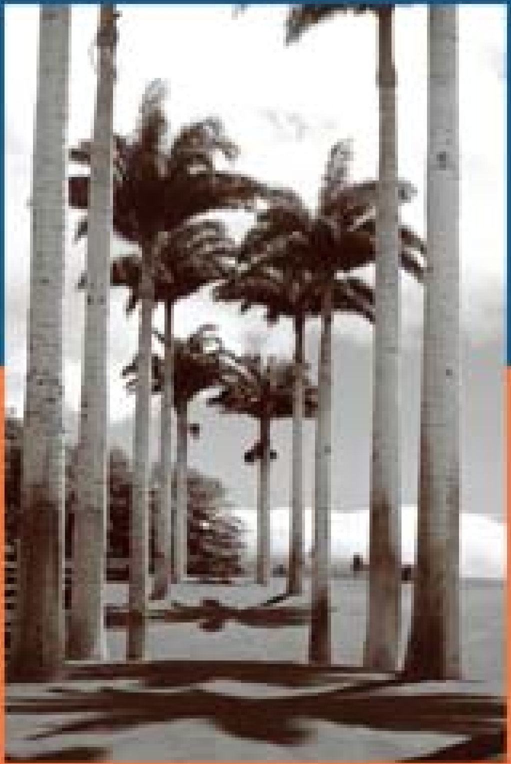
COLLEGE ALGEBRA

with Modeling and Visualization



GARY
ROCKSWOLD

THIRD EDITION

3.1

Quadratic Functions and Models

- Learn basic concepts about quadratic functions and their graphs.
- Complete the square and apply the vertex formula.
- Graph a quadratic function by hand.
- Solve applications and model data.

Basic Concepts

Recall that a linear function can be written as $f(x) = ax + b$ (or $f(x) = mx + b$). The formula for a quadratic function is different from that of a linear function because it contains an x^2 term.

$$f(x) = 3x^2 + 3x + 5 \qquad g(x) = 5 - x^2$$

QUADRATIC FUNCTION

Let a, b, and c be real numbers with $a \neq 0$. A function represented by

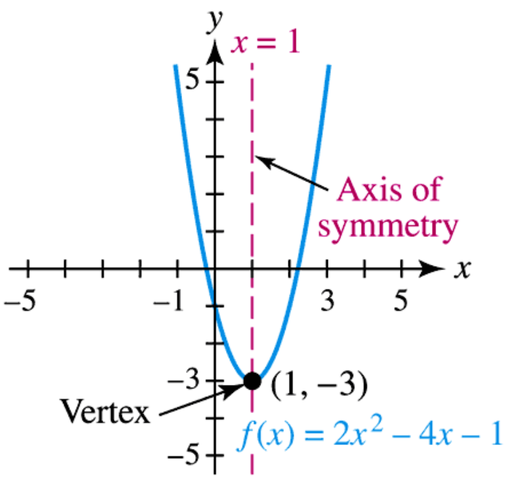
$$f(x) = ax^2 + bx + c$$

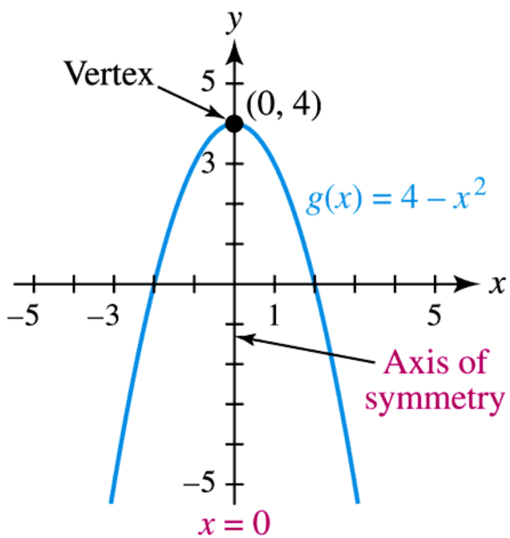
is a **quadratic function**.

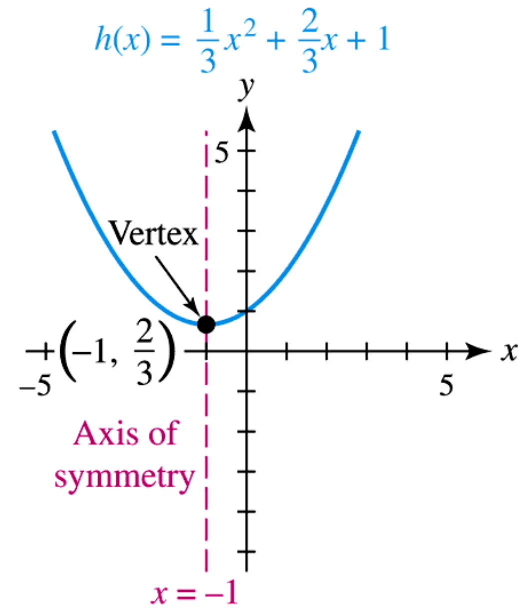
Quadratic Function

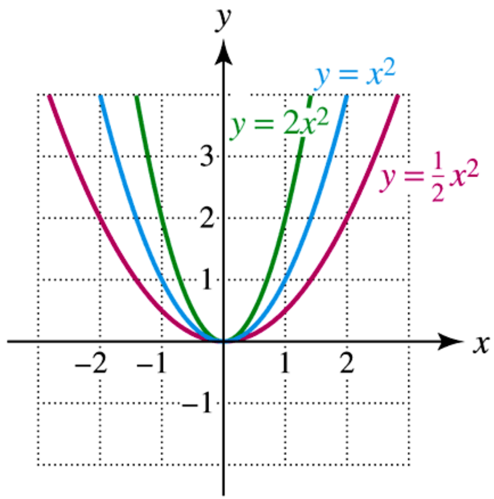
- The graph of a quadratic function is a **parabola**—a U shaped graph that opens either upward or downward.
- A parabola opens upward if *a* is positive and opens downward if *a* is negative.
- The highest point on a parabola that opens downward and the lowest point on a parabola that opens upward is called the **vertex**.
- The vertical line passing through the vertex is called the **axis of symmetry.**
- The leading coefficient *a* controls the width of the parabola. Larger values of $|a|$ result in a narrower parabola, and smaller values of $|a|$ result in a wider parabola.

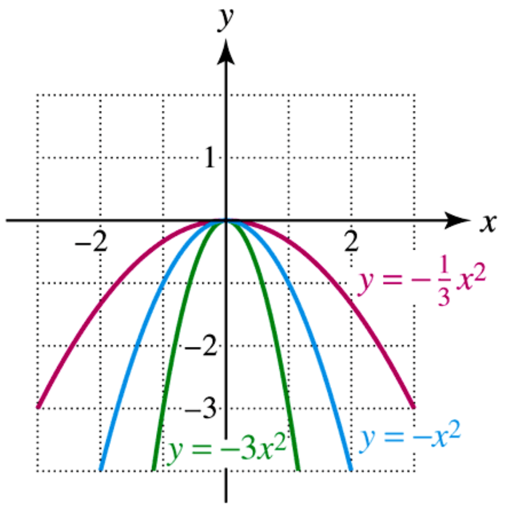
Examples of different parabolas









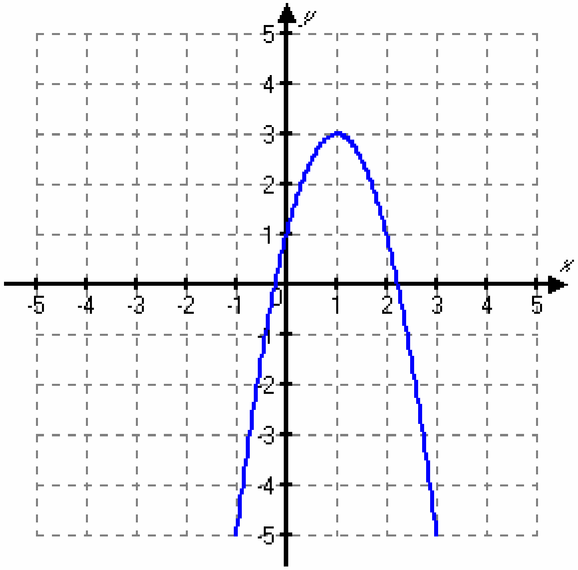


Example

Use the graph of the quadratic function shown to determine the sign of the leading coefficient, its vertex, and the equation of the axis of symmetry.

Solution

Leading coefficient: The graph opens downward, so the leading coefficient a is negative.



Vertex: The vertex is the highest point on the graph and is located at (1, 3).

Axis of symmetry: Vertical line through the vertex with equation $x = 1$.

VERTEX FORM

The parabolic graph of $f(x) = a(x - h)^2 + k$ with $a \neq 0$ has vertex (h, k). Its graph opens upward when $a > 0$ and opens downward when $a < 0$.

Example

Write the formula $f(x) = x^2 + 10x + 23$ in vertex form by completing the square.

Solution

$$y = x^2 + 10x + 23$$ Given formula

$$y - 23 = x^2 + 10x$$ Subtract 23 from each side.

$$y - 23 + 25 = x^2 + 10x + 25$$ Let $k = 10$; add $(10/2)^2 = 25$.

$$y + 2 = (x+5)^2$$ Factor perfect square trinomial.

$$y = (x+5)^2 - 2$$ Subtract 2.

VERTEX FORMULA

The *vertex* of the graph of $f(x) = ax^2 + bx + c$ with $a \neq 0$ is the point $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

Example Find the vertex of the graph of $f(x) = \frac{1}{2}x^2 - 4x + 8$ symbolically. Support your answer graphically and numerically.

Solution

$f(x) = \frac{1}{2}x^2 - 4x + 8$ $a = 1/2$, $b = -4$, and $c = 8$.

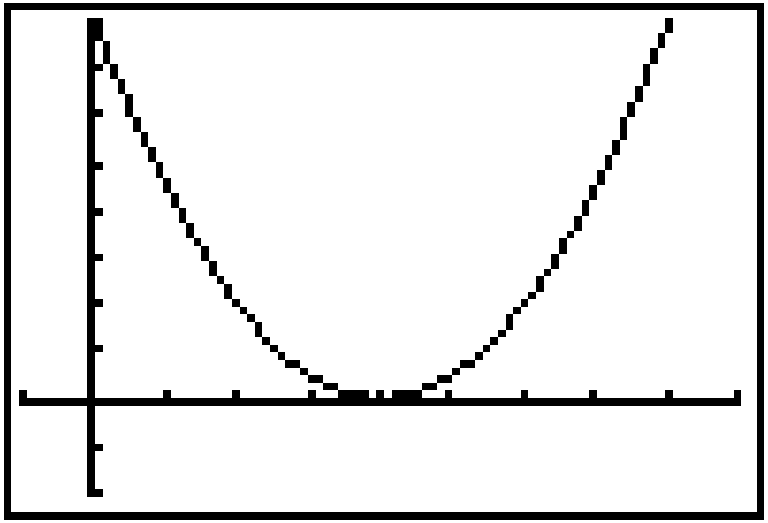
x-coordinate of vertex: $x = -\frac{b}{2a} = -\frac{(-4)}{2\left(\frac{1}{2}\right)} = \frac{4}{1} = 4$

y-coordinate evaluate *f*(4): $f(x) = \frac{1}{2}(4)^2 - 4(4) + 8 = 0$

The vertex is (4, 0).

Solution continued

Graphically and Numerically



X	Y1
0	8
1	4.5
2	2
3	.5
4	0
5	.5
6	2

X=0

axis of symmetry

Example

Use the vertex formula to write $f(x) = -3x^2 - 3x + 1$ in vertex form.

Solution

1. Begin by finding the vertex.

$$x = -\frac{b}{2a}$$

$$= -\frac{(-3)}{2(-3)}$$

$$= -\frac{1}{2}$$

2. Find *y*.

$$f\left(-\frac{1}{2}\right) = -3\left(\frac{-1}{2}\right)^2 - 3\left(\frac{-1}{2}\right) + 1 = \frac{7}{4}$$

The vertex is: $\left(-\frac{1}{2}, \frac{7}{4}\right)$

Vertex form:

$$f(x) = -3\left(x + \frac{1}{2}\right)^2 + \frac{7}{4}$$

Example

Graph the quadratic equation $g(x) = -3x^2 + 24x - 49$.

Solution

The formula is not in vertex form, but we can find the vertex.

$$x = -\frac{b}{2a} = -\frac{24}{2(-3)} = 4$$

The y-coordinate of the vertex is:

$$g(4) = -3(4)^2 + 24(4) - 49 = -1$$

The vertex is at (4, –1). The axis of symmetry is $x = 4$, and the parabola opens downward because the leading coefficient is negative.

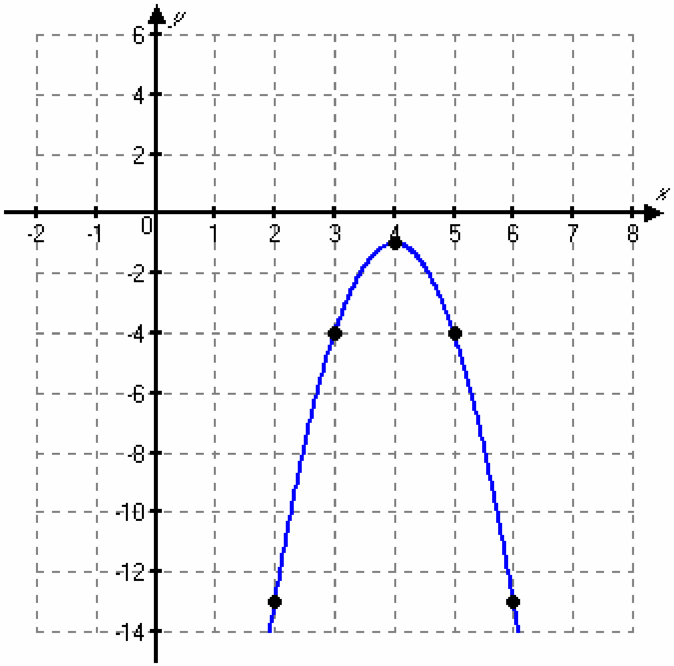
Solution continued

Graph: $g(x) = -3x^2 + 24x - 49$

Table of Values

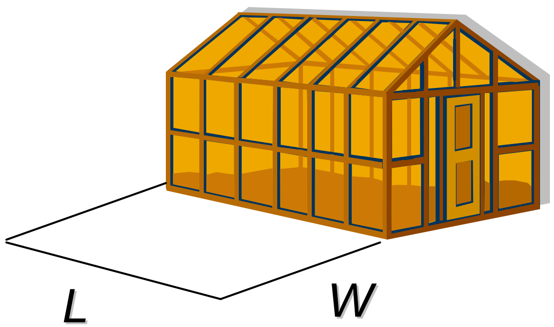
x	y
2	–13
3	–4
4	–1
5	–4
6	–13

vertex



Applications and Models Example

A junior horticulture class decides to enclose a rectangular garden, using a side of the greenhouse as one side of the rectangle. If the class has 32 feet of fence, find the dimensions of the rectangle that give the maximum area for the garden.



Solution

Let w be the width and L be the length of the rectangle. Because the 32-foot fence does not go along the greenhouse, if follows that

$$W + L + W = 32 \quad \text{or} \quad L = 32 - 2W$$

Solution continued

The area of the garden is the length times the width. $A = LW$

$$= (32 - 2W)W$$

$$= 32W - 2W^2$$

This is a parabola that opens downward, and by the vertex formula, the maximum area occurs when

$$W = -\frac{32}{2(-2)} = 8 \text{ feet}$$

The corresponding length is
$L = 32 - 2W = 32 - 2(8) = 16$ feet.

The dimensions are 8 feet by 16 feet.

Example

A model rocket is launched with an initial velocity of v_o = 150 feet per second and leaves the platform with an initial height of h_o = 10 feet.

a) Write a formula $s(t)$ that models the height of the rocket after t seconds.

b) How high is the rocket after 3 seconds?

c) Find the maximum height of the baseball. Support your answer graphically.

Solution

a) $s(t) = -16t^2 + v_o t + h_o$

$$= -16t^2 + 150t + 10$$

Solution continued $s(t) = -16t^2 + 150t + 10$

b) $s(3) = -16(3)^2 + 150(3) + 10 = 316$

The rocket is 316 feet high after 3 seconds.

c) Because *a* is negative, the vertex is the highest point on the graph, with an *t*-coordinate of

$$t = -\frac{b}{2a} = -\frac{150}{2(-16)} = 4.6875 \approx 4.7$$

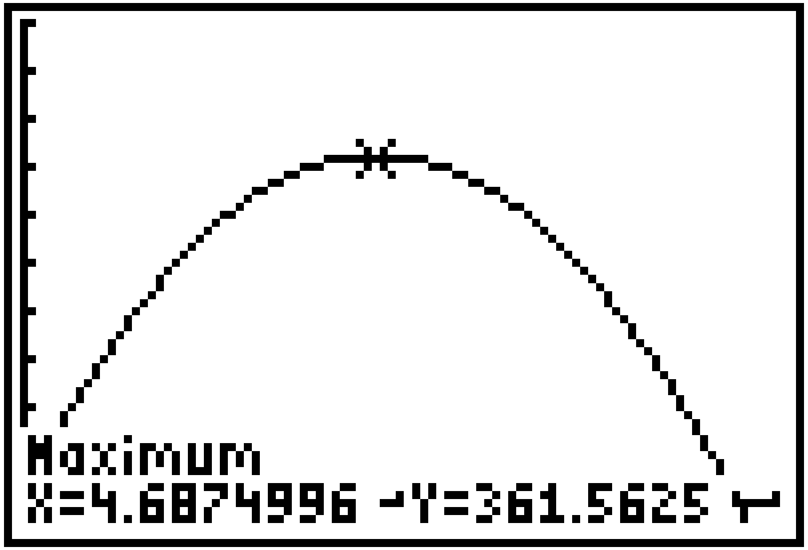
The *y*-coordinate is:

$$s(4.7) = -16(4.7)^2 + 150(4.7) + 10 = 361.56 \text{ feet}$$

The vertex is at (4.7, 361.6).

Solution continued

Graphical support is shown below.



3.2

Quadratic Equations and Problem Solving

- Understand basic concepts about quadratic equations
- Use factoring, the square root property, completing the square, and the quadratic formula to solve quadratic equations
- Understand the discriminant
- Solve problems involving quadratic equations

QUADRATIC EQUATION

A **quadratic equation** in one variable is an equation that can be written in the form

$$ax^2 + bx + c = 0,$$

where a, b, and c are real numbers with $a \neq 0$.

Solving Quadratic Equations

The are four basic symbolic strategies in which quadratic equations can be solved.

- Factoring
- Square root property
- Completing the square
- Quadratic formula

Factoring

A common technique used to solve equations that is based on the *zero-product property.*

Example $2x^2 - 4x - 5 = 1$

Solution

$$2x^2 - 4x - 5 = 1$$

$$2x^2 - 4x - 6 = 0$$

$$x^2 - 2x - 3 = 0$$

$$(x-3)(x+1) = 0$$

$$x - 3 = 0 \quad \text{or} \quad x + 1 = 0$$

$$x = 3 \quad \text{or} \quad x = -1$$

SQUARE ROOT PROPERTY

Let k be a nonnegative number. Then the solutions to the equation

$$x^2 = k$$

are given by $x = \pm\sqrt{k}$.

Example

A rescue helicopter hovers 68 feet above a jet ski in distress and drops a life raft. The height in feet of the raft above the water is given by

$$h(t) = -16t^2 + 68.$$

Determine how long it will take for the raft to hit the water after being dropped from the helicopter.

Solution (continued on next slide)

The raft will hit the water when its height is 0 feet above the water.

Solution continued

$$-16t^2 + 68 = 0$$

$$-16t^2 = -68$$

$$t^2 = \frac{-68}{-16}$$

$$t^2 = \frac{17}{4}$$

$$t = \pm\sqrt{\frac{17}{4}}$$

$$t \approx \pm 2.1$$

The life raft will hit about 2.1 seconds after it is dropped.

Completing the Square

Completing the square is useful when solving quadratic equations that do not factor easily.

If a quadratic equation can be written in the form $x^2 + kx = d$, where *k* and *d* are constants, then the equation can be solved using

$$x^2 + kx + \left(\frac{k}{2}\right)^2 = \left(x + \frac{k}{2}\right)^2.$$

Example Solve $2x^2 + 6x = 7$.

Solution

$$2x^2 + 6x = 7$$

$$x^2 + 3x = \frac{7}{2}$$

$$x^2 + 3x + \left(\frac{3}{2}\right)^2 = \frac{7}{2} + \left(\frac{3}{2}\right)^2$$

$$\left(x + \frac{3}{2}\right)^2 = \frac{23}{4}$$

$$x + \frac{3}{2} = \pm\sqrt{\frac{23}{4}}$$

$$x = -\frac{3}{2} \pm \sqrt{\frac{23}{4}} = -\frac{3}{2} \pm \frac{\sqrt{23}}{2}$$

QUADRATIC FORMULA

The solutions to the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example

Solve the equation $2x^2 - 5x - 9 = 0.$

Solution

Let $a = 2$, $b = -5$, and $c = -9$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{5 \pm \sqrt{(-5)^2 - 4(2)(-9)}}{2(2)}$$

$$x = \frac{5 \pm \sqrt{97}}{4}$$

QUADRATIC EQUATIONS AND THE DISCRIMINANT

To determine the number of real solutions to $ax^2 + bx + c = 0$, with $a \neq 0$, evaluate the discriminant $b^2 - 4ac$.

1. If $b^2 - 4ac > 0$, there are two real solutions.
2. If $b^2 - 4ac = 0$, there is one real solution.
3. If $b^2 - 4ac < 0$, there are no real solutions.

Example

Use the discriminant to determine the number of solutions to the quadratic equation $-3x^2 - 6x + 15 = 0.$

Solution $b^2 - 4ac = (-6)^2 - 4(-3)(15) = 216$

Since $b^2 - 4ac > 0$ the equation has two real solutions.

Modeling Projectile Motion

Example The following table shows the height of a toy rocket launched in the air.

Height of a toy rocket			
t (sec)	0	1	2
s(*t*) feet	12	36	28

a) Use $s(t) = -16t^2 + v_o t + h_o$ to model the data.

b) After how many seconds did the toy rocket strike the ground?

Solution continued

a) If $t = 0$, then $s(0) = 12$, so $s(t) = -16t^2 + v_0 t + 12.$ The value of v_o can be found by noting that when $t = 2$, $s(2) = 28$. Substituting gives the following result.

$$\begin{aligned} -16(2)^2 + v_0(2) + 12 &= 28 \\ 2v_0 &= 80 \\ v_0 &= 40 \end{aligned}$$

Thus $s(t) = -16t^2 + 40t + 12$ models the height of the toy rocket.

Solution continued

b) The rocket strikes the ground when $s(t) = 0$, or when $-16t^2 + 40t + 12 = 0$.

$$-16t^2 + 40t + 12 = 0$$

$$4t^2 - 10t - 3 = 0$$

Using the quadratic formula, where $a = 4$, $b = -10$ and $c = -3$ we find that

$$x \approx 2.8 \quad \text{or} \quad x \approx -0.3$$

- Only the positive solution is possible, so the toy rocket reaches the ground after approximately 2.8 seconds.

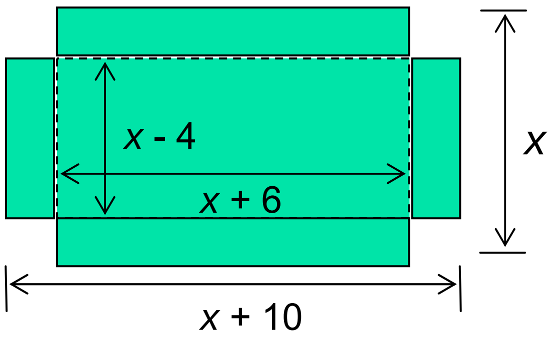
Example

A box is is being constructed by cutting 2 inch squares from the corners of a rectangular sheet of metal that is 10 inches longer than it is wide. If the box has a volume of 238 cubic inches, find the dimensions of the metal sheet.

Solution

Step 1: Let x be the width and $x + 10$ be the length.

Step 2: Draw a picture.



Solution continued

Since the height times the width times the length must equal the volume, or 238 cubic inches, the following can be written

$$2(x-4)(x+6)=238 \quad \text{or}$$

$$(x-4)(x+6)=119$$

Step 3: Write the quadratic equation in the form $ax^2 + bx + c = 0$ and factor.

$$x^2+2x-24=119$$

$$x^2+2x-143=0$$

$$(x+13)(x-11)=0$$

$$x=-13 \quad \text{or} \quad x=11$$

Solution continued

The dimensions can not be negative, so the width is 11 inches and the length is 10 inches more, or 21 inches.

Step 4: After the 2 square inch pieces are cut out, the dimensions of the bottom of the box are 11 – 4 = 7 inches by 21 – 4 = 17 inches. The volume of the box is then $2 \cdot 7 \cdot 17 = 238$, which checks.