Core Connections Geometry
Checkpoint Materials

Notes to Students (and their Teachers)

Students master different skills at different speeds. No two students learn exactly the same way at the same time. At some point you will be expected to perform certain skills accurately. Most of the Checkpoint problems incorporate skills that you should have developed in previous courses. If you have not mastered these skills yet it does not mean that you will not be successful in this class. However, you may need to do some work outside of class to get caught up on them.

Starting in Chapter 1 and finishing in Chapter 11, there are 13 problems designed as Checkpoint problems. Each one is marked with an icon like the one above and numbered according to the chapter that it is in. After you do each of the Checkpoint problems, check your answers by referring to this section. If your answers are incorrect, you may need some extra practice to develop that skill. The practice sets are keyed to each of the Checkpoint problems in the textbook. Each has the topic clearly labeled, followed by the answers to the corresponding Checkpoint problem and then some completed examples. Next, the complete solution to the Checkpoint problem from the text is given, and there are more problems for you to practice with answers included.

Remember, looking is not the same as doing! You will never become good at any sport by just watching it, and in the same way, reading through the worked examples and understanding the steps is not the same as being able to do the problems yourself. How many of the extra practice problems do you need to try? That is really up to you. Remember that your goal is to be able to do similar problems on your own confidently and accurately. This is your responsibility. You should not expect your teacher to spend time in class going over the solutions to the Checkpoint problem sets. If you are not confident after reading the examples and trying the problems, you should get help outside of class time or talk to your teacher about working with a tutor.

Another source for help with the Checkpoint problems and other topics in *Core Connections Geometry* is the *Parent Guide with Extra Practice*. This resource is available for download free of charge at www.cpm.org.

Checkpoint Topics

1. Solving Linear Equations
2. Solving Linear Systems of Equations
3. Linear Equations from Multiple Representations
4. Finding Areas and Perimeters of Complex Shapes

5A. Multiplying Polynomials and Solving Quadratics

5B. Writing Equations for Arithmetic and Geometric Sequences

6. Solving Proportional Equations and Similar Figures
7. Solving with Trigonometric Ratios and the Pythagorean Theorem
8. Angle Relationships in Geometric Figures

9A. Probabilities with Unions, Intersections, and Complements

9B. Exponential Functions

10. Finding Angles in and Areas of Regular Polygons
11. Volumes and Surface Areas of Prisms and Cylinders

Checkpoint 1

Problem 1-127

Solving Linear Equations

Answers to problem 1-127: a. $x = -2$, b. $x = 1\frac{1}{2}$, c. $x = 3$, d. no solution

Equations may be solved in a variety of ways. Commonly, the first steps are to remove parenthesis using the Distributive Property and then simplify by combining like terms. Next isolate the variable on one side and the constant terms on the other. Finally, divide to find the value of the variable. Note: When the process of solving an equation ends with different numbers on each side of the equal sign (for example, $2 = 4$), there is *no solution* to the problem. When the result is the same expression or number on each side of the equation (for example, $x + 3 = x + 3$) it means that *all real numbers* are solutions.

Example 1: Solve $4x+4x-3=6x+9$

Solution:

$4x+4x-3=6x+9$	problem
$8x-3=6x+9$	simplify
$2x=12$	add 3, subtract $6x$ on each side
$x=6$	divide

Check:
$$\begin{aligned} 4(6)+4(6)-3&=6(6)+9\\ 24+24-3&=36+9\\ 48-3&=45\\ 45&=45 \end{aligned}$$

Example 2: Solve $-4x+2-(-x+1)=-3+(-x+5)$

Solution:

$-4x+2-(-x+1)=-3+(-x+5)$	problem
$-4x+2+x-1=-3-x+5$	remove parenthesis (distribute)
$-3x+1=-x+2$	simplify
$-2x=1$	add x, subtract 1 from each side
$x=-\frac{1}{2}$	divide

Check:
$$\begin{aligned} -4\left(-\tfrac{1}{2}\right)+2-\left(-\left(-\tfrac{1}{2}\right)+1\right)&=-3+\left(-\left(-\tfrac{1}{2}\right)+5\right)\\ 2+2-\left(\tfrac{1}{2}+1\right)&=-3+\left(\tfrac{1}{2}+5\right)\\ 4-\left(1\tfrac{1}{2}\right)&=-3+\left(5\tfrac{1}{2}\right)\\ 2\tfrac{1}{2}&=2\tfrac{1}{2} \end{aligned}$$

Now we can go back and solve the original problems.

a.
$$\begin{aligned} 3x+7&=-x-1 \\ 4x&=-8 \\ x&=-2 \end{aligned}$$

b.
$$\begin{aligned} 1-2x+5&=4x-3 \\ -2x+6&=4x-3 \\ 9&=6x \\ 1\tfrac{1}{2}&=x \end{aligned}$$

c.
$$\begin{aligned} -2x-6&=2-4x-(x-1) \\ -2x-6&=2-4x-x+1 \\ -2x-6&=-5x+3 \\ 3x&=9 \\ x&=3 \end{aligned}$$

d.
$$\begin{aligned} 3x-4+1&=-2x-5+5x \\ 3x-3&=3x-5 \\ -3&=-5 \\ -3&\neq-5 \Rightarrow \text{no solution} \end{aligned}$$

Here are some more to try. Solve each equation.

1. $2x-3=-x+3$
2. $3x+2+x=x+5$
3. $6-x-3=4(x-2)$
4. $4x-2-2x=x-5$
5. $-(x+3)=2x-6$
6. $-x+2=x-5-3x$
7. $1+3x-x=x-4+2x$
8. $5x-3+2x=x+7+6x$
9. $4y-8-2y=4$
10. $-x+3=6$
11. $-2+3y=y-2-4y$
12. $2(x-2)+x=5$
13. $-x-3=2x-6$
14. $10=x+5+x$
15. $2x-1-1=x-3-(-5+x)$
16. $3+3x-x+2=3x+4$
17. $-4+3x-1=2x+1+2x$
18. $2x-7=-x-1$
19. $7=3x-4-(x+2)$
20. $5y+(-y-2)=4+y$

Answers

1. $x=2$	2. $x=1$	3. $x=2\frac{1}{5}$	4. $x=-3$
5. $x=1$	6. $x=-7$	7. $x=5$	8. no solution
9. $y=6$	10. $x=-3$	11. $y=0$	12. $x=3$
13. $x=1$	14. $x=2\frac{1}{2}$	15. $x=2$	16. $x=1$
17. $x=-6$	18. $x=2$	19. $x=6\frac{1}{2}$	20. $y=2$

Checkpoint 2

Problem 2-113

Solving Linear Systems of Equations

Answers to problem 2-113: a. (–2, 5), b. (1, 5), c. (–12, 14), d. (2, 2)

When two equations are both in $y = mx + b$ form it is convenient to use the Equal Values Method to solve for the point of intersection. Set the two equations equal to each other to create an equation in one variable and solve for x. Then use the x-value in either equation to solve for y.

If one of the equations has a variable by itself on one side of the equation, then that expression can replace the variable in the second equation. This again creates an equation with only one variable. This is called the Substitution Method. See Example 1 below.

If both equations are in standard form (that is $ax + by = c$), then adding or subtracting the equations may eliminate one of the variables. Sometimes it is necessary to multiply before adding or subtracting so that the coefficients are the same or opposite. This is called the Elimination Method. See Example 2.

Sometimes the equations are not convenient for substitution or elimination. In that case one of both of the equations will need to be rearranged into a form suitable for the previously mentioned methods.

Example 1: Solve the following system.

$$\begin{aligned} 4x + y &= 8 \\ x &= 5 - y \end{aligned}$$

Solution: Since x is alone in the second equation, substitute $5 - y$ in the first equation, then solve as usual.

$$\begin{aligned} 4(5 - y) + y &= 8 \\ 20 - 4y + y &= 8 \\ 20 - 3y &= 8 \\ -3y &= -12 \\ y &= 4 \end{aligned}$$

Then substitute $y = 4$ into either original equation to find x. Using the second equation $x = 1$ so the solution is $(1, 4)$.

$$\begin{aligned} x &= 5 - 4 \\ x &= 1 \end{aligned}$$

Example 2: Solve the following system. $\begin{array}{l} -2x+y=-7 \\ 3x-4y=8 \end{array}$

Solution: If we add or subtract the two equations no variable is eliminated. Notice, however, that if everything in the top equation is multiplied by 4, then when the two equations are added together, the y-terms are eliminated.

$$\begin{array}{l} -2x+y=-7 \\ 3x-4y=8 \end{array} \Rightarrow \begin{array}{c} 4(-2x+y=-7) \\ 3x-4y=8 \end{array} \Rightarrow \begin{array}{c} -8x+4y=-28 \\ \underline{\ \ 3x-4y=8\ \ } \\ -5x+0=-20 \\ x=4 \end{array}$$

Substitute $x=4$ into the first equation: $-2(4)+y=-7 \Rightarrow -8+y=-7 \Rightarrow y=1$
The solution is $(4,1)$.

Now we can go back and solve the original problems.

a. $\begin{array}{l} y=3x+11 \\ x+y=3 \end{array}$

Using substitution:

$$\begin{aligned} x+(3x+11)&=3 \\ 4x+11&=3 \\ 4x&=-8 \\ x&=-2 \\ y&=3(-2)+11=5 \end{aligned}$$

The answer is (–2, 5).

b. $\begin{array}{l} y=2x+3 \\ x-y=-4 \end{array}$

Using substitution:

$$\begin{aligned} x-(2x+3)&=-4 \\ x-2x-3&=-4 \\ -x-3&=-4 \\ -x&=-1 \\ x&=1 \\ y&=2(1)+3=5 \end{aligned}$$

The answer is (1, 5).

c. $\begin{array}{l} x+2y=16 \\ x+y=2 \end{array}$

Subtracting the second equation from the first eliminates x.

$$\begin{array}{r} x+2y=16 \\ \underline{-(x+y=2)} \\ y=14 \\ x+14=2 \\ x=-12 \end{array}$$

The answer is (–12, 14).

d. $\begin{array}{l} 2x+3y=10 \\ 3x-4y=-2 \end{array}$

Multiplying the top by 4, the bottom by 3, and adding the equations eliminates y.

$$\begin{array}{r} 8x+12y=40 \\ \underline{9x-12y=-6} \\ 17x\qquad=34 \\ x=2 \\ 2(2)+3y=10 \\ 3y=6 \\ y=2 \end{array}$$

The answer is (2, 2).

Here are some more to try. Solve each system of equations.

1. $y = -3x$
   $4x + y = 2$
2. $y = 7x - 5$
   $2x + y = 13$
3. $x = -5y - 4$
   $x - 4y = 23$
4. $x + y = -4$
   $-x + 2y = 13$
5. $3x - y = 1$
   $-2x + y = 2$
6. $2x + 5y = 1$
   $2x - y = 19$
7. $x + y = 10$
   $y = x - 4$
8. $y = 5 - x$
   $4x + 2y = 10$
9. $3x + 5y = 23$
   $y = x + 3$
10. $y - x = 4$
    $2y + x = 8$
11. $2x - y = 4$
    $\frac{1}{2}x + y = 1$
12. $-4x + 6y = -20$
    $2x - 3y = 10$
13. $x = \frac{1}{2}y + \frac{1}{2}$
    $2x + y = -1$
14. $a = 2b + 4$
    $b - 2a = 16$
15. $y = 3 - 2x$
    $4x + 2y = 5$
16. $6x - 2y = -16$
    $4x + y = 1$
17. $4x - 4y = 14$
    $2x - 4y = 8$
18. $3x + 2y = 12$
    $5x - 3y = -37$
19. $x + y = 5$
    $2y - x = -2$
20. $2y = 10 - x$
    $3x - 2y = -2$
21. $2x - 3y = 50$
    $7x + 8y = -10$
22. $3x = y - 2$
    $6x + 4 = 2y$
23. $y = -\frac{2}{3}x + 4$
    $y = \frac{1}{3}x - 2$
24. $5x + 2y = 9$
    $2x + 3y = -3$

Answers:

1. $(2, -6)$
2. $(2, 9)$
3. $(11, -3)$
4. $(-7, 3)$
5. $(3, 8)$
6. $(8, -3)$
7. $(7, 3)$
8. $(0, 5)$
9. $(1, 4)$
10. $(0, 4)$
11. $(2, 0)$
12. infinite solutions
13. $(0, -1)$
14. $(-12, -8)$
15. no solution
16. $(-1, 5)$
17. $(3, -\frac{1}{2})$
18. $(-2, 9)$
19. $(4, 1)$
20. $(2, 4)$
21. $(10, -10)$
22. infinite solutions
23. $(6, 0)$
24. $(3, -3)$

Checkpoint 3

Problem 3-111

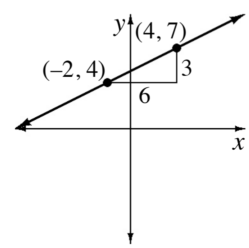
Linear Equations from Multiple Representations

Answers to problem 3-111: a. $y=-\frac{1}{2}x+4$, b. $y=2x-1$, c. $y=\frac{2}{5}x+\frac{7}{5}$,
d. $C=15+7(t-1)=8+7t$

Linear equations are equations of the form $y = mx + b$. The slope or rate of change is represented by m and the y-intercept or starting value is represented by b. Horizontal lines have a slope of zero and an equation of the form $y = k$. Vertical lines have undefined slope and an equation of the form $x = h$. Parallel lines have the same slope and perpendicular lines have slopes that are opposite reciprocals.

Example 1: Find the equation of the line passing through (–2, 4) and (4, 7). What is the slope of any line perpendicular to this line?

Solution: The slope is $m=\frac{\text{vertical change}}{\text{horizontal change}}=\frac{3}{6}=\frac{1}{2}$ and can be seen in the generic slope triangle at right. Since the line slants upward (when reading from left to right), the slope is positive. The equation of a line is $y = mx + b$ and substituting $\frac{1}{2}$ for b it becomes $y=\frac{1}{2}x+b$. Next choose either given point and substitute for x and y. Choosing (–2, 4), the equation becomes $4=\frac{1}{2}(-2)+b \Rightarrow b=5$. The equation of the line is: $y=\frac{1}{2}x+5$. The slope of any line perpendicular to this line would be $m = -2$.



Note: Some people prefer to use formulas that represent the generic slope triangle.

$$\text{slope} = \frac{y_2-y_1}{x_2-x_1}=\frac{7-(4)}{4-(-2)}=\frac{3}{6}=\frac{1}{2}$$

Notice that x_2-x_1 and y_2-y_1 represent the lengths of the horizontal and vertical legs respectively.

Example 2: The cost to rent a jet ski on Evantown Lake is \$30 plus \$7.50 per hour. Write an equation that represents the cost for various rental hours. Be sure to define your variables.

Solution: The prices are fixed but the hours and total cost vary. Let C = total cost and h = the hours. A 5-hour rental would cost \$30 + \$7.50(5), so in general $C = 30 + 7.50h$.

Now we can go back and solve the original problems.

a. Using the slope triangle formed by the line and the x- and y-axes, $m = -\frac{4}{8} = -\frac{1}{2}$. The y-intercept is the point $(0, 4)$ so $b = 4$. The equation of the line is $y = -\frac{1}{2}x + 4$.

b. The given line has slope $-\frac{1}{2}$ so the perpendicular line has opposite reciprocal slope of $m = 2$. Using the $y = mx + b$ equation of a line with $m = 2$ and $(x, y) = (-1, -3)$ we have $-3 = 2(-1) + b \Rightarrow b = -1$. The equation of the perpendicular line is $y = 2x - 1$.

c. Using a generic slope triangle or the formula, $m = \frac{2}{5}$. Choosing $(x, y) = (-1, 1)$ and using $y = mx + b \Rightarrow 1 = \frac{2}{5}(-1) + b \Rightarrow b = \frac{7}{5}$. The equation is $y = \frac{2}{5}x + \frac{7}{5}$.

d. The parking charges for 6 hours would be $\$15(1) + \$7(6-1)$ so in general $C = 15 + 7(t-1)$ which can also be written as $C = 8 + 7t$.

Here are some more to try. Write an equation for each graphed line.

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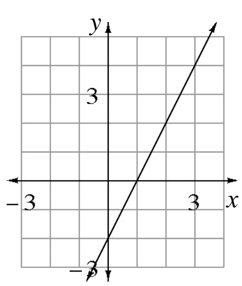

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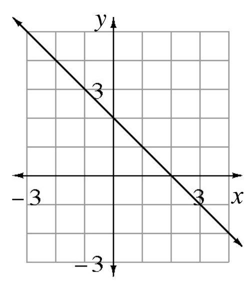

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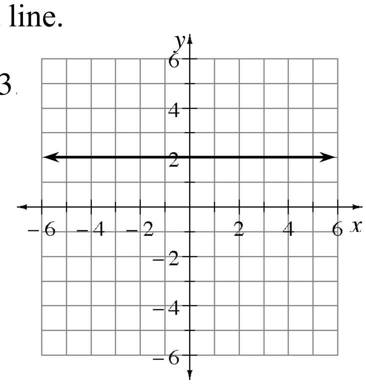

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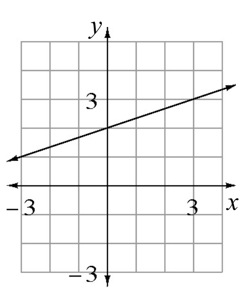

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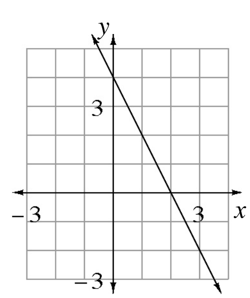

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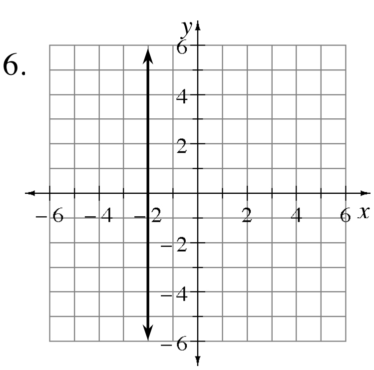


Given each description below, write an equation of the line.

7. Passing through $(1, 1)$ and $(0, 4)$.

8. Passing through $(-2, 3)$ and $(3, 5)$.

9. Perpendicular to the line $y = 2x - 2$ and passing through $(-3, 5)$.

10. Perpendicular to the line $y = x - 2$ and passing through $(-2, 3)$.

11. Passing through $(2, -1)$ and $(3, -3)$.

12. Passing through $(4, 5)$ and $(-2, -4)$.

13. Perpendicular to the line $y = -\frac{3}{2}x + 3$ and passing through $(2, -1)$.

14. Perpendicular to the line $3x - 4y = 12$ and passing through $(4, -2)$.

15. Passing through $(-3, -2)$ and $(5, -2)$.

16. Passing through $(4, 5)$ and $(4, -4)$.

Write a linear equation to represent each situation. Be sure to define your variables.

17. The cost of attending the state fair with a \$10 admission fee and cost of \$1.50 per ride.

18. The population of Salem that is currently 15,375 but is decreasing by 27 people per year.

19. The weight of Karen who currently weighs 105 pounds but is gaining two pounds per month.

20. The value of Miguel's bank account that currently has \$3275 and he is saving \$35 per week.

21. Paula's distance from home as her mother drives her home at 50 miles per hour from a camp that is located 250 miles away.

22. The perimeter of a rectangle with length 3 cm more than twice the width.

23. The cost to rent a sailboat that is advertised as \$75 for the first 2 hours and \$15 for each additional hour.

24. The total ticket receipts for a play with \$5 admission for students and \$9 admission for adults if there were 40 more student tickets sold than adult tickets.

Answers:

1. $y = 2x - 2$
2. $y = -x + 2$
3. $y = 2$
4. $y = \frac{1}{2}x + 2$
5. $y = -2x + 4$
6. $x = -2$
7. $y = -3x + 4$
8. $y = \frac{2}{5}x + \frac{19}{5}$
9. $y = -\frac{1}{2}x + \frac{7}{2}$
10. $y = -x + 1$
11. $y = -2x + 3$
12. $y = \frac{3}{2}x - 1$
13. $y = \frac{2}{3}x - \frac{7}{3}$
14. $y = -\frac{4}{3}x + \frac{10}{3}$
15. $y = -2$
16. $x = 4$

For answers 17 through 24 different variables are possible but all variables should be defined.

17. $c = 10 + 1.5n$
18. $p = 15375 - 27y$
19. $w = 105 + 2m$
20. $v = 3275 + 35w$
21. $d = 250 - 50h$
22. $p = 2w + 2(2w + 3) = 6w + 6$
23. $c = 75 + 15(h - 2)$
24. $r = 5(a + 40) + 9a$

Checkpoint 4

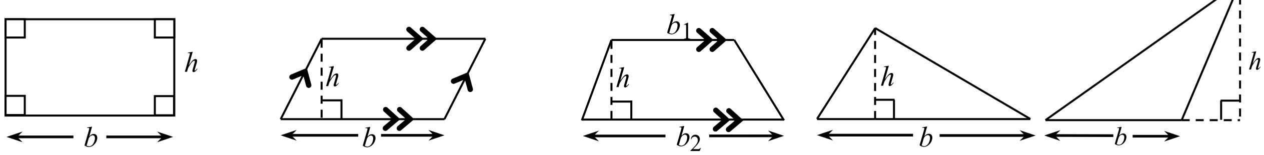
Problem 4-44

Finding the Areas and Perimeters of Complex Shapes

Answers to problem 4-44: a. 144 cm^2, 52 cm; b. 696.67 m^2, 114.67 m; c. 72 cm^2, 48 cm;
d. 130 square units, 58 units

Area is the number of square units in a flat region. The formulas to calculate the area of several kinds of polygons are:

RECTANGLE PARALLELOGRAM TRAPEZOID TRIANGLE

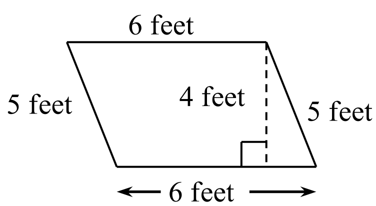


$A = bh$ $A = bh$ $A = \frac{1}{2}(b_1 + b_2)h$ $A = \frac{1}{2}bh$

Perimeter is the distance around a figure on a flat surface. To calculate the perimeter of a polygon, add together the length of each side.

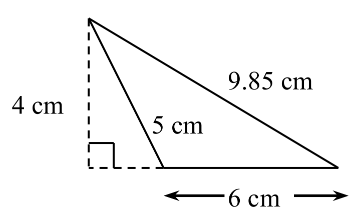
For complex figures, divide the figure into more recognizable parts. Then find the sum of the area of the parts. When finding the perimeter of a complex region, be sure that the sum only includes the edges on the outside of the region.

Example 1:
Calculate the area and perimeter.



parallelogram

Example 2:
Calculate the area and perimeter.



triangle

Solutions:

Example 1:

$A = bh = 6 \cdot 4 = 24 \text{ feet}^2$

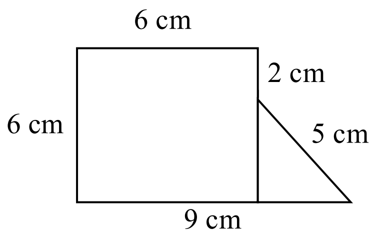
$P = 6 + 6 + 5 + 5 = 22 \text{ feet}$

Example 2:

$A = \frac{1}{2}bh = \frac{1}{2} \cdot 6 \cdot 4 = 12 \text{ cm}^2$

$P = 6 + 5 + 9.85 = 20.85 \text{ cm}$

Example 3: Calculate the area and perimeter.



Solution:

Area of square plus triangle:
$A = s^2 + \frac{1}{2}bh = 6^2 + \frac{1}{2} \cdot 3 \cdot 4 = 42 \text{ cm}^2$

Add all sides for perimeter:
$6 + 6 + 2 + 5 + 9 = 28$ cm

Now we can go back and solve the original problems.

a. Parallelogram: $A = bh = 16 \cdot 9 = 144 \text{ cm}^2$; $P = 16 + 16 + 10 + 10 = 52$ cm

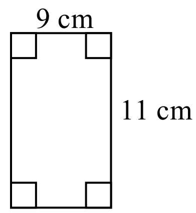
b. Trapezoid: $A = \frac{1}{2}(b_1 + b_2)h = \frac{1}{2}(25 + 44.67) \cdot 10 = 696.67 \text{ m}^2$;
$P = 21 + 25 + 24 + 44.67 = 114.67$ m

c. Rectangle complex: First determine the lengths of the missing sides. Adding them clockwise P = 12 + 2 + 3 + 5 + 3 + 2 + 9 + 7 + 3 + 2 = 48 cm.
To find the area, imagine two vertical lines that divide the shape into four rectangles–one small rectangle on the left, two small rectangles on the right and a large rectangle in the middle. Each of the small rectangles has a base of 3 cm and a height of 2 cm. The middle rectangle has a base of 6 cm and a height of 9 cm.
Total area = area of 3 small rectangle + area of larger rectangle.
$A = 3bh_{\text{small}} + bh_{\text{big}} = 3(3 \cdot 2) + (6 \cdot 9) = 72 \text{ cm}^2$.

d. Trapezoid–rectangle: P = 23 + 10 + 4 + 2 + 3 + 2 + 4 + 10 = 58 units.
$A = \frac{1}{2}h(b_1 + b_2) - bh = \frac{1}{2} \cdot 8(23 + 11) - 2 \cdot 3 = 130 \text{ units}^2$

Here are some more to try. Find the area and perimeter of each figure. Note: All angles that look like right angles can be assumed to be right angles.

1.



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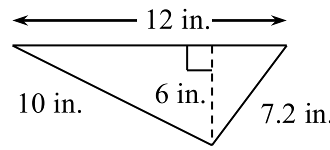
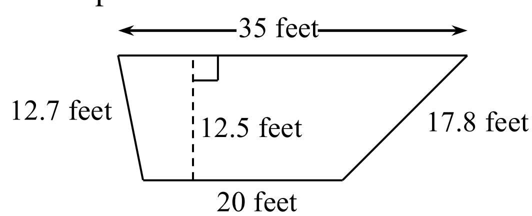
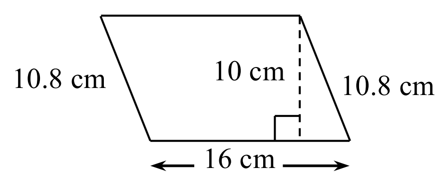


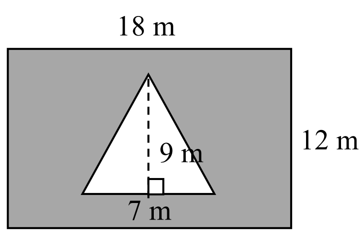
3. Trapezoid



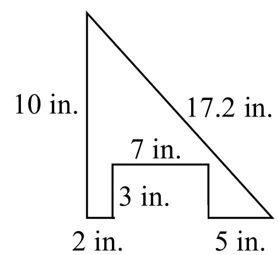
4. Parallelogram



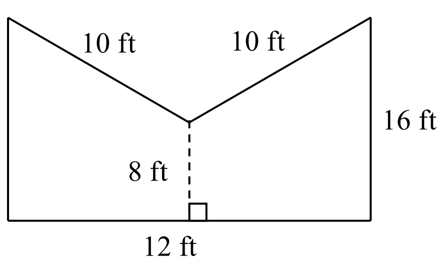
5. Find the area of the shaded region.



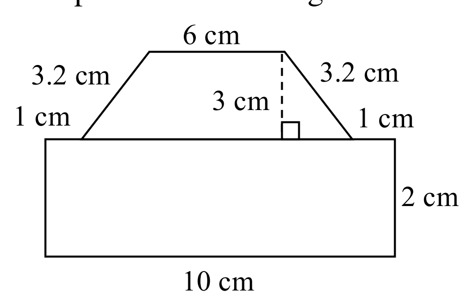
6.



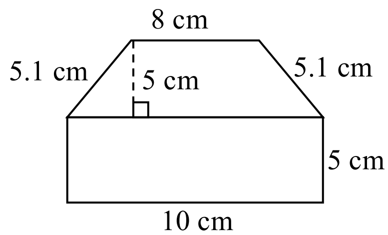
7.



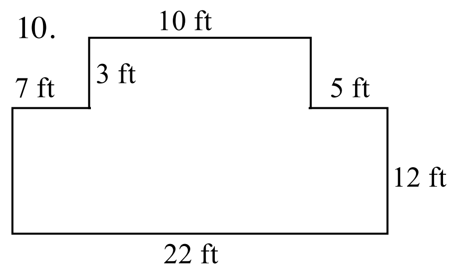
8. Trapezoid on a rectangle



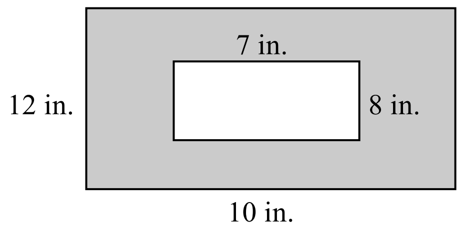
9.



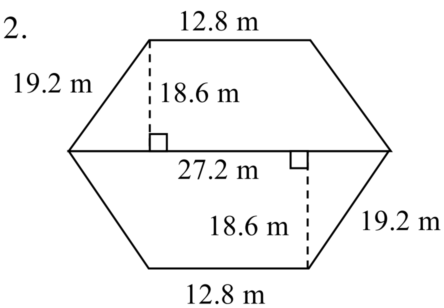
10.



11. Find the area of the shaded region.



12.



Answers:

1. 99 cm^2, 40 cm
2. 36 in.^2, 29.2 in.
3. 343.75 feet^2, 85.5 feet
4. 160 cm^2, 53.6 cm
5. A = 184.5 m^2
6. A = 49 in.^2, P = 47.2 in.
7. A = 144 ft^2, P = 64 ft
8. A = 41 cm^2, P = 28.4 cm
9. A = 95 cm^2, P = 38.2 cm
10. A = 294 ft^2, P = 74 ft
11. A = 64 in.^2
12. A = 744 m^2, P = 102.4 m

Checkpoint 5A

Problem 5-104

Multiplying Polynomials and Solving Quadratics

Answers to problem 5-104: a. $2x^2+6x$, b. $3x^2-7x-6$, c. $x=7$ or 1 , d. $y=5$ or -3

Polynomials can be multiplied (changed from the area written as a product to the area written as a sum) by using the Distributive Property or generic rectangles.

Example 1: Multiply $-5x(-2x+y)$.

Solution: Using the Distributive Property $\underbrace{-5x(-2x+y)}_{\text{area as a product}} = -5x\cdot -2x + -5x \cdot y = \underbrace{10x^2-5xy}_{\text{area as a sum}}$

Example 2: Multiply $(x-3)(2x+1)$.

Solution: Although the Distributive Property may be used, for this problem and other more complicated ones, it is beneficial to use a generic rectangle to find all the parts.

$$\begin{array}{c|c|c|} \cline{2-3} -3 & & \\ \cline{2-3} x & & \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c}{2x} & \multicolumn{1}{c}{+1} \end{array} \Rightarrow \begin{array}{c|c|c|} \cline{2-3} -3 & -6x & -3 \\ \cline{2-3} x & 2x^2 & x \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c}{2x} & \multicolumn{1}{c}{+1} \end{array} \Rightarrow \underbrace{(x-3)(2x+1)}_{\text{area as a product}} = \underbrace{2x^2-5x-3}_{\text{area as a sum}}$$

Solving quadratics first required factoring the polynomials to change the sum into a product. It is the reverse of multiplying polynomials and using a generic rectangle is helpful. Once the expression is factored, then the factors can be found using the Zero Product Property.

Example 3: Solve $x^2+7x+12=0$.

Solution: First, factor the polynomial. Sketch a generic rectangle with 4 sections.

Write the x^2 and the 12 along one diagonal.

Find two terms whose product is $12\cdot x^2 = 12x^2$ and whose sum is $7x$. That is, $3x$ and $4x$. This is the same as a Diamond Problem.

Write these terms as the other diagonal.

Find the base and height of the rectangle by using the partial areas.

Write the factored equation. $x^2+7x+12=(x+3)(x+4)=0$

Then, using the Zero Product Property, we know that either $(x + 3)$ or $(x + 4)$ is equal to zero (since their product is zero).
This means that $x = -3$ or $x = -4$.

$$\begin{array}{|c|c|} \hline & 12 \\ \hline x^2 & \\ \hline \end{array} \;\downarrow\; \begin{array}{|c|c|} \hline \mathbf{3x} & 12 \\ \hline x^2 & \mathbf{4x} \\ \hline \end{array} \;\downarrow\; \begin{array}{c|c|c|} \cline{2-3} 3 & 3x & 12 \\ + & & \\ x & x^2 & 4x \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{2}{c}{x \;+\; 4} \end{array}$$

Example 4: Solve $2x^2 + x = 6$.

Solution: In order to factor and use the Zero Product Property, the equation must be equal to zero. First, rearrange the equation to $2x^2 + x - 6 = 0$. Then factor the expression on the left side.

Sketch a generic rectangle with 4 sections.

Write $2x^2$ and –6 along one diagonal.

Find two terms whose product is $-12x^2$ and whose sum is $1x$. That is, $4x$ and $-3x$.

Write these terms as the other diagonal.

Find the base and height of the rectangle.

$$\begin{array}{|c|c|}\hline \mathbf{-3x} & -6 \\ \hline 2x^2 & \mathbf{4x} \\ \hline \end{array} \longrightarrow \begin{array}{c|c|c|}\cline{2-3} -3 & -3x & -6 \\ \cline{2-3} 2x & 2x^2 & 4x \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c}{x} & \multicolumn{1}{c}{+\ 2} \end{array}$$

Write the factored equation. $2x^2 + x - 6 = (2x-3)(x+2) = 0$

Then use the Zero Product Property and finish solving.

$$2x - 3 = 0 \qquad\qquad x + 2 = 0$$
$$2x = 3 \quad \text{and} \quad x = -2$$
$$x = \tfrac{3}{2}$$

Now we can go back and solve the original problems.

a. Using the Distributive Property: $2x(x+3) = 2x \cdot x + 2x \cdot 3 = 2x^2 + 6x$

b. Using generic rectangles:

$$\begin{array}{c|c|c|}\cline{2-3} -3 & & \\ \cline{2-3} x & & \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c}{3x} & \multicolumn{1}{c}{+2} \end{array} \Rightarrow \begin{array}{c|c|c|}\cline{2-3} -3 & -9x & -6 \\ \cline{2-3} x & 3x^2 & 2x \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c}{3x} & \multicolumn{1}{c}{+2} \end{array} \Rightarrow 3x^2 - 7x - 6$$

c. $x^2 - 8x + 7 = 0$

The two terms whose product is $7x^2$ and whose sum is $-8x$ are $-1x$ and $-7x$.

$$\begin{array}{|c|c|}\hline \mathbf{-1x} & 7 \\ \hline x^2 & \mathbf{-7x} \\ \hline \end{array} \longrightarrow \begin{array}{c|c|c|}\cline{2-3} -1 & -1x & 7 \\ \cline{2-3} x & x^2 & -7x \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c}{x} & \multicolumn{1}{c}{-\ 7} \end{array}$$

$$x^2 - 8x + 7 = (x-1)(x-7) = 0$$
$$x - 1 = 0 \Rightarrow x = 1$$
$$x - 7 = 0 \Rightarrow x = 7$$

d. $y^2 - 2y = 15$

First, rewrite the problem to: $y^2 - 2y - 15 = 0$.

The two terms whose product is $-15y^2$ and whose sum is $-2y$ are $-5y$ and $3y$.

$$\begin{array}{|c|c|}\hline \mathbf{-5y} & -15 \\ \hline y^2 & \mathbf{3y} \\ \hline \end{array} \longrightarrow \begin{array}{c|c|c|}\cline{2-3} -5 & -5y & -15 \\ \cline{2-3} y & y^2 & 3y \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c}{y} & \multicolumn{1}{c}{+\ 3} \end{array}$$

$$y^2 - 2y - 15 = (y-5)(y+3) = 0$$
$$y - 5 = 0 \Rightarrow y = 5$$
$$y + 3 = 0 \Rightarrow y = -3$$

Here are some more to try. Multiply the expressions in problems 1 through 15 and solve the equations in problems 16 through 30.

1.	$2x(x-1)$	2.	$(3x+2)(2x+7)$	3.	$(2x-1)(3x+1)$
4.	$2y(x-1)$	5.	$(2y-1)(3y+5)$	6.	$(x+3)(x-3)$
7.	$3y(x-y)$	8.	$(2x-5)(x+4)$	9.	$(3x+7)(3x-7)$
10.	$(4x+3)^2$	11.	$(x+y)(x+2)$	12.	$(x-1)(x+y+1)$
13.	$(2y-3)^2$	14.	$(x+2)(x+y-2)$	15.	$2(x+3)(3x-4)$
16.	$x^2+5x+6=0$	17.	$2x^2+5x+3=0$	18.	$3x^2+4x+1=0$
19.	$x^2-10x+25=0$	20.	$x^2+15x+44=0$	21.	$x^2-6x=7$
22.	$x^2-11x=-24$	23.	$x^2=4x+32$	24.	$4x^2+12x+9=0$
25.	$12x^2+11x=5$	26.	$x^2=-x+72$	27.	$3x^2=20x+7$
28.	$x^2-11x+28=0$	29.	$3x^2-5=-2x$	30.	$6x^2-2=x$

Answers:

1.	$2x^2-2x$	2.	$6x^2+25x+14$	3.	$6x^2-x-1$
4.	$2xy-2y$	5.	$6y^2+7y-5$	6.	x^2-9
7.	$3xy-3y^2$	8.	$2x^2+3x-20$	9.	$9x^2-49$
10.	$16x^2+24x+9$	11.	$x^2+xy+2x+2y$	12.	$x^2+xy-y-1$
13.	$4y^2-12y+9$	14.	$x^2+xy+2y-4$	15.	$6x^2+10x-24$
16.	$x=-2,-3$	17.	$x=-\frac{3}{2},-1$	18.	$x=-\frac{1}{3},-1$
19.	$x=5$	20.	$x=-11,-4$	21.	$x=7,-1$
22.	$x=3,8$	23.	$x=8,-4$	24.	$x=-\frac{3}{2}$
25.	$x=\frac{1}{3},-\frac{5}{4}$	26.	$x=-9,8$	27.	$x=-\frac{1}{3},7$
28.	$x=4,7$	27.	$x=-\frac{5}{3},1$	30.	$x=-\frac{1}{2},\frac{2}{3}$

Checkpoint 5B

Problem 5-136

Writing Equations for Arithmetic and Geometric Sequences

Answers to problem 5-136: a. E $t(n) = -2 + 3n$, R $t(0) = -2$, $t(n+1) = t(n) + 3$;
b. E $t(n) = 6(\frac{1}{2})^n$, R $t(0) = 6$, $t(n+1) = \frac{1}{2}t(n)$; c. $t(n) = 24 - 7n$,
d. $t(n) = 5(1.2)^n$; e. $t(4) = 1620$

An ordered list of numbers such as: 4, 9, 16, 25, 36, … creates a sequence. The numbers in the sequence are called terms. One way to identify and label terms is to use function notation. For example, if $t(n)$ is the name of the sequence above, the first term is 4 and the third term is 16. This is written $t(1) = 4$ and $t(3) = 16$. Some books use subscripts instead of function notation. In this case $t_1 = 4$ and $t_3 = 16$.

The initial value is *not* part of the sequence. It is only a reference point and is useful for writing a rule for the sequence. For the sequence above, the initial value, $t(0)$ or t_0 , is the value that would come before 4, which is –1. When writing a sequence, start by writing the first term after the initial value, $t(1)$ or t_1 . When writing the rule, use the initial value, $t(0)$ or t_0 .

Arithmetic sequences have a common difference between the terms. The rule for the values in an arithmetic sequences can be found by $t(n) = a + dn$ where a = the initial value, d = the common difference and n = the number of terms after the initial value.

Geometric sequences have a common ratio between the terms. The rule for the values in a geometric sequence may be found by $t(n) = ar^n$ where a = the initial value, r = the common ratio and n = the number of terms after the initial value.

Example 1: Find a rule for the sequence: –2, 4, 10, 16, …

Solution: There is a common difference between the terms $(d = 6)$ so it is an arithmetic sequence. Work backward to find the initial value: $a = -2 - 6 = -8$.
Now use the general rule: $t(n) = a + dn = -8 + 6n$.

Example 2: Find a rule for the sequence: 81, 27, 9, 3, …

Solution: There is a common ratio between the terms $(r = \frac{1}{3})$ so it is a geometric sequence.
Work backward to find the initial value: $a = 81 \div \frac{1}{3} = 243$.
Now use the general rule: $t(n) = ar^n = 243(\frac{1}{3})^n$.

A rule such as $t(n) = 5 - 7n$ is called an explicit rule because any term can be found by substituting the appropriate value for n into the rule. To find the 10^{th} term after the initial value, $t(10)$, substitute 10 for n. $t(10) = 5 - 7(10) = -65$.

A second way to find the terms in a sequence is by using a recursive formula. A recursive formula tells first term or the initial value and then how to get from one term to the next.

Example 3: Write the first five terms of the sequence determined by $b_1 = 8,\ b_{n+1} = b_n \cdot \frac{1}{2}$ (using subscript notation).

Solution: $b_1 = 8$ tells you the first term and $b_{n+1} = b_n \cdot \frac{1}{2}$ tells you to multiply by $\frac{1}{2}$ to get from one term to the next.

$b_1 = 8$ $\qquad b_2 = b_1 \cdot \frac{1}{2} = 8 \cdot \frac{1}{2} = 4$ $\qquad b_3 = b_2 \cdot \frac{1}{2} = 4 \cdot \frac{1}{2} = 2$

$b_4 = b_3 \cdot \frac{1}{2} = 2 \cdot \frac{1}{2} = 1$ $\qquad b_5 = b_4 \cdot \frac{1}{2} = 1 \cdot \frac{1}{2} = \frac{1}{2}$

The sequence is: $8, 4, 2, 1, \frac{1}{2}, \ldots$

Now we can go back and solve the original problems.

a. It is an arithmetic sequence $(d = 3)$. Working backward the initial value is $1 - 3 = -2$. Using the general formula the explicit rule: $t(n) = a + dn = -2 + 3d$.

   A possible recursive rule is $t(0) = -2,\ t(n+1) = t(n) + 3$.

b. It is a geometric sequence $(r = \frac{1}{2})$. Working backward the initial value is $3 \div \frac{1}{2} = 6$. Using the general formula for the explicit rule: $t(n) = ar^n = 6(\frac{1}{2})^n$.

   A possible recursive rule is $t(0) = 6,\ t(n+1) = \frac{1}{2}t(n)$.

c. $t(2)$ is halfway between $t(1)$ and $t(3)$ so $t(2) = 10$. This means $d = -7$ and the initial value is 24. Using the general formula the explicit rule: $t(n) = a + dn = 24 - 7d$.

d. The common ratio $r = \frac{8.64}{7.2} = 1.2$ so $t(1) = \frac{7.2}{1.2} = 6,\ t(0) = \frac{6}{1.2} = 5$. Using an initial value of 5 and a common ratio of 1.2 in the general formula for the explicit rule: $t(n) = ar^n = 5(1.2)^n$.

e. The common difference is the difference in the values divided by the number of terms. $d = \frac{t(12)-t(7)}{12-7} = \frac{116-1056}{5} = -188$. Working backward three terms: $t(4) = 1056 - 3(-188) = 1620$.

Here are some more to try.

Write the first 6 terms of each sequence.

1. $t(n) = 5n + 2$
2. $t(n) = 6(-\frac{1}{2})^n$
3. $t(n) = -15 + \frac{1}{2}n$
4. $t_n = -3 \cdot 3^{n-1}$
5. $t(1) = 3,\ t(n+1) = t(n) - 5$
6. $t_1 = \frac{1}{3},\ t_{n+1} = \frac{1}{3}t_n$

For each sequence, write an explicit and recursive rule.

7. 10, 50, 250, 1250, …
8. 4, 8, 12, 16, …
9. –2, 5, 12, 19, …
10. 16, 4, 1, $\frac{1}{4}$, ...
11. –12, 6, –3, $\frac{3}{2}$, …
12. $\frac{5}{6}, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}$, ...

For each sequence, write an explicit rule.

13. A geometric sequence

n	$t(n)$
0	
1	15
2	45
3	
4	

14. An arithmetic sequence

n	$t(n)$
0	27
1	15
2	
3	
4	

15. An arithmetic sequence

n	$t(n)$
1	
2	$3\frac{1}{3}$
3	
4	
5	$4\frac{1}{3}$

16. A geometric sequence

n	$t(n)$
1	
2	
3	–24
4	48
5	

Solve each problem.

17. An arithmetic sequence has $t(3) = 52$ and $t(10) = 108$. Find a rule for $t(n)$ and find $t(100)$.
18. An arithmetic sequence has $t(1) = -17$, $t(2) = -14$ and $t(n) = 145$. What is the value of n?
19. An arithmetic sequence has $t(61) = 810$ and $t(94) = 1239$. Find a rule for $t(n)$.
20. A geometric sequence has $t(4) = 12$ and $t(7) = 324$. Find the common ratio and a rule for $t(n)$.

Answers:

1. $7, 12, 17, 22, 27, 32$
2. $-3, \frac{3}{2}, -\frac{3}{4}, \frac{3}{8}, -\frac{3}{16}, \frac{3}{32}$
3. $-14\frac{1}{2}, -14, -13\frac{1}{2}, -13, -12\frac{1}{2}, -12$
4. $-3, -9, -27, -81, -243, -729$
5. $3, -2, -7, -12, -17, -22$
6. $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}, \frac{1}{729}$

Rules for problems 7 through 20 may vary.

7. $t(n) = 2 \cdot 5^n$; $t(0) = 2$, $t(n+1) = 5t(n)$
8. $t(n) = 4n$; $t(0) = 0$, $t(n+1) = t(n) + 4$
9. $t(n) = -9 + 7n$; $t(0) = -9$, $t(n+1) = t(n) + 7$
10. $t(n) = 64(\frac{1}{4})^n$; $t(0) = 64$, $t(n+1) = \frac{1}{4}t(n)$
11. $t(n) = 24(-\frac{1}{2})^n$; $t(0) = 24$, $t(n+1) = -\frac{1}{2}t(n)$
12. $t(n) = 1 - \frac{1}{6}n$; $t(0) = 1$, $t(n+1) = t(n) - \frac{1}{6}$
13. $t(n) = 5 \cdot 3^n$
14. $t(n) = 27 - 12n$
15. $t(n) = 2\frac{2}{3} + \frac{1}{3}n$
16. $t(n) = 3(-2)^n$
17. $t(n) = 28 + 8n$; $t(100) = 828$
18. $n = 55$
19. $t(n) = 17 + 13n$
20. $r = 3$; $t(n) = \frac{4}{27}(3)^n$

Checkpoint 6

Problem 6-50

Solving Proportional Equations and Similar Figures

Answers to problem 6-50: a. $y=\frac{13}{4}$, b. $y=-2$, c. $4\frac{2}{3}$ inches, d. $x=4$

A proportion is an equation stating that two ratios are equal. To solve a proportion, begin by eliminating the fractions. Multiply both sides of the proportion by one or both of the denominators. Then solve the resulting equation in the usual way. Setting up the ratios using the unit words consistently helps to proportional situations. To solve problems with similar triangles the corresponding sides must be consistently used in the ratios.

Example 1: Solve: $\frac{x}{x+1}=\frac{3}{5}$

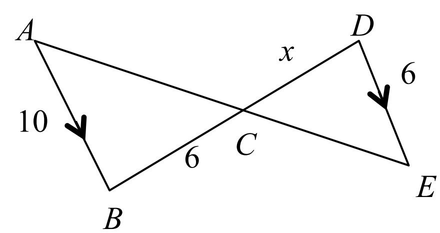
Solution: Multiply by 5 and $(x+1)$ on both sides of the equation.

$5(x+1)\frac{x}{x+1}=\frac{3}{5}(5)(x+1)$

since $\frac{(x+1)}{(x+1)}=1$ and $\frac{5}{5}=1$, we have $5x=3(x+1)$

$5x=3x+3 \Rightarrow 2x=3 \Rightarrow x=\frac{3}{2}$

Example 2: Solve for x in the diagram at right.



Solution: With two sets of congruent angles (alternating interior and vertical), $\Delta ABC \sim \Delta EDC$ and the ratio of corresponding sides is proportional.
$\frac{6}{10}=\frac{x}{6} \Rightarrow 10x=36 \Rightarrow x=3.6$

Now we can go back and solve the original problems.

a. $\frac{7-y}{5}=\frac{3}{4}$

$20(\frac{7-y}{5})=20(\frac{3}{4})$

$4(7-y)=15$

$28-4y=15$

$-4y=-13 \Rightarrow y=\frac{13}{4}$

b. $\frac{3}{y}=\frac{6}{y-2}$

$y(y-2)\frac{3}{y}=y(y-2)\frac{6}{y-2}$

$(y-2)3=(y)6$

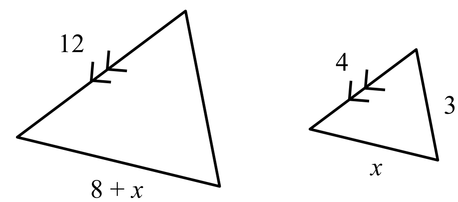
$3y-6=6y$

$-6=3y \Rightarrow y=-2$

c. $\frac{\text{inches}}{\text{months}} : \frac{1\frac{3}{4}}{4\frac{1}{2}} = \frac{x}{12}$

$$4\tfrac{1}{2}x = 1\tfrac{3}{4}(12)$$
$$\tfrac{9}{2}x = 21$$
$$x = \tfrac{2}{9}(21) = \tfrac{42}{9} = 4\tfrac{2}{3} \text{ in}$$

d. Since the triangles are similar, the lengths of corresponding sides are proportional. Separating the figure into two triangles makes it easier to write the ratios.



$$\tfrac{12}{8+x} = \tfrac{4}{x} \Rightarrow 12x = 4(8+x)$$
$$12x = 32 + 4x \Rightarrow 8x = 32 \Rightarrow x = 4$$

Here are some more to try. Solve for the variable.

1. $\frac{2y-1}{15} = \frac{y}{10}$
2. $\frac{5}{8} = \frac{x}{100}$
3. $\frac{8-x}{x} = \frac{3}{2}$
4. $\frac{4x}{5} = \frac{x-2}{7}$
5. $\frac{9-x}{6} = \frac{24}{2}$
6. $\frac{1}{t} = \frac{5}{t+1}$
7. $\frac{4}{m} = \frac{m}{9}$
8. $\frac{3x}{4} = \frac{x+1}{6}$

Use proportions to solve each problem.

9. A rectangle has length 10 feet and width six feet. It is enlarged to a similar rectangle with length 18 feet. What is the new width?

10. If 300 vitamins cost $5.75, what should 500 vitamins cost?

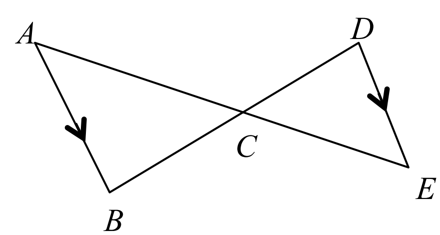
11. The tax on a $400 statue is $34. What should be the tax on a $700 statue?

12. If a basketball player made 72 of 85 free throws, how many free throws could she expect to make in 200 attempts?

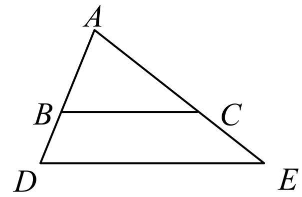
13. A cookie recipe uses $\frac{1}{2}$ teaspoon of vanilla with $\frac{3}{4}$ cups of flour. How much vanilla should be used with five cups of flour?

14. The length of a rectangle is four centimeters more than the width. If the ratio of length to width is seven to five, find the dimensions of the rectangle.

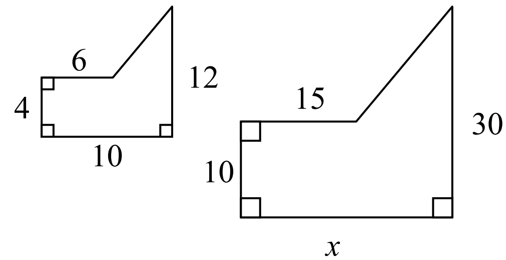
15. Use the similar triangles at right. If, $AB = 14$, $AC = 16$, and $DE = 12$, find CE.



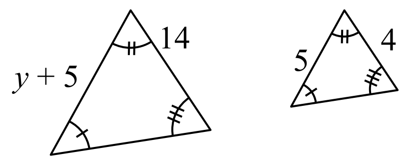
16. In the diagram at right, ΔABC and ΔADE are similar. If $AB = 6$, $BD = 4$, and $BC = 7$, then what is DE? Start by drawing two separate triangles and labeling the dimensions.



17. The two shapes at right are similar. Find the value of x. Show all work.



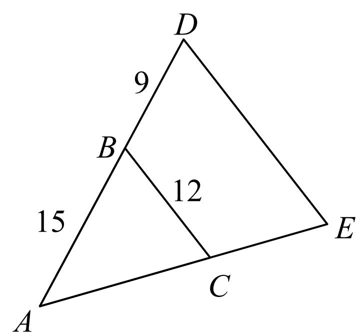
18. Examine the triangles at right. Solve for y.



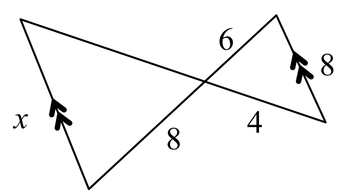
19. In the diagram at right, $\Delta ABC \sim \Delta ADE$.

a. Draw each triangle separately on your paper. Be sure to include all measurements in your diagrams.

b. Find the length of $\overline{DE}$.



20. Use your knowledge of similar triangles to solve for x.



Answers:

1. $y = 2$	2. $x = 62.5$	3. $x = 3\frac{1}{5}$	4. $x = -\frac{10}{23}$
5. $x = -63$	6. $t = \frac{1}{4}$	7. $m = \pm 6$	8. $x = \frac{2}{7}$
9. 10.8 ft	10. \$9.58	11. \$59.50	12. about 169
13. $3\frac{1}{3}$ tsp	14. 10 cm x 14 cm	15. ≈ 13.7 units	16. $11\frac{2}{3}$ units
17. 25 units	18. $12\frac{1}{2}$ units	19. 19.2 units	20. $10\frac{2}{3}$ units

Checkpoint 7

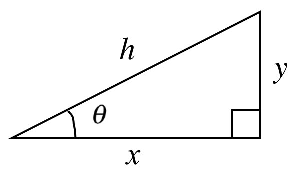
Problem 7-136

Solving with Trigonometric Ratios and the Pythagorean Theorem

Answers to problem 7-136: a. 23.83 ft, b. $x \approx 7$ yds, c. $x \approx 66.42^\circ$, d. ≈ 334.57 ft

Three trigonometric ratios and the Pythagorean Theorem can used to solve for the missing side lengths and angle measurements in any right triangle.

In the triangle below, when the sides are described relative to the angle θ, the opposite leg is y and the adjacent leg is x. The hypotenuse is h regardless of which acute angle is used.



$$\tan\theta = \frac{\text{opposite leg}}{\text{adjacent leg}} = \frac{y}{x}$$

$$\sin\theta = \frac{\text{opposite leg}}{\text{hypotenuse}} = \frac{y}{h}$$

$$\cos\theta = \frac{\text{adjacent leg}}{\text{hypotenuse}} = \frac{x}{h}$$

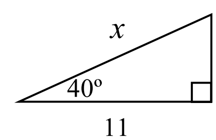
Also for the triangle above, from the Pythagorean Theorem: $h^2 = x^2 + y^2$.

Example 1: A rectangle has a diagonal of 16 cm and one side of 11 cm. What is the length of the other side?

Solution: The diagonal represents the hypotenuse of a right triangle and the one given side represents one of the legs. Using the Pythagorean Theorem:

$$h^2 = x^2 + y^2 \Rightarrow 16^2 = 11^2 + y^2 \Rightarrow 256 = 121 + y^2$$
$$135 = y^2 \Rightarrow y = \sqrt{135} \approx 11.62 \text{ cm}$$

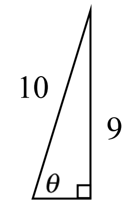
Example 2: Solve for x in the triangle at right.



Solution: Based on the 40º angle, 11 is the adjacent side and x is the hypotenuse. Use the cosine ratio to solve.

$$\cos 40^\circ = \frac{11}{x}$$
$$x\cos 40^\circ = 11 \Rightarrow x = \frac{11}{\cos 40^\circ} \approx 14.36 \text{ units}$$

Example 3: A ten-foot ladder is leaning against the side of a house. If the top of the ladder touches the house nine feet above the ground, what is the angle made by the ladder and the ground?

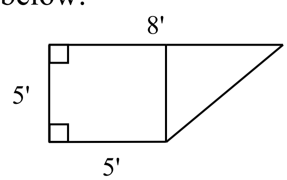


Solution: Make a diagram of the situation similar to the one at right. The ladder (10) is the hypotenuse and the house (9) is the opposite leg. Using the sine ratio, $\sin\theta = \frac{9}{10}$. To find θ, "undo" the sine function with the inverse sine function ($\sin^{-1} x$) as follows:

$$\sin\theta = \tfrac{9}{10}$$
$$\sin^{-1}(\sin\theta) = \sin^{-1}\left(\tfrac{9}{10}\right)$$
$$\theta = \sin^{-1}\left(\tfrac{9}{10}\right)$$
$$\theta \approx 64.2^\circ$$

Now we can go back and solve the original problems.

a. Separate the trapezoid into a rectangle and a triangle as shown below:



The legs of the triangle are 5 and 3. The hypotenuse or unlabeled side of the trapezoid is found by $h^2 = 5^2 + 3^2 = 34$.
So $h = \sqrt{34} \approx 5.83$.
The perimeter is:
$5 + 5 + 8 + 5.83 = 23.83$ feet

b. $\tan 35^\circ = \frac{x}{10}$

$x = 10 \tan 35^\circ \approx 7.0$ yds

c. $\cos x = \frac{60}{150}$

$\cos^{-1}(\cos x) = \cos^{-1}(\frac{60}{150})$

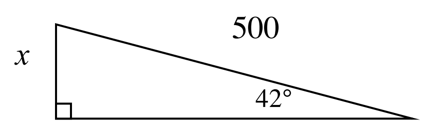
$x = \cos^{-1}(\frac{60}{150})$

$x \approx 66.4^\circ$

d. Using the possible diagram below:

$\sin 42^\circ = \frac{x}{500}$

$x = 500 \sin 42^\circ \approx 335$ ft



Here are some more to try. Use the right triangle trigonometric ratios or the Pythagorean Theorem to solve for the variable(s).

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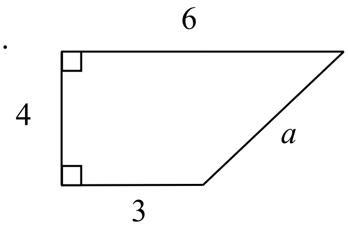

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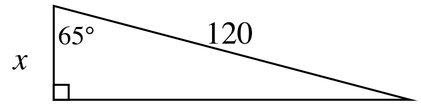

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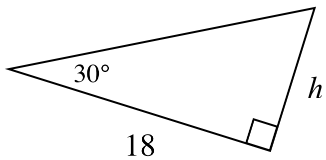

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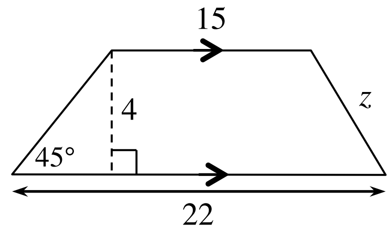

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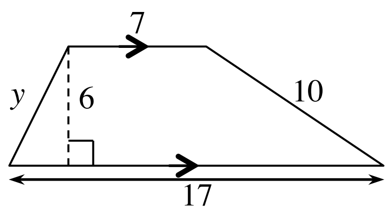

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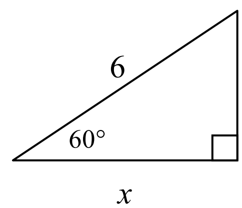

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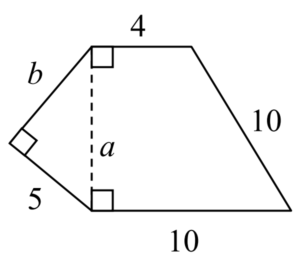

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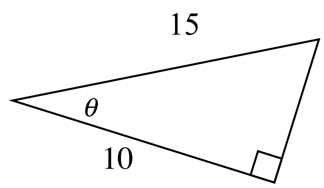

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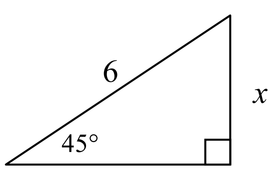

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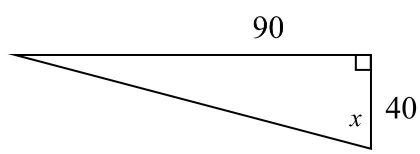

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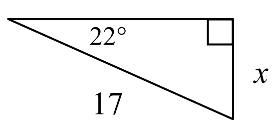

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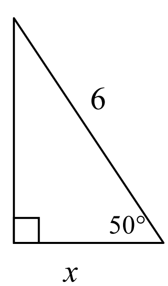

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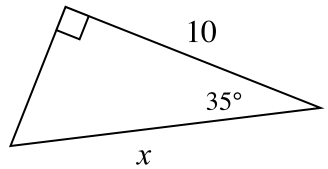

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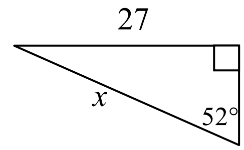

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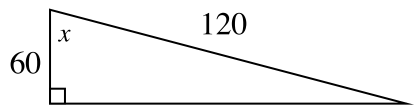

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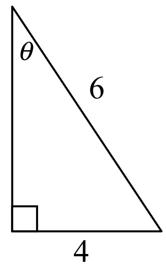


Draw a diagram and solve each of the following problems.

17. The base of a 12-foot ladder is six feet from the wall. How high on the wall does the ladder touch?

18. A garden gate has a six-foot by four-foot rectangular frame that is strengthened by a diagonal brace. How long is the brace?

19. What is the distance between (–6, –6) and (–3, 2)?

20. What is the length of the hypotenuse of the right triangle with coordinates: (–2, –1), (–6, 5), and (4, 3)?

21. From the takeoff point, the launch crew of a hot air balloon can see the balloon in the sky at an elevation of 13°. The pilot tells the crew that the passenger basket is now 1200 feet above the ground. What is the current ground distance from the crew to the balloon?

22. Federal standards require the angle ramp for wheel chairs to be less than 5°. If the length of a ramp is 20 feet and the vertical rise is 15 inches, does it meet federal standards?

23. If an eight-foot stop sign casts a 10-foot shadow, what is the angle of elevation to the top of the sign?

24. Mayfield High School's flagpole is 15 feet high. Using a clinometer, Tamara measures an angle of 11.3° to the top of the pole. Tamara is 62 inches tall. How far from the flagpole is Tamara standing?

Answers:

1. 5 units	2. ≈ 50.7 units	3. ≈ 10.4 units
4. 5 units	5. $\sqrt{40} \approx 6.32$ units	6. 3 units
7. $a = 8$, $b = \sqrt{39} \approx 6.24$ units	8. $\approx 48.2^{\circ}$	9. $3\sqrt{2} \approx 4.24$ units
10. $\approx 66.0^{\circ}$	11. ≈ 6.37 units	12. ≈ 3.86 units
13. ≈ 12.2 units	14. ≈ 34.3 units	15. 60°
16. $\approx 41.8^{\circ}$	17. $\sqrt{108} \approx 10.4$ ft	18. $\sqrt{52} \approx 7.2$ ft
19. $\sqrt{73} \approx 8.5$ units	20. $\sqrt{104} \approx 10.2$ units	21. ≈ 5198 ft
22. 3.6°, yes	23. $\approx 38.7^{\circ}$	24. ≈ 590.5 in or 49.2 ft

Checkpoint 8

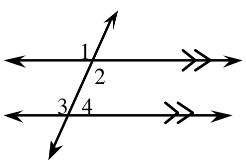
Problem 8-124

Angle Relationships in Geometric Figures

Answers to problem 8-124: a. supplementary angles sum to 180º; $x = 26°$
b. alternate exterior angles are congruent; $x = 5°$
c. Triangle Angle Sum Theorem; $x = 15°$
d. exterior angle equals sum of remote interior angles; $x = 35°$

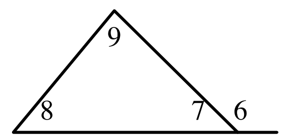
Illustrated below are several common relationships between angles in triangles and lines.

Parallel lines



- corresponding angles are equal: $m\angle 1 = m\angle 3$
- alternate interior angles are equal: $m\angle 2 = m\angle 3$
- $m\angle 2 + m\angle 4 = 180°$

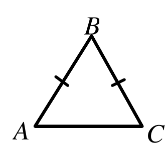
Triangles

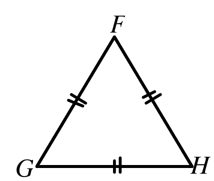


- $m\angle 7 + m\angle 8 + m\angle 9 = 180°$
- $m\angle 6 = m\angle 8 + m\angle 9$
  (exterior angle = sum remote interior angles)

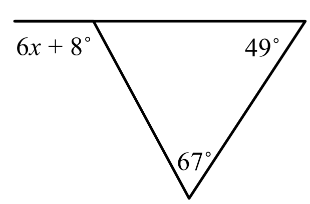
Also shown in the above figures:
- vertical angles are equal: $m\angle 1 = m\angle 2$
- linear pairs are supplementary: $m\angle 3 + m\angle 4 = 180°$
  and $m\angle 6 + m\angle 7 = 180°$

In addition, an isosceles triangle, ΔABC, has $\overline{BA} = \overline{BC}$ and $m\angle A = m\angle C$. An equilateral triangle, ΔGFH, has $\overline{GF} = \overline{FH} = \overline{HG}$ and $m\angle G = m\angle F = m\angle H = 60°$.





Example 1: Solve for x.

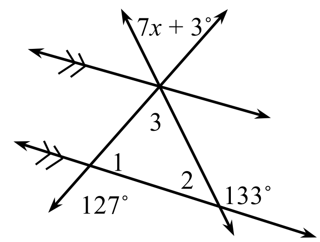


Solution: Use the Exterior Angle Theorem:

$$6x + 8° = 49° + 67°$$

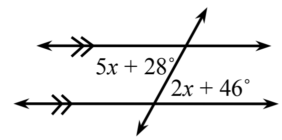
$$6x° = 108° \Rightarrow x = \frac{108°}{6} \Rightarrow x = 18°$$

Example 2: Solve for x.



Solution: There are a number of relationships in this diagram. First, $\angle 1$ and the 127º angle are supplementary, so we know that $m\angle 1 + 127° = 180°$ so $m\angle 1 = 53°$. Using the same idea, $m\angle 2 = 47°$. Next, $m\angle 3 + 53° + 47° = 180°$, so $m\angle 3 = 80°$. Because angle 3 forms a vertical pair with the angle marked $7x + 3°$, $80° = 7x + 3°$, so $x = 11°$.

Example 3: Find the measure of the acute alternate interior angles.



Solution: Parallel lines mean that alternate interior angles are equal, so $5x + 28° = 2x + 46° \Rightarrow 3x = 18° \Rightarrow x = 6°$. Use either algebraic angle measure: 2(6º) + 46º = 58º for the measure of the acute angle.

Now we can go back and solve the original problems.

a. Supplementary angles sum to 180º

$$4x - 3° + 3x + 1° = 180°$$
$$7x - 2° = 180°$$
$$7x = 182°$$
$$x = 26°$$

b. Alt. exterior angles are congruent

$$5x + 6° = 2x + 21°$$
$$3x = 15°$$
$$x = 5°$$

c. Triangle Angle Sum Theorem

$$4x + 28° + x + 19° + 3x + 13° = 180°$$
$$8x + 60° = 180°$$
$$8x = 120°$$
$$x = 15°$$

d. Ext. angle = sum of remote int. angles

$$4x - 10° = 40° + 90°$$
$$4x = 140°$$
$$x = 35°$$

Here are some more to try. For each diagram, name the relationship used and solve for the variable.

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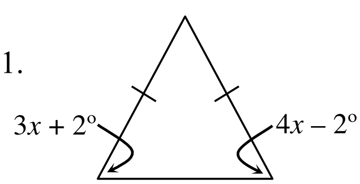

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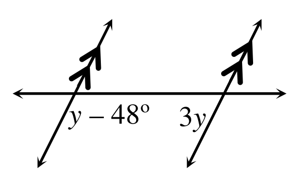

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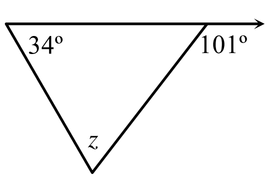

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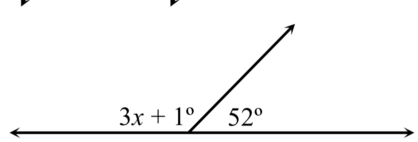

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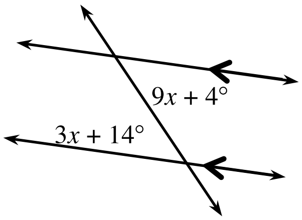

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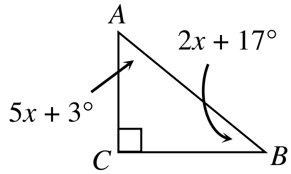

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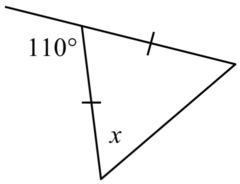

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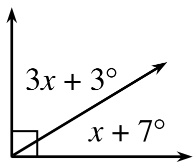

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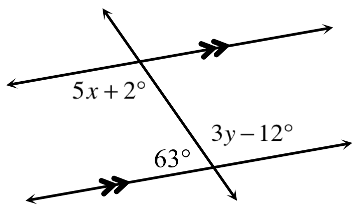

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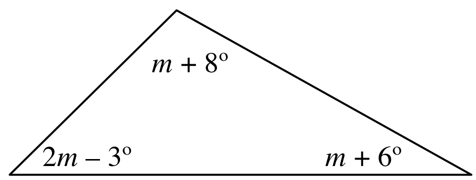

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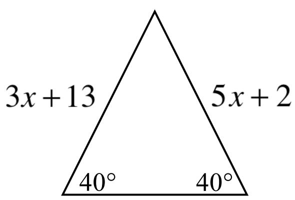

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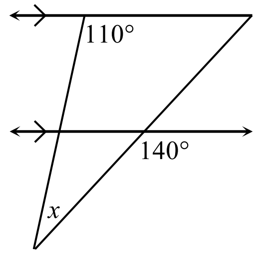
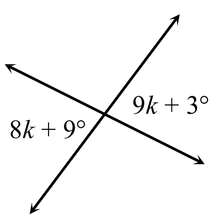
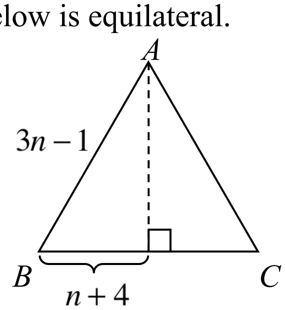


13.



14. ΔABC below is equilateral.



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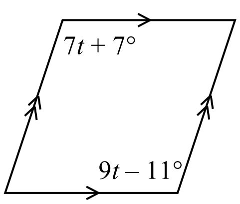

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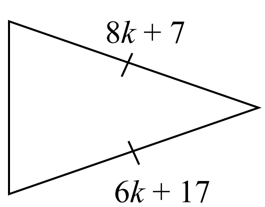

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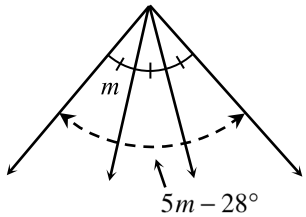

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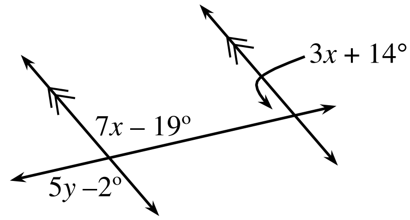

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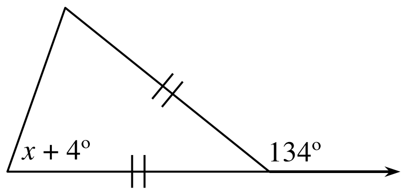

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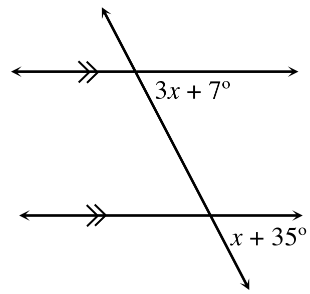


Answers: (explanations may vary)

1. congruent base angles in isosceles triangle; $x = 4^\circ$
2. supplementary same-side interior angles; $y = 57^\circ$
3. exterior angle sum; $z = 67^\circ$
4. supplementary angles; $x = 42\frac{1}{3}^\circ$
5. congruent alternate interior angles; $x = 1\frac{2}{3}^\circ$
6. triangle angle sum; $x = 10^\circ$
7. isosceles triangle and exterior angle sum; $x = 55^\circ$
8. complementary angles; $x = 20^\circ$
9. supplementary angles and congruent alternating interior angles; $y = 43^\circ, x = 23^\circ$
10. triangle angle sum; $m = 42\frac{1}{4}^\circ$
11. isosceles triangle; $x = 5\frac{1}{2}$
12. congruent corresponding angles and exterior angle sum; $x = 30^\circ$
13. congruent vertical angles; $k = 6^\circ$
14. congruent sides on equilateral triangle; $n = 9$
15. two pairs of same-side interior angles; $t = 9^\circ$
16. isosceles triangle; $k = 5$
17. congruent angles and angle sum; $m = 14^\circ$
18. congruent vertical angles, supplementary same-side interior; $x = 18.5^\circ, y = 22.5^\circ$
19. isosceles triangle and exterior angle sum; $x = 63^\circ$
20. congruent corresponding angles; $x = 14^\circ$

Checkpoint 9A

Problem 9-73

Probabilities with Unions, Intersections, and Complements

Answers to problem 9-73:

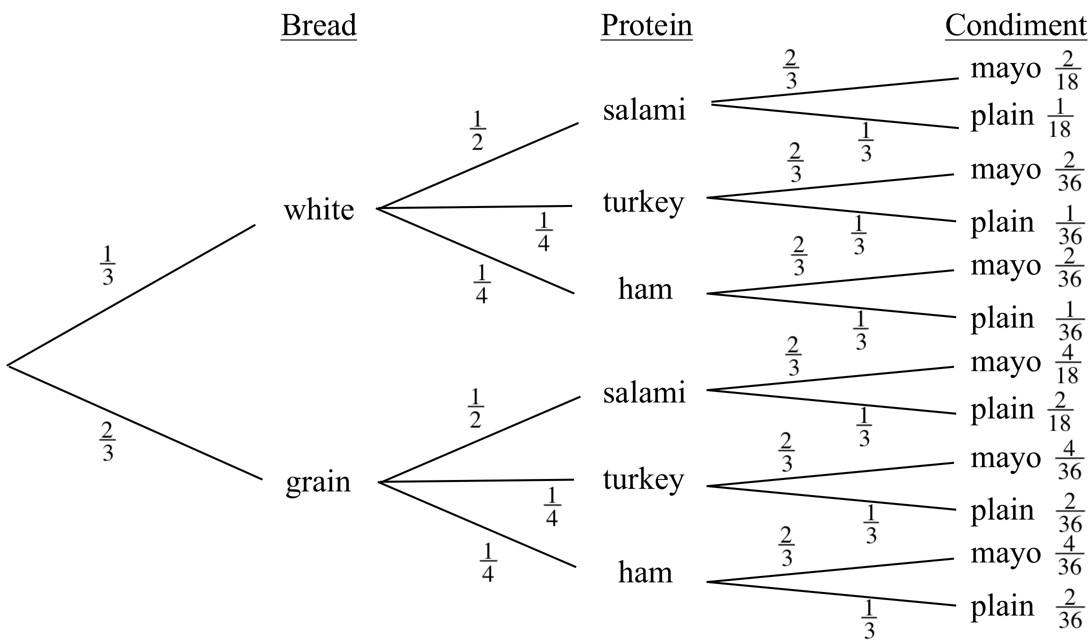
a. See tree diagram below.

b. {WSM, WSP, WTM, WHM, GSM, GSP, GTM, GHM},
$\frac{2}{18}+\frac{1}{18}+\frac{2}{36}+\frac{2}{36}+\frac{4}{18}+\frac{2}{18}+\frac{4}{36}+\frac{4}{36}=\frac{30}{36}\approx 83.3\%$

c. {WTP, WHP, GTP, GHP},
$\frac{2}{36}+\frac{2}{36}+\frac{1}{36}+\frac{1}{36}=\frac{6}{36}\approx 16.7\%$

d. 100% – 83.3% = 16.7%

e. {WSM, GSM}



See the Lesson 1.2.1 Math Notes box for definitions of probability terms. There are several models for listing all of the outcomes of a probabilistic situation and showing their probabilities, such as, systematic lists, area models, and tree diagrams. See the Lesson 4.2.3 Math Notes box for an explanation of some of these probability models. An area model is best used when the situation involves exactly two events and the events are independent. See the Lesson 4.1.5 Math Notes box for more information about independent events.

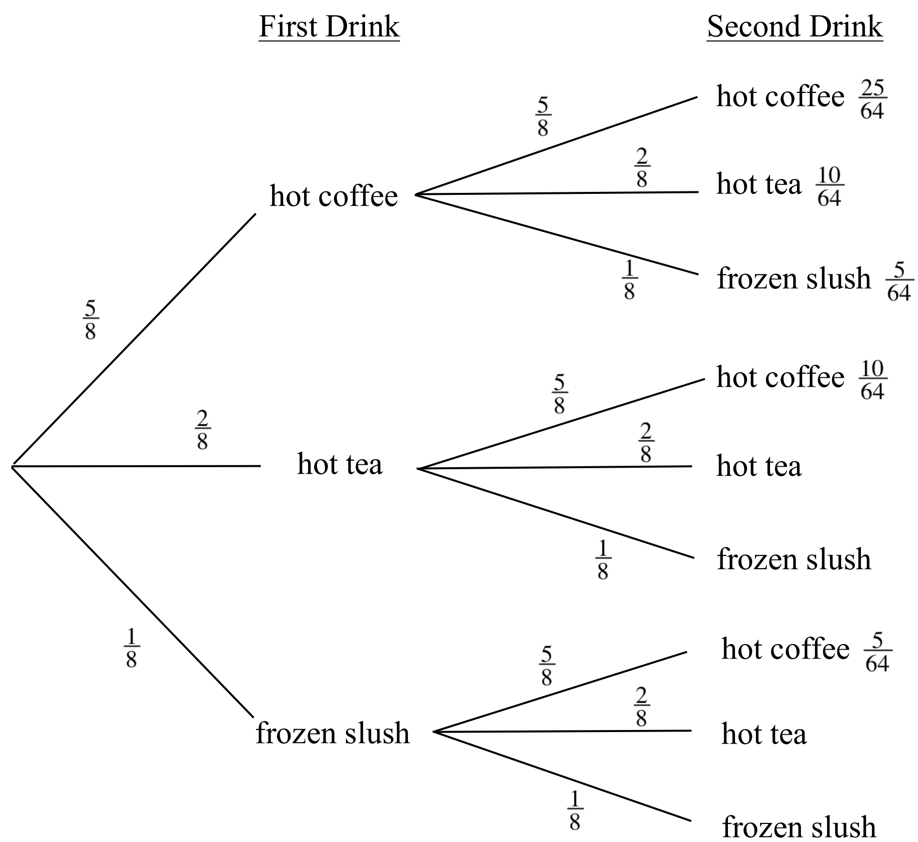
Outcomes from probabilistic situations are called events. Several events can be combined using unions or intersections. See the Lesson 4.2.4 Math Notes box for more information about unions and intersections.

Example 1: Howard can never remember what kind of drink he is supposed to get for his wife from the coffee cart on the street. The coffee cart has 5 different hot coffee drinks, 2 different hot tea drinks, and a frozen coffee slush. Harold decides to randomly choose two drinks. When he sees his wife, he learns that she wanted a hot coffee drink. What is the probability (as a percent) that Harold chose at least one hot coffee drink? Make an area model or a tree diagram to justify your solution.

Solution: An area model can be used since there are exactly two events that are independent. The two events are picking the first drink, and picking the second drink. We are only concerned with the events that include a hot coffee drink.

		First Drink		
		hot coffee $\frac{5}{8}$	hot tea $\frac{2}{8}$	frozen coffee slush $\frac{1}{8}$
Second Drink	hot coffee $\frac{5}{8}$	$\frac{25}{64}$	$\frac{10}{64}$	$\frac{5}{64}$
	hot tea $\frac{2}{8}$	$\frac{10}{64}$		
	frozen coffee slush $\frac{1}{8}$	$\frac{5}{64}$		

A tree diagram could have also been used.



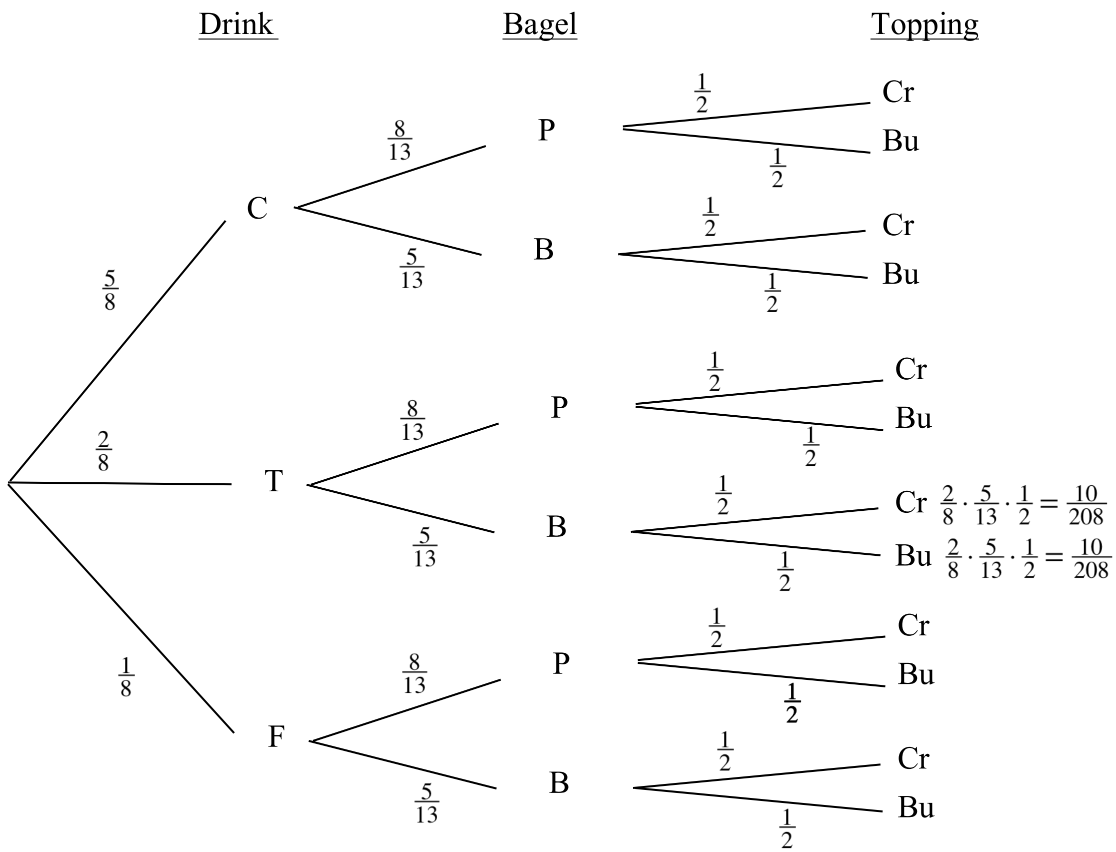
P(at least one hot coffee drink) = $\frac{25}{64}+\frac{10}{64}+\frac{5}{64}+\frac{10}{64}+\frac{5}{64}=\frac{55}{64}\approx 85.9\%$

There is an 85.9% chance that Howard chose at least one hot coffee drink.

Example 2: Denise is also at the coffee cart from Example 1. In addition to one drink, she chooses a bagel. The cart has 8 plain bagels and 5 blueberry bagels. She can top her bagel with cream cheese or butter. If she randomly chooses a drink, a bagel, and a topping, what are all the possible combinations in the sample space? Use the abbreviation C, T or F for the drink, and P or B for the bagel, and Cr or Bu for the topping.

Denise likes hot tea and blueberry bagels. Which outcomes are in the union of the events {hot tea} and {blueberry bagel}? Which outcomes are in the event {hot tea and blueberry bagel}. Are these two events the same? Explain why or why not.

Solution: An area model cannot be used since there are three events (drink, bagel, and topping). The following tree diagram can be used to represent the sample space of all possible outcomes:



The union of {hot tea} and {blueberry bagel} is any outcome in the sample space above that contains T *or* B or both: {CBCr, CBBu, TPCr, TPBu, TBCr, TBBu, FBCr, FBBu}.

The event {hot tea and blueberry bagel} is different. {hot tea and blueberry bagel} is the *intersection* of {hot tea} and {blueberry bagel} and contains any combination with both T *and* B. Thus, {hot tea and blueberry bagel} = {TBCr, TBBu}.

Example 3: Denise is randomly handed a drink-bagel-topping combination by the coffee cart owner. Use a complement to find the probability that Denise was not handed a hot tea with a blueberry bagel.

Solution: P(hot tea and blueberry bagel) = $\frac{10}{208}+\frac{10}{208}=\frac{20}{208}\approx 9.6\%$ from the tree diagram in Example 2. The complement is the probability that Denise does not get a hot tea and blueberry bagel: P(not hot tea and blueberry bagel) = 100% – 9.6% = 90.4%.

There is a 90.4% chance that Denise was not handed a hot tea with a blueberry bagel.

Example 4: Zoey's grandfather has purchased a kit of 75 assorted pieces of sea life for her. The kit contains equal numbers of gastropod seashells, bivalve seashells, sand dollars, sea fans, and starfish. Each type of sea life is divided into equal numbers of large, medium, and small objects. Zoey is creating a piece of display art with the sea life.

To start the artwork, Zoey selects a random piece of sea life. What is the probability that she picks a small seashell?

Solution: There are 5 small gastropod seashells and 5 small bivalve seashells.

$$\text{P(small seashell)} = \frac{\text{number of successful outcomes}}{\text{total number of possible outcomes}} = \frac{10}{75} = \frac{2}{15} \approx 13.3\%$$

Alternatively, an area model, like the one below, could be used.

	gastropod seashells $\frac{1}{5}$	bivalve seashells $\frac{1}{5}$	sand dollar $\frac{1}{5}$	sea fans $\frac{1}{5}$	starfish $\frac{1}{5}$
small $\frac{1}{3}$	$\frac{1}{15}$	$\frac{1}{15}$			
medium $\frac{1}{3}$					
large $\frac{1}{3}$					

$$\text{P(small seashell)} = \frac{1}{15}+\frac{1}{15}=\frac{2}{15}\approx 13.3\%$$

What is the probability that she does *not* pick a small seashell?

Solution: The probability she does *not* pick a small seashell is the complement of the probability that Zoey *does* pick a small seashell.

P(not small seashell) = 1 – P(small seashell)

$= 1-\frac{2}{15}=\frac{13}{15}\approx 86.7\%$ OR $100\%-13.3\%=86.7\%$

What is the probability she picks a large piece *or* a seashell?

Solution: Use an area model, as shown below, or a tree diagram.

	gastropod seashells $\frac{1}{5}$	bivalve seashells $\frac{1}{5}$	sand dollar $\frac{1}{5}$	sea fans $\frac{1}{5}$	starfish $\frac{1}{5}$
small $\frac{1}{3}$	$\frac{1}{15}$	$\frac{1}{15}$			
medium $\frac{1}{3}$	$\frac{1}{15}$	$\frac{1}{15}$			
large $\frac{1}{3}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$

P(large piece or seashell) = $9(\frac{1}{15}) = \frac{9}{15} = 60\%$

Alternatively, the Addition Rule could be used:
P(large piece or seashell) = P(large piece) + P(seashell) – P(large piece and seashell)
P(large piece or seashell) = $\frac{1}{3} + \frac{2}{5} - \frac{2}{15}$
$= \frac{9}{15} = 60\%$

Now we can go back and solve the original problems.

a. There are three probabilistic situations, or events, in this situation: choosing a bread (white or grain), choosing a protein (salami, turkey, or ham), and choosing a condiment (mayonnaise or plain). Area models are best used when there are two events, so an area model is not a good choice in this situation. See the answer for part (b) for the tree diagram.

b. The following sandwiches have salami or mayonnaise (or both) on them: {WSM, WSP, WTM, WHM, GSM, GSP, GTM, GHM}.

To find the probability of any of the sandwiches above, probabilities are added to each branch of the tree diagram, as shown in the part (b) answer above. For example, since 12 of the 36 sandwiches are made with white bread, the probability of randomly selecting a sandwich with white bread is $\frac{12}{36} = \frac{1}{3}$. Once white bread is chosen, the probability of choosing salami is $\frac{1}{2}$. The remaining half of the white-bread sandwiches are split evenly between turkey and ham, so the probability of choosing turkey is $\frac{1}{4}$ and the probability of choosing ham is $\frac{1}{4}$. In this manner, the remainder of the probabilities can be added to the remaining branches. *Note that the sum of the probabilities for any set of branches is always 1.* For example, for white bread, the probabilities of the branches for salami, turkey, and ham add to 1: $\frac{1}{2}+\frac{1}{4}+\frac{1}{4}=1$.

Solution continues on next page →

Solution continued from previous page.

To find the probability of any one outcome (any one sandwich), multiply the probabilities across all the branches. For example the probability of white-salami-mayonnaise, or WSM, is $\frac{1}{3} \cdot \frac{1}{2} \cdot \frac{2}{3} = \frac{2}{18}$. The probabilities for individual outcomes are indicated in the far right column of the tree diagram.

The probability that Wade randomly picks a sandwich he likes is the sum of the probability of all the outcomes that are successes for Wade:

$\frac{2}{18} + \frac{1}{18} + \frac{2}{36} + \frac{2}{36} + \frac{4}{18} + \frac{2}{18} + \frac{4}{36} + \frac{4}{36} = \frac{30}{36} \approx 83.3\%$

c. Successes for Madison are any sandwich that has neither salami nor mayonnaise. That is, {WTP, WHP, GTP, GHP}. Using the tree diagram from part (b) above, the probability of each of these individual outcomes can be found. For example, P(WTP) = $\frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{36}$.

The probability Madison picks a sandwich she likes is the sum of the probabilities of the sandwiches she considers a success:

$\frac{2}{36} + \frac{2}{36} + \frac{1}{36} + \frac{1}{36} = \frac{6}{36} \approx 16.7\%$

d. The sum of the probabilities of all the outcomes in any probabilistic situation is 1 or 100%. Madison likes any sandwich that Wade does not like. Since the probability of all the outcomes is 100%, and the probability of the outcomes that are success for Wade are about 83.3%, the probability for Madison's success must be 100% – 83.3% = 16.7%.

e. Intersections are the set of outcomes in which both the first *and* the second event must occur. The outcomes in which both {salami} and {mayonnaise} occur are {WSM, GSM}.

Unions, on the other hand, are the set of outcomes in which either the first *or* the second event must occur. Wade's favorite sandwiches in part (b) were the union of {salami} or {mayonnaise}.

Here are some more to try.

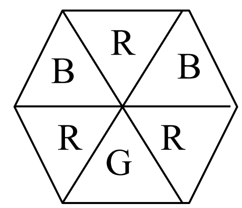
1. As you open the ice chest during your family picnic you find that there are 10 bottles of fruit juice and 10 sandwiches. Unfortunately the bottle labels came off and the sandwiches are not labeled. Your mother says that there are 5 strawberry, 3 grape, and 2 raspberry juices. There are 4 turkey, 3 roast beef, and 3 tuna sandwiches. You are not really picky, but you are hungry and you really do not want grape juice and tuna together.

   a. Draw a diagram to show the sample space for this situation.

   b. What is the probability of picking a grape juice and tuna combination?

   c. What is the probability of picking a lunch with strawberry juice or turkey?

2. A carnival game has a spinner like the one at right. The sections are all the same size.



   R = red   B = blue   G = green

   a. Draw a diagram to show the sample space for spinning twice.

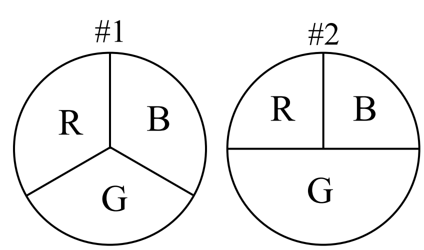
   b. What is the probability of spinning the same color twice?

3. There is a 25% chance that you will have to work tonight and cannot study for the big math test. If you study, then you have an 80% chance of earning a good grade. If you do not study, you only have a 30% chance of earning a good grade.

   a. Draw a diagram to represent this situation.

   b. Calculate the probability of earning a good grade on the math test.

4. For the spinners at right, assume that sections of spinner #1 are all the same size and the R and B sections of spinner #2 are each half the size of section G.



   R = red   B = blue   G = green

   a. Draw a diagram for the sample space for spinning both spinners.

   b. What is the probability of spinning green on both spinners?

   c. What is the probability of spinning the same color on both spinners?

   d. What is the probability of *not* spinning green on both spinners?

5. Judy's pencil box contains two red, one blue, and three yellow pencils and one yellow and two red erasers. If she randomly picks out one pencil and one eraser, find the following probabilities.

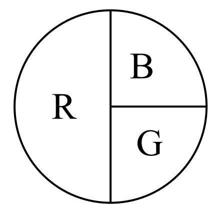
   a. Getting a yellow pencil and red eraser.

   b. Getting a yellow pencil or a red eraser.

   c. Not getting the yellow pencil and red eraser combination.

6. A baseball player gets a hit 40% of the time if the weather is good but only 20% of the time if it is cold or windy. The weather forecast is 70% chance of nice weather, 20% chance of cold weather, and 10% chance of windy. What is the probability of the player getting a hit?

7. The State Fair has the following carnival game.

   Pay $5 to spin the wheel at right. Assume that sections B and G are each half the size of section R.

   R = red B = blue G = green

   - If red comes up on the first spin, you win a stuffed animal.
   - If blue comes up on the first spin, you spin again. If blue comes up again you win $20. Otherwise you win nothing.
   - If green comes up on the first spin, you spin again. If green comes up a second time, you spin again. If green comes up a third time you win $150. Otherwise you win nothing.

   a. Draw a diagram to represent this situation.

   b. What is the probability of winning something in the game?

8. Evan gets up at 4 a.m. each day to deliver newspapers. He does not want to wake up his brother so he gets dressed in the dark and cannot see what clothes he is choosing. In his closet there are three pairs of shoes – one pair of dress shoes and two pairs of tennis shoes. There are also five pairs of pants – three pairs of jeans and two pairs of slacks. In the dresser are eight school shirts – three are blue, two are green and three are Hawaiian. If Evan randomly chooses a pair of shoes, a pair of pants, and a shirt, compute the following probabilities.

   a. Evan wears tennis shoes, jeans, and a Hawaiian shirt.

   b. Evan wears tennis shoes and jeans or tennis shoes and a Hawaiian shirt.

   c. Evan does not wear a dress shoes, slacks, blue shirt combination.

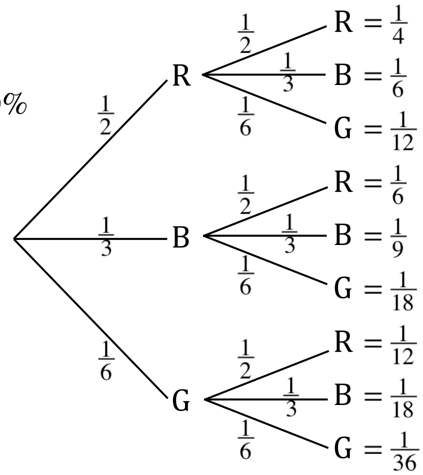
Answers:

1. a.

Drink \ Sandwich	TY(0.4)	RB(0.3)	TU(0.3)
S(0.5)	0.20	0.15	0.15
G(0.3)	0.12	0.09	0.09
R(0.2)	0.08	0.06	0.06

b. 0.09 = 9%

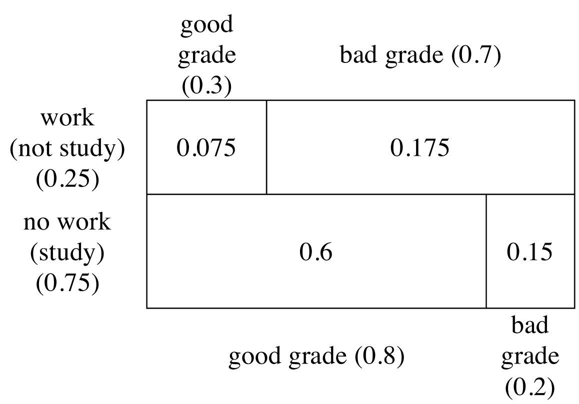
c. 0.08 + 0.12 + 0.20 + 0.15 + 0.15 = 0.70 = 70%



2. a. See tree diagram at right.

b. $\{RR\} + \{BB\} + \{GG\} = \frac{1}{4}+\frac{1}{9}+\frac{1}{36}=\frac{7}{18}$

3. a.



b. 0.075 + 0.60 = 0.675 = 67.5%

4. a.

#1 \ #2	$R\left(\frac{1}{4}\right)$	$B\left(\frac{1}{4}\right)$	$G\left(\frac{1}{2}\right)$
$R\left(\frac{1}{3}\right)$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$
$B\left(\frac{1}{3}\right)$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$
$G\left(\frac{1}{3}\right)$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$

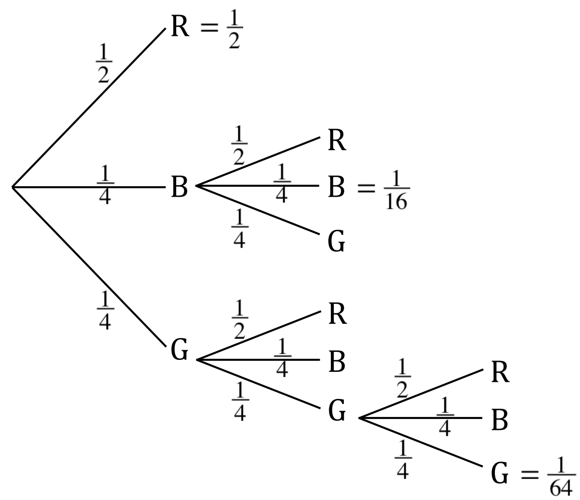
b. $\frac{1}{6}$ c. $\frac{1}{12}+\frac{1}{12}+\frac{1}{6}=\frac{1}{3}$ d. $1-\frac{1}{6}=\frac{5}{6}$

5. a. $\frac{1}{2}\cdot\frac{2}{3}=\frac{1}{3}$ b. $\frac{2}{9}+\frac{2}{18}+\frac{1}{6}+\frac{1}{3}=\frac{5}{6}$ c. $1-\frac{1}{3}=\frac{2}{3}$

6. $(0.7)(0.4) + (0.2)(0.2) + (0.2)(0.1) = 0.28 + 0.04 + 0.02 = 0.34$

7. a. See tree diagram at right.

   b. $\frac{1}{2}+\frac{1}{16}+\frac{1}{64}=\frac{37}{64}\approx 57.8\%$



8. a. $\frac{3}{20}$

   b. $\frac{2}{3}\cdot\frac{3}{5}+\frac{2}{3}\cdot\frac{3}{8}\cdot\frac{2}{5}=\frac{2}{5}+\frac{2}{20}=\frac{1}{2}$

   c. $1-\frac{1}{3}\cdot\frac{2}{5}\cdot\frac{3}{8}=\frac{19}{20}$

Checkpoint 9B

Problem 9-107

Exponential Functions

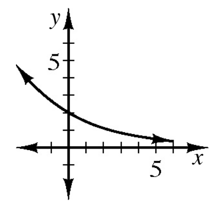
Answers to problem 9-107:

a. See graph at right.

b. $f(x) = 10(2.3)^x$

c. $y = 42,000(0.75)^5 = 9967$

d. $60 = 25(b)^{10}$, $b = 1.09$, 9% increase



An exponential function is an equation of the form $y = ab^x$ (with $b \geq 0$).

In many cases a represents a starting or initial value, b represents the multiplier or growth/decay factor, and x represents the time. If something is increasing by a percent then the multiplier b is always found by adding the percent increase (as a decimal) to the number 1. If something is decreasing by a percent then the multiplier b is always found by subtracting the percent from the number 1. From a table, the multiplier can be calculated using the ratio of the y-values corresponding to one integer x-value divided by the preceding x-value.

Example 1: Graph $y = 3 \cdot 2^x$.

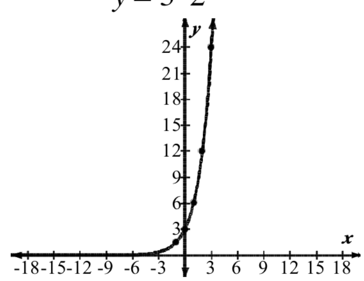
Solution:
First, make a table of values.

x	–1	0	1	2	3
y	1.5	3	6	12	24

This is called an increasing exponential curve.

Then, plot the points and connect them to form a smooth curve.

$y = 3 \cdot 2^x$



Example 2: The ticket prices at African Safari Land have increased annually according to the table at right. Write an equation that represents the cost over various years.

Year	Price
0	\$50
1	\$55
2	\$60.50
3	\$66.55

Solution: The initial value is $a = 50$. The multiplier is the ratio of one y-value divided by the previous one: $b = \frac{55}{50} = \frac{60.50}{55} = \frac{66.55}{60.50} = 1.1$.
The prices are increasing by 10% each year.
The equation is $y = 50(1.1)^x$.

Example 3: A house that was worth \$200,000 in 2005 was only worth \$150,000 in 2010. Write an equation to represent the value since 2005 and tell the percent of decrease.

Solution: The equation to use is $y = ab^x$. The given initial value is a = 200,000. The other given value is y = 150,000 when x = 5. Substituting these into the equation and solving for b we get:

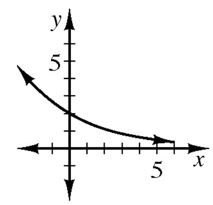
$$150,000 = 200,000b^5$$
$$0.75 = b^5$$
$$b = \sqrt[5]{0.75} \approx 0.944$$

The equation is $y = 200000(0.944)^x$. $1 - 0.944 = 0.056 = 5.6\%$ decrease.

Now we can go back and solve the original problems.

a. Make a table of values. Plot the points and connect them to form a smooth curve. See table below and graph at right.

x	–1	0	1	2	3
y	2.7	2	1.5	1.1	0.8



b. The multiplier $b = \frac{52.9}{23} = 2.3$. The starting value for $x = 0$ can be determined by working backwards from the value of $x = 1$. $a = \frac{23}{2.3} = 10$. The equation represented by the table is: $f(x) = 10(2.3)^x$.

c. $y = ab^x$ $\quad a = 42,000,\ b = 1 - 0.25 = 0.75,\ x = 5$

$y = 42,000(0.75)^x \Rightarrow 42,000(0.75)^5 \approx 9967$

d. $y = ab^x$ $\quad a = 25,\ y = 60,\ x = 10$

$60 = 25b^{10} \Rightarrow \frac{60}{25} = b^{10} \Rightarrow b = \sqrt[10]{\frac{60}{25}} \approx 1.09$ That is a 9% increase.

$y = 25(1.09)^x$

Here are some more to try. Make a table of values and draw a graph of each exponential function.

1. $y = 4(0.5)^x$
2. $y = 2(3)^x$
3. $f(x) = 5(1.2)^x$
4. $f(x) = 10\left(\frac{2}{3}\right)^x$

Write an equation to represent the information in each table.

5.

x	$f(x)$
0	1600
1	2000
2	2500
3	3125

6.

x	y
1	40
2	32
3	25.6

7.

x	$f(x)$
0	1.8
1	5.76
2	18.432

8.

x	y
0	5
1	35
2	245

Write a possible context based on each equation.

9. $y = 32,500(0.85)^x$

10. $f(x) = 2.75(1.025)^x$

Write and use an exponential equation to solve each problem.

11. A powerful computer is purchased for $1500, but on the average loses 20% of its value each year. How much will it be worth 4 years from now?

12. If a gallon of milk costs $3 now and the price is increasing 10% per year, how long before milk costs $10 a gallon? (Note that guess and check will be required to solve the equation after it is written.)

13. Dinner at your grandfather's favorite restaurant now costs $25.25 and has been increasing steadily at 4% per year. How much did it cost 35 *years ago* when he was courting your grandmother?

14. The number of bacteria present in a colony is 280 at 12 noon and the bacteria grows at a rate of 18% per hour. How many will be present at 10 p.m.?

15. A house purchased for $226,000 has lost 4% of its value each year for the past five years. What is it worth now?

16. A 1970 comic book has appreciated 10% per year and originally sold for $0.35. What will it be worth in 2020?

17. A compact car depreciates at 15% per year. Six year ago it was purchased for $21,000. What is it worth now?

18. Inflation is at a rate of 7% per year. Today Janelle's favorite bread costs $4.79. What would it have cost ten years ago?

19. Ryan's motorcycle is now worth \$2500. It has decreased in value 12% each year since it was purchased. If he bought it four years ago, what did it cost new?

20. The cost of a High Definition television now averages \$900, but the cost is decreasing about 12% per year. In how many years will the cost be under \$400?

21. A two-bedroom house in Nashville is worth \$110,000. If it appreciates at 2.5% per year, when will it be worth \$200,000?

22. Last year the principal's car was worth \$28,000. Next year it will be worth \$25,270. What is the annual rate of depreciation? What is the car worth now?

23. A concert has been sold out for weeks, and as the date of the concert draws closer, the price of the ticket increases. The cost of a pair of tickets was \$150 yesterday and is \$162 today. Assume that the cost continues to increase at this rate exponentially.

    a. What is the daily rate of increase? What is the multiplier?

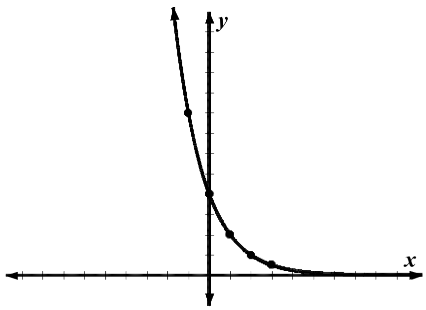
    b. What will be the ticket cost one week from now, the day before the concert?

    c. What was the cost two weeks ago?

Answers:

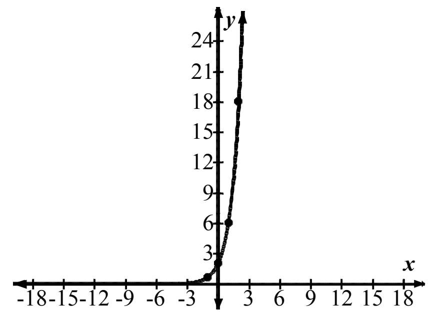
1.

x	–1	0	1	2	3
y	8	4	2	1	0.5



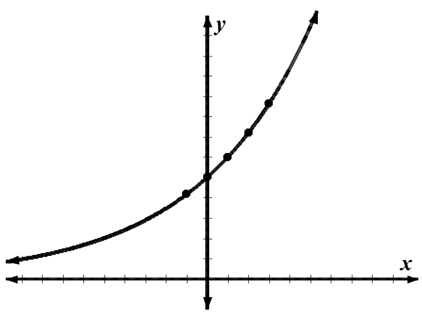
2.

x	–1	0	1	2	3
y	2/3	2	6	18	54



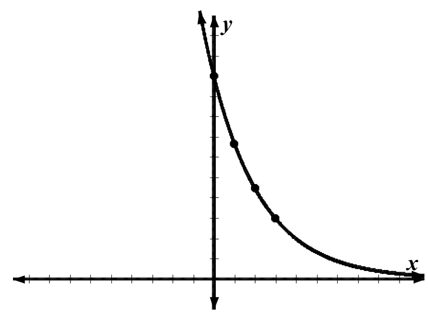
3.

x	–1	0	1	2	3
y	4.17	5	6	7.2	8.64



4.

x	–1	0	1	2	3
y	15	10	6.67	4.44	2.96



5. $1600(1.25)^x$ 6. $40(0.8)^x$ 7. $1.8(3.2)^x$ 8. $5 \cdot 7^x$

9. Possible answer: If a \$32,500 boat loses 15% of its value each year, what will it be worth x years from now?

10. Possible answer: A soda at the movies now costs \$2.75. If the cost is increasing 2.5% per year, what will it cost x years from now?

11. \$614.40 12. ≈ 12-13 years

13. \$6.40 (Note that answers of \$6.05 used $b = 0.96$ which is incorrect.)

14. ≈ 1465 15. \$184,274 16. \$41.09

17. ≈ \$7920 18. ≈ \$2.43 19. ≈ \$4169

20. ≈ 6-7 years 21. ≈ 24-25 years 22. 5%, ≈ \$26,600

23. a. 8%, 1.08 b. ≈ \$277.64 c. ≈ \$55.15

Checkpoint 10

Problem 10-156

Finding Angles in and Areas of Regular Polygons

Answers to problem 10-156: a. 162°; b. 16 sides; c. ≈ 120.8 cm^2

The sum of the measures of a polygon with n sides is $(n - 2)180°$ and therefore each angle in a regular polygon with n sides measures $\frac{(n-2)180°}{n}$.

The sum of the exterior angles of any polygon is 360°.

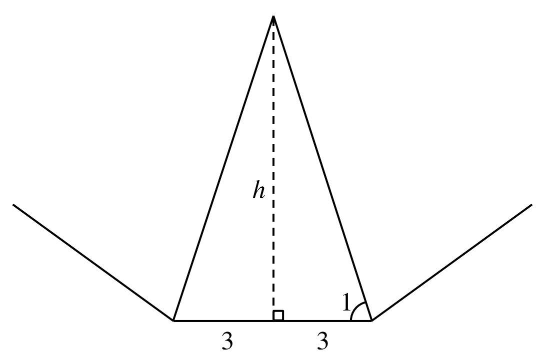
To find the area of a regular polygon, use the angles and side length of one of the identical triangles that make up the polygon to find the area of the triangle. Then multiply by the number of identical triangles to determine the area of the polygon.

Example 1: What is the measure of each interior angle of a regular 10-gon?

Solution: Using $n = 10$ in the formula given above, $\frac{(n-2)180°}{n} = \frac{8 \cdot 180°}{10} = 144°$.

Example 2: If the regular 10-gon in Example 1 has a side length of 6 inches, what is the area of the polygon?

Solution: The regular 10-gon is made is made up of 10 identical isosceles triangles like the one at right. From Example 1, each interior angle is 144°. The base angle in the triangle is half of the interior angle so $m\angle 1 = 72°$. We can use the tangent ratio to find the height of the triangle: $\tan 72° = \frac{h}{3} \Rightarrow h = 3\tan 72° \approx 9.23$. The area of the triangle is $\frac{1}{2} \cdot 6 \cdot 9.23 \approx 27.69$.



Therefore the area of the 10-gon is $10(27.69) \approx 276.9\ \text{in}^2$.

Now we can go back and solve the original problems.

a. $\frac{(n-2)180°}{n} = \frac{18 \cdot 180°}{20} = 162°$

b. Method 1 (using the formula): $157.5° = \frac{(n-2)180°}{n} \Rightarrow 157.5°\,n = (n-2)180°$

$\Rightarrow 157.5°\,n = 180°\,n - 360° \Rightarrow -22.5°\,n = -360° \Rightarrow n = 16$

Method 2 (using the exterior angle): If the interior angle is 157.5° then the exterior angle is 180°-157.5° = 22.5°. Since the sum of the exterior angles of a polygon is 360°, 360° ÷ 22.5° = 16 sides.

c. The octagon is made up of 8 identical triangles. Each interior angle of the polygon measures $\frac{6 \cdot 180°}{8} = 135°$. Using a diagram as in example 2 above, $m\angle 1 = \frac{135°}{2} = 67.5°$. The base of the small triangle is 2.5 cm. To find the height, $\tan 67.5° = \frac{h}{2.5} \Rightarrow h = 2.5 \tan 67.5° \approx 6.04$. The area of the large triangle is $\frac{1}{2} \cdot 5 \cdot 6.04 \approx 15.1$. Eight of these triangles make up the octagon so the area of the octagon is $8 \cdot 15.1 \approx 120.8 \text{ cm}^2$.

Here are some more to try. Find the measure of the angle of each regular polygon.

1. Interior angle, 12 sides
2. Interior angle, 15 sides
3. Interior angle, 7 sides
4. Interior angle, 60 sides
5. Exterior angle, 10 sides
6. Exterior angle, 20 sides

Answer each of the following questions.

7. What is the measure of each interior angle of a regular polygon with 16 sides?
8. What is the measure of each exterior angle of a regular polygon with 16 sides?
9. What is the area of a regular polygon with 16 sides and side length 4 inches?
10. Each interior angle of a regular polygon measures $156°$. How many sides does it have?
11. What is the area of a regular pentagon with side length 10 feet?
12. Each exterior angle of a regular polygon measures 15°. How many sides does it have?
13. Each interior angle of a regular polygon measures 165.6°. How many sides does it have?
14. What is the area of a regular octagon with side length 1 meter?
15. Each exterior angle of a regular polygon measures $13\frac{1}{3}°$. How many sides does it have?
16. What is the area of a regular polygon with 15 sides and a side length of 4 inches?

Answers:

1.	150°	2.	156°	3.	$128\frac{4}{7}{}^{\circ}$
4.	174°	5.	36°	6.	18°
7.	157.5°	8.	22.5°	9.	$\approx 321.7\text{ in}^2$
10.	15 sides	11.	$\approx 172.0\text{ ft}^2$	12.	24 sides
13.	25 sides	14.	$\approx 4.83\text{ m}^2$	15.	27 sides
16.	$\approx 282.3\text{ in}^2$				

Checkpoint 11

Problem 11-102

Volumes and Surface Areas of Prisms and Cylinders

Answers to problem 11-102: a. $V = 2100\ \text{units}^3$, $SA \approx 1007.34\ \text{units}^2$

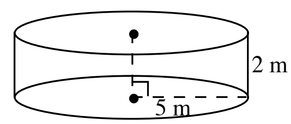
b. $V = 1000\pi \approx 3141.59\ \text{cm}^3$, $SA = \frac{1100\pi}{3} + 240 \approx 1391.92\ \text{cm}^2$

c. $V = 60\ \text{in.}^3$, $SA = 144\ \text{in.}^2$

The volume (in cubic units) of a prism or cylinder is found by multiplying the area of the base by the height: $V = Bh$.

The surface area (in square units) of either of these solids is found by adding the areas of the bases and the area of the lateral faces (sides).

Example 1: Find the volume and surface area of the cylinder at right.



Solution: To find the volume we need to calculate the area of the base, which in this case is a circle with radius 5 m.

$B = \pi r^2 = \pi(5)^2 = 25\pi$

$V = Bh = (25\pi)(2) = 50\pi \approx 157.08\ \text{m}^3$

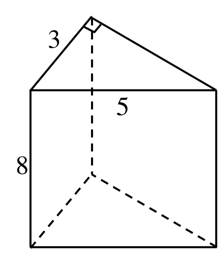
To find the surface area we need to add the area of the two bases and the area of the side. The unrolled side is a rectangle with a base length equal to the circumference of the circle and a height equal to the height of the cylinder.

2(area of the base) + area of the side

$2(\pi r^2) + \pi 2r(h)$

$2\pi \cdot 25 + \pi \cdot 10 \cdot 2 = 70\pi \approx 219.91\ \text{m}^2$

Example 2: Find the volume and surface area of the prism at right.



Solution: To find the volume we need to calculate the area of the base, which in this case is a triangle. First use the Pythagorean theorem to find the second leg of the right triangular base.
Since $3^2 + \text{leg}^2 = 5^2$, leg = 4.

$B = \frac{1}{2}bh = \frac{1}{2}(3)(4) = 6 \qquad V = Bh = (6)(8) = 48\ \text{units}^3$

To find the surface area we must add together the area of the two bases and the area of the three rectangular sides. The three rectangular sides all have a height of 8 units and the three bases are the lengths of the sides of the triangle.

2(area of the base) + area of side #1 + area of side #2 + area of side #3

$2(\frac{1}{2} \cdot 3 \cdot 4) \quad + \quad 5 \cdot 8 \quad + \quad 3 \cdot 8 \quad + \quad 4 \cdot 8 = 108\ \text{units}^2$

Now we can go back and solve the original problems.

a. The bases are a trapezoid. $B = \frac{1}{2}h(b_1 + b_2) = \frac{1}{2} \cdot 15(8+12) = 150$.

To calculate the volume: $V = Bh = 150 \cdot 14 = 2100 \text{ units}^3$

To find the surface area we must add together the area of the two bases and the area of the four rectangular sides. The four rectangular sides all have a height of 14 units and the four bases are the lengths of the sides of the trapezoid. To find the missing length of the trapezoid we need to use the Pythagorean Theorem.

Use the triangular part of the trapezoid:

$(12-8)^2 + 15^2 = \text{hypotenuse}^2 \Rightarrow 241 = \text{hypotenuse}^2$

The length of the unlabeled side $= \sqrt{241}$.

2(area of the base) + area of side #1 + area of side #2 + area of side #3 + area of side #4

$2(\frac{1}{2} \cdot 15(8+12)) \;+\; 12 \cdot 14 \;+\; 15 \cdot 14 \;+\; 8 \cdot 14 \;+\; \sqrt{241} \cdot 14 \;\approx 1007.34 \text{ units}^2$

b. The base is $\frac{300°}{360°} = \frac{5}{6}$ of a circle. $B = \frac{5}{6}\pi r^2 = \frac{5}{6}\pi \cdot 10^2 = \frac{500\pi}{6}$

To find the volume: $V = Bh = (\frac{500\pi}{6})(12) = 1000\pi \approx 3141.59 \text{ cm}^3$

To find the surface area we must add together the area of the two bases and the area of the three rectangular sides. The three rectangular sides all have a height of 12 cm and the three bases are two radii and $\frac{5}{6}$ the circumference of the circle.

2(area of the base) + area of side #1 + area of side #2 + area of side #3

$2(\frac{5}{6}\pi \cdot 10^2) \;+\; 10 \cdot 12 \;+\; 10 \cdot 12 \;+\; \frac{5}{6}\pi \cdot 20 \cdot 12 \;= \frac{1100\pi}{3} + 240 \approx 1391.92 \text{ cm}^2$

c. The base is a square with a square cut out. $B = 4^2 - 2^2 = 12$

To find the volume: $V = Bh = 12 \cdot 5 = 60 \text{ in.}^3$

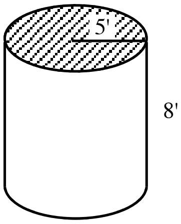
To find the surface area we must add together the area of the two bases and the area of the four identical external rectangular sides and the four identical internal rectangular sides.

2(area of the base) + 4(area of external side) + 4(area of internal side)

$2(4^2 - 2^2) \;+\; 4(4 \cdot 5) \;+\; 4(2 \cdot 5) \;= 144 \text{ in.}^2$

Here are some more to try. Compute the volume and surface area of each solid.

1.



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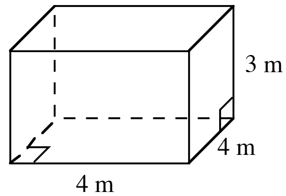

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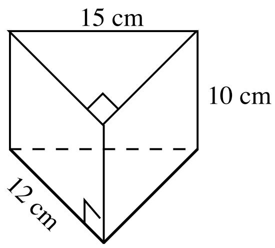

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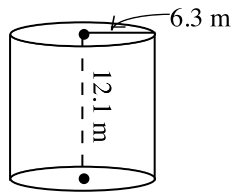

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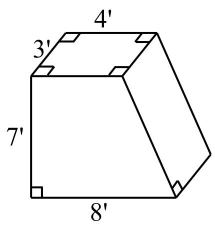

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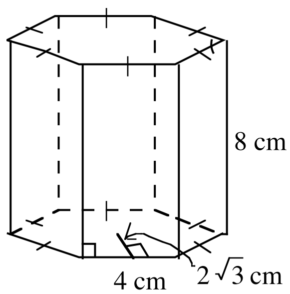

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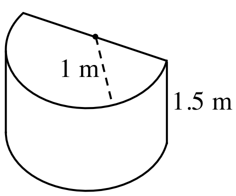

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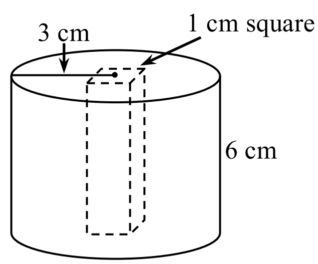

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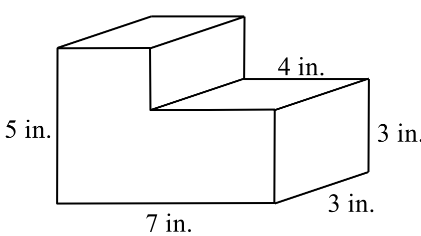

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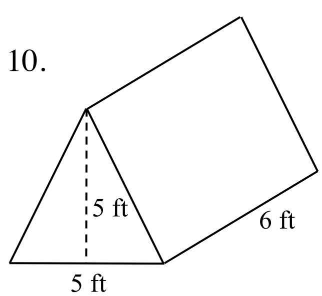

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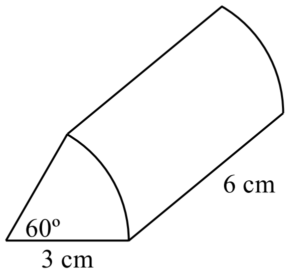

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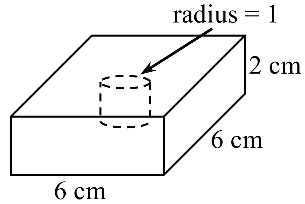

Answers:

1. $V = 200\pi \approx 628.32\ \text{ft}^3$; $SA = 130\pi \approx 408.41\ \text{ft}^2$
2. $V = 48\ \text{m}^3$; $SA = 80\ \text{m}^2$
3. $V = 540\ \text{cm}^3$; $SA = 468\ \text{cm}^2$
4. $V \approx 1508.75\ \text{m}^3$; $SA \approx 728.35\ \text{m}^2$
5. $V = 126\ \text{ft}^3$; $SA \approx 165.19\ \text{ft}^2$
6. $V \approx 332.6\ \text{cm}^3$; $SA \approx 275.14\ \text{cm}^2$
7. $V \approx 2.36\ \text{m}^3$; $SA \approx 10.85\ \text{m}^2$
8. $V \approx 163.65\ \text{cm}^3$; $SA \approx 191.65\ \text{cm}^2$
9. $V = 81\ \text{in.}^3$; $SA = 126\ \text{in.}^2$
10. $V = 75\ \text{ft}^3$; $SA \approx 122.08\ \text{ft}^2$
11. $V \approx 28.27\ \text{cm}^3$; $SA \approx 64.27\ \text{cm}^2$
12. $V \approx 65.72\ \text{cm}^3$; $SA \approx 126.28\ \text{cm}^2$