COORDINATE-FREE QUANTIZATION OF FIRST-CLASS CONSTRAINED SYSTEMS

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Abstract

The coordinate-free formulation of canonical quantization, achieved by a flat-space Brownian motion regularization of phase-space path integrals, is extended to a special class of closed first-class constrained systems that is broad enough to include Yang-Mills type theories with an arbitrary compact gauge group. Central to this extension are the use of coherent state path integrals and of Lagrange multiplier integrations that engender projection operators onto the subspace of gauge invariant states.

1 Introduction

It is well known that the canonical quantization procedure is consistent only in Cartesian coordinates [1]. For most physically relevant systems, it turns out to be possible to find a Cartesian system of axes and, hence, successfully apply canonical quantization. Nevertheless, the Hamiltonian dynamics of a classical system apparently exhibits, at first sight, a larger symmetry than the associated canonically quantized system. Indeed, Hamiltonian equations of motion are covariant under canonical transformations, while the Heisenberg equations of motion are covariant under unitary transformations. Unitary transformations preserve the spectrum of the canonical quantum operators, while in the classical case canonical transformations do not generally preserve the range of the canonical variables.

It is worth mentioning in this regard that the old Bohr-Sommerfeld quantization postulate

$$\oint pdq = 2\pi\hbar(n + 1/2), \quad n = 1, 2, \ldots \tag{1.1}$$

is invariant with respect to canonical transformations

$$p \to \bar{p}(p, q), \quad q \to \bar{p}(q, p) \tag{1.2}$$

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because

$$\oint pdq = \oint \bar{p}d\bar{q} \ . \tag{1.3}$$

As a consequence, since the result is identical in all canonical coordinate systems, the Bohr-Sommerfeld quantization is in fact "coordinate-free". The characteristic properties of the quantum theory, like the energy spectrum, will be independent of the choice of canonical coordinates. In this respect, the old quantum dynamics enjoys the same symmetry as classical dynamics.

In contrast to the Bohr-Sommerfeld procedure, canonical quantization leads to a result that is not covariant with respect to the initial choice of canonical coordinates. For example, for a single degree of freedom, the coherent-state phase space path integral representation of the evolution operator

$$\begin{aligned} \langle p'', q'', t | p', q' \rangle &= \langle p'', q'' | e^{-it\mathcal{H}/\hbar} | p', q' \rangle \\ &= \int \prod_{\tau=0}^{t} \left(\frac{dp(\tau)dq(\tau)}{2\pi\hbar} \right) \exp \frac{i}{\hbar} \int_0^t d\tau \, [p\dot{q} - h(p,q)] \ , \qquad (1.4) \\ \mathcal{H} &= \textstyle\int h(p,q) \, |p,q\rangle\langle p,q| \, dp \, dq/(2\pi\hbar) \ , \qquad (1.5) \end{aligned}$$

is not covariant with respect to canonical transformations, although the measure, being the product of local Liouville measures at each moment of time, is invariant under canonical transformations. The contradiction follows from the observation that all classical dynamical systems with positive energy and one degree of freedom are equivalent to, say, a free particle ($h = \tilde{p}^2/2$ after a suitable canonical transformation). Making such a canonical transformation in (1.4), we seem to arrive at the same conclusion for quantum systems because the integral (1.4) is formally invariant. Such a conclusion is certainly incorrect.

Coordinate-free quantization

To resolve this paradox, it has been proposed [2] to interpret the ill-defined path integral (1.4) by means of the regularized expression

$$\int \mathcal{D}p\mathcal{D}q \, (\,\cdot\,) \rightarrow \lim_{\nu\to\infty} \mathcal{M}_\nu \int \mathcal{D}p\mathcal{D}q \, (\,\cdot\,) \, e^{-\frac{1}{2\nu}\int_0^t d\tau (\dot{p}^2 + \dot{q}^2)} \ , \tag{1.6}$$

where $\mathcal{M}_\nu$ is a suitable normalization and the limit $\nu \to \infty$ must be taken *after* evaluation of the path integral. Various factors in (1.6) combine to give a Wiener measure on the two-dimensional phase space. In contrast to the integral (1.4), the coherent-state path integral with the regularized measure can be regarded as a sum over trajectories of a Brownian particle whose flat, two-dimensional configuration space is the original phase space of the system.

The spectrum E of the system can be obtained from the pole structure of the Fourier transform of the trace of the transition amplitude

$$Z_t = \mathrm{tr} e^{-it\mathcal{H}/\hbar} = \sum_E e^{-itE/\hbar} = \int (dp'dq'/2\pi\hbar)\langle p', q', t|p', q'\rangle \,, \tag{1.7}$$

where $\langle p', q', t|p', q'\rangle$ is given by the corresponding path integral. Under canonical transformations (1.2) the Brownian motion on a flat two-dimensional phase space remains such a Brownian motion, and if one interprets the stochastic integral $\int pdq$ in the Stratonovich sense, then the spectrum of the system is invariant under canonical coordinate transformations.

In other words, the coherent-state path integral regularized with the help of the Wiener measure (1.6) provides a "coordinate-free" description of quantum theory [2]. Such a regularization procedure applies to general theories without constraints.

Gauge theories

Hamiltonian path integrals are often used to quantize gauge theories [3]. We now have in mind a system of J degrees of freedom $p = \{p_j\}$, $q = \{q^j\}$, $1 \le j \le J$. A main feature of gauge systems is the existence of nonphysical canonical variables. In the standard formulation, the formal path integral (1.4) is divergent because the Hamiltonian action for gauge systems is invariant with respect to transformations

$$q \to q^\omega, \quad p \to p^\omega \tag{1.8}$$

whose parameters ω depend on the time, that is, there are orbits traversed by the gauge transformations (1.8) in the phase space along which the action is constant and traditionally have an infinite volume. The nonphysical variables can be associated with these "gauge" directions in phase space.

To factor out such divergencies of the path integral, one should integrate out the nonphysical variables and obtain a measure on the physical phase space

$$[PS]_{ph} = [PS]/\mathcal{G} \;; \tag{1.9}$$

here $\mathcal{G}$ consists of all transformations (1.8). Technically, the procedure amounts to a canonical transformation such that the generators of (1.8) become some elements of a new set of canonical momenta [3]. This canonical transformation introduces explicit symplectic coordinates p^* and q^* on the physical phase space (1.9). However, it is important to realize that the canonical coordinates on $[PS]_{ph}$ are themselves defined only up to a canonical transformation, i.e., the parametrization of the physical phase space is not unique. As we have argued above, the formal integral in the Hamiltonian path integral cannot provide a genuine invariance with respect to canonical transformations. In the framework of gauge theories, this invariance implies gauge invariance because the spectrum of a gauge theory cannot depend on one or another particular parametrization of the physical phase space.

Thus, the regularization of the path integral measure with the help of a Wiener measure and the invariance under canonical coordinate transformations it offers should be extended to gauge theories. The aim of this letter is to address this problem. Hereafter, we use units where $\hbar = 1$.

2 The projection method

Special constraint class

Let $\varphi_a = \varphi_a(p,q)$ be a set of independent closed first-class constraints, i.e.

$$\{\varphi_a, \varphi_b\} = f_{abc}\varphi_c \ , \tag{2.1}$$

and for convenience we also suppose that the Poisson bracket of φ_a with the system Hamiltonian vanishes. The constraints generate gauge transformations on phase space which in their infinitesimal form are given by

$$p \quad \to p + \delta p = p + \delta\omega^a\{p, \varphi_a\} \equiv p^{\delta\omega} \tag{2.2}$$
$$q \quad \to q + \delta q = q + \delta\omega^a\{q, \varphi_a\} \equiv q^{\delta\omega} \ , \tag{2.3}$$

for general $\{\omega^a\}$. From (2.2) and (2.3) it follows that the infinitesimal gauge transformations generated by the constraints are also infinitesimal canonical transformations

$$\{p^{\delta\omega}, q^{\delta\omega}\} = \{p, q\} + O(\delta\omega^2) \ . \tag{2.4}$$

A finite gauge transformation can be obtained by applying the operator $\exp[-(\omega^a ad\,\varphi_a)]$, $ad\,\varphi_a = \{\varphi_a,\ \cdot\}$, to phase space variables.

As noted at the outset, canonical quantization singles out Cartesian coordinates for special attention. We formulate a special class of closed first-class constraint systems—which we shall refer to as constraints of "Yang-Mills type"—in such a favored set of coordinates. Specifically, we choose

$$\varphi_a(p,q) = f_a^j(q)p_j \equiv (f_a(q), p) \,, \tag{2.5}$$

where (,) denotes a scalar product in a Euclidean space, and $f_a(q)$ *are linear functions of q chosen so that the constraints (2.5) are of the first class*, i.e. they satisfy (2.1). With this choice, the gauge transformations (1.8) are linear canonical transformations. It follows for such constraints that

$$p_j\{\varphi_a, q^j\} = \varphi_a(p,q) \tag{2.6}$$

holds as an identity, which we shall find useful. We also assume that there is no operator ordering ambiguity in the constraints after quantization. This situation is in fact entirely realized for a gauge theory based on a compact semi-simple gauge group [2].

[2] The formalism applies also to gauge groups being the direct product a semi-simple and some number of Abelian groups.

Such constraints enjoy an additional useful property. If

$$|p,q\rangle \equiv e^{-iq^jP_j}e^{ip_jQ^j}|0\rangle\,, \tag{2.7}$$

where $|0\rangle$ is the ground state of an harmonic oscillator, i.e, $(Q^j+iP_j)\,|0\rangle = 0$ for all j, denotes the coherent states in the same Cartesian coordinates, then it follows that

$$e^{-i\Omega^a\hat{\varphi}_a(P,Q)}\,|p,q\rangle = |p^\Omega,q^\Omega\rangle\,, \tag{2.8}$$

namely the action of any finite gauge transformation is to map one coherent state into another. Here $\{\hat{\varphi}_a\}$ denote the constraint operators that generate the gauge transformations.

Coherent state propagator

The total Hilbert space of a gauge system can always be split into an orthogonal sum of a subspace formed by gauge invariant states and a subspace that consists of gauge variant states. Therefore an averaging over the gauge group automatically leads to a projection operator onto the physical subspace of gauge invariant states. The physical transition amplitude is obtained from the unconstrained propagator by averaging the latter over the gauge group,

$$\begin{aligned}
\langle p'',q'',t|p',q'\rangle^{ph} &\equiv \int\limits_G \frac{d\mu(\omega)}{Vol\ G}\langle p'',q'',t|e^{-i\omega^a\hat{\varphi}_a}|p',q'\rangle && (2.9)\\
&\equiv \langle p'',q'',t|\hat{P}_G|p',q'\rangle && (2.10)\\
&= \int (d^Jpd^Jq/(2\pi)^J)\langle p'',q'',t|p,q\rangle\langle p,q|\hat{P}_G|p',q'\rangle\ , && (2.11)
\end{aligned}$$

which is a quantum implementation of the classical initial value equation for first-class constraints. Here $d\mu(\omega)$ is the invariant measure on the space of gauge group parameters, and $Vol\ G = \int_G d\mu(\omega) < \infty$ is the gauge group volume. In what follows we also adopt a shorthand notation for the normalized Haar measure

$$\delta\omega \equiv \frac{d\mu(\omega)}{Vol\ G}\,, \qquad \int\limits_G \delta\omega = 1\ . \tag{2.12}$$

The operator $\hat{P}_G$ is a projection operator onto the gauge invariant subspace. Its kernel is determined as the gauge group average of the unit operator kernel

$$\langle p'',q''|p',q'\rangle^{ph} \equiv \langle p'',q''|\hat{P}_G|p',q'\rangle = \int\limits_G \delta\omega\,\langle p'',q''|e^{-i\omega_a\hat{\varphi}_a}|p',q'\rangle\ . \tag{2.13}$$

For some gauge systems, it can be calculated explicitly as well as the kernel (2.9) [4].

The path integral based on the projective method

Applying the projective formula (2.9) to an infinitesimal transition amplitude $t \to \epsilon = t/N$ and making a convolution of N physical infinitesimal evolution operator kernels, we arrive at the following representation of the amplitude (2.9)

$$\begin{aligned}\langle p'', q'', t | p', q' \rangle^{ph} = & \int \prod_{l=1}^{N-1} (dp_l^J dq_l^J / (2\pi)^J) \langle p'', q'', \epsilon | p_{N-1}, q_{N-1} \rangle^{ph} \\ \times & \langle p_{N-1}, q_{N-1}, \epsilon | p_{N-2}, q_{N-2} \rangle^{ph} \cdots \langle p_1, q_1, \epsilon | p', q' \rangle^{ph} . \end{aligned} \tag{2.14}$$

In the continuum limit, where $N \to \infty,\ \epsilon \to 0$, while the product $t = N\epsilon$ is kept fixed, the convolution (2.14) of the kernels (2.9) ($t = \epsilon$) results in the coherent state path integral [5]

$$\langle p'', q'', t | p', q' \rangle^{ph} = \mathcal{M} \int \mathcal{D}C(\omega) \mathcal{D}p \mathcal{D}q \, e^{iS_H} , \tag{2.15}$$

$$S_H = \int_0^t dt' [(p, \dot{q}) - \omega^a \varphi_a(p, q) - h(p, q)] , \tag{2.16}$$

where $\mathcal{D}C(\omega) = \prod_t \delta\omega(t)$ is a formal (normalized) measure for the gauge group average parameters (cf (2.9)), and the symbol $h(p, q)$ is defined in (1.5). Thus, the gauge group averaging parameters ω^a become the Lagrange multipliers of the classical theory in the continuum limit.

A relation between the path integral (2.15) and the projective formula (2.9) is found in the boundary condition for the path integral. Recall that the integral (2.15) is taken over phase space trajectories that obey the boundary conditions

$$p(0) = p' , \quad q(0) = q' ; \tag{2.17}$$

$$p(t) = p'' , \quad q(t) = q'' . \tag{2.18}$$

It is not hard to find a gauge transformation such that

$$(p^\omega, \dot{q}^\omega) - \omega^a \varphi_a(p^\omega, q^\omega) = (p, \dot{q}) . \tag{2.19}$$

It is equivalent to solving a linear equation

$$\dot{q}^\omega + \omega^a f_a(q^\omega) = \dot{q} . \tag{2.20}$$

Having found q^ω one easily determines p^ω as its canonical momenta.

The path integral measure is formally invariant under canonical transformations and, hence, the explicit dependence on the Lagrange multipliers of the action S_H disappears after the canonical transformation constructed above. The residual coherent state path integral represents a transition amplitude in the unconstrained Hilbert space. However the integral $\int \mathcal{D}C(\omega)$ cannot be factored out because a nontrivial dependence on the Lagrange multipliers survives at the boundaries. To maintain the boundary conditions (2.17) and (2.18), one can, say, require

$$p^\omega(t) = p'' \ , \quad q^\omega(t) = q'' \ . \tag{2.21}$$

Then it is impossible to satisfy the boundary condition (2.17) because equation (2.20) admits only one boundary condition, say, at the final time point. Thus, after the canonical transformation the path integral must be taken with boundary conditions that depends on ω_a

$$p^\omega(0) = p'^\Omega \ , \quad q^\omega(0) = q'^\Omega \ , \quad \Omega = \Omega[\omega] \ , \tag{2.22}$$

that is, one gauge group average "survives" the canonical transformation that removes the Lagrange multipliers from the action and provides the equivalence of the path integral (2.15) to the projective representation (2.9).

3 Gauge fixing and the path integral over physical phase space

In practice, it often turns out to be useful to integrate out the nonphysical phase-space variables associated with pure gauge degrees of freedom and work with the path integral over the physical phase space (1.9). For this purpose one usually fixes a gauge [3]

$$\chi_a(q) = 0 \ . \tag{3.1}$$

By a necessary assumption, each gauge orbit q^ω must intersect the gauge condition surface (3.1) (at least) once. Under this assumption a generic configuration q can be parametrized via lifting it onto the gauge condition surface along a gauge orbit passing through q

$$q = q_\chi^\theta(q^*) \ , \tag{3.2}$$

where θ_a parametrizes the lift along a gauge orbit, and points $q = q_\chi(q^*)$ form the surface (3.1), i.e., q^* parametrizes the surface (3.1).

In the curvilinear coordinates (3.2) associated with the chosen gauge condition, the constraints are linear combinations of canonical momenta for θ_a, and the Poisson bracket of the canonical variables p^* and q^* with the constraints vanishes, that is, p^* and q^* are gauge invariant according to (2.2) and (2.3). The θ-dependence of the action can be absorbed by a shift of the Lagrange multipliers ω^a on a suitable linear combination of the velocities $\dot\theta_a$ because the canonical one-form assumes the form

$$p\dot q + \omega^a\varphi_a = p^*\dot q^* + p_\theta^a\dot\theta_a + \omega^a\varphi_a \tag{3.3}$$

and the Hamiltonian is gauge invariant (the θ_a's are cyclic variables).

The integral over θ_a yields the gauge group volume that cancels the one sitting in the measure $\mathcal{D}C(\omega)$. Finally, the integrals over ω^a and p_θ^a can also be done, and one ends up with the integral over physical phase space spanned by local symplectic coordinates p^*, q^*.

This result is usually achieved by a formal restriction of the path integral measure support in (2.15) to a subspace of the constraint surface $\varphi_a(p,q) = 0$ selected by the gauge (supplementary) condition (3.1) [3]:

$$\mathcal{D}p\mathcal{D}q\mathcal{D}C(\omega)e^{-i\int dt\omega^a\varphi_a} \to \mathcal{D}p\mathcal{D}q\prod_t\left(\Delta_{FP}\prod_a\delta(\chi_a)\delta(\varphi_a)\right) , \tag{3.4}$$

where $\Delta_{FP} = \det\{\varphi_a, \chi_b\}$ is the Faddeev-Popov determinant. After the canonical transformation associated with (3.2) the Faddeev-Popov measure assumes the form [3]

$$\mathcal{D}p^*\mathcal{D}q^*\mathcal{D}p^\theta\mathcal{D}\theta\prod_t\delta(p^\theta)\delta(\theta) , \tag{3.5}$$

and the integration over the nonphysical variables p^θ and θ becomes trivial.

Two important observations are in order. First, the procedure (3.4) corresponds to a canonical quantization *after* the elimination of all nonphysical degrees of freedom (the so called reduced phase-space quantization). As shown above, the physical variables are associated with curvilinear coordinates, while canonical quantization is consistent only in Cartesian coordinates. As a result canonical quantization and the elimination of nonphysical degrees of freedom generally do *not* commute [7]. In other words, the procedure (3.4) is not, in general, equivalent to the Dirac quantization scheme [6] where nonphysical degrees of freedom are removed after quantization.

Second, the geometry and topology of gauge orbits may happen to be such that there exists no unique gauge condition [8], meaning that for any given χ_a the system

$$\chi_a(q) = \chi_a(q^{\omega_s}) = 0 \tag{3.6}$$

always admits nontrivial solutions with respect to ω_s^a. From the geometrical point of view, the latter implies that the gauge orbit q^ω intersects the gauge fixing surface more than once, namely, at points q^{ω_s}. Discrete gauge transformations associated with the gauge variables ω_s^a do not reduce the number of physical degrees of freedom, but they do reduce the "volume" of the physical configuration and phase spaces. Therefore the formal measure $\mathcal{D}p^*\mathcal{D}q^*$ can no longer be Euclidean and the corresponding path integral should be modified. If the residual discrete gauge transformations are explicitly known, then in such cases it appears to be possible to find a modified path integral formalism that is equivalent to the Dirac method [9].

Finally we remark that the Liouville measure $\mathcal{D}p^*\mathcal{D}q^* = \prod_t dp^*(t)dq^*(t)$ is invariant with respect to canonical transformations. This freedom in the path integral over physical phase space can be interpreted as gauge invariance. Indeed, another choice of a gauge condition (3.1) would induce another parametrization of the physical phase space that is equivalent to the former via a canonical transformation. On the other hand, we have argued in Section 1 that the formal invariance of the Liouville measure in the path integral is not sufficient to ensure the invariance of the quantum theory with respect to canonical transformations. In the framework of gauge systems, it implies that, to achieve gauge invariance of the path integral over physical phase space, the measure should be regularized *before* integrating out pure gauge degrees of freedom with the help of a canonical transformation associated with a chosen parametrization of the physical phase space by local symplectic coordinates.

In the next section we propose a generalization of the path integral measure regularization with a Wiener measure to gauge theories.

4 The Wiener measure for gauge theories

The Wiener measure regularized phase space path integral for a general phase function $G(p,q)$ is given by

$$\begin{aligned}
&\lim_{\nu\to\infty} \mathcal{M}_\nu \int \exp\{i{\textstyle\int_0^T}[p_j\dot{q}^j + \dot{G}(p,q) - h(p,q)]\,dt\} \\
&\qquad\qquad \times \exp\{-(1/2\nu){\textstyle\int_0^T}[\dot{p}^2+\dot{q}^2]\,dt\}\,\mathcal{D}p\,\mathcal{D}q \\
&\quad = \lim_{\nu\to\infty}(2\pi)^J e^{J\nu T/2}\int \exp\{i{\textstyle\int_0^T}[p_j dq^j + dG(p,q) - h(p,q)dt]\}\,d\mu_W^\nu(p,q) \\
&\quad = \langle p'', q''|e^{-i\mathcal{H}T}|p',q'\rangle \ \ ,
\end{aligned} \tag{4.1}$$

where the last relation involves a coherent state matrix element. In this expression we note that $\int p_j\,dq^j$ is a *stochastic integral*, and as such we need to give it a definition. As it stands both the Itô (nonanticipating) rule and the Stratonovich (midpoint) rule of definition for stochastic integrals yield the same result (since $dp_j(t)dq^k(t) = 0$ is a valid Itô rule in these coordinates). Under any change of canonical coordinates, we consistently will interpret this stochastic integral in the Stratonovich sense because it will then obey the ordinary rules of calculus.

Why does the representation of the propagator as well as the Hamiltonian operator involve coherent states

$$|p,q\rangle \equiv e^{-iG(p,q)}e^{-iq^jP_j}e^{ip_jQ^j}|0\rangle\ , \qquad (Q^j+iP_j)|0\rangle = 0\ ? \tag{4.2}$$

One simple argument is as follows. The Wiener measure is on a flat *phase space*, and is pinned at both ends thus resulting in the boundary conditions $p(T), q(T) = p'', q''$ and $p(0), q(0) = p', q'$. Holding this many end points fixed is incompatible with a Schrödinger representation, which holds just $q(T)$ and $q(0)$ fixed, or with a momentum space representation, which holds just $p(T)$ and $p(0)$ fixed. It turns out, as a consequence of the Wiener measure regularization, that the propagator is *forced* to be in a coherent state representation. We also emphasize the covariance of this expression under canonical coordinate transformations. In particular, if $\overline{p}d\overline{q} = pdq + dF(\overline{q},q)$ characterizes a canonical transformation from the variables p,q to $\overline{p},\overline{q}$, then with the Stratonovich rule the path integral becomes

$$\langle \overline{p}'', \overline{q}''|e^{-i\mathcal{H}T}|\overline{p}', \overline{q}'\rangle$$

$$= \lim_{\nu\to\infty} (2\pi)^J e^{J\nu T/2} \int \exp\{i\textstyle\int_0^T [\overline{p}_j d\overline{q}^j + d\overline{G}(\overline{p},\overline{q}) - \overline{h}(\overline{p},\overline{q})dt]\}\, d\mu_W^\nu(\overline{p},\overline{q})$$

$$= \lim_{\nu\to\infty} \mathcal{M}_\nu \int \exp\{i\textstyle\int_0^T [\overline{p}_j \dot{\overline{q}}^j + \dot{\overline{G}}(\overline{p},\overline{q}) - \overline{h}(\overline{p},\overline{q})dt]\}$$

$$\times \exp\{-(1/2\nu)\textstyle\int_0^T [d\sigma(\overline{p},\overline{q})^2/dt^2]\, dt\}\, \mathcal{D}\overline{p}\, \mathcal{D}\overline{q}\,, \tag{4.3}$$

where $\overline{G}$ incorporates both F and G. In this expression we have set $d\sigma(\overline{p},\overline{q})^2 = dp^2 + dq^2$, namely, the new form of the flat metric in curvilinear phase space coordinates. We emphasize that this path integral regularization involves Brownian motion on a flat space whatever choice of coordinates is made. Our transformation has also made use of the formal – and in this case valid – invariance of the Liouville measure.

If we have auxiliary terms in the classical action representing constraints, then the expression of interest would seem to be

$$\lim_{\nu\to\infty} \mathcal{M}_\nu \int \exp\{i\textstyle\int_0^T [p_j \dot{q}^j - h(p,q) - \omega^a \varphi_a(p,q)]\, dt\}$$

$$\times \exp\{-(1/2\nu)\textstyle\int_0^T [\dot{p}^2 + \dot{q}^2]\, dt\}\, \mathcal{D}p\, \mathcal{D}q\, \mathcal{D}C(\omega)\ , \tag{4.4}$$

where the formal measure $\mathcal{D}C(\omega) = \prod_t \delta\omega(t)$ may be proposed. We expect some expression of this sort to hold; however, the explicit proposal in (4.4) is incorrect as we now proceed to demonstrate.

According to the discussion of the previous sections it is clear that the physical propagator may also be given by

$$\lim_{\nu\to\infty} \mathcal{M}_\nu \int_G \delta\Omega \int \exp\{i\textstyle\int [p_j \dot{q}^j - h(p,q)]\, dt\} \exp\{-(1/2\nu)\textstyle\int [\dot{p}^2 + \dot{q}^2]\, dt\}\, \mathcal{D}p\mathcal{D}q\ ; \tag{4.5}$$

here all the paths satisfy $p(T), q(T) = p'', q''$ and $p(0), q(0) = p'^{\Omega}, q'^{\Omega}$, where following the notation introduced in Section 2, we define

$$p^\Omega = e^{-\Omega^a ad\, \varphi_a} p\ , \qquad\qquad q^\Omega = e^{-\Omega^a ad\, \varphi_a} q\ . \tag{4.6}$$

In short, we have used the fact that the unitary operators representing the finite gauge group transformations satisfy the condition (2.8) mapping any coherent state into another coherent state.

Based on the mapping property (4.6), we can give another formulation to the path integral (4.5). With the Wiener measure regularization present, the path integral for any finite ν is well defined, and as such we are free to change variables of integration. In particular, let us make a canonical change of variables so that

$$p(t) \to e^{\int_t^T ds\omega^a(s) ad\, \varphi_a} p(t)\ ,$$

$$q(t) \to e^{\int_t^T ds\omega^a(s) ad\, \varphi_a} q(t)\ , \tag{4.7}$$

where ω^a are functions of time subject only to the requirement that

$$\textstyle\int_0^T \omega^a(s)\, ds \equiv \Omega^a\ . \tag{4.8}$$

Clearly, there are infinitely many functions ω^a that will satisfy such a criterion, and in a certain sense we will be led to average over "all" of them. Note what this change of variables accomplishes. In the new variables, whatever the choice of ω^a may be, the final values remain unchanged, $p(T), q(T) = p'', q''$, while the initial values have become $p(0), q(0) = p', q'$ since $(p'^{\Omega})^{-\Omega} \equiv p'$ and $(q'^{\Omega})^{-\Omega} \equiv q'$. Thus we have transformed all the gauge dependence from the initial points p'^{Ω}, q'^{Ω} and have distributed it throughout the time interval T. This discussion is reminiscent of that in Sections 2 and 3.

It should be remarked that the condition (4.8) may also be avoided if so desired. Suppose we drop the condition (4.8). Let $\bar{\Omega}^a$ be the value of the integral in the right-hand side of (4.8). Since the integral (4.5) involves the average over the gauge orbit that goes through the initial point p', q', the explicit dependence of the boundary condition on $\bar{\Omega}^a$ at the initial time can be removed by an appropriate shift of the average parameters Ω^a. The initial boundary condition remains intact $p(0), q(0) = p'^{\Omega}, q'^{\Omega}$ in contrast to the case when the condition (4.8) is imposed. Nevertheless, we proceed on the basis of (4.8).

Let us next see what are the consequences for the path integral of such a change of integration variables. We first observe that

$$\begin{aligned} \dot{p}(t) &\to \dot{p}(t) - \omega^a ad\, \varphi_a p(t) = \dot{p}(t) - \omega^a\{\varphi_a, p\}(t) \ , \\ \dot{q}(t) &\to \dot{q}(t) - \omega^a ad\, \varphi_a q(t) = \dot{q}(t) - \omega^a\{\varphi_a, q\}(t) \ . \end{aligned} \tag{4.9}$$

Such a change leads to a new form for the path integral given by

$$\begin{aligned} &\lim_{\nu\to\infty} \mathcal{M}_\nu \int_G \delta\Omega \int \exp\{i{\textstyle\int}[p_j(\dot{q}^j - \omega^a\{\varphi_a, q^j\}) - h(p,q)]\, dt\} \\ &\times \exp\{-(1/2\nu){\textstyle\int}[(\dot{p} - \omega^a\{\varphi_a, p\})^2 + (\dot{q} - \omega^a\{\varphi_a, q\})^2]\, dt\}\, \mathcal{D}p\, \mathcal{D}q \ . \end{aligned} \tag{4.10}$$

This relation holds because the formal measure remains invariant under this canonical transformation of coordinates. We recall that in this form the fixed end points are $p(T), q(T) = p'', q''$ and $p(0), q(0) = p', q'$. This equation is true for any choice of ω^a which fulfills the required integral condition (4.8), and *a fortiori* it is still true if we average (4.10) over "all" functions which satisfy the required integral condition. In so doing let us at the same time incorporate the integral over Ω and simply average over "all" functions ω^a directly without any condition on the overall integral value. For now let us continue to treat such an average in a formal manner; we will return to the question of a proper average at a later stage. Thus we may replace (4.10) by

$$\begin{aligned} &\lim_{\nu\to\infty} \mathcal{M}_\nu \int \exp\{i{\textstyle\int}[p_j(\dot{q}^j - \omega^a\{\varphi_a, q^j\}) - h(p,q)]\, dt\} \\ &\times \exp\{-(1/2\nu){\textstyle\int}[(\dot{p} - \omega^a\{\varphi_a, p\})^2 + (\dot{q} - \omega^a\{\varphi_a, q\})^2]\, dt\}\, \mathcal{D}p\, \mathcal{D}q\, \mathcal{D}C(\omega) \ , \end{aligned} \tag{4.11}$$

where $C(\omega)$ denotes a measure which averages over all functions ω^a as required. Since the object under discussion is manifestly gauge invariant, it is noteworthy that we can explicitly display such invariance under the gauge transformations

$$\delta p = \{\varphi_a, p\}\delta\lambda^a \ , \qquad \delta q = \{\varphi_a, q\}\delta\lambda^a \ , \qquad \delta\omega^a = \delta\dot{\lambda}^a - f_{abc}\omega^b\delta\lambda_c \ , \tag{4.12}$$

for general infinitesimal functions $\delta\lambda^a(t)$ which vanish at the end points, and for which the indicated path integral is invariant for all values of ν, hence in the limit. Although the path integral is invariant under the gauge transformations indicated, the reader should not jump to the conclusion that the path integral diverges. In fact, the integral over the gauge functions ω^a is an *average*, that is, $\int\mathcal{D}C(\omega)$ is finite, as we have stressed, and for a bounded integrand no divergences are possible.

Equation (4.11) represents a manifestly gauge invariant expression that is covariant under a general canonical change of variables. For the class of constraints under discussion, we can also present another useful expression. Using the identity (2.6) leads to the equivalent relation

$$\lim_{\nu\to\infty}\mathcal{M}_\nu\int\exp\{i{\textstyle\int}[p_j\dot{q}^j-\omega^a\varphi_a(p,q)-h(p,q)]\,dt\} \tag{4.13}$$
$$\times\exp\{-(1/2\nu){\textstyle\int_0^T}[(\dot{p}-\omega^a\{\varphi_a,p\})^2+(\dot{q}-\omega^a\{\varphi_a,q\})^2]\,dt\}\,\mathcal{D}p\,\mathcal{D}q\,\mathcal{D}C(\omega)\ ,$$

and once again we recognize the parameters $\{\omega^a\}$ as the Lagrange mutipliers of the classical theory.

Additionally, we observe that the drift terms in the Wiener measure cannot be neglected. For the Brownian motion we have the Itô rule $dp(t)^2=\nu dt$, and the connected expectation value $E(p(t)p(s))_{\rm conn}=\nu s(1-t/T)$ for $s<t$. Thus the (Stratonovich) stochastic integral $(1/\nu)\int\omega^a\{\varphi_a,p\}\,dp$ and the term $(1/2\nu)\int[\omega^a\{\varphi_a,p\}]^2\,dt$ are both of order unity for all values of ν since in the general case $\omega^a\{\varphi_a,p\}\simeq p$. A similar discussion holds for q as well. It is for this reason that our initial naive proposal (4.4) is not acceptable.

Choice of measure for the gauge variables

Finally we take up the question of the choice of the measure $C(\omega)$ and its associated integral. Although we have loosely stated that $\mathcal{D}C(\omega)=\prod_t\delta\omega(t)$ and that we should integrate over all functions ω^a, this is still an imprecise concept. Despite appearances, there is actually a great deal of choice in this measure. This freedom arises because the only real requirement on this measure is that it simulates a *single group invariant integral over the initial parameters* p'^{Ω},q'^{Ω} as discussed in (4.5). We shall consider two possible choices. The first one will provide us with a manifestly gauge invariant measure, while the second choice gives an example of a gauge noninvariant measure which nonetheless leads to the gauge invariant transition amplitude. The latter amounts to some specific gauge fixing that is manifestly *free* of any Gribov problem.

To define an appropriate measure that is invariant under general gauge transformations, we appeal to the classical theory of Kolmogorov [10] on stochastic processes, which will ensure that we obtain a well-defined probability measure on the gauge path space. Kolmogorov's theorem asserts that an underlying probability measure on paths exists provided the set of multi-time joint probability densities satisfies certain basic consistency conditions. To show the needed consistency let us again use $\delta\omega$ as the normalized Haar measure (2.12) for the compact semi-simple gauge group under consideration. Then let us introduce a stochastic process defined by the following set of multi-time joint probability densities

$$\mathcal{P}_n(\omega_n, t_n; \ldots; \omega_2, t_2; \omega_1, t_1) \equiv 1 \tag{4.14}$$

for all $n \geq 1$. Here $T \geq t_n > \cdots t_2 > t_1 \geq 0$. The left-hand side of this equation is the joint probability density for the gauge field to have value ω_1 at time t_1, value ω_2 at time t_2, etc. In this terminology $\omega = \{\omega^a\}$. The right-hand side of this joint probability density relation is simply unity, meaning that *any* set of values at *any* set of distinct times is equally likely. This is the proper mathematical statement of a uniform average over all gauge paths. Consistency of the given joint probability densities is simply the trivial observation that

$$\begin{aligned} \int \mathcal{P}_n(\omega_n, t_n; \ldots; \omega_r, t_r; \ldots; \omega_1, t_1)\, \delta\omega_r \\ &= 1 \\ &= \mathcal{P}_{n-1}(\omega_n, t_n; \ldots; \omega_{r+1}, t_{r+1}; \omega_{r-1}, t_{r-1}; \ldots; \omega_1, t_1)\ , \end{aligned} \tag{4.15}$$

for any choice or r, $n \geq r \geq 1$, and all n, $n \geq 2$; for $n = 1$ the last line should be ignored. The evident consistency of this set of joint probability densities is then sufficient to guarantee for us a (countably additive) probability measure on gauge fields, which we denote by $\rho(\omega)$, that exhibits these joint probability distributions.

Accepting this choice for the integration over gauge fields leads to the fact that the physical propagator may be given the mathematically well-defined formulation

$$\begin{aligned} &\langle p'', q''| e^{-i\mathcal{H}T} |p', q'\rangle^{ph} \\ &= \lim_{\nu\to\infty} (2\pi)^J e^{J\nu T/2} \int \exp\{i{\textstyle\int}[p_j dq^j + dG(p,q) - \omega^a \varphi_a(p,q)dt - h(p,q)dt]\} \\ &\qquad\qquad \times d\mu_W^\nu(p, q, \omega)\, d\rho(\omega)\ ; \end{aligned} \tag{4.16}$$

here we have added ω to the argument of μ_W^ν to acknowledge the presence of the drift terms. The result only depends on the initial and final values of p and q since we have integrated over the set of gauge paths without any boundary conditions; this result is still invariant under continuous and differential gauge transformations (4.12).

Finally we note that the relation between the physical Hamiltonian operator and the classical expression $h(p, q)$ is given by

$$\mathcal{H}_{ph} \equiv \int h(p,q)\, |p,q\rangle^{ph\ ph}\langle p,q|\, d^Jp\, d^Jq/(2\pi)^J\ , \tag{4.17}$$

where the physical coherent state $|p, q\rangle^{ph}$ is obtained by the average of the coherent state (2.8) over the group G with the normalized measure $\delta\Omega$.

Formally, the measure $d\rho(\omega)$ constructed above comes naturally from the convolution formula (2.14) where each infinitesimal transition amplitude is to be replaced by the corresponding infinitesimal amplitude (4.5) with the Wiener measure. In this construction the projection operator is inserted at each moment of time, that is, formally, $d\rho(\omega) =$

$\prod_t \delta\omega(t)$. Clearly, this formal measure satisfies the conditions (4.14) and (4.15), and in addition it is manifestly gauge invariant and normalized $\int d\rho(\omega) = 1$.

However, from the calculational point of view the measure $\rho(\omega)$ is not always convenient. Sometimes it is also useful to have a measure for the gauge variables that is not explicitly gauge invariant (gauge fixing). A conventional gauge fixing discussed in Section 3 may suffer from Gribov ambiguities. Next we show an example of a Gaussian probability measure free of such a disease.

Since we want the measure to have at least one average over the group manifold G, it is natural to assume that for any time slice the measure must be the group invariant measure, but what is at our disposal is the relationship of the functions at neighboring points of time. As one set of examples, it would suffice to restrict our integration to the set, or even a subset, of *continuous functions*. A natural way to achieve it is to choose $\mathcal{D}C(\omega)$ to be a Wiener measure on the manifold G

$$\mathcal{D}C(\omega) = d\rho_W(\omega) = \mathcal{N} \exp[-\tfrac{1}{2}\textstyle\int g_{ab}(\omega)\dot{\omega}^a\dot{\omega}^b\, dt]\, \Pi_t\, \delta\omega(t)\ . \tag{4.18}$$

Here the metric $g_{ab}(\omega)$ is the positive-definite metric associated with a homogeneous space determined by the compact semi-simple gauge group. The measure can also be regarded as the imaginary time quantum dynamics of a free particle propagating on the compact homogeneous manifold G.

Let us now establish a relation between the projection formula (4.5) and (4.13) with the choice (4.18) for the measure. Let g_ω be an element of the gauge group in a matrix representation. Then the action in the exponential in (4.18) can also be rewritten as

$$S_W = -c\, \mathrm{tr} \int_0^T (\dot{g}_\omega g_\omega^{-1})^2/2dt\ , \tag{4.19}$$

where $c = 1/\mathrm{tr}(1)$ is a normalization factor. Consider a transition amplitude of a free particle on the manifold G

$$K_T(g_\Omega, g_{\Omega'}) = \mathcal{N} \int_{g_\omega(0)=g_{\Omega'}}^{g_\omega(T)=g_\Omega} \prod_{t=0}^{T} \delta\omega(t) e^{-S_W}\ , \tag{4.20}$$

normalized so as to satisfy

$$K_T(g_{\Omega''}, g_{\Omega'}) = \int K_{T-t}(g_{\Omega''}, g_\Omega) K_t(g_\Omega, g_{\Omega'})\, \delta\Omega\,. \tag{4.21}$$

Due to the global invariance of the action with respect to the left and right shifts, $g_\omega \to g_0 g_\omega$ and $g_\omega \to g_\omega g_0$, the amplitude (4.20) is also invariant under these transformations

$$K_T(g_\Omega, g_{\Omega'}) = K_T(g_0 g_\Omega, g_0 g_{\Omega'}) = K_T(g_\Omega g_0, g_{\Omega'} g_0)\ . \tag{4.22}$$

From (4.22) we deduce the identity

$$\int_G \delta\Omega'' K_T(g_{\Omega''}, g_{\Omega'}) = \int_G \delta\Omega' K_T(g_{\Omega''}, g_{\Omega'}) = 1\ , \tag{4.23}$$

which can be easily seen from the Feynman-Kac representation of the transition amplitude (4.20) as a spectral sum. The integral (4.23) determines an action of the evolution operator on the ground state of the system. So, only the ground state will contribute to the integral. We naturally assume that the Casimir energy (the ground state energy) can always be subtracted and included into the path integral normalization.

Now we insert the identity (4.23) into the measure of the path integral (4.5) and then proceed with the change of variables (4.7). Since in the identity (4.23) either Ω'' or Ω' is a free parameter, we can always choose it to coincide with the parameter Ω of the G-average in (4.5). Substituting the path integral representation of K_T (4.20) in the appropriately transformed integral (4.5), we arrive at the expression (4.13) with the Wiener measure (4.18) for the gauge variables.

Typically we encounter Wiener measures that are pinned at either the initial time or at both end points; in the present case, the measure for gauge variables is neither pinned at the initial nor the final time as seen from the derivation of (4.13). Since the group is compact, the group volume is finite and we may therefore normalize such a Wiener measure that is not pinned; our normalization is such that

$$\int \mathcal{D}C(\omega) = \int\limits_G \delta\Omega''\delta\Omega' K_T(g_{\Omega''}, g_{\Omega'}) = 1 \ . \tag{4.24}$$

In that case the formal measure $\mathcal{D}C(\omega)$ is actually a well-defined (countably additive) probability measure which we denote by $d\rho_W(\omega)$. With this choice we note that the physical propagator may also be given the well-defined definition (4.16) where $d\rho(\omega) \to d\rho_W(\omega)$. The result only depends on the initial and final values of p, q since we have integrated over the set of continuous ω^a paths without any boundary conditions.

The measure is not invariant under the gauge transformations (4.12), nonetheless the transition amplitude is gauge invariant because the measure provides the necessary projection onto gauge invariant states. In contrast to the conventional procedure of section 3, there is no explicit gauge condition imposed on the system of phase space variables, and hence the Gribov problem is avoided.

One should add that two such propagators, one from $t = 0$ to $t = T$ and the second from $t = T$ to $t = 2T$, for example, seems to not compose to a propagator of the same form as (4.16) due to the discontinuity of paths at the interface. However, the resultant propagator is nonetheless correct; it simply involves another acceptable form for the measure $\mathcal{D}C(\omega)$.

Conclusion

With (4.13) and two choices of the measure for the gauge variables, we have arrived at our coordinate-free and mathematically well-defined formulation for the path integral representation of the special class of first-class constraints that was our goal.

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