CONVERTING REPEATING DECIMALS TO FRACTIONS

LESSON 3-D

Convert repeating decimals to fractions.

A number that can be expressed as a fraction of two integers is called a **rational number**. Every rational number can be written as a decimal number. The decimal numbers will either terminate (end) or repeat.

Terminating Decimals			Repeating Decimals		
$\frac{3}{4} = 0.75$	$\frac{19}{10} = 1.9$	$\frac{394}{1000} = 0.394$	8.16161616... is $8.\overline{16}$	0.3333... is $0.\overline{3}$	5.0424242... is $5.0\overline{42}$

The bar goes above the 16 because both digits repeat.

The bar goes above the 3 because it is the number that repeats.

The bar goes above the 42 because they are the only two digits that repeat.

Converting a repeating decimal to a fraction can be done by creating an equation or system of equations and then solving those equations.

Converting a Repeating Decimal to a Fraction

1. Let x equal the repeating portion of the decimal number.
2. Multiply both sides of the equation by a power of ten to move the repeating digit(s) to the left side of the decimal point.
3. Subtract x from both sides of the equation.
4. Solve for x.

EXAMPLE 1 **Convert $0.\overline{4}$ to a fraction.**

SOLUTION

Let x equal the repeating decimal. $x = 0.\overline{4}$ or $x = 0.4444...$

Multiply both sides of the equation by 10. This moves the repeating digit to the left side of the decimal point.

$$10x = 10(0.4444...)$$
$$10x = 4.444...$$

Subtract x from both sides of the equation.

$$\begin{array}{r} 10x = 4.444... \\ \underline{-x \quad -0.444...} \\ 9x = 4 \end{array}$$

Remember that x is equal to 0.444...

Divide both sides of the equation by 9.

$$x = \frac{4}{9}$$

$$\boxed{0.\overline{4} = \frac{4}{9}}$$

$\frac{4}{9}$ is the fraction equal to $0.\overline{4}$

EXAMPLE 2 **David had $1.\overline{18}$ ounces of silver. What is this amount as a fraction?**

SOLUTION

Let x equal the repeating decimal. $x = 0.1818...$

Ignore the whole number while finding the fraction.

Multiply both sides of the equation by 100. This moves the repeating digit to the left side of the decimal point.

$$100x = 100(0.1818...)$$
$$100x = 18.1818...$$

Remember that x is equal to 0.1818...

Subtract x from both sides of the equation.

$$\begin{array}{rl} 100x = & 18.1818... \\ -x & -0.1818... \\ \hline 99x = & 18 \end{array}$$

Divide both sides of the equation by 99. $x = \frac{18}{99} = 1\frac{18}{99}$

Now that the fraction is found, add the whole number back in.

$1.\overline{18}$ ounces of silver is equivalent to $1\frac{18}{99}$ ounces.

When the repeating decimal digits fall after digits that do not repeat, you need to set up a system of equations to find the equivalent fractions.

EXAMPLE 3 **What is $0.8\overline{3}$ as a fraction?**

SOLUTION

Let x equal the repeating decimal. $x = 0.8\overline{3}$ or $x = 0.8333...$

Multiply both sides of the equation by 100. This moves the repeating digit (3) to the left side of the decimal point.

$$100x = 100(0.8333...)$$
$$100x = 83.333...$$

Create another equation by multiplying both sides of the original equation by a different power of 10. This will allow you to subtract the repeating part of the decimal.

$$10x = 8.3333...$$

Any other power of 10 will work.

Subtract the equations from one another.

$$\begin{array}{rl} 100x = & 83.3333... \\ -10x & -8.3333... \\ \hline 90x = & 75 \end{array}$$

Divide both sides of the equation by 90. $x = \frac{75}{90} = \frac{5}{6}$

$0.8\overline{3} = \frac{5}{6}$

EXERCISES

Label each of the following decimals with the term: terminating decimal or repeating decimal.

1. 5.45

2. $7.\overline{6}$

3. $2.08\overline{4}$

4. 6.32

5. $0.8\overline{58}$

6. 23.769

Match the following repeating decimal with its equivalent fraction value in the box.

7. $0.\overline{18}$

8. $0.\overline{6}$

9. $0.1\overline{6}$

10. $0.41\overline{6}$

11. $0.\overline{1}$

12. $0.6\overline{3}$

<u>**Rational Numbers**</u>

$\frac{2}{3}$ $\frac{1}{9}$ $\frac{1}{6}$ $\frac{5}{12}$ $\frac{2}{11}$ $\frac{19}{30}$

Convert each repeating decimal into a fraction in simplest form.

13. $0.\overline{2}$

14. $0.\overline{5}$

15. $0.\overline{15}$

16. $0.\overline{63}$

17. $0.1\overline{2}$

18. $0.04\overline{3}$

19. $0.\overline{414}$

20. $0.36\overline{3}$

21. $0.\overline{162}$

22. In what situations must you set up multiple equations when converting a repeating decimal to a fraction? Write an example of one such type of decimal number.

23. Why are powers of 10 chosen as the multipliers for converting decimals to fractions?

24. Hank has a snake which weighs $3.\overline{7}$ ounces. Jill has a lizard which weighs $3\frac{4}{5}$ ounces. Whose reptile weighs more? Support your solution with calculations.

25. Megan ran a mile in $9.\overline{45}$ minutes. Janeen ran a mile in $9\frac{4}{9}$ minutes. Anna ran a mile in $9\frac{13}{30}$ minutes.

a. Put the runners' times in order from fastest to slowest.

b. Who was the fastest?

26. When a single digit repeats after the decimal point (i.e., $0.\overline{1}$, $0.\overline{2}$, $0.\overline{3}$, $0.\overline{4}$, etc), what do you notice about the denominators of the equivalent fractions?

MULTIPLICATION PROPERTIES OF EXPONENTS

LESSON 3-E

Simplify expressions involving multiplication using properties of exponents.

When a numerical expression is the product of a repeated factor, it can be written using a **power**. A power consists of two parts, the **base** and the **exponent**. The base of the power is the repeated factor. The exponent shows the number of times the factor is repeated.

POWERS, BASES AND EXPONENTS

exponent

$$\text{base} \rightarrow \underbrace{3^4}_{\text{power}} = \overbrace{3 \times 3 \times 3 \times 3}^{\text{expanded form}}$$

It is important to know how to read powers correctly.

Power	Reading the Expression	Expanded Form	Value
5^2	"five to the second power" or "five **squared**"	5 × 5	25
6^3	"six to the third power" or "six **cubed**"	6 × 6 × 6	216
2^4	"two to the fourth power"	2 × 2 × 2 × 2	16

EXPLORE! **EXPAND IT**

Use expanded form to discover two different exponent multiplication properties.

Step 1: Write each of the following products in expanded form.

a. $5^3 \cdot 5^4$ **b.** $4^2 \cdot 4^8$ **c.** $x^3 \cdot x^2$

Step 2: Rewrite each of the products in **Step 1** as a single term with one base and one exponent.

Step 3: What relationship do you see between the original bases and the single term's base? What about the original exponents and the single term's exponent?

Step 4: Based on your findings, write a statement explaining how to find the product of two powers with the same base WITHOUT writing the terms in expanded form.

Step 5: Write each of the following powers in expanded form. Then rewrite the power as a single term. The first one is done for you.

a. $(3^2)^4 \rightarrow (3^2)(3^2)(3^2)(3^2) \rightarrow \underbrace{3 \cdot 3}\cdot\underbrace{3 \cdot 3}\cdot\underbrace{3 \cdot 3}\cdot\underbrace{3 \cdot 3} \rightarrow 3^8$

b. $(7^3)^2$

c. $(x^3)^5$

Step 6: What is the relationship between the final exponent and the power to a power? Based on your findings, write a statement explaining how to find the power of a power WITHOUT expanding the power.

You can also simplify a power of a product. Look at $(df)^3$.

Written in expanded form: $(df)(df)(df)$

Group like variables together: $(d \cdot d \cdot d)(f \cdot f \cdot f)$

Simplify: d^3f^3

MULTIPLICATION PROPERTIES OF EXPONENTS

Product of Powers

To multiply two powers with the same base, add the exponents.

$$a^m \cdot a^n = a^{m+n}$$

Power of a Power

To find the power of a power, multiply the exponents.

$$(a^m)^n = a^{mn}$$

Power of a Product

To find the power of a product, find the power of each factor and multiply.

$$(ab)^n = a^n b^n$$

EXAMPLE 1 **Simplify each of the following.**

a. $y^3x^2y^6x$ **b.** $(b^3w^2)^4$ **c.** $(5p^4)(2p^3)$

On a term with no exponent, the exponent is 1.

SOLUTIONS

a. Group like variables together. $y^3x^2y^6x = y^3y^6 \cdot x^2x$

Add exponents with the same base. $y^{3+6}x^{2+1} = y^9x^3$

b. Distribute the exponent to each base. $(b^3w^2)^4 = (b^3)^4(w^2)^4$

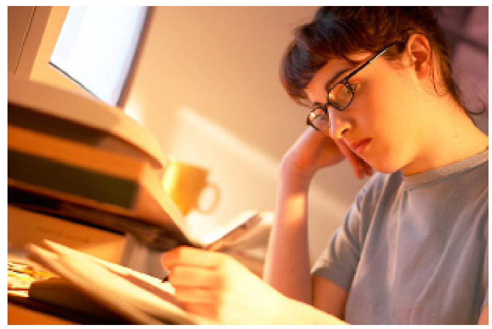
Multiply exponents. $(b^3)^4(w^2)^4 = b^{12}w^8$

c. Group like terms together. $(5p^4)(2p^3) = (5 \cdot 2)(p^4p^3)$

Multiply coefficients. Add exponents with the same base. $(5 \cdot 2)(p^4p^3) = 10p^7$

A simplified expression should have:

- each base appear exactly once,
- no powers to powers,
- no numeric values with powers, and
- fractions written in simplest form.



EXAMPLE 2 **Simplify each of the following.**

a. $6x^2y^4z^3 \cdot 3x^5z^2$ **b.** $(4m^3w)^2(5m^2w^2)^3$

SOLUTIONS

a. Rewrite the expression. $6x^2y^4z^3 \cdot 3x^5z^2$

Group like terms together. $6 \cdot 3 \cdot x^2 \cdot x^5 \cdot y^4 \cdot z^3 \cdot z^2$

Multiply coefficients. Add exponents with the same base. $18x^7y^4z^5$

b. Rewrite the expression. $(4m^3w)^2(5m^2w^2)^3$

Distribute the exponent to each base. $4^2(m^3)^2(w)^2 \cdot 5^3(m^2)^3(w^2)^3$

Find the values of the coefficients with exponents. $16(m^3)^2(w)^2 \cdot 125(m^2)^3(w^2)^3$

Multiply exponents. $16m^6w^2 \cdot 125m^6w^6$

Multiply coefficients. Add exponents with the same base. $2000m^{12}w^8$

EXERCISES

Simplify.

1. x^3x^2

2. $(y^5)^2$

3. $(pq)^5$

4. $(4x^5)^3$

5. $(w^2y^4z^6)(w^5y^3z)$

6. $(2a^6b)(3a^3b^3)$

7. $(5gh^2)^2$

8. $(9x^4y^5)(-2x^2y^7)$

9. $(0.5f^2d^9)^3$

10. A farmer wants to fence in a square pasture for his sheep. The length of one side of the pasture is represented by kp^3. What term represents the area of the pasture?

11. Write and solve a problem that requires you to multiply exponents.

12. Write and solve a problem that requires you to add exponents.

13. Jake, Joe and Jenny attempted to rewrite $(5^2)(4^3)$ as a single power. Who got the correct answer? How do you know?

Jake

$(5^2)(4^3) = 20^5$

Joe

$(5^2)(4^3)$ cannot be written as a single power.

Jenny

$(5^2)(4^3) = 9^5$

Simplify.

14. $(3x^2)^3(4x^5)^2$

15. $(-4y^5w^2)^2(2y^4)^3$

16. $(5p^3)(5p^2)^3$

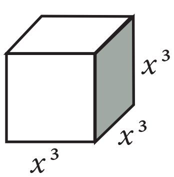
17. $(2x^2)^3(3x^4)^2(-2x^3)^3$

18. $(-2y^2)(-3xy^4)^2(5x^6)^2$

19. $(4a^2b)^3(5a^4b^5)^2$

Write the volume of each solid as a single term.

20.



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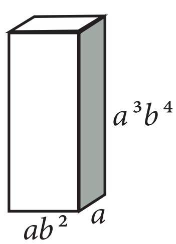

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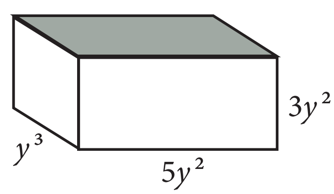


23. The length of a rectangle is five times its width. If its width is x^3 units, what is the area of the rectangle?

24. The width of a rectangular prism is four times its length. The height is three times its length. If its length is m units, what is the volume of the prism?

DIVISION PROPERTIES OF EXPONENTS

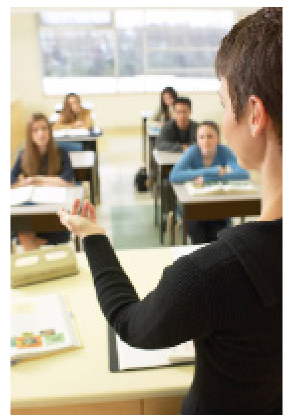
LESSON 3-F

Use properties of exponents to simplify expressions involving division.

You discovered that when two powers with the same base are multiplied, the base remains the same and the exponents are added together. Examine the division problems below.

$$\frac{5^7}{5^3} = \frac{\cancel{5} \cdot \cancel{5} \cdot \cancel{5} \cdot 5 \cdot 5 \cdot 5 \cdot 5}{\cancel{5} \cdot \cancel{5} \cdot \cancel{5}} = 5^4 \qquad \frac{x^6}{x^4} = \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x}} = x^2$$

Simplify the fractions by cancelling common factors.



What is the relationship of the original exponents and the resulting exponent?

$$\frac{5^7}{5^3} = 5^4 \qquad \frac{x^6}{x^4} = x^2$$

When two powers with the same base are divided, the base remains the same and the exponents are subtracted.

You can also simplify a power of a quotient or fraction by "distributing" the power to both the numerator and denominator.

$$\left(\frac{3}{4}\right)^3 = \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{3^3}{4^3} = \frac{27}{64}$$

<u>Division Properties of Exponents</u>

<u>Quotient of Powers</u>

To divide two powers with the same base, subtract the exponents.

$$\frac{a^m}{a^n} = a^{m-n}$$

<u>Power of a Quotient</u>

To find the power of a quotient, find the power of each factor and divide.

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

EXAMPLE 1

Simplify each of the following.

a. $\dfrac{x^5y^8}{x^2y}$ **b.** $\left(\dfrac{3m^4}{5}\right)^2$

SOLUTIONS

a. Group powers that have the same base. $\left(\dfrac{x^5}{x^2}\right)\left(\dfrac{y^8}{y}\right)$

Subtract exponents. $x^{5-2}y^{8-1} = x^3y^7$

b. Distribute the power to each base. $\left(\dfrac{3m^4}{5}\right)^2 = \dfrac{3^2 \cdot (m^4)^2}{5^2}$

Multiply powers of powers and evaluate coefficients. $\dfrac{9m^8}{25}$

EXPLORE! **GOING NEGATIVE**

Step 1: Copy the tables below. Find the value of each power with a calculator. If the value of the power is less than 1, write the power as a fraction.

Power	Value
2^4	
2^3	
2^2	
2^1	

Power	Value
2^{-4}	
2^{-3}	
2^{-2}	
2^{-1}	

Step 2: What do you notice about the powers with opposite exponents (i.e. 2^2 and 2^{-2})?

Step 3: Use your observation from **Step 2** to predict the value of each power below.

a. Given that $4^2 = 16$, what is the value of 4^{-2}?

b. Given that $3^5 = 243$, what is the value of 3^{-5}?

c. Given that $6^{-3} = \dfrac{1}{216}$, what is the value of 6^3?

Step 4: Look at the statements below. What is the value of each expression (write without an exponent)?

a. $\dfrac{5^2}{5^2} = \dfrac{25}{25} = ?$ **b.** $\dfrac{2^3}{2^3} = ?$ **c.** $\dfrac{3^4}{3^4} = ?$

Step 5: Notice that $\dfrac{5^2}{5^2} = 5^{2-2} = 5^0$, $\dfrac{2^3}{2^3} = 2^0$ and $\dfrac{3^4}{3^4} = 3^0$ using the Division Property of Exponents.

Based on your findings in **Step 4**, what is the value of 5^0, 2^0 and 3^0?

Step 6: Use your calculator to raise other numbers to a power of 0. Try whole numbers and decimal values. What can you conclude?

More Properties of Exponents

Negative Exponents

For any nonzero number, a and integer n, the expression a^{-n} is the reciprocal of a^n. Also, a^n is the reciprocal of a^{-n}.

$$a^{-n} = \frac{1}{a^n} \text{ and } a^n = \frac{1}{a^{-n}}$$

Zero Exponents

Any nonzero number, a, raised to the zero power is 1.

$$a^0 = 1$$

An expression is in simplest form only if it has no negative or zero exponents. As shown in the blue box above, an exponent with its base can be moved to the opposite side of the fraction bar to change its sign.

EXAMPLE 2

Simplify each of the following.

a. $\left(\frac{4p^5k}{3}\right)^0$ **b.** $\frac{x^2y^{-4}}{z^{-3}}$ **c.** $\frac{6a^2b^{-6}c^8}{-2a^2b^{-5}c^{-1}}$

Solutions

a. Any term to the power of 0 equals 1. $\left(\frac{4p^5k}{3}\right)^0 = 1$

b. Write as separate factors. $\left(\frac{x^2y^{-4}}{z^{-3}}\right) = \left(\frac{x^2}{1}\right)\left(\frac{y^{-4}}{1}\right)\left(\frac{1}{z^{-3}}\right)$

Negative exponents move a power to the opposite side of the fraction bar.

Use the rules for negative exponents. $\left(\frac{x^2}{1}\right)\left(\frac{1}{y^4}\right)\left(\frac{z^3}{1}\right)$

Multiply factors. $\frac{x^2z^3}{y^4}$

c. Write as separate factors. Group like bases. $\frac{6a^2b^{-6}c^8}{-2a^2b^{-5}c^{-1}} = \left(\frac{6}{-2}\right)\left(\frac{a^2}{a^2}\right)\left(\frac{b^{-6}}{b^{-5}}\right)\left(\frac{c^8}{c^{-1}}\right)$

Subtract exponents on like bases. $\left(\frac{6}{-2}\right)(a^{2-2})(b^{-6-(-5)})(c^{8-(-1)})$

Simplify. $(-3)(a^0)(b^{-1})(c^9)$

Use rules for zero and negative exponents. $(-3)(1)\left(\frac{1}{b}\right)c^9$

Multiply. $\frac{-3c^9}{b}$

EXERCISES

Simplify.

1. $\dfrac{8^{12}}{8^5}$

2. $\dfrac{x^5}{x^2}$

3. $\dfrac{a^6b^9}{a^3b^5}$

4. $\dfrac{2w^5v^4}{10wv^2}$

5. $\left(\dfrac{d^2}{g^3}\right)^5$

6. $\left(\dfrac{2y^3}{3}\right)^3$

7. $(4yh^2)^0$

8. $\left(\dfrac{5x^{11}}{3w^{-4}}\right)^0$

9. 5^{-2}

10. 2^{-4}

11. $\dfrac{k^{-3}m^2}{n^{-7}}$

12. $7p^{-2}q^{-5}$

13. A contractor plans to build a rectangular office complex. The floor area on the first floor of the complex is represented by $15x^5y^9$. The length of the complex is represented by $3x^2y^5$. What expression represents the width?

14. Write and solve a problem that requires you to subtract exponents.

15. A truckload of cement weighs approximately 4^7 pounds. The driver of the truck weighs 4^4 pounds. The truckload of cement weighs how many times more than the driver?

Simplify.

16. $\dfrac{-3m^5}{m^{11}}$

17. $\dfrac{24x^7y^{-4}}{4x^{-3}y^2}$

18. $\dfrac{10p^2w^6}{6p^{-2}w^6}$

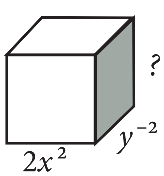
19. $\left(\dfrac{r^{-2}t^0}{n^{-5}}\right)^3$

20. $\left(\dfrac{6y^2}{z^{-3}}\right)^2$

21. $\left(\dfrac{2q^{-2}w^3}{3x^{-4}y^5}\right)^{-1}$

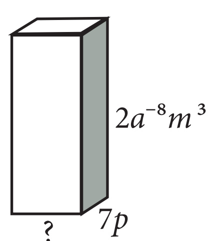
Find the missing dimension when given the volume. Write the missing dimension in simplest form.

22.



Volume = $10x^4y^{-5}z$

23.



Volume = $14a^{-3}m^8p^{-1}$

SCIENTIFIC NOTATION

LESSON 3-G

Express numbers in scientific notation and standard notation.

The table below shows the approximate distance to the sun from each planet in our solar system.

Planet	Approximate Distance to the Sun (in miles)	Distance Written in Scientific Notation
Mercury	36,000,000	3.6×10^7
Venus	67,000,000	6.7×10^7
Earth	93,000,000	9.3×10^7
Mars	142,000,000	1.42×10^8
Jupiter	484,000,000	4.84×10^8
Saturn	888,000,000	8.88×10^8
Uranus	1,784,000,000	1.784×10^9
Neptune	2,799,000,000	2.799×10^9

The approximate distances are written in both standard notation and scientific notation. It is useful to write numbers in **scientific notation** when they are very large or very small. Examine the numbers in scientific notation. How would you describe a number in scientific notation?

Scientific notation uses powers of 10 since numbers are in a base-ten system. Each time you multiply a decimal value by 10, you move the decimal point one place to the right. Each time you divide by 10, the decimal point moves one place to the left. Look at the difference between 4.5×10^4 and 4.5×10^{-4}.

Multiplying by Tens

$$4.5 \xrightarrow{\times 10} 45 \xrightarrow{\times 10} 450 \xrightarrow{\times 10} 4500 \xrightarrow{\times 10} 45000$$

$$45000 = 4.5 \times 10^4$$

Dividing by Tens

$$4.5 \xrightarrow{\div 10} 0.45 \xrightarrow{\div 10} 0.045 \xrightarrow{\div 10} 0.0045 \xrightarrow{\div 10} 0.00045$$

$$0.00045 = 4.5 \times 10^{-4}$$

Dividing by 10 is the same as multiplying by 10^{-1}.

When 4.5 was multiplied by 10 four times, the decimal point moved four places to the right. When 4.5 was divided by 10 four times, the decimal point moved four places to the left.

Scientific Notation

Scientific notation is an exponential expression using a power of 10 where $1 \le |N| < 10$ and P is an integer.

$$N \times 10^P$$

EXAMPLE 1 **Convert 52,000 to scientific notation.**

Solution

Move the decimal point to the **left** until it creates a number whose absolute value is equal to or larger than 1 and less than 10.

52,000 → 5.2

Count how many places the decimal point was moved in the number above.

52000.

The decimal point was moved 4 places.

The decimal point was moved 4 places to the **left**. Since the decimal point was moved left, the P value is positive.

$52{,}000 = 5.2 \times 10^4$.

EXAMPLE 2 **Convert 0.00492 to scientific notation.**

Solution

Move the decimal point to the **right** until it creates an absolute value equal to or larger than 1 and less than 10.

0.00492 → 4.92

Count how many places the decimal point was moved in the number above.

0.00492

The decimal point was moved 3 places.

The decimal point was moved 3 places to the **right**. Since the decimal point was moved right, the P value is negative.

$0.00492 = 4.92 \times 10^{-3}$.

There are times when numbers are written in scientific notation and you want to convert them to standard notation. You must expand the number by multiplying by the given power of ten. The exponent on the 10 gives the number of digits the decimal point is moved to the right or left.

EXAMPLE 3 **Write each of the following numbers in standard notation.**
a. 3.8×10^5 **b.** 6.12×10^{-3}

SOLUTIONS

a. Move the decimal five places to the right. 3.8

Fill in empty spaces with zeros. $3.8 \times 10^5 = 380{,}000$

b. Move the decimal three places to the left. 6.12

Fill in empty spaces with zeros. $6.12 \times 10^{-3} = 0.00612$

You may need to compare numbers in scientific and standard notation. Convert each number to standard notation and then compare the numbers as asked.

EXAMPLE 4 **Four truck drivers kept track of their mileage for the year. The chart below shows the number of miles each one drove. List the drivers in order from the least mileage driven to the greatest mileage driven.**

Name	Mileage
Sam	5.41×10^4
Tom	3×10^4
Pete	25,000
Juan	1.1×10^5

SOLUTION Convert each number to standard notation.

Name	Mileage	Standard Notation
Sam	5.41×10^4	54,100
Tom	3×10^4	30,000
Pete	25,000	25,000
Juan	1.1×10^5	110,000

List the drivers from least mileage to greatest mileage. Pete, Tom, Sam, Juan

EXERCISES

Write each large or small number in scientific notation.

1. 0.000058 **2.** 9,700 **3.** 2,000,000,000

4. 0.00921 **5.** 0.0000007 **6.** 81,400,000

7. 560 **8.** 0.001092 **9.** 750,000

Write the following numbers in standard notation.

10. 2.3×10^7

11. 7.1×10^{-3}

12. 6.04×10^3

13. 8×10^{-4}

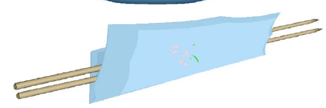
14. 4.317×10^5

15. 9.45×10^{-2}

16. According to most-expensive.net, *Spider-Man 3* cost more to produce than any other movie before it. The budget was \$258,000,000. Write this number in scientific notation.

17. Maryland's population in 2009 could be approximated by 5.7×10^6. Write this number in standard notation.

18. Jillian ate 6.1×10^5 milligrams of rice. Chris ate 4×10^6 milligrams of rice. Who ate more?



19. List the following numbers in order from least to greatest.

0.091, 9.9×10^{-3}, 0.06, 8×10^{-2}

20. The approximate heights of four Oregon mountains are listed in the table below. List the mountains in order from the shortest to the tallest.

Name	Height (feet)
South Sister	10,300
Mt. Hood	1.1×10^4
Mt. McLoughlin	9,300
Mt. Jefferson	1.02×10^4

21. The diameter of Jupiter (the largest planet) is approximately 1.4×10^5 kilometers. What is this distance in standard notation?

22. Owen weighed 3 different bugs. List the bugs in order from lightest to heaviest.

Bug	Weight (ounces)
Fly	2.5×10^{-4}
Spider	7.1×10^{-5}
Bee	4×10^{-3}

23. Jamal determined that he is approximately 1.04×10^{-3} miles tall. How many places, and in which direction, must you move the decimal to determine his height in standard notation?

24. In what types of jobs might people work with very large or very small numbers? Give examples of the values they may work with.

APPLICATIONS OF SCIENTIFIC NOTATION

LESSON 3-H

Compute with numbers in scientific notation.

EXPLORE! **POPULATIONS**

The table below shows the estimated 2005 populations of ten countries based on the approximations of the U.N. World Population Prospects.

Country / Territory	Approximate Population July 2005 UN estimate
People's Republic of China	1.3×10^9
India	1.1×10^9
United States of America	3×10^8
Mexico	1.1×10^8
France	6.5×10^7
Kenya	3.4×10^7
Uzbekistan	2.6×10^7
Nicaragua	5.5×10^6
Iceland	3×10^5
Tonga	1×10^5

Step 1: Write the population of the People's Republic of China and the population of France in standard notation.

Step 2: How many times larger is the population of China than the population of France?

Step 3: Look at the equivalence statement: $\left(\frac{1.3 \times 10^9}{6.5 \times 10^7}\right) = \left(\frac{1.3}{6.5}\right) \times \left(\frac{10^9}{10^7}\right)$. Simplify each factor.

Write your answer as a decimal.

$\square \times \square$

Remember to use the Division Property of Exponents.

Step 4: Is your answer in **Step 3** in scientific notation? If not, change it so that it is in scientific notation. How does your answer compare to your answer in **Step 2**?

Step 5: You have either: (1) converted to standard notation and then divided or (2) divided the factors while in scientific notation. Which of the two methods for dividing numbers in scientific notation do you like better. Why?

Step 6: Use either method in **Step 5** to answer the following questions. Write each answer in standard and scientific notation.

a. How many times larger is the population of India compared to Nicaragua?

b. How many times larger is the population of the USA compared to Iceland?

c. According to the US Department of Agriculture, the average American consumes 1.6×10^2 pounds of sugar each year. Approximately how many pounds of sugar are consumed in America in one year?

d. In 2005, if the land in Mexico was split evenly among the population, each person would have 1.9×10^5 square feet. Approximately how many square feet of land is there in Mexico? Write your answer in both standard notation and scientific notation.

Computing with numbers in scientific notation can be completed using different methods. One method involves converting numbers in scientific notation to standard notation and then completing the computation. Another method involves using exponent properties to simplify and compute. In **Examples 1 and 2**, you will see how a division problem and a multiplication problem can be solved using each method.

EXAMPLE 1

Find the value of $\frac{8.6 \times 10^4}{4.3 \times 10^{-3}}$. Write the answer in scientific and standard notation.

SOLUTION

METHOD 1 – Working in Standard Notation

Convert both numbers to standard notation. $\frac{8.6 \times 10^4}{4.3 \times 10^{-3}} = \frac{86000}{0.0043}$

Divide. $\frac{86000}{0.0043} = 20{,}000{,}000$

Write the answer in scientific notation. $20{,}000{,}000 = 2 \times 10^7$

METHOD 2 – Using Exponent Properties

Group factors. $\frac{8.6 \times 10^4}{4.3 \times 10^{-3}} = \left(\frac{8.6}{4.3}\right)\left(\frac{10^4}{10^{-3}}\right)$

Divide each factor. Subtract exponents when dividing with like bases.

$$\left(\frac{8.6}{4.3}\right)\left(\frac{10^4}{10^{-3}}\right) = 2 \times 10^{4-(-3)}$$
$$= 2 \times 10^7$$

Write the answer in standard notation. $2 \times 10^7 = 20{,}000{,}000$

EXAMPLE 2

Find the value of $(2.4 \times 10^5)(6 \times 10^9)$. Write the answer in scientific and standard notation.

SOLUTION

METHOD 1 – Working in Standard Notation

Convert both numbers to standard notation.	$(2.4 \times 10^5)(6 \times 10^9)$ $= 240000 \times 6000000000$
Multiply.	240000×6000000000 $= 1{,}440{,}000{,}000{,}000{,}000$
Write the answer in scientific notation.	$1{,}440{,}000{,}000{,}000{,}000 = 1.44 \times 10^{15}$

METHOD 2 – Using Exponent Properties

Group like factors.	$(2.4 \times 10^5)(6 \times 10^9) = (2.4 \times 6)(10^5 \times 10^9)$
Multiply each factor. Add exponents when multiplying with like bases.	$(2.4 \times 6)(10^5 \times 10^9) = 14.4 \times 10^{14}$
Convert to scientific notation. In this case the decimal point must be moved so the leading value is less than 10.	$14.4 \times 10^{14} = 1.44 \times 10^{15}$
Write the answer in standard notation.	$1.44 \times 10^{15} = 1{,}440{,}000{,}000{,}000{,}000$

Computing with Numbers in Scientific Notation

Method 1 - Working in Standard Notation

1. Convert numbers to standard notation.
2. Perform calculations.
3. Convert answer back to scientific notation when necessary.

Method 2 - Using Exponent Properties

1. Group like factors.
2. Multiply or divide. Use properties of exponents when finding products or quotients of powers of 10.
3. If necessary, convert to scientific notation so the absolute value of the leading number is greater than or equal to 1 and less than 10.
4. Convert answer to standard notation when necessary.

EXERCISES

Find each product. Write each answer in scientific and standard notation.

1. $(3 \times 10^6)(2 \times 10^3)$

2. $(1.5 \times 10^{-5})(5 \times 10^{-3})$

3. $(5.4 \times 10^3)(3 \times 10^7)$

4. $(-7 \times 10^4)(2 \times 10^{-6})$

5. $(8.6 \times 10^5)(3.1 \times 10^3)$

6. $(2.2 \times 10^{-10})(8 \times 10^{-7})$

7. Which method(s) from this lesson did you use to find the products? Why did you choose this method for multiplying?

Find each quotient. Write each answer in scientific and standard notation.

8. $\dfrac{9 \times 10^9}{3 \times 10^4}$

9. $\dfrac{7.7 \times 10^9}{1.1 \times 10^1}$

10. $\dfrac{2.55 \times 10^{-2}}{5 \times 10^{-5}}$

11. $\dfrac{4.96 \times 10^3}{6.2 \times 10^7}$

12. $\dfrac{1.8 \times 10^2}{9 \times 10^{-8}}$

13. $\dfrac{1.6 \times 10^1}{6.4 \times 10^{-5}}$

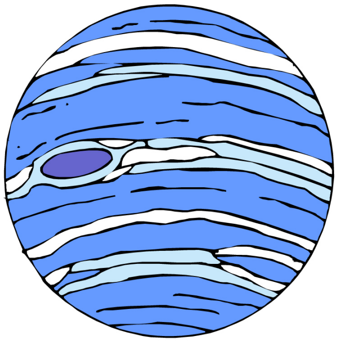
14. Which method(s) from this lesson did you use to find the quotients? Why did you choose this method for dividing?

15. The population of Russia in 2005 was approximately 1.4×10^8 people. The population of Greenland was approximately 5.6×10^4 people. How many times more people were in Russia than Greenland? Write your answer in scientific and standard notation.

16. There were approximately 3.1×10^8 people in the United States of America in 2010. The average person drank 3.3×10^3 ounces of soda.

a. Approximately how many ounces of soda were drunk in the USA in 2010?

b. The average global citizen (outside of the USA) drank 6.8×10^2 ounces of soda in 2010. Approximately how many times more ounces did an American drink per year compared to a global citizen?



17. The sun has a mass of approximately 2×10^{30} *kg*. The planet Neptune has a mass of approximately 1×10^{26} *kg*. About how many times more mass does the sun have compared to Neptune? Write your answer in standard notation.

18. Both Liam and Micah performed the same calculation. Liam got an answer of 24×10^8. Micah arrived at an answer of 2.4×10^9.

a. Are their answers equal to each other? How do you know?

b. Which answer is in scientific notation? Why is the other answer not in scientific notation?

19. In 2008, the average American produced 4.5 pounds of garbage per day (including recycled garbage). The population of the United States at that time was approximately 3×10^8 people. How much garbage was produced in America during 2008? Assume there are 365 days in one year.

20. Mr. Knight's class wanted to determine how many people lived in Mexico, the United States and Canada combined. They discovered that in 2005 Canada had an approximate population of 3.2×10^7 people, Mexico had approximately 1.1×10^8 people and the USA had 3×10^8 people.

a. Frank said he added together all of the front numbers (3.2, 1.1 and 3) and then added the exponents to get a total of 7.3×10^{23} people. Does Frank's method work? Why or why not?

b. What was the approximate population of Canada, Mexico and the USA in 2005? How did you find this total?

EXPONENTS AND ROOTS

LESSON 3-J

Use roots to solve equations with exponents.

The standard size of basketball used in the United States has a radius of 4.7 inches. You can use geometric formulas to find the surface area and volume of a basketball. Use 3.14 for π.

Surface Area $= 4\pi r^2$

$4(3.14)(4.7)^2 \approx 277.5$ inches^2

Volume $= \frac{4}{3}\pi r^3$

$\frac{4}{3}(3.14)(4.7)^3 \approx 434.7$ inches^3

There are times when you may know the surface area or volume of an object but not know other key information about the object (like the radius). You can solve equations for variables with exponents using roots. You have likely used the most common root, called the square root, in previous math courses. A **square root** is one of the two equal factors of a number. The square root symbol, $\sqrt{\ }$, is used to show the positive value of a square root. A **perfect square** is a number whose square root is an integer.

A **cube root** is one of the three equal factors of a number. The mathematical symbol for a cube root is $\sqrt[3]{\ }$. A **perfect cube** is a number whose cube root is an integer. Look at the square roots of some perfect squares and cube roots of some perfect cubes in the tables below.

Perfect Squares	*Square Root*
$1 \cdot 1 = 1$	$\sqrt{1} = 1$
$2 \cdot 2 = 4$	$\sqrt{4} = 2$
$3 \cdot 3 = 9$	$\sqrt{9} = 3$
$4 \cdot 4 = 16$	$\sqrt{16} = 4$
$5 \cdot 5 = 25$	$\sqrt{25} = 5$
$6 \cdot 6 = 36$	$\sqrt{36} = 6$
$7 \cdot 7 = 49$	$\sqrt{49} = 7$
$8 \cdot 8 = 64$	$\sqrt{64} = 8$
$9 \cdot 9 = 81$	$\sqrt{81} = 9$
$10 \cdot 10 = 100$	$\sqrt{100} = 10$

Perfect Cubes	*Cube Root*
$1 \cdot 1 \cdot 1 = 1$	$\sqrt[3]{1} = 1$
$2 \cdot 2 \cdot 2 = 8$	$\sqrt[3]{8} = 2$
$3 \cdot 3 \cdot 3 = 27$	$\sqrt[3]{27} = 3$
$4 \cdot 4 \cdot 4 = 64$	$\sqrt[3]{64} = 4$
$5 \cdot 5 \cdot 5 = 125$	$\sqrt[3]{125} = 5$
$6 \cdot 6 \cdot 6 = 216$	$\sqrt[3]{216} = 6$
$7 \cdot 7 \cdot 7 = 343$	$\sqrt[3]{343} = 7$
$8 \cdot 8 \cdot 8 = 512$	$\sqrt[3]{512} = 8$
$9 \cdot 9 \cdot 9 = 729$	$\sqrt[3]{729} = 9$
$10 \cdot 10 \cdot 10 = 1000$	$\sqrt[3]{1000} = 10$

Roots are the inverse operation of the corresponding powers. When solving equations, you can "undo" an exponent by using a root. For example, to "undo" x^4 you can use a fourth root ($\sqrt[4]{}$).

When you use a root on a value, the answer may be positive, negative or both. For example, if $x^2 = 9$, then x can equal either 3 or −3 because $(3)(3) = 9$ and $(-3)(-3) = 9$. Look at the following rules below:

- When finding the root of a term with an even exponent, the answer will be both negative and positive (shown using the symbol ±).
- When finding the root of a term with an odd exponent, the answer will have the same sign as the term under the root.
- In application settings, negative answers may not always make sense. Look closely at the situation to determine if only a positive answer is needed.

Using Roots to Solve Equations with Exponents

1. Isolate the variable with the exponent.
2. Use inverse operations to undo the exponent with the corresponding root.
3. Determine if the answer should be positive, negative or both based on the original exponent and the application setting.

EXAMPLE 1

A child plays with a ball that has a surface area of 314 square inches. What is the radius of the ball? Use 3.14 for π.

Solution

Write the formula for surface area from the beginning of the lesson. $\quad$ Surface Area $= 4\pi r^2$

The ≈ sign is used because 3.14 is an approximation of π.

Substitute given information. $\quad 314 \approx 4(3.14)r^2$

Divide both sides of the equation by 4(3.14). $\quad \dfrac{314}{4(3.14)} \approx \dfrac{\cancel{4(3.14)}r^2}{\cancel{4(3.14)}}$

Square root both sides of the equation. $\quad \sqrt{25} \approx \sqrt{r^2}$

The radius of the ball is 5 inches. $\quad 5 \approx r$

A negative answer does not make sense in this situation.

EXAMPLE 2 **Solve $\frac{x^2}{2} + 11 = 43$ for x. Include all answers.**

SOLUTION

Subtract 11 from both sides of the equation.
$$\frac{x^2}{2} + \cancel{11} = 43$$
$$\quad\ -\cancel{11} \quad -11$$

Multiply both sides of the equation by 2.
$$\cancel{2} \cdot \frac{x^2}{\cancel{2}} = 32 \cdot 2$$

Square root both sides of the equation.
$$\sqrt{x^2} = \sqrt{64}$$

Include both positive and negative answers.
$$x = \pm 8$$

✓ $(-8)(-8) = 64$
and
$(8)(8) = 64$

EXAMPLE 3 **Solve $2x^3 - 5 = -59$ for x. Include all answers.**

SOLUTION

Add 5 to both sides of the equation.
$$2x^3 - \cancel{5} = -59$$
$$\quad +\cancel{5} \quad +5$$

Divide both sides of the equation by 2.
$$\frac{\cancel{2}x^3}{\cancel{2}} = \frac{-54}{2}$$

Cube root both sides of the equation.
$$\sqrt[3]{x^3} = \sqrt[3]{-27}$$
$$x = -3$$

✓ $(-3)(-3)(-3) = (-27)$

When solving equations with exponents, answers may not always be integers. When this is the case, round to the specified place value.

EXAMPLE 4 **Finley has a spherical water balloon filled with water. She knows there are 56 cubic inches of water in the balloon. What is the approximate radius, r, of the balloon? Round the answer to the nearest hundredth.**

SOLUTION

Write the formula for volume of a sphere.
$$\text{Volume} = \frac{4}{3}\pi r^3$$

Input given information.
$$56 \approx \frac{4}{3}(3.14)r^3$$

Multiply both sides of the equation by the reciprocal of $\frac{4}{3}$.
$$\frac{3}{4} \cdot 56 \approx \frac{4}{3}(3.14)r^3 \cdot \frac{3}{4}$$

Divide both sides of the equation by 3.14.
$$\frac{42}{3.14} \approx \frac{\cancel{3.14}r^3}{\cancel{3.14}}$$
$$13.38 \approx r^3$$

Cube root both sides of the equation.
$$\sqrt[3]{13.38} \approx \sqrt[3]{r^3}$$
$$2.37 \approx r$$

The radius of the balloon is approximately 2.37 inches.

EXERCISES

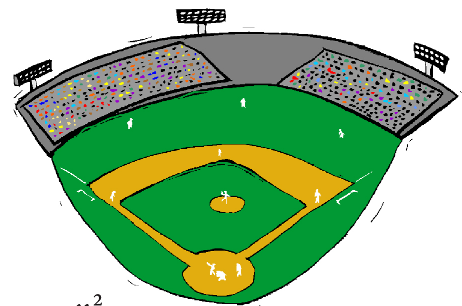
1. Match the inverse operations.

a. x^2	**A.** $-$
b. x^5	**B.** $\sqrt{}$
c. $\div$	**C.** $\sqrt[5]{}$
d. $\sqrt[3]{}$	**D.** $\times$
e. $+$	**E.** x^3

2. Listed below is the last calculation needed to solve an equation. Determine what sign the solution will have (+, – or ±).

a. $\sqrt[5]{32}$ **b.** $\sqrt{121}$ **c.** $\sqrt[3]{-343}$ **d.** $\sqrt[4]{49}$

3. Jennifer used a square root to solve a math problem that helped her find the distance between home plate and 2nd base on a baseball field. She said her approximate answer was ±127.3 feet. Her teacher said her answer was almost correct. What is wrong with her answer?



Solve each equation. Include all answers.

4. $x^3 = 512$ **5.** $x^2 + 4 = 53$ **6.** $\frac{x^2}{5} = 20$

7. $10 + 7x^3 = -46$ **8.** $4x^2 - 20 = 556$ **9.** $\frac{2}{3}x^3 - 1 = 17$

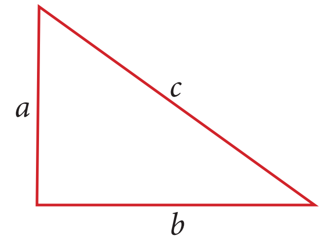
10. $\frac{x^3}{10} + 12 = 112$ **11.** $x^5 - 1 = 242$ **12.** $5x^2 - 30.5 = 149.5$

13. The surface area of a sphere is found using the formula SA = $4\pi r^2$ where r is the radius. A playground ball has a surface area of 113.04 square inches. What is the radius of the ball? Use 3.14 for π.

14. The volume of a sphere is found using the formula V = $\frac{4}{3}\pi r^3$ where r is the radius. A large globe has a volume of 3,052.08 cubic inches. What is the radius of the globe? Use 3.14 for π.

15. The Pythagorean Theorem states that in a right triangle, the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse. Use this to solve for the missing side lengths.

$$\boxed{a^2 + b^2 = c^2}$$



a. If $a = 12$ and $c = 13$, find b.
b. If $b = 15$ and $c = 25$, find a.
c. If $a = 7$ and $b = 24$, find c.

Solve each equation. Include all answers. Round each answer to the nearest hundredth.

16. $x^3 = 88$

17. $7x^2 = 77$

18. $2x^3 - 4 = 20$

19. $5x^3 - 20 = -130$

20. $\frac{x^2}{4} = 260$

21. $\frac{1}{2}x^2 + 16 = 400$

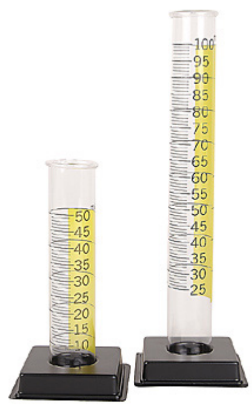
22. The volume of a cube is found using the formula $V = s^3$ where s is the length of a side. A cube has a volume of 250 cubic feet. What is the approximate length of a side? Round to the nearest hundredth.

VOLUME OF CYLINDERS

LESSON 3-K

Find the volume of cylinders and solve real-world problems involving cylinders.

Volume is the number of cubic units needed to fill a three-dimensional figure. A **cylinder** is a solid with two congruent and parallel bases that are circles. Just as with prisms, the volume of a cylinder is found by multiplying the area of its base by its height. Volume is written using cubic units (e.g., in^3, cm^3, ft^3, m^3).



VOLUME OF A CYLINDER

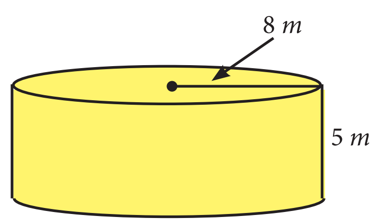
The volume of a cylinder is equal to the product of the area of the base (B) and the height (h).

$$V = Bh$$

$$V = \pi r^2 h$$

EXAMPLE 1

Find the volume of the cylinder. Use 3.14 for π.



Since 3.14 is an approximation of π, the answer will be an approximation.

SOLUTION

Write the volume formula for a cylinder.	$V = \pi r^2 h$
Substitute all known values for the variables.	$V \approx (3.14)(8)^2(5)$
Find the value of the power.	$V \approx (3.14)(64)(5)$
Multiply.	$V \approx 1004.8$

The volume of the cylinder is about 1,004.8 cubic meters.

The missing dimensions of a cylinder can be found if the volume and one other measurement is provided. First substitute all known values into the volume formula, then use inverse operations to find the missing value.

EXAMPLE 2

The volume of a cylindrical water cooler is 1,695.6 cubic inches. The cooler has a radius of 6 inches. Find the height of the cooler. Use 3.14 for π.

SOLUTION

Write the volume formula for a cylinder. $V = \pi r^2 h$

Substitute all known values for the variables. $1695.6 \approx (3.14)(6^2)h$

Find the value of the power. $1695.6 \approx 3.14(36)h$

Multiply. $1695.6 \approx 113.04h$

Divide both sides of the equation by 113.04. $\frac{1695.6}{113.04} \approx \frac{\cancel{113.04}h}{\cancel{113.04}}$

$15 \approx h$

The height of the water cooler is about 15 inches.

EXAMPLE 3

Find the radius of a cylinder with an approximate volume of 3,014.4 and a height of 15 inches. Use 3.14 for π.

SOLUTION

Write the volume formula for a cylinder. $V = \pi r^2 h$

Substitute all known values for the variables. $3014.4 \approx 3.14r^2 15$

Multiply. $3014.4 \approx 47.1r^2$

Divide both sides of the equation by 47.1. $\frac{3014.4}{47.1} \approx \frac{\cancel{47.1}r^2}{\cancel{47.1}}$

$64 \approx r^2$

Square root both sides of the equation. $\sqrt{64} \approx \sqrt{r^2}$

Simplify. $8 = r$

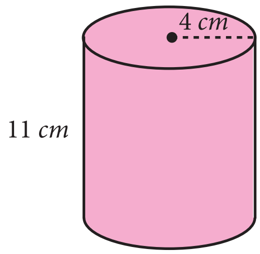
Only the positive root is appropriate in this situation.

The radius of the cylinder is about 8 inches.

EXERCISES

Find the volume of each cylinder. Use 3.14 for π. Round to the nearest hundredth.

1.



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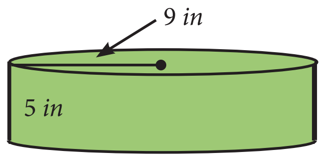

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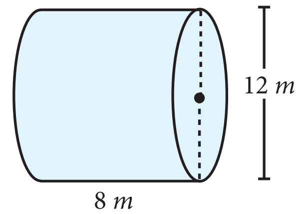
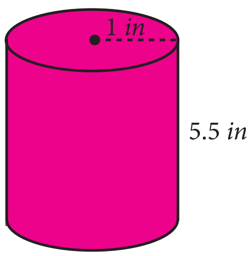


Find the volume of each solid. Use 3.14 for π.

4.



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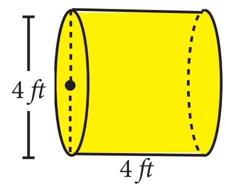

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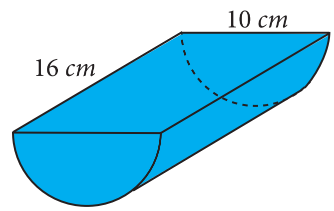


7. Kambree filled her circular swimming pool. The radius of the pool is 10 feet. The height is 4 feet. The instructions recommend filling the pool to the 3 foot mark.

a. Calculate the volume of water the pool contains when Kambree follows the recommendations.

b. Why would it be recommended not to fill the pool to the very top?

8. Which solid can hold more: a cylinder with a diameter and height of 6 inches or a square prism with side lengths of 6 inches? Verify the answer with calculations.

9. A round cement stepping stone has a radius of 1 foot and is 2 inches thick.

a. Convert the radius to inches.

b. How many cubic inches of cement does it take to make one stepping stone?

c. How many cubic inches of cement would it take to make 45 stepping stones?

10. A grain silo is made of cylindrical sections that have a 7.2 *m* diameter and are 2.4 *m* tall. The cylindrical sections are stacked on top of one another to form a silo. Orville wants to be able to store 500 cubic meters of grain. How many sections will Orville need to buy?

11. Recycling is important for the environment. A cylindrical rain barrel used to recycle water can hold 15,197.6 cubic inches of water. It has a radius of 11 inches. How tall is the barrel?

12. A cylindrical juice pitcher has a radius of 3 inches and is 10 inches tall.

a. Find the volume of the pitcher.

b. If the pitcher is half full, how much juice is in it?

13. A round sugar container has a volume of 3,140 cubic centimeters. The radius of the container is 10 *cm*.

a. Find the height of the container.

b. Could 3,140 sugar cubes fit in the container if each cube measures 1 *cm* on each side? Why or why not?

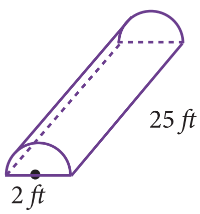
14. A cylindrical vase for flowers holds about 125.6 cubic inches of water. The vase is 10 inches tall. Find the radius of the vase.

15. A cylindrical diesel tank holds approximately 226.08 cubic feet of diesel fuel. The tank is 8 feet long. What is the radius of the tank?

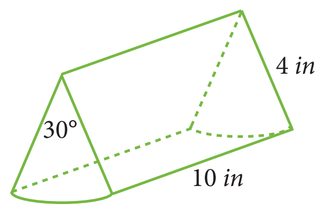
16. Becky baked cookies. She wants to ship some to her brother who is overseas with the military. She has a cylindrical can with a volume of 1130.4 cm^3. The can is 22.5 *cm* tall. Assuming each cookie is circular and lays parallel to the base of the can, what is the largest diameter a cookie can have?

Find the volume of each solid. Use 3.14 for π. Round to the nearest hundredth.

17.



18.



19. Brenden has a cylinder with a radius that is 18 *cm*. It is 25 *cm* tall.

a. Find the volume of Brenden's cylinder.

b. What is the radius of a cylinder that is half the volume of Brenden's cylinder and is 50 *cm* tall?

VOLUME OF CONES

LESSON 3-L

Find the volume of cones.
Solve real-world problems involving cones.

EXPLORE! **CONES IN A CYLINDER**

Rita is making snow cones for her friends. She has four cylindrical glasses that she is using to take the crushed ice outside to her friends that are waiting to fill their snow cone cups. Each snow cone cup is as tall as one of the glasses and has the same size circular base. Rita wants to determine if there is a relationship between a cone and a cylinder. She needs to make sure she has enough ice for 12 snow cones. Complete the Explore! to find if Rita will have enough ice for the snow cones.

Step 1: Make or find a cone and a cylinder that have congruent bases and are the same height.

Step 2: Estimate how many times larger the volume of the cylinder is compared to the volume of the cone.

Step 3: Fill the cone with rice, beans or popcorn kernels. Pour the contents of the cone into the cylinder. Repeat until the cylinder is full.

Step 4: How many times did you need to empty the cone in order to fill the cylinder?

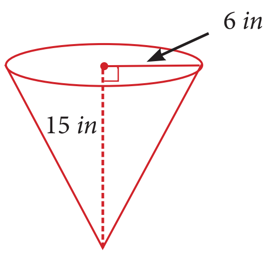
Step 5: What fraction is the volume of the cone compared to the volume of the cylinder?

Step 6: What is the formula for the volume of a cylinder?

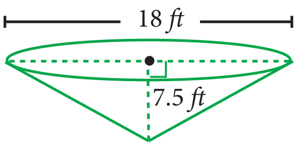
Step 7: Combine the answers from **Step 5** and **Step 6** to write a formula that can be used to find the volume of a cone.

Step 8: Use your formula to find the volume of each cone. Use 3.14 for π. Round to the nearest hundredth.

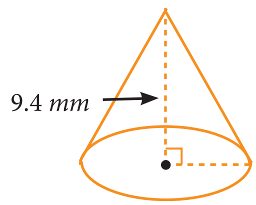
a.



b.



c. $r = 2.1$ *mm*



Step 9: Will Rita have enough crushed ice for 12 snow cones if she fills the four cylinders?

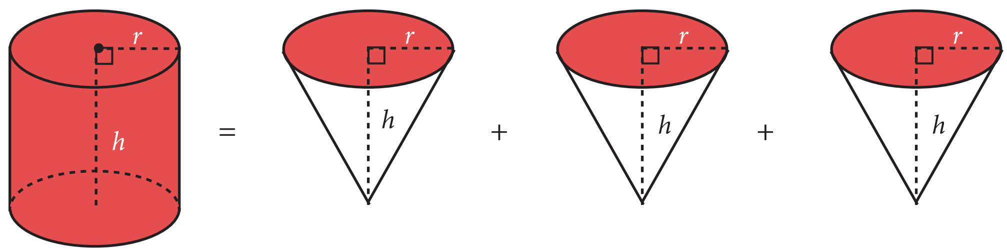
Volume of a Cone

The volume of a cone is equal to one-third of the product of the area of the base (B) and height (h).

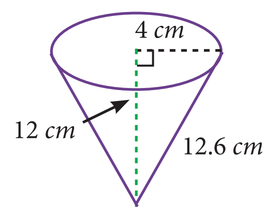
$$V = \frac{1}{3}Bh \quad \textbf{or} \quad V = \frac{Bh}{3}$$

$$V = \frac{1}{3}\pi r^2 h \quad \textbf{or} \quad \frac{\pi r^2 h}{3}$$

It takes the volume of three cones to equal the volume of one cylinder when they have congruent bases and heights. The height, h, of a cone is how tall a cone is from its vertex to the center of its base. A pyramid and a prism with congruent bases and heights have the same relationship.



EXAMPLE 1 **Find the volume of the cone. Use 3.14 for π.**



SOLUTION

Write the volume formula for a cone.	$V = \frac{1}{3}\pi r^2 h$
Substitute all known values for the variables.	$V \approx \frac{1}{3}(3.14)(4)^2(12)$
Find the value of the power.	$V \approx \frac{1}{3}(3.14)(16)(12)$
Multiply.	$V \approx 200.96$

The volume of the cone is about 200.96 cm^3.

EXAMPLE 2 **Chantel helped with her sister's party. Each child received a party hat full of treats. Each hat had a volume of 65.94 cubic inches. The radius of each hat was 3 inches. About how tall was each party hat?**

SOLUTION

Write the volume formula for a cone.	$V = \frac{1}{3}\pi r^2 h$
Substitute all known values for the variables.	$65.94 \approx \frac{1}{3}(3.14)(3)^2 h$
Find the value of the power.	$65.94 \approx \frac{1}{3}(3.14)(9)(h)$
Multiply.	$65.94 \approx 9.42h$
Divide both sides of the equation by 9.42.	$\frac{65.94}{9.42} \approx \frac{\cancel{9.42}h}{\cancel{9.42}}$
	$7 \approx h$

Each party hat was about 7 inches tall.

EXAMPLE 3 **A cone-shaped popcorn container holds 314 cubic inches of popped corn. The container is 12 inches tall. Find the radius of the conical container. Use 3.14 for π.**

SOLUTION

Write the volume formula for a cone. $V = \frac{1}{3}\pi r^2 h$

Substitute all known values for the variables. $314 \approx \frac{1}{3}(3.14)r^2(12)$

Multiply. $314 \approx 12.56r^2$

Divide by 12.56 on both sides of the equation. $\frac{314}{12.56} \approx \frac{\cancel{12.56}r^2}{\cancel{12.56}}$

Square root both sides of the equation. $\sqrt{25} \approx \sqrt{r^2}$

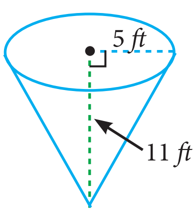
$5 \approx r$

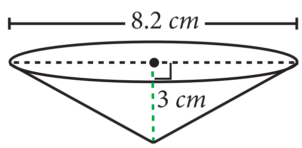
Only the positive root is used in this situation.

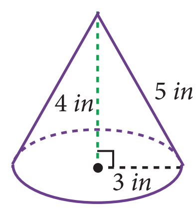
The radius of the popcorn container is about 5 inches.

EXERCISES

Find the volume of each cone. Use 3.14 for π. Round answers to the nearest whole number.

1.


2.


3.


4. A cone has the same height as a cylinder. The areas of the bases of the two solids are the same. How many cones does it take to fill the cylinder?

5. The volume of a cylinder is 18 cubic meters. What is the volume of a cone with a congruent base and height?

6. A cylinder holds 87 gallons of water. How many gallons of water does a cone with the same radius and height hold?

7. Sketch a diagram of a cone with a radius of 9 yards and a height of 12 yards. Find the volume of the cone.

Find the volume of each cone with the given radius and height. Use 3.14 for π. Round to the nearest hundredth.

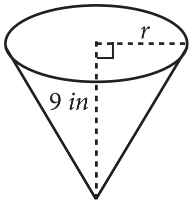
8. $r = 3\ m$
$h = 5\ m$

9. $r = 6\ cm$
$h = 2\ cm$

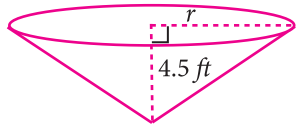
10. $d = 20\ in$
$h = 7\ in$

Find the length of the radius in each cone. Use 3.14 for π. Round to the nearest whole number.

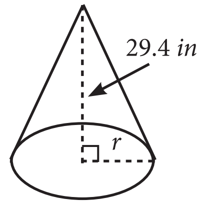
11. V ≈ 339.12 in^3

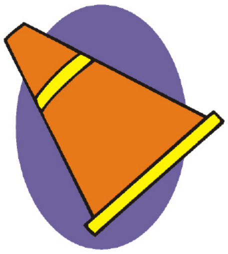


12. V ≈ 169.56 ft^3



13. V ≈ 30.77 in^3





14. An orange cone is often used to alert drivers in a construction zone to slow down. A construction cone is 24 inches tall. The circular base has a diameter of 18 inches. Find the volume of the cone. Use 3.14 for π.

15. An icicle is shaped like a cone. It has a diameter of 1 inch. It is 2 feet long.
- **a.** Convert all measurements to inches.
- **b.** Draw a diagram and label it.
- **c.** Find the volume of the icicle. Use 3.14 for π.

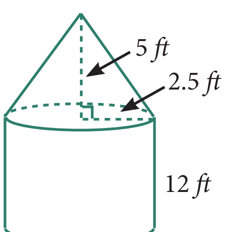
16. A candle maker makes cone-shaped candles. The candle maker uses 9.42 cubic inches of wax to make each candle. The radius of each candle is 1.5 inches. Find the height of the candles. Use 3.14 for π.

17. The conical paper cups next to a water cooler each hold 3.14 cubic inches of water. The radius of each paper cup is 1 inch. Find the height of the cup. Use 3.14 for π.

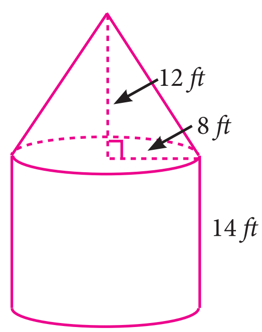
18. Frank is setting up a conical teepee. The box says it is 10 feet tall and has a volume of approximately 150.72 cubic feet. He needs to find the diameter of the teepee to find a space large enough to set it up.
- **a.** Find the radius of the teepee.
- **b.** What is the diameter of the teepee?
- **c.** If Frank has a square yard that has a perimeter of 28 feet, will the teepee fit in it?

Find the volume of each composite solid. Use 3.14 for π. Round to the nearest hundredth.

19.



20.



VOLUME OF SPHERES

LESSON 3-M

Find the volume of spheres and solve real-world problems involving spheres.

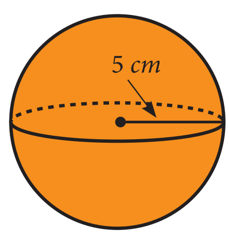
Many sports involve spheres. Baseballs, softballs, volleyballs, dodge balls, soccer balls, tennis balls and basketballs are all spheres. A **sphere** is a round, curved, closed three-dimensional solid. A sphere has no edges, sides or vertices. Every point on the surface of the sphere is an equal distance from the center of the sphere. This distance is the radius of the sphere.

VOLUME OF A SPHERE

The volume of a sphere is equal to four-thirds the product of pi (π) and the cube of its radius (r^3).

$$V = \frac{4}{3}\pi r^3 \quad \textbf{or} \quad V = \frac{4\pi r^3}{3}$$

EXAMPLE 1 **Find the volume of the sphere. Use 3.14 for π.**



SOLUTION

Write the volume formula for a sphere.	$V = \frac{4}{3}\pi r^3$
Substitute known values for the variables.	$V \approx \frac{4}{3}(3.14)(5)^3$
Find the value of the power.	$V \approx \frac{4}{3}(3.14)(125)$
Multiply.	$V \approx 523.33$

The volume of the sphere is about 523.33 cm^3.

EXAMPLE 2 **A water tower has a spherical tank. The diameter of the tank is 30 meters. How much water can the tank hold? Use 3.14 for π.**

SOLUTION

Find the radius of the tank.	Diameter ÷ 2 = 30 ÷ 2 = 15
Write the volume formula for a sphere.	$V = \frac{4}{3}\pi r^3$
Substitute known values for the variables.	$V \approx \frac{4}{3}(3.14)(15)^3$
Find the value of the power.	$V \approx \frac{4}{3}(3.14)(3375)$
Multiply.	$V \approx 14{,}130$

The tank can hold approximately 14,130 cubic meters of water.

EXAMPLE 3 **A bouncy ball has a volume of 113.04 cubic centimeters. Find the radius of the ball. Use 3.14 for π.**

SOLUTION

Write the volume formula for a sphere.	$V = \frac{4}{3}\pi r^3$
Substitute known values for the variables.	$113.04 \approx \frac{4}{3}(3.14)r^3$
Multiply.	$113.04 \approx 4.19r^3$
Divide both sides of the equation by 4.19.	$\frac{113.04}{4.19} \approx \frac{\cancel{4.19}r^3}{\cancel{4.19}}$
	$27 \approx r^3$
Cube root both sides of the equation.	$\sqrt[3]{27} \approx \sqrt[3]{r^3}$
Simplify.	$3 \approx r$

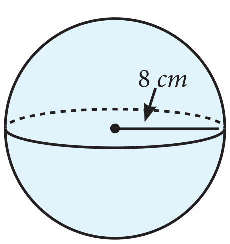
This is a rounded answer. Rounding can make cubic roots easier to calculate.

The radius of the bouncy ball is close to 3 *cm*.

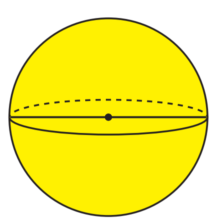
EXERCISES

Find the volume of each sphere. Use 3.14 for π. Round to the nearest hundredth.

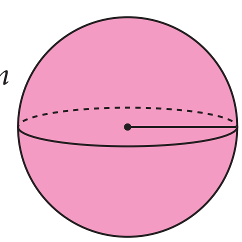
1.



2. $d = 15$ *in*



3. $r = 4$ *m*



4. A standard basketball used by professionals has a radius close to 12 *cm*. Find the approximate volume of a standard basketball.

5. A spherical liquid soap container has a diameter of 5 *in*. How much soap can the container hold?

6. A bowl is shaped like a hemisphere, which is half of a sphere. The diameter of the bowl is 8 inches. How much water will the bowl hold?

7. Karissa is having a party. She has a dozen balloons that are spheres when they are inflated. She wants to fill them with helium, but she does not know how much helium she needs to buy. The radius of each balloon is 6 inches when it is inflated properly.

a. Calculate the volume of one balloon. Use 3.14 for π. Round to the nearest hundredth, as needed.

b. How much total helium will it take to fill all of the balloons?

c. If helium can only be purchased in whole number units, how much helium will Karissa need to buy?

8. Tennis balls are sold in sets of three inside a cylindrical can. Each tennis ball has a diameter of 2.5 inches. Assume the balls touch the can on the sides, top and bottom.

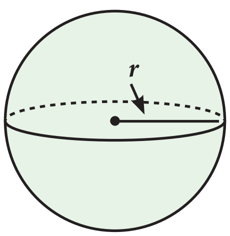
a. Calculate the volume of one tennis ball.

b. Calculate the volume of the cylindrical can.

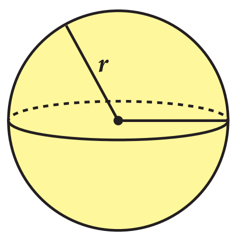
c. How many cubic inches are not used by the tennis balls inside the cylinder?

Find each missing measure. Use 3.14 for π.

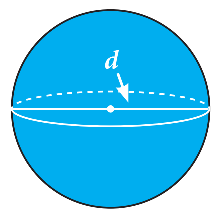
9. Volume ≈ 904.32 cm^3



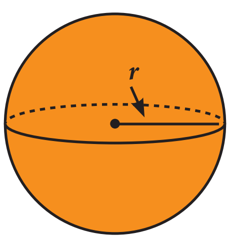
10. Volume ≈ 33.49 ft^3



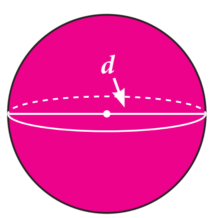
11. Volume ≈ 267.95 in^3



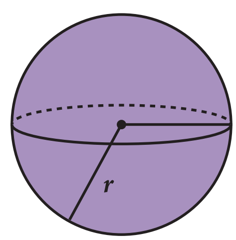
12. Volume ≈ 7234.56 m^3



13. Volume ≈ 523.33 yd^3



14. Volume ≈ 1436.03 in^3



15. A bouncy ball has a volume of 4.187 cubic inches. What is the diameter of the ball?

16. Graysen owns a world globe that has a volume of 3,052.08 cubic inches. What is the diameter of the globe?

17. A beach ball holds 800 cubic inches of air. What is the radius of the ball? Round to the nearest hundredth.

18. A spherical piece of candy has a chocolate outside with caramel in the middle. The diameter of the whole piece of candy is 3 centimeters. The diameter of the caramel filling is 2 centimeters.

a. Find the volume of one whole piece of candy.
b. Find the volume of the caramel center.
c. What is the volume of chocolate used in each piece of candy?
d. The candy comes in a package that states the volume of the candy is approximately 255 cubic centimeters. About how many pieces of candy are in the package?

19. A small beach ball has a radius of 10 inches. A larger beach ball has a volume that is twice the volume of the smaller ball. What is the radius of the large beach ball?

20. The equator is an imaginary line on the Earth's surface which divides the Earth into two equal hemispheres. It is approximately 24,901.55 miles long. Assume the earth is perfectly round and use the length of the equator to find the volume of the earth.

21. Abbigail shipped a globe to her sister. The globe fit snuggly in a box, touching the sides, bottom and top. The globe has a volume of 1,436 cubic inches. Find the volume of the box.

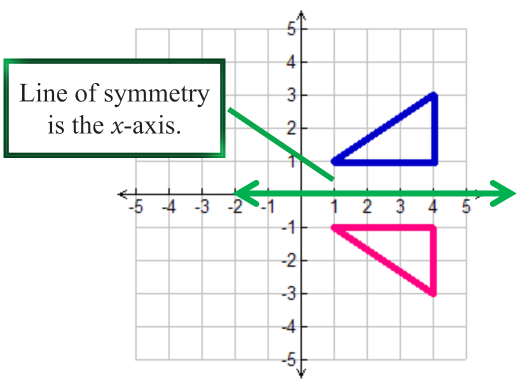
TRANSFORMATIONS

LESSON 3-N

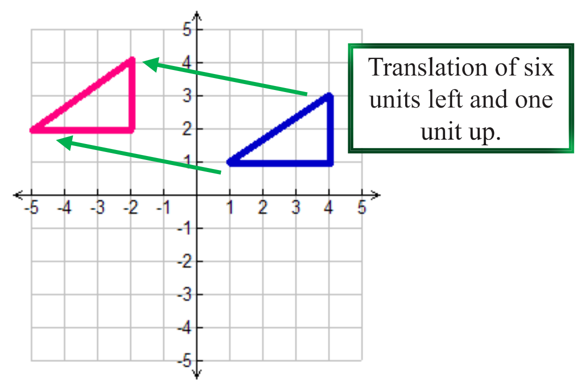
Perform single transformations on a figure including reflections, rotations, translations and dilations.

A **transformation** is the movement of a point or figure that changes its position or size. The original figure is called the **pre-image** and the resulting figure is the **image**. In this lesson, transformations are drawn on a coordinate plane. Four basic transformations are shown below. The pre-image is blue and the image is pink.

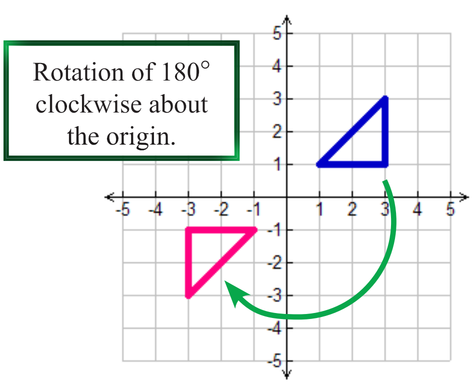
A **reflection** flips a figure over a line.
The figures will be mirror images of each other.



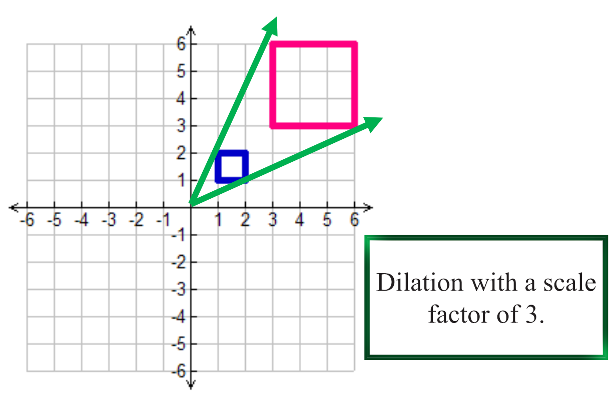
A **translation** slides a figure to a new position without turning it.



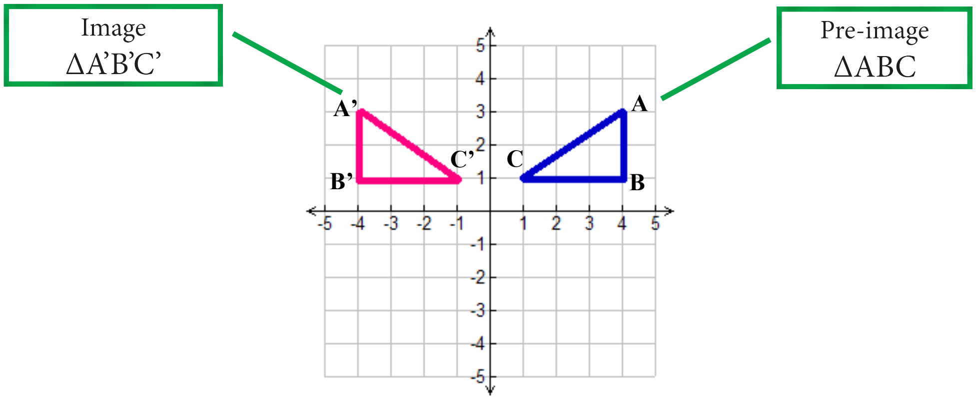
A **rotation** turns a figure about a fixed point, often the origin (0, 0).



A **dilation** changes the size of a figure but not the shape. A dilation creates similar figures.

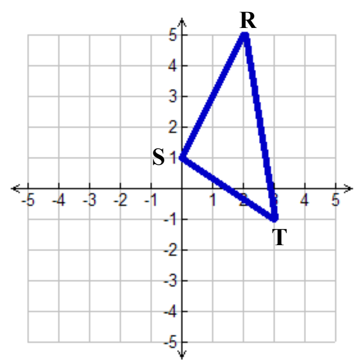


On a coordinate plane, the vertices of a figure are often labeled with letters. After a transformation occurs, the new image has vertices that are labeled with the same letter but an apostrophe is added. For example, if ΔABC is reflected over the *y*-axis, then the image is labeled ΔA'B'C'. This is read "Triangle A prime, B prime, C prime".

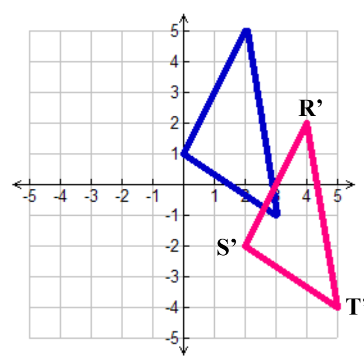


EXAMPLE 1 **A triangle with coordinates R(2, 5), S(0, 1) and T(3, –1) is translated 2 units right and 3 units down. What are the coordinates of ΔR'S'T'?**

SOLUTION Graph the original figure.



Translate each point 2 units right (add 2 to the *x*-values) and 3 units down (subtract 3 from the *y*-value).

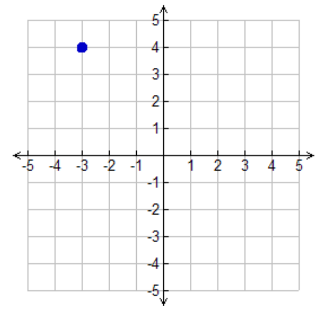


Write the ordered pairs for the coordinates of ΔR'S'T'.

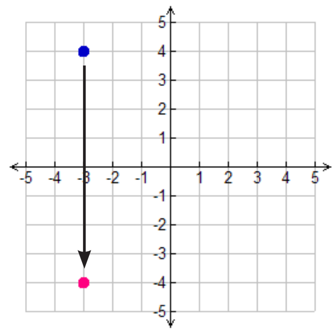
R'(4, 2) S'(2, –2) T'(5, –4)

EXAMPLE 2 **The point (−3, 4) is reflected over the *x*-axis. What are the coordinates of its image?**

SOLUTION Graph the point on a coordinate plane.

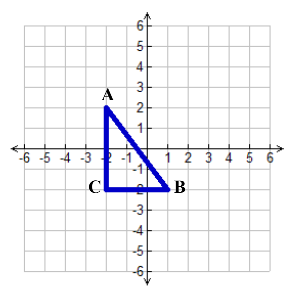


Reflect the point over the *x*-axis.



The coordinates of the image are (−3, −4).

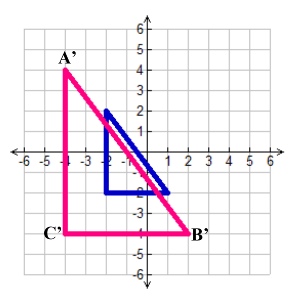
EXAMPLE 3 **ΔABC is drawn below. Find its image under a dilation with a scale factor of 2. Graph the image.**



SOLUTION Multiply the coordinates of each point by the scale factor of 2.

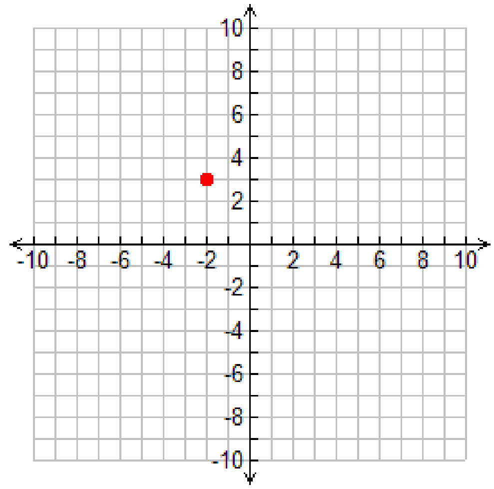
A(−2, 2) ⟶ A'(−4, 4)
B(1, −2) ⟶ B'(2, −4)
C(−2, −2) ⟶ C'(−4, −4)

Graph the image.



EXERCISES

Suppose a transformation moves a point (−2, 3). Write the coordinates for the image after each transformation.



1. A reflection over the *x*-axis.

2. A reflection over the *y*-axis.

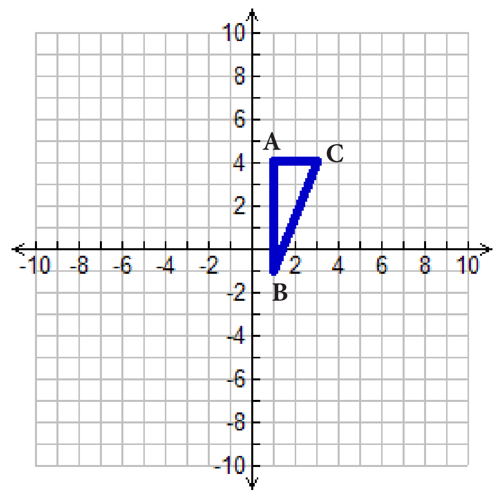
3. A translation down 3 units.

4. A translation right 1 unit and up 4 units.

5. A dilation with a scale factor of 5.

6. A rotation of 180° about the origin.

Triangle ABC has the coordinates A(1, 4), B(1, −1) and C(3, 4). For each transformation below, graph the resulting image and give the coordinates of the vertices of the image.



7. Reflection over the *y*-axis.

8. Translation of 4 units right.

9. Translation of 5 units left and 3 units down.

10. Dilation with a scale factor of 3.

11. Dilation with a scale factor of $\frac{1}{2}$.

12. A reflection over the *x*-axis followed by a translation of 2 units up.

Graph the original figure and the image using the given transformation. Label the vertices.

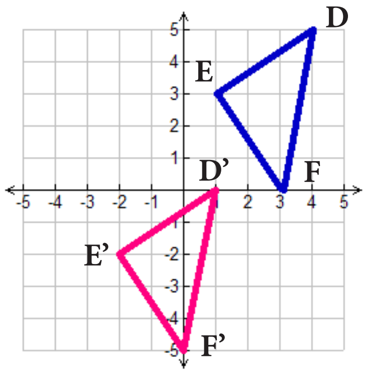
13. Square A(−1, 2), B(1, 2), C(1, 4) and D(−1, 4)
A reflection over the *x*-axis.

14. Rectangle J(0, 1), K(0, 3), L(5, 3) and M(5, 1)
A translation of 1 unit to the right and 2 units down.

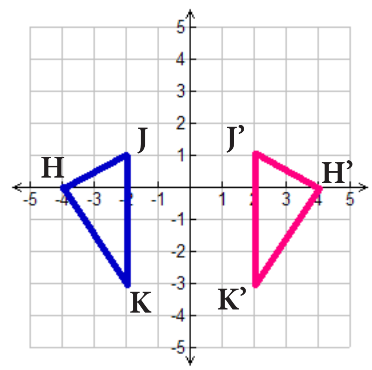
15. Triangle M(2, −1), N(−1, 0) and P(3, 0)
A dilation with a scale factor of 3.

16. Line segment G(−2, 3) and H(−4, 5)
A rotation of 180° clockwise around the origin.

17. Describe the translation that maps ΔDEF onto ΔD'E'F'.



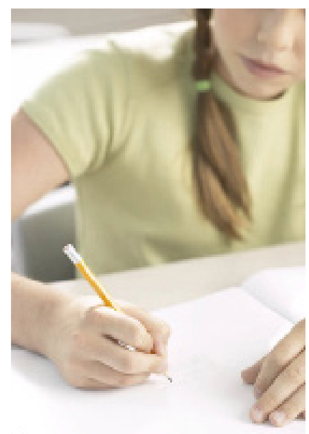
18. Describe the reflection that maps ΔHJK onto ΔH'J'K'.



19. Shanequa graphed a triangle with the coordinates R(−2, 1), S(2, 1) and T(0, −3). She wanted to graph a dilation of the figure with a scale factor of 2 so she graphed the points R'(−2, 2), S'(2, 2) and T'(0, −6).

a. Is the ΔR'S'T' a dilation of ΔRST? Why or why not?

b. If Shanequa was wrong, correct her answer by giving the correct coordinates for ΔR'S'T'.



20. What are the coordinates for the image of the point (3, −5) after it is reflected over the *y*-axis and then translated 4 units up?

21. What are the coordinates for the image of the point (−2, −1) after it is translated 6 units left, 2 units down and then reflected over the *x*-axis?

22. Write a series of three translations so the final image of a point will end up at its original location at the end of the series.

23. Tim rotated a figure 360° about the origin. How does the pre-image compare to the image?

TRANSFORMATIONS AND CONGRUENCE

LESSON 3-O

Understand the relationship between the pre-image and the image of a transformation.

Two figures are congruent if they are the exact same shape and the exact same size. Figures are similar if they are the same shape but different sizes.

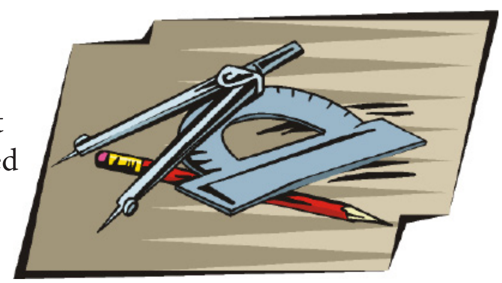
In the **Explore!** you will draw the image of a triangle given a transformation and then determine if the image is congruent or similar to the original (or pre-image) triangle.

EXPLORE! **CONGRUENT OR SIMILAR?**

Step 1: List four types of transformations on four separate lines of your paper.

Step 2: Each type of transformation forms an image that is either similar or congruent to the original figure. Make a conjecture (an educated guess) for each transformation you listed in **Step 1** about whether the image will be congruent or similar to the original figure.

Step 3: Use ΔABC as the pre-image for **Steps 4-7**: A(2, 5), B(4, 3), and C(3, 0). Use graph paper to make five coordinate planes (each from −10 to 10 on both the x- and y-axes). Graph ΔABC on the first coordinate plane.



Step 4: One type of transformation is a translation. On the second coordinate plane, graph ΔA'B'C' by shifting ΔABC 3 units left and 2 units up. What do you notice about the image compared to the pre-image? Use the word "similar" or "congruent" in your observation.

Step 5: Another type of transformation is a reflection. On the third coordinate plane, graph ΔA'B'C' by reflecting ΔABC over the x-axis. What do you notice about the image compared to the pre-image? Use the word "similar" or "congruent" in your observation.

Step 6: Dilations are another type of transformation. On the fourth coordinate plane, graph ΔA'B'C' using a scale factor of 2. What do you notice about the image compared to the pre-image? Use the word "similar" or "congruent" in your observation.

Step 7: The last type of transformation is a rotation. On the last coordinate plane, graph ΔA'B'C' by rotating ΔABC 180° clockwise about the origin. What do you notice about the image compared to the pre-image? Use the word "similar" or "congruent" in your observation.

Step 8: Compare your observations with your original conjectures. Did your work in **Steps 4-7** support your conjectures? If not, rewrite your conjecture(s) to match your work.

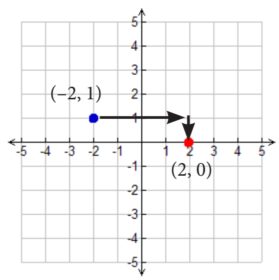
Congruency and Similarity with Transformations

Under a translation, reflection or rotation, the original figure and its image are <u>congruent</u>.

Under a dilation, the original figure and its image are <u>similar</u>.

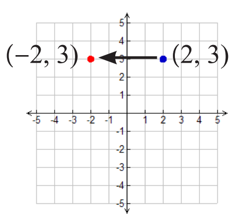
Transformations can be described using transformation rules. The rules describe the effect of the transformation on the ordered pair.

Translation – The rule shows the amount being added or subtracted to the x- and y-coordinates.
Example: shift right 4 units and down 1 unit: $(x, y) \longrightarrow (x + 4, y - 1)$



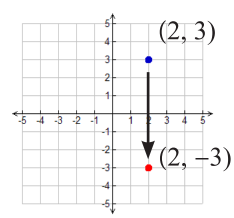
Reflection – The rule changes the sign of the x- or y- coordinate affected by the reflection.

<u>y-axis reflection</u>
When a figure is reflected over the y-axis, the x-coordinate changes sign.



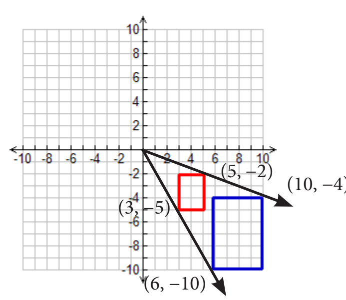
$(x, y) \longrightarrow (-x, y)$

<u>x-axis reflection</u>
When a figure is reflected over the x-axis, the y-coordinate changes sign.



$(x, y) \longrightarrow (x, -y)$

Dilation – The rule shows multiplying the x- and y-coordinates by the scale factor.
Example: scale factor of $\frac{1}{2}$: $(x, y) \longrightarrow (\frac{1}{2}x, \frac{1}{2}y)$



EXAMPLE 1

State the type of transformation described using each rule. Then state if the image will be congruent or similar to its pre-image.

a. $(x, y) \longrightarrow (x - 3, y + 4)$ **b.** $(x, y) \longrightarrow (4x, 4y)$

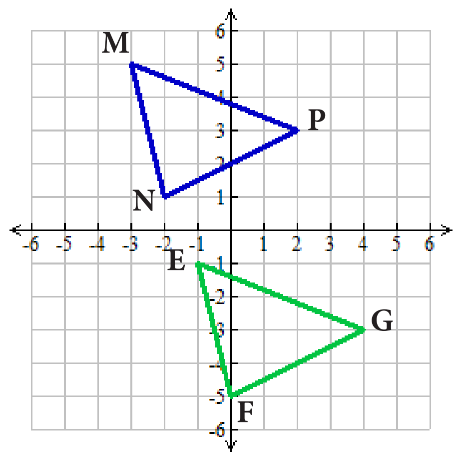
SOLUTIONS

a. $(x - 3)$ means shifting 3 units left and $(y + 4)$ means shifting 4 units up. This is a translation. Translations form congruent figures.

b. A transformation that multiplies the coordinates of a figure by a scale factor (in this case, 4), is a dilation. Dilations form similar figures.

EXAMPLE 2

ΔEFG was formed by a single transformation of ΔMNP.



a. Are ΔEFG and ΔMNP congruent or similar?
b. Write a translation rule that maps ΔMNP onto ΔEFG.

SOLUTIONS

a. ΔEFG and ΔMNP are congruent because they are the same shape and same size.
b. ΔMNP is moved 2 units to the right and 6 units down to make ΔEFG. This transformation rule can be written $(x, y) \longrightarrow (x + 2, y - 6)$.

EXERCISES

Write the transformation rule for each transformation.

1. A reflection over the y-axis.

2. A translation 4 units up.

3. A dilation with a scale factor of 3.

4. A reflection over the x-axis.

5. A translation 2 units left and 8 units down.

6. A dilation with a scale factor of 0.25.

Name the type of transformation given by each rule. State whether the transformation creates similar or congruent figures.

7. $(x, y) \longrightarrow (7x, 7y)$

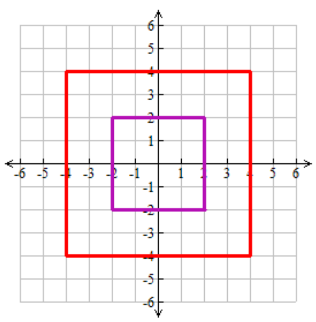
8. $(x, y) \longrightarrow (x - 1, y)$

9. $(x, y) \longrightarrow (-x, y)$

10. $(x, y) \longrightarrow (x + 5, y - 4)$

11. Which transformation rule does not always create a congruent figure?

12. Jacob shifted a figure five units down and two units left. He predicted that the resulting figure would be similar to, but not congruent to, his original shape. Do you agree or disagree? Explain your answer.



13. The red square at the left was graphed first. A transformation was performed on the red square to form the purple square.

a. Write a transformation rule that maps the red square onto the purple square.

b. What type of transformation is this?

c. Are the figures similar or congruent?

14. Write a transformation rule that creates a figure congruent to its original.

15. Write a transformation rule that creates a figure similar, but not congruent, to its pre-image.

16. Use the triangle formed by the vertices P(−3, 4), A(0, 5) and N(2, 1).

a. ΔP'A'N' is a triangle formed by using the transformation rule $(x, y) \longrightarrow (x, -y)$. Graph ΔP'A'N'.

b. What type of transformation is ΔPAN $\longrightarrow$ ΔP'A'N'?

c. Are ΔPAN and ΔP'A'N' congruent or similar?

17. Nachelle wrote the transformation rule $(x, y) \longrightarrow (2x, 3y)$. She says it will produce figures that are similar to one another but not congruent. Do you agree or disagree with Nachelle's statement? Support your answer by graphing an original figure and its image using the transformation rule.

18. Transformation rules can be written for rotations in 90° increments about the origin. Write a transformation rule for each of the following.

a. Rotation of 90° clockwise about the origin.

b. Rotation of 180° about the origin.

c. Rotation of 90° counterclockwise about the origin.

19. Rectangle ABCD is transformed to form Rectangle EFGH. The coordinates for ABCD are A(2, 3), B(2, 5), C(5, 5) and D(5, 3). The coordinates for EFGH are E(−2, 3), F(−2, 5), G(−5, 5) and H(−5, 3). Describe how the original figure and the new figure are related (transformation type and congruence/similarity).

20. Jeff said he could reflect a figure and then translate the reflected figure. He said the ending image would be similar, but not congruent, to the pre-image. Is he correct? Support your answer with reasoning and/or an example.

COMPOSITION OF TRANSFORMATIONS

LESSON 3-P

Perform multiple transformations on a figure including a combination of reflections, rotations, translations and dilations.

EXPLORE! **ON A TRIP**

Step 1: Graph the point A(3, 3) on a coordinate plane.

Step 2: Let point B be the image of point A under the transformation $(x, y) \longrightarrow (x + 1, y - 5)$. Graph point B and give the ordered pair for its location.

Step 3: Let point C be a reflection of point B over the y-axis. Graph point C and give its ordered pair.

Step 4: Let point D be a rotation of point C 90° clockwise about the origin. Graph point D and give its ordered pair.

Step 5: Let point E be the image of point D under the transformation $(x, y) \longrightarrow (\frac{1}{2}x, \frac{1}{2}y)$. Graph point E and give its ordered pair.

Step 6: Write a translation rule that moves point E back to the original location (point A).

Step 7: Create your own "trip" around the coordinate plane for someone else to follow. Start by giving the ordered pair for a point you choose to be point S. Give four transformation directions that will move the points to four different locations around the coordinate plane S → T → U → V → W.

Step 8: Create an answer key for your path by giving an ordered pair for points T, U, V and W.

A series of transformations is called a **composition of transformations**. The transformations that are included in the sequence will determine whether or not the image is similar or congruent to the original figure.

Transformations that create congruent figures include translations, reflections and rotations. Any combination of these transformations on a figure create a congruent image. If a dilation occurs on a figure during a transformation sequence, that image will be similar to the pre-image.

On a coordinate plane, you can prove two figures are similar or congruent if you can write a sequence of transformations that maps one figure onto the other. If the sequence does not include a dilation, the figures are congruent.

You can show that two figures are congruent if you can map one onto another using transformation rules that include reflections, rotations and/or translations.

EXAMPLE 1

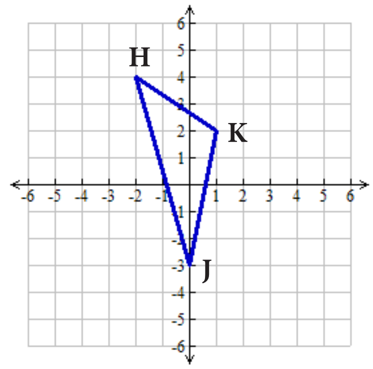
Use the triangle formed by the points H(−2, 4), K(1, 2) and J(0, −3).

a. Graph ΔH'K'J' by reflecting ΔHKJ over the *x*-axis and then following the translation rule $(x, y) \longrightarrow (x - 4, y + 3)$.

b. Are the figures congruent or similar to each other? How do you know?

SOLUTIONS

a. Graph the original figure, ΔHKJ, on a coordinate plane.



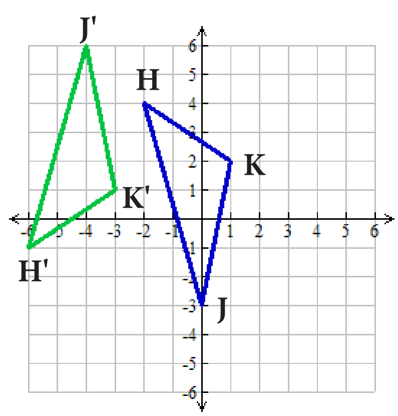
Reflection over the *x*-axis is $(x, y) \longrightarrow (x, -y)$

$H(-2, 4) \longrightarrow (-2, -4)$
$K(1, 2) \longrightarrow (1, -2)$
$J(0, -3) \longrightarrow (0, 3)$

Translation left 4 units and up 3 units.
$(x, y) \longrightarrow (x - 4, y + 3)$

$(-2, -4) \longrightarrow H'(-6, -1)$
$(1, -2) \longrightarrow K'(-3, 1)$
$(0, 3) \longrightarrow J'(-4, 6)$

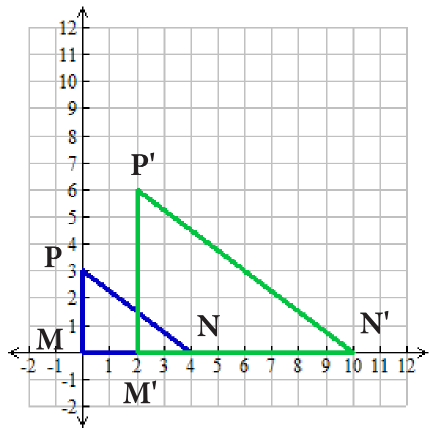
Graph the resulting figure, ΔH'K'J', on the coordinate plane.



b. The figures are congruent because ΔH'K'J' is a result of two transformations (reflection and translation) on ΔHKJ. Each of these transformations creates congruent images from its pre-image.

You can show that two triangles are similar if you can map one onto another using transformation rules that include at least one dilation.

EXAMPLE 2 **Write a series of two transformations that shows ΔMNP is similar to ΔM'N'P'.**



SOLUTION The final triangle appears to be longer and moved to the right. This can be done using a dilation and a translation.

Record the original coordinates.	M(0, 0), N(4, 0), P(0, 3)
The image appears larger. Find the scale factor by looking at the ratio of one side of ΔM'N'P' and the corresponding side ΔMNP.	$\frac{M'N'}{MN} = \frac{8}{4} = 2$
Multiply each vertex of ΔMNP by a scale factor of 2.	$(x, y) \longrightarrow (2x, 2y)$ M(0, 0) ⟶ (0, 0) N(4, 0) ⟶ (8, 0) P(0, 3) ⟶ (0, 6)
The point at the origin is shifted 2 units to the right. Add 2 units to each x-coordinate for a translation rule of $(x, y) \longrightarrow (x + 2, y)$.	$(x, y) \longrightarrow (x + 2, y)$ (0, 0) ⟶ M'(2, 0) (8, 0) ⟶ N'(10, 0) (0, 6) ⟶ P'(2, 6)

Since these points match ΔM'N'P' on the graph above, this series of transformations (dilation $(x, y) \longrightarrow (2x, 2y)$ and translation $(x, y) \longrightarrow (x + 2, y)$) maps ΔMNP onto ΔM'N'P'.

The two figures are similar (not congruent) because a dilation was part of the transformation sequence that made the triangle larger.

EXERCISES

Start with the point Q(1, 5) for Exercises 1-5. Write the ordered pair for the final location of the given point after completing the series of transformations in the order listed.

1. $(x, y) \longrightarrow (x - 1, y + 3)$, reflection over the x-axis

2. A 180° rotation about the origin, $(x, y) \longrightarrow (-x, y)$

3. $(x, y) \longrightarrow (x + 3, y - 7)$, $(x, y) \longrightarrow (x - 2, y + 4)$, $(x, y) \longrightarrow (2x, 2y)$

4. Reflection over the y-axis, reflection over the x-axis, $(x, y) \longrightarrow (x, y + 6)$

5. $(x, y) \longrightarrow (4x, 4y)$, $(x, y) \longrightarrow (\frac{1}{2}x, \frac{1}{2}y)$, 90° rotation clockwise about the origin

6. Which composition of transformations in **Exercises 1-5** form figures that are congruent to their original figures? Explain how you know.

7. Write a series of two transformations that would form similar (but not congruent) figures. Explain how you can know the figures will be similar without graphing.

Determine if each series of transformations forms a similar figure or congruent figure.

8. Translation, Reflection, Reflection

9. Reflection, Rotation, Dilation (scale factor of 3)

10. Rotation, Rotation, Translation

11. Dilation (scale factor of 0.5), Reflection, Translation

12. Write a series of transformations that includes a reflection followed by a translation to map A(−1, 3) onto A'(4, 5).

13. Write a series of transformations that includes a translation followed by a dilation to map B(0, 0) onto B'(4, −6).

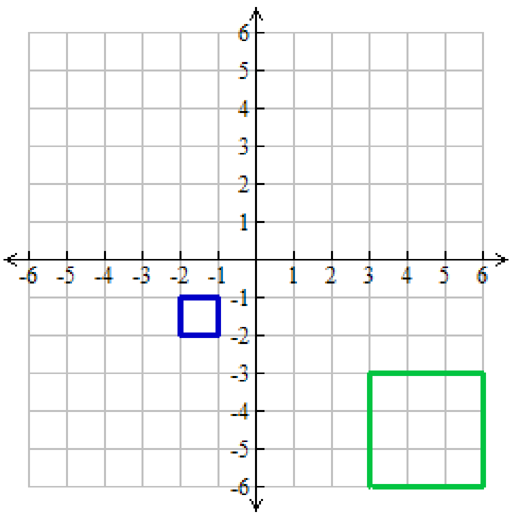
14. What dilation scale factor can be used to create an image that is congruent to its pre-image?

15. Graph a triangle with any three points in the third quadrant (all coordinates should have negative values). Write a translation rule followed by a reflection and graph your new image. Are the figures similar or congruent? Explain your reasoning.

16. Larry graphed a figure in the first quadrant of a coordinate plane. He performed two transformations. The resulting figure was twice as large as the original figure and all the x-values were now negative. Write a possible series of transformations that he might have performed.

Show that each set of figures is either similar or congruent by writing a series of transformation rules that maps the blue figure onto the green figure.

17.



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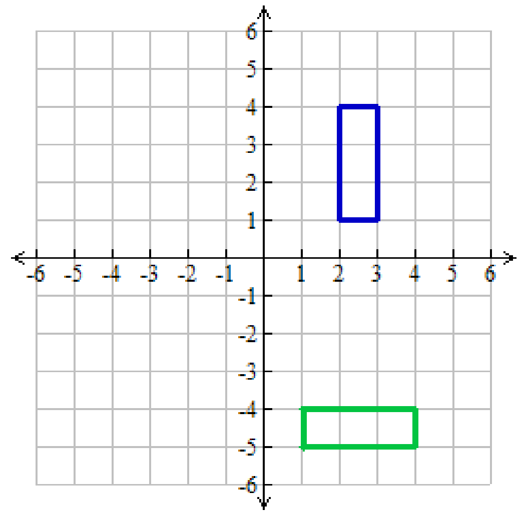

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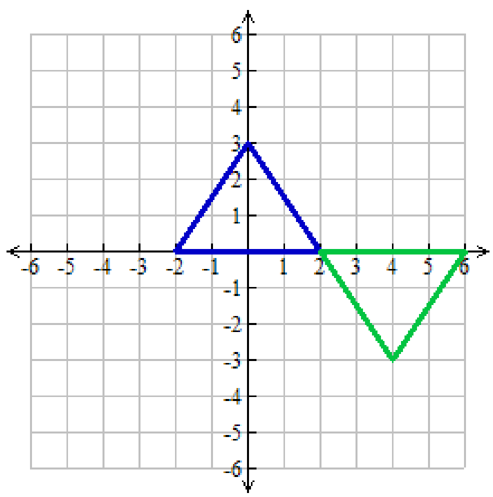

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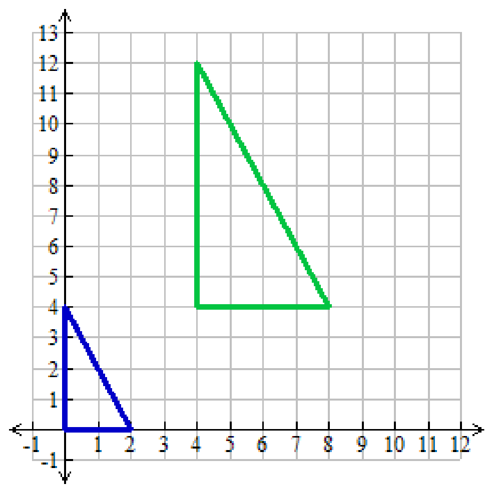

21. Graph two similar triangles on a coordinate plane. Label them and record the ordered pairs for each vertex. Write a series of three transformations that would map the original figure onto the resulting figure.

22. Is there always just one unique series of transformations that can be written to map a figure onto another? Explain your reasoning.

23. N(2, 4) was mapped onto N'(−3, −5). Evan said this was a 180° rotation followed by $(x, y) \longrightarrow (x + 1, y - 1)$. Vicky said this was a reflection over the y-axis followed by $(x, y) \longrightarrow (x - 1, y - 9)$. Who is correct? Support your answer with evidence.

24. Write two different series of at least two different transformations that move point A(1, 2) to B(−3, 4).