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Sample

Chapter 1 Complex numbers

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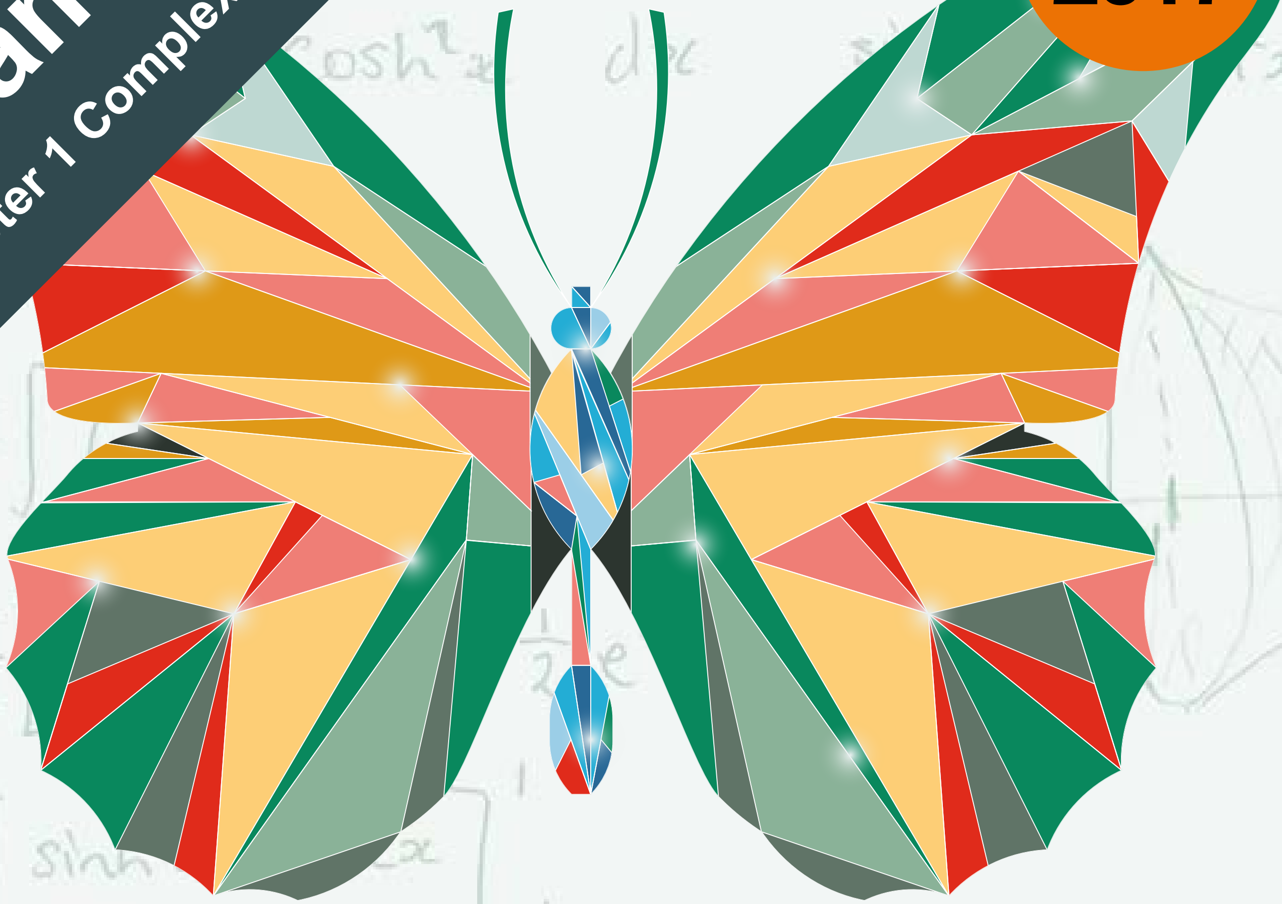

Edexcel AS and A level Further Mathematics

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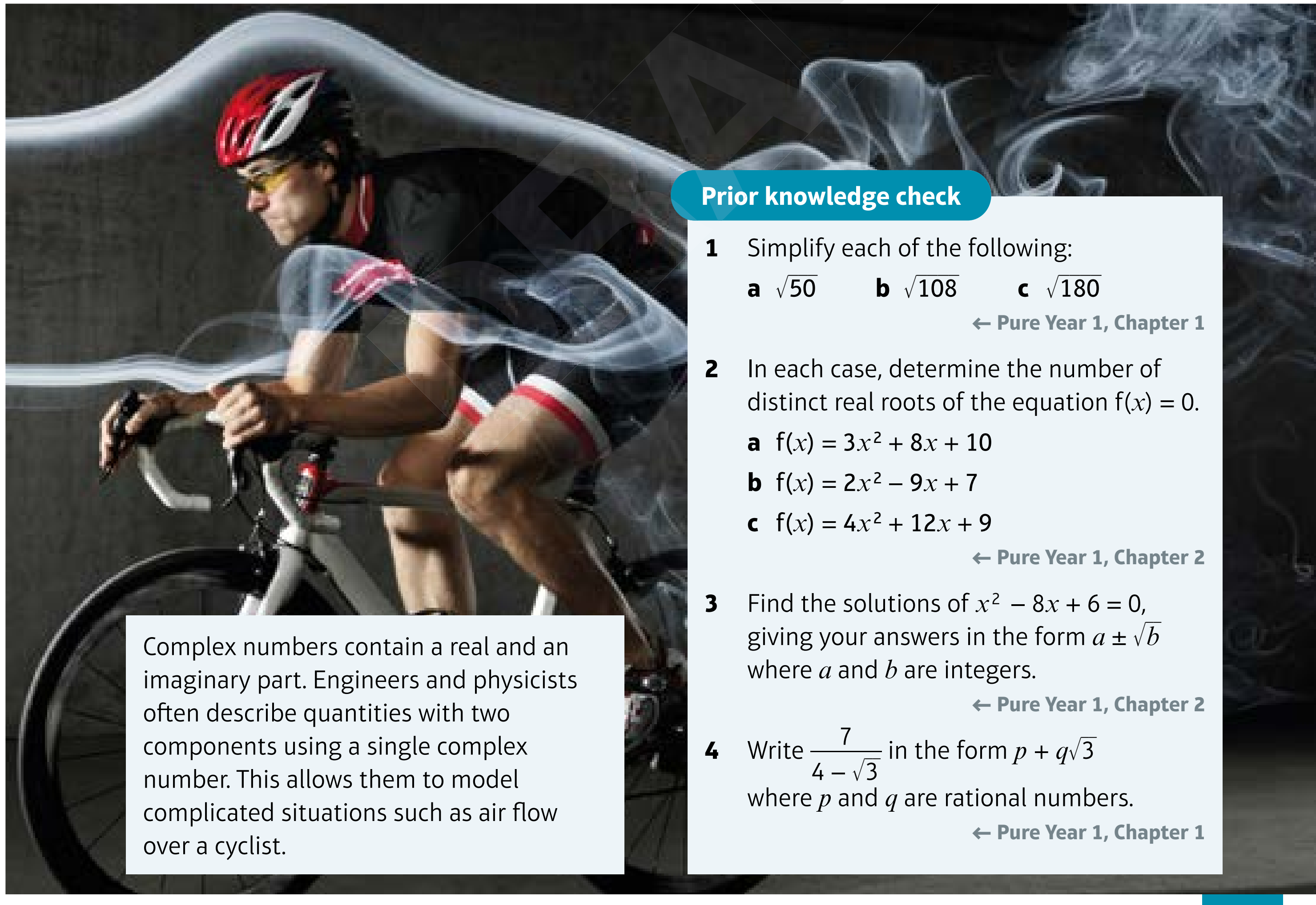
Sample material

Complex numbers

1

Objectives

- Understand and use the definitions of imaginary and complex numbers → page 2
- Add and subtract complex numbers → pages 2–3
- Multiply complex numbers → pages 5–6
- Understand the definition of a complex conjugate → pages 6–8
- Divide complex numbers → pages 7–8
- Solve quadratic equations that have complex roots → pages 8–10
- Solve cubic or quartic equations that have complex roots → pages 10–14



Complex numbers contain a real and an imaginary part. Engineers and physicists often describe quantities with two components using a single complex number. This allows them to model complicated situations such as air flow over a cyclist.

Prior knowledge check

1 Simplify each of the following:

a $\sqrt{50}$ **b** $\sqrt{108}$ **c** $\sqrt{180}$

← Pure Year 1, Chapter 1

2 In each case, determine the number of distinct real roots of the equation $f(x) = 0$.

a $f(x) = 3x^2 + 8x + 10$

b $f(x) = 2x^2 - 9x + 7$

c $f(x) = 4x^2 + 12x + 9$

← Pure Year 1, Chapter 2

3 Find the solutions of $x^2 - 8x + 6 = 0$, giving your answers in the form $a \pm \sqrt{b}$ where a and b are integers.

← Pure Year 1, Chapter 2

4 Write $\frac{7}{4 - \sqrt{3}}$ in the form $p + q\sqrt{3}$ where p and q are rational numbers.

← Pure Year 1, Chapter 1

1.1 Imaginary and complex numbers

The quadratic equation $ax^2 + bx + c = 0$ has solutions given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If the expression under the square root is negative, there are no real solutions.

Links For the equation $ax^2 + bx + c = 0$, the **discriminant** is $b^2 - 4ac$.

- If $b^2 - 4ac > 0$, there are two distinct real roots.
- If $b^2 - 4ac = 0$, there are two equal real roots.
- If $b^2 - 4ac < 0$, there are no real roots.

← Pure Year 1, Section 2.5

You can find solutions to the equation in all cases by extending the number system to include $\sqrt{-1}$. Since there is no real number that squares to produce −1, the number $\sqrt{-1}$ is called an **imaginary number**, and is represented using the letter i. Sums of real and imaginary numbers, for example 3 + 2i, are known as **complex numbers**.

- **$\mathrm{i} = \sqrt{-1}$**
- **An imaginary number is a number of the form $b\mathrm{i}$, where $b \in \mathbb{R}$.**
- **A complex number is written in the form $a + b\mathrm{i}$, where $a, b \in \mathbb{R}$.**

Notation The set of all complex numbers is written as $\mathbb{C}$.
For the complex number $z = a + b\mathrm{i}$:

- $\mathrm{Re}(z) = a$ is the real part
- $\mathrm{Im}(z) = b$ is the imaginary part

Example 1

Write each of the following in terms of i:

a $\sqrt{-36}$ **b** $\sqrt{-28}$

a $\sqrt{-36} = \sqrt{36 \times (-1)} = \sqrt{36}\sqrt{-1} = 6\mathrm{i}$

b $\sqrt{-28} = \sqrt{28 \times (-1)} = \sqrt{28}\sqrt{-1}$
$= \sqrt{4}\sqrt{7}\sqrt{-1} = (2\sqrt{7})\mathrm{i}$

You can use the rules of surds to manipulate imaginary numbers.

Watch out An alternative way of writing $(2\sqrt{7})\mathrm{i}$ is $2\mathrm{i}\sqrt{7}$. Avoid writing $2\sqrt{7}\mathrm{i}$ as this can easily be confused with $2\sqrt{7\mathrm{i}}$.

In a complex number, the real part and the imaginary part cannot be combined to form a single term.

- **Complex numbers can be added or subtracted by adding or subtracting their real parts and adding or subtracting their imaginary parts.**
- **You can multiply a real number by a complex number by multiplying out the brackets in the usual way.**

Example 2

Simplify each of the following, giving your answers in the form $a + b\mathrm{i}$, where $a, b \in \mathbb{R}$.

a $(2 + 5\mathrm{i}) + (7 + 3\mathrm{i})$ **b** $(2 - 5\mathrm{i}) - (5 - 11\mathrm{i})$ **c** $2(5 - 8\mathrm{i})$ **d** $\frac{10 + 6\mathrm{i}}{2}$

a $(2 + 5\mathrm{i}) + (7 + 3\mathrm{i}) = (2 + 7) + (5 + 3)\mathrm{i}$
$= 9 + 8\mathrm{i}$

Add the real parts and add the imaginary parts.

b $(2 - 5\mathrm{i}) - (5 - 11\mathrm{i}) = (2 - 5) + (-5 - (-11))\mathrm{i}$
$= -3 + 6\mathrm{i}$

Subtract the real parts and subtract the imaginary parts.

c $2(5 - 8\mathrm{i}) = (2 \times 5) - (2 \times 8)\mathrm{i} = 10 - 16\mathrm{i}$

$2(5 - 8\mathrm{i})$ can also be written as $(5 - 8\mathrm{i}) + (5 - 8\mathrm{i})$.

d $\frac{10 + 6\mathrm{i}}{2} = \frac{10}{2} + \frac{6}{2}\mathrm{i} = 5 + 3\mathrm{i}$

First separate into a real part and an imaginary part.

Exercise 1A

Do not use your calculator in this exercise.

1 Write each of the following in the form $b\mathrm{i}$ where b is a real number.

a $\sqrt{-9}$ **b** $\sqrt{-49}$ **c** $\sqrt{-121}$ **d** $\sqrt{-10\,000}$ **e** $\sqrt{-225}$
f $\sqrt{-5}$ **g** $\sqrt{-12}$ **h** $\sqrt{-45}$ **i** $\sqrt{-200}$ **j** $\sqrt{-147}$

2 Simplify, giving your answers in the form $a + b\mathrm{i}$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

a $(5 + 2\mathrm{i}) + (8 + 9\mathrm{i})$ **b** $(4 + 10\mathrm{i}) + (1 - 8\mathrm{i})$
c $(7 + 6\mathrm{i}) + (-3 - 5\mathrm{i})$ **d** $\left(\frac{1}{2} + \frac{1}{3}\mathrm{i}\right) + \left(\frac{5}{2} + \frac{5}{3}\mathrm{i}\right)$
e $(20 + 12\mathrm{i}) - (11 + 3\mathrm{i})$ **f** $(2 - \mathrm{i}) - (-5 + 3\mathrm{i})$
g $(-4 - 6\mathrm{i}) - (-8 - 8\mathrm{i})$ **h** $(3\sqrt{2} + \mathrm{i}) - (\sqrt{2} - \mathrm{i})$
i $(-2 - 7\mathrm{i}) + (1 + 3\mathrm{i}) - (-12 + \mathrm{i})$ **j** $(18 + 5\mathrm{i}) - (15 - 2\mathrm{i}) - (3 + 7\mathrm{i})$

3 Simplify, giving your answers in the form $a + b\mathrm{i}$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

a $2(7 + 2\mathrm{i})$ **b** $3(8 - 4\mathrm{i})$
c $2(3 + \mathrm{i}) + 3(2 + \mathrm{i})$ **d** $5(4 + 3\mathrm{i}) - 4(-1 + 2\mathrm{i})$
e $\frac{6 - 4\mathrm{i}}{2}$ **f** $\frac{15 + 25\mathrm{i}}{5}$
g $\frac{9 + 11\mathrm{i}}{3}$ **h** $\frac{-8 + 3\mathrm{i}}{4} - \frac{7 - 2\mathrm{i}}{2}$

(P) **4** Write in the form $a + b\mathrm{i}$, where a and b are simplified surds.

a $\frac{4 - 2\mathrm{i}}{\sqrt{2}}$ **b** $\frac{2 - 6\mathrm{i}}{1 + \sqrt{3}}$

5 Given that $z = 7 - 6\mathrm{i}$ and $w = 7 + 6\mathrm{i}$, find, in the form $a + b\mathrm{i}$, where $a, b \in \mathbb{R}$

a $z - w$ **b** $w + z$

Notation Complex numbers are often represented by the letter z or the letter w.

(E) **6** Given that $z_1 = a + 9\mathrm{i}$, $z_2 = -3 + b\mathrm{i}$ and $z_2 - z_1 = 7 + 2\mathrm{i}$, find a and b where $a, b \in \mathbb{R}$. **(2 marks)**

(P) **7** Given that $z_1 = 4 + \mathrm{i}$ and $z_2 = 7 - 3\mathrm{i}$, find, in the form $a + b\mathrm{i}$, where $a, b \in \mathbb{R}$

a $z_1 - z_2$ **b** $4z_2$ **c** $2z_1 + 5z_2$

(P) **8** Given that $z = a + b\mathrm{i}$ and $w = a - b\mathrm{i}$, show that:

a $z + w$ is always real **b** $z - w$ is always imaginary

You can use complex numbers to find solutions to any quadratic equation with real coefficients.

- **If $b^2 - 4ac < 0$ then the quadratic equation $ax^2 + bx + c = 0$ has two distinct complex roots.**

Example 3

Solve the equation $z^2 + 9 = 0$.

$z^2 = -9$
$z = \pm\sqrt{-9} = \pm\sqrt{9 \times -1} = \pm\sqrt{9}\sqrt{-1} = \pm 3\text{i}$
$z = +3\text{i},\ z = -3\text{i}$

Note that just as $z^2 = 9$ has two roots +3 and −3, $z^2 = -9$ also has two roots +3i and −3i.

Example 4

Solve the equation $z^2 + 6z + 25 = 0$.

Method 1 (Completing the square)
$z^2 + 6z = (z + 3)^2 - 9$
$z^2 + 6z + 25 = (z + 3)^2 - 9 + 25 = (z + 3)^2 + 16$
$(z + 3)^2 + 16 = 0$
$(z + 3)^2 = -16$
$z + 3 = \pm\sqrt{-16} = \pm 4\text{i}$
$z = -3 \pm 4\text{i}$
$z = -3 + 4\text{i}, \quad z = -3 - 4\text{i}$

Because $(z + 3)^2 = (z + 3)(z + 3) = z^2 + 6z + 9$

$\sqrt{-16} = \sqrt{16 \times (-1)} = \sqrt{16}\sqrt{-1} = 4\text{i}$

Online Solve this equation quickly using your calculator.

Method 2 (Quadratic formula)
$z = \dfrac{-6 \pm \sqrt{6^2 - 4 \times 1 \times 25}}{2}$
$= \dfrac{-6 \pm \sqrt{-64}}{2}$
$\sqrt{-64}$ 0= $\pm 8\text{i}$
$z = \dfrac{-6 \pm 8\text{i}}{2} = -3 \pm 4\text{i}$
$z = -3 + 4\text{i}, \quad z = -3 - 4\text{i}$

Using $z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\sqrt{-64} = \sqrt{64 \times (-1)} = \sqrt{64}\sqrt{-1} = 8\text{i}$

Exercise 1B

Do not use your calculator in this exercise.

1 Solve each of the following equations. Write your answers in the form $\pm b\text{i}$.

a $z^2 + 121 = 0$ **b** $z^2 + 40 = 0$ **c** $2z^2 + 120 = 0$
d $3z^2 + 150 = 38 - z^2$ **e** $z^2 + 30 = -3z^2 - 66$ **f** $6z^2 + 1 = 2z^2$

2 Solve each of the following equations.
Write your answers in the form $a \pm b\text{i}$.

a $(z - 3)^2 - 9 = -16$
b $2(z - 7)^2 + 30 = 6$
c $16(z + 1)^2 + 11 = 2$

Hint The left-hand side of each equation is in completed square form already. Use inverse operations to find the values of z.

3 Solve each of the following equations. Write your answers in the form $a \pm b\text{i}$.

a $z^2 + 2z + 5 = 0$ **b** $z^2 - 2z + 10 = 0$ **c** $z^2 + 4z + 29 = 0$
d $z^2 + 10z + 26 = 0$ **e** $z^2 + 5z + 25 = 0$ **f** $z^2 + 3z + 5 = 0$

4 Solve each of the following equations. Write your answers in the form $a \pm b\text{i}$.

a $2z^2 + 5z + 4 = 0$ **b** $7z^2 - 3z + 3 = 0$ **c** $5z^2 - z + 3 = 0$

5 The solutions to the quadratic equation $z^2 - 8z + 21 = 0$ are z_1 and z_2.
Find z_1 and z_2, giving each answer in the form $a \pm \text{i}\sqrt{b}$.

E/P 6 The equation $z^2 + bz + 11 = 0$, where $b \in \mathbb{R}$, has distinct complex roots.
Find the range of possible values of b. **(3 marks)**

1.2 Multiplying complex numbers

You can multiply complex numbers using the same technique that you use for multiplying brackets in algebra. You can use the fact that $\text{i} = \sqrt{-1}$ to simplify powers of i.

- **$\text{i}^2 = -1$**

Example 5

Express each of the following in the form $a + b\text{i}$, where a and b are real numbers.

a $(2 + 3\text{i})(4 + 5\text{i})$ **b** $(7 - 4\text{i})^2$

a $(2 + 3\text{i})(4 + 5\text{i}) = 2(4 + 5\text{i}) + 3\text{i}(4 + 5\text{i})$
$= 8 + 10\text{i} + 12\text{i} + 15\text{i}^2$
$= 8 + 10\text{i} + 12\text{i} - 15$
$= (8 - 15) + (10\text{i} + 12\text{i})$
$= -7 + 22\text{i}$

Multiply the two brackets as you would with real numbers.

Use the fact that $\text{i}^2 = -1$.

Add real parts and add imaginary parts.

b $(7 - 4\text{i})^2 = (7 - 4\text{i})(7 - 4\text{i})$
$= 7(7 - 4\text{i}) - 4\text{i}(7 - 4\text{i})$
$= 49 - 28\text{i} - 28\text{i} + 16\text{i}^2$
$= 49 - 28\text{i} - 28\text{i} - 16$
$= (49 - 16) + (-28\text{i} - 28\text{i})$
$= 33 - 56\text{i}$

Multiply out the two brackets as you would with real numbers.

Use the fact that $\text{i}^2 = -1$.

Add real parts and add imaginary parts.

Example 6

Simplify: **a** i^3 **b** i^4 **c** $(2\text{i})^5$

a $\text{i}^3 = \text{i} \times \text{i} \times \text{i} = \text{i}^2 \times \text{i} = -\text{i}$
b $\text{i}^4 = \text{i} \times \text{i} \times \text{i} \times \text{i} = \text{i}^2 \times \text{i}^2 = (-1) \times (-1) = 1$
c $(2\text{i})^5 = 2\text{i} \times 2\text{i} \times 2\text{i} \times 2\text{i} \times 2\text{i}$
$= 32(\text{i} \times \text{i} \times \text{i} \times \text{i} \times \text{i}) = 32(\text{i}^2 \times \text{i}^2 \times \text{i})$
$= 32 \times (-1) \times (-1) \times \text{i} = 32\text{i}$

$\text{i}^2 = -1$

$(2\text{i})^5 = 2^5 \times \text{i}^5$
First work out $2^5 = 32$.

Exercise 1C

Do not use your calculator in this exercise.

1 Simplify each of the following, giving your answers in the form $a + bi$.

a $(5 + i)(3 + 4i)$ b $(6 + 3i)(7 + 2i)$ c $(5 - 2i)(1 + 5i)$

d $(13 - 3i)(2 - 8i)$ e $(-3 - i)(4 + 7i)$ f $(8 + 5i)^2$

g $(2 - 9i)^2$ h $(1 + i)(2 + i)(3 + i)$

i $(3 - 2i)(5 + i)(4 - 2i)$ j $(2 + 3i)^3$

Hint For part **h**, begin by multiplying the first pair of brackets.

(P) 2 a Simplify $(4 + 5i)(4 - 5i)$, giving your answer in the form $a + bi$.

b Simplify $(7 - 2i)(7 + 2i)$, giving your answer in the form $a + bi$.

c Comment on your answers to parts **a** and **b**.

d Prove that $(a + bi)(a - bi)$ is a real number for any real numbers a and b.

(P) 3 Given that $(a + 3i)(1 + bi) = 31 - 38i$, find two possible pairs of values for a and b.

4 Write each of the following in its simplest form.

a i^6 b $(3i)^4$ c $i^5 + i$ d $(4i)^3 - 4i^3$

(P) 5 Express $(1 + i)^6$ in the form $a - bi$, where a and b are integers to be found.

(P) 6 Find the value of the real part of $(3 - 2i)^4$.

(P) 7 $f(z) = 2z^2 - z + 8$

Find: a $f(2i)$ b $f(3 - 6i)$

Problem-solving You can use the binomial theorem to expand $(a + b)^n$. ← Pure Year 1, Section 8.3

(E/P) 8 $f(z) = z^2 - 2z + 17$

Show that $z = 1 - 4i$ is a solution to $f(z) = 0$. **(2 marks)**

9 a Given that $i^1 = i$ and $i^2 = -1$, write i^3 and i^4 in their simplest forms.

b Write i^5, i^6, i^7 and i^8 in their simplest forms.

c Write down the value of:

i i^{100} ii i^{253} iii i^{301}

Challenge

a Expand $(a + bi)^2$.

b Hence, or otherwise, find $\sqrt{40 - 42i}$, giving your answer in the form $a - bi$, where a and b are positive integers.

Notation The **principal square root** of a complex number, $\sqrt{z}$, has a positive real part.

1.3 Complex conjugation

- **For any complex number $z = a + bi$, the complex conjugate of the number is defined as $z^* = a - bi$.**

Notation Together z and z^* are called a **complex conjugate pair**.

Example 7

Given that $z = 2 - 7i$:

a write down z^* b find the value of $z + z^*$ c find the value of zz^*

a $z^* = 2 + 7i$ — Change the sign of the imaginary part from – to +.

b $z + z^* = (2 - 7i) + (2 + 7i)$
$= (2 + 2) + (-7 + 7)i = 4$ — Note that $z + z^*$ is real.

c $zz^* = (2 - 7i)(2 + 7i)$
$= 2(2 + 7i) - 7i(2 + 7i)$
$= 4 + 14i - 14i - 49i^2$ — Remember $i^2 = -1$.
$= 4 + 49 = 53$ — Note that zz^* is real.

For any complex number z, the product of z and z^* is a real number. You can use this property to **divide two complex numbers**. To do this, you multiply both the numerator and the denominator by the complex conjugate of the denominator and then simplify the result.

Links The method used to divide complex numbers is similar to the method used to rationalise a denominator when simplifying surds ← Pure Year 1, Section 1.6

Example 8

Write $\frac{5 + 4i}{2 - 3i}$ in the form $a + bi$.

$\frac{5 + 4i}{2 - 3i} = \frac{5 + 4i}{2 - 3i} \times \frac{2 + 3i}{2 + 3i}$ — The complex conjugate of the denominator is $2 + 3i$. Multiply both the numerator and the denominator by the complex conjugate.

$= \frac{(5 + 4i)(2 + 3i)}{(2 - 3i)(2 + 3i)}$ — zz^* is real, so $(2 - 3i)(2 + 3i)$ will be a real number.

$(5 + 4i)(2 + 3i) = 5(2 + 3i) + 4i(2 + 3i)$
$= 10 + 15i + 8i + 12i^2$
$= -2 + 23i$

$(2 - 3i)(2 + 3i) = 2(2 + 3i) - 3i(2 + 3i)$
$= 4 + 6i - 6i - 9i^2 = 13$

$\frac{5 + 4i}{2 - 3i} = \frac{-2 + 23i}{13} = -\frac{2}{13} + \frac{23}{13}i$ — Divide each term in the numerator by 13.

Online Divide complex numbers quickly using your calculator.

Exercise 1D

Do not use your calculator in this exercise.

1 Write down the complex conjugate z^* for:

a $z = 8 + 2i$ b $z = 6 - 5i$ c $z = \frac{2}{3} - \frac{1}{2}i$ d $z = \sqrt{5} + i\sqrt{10}$

2 Find $z + z^*$ and zz^* for:

a $z = 6 - 3i$ b $z = 10 + 5i$ c $z = \frac{3}{4} + \frac{1}{4}i$ d $z = \sqrt{5} - 3i\sqrt{5}$

3 Write each of the following in the form $a + bi$.

a $\frac{3 - 5i}{1 + 3i}$ b $\frac{3 + 5i}{6 - 8i}$ c $\frac{28 - 3i}{1 - i}$ d $\frac{2 + i}{1 + 4i}$

4 Write $\frac{(3-4i)^2}{1+i}$ in the form $x + iy$ where $x, y \in \mathbb{R}$.

5 Given that $z_1 = 1 + i$, $z_2 = 2 + i$ and $z_3 = 3 + i$, write each of the following in the form $a + bi$.

a $\frac{z_1 z_2}{z_3}$ **b** $\frac{(z_2)^2}{z_1}$ **c** $\frac{2z_1 + 5z_3}{z_2}$

E 6 Given that $\frac{5+2i}{z} = 2 - i$, find z in the form $a + bi$. **(2 marks)**

7 Simplify $\frac{6+8i}{1+i} + \frac{6+8i}{1-i}$, giving your answer in the form $a + bi$.

8 $w = \frac{4}{8 - i\sqrt{2}}$

Express w in the form $a + bi\sqrt{2}$, where a and b are rational numbers.

9 $w = 1 - 9i$

Express $\frac{1}{w}$ in the form $a + bi$, where a and b are rational numbers.

10 $z = 4 - i\sqrt{2}$

Use algebra to express $\frac{z+4}{z-3}$ in the form $p + qi\sqrt{2}$, where p and q are rational numbers.

E/P 11 The complex number z satisfies the equation $(4 + 2i)(z - 2i) = 6 - 4i$.
Find z, giving your answer in the form $a + bi$ where a and b are rational numbers. **(4 marks)**

E/P 12 The complex numbers z_1 and z_2 are given by $z_1 = p - 7i$ and $z_2 = 2 + 5i$ where p is an integer.
Find $\frac{z_1}{z_2}$ in the form $a + bi$ where a and b are rational, and are given in terms of p. **(4 marks)**

E 13 $z = \sqrt{5} + 4i$
z^* is the complex conjugate of z.
Show that $\frac{z}{z^*} = a + bi\sqrt{5}$, where a and b are rational numbers to be found. **(4 marks)**

E/P 14 The complex number z is defined by $z = \frac{p+5i}{p-2i}$, $p \in \mathbb{R}$, $p > 0$.
Given that the real part of z is $\frac{1}{2}$,

a find the value of p **(4 marks)**

b write z in the form $a + bi$, where a and b are real. **(1 mark)**

1.4 Roots of quadratic equations

- **For real numbers a, b and c, if the roots of the quadratic equation $az^2 + bz + c = 0$ are complex, then they occur as a conjugate pair.**

Another way of stating this is that for a real-valued quadratic function f(z), if z_1 is a root of f(z) = 0 then z_1^* is also a root. You can use this fact to find one root if you know the other, or to find the original equation.

- **If the roots of a quadratic equation are α and β, then you can write the equation as $(z - \alpha)(z - \beta) = 0$ or $z^2 - (\alpha + \beta)z + \alpha\beta = 0$**

Notation Roots of complex-valued polynomials are often written using Greek letters such as α (alpha), β (beta) and γ (gamma).

Example 9

Given that $\alpha = 7 + 2i$ is one of the roots of a quadratic equation with real coefficients,

a state the value of the other root, β

b find the quadratic equation

c find the values of $\alpha + \beta$ and $\alpha\beta$ and interpret the results.

a $\beta = 7 - 2i$

α and β will always be a complex conjugate pair.

b $(z - \alpha)(z - \beta) = 0$

$(z - (7 + 2i))(z - (7 - 2i)) = 0$

$z^2 - z(7 - 2i) - z(7 + 2i) + (7 + 2i)(7 - 2i) = 0$

$z^2 - 7z + 2iz - 7z - 2iz + 49 - 14i + 14i - 4i^2 = 0$

$z^2 - 14z + 49 + 4 = 0$

$z^2 - 14z + 53 = 0$

The quadratic equation with roots α and β is $(z - \alpha)(z - \beta) = 0$

Collect like terms. Use the fact that $i^2 = -1$.

c $\alpha + \beta = (7 + 2i) + (7 - 2i)$
$= (7 + 7) + (2 + (-2))i = 14$

The coefficient of z in the above equation is $-(\alpha + \beta)$.

$\alpha\beta = (7 + 2i)(7 - 2i) = 49 - 14i + 14i - 4i^2$
$= 49 + 4 = 53$

The constant term in the above equation is $\alpha\beta$.

Problem-solving

For $z = a + bi$, you should learn the results:

$z + z^* = 2a$

$zz^* = a^2 + b^2$

You can use these to find the quadratic equation quickly.

Exercise 1E

1 The roots of the quadratic equation $z^2 + 2z + 26 = 0$ are α and β.
Find: **a** α and β **b** $\alpha + \beta$ **c** $\alpha\beta$

2 The roots of the quadratic equation $z^2 - 8z + 25 = 0$ are α and β.
Find: **a** α and β **b** $\alpha + \beta$ **c** $\alpha\beta$

E 3 Given that $2 + 3i$ is one of the roots of a quadratic equation with real coefficients,

a write down the other root of the equation **(1 mark)**

b find the quadratic equation, giving your answer in the form $az^2 + bz + c = 0$ where a, b and c are real constants. **(3 marks)**

E 4 Given that $5 - i$ is a root of the equation $z^2 + pz + q = 0$, where p and q are real constants,

a write down the other root of the equation **(1 mark)**

b find the value of p and the value of q. **(3 marks)**

E/P 5 Given that $z_1 = -5 + 4i$ is one of the roots of the quadratic equation $z^2 + bz + c = 0$, where b and c are real constants, find the values of b and c. **(4 marks)**

E/P 6 Given that $1 + 2i$ is one of the roots of a quadratic equation with real coefficients, find the equation giving your answer in the form $z^2 + bz + c = 0$ where b and c are integers to be found. **(4 marks)**

E/P **7** Given that $3 - 5i$ is one of the roots of a quadratic equation with real coefficients, find the equation giving your answer in the form $z^2 + bz + c = 0$ where b and c are real constants. **(4 marks)**

E/P **8** $z = \dfrac{5}{3 - i}$

a Find z in the form $a + bi$, where a and b are real constants. **(1 mark)**

Given that z is a complex root of the quadratic equation $x^2 + px + q = 0$, where p and q are real integers,

b find the value of p and the value of q. **(4 marks)**

E/P **9** Given that $z = 5 + qi$ is a root of the equation $z^2 - 4pz + 34 = 0$, where p and q are positive real constants, find the value of p and the value of q. **(4 marks)**

1.5 Solving cubic and quartic equations

You can generalise the rule for the roots of quadratic equations to any polynomial with real coefficients.

- **If f(z) is a polynomial with real coefficients, and z_1 is a root of f(z) = 0, then z_1* is also a root of f(z) = 0.**

Hint Note that if z_1 is real, then $z_1^* = z_1$.

You can use this property to find roots of cubic and quartic equations with real coefficients.

- **An equation of the form $az^3 + bz^2 + cz + d = 0$ is called a cubic equation, and has three roots.**
- **For a cubic equation with real coefficients, either**
  - **all three roots are real, or**
  - **one root is real and the other two roots form a complex conjugate pair.**

Watch out A real-valued cubic equation might have two, or three, repeated real roots.

Example 10

Given that -1 is a root of the equation $z^3 - z^2 + 3z + k = 0$,

a find the value of k **b** find the other two roots of the equation.

a If -1 is a root,
$(-1)^3 - (-1)^2 + 3(-1) + k = 0$
$-1 - 1 - 3 + k = 0$
$k = 5$

b -1 is a root of the equation, so $z + 1$ is a factor of $z^3 - z^2 + 3z + 5$.

$$\begin{array}{r} z^2 - 2z + 5 \\ z + 1 \overline{)\, z^3 - z^2 + 3z + 5} \\ \underline{z^3 + z^2} \\ -2z^2 + 3z \\ \underline{-2z^2 - 2z} \\ 5z + 5 \\ \underline{5z + 5} \\ 0 \end{array}$$

Problem-solving Use the factor theorem to help: if f(α) = 0, then α is a root of the polynomial and $z - \alpha$ is a factor of the polynomial.

Use long division (or another method) to find the quadratic factor.

$z^3 - z^2 + 3z + 5 = (z + 1)(z^2 - 2z + 5) = 0$

Solving $z^2 - 2z + 5 = 0$

$z^2 - 2z = (z - 1)^2 - 1$
$z^2 - 2z + 5 = (z - 1)^2 - 1 + 5 = (z - 1)^2 + 4$
$(z - 1)^2 + 4 = 0$
$(z - 1)^2 = -4$
$z - 1 = \pm\sqrt{-4} = \pm 2i$
$z = 1 \pm 2i$
$z = 1 + 2i,\ z = 1 - 2i$

So the other two roots of the equation are $1 + 2i$ and $1 - 2i$.

The other two roots are found by solving the quadratic equation.

Solve by completing the square. Alternatively, you could use the quadratic formula.

The quadratic equation has complex roots, which must be a conjugate pair.

You could write the equation as $(z + 1)(z - (1 + 2i))(z - (1 - 2i)) = 0$

- **An equation of the form $az^4 + bz^3 + cz^2 + dz + e = 0$ is called a quartic equation, and has four roots.**
- **For a quartic equation with real coefficients, either**
  - **all four root are real, or**
  - **two roots are real and the other two roots form a complex conjugate pair, or**
  - **two roots form a complex conjugate pair and the other two roots also form a complex conjugate pair.**

Watch out A real-valued quartic equation might have repeated real roots or repeated complex roots.

Example 11

Given that $3 + i$ is a root of the quartic equation $2z^4 - 3z^3 - 39z^2 + 120z - 50 = 0$, solve the equation completely.

Another root is $3 - i$.

So $(z - (3 + i))(z - (3 - i))$ is a factor of $2z^4 - 3z^3 - 39z^2 + 120z - 50$
$(z - (3 + i))(z - (3 - i)) = z^2 - z(3 - i) - z(3 + i) + (3 + i)(3 - i)$
$= z^2 - 6z + 10$

So $z^2 - 6z + 10$ is a factor of $2z^4 - 3z^3 - 39z^2 + 120z - 50$.
$(z^2 - 6z + 10)(az^2 + bz + c) = 2z^4 - 3z^3 - 39z^2 + 120z - 50$

Consider $2z^4$:
The only z^4 term in the expansion is $z^2 \times az^2$, so $a = 2$.
$(z^2 - 6z + 10)(2z^2 + bz + c) = 2z^4 - 3z^3 - 39z^2 + 120z - 50$

Consider $-3z^3$:
The z^3 terms in the expansion are $z^2 \times bz$ and $-6z \times 2z^2$,
so $bz^3 - 12z^3 = -3z^3$
$b - 12 = -3$
so $b = 9$
$(z^2 - 6z + 10)(2z^2 + 9z + c) = 2z^4 - 3z^3 - 39z^2 + 120z - 50$

Complex roots occur in conjugate pairs.

If α and β are roots of f(z) = 0, then $(z - \alpha)(z - \beta)$ is a factor of f(z).

You can work this out quickly by noting that $(z - (a + bi))(z - (a - bi)) = z^2 - 2az + a^2 + b^2$

Problem-solving It is possible to factorise a polynomial without using a formal algebraic method. Here, the polynomial is factorised by 'inspection'. By considering each term of the quartic separately, it is possible to work out the missing coefficients.

Consider −50:

The only constant term in the expansion is $10 \times c$, so $c = -5$.

$2z^4 - 3z^3 - 39z^2 + 120z - 50 = (z^2 - 6z + 10)(2z^2 + 9z - 5)$

Solving $2z^2 + 9z - 5 = 0$:

$(2z - 1)(z + 5) = 0$

$z = \frac{1}{2}, z = -5$

So the roots of $2z^4 - 3z^3 - 39z^2 + 120z - 50 = 0$ are $\frac{1}{2}$, -5, $3 + \text{i}$ and $3 - \text{i}$

You can check this by considering the z and z^2 terms in the expansion.

Example 12

Show that $z^2 + 4$ is a factor of $z^4 - 2z^3 + 21z^2 - 8z + 68$.

Hence solve the equation $z^4 - 2z^3 + 21z^2 - 8z + 68 = 0$.

Using long division:

$$\begin{array}{r l}
 & z^2 - 2z + 17 \\
z^2 + 4 \,) & z^4 - 2z^3 + 21z^2 - 8z + 68 \\
 & z^4 \qquad\quad + 4z^2 \\
 & \quad -2z^3 + 17z^2 - 8z \\
 & \quad -2z^3 \qquad\quad - 8z \\
 & \qquad\quad 17z^2 \qquad + 68 \\
 & \qquad\quad 17z^2 \qquad + 68 \\
 & \qquad\qquad\qquad\quad 0
\end{array}$$

So $z^4 - 2z^3 + 21z^2 - 8z + 68 = (z^2 + 4)(z^2 - 2z + 17) = 0$

Either $z^2 + 4 = 0$ or $z^2 - 2z + 17 = 0$

Solving $z^2 + 4 = 0$:

$z^2 = -4$

$x = \pm 2\text{i}$

Solving $z^2 - 2z + 17 = 0$:

$(z - 1)^2 + 16 = 0$

$(z - 1)^2 = -16$

$z - 1 = \pm 4\text{i}$

$z = 1 \pm 4\text{i}$

So the roots of $z^4 - 2z^3 + 21z^2 - 8z + 68 = 0$ are 2i, -2i, $1 + 4\text{i}$ and $1 - 4\text{i}$

Alternatively, the quartic can be factorized by inspection:

$z^4 - 2z^3 + 21z^2 - 8z + 68$
$= (z^2 + 4)(az^2 + bz + c)$

$a = 1$, as the leading coefficient is 1.

The only z^3 term is formed by $z^2 \times bz$ so $b = -2$.

The constant term is formed by $4 \times c$, so $4c = 68$, and $c = 17$.

Solve by completing the square. Alternatively, you could use the quadratic formula.

Watch out You could also use your calculator to solve $z^2 - 2z + 17 = 0$. You should still write down the equation you are solving, and both roots.

Exercise 1F

(E) **1** $f(z) = z^3 - 6z^2 + 21z - 26$

a Show that $f(2) = 0$. **(1 mark)**

b Hence solve $f(z) = 0$ completely. **(3 marks)**

(E) **2** $f(x) = 2z^3 + 5z^2 + 9z - 6$

a Show that $f\left(\frac{1}{2}\right) = 0$. **(1 mark)**

b Hence write $f(z)$ in the form $(2z - 1)(z^2 + bz + c)$, where b and c are real constants to be found. **(2 marks)**

c Use algebra to solve $f(z) = 0$ completely. **(2 marks)**

(E/P) **3** $g(x) = 2z^3 - 4z^2 - 5z - 3$

Given that $z = 3$ is a root of the equation $g(z) = 0$, solve $g(z) = 0$ completely. **(4 marks)**

(E) **4** $p(z) = z^3 + 4z^2 - 15z - 68$

Given that $z = -4 + \text{i}$ is a solution to the equation $p(z) = 0$,

a show that $z^2 + 8z + 17$ is a factor of $p(z)$ **(2 marks)**

b hence solve $p(z) = 0$ completely. **(2 marks)**

(E) **5** $f(z) = z^3 + 9z^2 + 33z + 25$

Given that $f(z) = (z + 1)(z^2 + az + b)$, where a and b are real constants,

a find the value of a and the value of b **(2 marks)**

b find the three roots of $f(z) = 0$ **(4 marks)**

c find the sum of the three roots of $f(z) = 0$. **(1 mark)**

(E/P) **6** $g(z) = z^3 - 12z^2 + cz + d = 0$, where $c, d \in \mathbb{R}$

Given that 6 and $3 + \text{i}$ are roots of the equation $g(z) = 0$,

a write down the other complex root of the equation **(1 mark)**

b find the value of c and the value of d. **(4 marks)**

(E/P) **7** $h(z) = 2z^3 + 3z^2 + 3z + 1$

Given that $2z + 1$ is a factor of $h(z)$, find the three roots of $h(z) = 0$. **(4 marks)**

(E/P) **8** $f(z) = z^3 - 6z^2 + 28z + k$

Given that $f(2) = 0$,

a find the value of k **(1 mark)**

b find the other two roots of the equation. **(4 marks)**

9 Find the four roots of the equation $z^4 - 16 = 0$.

(E) **10** $f(z) = z^4 - 12z^3 + 31z^2 + 108z - 360$

a Write $f(z)$ in the form $(z^2 - 9)(z^2 + bz + c)$, where b and c are real constants to be found. **(2 marks)**

b Hence find all the solutions to $f(z) = 0$. **(3 marks)**

P **11** $g(z) = z^4 + 2z^3 - z^2 + 38z + 130$

Given that $g(2 + 3i) = 0$, find all the roots of $g(z) = 0$.

E/P **12** $f(z) = z^4 - 10z^3 + 71z^2 + Qz + 442$, where Q is a real constant.

Given that $z = 2 - 3i$ is a root of the equation $f(z) = 0$,

a show that $z^2 - 6z + 34$ is a factor of $f(z)$ **(4 marks)**

b find the value of Q **(1 mark)**

c solve completely the equation $f(z) = 0$. **(2 marks)**

Challenge

Three of the roots of the equation $az^5 + bz^4 + cz^3 + dz^2 + ez + f = 0$ are -2, $2i$ and $1 + i$. Find the values of a, b, c, d, e and f.

Mixed exercise 1

1 Given that $z_1 = 8 - 3i$ and $z_2 = -2 + 4i$, find, in the form $a + bi$, where $a, b \in \mathbb{R}$,

a $z_1 + z_2$

b $3z_2$

c $6z_1 - z_2$

E/P **2** The equation $z^2 + bz + 14 = 0$, where $b \in \mathbb{R}$ has no real roots.

Find the range of possible values of b. **(3 marks)**

3 The solutions to the quadratic equation $z^2 - 6z + 12 = 0$ are z_1 and z_2.

Find z_1 and z_2, giving each answer in the form $a \pm i\sqrt{b}$.

E/P **4** By using the binomial expansion, or otherwise, show that $(1 + 2i)^5 = 41 - 38i$. **(3 marks)**

E **5** $f(z) = z^2 - 6z + 10$

Show that $z = 3 + i$ is a solution to $f(z) = 0$. **(2 marks)**

6 $z_1 = 4 + 2i$, $z_2 = -3 + i$

Express, in the form $a + bi$, where $a, b \in \mathbb{R}$,

a z_1^* **b** $z_1 z_2$ **c** $\dfrac{z_1}{z_2}$

7 Write $\dfrac{(7 - 2i)^2}{1 + i\sqrt{3}}$ in the form $x + iy$ where $x, y \in \mathbb{R}$.

E/P **8** Given that $\dfrac{4 - 7i}{z} = 3 + i$, find z in the form $a + bi$, where $a, b \in \mathbb{R}$. **(2 marks)**

9 $z = \dfrac{1}{2 + i}$

Express in the form $a + bi$, where $a, b \in \mathbb{R}$

a z^2

b $z - \dfrac{1}{z}$

E/P **10** Given that $z = a + bi$, show that $\dfrac{z}{z^*} = \left(\dfrac{a^2 - b^2}{a^2 + b^2}\right) + \left(\dfrac{2ab}{a^2 + b^2}\right)i$ **(4 marks)**

E/P **11** The complex number z is defined by $z = \dfrac{3 + qi}{q - 5i}$, where $q \in \mathbb{R}$.

Given that the real part of z is $\frac{1}{13}$,

a find the possible values of q **(4 marks)**

b write the possible values of z in the form $a + bi$, where a and b are real constants. **(1 mark)**

E/P **12** Given that $z = x + iy$, find the value of x and the value of y such that

$z + 4iz^* = -3 + 18i$

where z^* is the complex conjugate of z. **(5 marks)**

13 $z = 9 + 6i$, $w = 2 - 3i$

Express $\frac{z}{w}$ in the form $a + bi$, where a and b are real constants.

E/P **14** The complex number z is given by $z = \dfrac{q + 3i}{4 + qi}$ where q is an integer.

Express z in the form $a + bi$ where a and b are rational and are given in terms of q. **(4 marks)**

E **15** Given that $6 - 2i$ is one of the roots of a quadratic equation with real coefficients,

a write down the other root of the equation **(1 mark)**

b find the quadratic equation, giving your answer in the form $az^2 + bz + c = 0$ where a, b and c are real constants. **(2 marks)**

E/P **16** Given that $z = 4 - ki$ is a root of the equation $z^2 - 2mz + 52 = 0$, where k and m are positive real constants, find the value of k and the value of m. **(4 marks)**

E/P **17** $h(z) = z^3 - 11z + 20$

Given that $2 + i$ is a root of the equation $h(z) = 0$, solve $h(z) = 0$ completely. **(4 marks)**

E/P **18** $f(z) = z^3 + 6z + 20$

Given that $f(1 + 3i) = 0$, solve $f(z) = 0$ completely. **(4 marks)**

E/P **19** $f(z) = z^3 + 3z^2 + kz + 48$

Given that $f(4i) = 0$,

a find the value of k **(2 marks)**

b find the other two roots of the equation. **(3 marks)**

E **20** $f(z) = z^4 - z^3 - 16z^2 - 74z - 60$

a Write $f(z)$ in the form $(z^2 - 5z - 6)(z^2 + bz + c)$, where b and c are real constants to be found. **(2 marks)**

b Hence find all the solutions to $f(z) = 0$. **(3 marks)**

E/P **21** $g(z) = z^4 - 6z^3 + 19z^2 - 36z + 78$

Given that $g(3 - 2i) = 0$, find all the roots of $g(z) = 0$. **(4 marks)**

E/P **22** $f(z) = z^4 - 4z^3 - 3z^2 + pz + 16$

Given that $f(4) = 0$,

a find the value of p **(1 mark)**

b solve completely the equation $f(z) = 0$. **(5 marks)**

Challenge

a Explain why a cubic equation with real coefficients cannot have a repeated complex root.

b By means of an example, show that a quartic equation with real coefficients can have a repeated complex root.

Summary of key points

1 • $\mathrm{i} = \sqrt{-1}$ and $\mathrm{i}^2 = -1$

• An **imaginary number** is a number of the form $b\mathrm{i}$, where $b \in \mathbb{R}$.

• A **complex number** is written in the form $a + b\mathrm{i}$, where $a, b \in \mathbb{R}$.

2 • Complex numbers can be added or subtracted by adding or subtracting their real parts and adding or subtracting their imaginary parts.

• You can multiply a real number by a complex number by multiplying out the brackets in the usual way.

3 If $b^2 - 4ac < 0$ then the quadratic equation $ax^2 + bx + c = 0$ has two distinct complex roots.

4 For any complex number $z = a + b\mathrm{i}$, the **complex conjugate** of the number is defined as $z^* = a - b\mathrm{i}$.

5 For real numbers a, b and c, if the roots of the quadratic equation $az^2 + bz + c = 0$ are complex, then they occur as a conjugate pair.

6 If the roots of a quadratic equation are α and β, then you can write the equation as

$(z - \alpha)(z - \beta) = 0$ or $z^2 - (\alpha + \beta)z + \alpha\beta = 0$

7 If f(z) is a polynomial with real coefficients, and z_1 is a root of f(z) = 0, then $z_1{}^*$ is also a root of f(z) = 0.

8 An equation of the form $az^3 + bz^2 + cz + d = 0$ is called a cubic equation, and has three roots.

For a cubic equation with real coefficients, either

• all three roots are real, or

• one root is real and the other two roots form a complex conjugate pair.

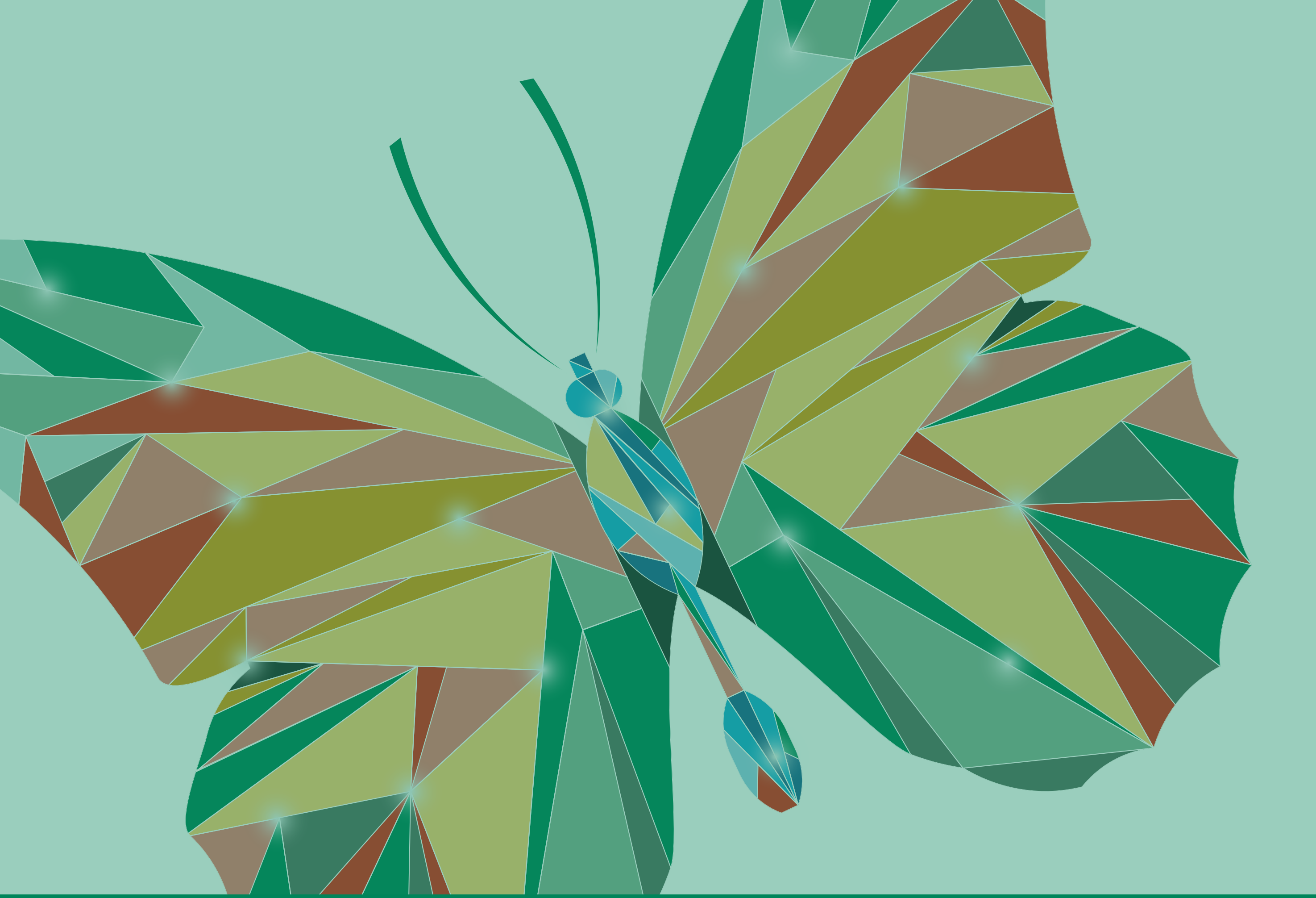
9 An equation of the form $az^4 + bz^3 + cz^2 + dz + e = 0$ is called a quartic equation, and has four roots.

For a quartic equation with real coefficients, either

• all four root are real, or

• two roots are real and the other two roots form a complex conjugate pair, or

• two roots form a complex conjugate pair and the other two roots also form a complex conjugate pair.



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