad}\mathbf{d})$ have the particular forms given in (13.36) and (13.33) respectively.

Solution: First substitute $(13.30)_2$ and $(13.30)_3$ into $(13.32)_2$ and simplify the result making use of the identity (13.22). This leads to

$$\rho c_i + e_{ijk}\tau_{kj} + e_{ipq}d_p\left[\frac{\partial}{\partial y_j}\left(\frac{\partial(\rho\psi)}{\partial G_{qj}}\right) - \frac{\partial(\rho\psi)}{\partial d_q}\right] = 0. \tag{13.38}$$

If we assume that the body couple has the form (13.9) (but not necessarily (13.10)), then (13.38) can be written as

$$e_{ijk}\tau_{kj} + e_{ipq}d_p\left[\frac{\partial}{\partial y_j}\left(\frac{\partial(\rho\psi)}{\partial G_{qj}}\right) - \frac{\partial(\rho\psi)}{\partial d_q} + g_q\right] = 0. \tag{13.39}$$

For the particular form (13.36) of the extra stress, one can verify that

$$e_{ipq}\tau_{qp} = e_{ipq}d_p(-\gamma_1 \overset{o}{d}_q - \gamma_2 D_{qr}d_r)$$

where we have set

$$\gamma_1 = \alpha_3 - \alpha_2, \qquad \gamma_2 = \alpha_6 - \alpha_5. \tag{13.40}$$

Therefore the angular momentum balance (13.39) can be written as

$$e_{ipq}d_p\left[-\gamma_1 \overset{o}{d}_q - \gamma_2 D_{qr}d_r + \frac{\partial}{\partial y_j}\left(\frac{\partial(\rho\psi)}{\partial G_{qj}}\right) - \frac{\partial(\rho\psi)}{\partial d_q} + g_q\right] = 0.$$

Since this has the form $e_{ipq}a_pb_q = 0$ or $\mathbf{a}\times\mathbf{b} = \mathbf{o}$, the vectors $\mathbf{a}$ and $\mathbf{b}$ must be parallel, i.e. $\mathbf{a} = \lambda\mathbf{b}$ for some scalar λ. Thus the preceding equation leads to

$$-\gamma_1 \overset{o}{d}_q - \gamma_2 D_{qr}d_r + \frac{\partial}{\partial y_j}\left(\frac{\partial(\rho\psi)}{\partial G_{qj}}\right) - \frac{\partial(\rho\psi)}{\partial d_q} + g_q = \lambda d_q \tag{13.41}$$

for some scalar field $\lambda(\mathbf{y}, t)$.

Equation (13.41) holds for any free energy $\psi(\mathbf{d}, \operatorname{grad}\mathbf{d})$. When ψ is the Frank free energy (13.33), a direct calculation gives the explicit formula

$$\frac{\partial}{\partial y_j}\left(\frac{\partial(\rho\psi)}{\partial G_{ij}}\right) - \frac{\partial(\rho\psi)}{\partial d_i} = (K_1-K_2)d_{j,ji} + K_2d_{i,jj} + (K_3-K_2)(d_jd_kd_{i,k})_{,j} - (K_3-K_2)d_jd_{k,j}d_{k,i}; \tag{13.42}$$

note that this reduces to

$$\frac{\partial}{\partial y_j}\left(\frac{\partial(\rho\psi)}{\partial G_{ij}}\right) - \frac{\partial(\rho\psi)}{\partial d_i} = K d_{i,jj} \tag{13.43}$$

in the one-parameter case $K = K_1 = K_2 = K_3$. Equation (13.43) can alternatively be derived directly from (13.35) confirming that the null lagrangian term in (13.35) does not contribute to the quantity represented by the lefthand side of (13.43).

In the special case of equilibrium, we have $\mathbf{D} = \mathbf{0}, \overset{o}{\mathbf{d}} = \dot{\mathbf{d}} - \mathbf{W}\mathbf{d} = \mathbf{o}$ and so the angular momentum equation (13.41) specializes to

$$\frac{\partial}{\partial y_j}\left(\frac{\partial(\rho\psi)}{\partial G_{qj}}\right) - \frac{\partial(\rho\psi)}{\partial d_q} + g_q = \lambda d_q. \tag{13.44}$$

The system of equations (13.44) together with $\mathbf{d}\cdot\mathbf{d} = 1$ is the complete set of differential equations governing the fields $\mathbf{d}(\mathbf{y}), \lambda(\mathbf{y})$. It may be observed that the equilibrium torque balance equation (13.44) is *not limited* to the special material characterized by (13.36) and (13.33). It holds for any free energy function $\psi(\mathbf{d}, \mathbf{G})$ provided only that the extra stress vanishes in equilibrium.

Problem 13.2. *Variational formulation of equilibrium problem:* Consider a liquid crystal body occupying a region $\mathcal{R}$. The director is specified, say $\mathbf{d}_0$, on a portion $\partial\mathcal{R}_{\text{anchoring}}$ of the boundary. A body couple field $\rho\mathbf{c} = \mathbf{d} \times \mathbf{g}$ acts on the body where $\mathbf{g}$ is conservative: i.e. there exists a potential $\beta(\mathbf{d})$ such that

$$\mathbf{g} = -\frac{\partial\beta}{\partial\mathbf{d}}.$$

For example, in the case of body couples induced by an electric field $\mathbf{E}$, one has

$$\beta(\mathbf{d}) = -\frac{1}{2}\epsilon_0\chi(\mathbf{E}\cdot\mathbf{d})^2 \quad \text{and therefore} \quad \mathbf{g} = \epsilon_0\chi(\mathbf{E}\cdot\mathbf{d})\mathbf{E}.$$

Consider the potential energy of the system defined by

$$\overline{\Phi} \;=\; \int_{\mathcal{R}} \Big(\psi(\mathbf{d},\mathbf{G}) \;+\; \beta(\mathbf{d})\Big) dV \tag{13.45}$$

where $\mathbf{G} = \operatorname{grad}\mathbf{d}$. The functional here is rather subtle in that we have two unknown vector fields $\mathbf{y}(\mathbf{x})$ and $\mathbf{d}(\mathbf{y})$ involved. We shall consider a limited problem statement where we treat $\mathbf{y}(\mathbf{x})$ as known. Thus we seek to minimize the functional (13.45) over the set of all unit vector fields $\mathbf{d}(\mathbf{y})$ which obey $\mathbf{d}(\mathbf{y}) = \mathbf{d}_0(\mathbf{y})$ on $\partial\mathcal{R}_{\text{anchoring}}$.

The requirement that $\mathbf{d}$ be a unit vector is a constraint that can be relaxed by introducing a lagrange multiplier $\lambda(\mathbf{y})$. Accordingly we consider the modified functional

$$\Phi\{\mathbf{d}\} \;=\; \int_{\mathcal{R}} \Big(\psi(\mathbf{d},\mathbf{G}) \;+\; \beta(\mathbf{d}) - \frac{1}{2}\lambda\,\mathbf{d}\cdot\mathbf{d}\Big) dV, \qquad \mathbf{G} = \operatorname{grad}\mathbf{d},$$

which is to be minimized over the set of all vector fields $\mathbf{d}(\mathbf{y})$ that obey the prescribed boundary condition on $\partial\mathcal{R}_{\text{anchoring}}$.

Proceeding in the usual way we calculate the first variation of Φ:

$$\delta\Phi\{\mathbf{d}\} \;=\; \int_{\mathcal{R}} \left(\frac{\partial\psi}{\partial d_i}\,\delta d_i + \frac{\partial\psi}{\partial G_{ij}}\,\delta d_{i,j} - g_i\delta d_i - \lambda d_i \delta d_i\right) dV$$

where we have used the fact that $\partial\beta/\partial\mathbf{d} = -\mathbf{g}$. By rewriting the second term we get

$$\delta\Phi\{\mathbf{d}\} \;=\; \int_{\mathcal{R}} \left(\frac{\partial\psi}{\partial d_i}\,\delta d_i + \left(\frac{\partial\psi}{\partial G_{ij}}\,\delta d_i\right)_{,j} - \frac{\partial}{\partial y_j}\left(\frac{\partial\psi}{\partial G_{ij}}\right)\delta d_i - g_i\delta d_i - \lambda d_i\delta d_i\right) dV$$

which after using the divergence theorem leads to

$$\delta\Phi\{\mathbf{d}\} \;=\; \int_{\mathcal{R}} \left[\frac{\partial\psi}{\partial d_i} - \frac{\partial}{\partial y_j}\left(\frac{\partial\psi}{\partial G_{ij}}\right) - g_i - \lambda d_i\right] \delta d_i \, dV \;+\; \int_{\partial\mathcal{R}} \frac{\partial\psi}{\partial G_{ij}}\, n_j \delta d_i \, dA.$$

Since $\delta\Phi\{\mathbf{d}\} = 0$ for all admissible variations δd_i this leads to the Euler Lagrange equation

$$\frac{\partial\psi}{\partial d_i} - \frac{\partial}{\partial y_j}\left(\frac{\partial\psi}{\partial G_{ij}}\right) - g_i - \lambda d_i \;=\; 0 \quad \text{on } \mathcal{R} \tag{13.46}$$

and the natural boundary condition

$$\frac{\partial\psi}{\partial G_{ij}}\, n_j = 0 \qquad \text{on } \partial\mathcal{R} - \partial\mathcal{R}_{\text{anchoring}}. \tag{13.47}$$

Thus the director field $\mathbf{d}(\mathbf{y})$ (and the Lagrange multiplier $\lambda(\mathbf{y})$) are found by solving the boundary value problem comprised of (13.46), (13.47) together with

$$\mathbf{d} = \mathbf{d}_0 \qquad \text{on } \partial\mathcal{R}_{\text{anchoring}}, \tag{13.48}$$

$$|\mathbf{d}| = 1 \qquad \text{on } \mathcal{R}. \tag{13.49}$$

Compare the Euler-Lagrange equation (13.46) with the torque equilibrium equation (13.44).

Problem 13.3. *An equilibrium problem in the absence of external forcing.* Consider a y_1, y_2, y_3- cartesian coordinate system. Suppose there are two infinite plates at $y_3 = 0$ and $y_3 = L$ and that the space between them is filled with a nematic liquid crystalline material. The plate surfaces are treated such that the director is anchored to each plate as follows:

$$\mathbf{d} = \mathbf{e}_1 \quad \text{at } y_3 = 0, \qquad \mathbf{d} = \mathbf{e}_2 \quad \text{at } y_3 = L.$$

Thus as y_3 increases from 0 to L the director changes its orientation from $\mathbf{e}_1$ to $\mathbf{e}_2$. Determine the director orientation throughout the body.

Solution: Based on the problem description it is natural to assume that the director always lies in the y_1, y_2-plane; and moreover, that it is independent of the y_1- and y_2-coordinates. Then, since $\mathbf{d}$ is a unit vector, we can write

$$\mathbf{d} = \cos\phi(y_3)\, \mathbf{e}_1 + \sin\phi(y_3)\, \mathbf{e}_2, \tag{i}$$

where the function $\phi(y_3)$ which characterizes the orientation of the director is to be determined. The prescribed anchoring boundary conditions tell us that

$$\phi(0) = 0, \qquad \phi(L) = \pi/2.$$

Evaluating (13.42) for the particular director field (i), and substituting the result into the field equation (13.44) leads to the pair of differential equations

$$\phi''(y_3) = 0, \quad \lambda = -\big(\phi'(y_3)\big)^2, \qquad 0 < y_3 < L.$$

Integrating the first of these and using the boundary conditions leads to the desired result

$$\phi(y_3) = \frac{\pi\, y_3}{2L}. \tag{ii}$$

Suppose we wish to calculate the contact couple acting on the lower plate. Evaluating the constitutive relation (13.30) for the couple stress $\mathbf{Z}$ in the case of the Frank free energy function (13.33), and then applying it to the particular director field (i) leads to

$$\mathbf{Z} = -K_2(\mathbf{d}\cdot \operatorname{curl}\mathbf{d})(\mathbf{I} - \mathbf{d}\otimes\mathbf{d})$$

or

$$[Z] = K_2 \begin{pmatrix} -\sin^2\phi & \sin\phi\cos\phi & 0 \\ \sin\phi\cos\phi & -\cos^2\phi & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Thus the contact couple $\mathbf{m}$ acting *on* the lower plate $y_3 = 0$ is

$$\{m\} = [Z]\{n\} = -K_2 \begin{pmatrix} -\sin^2\phi & \sin\phi\cos\phi & 0 \\ \sin\phi\cos\phi & -\cos^2\phi & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{Bmatrix} 0 \\ 0 \\ -1 \end{Bmatrix} = K_2 \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}.$$

As one might expect, $\mathbf{m}$ has only a $\mathbf{e}_3$-component.

Remark: Alternatively one can use a variational approach to derive (ii) as follows. One readily finds from (i) that

$$\operatorname{curl}\mathbf{d} = -\phi'(y_3)\,\mathbf{d},$$

and therefore that $\mathbf{d}\cdot\operatorname{curl}\mathbf{d} = -\phi'$ and $\mathbf{d}\times\operatorname{curl}\mathbf{d} = \mathbf{o}$. Thus the Frank free energy (13.33) reduces to

$$\psi = \frac{1}{2}K_2\big(\phi'\big)^2.$$

The total potential energy is therefore

$$\Phi\{\phi\} = \int_0^L \frac{1}{2}K_2\big(\phi'\big)^2\, dy_3.$$

Minimizing this with respect to all variations $\delta\phi$ with $\delta\phi(0) = \delta(L) = 0$ leads to the differential equation

$$\phi''(y_3) = 0, \qquad 0 < y_3 < L.$$

Problem 13.4. *An equilibrium problem in the presence of an electric field.* Consider a y_1, y_2, y_3- cartesian coordinate system. Suppose there are two infinite plates at $y_3 = 0$ and $y_3 = L$ and that the space between them is filled with a nematic liquid crystalline material. The plate surfaces are treated so that the director is anchored at each plate as follows:

$$\mathbf{d} = \mathbf{e}_1 \quad \text{at } y_3 = 0, \qquad \mathbf{d} = \mathbf{e}_1 \quad \text{at } y_3 = L. \tag{i}$$

Thus the director is anchored in the same direction $\mathbf{e}_1$ at both plates. In the absence of any external forcing, the director would therefore be in the $\mathbf{e}_1$ direction everywhere. Suppose there is an external electric field $\mathbf{E} = E\mathbf{e}_2$. Thus, if not for the boundary conditions, the director would align with the electric field and be in the $\mathbf{e}_2$ direction everywhere. The boundary condition (i) opposes this. We are asked to determine the director field.

Solution: Based on the problem description, we assume that at all locations the director lies in the y_1, y_2-plane; and moreover, that it is independent of the y_1 and y_2 coordinates: $\mathbf{d}(\mathbf{y}) = \mathbf{d}(z)$ where it is convenient to set $y_3 = z$. Accordingly, since $\mathbf{d}$ is a unit vector, we can write

$$\mathbf{d} = \cos\phi(z)\,\mathbf{e}_1 + \sin\phi(z)\,\mathbf{e}_2. \tag{ii}$$

The angular distribution of the director $\phi(z)$ is to be determined. Specializing the expression for the body couple induced by an electric field given in Section 13.2 to the current setting leads to

$$\rho\mathbf{c} = \mathbf{d} \times \mathbf{g}, \qquad \mathbf{g} = \epsilon_0 \chi E^2 \sin\phi\,\mathbf{e}_2. \tag{iii}$$

We differentiate (ii) and calculate the components $d_{i,j}$. We then evaluate the righthand side of (13.42). Substituting the result, together with (iii), into (13.44) leads to the pair of differential equations

$$\left.\begin{array}{rcl} -\quad K_2 \sin\phi\,\phi'' - K_2\cos\phi\,(\phi')^2 & = & \lambda\cos\phi, \\[2ex] K_2\,\cos\phi\,\phi'' - K_2\sin\phi\,(\phi')^2 + \epsilon_0\chi E^2\sin\phi & = & \lambda\sin\phi. \end{array}\right\}$$

Eliminating the lagrange multiplier λ from these equations leads to

$$K_2\phi'' + \epsilon_0\chi E^2 \cos\phi\sin\phi = 0, \qquad 0 < z < L. \tag{iv}$$

From (i) and (ii) we have the associated boundary conditions

$$\phi(0) = \phi(L) = 0. \tag{v}$$

Note that $(iv), (v)$ is an eigenvalue problem. Clearly, $\phi(z) = 0$ is one solution of the problem. For certain values of the field strength E (eigenvalues) there may be other solutions (eigenfunctions). Perhaps the two questions of greatest interest are determining whether the fundamental solution $\phi(z) = 0$ is stable for some range of field strengths $0 < E < E_{\text{crit}}$ and finding E_{crit}. We shall address this next.

The need to examine stability suggest that we turn to a variational formulation of the problem. For the admissible set of director fields we take those that have the form $\mathbf{d} = \cos\phi(z)\,\mathbf{e}_1 + \sin\phi(z)\,\mathbf{e}_2$ where $\phi(z)$ is an arbitrary smooth function except for satisfying the boundary conditions $\phi(0) = \phi(L) = 0$. Note that $\text{curl}\,\mathbf{d} = -\phi'(z)\,\mathbf{d}$, and therefore that $\mathbf{d}\cdot\text{curl}\,\mathbf{d} = -\phi'$ and $\mathbf{d}\times\text{curl}\,\mathbf{d} = \mathbf{o}$. Thus the Frank free energy (13.33) reduces to

$$\psi = \frac{1}{2}K_2\big(\phi'\big)^2.$$

The energy density of the electric field is

$$\beta(\mathbf{d}) = -\frac{1}{2}\epsilon_0\chi(\mathbf{E}\cdot\mathbf{d})^2 = -\frac{1}{2}\epsilon_0\chi(E\mathbf{e}_2\cdot\mathbf{d})^2 = -\frac{1}{2}\epsilon_0\chi E^2\sin^2\phi.$$

Thus the total potential energy is given by

$$\Phi\{\phi\} = \int_0^L \left[\frac{1}{2}K_2(\phi')^2 - \frac{1}{2}\epsilon_0\chi E^2\sin^2\phi\right]dz \tag{vi}$$

and is defined for all smooth functions $\phi(z)$ with $\phi(0) = \phi(L) = 0$.

It is convenient to let $\phi_0(z)$ denote the fundamental solution, so that in the present problem $\phi_0(z) = 0,\ 0 < z < L$. To examine the stability of ϕ_0 we calculate the energy difference between this solution and an arbitrary neighboring function. Thus we calculate $\Phi\{\phi\} - \Phi\{\phi_0\}$ for arbitrary functions ϕ which are close to ϕ_0:

$$\Phi\{\phi\} - \Phi\{\phi_0\} = \int_0^L \left[\frac{1}{2}K_2(\phi')^2 - \frac{1}{2}\epsilon_0\chi E^2\phi^2\right]dz \tag{vii}$$

where we have used the fact that $|\phi - \phi_0| = |\phi| << 1$ to replace $\sin\phi$ by ϕ. Consider functions ϕ which have the Fourier representation

$$\phi(z) = \sum_{n=1}^{\infty}\phi_n\,\sin\frac{n\pi z}{L}. \tag{viii}$$

Observe that all such functions satisfy the prescribed anchoring conditions $\phi(0) = \phi(L) = 0$. Since

$$\int_0^L \sin\frac{n\pi z}{L}\sin\frac{m\pi z}{L}\,dz = \int_0^L \cos\frac{n\pi z}{L}\cos\frac{m\pi z}{L}\,dz = \begin{cases} 0, & n \neq m, \\ L/2, & n = m, \end{cases}$$

it follows that

$$\int_0^L [\phi'(y)]^2\,dy = \frac{L}{2}\sum_{n=1}^{\infty}\left(\frac{n\pi\phi_n}{L}\right)^2, \qquad \int_0^L \phi^2(y)\,dy = \frac{L}{2}\sum_{n=1}^{\infty}(\phi_n)^2.$$

Therefore when the functional (vii) is evaluated at the function $(viii)$ we get

$$\Phi\{\phi\} - \Phi\{\phi_0\} = \frac{L}{4}\sum_{n=1}^{\infty}(\phi_n)^2\left(K_2\left[\frac{\pi n}{L}\right]^2 - \epsilon_0\chi E^2\right).$$

If the fundamental solution is to be stable we must have $\Phi\{\phi\} - \Phi\{\phi_0\} > 0$. For this to hold the coefficient of $(\phi_n)^2$ in the preceding equation must be positive for each n since otherwise, one can always choose a set of parameters ϕ_n such that $\Phi\{\phi\} - \Phi\{\phi_0\} < 0$. Therefore we must have

$$K_2\left[\frac{\pi n}{L}\right]^2 > \epsilon_0\chi E^2 \qquad \text{for all } n = 1, 2, \ldots$$

from which we see that the tightest bound on E is obtained by taking $n = 1$. Consequently

$$E_{\rm crit} = \frac{\pi}{L}\left(\frac{K_2}{\epsilon_0\chi}\right)^{1/2}.$$

Thus the fundamental solution $\phi(z) = 0$ is stable for $E < E_{\rm crit}$ and unstable for $E > E_{\rm crit}$, or equivalently, the director orientation will be $\mathbf{d} = \mathbf{e}_1$ for $E < E_{\rm crit}$ and differ from $\mathbf{e}_1$ for $E > E_{\rm crit}$. This transition in behavior at $E = E_{\rm crit}$ is referred to as the Freedericksz transition. The director configuration at $E = E_{\rm crit}$ is given by the corresponding eigenfunction.

Problem 13.5. Define and calculate some notion of "effective shear viscosity" of a liquid crystal.

Solution: Consider the uniform steady shear flow

$$\mathbf{v}(\mathbf{y}) = k(\mathbf{e}_1 \otimes \mathbf{e}_3)\mathbf{y} = ky_3\mathbf{e}_1 \tag{13.50}$$

with respect to a fixed orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Note that the particle velocity is in the $\mathbf{e}_1$ direction and its magnitude increases linearly with y_2; moreover, observe that $\dot{\mathbf{v}} = \mathbf{o}$ and $\operatorname{grad}\mathbf{v} =$ constant. In addition, suppose that

$$\operatorname{grad}\mathbf{d} = \mathbf{0}, \qquad \dot{\mathbf{d}} = \mathbf{o}$$

so that the director field $\mathbf{d}(\mathbf{y}, t)$ is uniform and time independent. We define the ratio between the shear stress T_{13} and shear rate k in this motion to be the shear viscosity.

We shall not address the question of whether such a motion is possible, i.e. we do not examine whether this motion is consistent with the field equations. This question is addressed in Problem 13.6.

Since the director field is spatially uniform $\operatorname{grad} \mathbf{d}$ vanishes and equations $(13.30)_2$ and (13.36) give the Cauchy stress to be

$$\mathbf{T} = -p\mathbf{I} + \boldsymbol{\tau} \tag{13.51}$$

where

$$\begin{aligned}\boldsymbol{\tau}(\mathbf{d}, \overset{o}{\mathbf{d}}, \mathbf{D}) \;=\;& \alpha_1 (\mathbf{d} \cdot \mathbf{D}\mathbf{d})\mathbf{d} \otimes \mathbf{d} \;+\; \alpha_2 \;\overset{o}{\mathbf{d}} \otimes \mathbf{d} \;+\; \alpha_3 \, \mathbf{d} \otimes \overset{o}{\mathbf{d}} \;+ \\ &+ \alpha_4 \mathbf{D} \;+\; \alpha_5 \mathbf{D}\mathbf{d} \otimes \mathbf{d} \;+\; \alpha_6 \, \mathbf{d} \otimes \mathbf{D}\mathbf{d}.\end{aligned} \tag{13.52}$$

Observe from (13.52) that if $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_5 = \alpha_6 = 0$, then (13.51) simplifies to the Newtonian constitutive relation

$$\mathbf{T} = -p\mathbf{I} + \alpha_4 \mathbf{D}$$

with viscosity $\alpha_4/2$.

Since $\dot{\mathbf{d}} = \mathbf{o}$ it follows from (13.5) that the co-rotational derivative of $\mathbf{d}$ is

$$\overset{o}{\mathbf{d}} = \; -\mathbf{W}\mathbf{d}. \tag{13.53}$$

The stretching and spin tensors associated with the velocity field (13.50) are

$$\mathbf{D} = \frac{k}{2}(\mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_1), \qquad \mathbf{W} = \frac{k}{2}(\mathbf{e}_1 \otimes \mathbf{e}_3 - \mathbf{e}_3 \otimes \mathbf{e}_1). \tag{13.54}$$

Substituting (13.53) and (13.54) into (13.52), and the result into (13.51), leads to

$$T_{13} = \frac{\alpha_4}{2}\, k \qquad \text{if it so happens that } \mathbf{d} = \mathbf{e}_2.$$

Thus if the director is perpendicular to the y_1, y_3-plane it has no effect on the relation between the shear stress and the shear rate. The effective viscosity is $\alpha_4/2$.

However if the director lies in the y_1, y_3-plane, i.e. if $\mathbf{d} \cdot \mathbf{e}_2 = 0$, then (13.51) - (13.54) leads to

$$T_{13} = \frac{1}{2}\Big(\alpha_4 + 2\alpha_1 d_1^2 d_3^2 + (\alpha_3 + \alpha_6) d_1^2 + (\alpha_5 - \alpha_2) d_3^2\Big) k$$

so that now the effective viscosity is $(\alpha_4 + 2\alpha_1 d_1^2 d_2^2 + (\alpha_3 + \alpha_6) d_1^2 + (\alpha_5 - \alpha_2) d_2^2)/2$ where the components d_1 and d_3 specify the orientation of the director with respect to the $\mathbf{e}_1$ and $\mathbf{e}_3$ directions and the α's are the Leslie viscosities.

Problem 13.6. *Elementary dynamical problem* [Stewart] Consider an infinite medium occupied by a nematic liquid crystal characterized by the constitutive relations (13.30), (13.33) and (13.36). With respect to some orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, the body undergoes a simple shearing flow

$$\mathbf{v}(\mathbf{y}) = ky_3\,\mathbf{e}_1 \qquad (i)$$

where $k > 0$ is a constant. Assume that the director field $\mathbf{d}(\mathbf{y}, t)$ is spatially uniform and that the director undergoes a steady motion. Determine the motion of the director, and since you will find multiple possible motions, examine their stability.

Solution: Since the director field is spatially uniform, using spherical polar coordinates we can describe the director by

$$\mathbf{d}(t) = \sum_{n=1}^{3} d_i\,\mathbf{e}_i = \cos\theta(t)\cos\phi(t)\,\mathbf{e}_1 + \cos\theta(t)\sin\phi(t)\,\mathbf{e}_2 + \sin\theta(t)\,\mathbf{e}_3. \qquad (ii)$$

Though we only seek steady motions (where $\dot{\theta} = \dot{\phi} = 0$), we shall not restrict our attention to such motions since we must examine the stability of the steady motions determined.

We start from the simplified form of the angular momentum equation (13.41). Note first that since $\mathbf{d}$ is spatially uniform in the present problem, all of its spatial derivatives $d_{i,j}$ vanish. Thus the righthand side of (13.42) is zero. Using this to simplify (13.41) leads to

$$\gamma_1 \overset{o}{\mathbf{d}} + \gamma_2 \mathbf{D}\mathbf{d} = -\lambda\mathbf{d}$$

where we have set $\mathbf{g} = \mathbf{o}$ since we assume there are no body couples and have set

$$\gamma_1 = \alpha_3 - \alpha_2, \qquad \gamma_2 = \alpha_6 - \alpha_5.$$

Taking the scalar product of this with $\mathbf{d}$ and recalling that $\overset{o}{\mathbf{d}}$ is perpendicular to $\mathbf{d}$ gives

$$\lambda = -\gamma_2\,\mathbf{D}\mathbf{d}\cdot\mathbf{d}.$$

Substituting this value of λ back into the preceding equation gives

$$\gamma_1 \overset{o}{\mathbf{d}} + \gamma_2 \mathbf{D}\mathbf{d} = \gamma_2(\mathbf{D}\mathbf{d}\cdot\mathbf{d})\mathbf{d}. \qquad (iv)$$

We note that because of the circular nature of the preceding calculation, in component form, equation (*iv*) leads to only two independent scalar equations. Finally, since $\overset{o}{\mathbf{d}} = \dot{\mathbf{d}} - \mathbf{W}\mathbf{d}$ we are led to

$$\gamma_1\dot{\mathbf{d}} = \gamma_2(\mathbf{D}\mathbf{d}\cdot\mathbf{d})\mathbf{d} - \gamma_2\mathbf{D}\mathbf{d} + \gamma_1\mathbf{W}\mathbf{d}. \qquad (\star)$$

The stretching and spin tensors associated with the simple shearing motion (i) are

$$\mathbf{D} = \frac{k}{2}\Big(\mathbf{e}_1\otimes\mathbf{e}_3 + \mathbf{e}_3\otimes\mathbf{e}_1\Big), \qquad \mathbf{W} = \frac{k}{2}\Big(\mathbf{e}_1\otimes\mathbf{e}_3 - \mathbf{e}_3\otimes\mathbf{e}_1\Big). \tag{v}$$

Because of (ii) and (v) we have

$$\mathbf{Dd} = \frac{k}{2}(d_3\mathbf{e}_1 + d_1\mathbf{e}_3), \qquad \mathbf{Wd} = \frac{k}{2}(d_3\mathbf{e}_1 - d_1\mathbf{e}_3), \qquad \mathbf{Dd}\cdot\mathbf{d} = kd_1d_3,$$

where the d_i's are the components of the director as given in (ii). Equation $(\star)$ can therefore be written explicitly as

$$\begin{aligned}\gamma_1\dot{d}_1\mathbf{e}_1 \quad &+\gamma_1\dot{d}_2\mathbf{e}_2 + \gamma_1\dot{d}_3\mathbf{e}_3 = \\ &= (\gamma_2 k d_1 d_3)(d_1\mathbf{e}_1 + d_2\mathbf{e}_2 + d_3\mathbf{e}_3) - \gamma_2\tfrac{k}{2}(d_3\mathbf{e}_1 + d_1\mathbf{e}_3) + \gamma_1\tfrac{k}{2}(d_3\mathbf{e}_1 - d_1\mathbf{e}_3),\end{aligned}$$

or as the three scalar equations

$$\left.\begin{aligned}\gamma_1\dot{d}_1 &= \frac{k}{2}\Big(2\gamma_2 d_1^2 d_3 - \gamma_2 d_3 + \gamma_1 d_3\Big),\\ \gamma_1\dot{d}_2 &= \frac{k}{2}\Big(2\gamma_2 d_1 d_2 d_3\Big),\\ \gamma_1\dot{d}_3 &= \frac{k}{2}\Big(2\gamma_2 d_1 d_3^2 - \gamma_2 d_1 - \gamma_1 d_1\Big).\end{aligned}\right\}$$

Finally since

$$d_1 = \cos\theta\cos\phi, \qquad d_2 = \cos\theta\sin\phi, \qquad d_3 = \sin\theta,$$

these equations can be simplified, leading to the pair of ordinary differential equations

$$\left.\begin{aligned}\gamma_1\,\dot\theta \quad &= \quad \frac{k}{2}\Big[(\gamma_2-\gamma_1)\sin^2\theta - (\gamma_2+\gamma_1)\cos^2\theta\Big]\cos\phi,\\ \gamma_1\cos\theta\,\dot\phi \quad &= \quad \frac{k}{2}(\gamma_2-\gamma_1)\sin\theta\sin\phi,\end{aligned}\right\}$$

where $\gamma_1 = \alpha_3 - \alpha_2$, $\gamma_2 = \alpha_6 - \alpha_5$. For algebraic simplicity we assume from hereon that[4] $\alpha_6 - \alpha_5 = \alpha_2 + \alpha_3$ whence we have

$$\left.\begin{aligned}\gamma_1\,\dot\theta \quad &= \quad k(\alpha_2\sin^2\theta - \alpha_3\cos^2\theta)\cos\phi,\\ \gamma_1\cos\theta\,\dot\phi \quad &= \quad k\alpha_2\sin\theta\sin\phi.\end{aligned}\right\} \tag{vi}$$

[4]There are reasons beyond mathematical simplicity to use this relation. It can be motivated by the Onsager relations, and in the liquid crystal context is known as the Parodi relation; see for example Stewart.

Recall from (13.37) that the dissipation inequality requires $\alpha_3 \geq \alpha_2$. We shall assume the strict inequality

$$\alpha_3 > \alpha_2. \tag{vii}$$

Steady motions: We now seek solutions for which $\dot{\theta}(t) = \dot{\phi}(t) = 0$. From the second of (vi) we find that necessarily $\sin\theta \sin\phi = 0$, and therefore either $\theta(t) = 0$ or $\phi(t) = 0$. In the former case the first of (vi) requires that $\phi(t) = \pi/2$, whereas in the latter case it requires that $\theta(t) = \pm\theta_0$ where θ_0 is found from $\tan^2\theta_0 = \alpha_3/\alpha_2$. Thus we have two steady motions:

$$\left.\begin{array}{lll} \text{Steady motion 1}: & \theta(t) = 0, & \phi(t) = \pi/2, \\[2ex] \text{Steady motion 2}: & \theta(t) = \pm\theta_0, & \phi(t) = 0, \end{array}\right\} \tag{viii}$$

where $\theta_0 \in [0, \pi/2]$ is given by

$$\tan\theta_0 = \sqrt{\alpha_3/\alpha_2}. \tag{ix}$$

Note that for the second solution to exist we must have $\alpha_3/\alpha_2 \geq 0$. When discussing the second solution we shall assume that the strict inequality

$$\alpha_2\alpha_3 > 0 \tag{x}$$

holds. Thus for the second solution we have two cases to consider: $\alpha_3 > \alpha_2 > 0$ and $0 > \alpha_3 > \alpha_2$. Observe that $0 < \theta_0 < \pi/4$ for $0 > \alpha_3 > \alpha_2$ and $\pi/4 < \theta_0 < \pi/2$ for $\alpha_3 > \alpha_2 > 0$.

In the first steady motion the director is aligned with $\mathbf{e}_2$ and so is perpendicular to the y_1, y_3-plane (and in particular to the particle velocity field $\mathbf{v}$). In the second steady motion the director lies in the y_1, y_3-plane and is inclined at an angle θ_0 to the particle velocity; see Figure 13.2.

We now examine the stability of these two steady motions. Consider the first steady motion. To examine its stability we consider a perturbed motion close to it,

$$\theta(t) = u(t), \qquad \phi(t) = \pi/2 + v(t), \tag{xi}$$

where the perturbations are small $|u|, |v| << 1$. Substituting (xi) into (vi) and linearizing for small u, v leads to the following pair of linear differential equations:

$$\left.\begin{array}{l} \gamma_1\,\dot{u} = k\alpha_3\,v, \\[2ex] \gamma_1\,\dot{v} = k\alpha_2\,u. \end{array}\right\}$$

By seeking solutions of the form $u(t) = u_0 \exp \mu t, v(t) = v_0 \exp \mu t$ and examining the sign of μ, we find that solutions grow exponentially if $\alpha_2\alpha_3 > 0$ and do know grow if $\alpha_2\alpha_3 < 0$; in the latter case the perturbed solution oscillates about the steady solution without decaying towards it. Therefore the first steady solution is unstable if $\alpha_2\alpha_3 > 0$ and "stable" if $\alpha_2\alpha_3 < 0$.

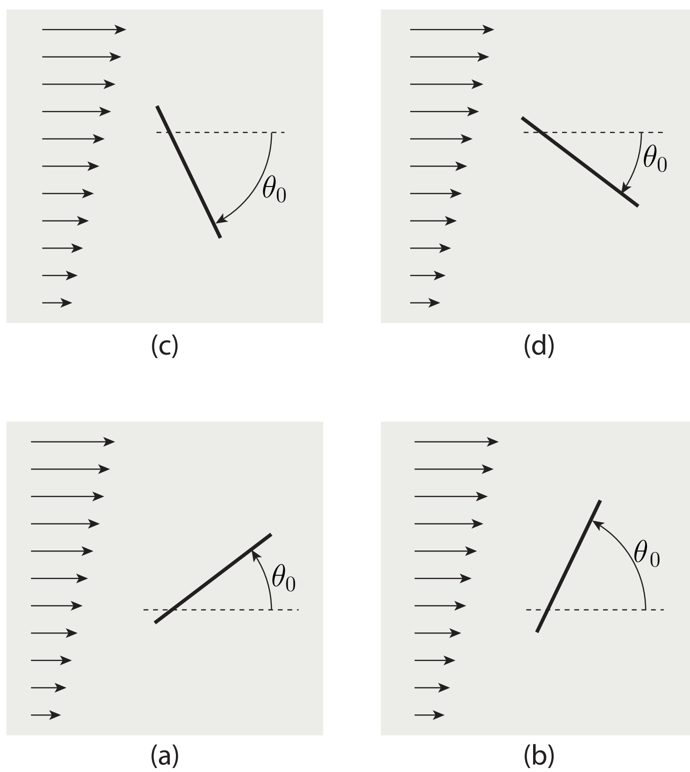

Figure 13.2: Four solution of second type: (a) Stable solution $\theta(t) = +\theta_0$. (Case $\alpha_2 < \alpha_3 < 0$ whence $0 < \theta_0 < \pi/4$.) (b) Unstable solution $\theta(t) = +\theta_0$. (Case $\alpha_3 > \alpha_2 > 0$ whence $\pi/4 < \theta_0 < \pi/2$.) (c) Stable solution $\theta(t) = -\theta_0$. (Case $\alpha_3 > \alpha_2 > 0$ whence $\pi/4 < \theta_0 < \pi/2$.) (d) Unstable solution $\theta(t) = -\theta_0$. (Case $\alpha_2 < \alpha_3 < 0$ whence $0 < \theta_0 < \pi/4$.)

Consider the second steady motion. We again consider perturbed motions close to it,

$$\theta(t) = \pm\theta_o + u(t), \qquad \phi(t) = v(t), \tag{xii}$$

where the perturbations are small $|u|, |v| << 1$. Substituting (xii) into (vi) and linearizing for small u, v leads to the following pair of linear differential equations:

$$\left.\begin{aligned} \gamma_1 \, \dot{u} &= \pm \frac{k}{2}(\alpha_2 + \alpha_3) \sin(2\theta_0) \, u, \\ \gamma_1 \cos\theta_0 \, \dot{v} &= \pm \, k\alpha_2 \sin(\theta_0) \, v. \end{aligned}\right\}$$

By seeking solutions of the form $u(t) = u_0 \exp \mu_1 t, v(t) = v_0 \exp \mu_2 t$ and examining the signs of the μ's, we find (keeping (vii) and (x) in mind) that for the case $+\theta_0$, that solutions grow exponentially if $\alpha_3 > \alpha_2 > 0$ and they decay to zero if $\alpha_2 < \alpha_3 < 0$. On the other hand for the case $-\theta_0$, solutions grow exponentially for $\alpha_2 < \alpha_3 < 0$ and decay to zero for $\alpha_3 > \alpha_2 > 0$. These are illustrated in Figure 13.2.

Problem 13.7. *Rotational inertia.* Consider the balance laws

$$\left.\begin{aligned} \tfrac{d}{dt} \int_{\mathcal{D}_t} \rho \mathbf{v} \, \mathrm{d}V_y &= \int_{\partial\mathcal{D}_t} \mathbf{t} \, \mathrm{d}A_y + \int_{\mathcal{D}_t} \rho \mathbf{b} \, \mathrm{d}V_y, \\ \tfrac{d}{dt} \int_{\mathcal{D}_t} \Big(\mathbf{y} \times \rho \dot{\mathbf{y}} \, + \, \mathbf{d} \times \sigma \dot{\mathbf{d}} \Big) \, \mathrm{d}V_y &= \int_{\partial\mathcal{D}_t} \mathbf{y} \times \mathbf{t} \, \mathrm{d}A_y + \int_{\mathcal{D}_t} \mathbf{y} \times \rho \mathbf{b} \, \mathrm{d}V_y + \\ &\quad + \int_{\partial\mathcal{D}_t} \mathbf{m} \, \mathrm{d}A_y + \int_{\mathcal{D}_t} \rho \mathbf{c} \, \mathrm{d}V_y, \end{aligned}\right\}$$

and the dissipation rate

$$\begin{aligned} \int_{\mathcal{D}_t} \rho \mathsf{D} \, dV_y &= \int_{\partial\mathcal{D}_t} \mathbf{t} \cdot \mathbf{v} \, \mathrm{d}A_y + \int_{\mathcal{D}_t} \rho \mathbf{b} \cdot \mathbf{v} \, \mathrm{d}V_y + \int_{\partial\mathcal{D}_t} \mathbf{m} \cdot \mathbf{w} \, \mathrm{d}A_y + \int_{\mathcal{D}_t} \rho \mathbf{c} \cdot \mathbf{w} \, \mathrm{d}V_y - \\ &\quad - \frac{d}{dt} \int_{\mathcal{D}_t} \rho \psi \, dV_y \; - \; \frac{d}{dt} \int_{\mathcal{D}_t} \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} \, dV_y \; - \; \frac{d}{dt} \int_{\mathcal{D}_t} \frac{1}{2} \sigma \dot{\mathbf{d}} \cdot \dot{\mathbf{d}} \, dV_y \end{aligned}$$

where we have included the effects of director rotational inertia via the terms involving $\sigma (\neq 0)$. Derive the field equations associated with the balance laws and an expression for the dissipation rate D.

Problem 13.8. In Section 12.2 we used the dissipation inequality to show that the extra stress $\boldsymbol{\tau}$ in a viscous fluid had to vanish in equilibrium. Can we do the same here? i.e. can we use (13.31) to show that

$$\boldsymbol{\tau}(\mathbf{d}, \overset{o}{\mathbf{d}}, \mathbf{D}) = \mathbf{0} \qquad \text{at} \quad \mathbf{D} = \mathbf{0}, \; \overset{o}{\mathbf{d}} = \mathbf{o} \; ?$$

Solution: Recall that $\overset{o}{\mathbf{d}} = \boldsymbol{\omega} \times \mathbf{d}$. Let $\mathbf{K}$ be the unique skew-symmetric tensor whose axial vector is $\boldsymbol{\omega}$:

$$K_{ik} = e_{ijk}\omega_j$$

and

$$\overset{o}{\mathbf{d}} = \mathbf{K}\mathbf{d}.$$

If $\mathbf{K}$ is non-singular then there is a one-to-one relation between the pairs $\{\mathbf{K}, \mathbf{d}\}$ and $\{\overset{o}{\mathbf{d}}, \mathbf{d}\}$. If this is the case then there is a function $\overline{\boldsymbol{\tau}}$ such that $\boldsymbol{\tau}(\mathbf{d}, \overset{o}{\mathbf{d}}, \mathbf{D}) = \overline{\boldsymbol{\tau}}(\mathbf{d}, \mathbf{K}, \mathbf{D})$. The dissipation inequality

$$\boldsymbol{\tau}(\mathbf{d}, \overset{o}{\mathbf{d}}, \mathbf{D}) \cdot \mathbf{D} - e_{ijk}\omega_j \tau_{ik}(\mathbf{d}, \overset{o}{\mathbf{d}}, \mathbf{D}) \geq 0$$

can then be written equivalently as

$$\overline{\boldsymbol{\tau}}(\mathbf{d}, \mathbf{K}, \mathbf{D}) \cdot \mathbf{D} - \overline{\boldsymbol{\tau}}(\mathbf{d}, \mathbf{K}, \mathbf{D}) \cdot \mathbf{K} \geq 0.$$

If we define a function $\overline{\mathsf{D}}$ by

$$\overline{\mathsf{D}}(\mathbf{d}, \mathbf{K}, \mathbf{D}) = \overline{\boldsymbol{\tau}}(\mathbf{d}, \mathbf{K}, \mathbf{D}) \cdot \mathbf{D} - \overline{\boldsymbol{\tau}}(\mathbf{d}, \mathbf{K}, \mathbf{D}) \cdot \mathbf{K}$$

then the function $\overline{\mathsf{D}}$ has the properties

$$\overline{\mathsf{D}}(\mathbf{d}, \mathbf{K}, \mathbf{D}) \geq 0, \qquad \overline{\mathsf{D}}(\mathbf{d}, \mathbf{0}, \mathbf{0}) = 0.$$

Thus $\overline{\mathsf{D}}(\mathbf{d}, \cdot, \cdot)$ has a minimum at $\mathbf{K} = \mathbf{0}, \mathbf{D} = \mathbf{0}$ and so $\partial\overline{\mathsf{D}}/\partial\mathbf{D} = \partial\overline{\mathsf{D}}/\partial\mathbf{K} = \mathbf{0}$ at $\mathbf{D} = \mathbf{0}, \mathbf{K} = \mathbf{0}$. Carrying out the differentiation leads to

$$\overline{\boldsymbol{\tau}}(\mathbf{d}, \mathbf{K}, \mathbf{D}) = \mathbf{0} \qquad \text{at} \quad \mathbf{D} = \mathbf{0}, \mathbf{K} = \mathbf{0}$$

and therefore that

$$\boldsymbol{\tau}(\mathbf{d}, \overset{o}{\mathbf{d}}, \mathbf{D}) = \mathbf{0} \qquad \text{at} \quad \mathbf{D} = \mathbf{0}, \overset{o}{\mathbf{d}} = \mathbf{o}.$$

The limitation to non-singular $\mathbf{K}$ is non-singular is a restriction on the preceding result.

Problem 13.9. *Frank free energy. Mathematical derivation.* Consider the free energy function $\psi(\mathbf{d}, \mathbf{G})$ of a nematic liquid crystal. Here $\mathbf{G} = \operatorname{grad} \mathbf{d}$ and $|\mathbf{d}| = 1$. Suppose that the spatial variation of the director field is slow, $|\mathbf{G}| << 1$, so that we expand $\psi(\mathbf{d}, \cdot)$ in a Taylor series retaining only terms that are quadratic or smaller. Since the molecules are assumed to be non-polarized there is no distinction between the two ends of the director and so the free energy must be invariant under the transformation $\mathbf{d} \to -\mathbf{d}$. We also assume centrosymmetry so that the free energy must also be invariant under the transformation $\mathbf{y} \to -\mathbf{y}$. Finally the energy must obey $\psi(\mathbf{Q}\mathbf{d}, \mathbf{Q}\mathbf{G}\mathbf{Q}^T) = \psi(\mathbf{d}, \mathbf{G})$ for all proper orthogonal $\mathbf{Q}$ which characterizes the rotational symmetry about $\mathbf{d}$. Determine the most general free energy function with the preceding characteristics.

Solution: Since $\mathbf{d}(\mathbf{y}, t)$ is a unit vector, taking the spatial gradient of $\mathbf{d} \cdot \mathbf{d} = 1$ shows that we must have

$$\mathbf{G}^T \mathbf{d} = \mathbf{o}.$$

First we normalize the energy so that it vanishes in the uniform state: $\psi(\mathbf{d}, \mathbf{0}) = 0$. Next we observe that the only linear terms in $\mathbf{G}$ that are possible are $\operatorname{div} \mathbf{d}$ and $\mathbf{d} \cdot \operatorname{curl} \mathbf{d}$. However these are both ruled out by the required invariances, the former because of invariance under $\mathbf{d} \to -\mathbf{d}$ and the latter due to invariance under $\mathbf{y} \to -\mathbf{y}$.

We now construct the most general expression for ψ that is quadratic in $\mathbf{G}$ and is consistent with the preceding requirements. It is convenient to decompose $\mathbf{G}$ into its symmetric and skew-symmetric parts:

$$\mathbf{S} = \frac{1}{2}\big(\operatorname{grad} \mathbf{d} + (\operatorname{grad} \mathbf{d})^T)\big), \qquad \mathbf{K} = \frac{1}{2}\big(\operatorname{grad} \mathbf{d} - (\operatorname{grad} \mathbf{d})^T\big).$$

First, note that there is a one-to-relation between the tensor $\mathbf{G}$ and the pair of tensors $\{\mathbf{S}, \mathbf{K}\}$. Second, just as in our discussion in Section 3.3 of classical kinematics where we had a one-to-one relation between the spin tensor (the skew-symmetric part of $\operatorname{grad} \mathbf{v}$) and the vorticity vector $\operatorname{curl} \mathbf{v}$, here we have a one-to-one relation between $\mathbf{K}$ and the vector $\operatorname{curl} \mathbf{d}$. Thus with no loss of generality we can write the free energy in the form $\psi = \psi(\mathbf{d}, \mathbf{S}, \operatorname{curl} \mathbf{d})$, which has to be quadratic in $\mathbf{G}$. It is convenient to write this as

$$\psi = \psi_1(\mathbf{d}, \mathbf{S}) + \psi_2(\mathbf{d}, \operatorname{curl} \mathbf{d}) + \psi_3(\mathbf{d}, \mathbf{S}, \operatorname{curl} \mathbf{d})$$

where ψ_1 is quadratic in $\mathbf{S}$, ψ_2 is quadratic in $\operatorname{curl} \mathbf{d}$, and ψ_3 involves the product of terms linear in $\mathbf{S}$ and linear in $\operatorname{curl} \mathbf{d}$.

The function ψ_1 must be mathematically identical to the strain energy function of a elastic material with rotational symmetry about the $\mathbf{d}$-direction. Thus from Section 8.8 (see also Chapter 4 of Volume I) it follows that ψ_1 must be a function of the scalar-valued invariants

$$\operatorname{tr} \mathbf{S}, \quad \operatorname{tr} \mathbf{S}^2, \quad \operatorname{tr} \mathbf{S}^3, \quad \operatorname{tr}\Big(\mathbf{S}(\mathbf{d} \otimes \mathbf{d})\Big), \quad \operatorname{tr}\Big(\mathbf{S}^2(\mathbf{d} \otimes \mathbf{d})\Big).$$

However since ψ_1 is to be quadratic in $\mathbf{S}$ it must therefore be a linear combination of

$$(\operatorname{tr}\mathbf{S})^2, \quad \operatorname{tr}\mathbf{S}^2, \quad \Big[\operatorname{tr}\Big(\mathbf{S}(\mathbf{d}\otimes\mathbf{d})\Big)\Big]^2, \quad \operatorname{tr}\Big(\mathbf{S}^2(\mathbf{d}\otimes\mathbf{d})\Big), \quad (\operatorname{tr}\mathbf{S})\operatorname{tr}\Big(\mathbf{S}(\mathbf{d}\otimes\mathbf{d})\Big).$$

Note that

$$\operatorname{tr}\Big(\mathbf{S}(\mathbf{d}\otimes\mathbf{d})\Big) = \mathbf{S}\mathbf{d}\cdot\mathbf{d} = \mathbf{G}^T\mathbf{d}\cdot\mathbf{d} = 0$$

and so the list of invariants reduces to

$$(\operatorname{tr}\mathbf{S})^2, \quad \operatorname{tr}\mathbf{S}^2, \quad \operatorname{tr}\Big(\mathbf{S}^2(\mathbf{d}\otimes\mathbf{d})\Big).$$

Since $\operatorname{tr}\mathbf{S} = \operatorname{div}\mathbf{d}$ we can write this list as

$$(\operatorname{div}\mathbf{d})^2, \quad \operatorname{tr}\mathbf{S}^2, \quad \operatorname{tr}\Big(\mathbf{S}^2(\mathbf{d}\otimes\mathbf{d})\Big).$$

Next one can readily verify that

$$\operatorname{tr}\mathbf{S}^2 = (d_{i,j}d_j)_{,i} - (d_{i,i}d_j)_{,j} + (\operatorname{div}\mathbf{d})^2 + \frac{1}{2}|\operatorname{curl}\mathbf{d}|^2.$$

If we calculate the volume integral of ψ, the first two terms on the righthand side of the preceding equation can be converted into surface integrals by using the divergence theorem and therefore they correspond to a "null lagrangian", i.e. terms that do not contribute to the Euler Lagrange equations associated with this energy. Thus we can drop these two terms in calculating the energy. Thus ψ_1 is a linear combination of

$$(\operatorname{div}\mathbf{d})^2, \quad (\operatorname{div}\mathbf{d})^2 + \frac{1}{2}|\operatorname{curl}\mathbf{d}|^2, \qquad \operatorname{tr}\Big(\mathbf{S}^2(\mathbf{d}\otimes\mathbf{d})\Big),$$

or equivalently of

$$(\operatorname{div}\mathbf{d})^2, \quad |\operatorname{curl}\mathbf{d}|^2, \qquad \operatorname{tr}\Big(\mathbf{S}^2(\mathbf{d}\otimes\mathbf{d})\Big)$$

Finally, since $\mathbf{S} = (\mathbf{G}+\mathbf{G}^T)/2$ and $\mathbf{G}^T\mathbf{d} = \mathbf{o}$ it can be readily shown that

$$\operatorname{tr}\Big(\mathbf{S}^2(\mathbf{d}\otimes\mathbf{d})\Big) = \frac{1}{4}|\mathbf{G}\mathbf{d}|^2 \qquad \text{and} \qquad |\mathbf{d}\times\operatorname{curl}\mathbf{d}|^2 = |\mathbf{G}\mathbf{d}|^2,$$

whence

$$\operatorname{tr}\Big(\mathbf{S}^2(\mathbf{d}\otimes\mathbf{d})\Big) = \frac{1}{4}|\mathbf{d}\times\operatorname{curl}\mathbf{d}|^2.$$

Thus ψ_1 is a linear combination of

$$(\operatorname{div}\mathbf{d})^2, \quad |\operatorname{curl}\mathbf{d}|^2, \qquad |\mathbf{d}\times\operatorname{curl}\mathbf{d}|^2.$$

However

$$|\mathbf{d}\times\operatorname{curl}\mathbf{d}|^2 = |\operatorname{curl}\mathbf{d}|^2 - |\mathbf{d}\cdot\operatorname{curl}\mathbf{d}|^2$$

and therefore ψ_1 can be expressed as

$$\rho\psi_1 = \mu_1(\mathrm{div}\,\mathbf{d})^2 \;+\; \mu_2(\mathbf{d}\cdot\mathrm{curl}\,\mathbf{d})^2 \;+\; \mu_3|\mathbf{d}\times\mathrm{curl}\,\mathbf{d}|^2.$$

While in theory the μ's could depend on the vector $\mathbf{d}$, the symmetry requirement $\mu_i(\mathbf{Qd}) = \mu_i(\mathbf{d})$ for all proper orthogonal $\mathbf{Q}$ requires that μ_i depend on $\mathbf{d}$ only through its length $|\mathbf{d}|$. However since $\mathbf{d}$ is a unit vector, it follows that the μ's are independent of $\mathbf{d}$.

We leave the details of the analysis of ψ_2 and ψ_3 as an exercise. For ψ_2, which is quadratic in $\mathrm{curl}\,\mathbf{d}$, one can show that rotational invariance about $\mathbf{d}$ implies that it be a linear combination of

$$(\mathbf{d}\cdot\mathrm{curl}\,\mathbf{d})^2, \qquad |\mathbf{d}\times\mathrm{curl}\,\mathbf{d}|^2.$$

Similarly for ψ_3, one can show that rotational symmetry about $\mathbf{d}$ and invariance under a change of sign of $\mathbf{d}$ imply that it only involves the term $|\mathbf{d}\times\mathrm{curl}\,\mathbf{d}|^2$.

Thus in summary we conclude that the free energy is given by

$$\rho\psi = \frac{1}{2}K_1(\mathrm{div}\,\mathbf{d})^2 \;+\; \frac{1}{2}K_2(\mathbf{d}\cdot\mathrm{curl}\,\mathbf{d})^2 \;+\; \frac{1}{2}K_3|\mathbf{d}\times\mathrm{curl}\,\mathbf{d}|^2$$

which is the Frank free energy (13.33).

Problem 13.10. *Frank free energy. Physical derivation.* The three basic modes of liquid crystal deformation are illustrated in Figure 13.3. Show that the first, second and third terms in the Frank free energy

$$\rho\psi(\mathbf{d},\mathrm{grad}\,\mathbf{d}) = \frac{1}{2}K_1(\mathrm{div}\,\mathbf{d})^2 + \frac{1}{2}K_2(\mathbf{d}\cdot\mathrm{curl}\,\mathbf{d})^2 + \frac{1}{2}K_3|\mathbf{d}\times\mathrm{curl}\,\mathbf{d}|^2 \tag{13.55}$$

are associated with the splay, twist and bend modes respectively. *Remark:* Note that $\mathbf{d}$ is parallel to $\mathrm{curl}\,\mathbf{d}$ in pure twist, while $\mathbf{d}$ is perpendicular to $\mathrm{curl}\,\mathbf{d}$ in pure bend.

Problem 13.11. *Frank free energy. Material Frame Indifference.* Show that the Frank free energy

$$\rho\psi(\mathbf{d},\mathrm{grad}\,\mathbf{d}) = \frac{1}{2}K_1(\mathrm{div}\,\mathbf{d})^2 + \frac{1}{2}K_2(\mathbf{d}\cdot\mathrm{curl}\,\mathbf{d})^2 + \frac{1}{2}K_3|\mathbf{d}\times\mathrm{curl}\,\mathbf{d}|^2$$

has the property $\psi(\mathbf{d},\mathbf{G}) = \psi(\mathbf{Qd},\mathbf{QGQ}^T)$ for all rotations $\mathbf{Q}$ as required by material frame indifference.

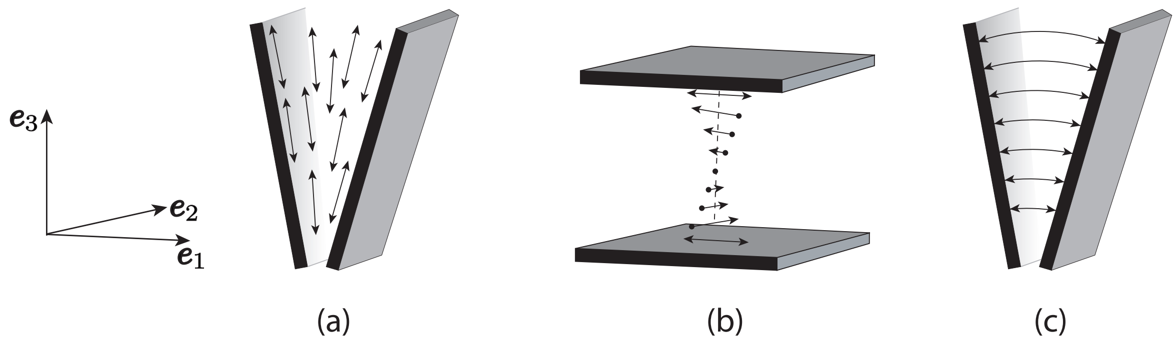

Figure 13.3: Schematic depiction of the three basic modes of distortion: (a) splay, (b) twist, and (c) bend. Consider a pair of flat plates that are initially parallel to each other, the space between them filled by a nematic liquid crystal, and which anchor the director in the respective directions (a) $\mathbf{e}_3$, (b) $\mathbf{e}_1$, and (c) $\mathbf{e}_1$. The plates are then rotated in opposite directions about the (a) $\mathbf{e}_2$-, (b) $\mathbf{e}_3$-, and (c) $\mathbf{e}_2$-axes respectively.

Problem 13.12. *The extra stress* $\boldsymbol{\tau}$*.* Derive the constitutive relation (13.36) for the extra stress, starting from the ansatz that $\boldsymbol{\tau}(\mathbf{d}, \overset{o}{\mathbf{d}}, \mathbf{D})$ is linear in $\overset{o}{\mathbf{d}}$ and $\mathbf{D}$ and that it vanishes in equilibrium. *Note:* This could, for example, be an approximation of a general constitutive relation for $\boldsymbol{\tau}(\mathbf{d}, \overset{o}{\mathbf{d}}, \mathbf{D})$ for motions that are not far from equilibrium, i.e. where $\overset{o}{\mathbf{d}}$ and $\mathbf{D}$ are small.

Solution Since we are told that the constitutive relation for the extra stress $\boldsymbol{\tau}(\mathbf{d}, \overset{o}{\mathbf{d}}, \mathbf{D})$ is linear in $\overset{o}{\mathbf{d}}$ and $\mathbf{D}$ and that it vanishes in equilibrium, we can write

$$\tau_{ij} = \mathcal{A}_{ijk} \overset{o}{d_k} + \mathcal{B}_{ijk\ell} D_{k\ell} \tag{13.56}$$

where $\mathcal{A}$ and $\mathcal{B}$ are functions of $\mathbf{d}$. For a nematic liquid crystal one must have rotational symmetry about the axis $\mathbf{d}$. Smith and Rivlin[5] have shown that in this event $\mathcal{A}$ and $\mathcal{B}$ must be comprised of linear combinations of products of δ_{ij} and d_i. The most general forms of $\mathcal{A}$

[5]G.F. Smith and R.S. Rivlin, The anisotropic tensors, *Quarterly Journal of Applied Mathematics*, volume 15, 1957, pp. 308-314.

and $\mathcal{B}$ are therefore

$$\begin{aligned}
\mathcal{A}_{ijk} &= \mu_1\delta_{ij}d_k + \mu_2\delta_{jk}d_i + \mu_3\delta_{ki}d_j + \mu_4 d_i d_j d_k, \\
\mathcal{B}_{ijk\ell} &= \mu_5\delta_{ij}\delta_{k\ell} + \mu_6\delta_{ik}\delta_{j\ell} + \mu_7\delta_{i\ell}\delta_{jk} \\
&+ \mu_8\delta_{ij}d_k d_\ell + \mu_9\delta_{jk}d_i d_\ell + \mu_{10}\delta_{ik}d_j d_\ell + \mu_{11}\delta_{i\ell}d_j d_k + \mu_{12}\delta_{j\ell}d_i d_k + \mu_{13}\delta_{k\ell}d_i d_j \\
&+ \mu_{14} d_i d_j d_k d_\ell.
\end{aligned} \tag{13.57}$$

If we substitute (13.57) into (13.56) and simplify the result, making use of $\mathbf{d}\cdot \overset{o}{\mathbf{d}} = 0$ and $\operatorname{tr}\mathbf{D} = 0$, we find that the terms involving μ_1 and μ_4 do not contribute to the extra stress. Moreover, the term involving μ_8 leads to a scalar multiple of the identity. This term can be absorbed into the pressure term $-p\,\mathbf{I}$ in the Cauchy stress $\mathbf{T}$, and so the term associated with μ_8 can be omitted from the extra stress. This leads to

$$\boldsymbol{\tau} = \alpha_1(\mathbf{d}\cdot\mathbf{Dd})\mathbf{d}\otimes\mathbf{d} + \alpha_2 \overset{o}{\mathbf{d}} \otimes \mathbf{d} + \alpha_3 \mathbf{d} \otimes \overset{o}{\mathbf{d}} + \alpha_4\mathbf{D} + \alpha_5\mathbf{Dd}\otimes\mathbf{d} + \alpha_6\mathbf{d}\otimes\mathbf{Dd}$$

where we have set $\alpha_1 = \mu_{14}, \alpha_2 = \mu_3, \alpha_3 = \mu_2, \alpha_4 = \mu_6 + \mu_7, \alpha_5 = \mu_{10} + \mu_{11}, \alpha_6 = \mu_9 + \mu_{12}$. This is precisely the form (13.36) given previously where the α's are material constants known as the Leslie viscosities.

References:

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2. P.J. Collings, *Liquid Crystals: Nature's Delicate Phase of Matter*, Princeton University Press, 1990.
3. P.G. de Gennes and J. Prost, *The Physics of Liquid Crystals*, Oxford, 1993.