Name ______________________ Class ______________ Date ________

21.2 Solving Equations by Factoring $ax^2 + bx + c$

Essential Question: How can you use factoring to solve quadratic equations in standard form for which $a \neq 1$?

Resource Locker

Explore Factoring $ax^2 + bx + c$ When $c > 0$

When you factor a quadratic expression in standard form $(ax^2 + bx + c)$, you are looking for two binomials, and possibly a constant numerical factor whose product is the original quadratic expression.

Recall that the product of two binomials is found by applying the Distributive Property, abbreviated sometimes as FOIL:

$$(2x + 5)(3x + 2) = \underbrace{6x^2}_{\text{F}} + \underbrace{4x}_{\text{O}} + \underbrace{15x}_{\text{I}} + \underbrace{10}_{\text{L}} = 6x^2 + 19x + 10$$

F The product of the coefficients of the first terms is a.

O, I The sum of the coefficients of the outer and inner products is b.

L The product of the last terms is c.

Because the a and c coefficients result from a single product of terms from the binomials, the coefficients in the binomial factors will be a combination of the factors of a and c. The trick is to find the combination of factors that results in the correct value of b.

Follow the steps to factor the quadratic $4x^2 + 26x + 42$.

(A) First, factor out the largest common factor of 4, 26, and 42 if it is anything other than 1.

$4x^2 + 26x + 42 = \boxed{}(2x^2 + 13x + 21)$

(B) Next, list the factor pairs of 2:

____

(C) List the factor pairs of 21:

____

(D) Make a table listing the combinations of the factors of a and c, and find the value of b that results from summing the outer and inner products of the factors.

Factors of 2	Factors of 21	Outer + inner
1 and 2	1 and 21	$(1)(21) + (2)(1) = 23$
1 and 2	____ and 7	____________
1 and 2	7 and 3	____________
1 and 2	____ and 1	____________

Ⓔ Copy the pair of factors that resulted in an outer + inner sum of 13 into the binomial factors. Be careful to keep the inner and outer factors from the table as inner and outer coefficients in the binomials.

$2x^2 + 13x + 21 = (\square x + \square)(\square x + \square)$

Ⓕ Replace the common factor of the original coefficients to complete the factorization of the original quadratic.

$4x^2 + 26x + 42 = \square(x + 3)(2x + 7)$

Reflect

1. **Critical Thinking** Explain why you should use negative factors of c when factoring a quadratic with $c > 0$ and $b < 0$.

2. **What If?** If none of the factor pairs for a and c result in the correct value for b, what do you know about the quadratic?

3. **Discussion** Why did you have to check each factor pair twice for the factors of c (3 and 7 versus 7 and 3) but only once for the factors of a (1 and 2, but not 2 and 1)? Hint: Compare the outer and inner sums of rows two and three in the table, and also check the outer and inner sums by switching the order of both pairs from row 2 (check 2 and 1 for a with 7 and 3 for c).

Explain 1 Factoring $ax^2 + bx + c$ When $c < 0$

Factoring $x^2 + bx + c$ when $c < 0$ requires one negative and one positive factor of c. The same applies for expressions of the form $ax^2 + bx + c$ as long as $a > 0$. When checking factor pairs, remember to consider factors of c in both orders, **and** consider factor pairs with the negative sign on either member of the pair of c factors.

When you find a combination of factors whose outer and inner product sum is equal to b, you have found the solution. Make sure you fill in the factor table systematically so that you do not skip any combinations.

If $a < 0$, factor out -1 from all three coefficients, or use a negative common factor, so that the factors of a can be left as positive numbers.

Example 1 **Factor the quadratic by checking factor pairs.**

(A) $6x^2 - 21x - 45$

Find the largest common factor of 6, 21, and 45, and factor it out, keeping the coefficient of x^2 positive.

$6x^2 - 21x - 45 = 3(2x^2 - 7x - 15)$

Factors of *a*	Factors of *c*	Outer Product + Inner Product
1 and 2	1 and −15	$(1)(-15) + (2)(1) = -13$
1 and 2	3 and −5	$(1)(-5) + (2)(3) = 1$
1 and 2	5 and −3	$(1)(-3) + (2)(5) = 7$
1 and 2	15 and −1	$(1)(-1) + (2)(15) = 29$
1 and 2	−1 and 15	$(1)(15) + (2)(-1) = 13$
1 and 2	−3 and 5	$(1)(5) + (2)(-3) = -1$
1 and 2	−5 and 3	$(1)(3) + (2)(-5) = -7$
1 and 2	−15 and 1	$(1)(1) + (2)(-15) = -29$

Use the combination of factor pairs that results in a value of −7 for *b*.
$2x^2 - 7x - 15 = (x - 5)(2x + 3)$

Replace the common factor of the original coefficients to factor the original quadratic.

$6x^2 - 21x - 45 = 3(x - 5)(2x + 3)$

(B) $20x^2 - 40x - 25$

Factor out common factors of the terms.

$20x^2 - 40x - 25 = \square(4x^2 - 8x - 5)$

Factors of *a*	Factors of *c*	Outer Product + Inner Product
1 and 4	1 and − 5	$(1)(-5) + (4)(1) = \square$
1 and 4	5 and −1	$\square + \square = \square$
1 and 4	−1 and 5	$\square + \square = \square$
1 and 4	−5 and 1	$\square + \square = \square$
2 and 2	1 and −5	$\square + \square = \square$
2 and 2	−1 and 5	$\square + \square = \square$

Use the combination of factor pairs that results in a value of $\square$ for *b*.

$4x^2 - 8x - 5 = (\square x + \square)(\square x + \square)$

Replace the common factor of the original coefficients to factor the original quadratic.

$20x^2 - 40x - 25 = \square(2x + 1)(2x - 5)$

Reflect

4. **What If?** Suppose a is a negative number. What would be the first step in factoring $ax^2 + bx + c$?

Your Turn

5. Factor. $-5x^2 + 8x + 4$

Explain 2 Solving Equations of the Form $ax^2 + bx + c = 0$ by Factoring

For a quadratic equation in standard form, $ax^2 + bx + c = 0$, factoring the quadratic expression into binomials lets you use the Zero Product Property to solve the equation, as you have done previously. If the equation is not in standard form, convert it to standard form by moving all terms to one side of the equation and combining like terms.

Example 2 **Change the quadratic equation to standard form if necessary and then solve by factoring.**

(A) $2x^2 + 7x - 2 = 4x^2 + 4$

Convert the equation to standard form:

Subtract $4x^2$ and 4 from both sides. $-2x^2 + 7x - 6 = 0$

Multiply both sides by -1. $2x^2 - 7x + 6 = 0$

Consider factor pairs for 2 and 6. Use negative factors of 6 to get a negative value for b.

Use the combination pair that results in a sum of -7 and write the equation in factored form. Then solve it using the Zero Product Property.

$$(x - 2)(2x - 3) = 0$$

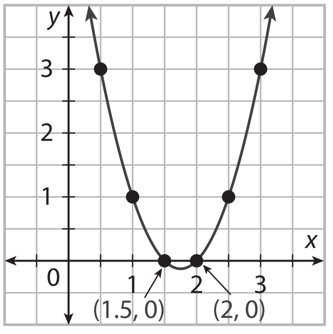
$$x - 2 = 0 \quad \text{or} \quad 2x - 3 = 0$$

$$x = 2 \qquad\qquad 2x = 3$$

$$x = \frac{3}{2} = 1.5$$

The solutions are 2 and $\frac{3}{2}$, or 1.5.

The solution can be checked by graphing the related function, $f(x) = 2x^2 - 7x + 6$, and finding the x-intercepts.



(B) $3(x^2 - 1) = -3x^2 + 2x + 5$

Write the equation in standard form and factor so you can apply the Zero Product Property.

$$\square x^2 - \square = -3x^2 + 2x + 5$$

$$\square x^2 - 2x - \square = 0$$

$$\square x^2 - x - 4 = 0$$

Use the combination pair that results in a sum of ____.

$$(x + 1)\left(\square x + \square\right) = 0$$

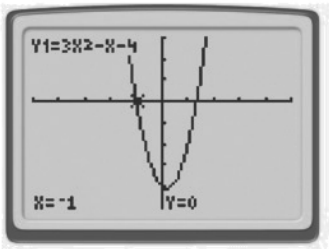
$$x + 1 = \square \quad \text{or} \quad 3x - 4 = 0$$

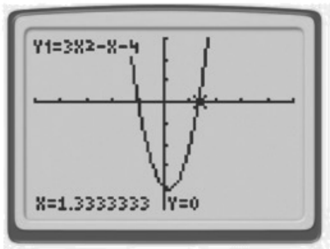
$$x = \square \qquad \square x = 4$$

$$x = \square$$

The solutions are -1 and $\frac{4}{3}$.

Use a graphing calculator to check the solutions.





Reflect

6. In the two examples, a common factor was divided out at the beginning of the solution, and it was not used again. Why didn't you include the common term again when solving x for the original quadratic equation?

Your Turn

7. $12x^2 + 48x + 45 = 0$

Explain 3 Solving Equation Models of the Form $ax^2 + bx + c = 0$ by Factoring

A projectile is an object moving through the air without any forces other than gravity acting on it. The height of a projectile at a time in seconds can be found by using the formula $h = -16t^2 + vt + s$, where v is in the initial upwards velocity in feet per second (and can be a negative number if the projectile is launched downwards) and s is starting height in feet. The a term of -16 accounts for the effect of gravity accelerating the projectile downwards and is the only appropriate value when measuring distance with feet and time in seconds.

To use the model to make predictions about the behavior of a projectile, you need to read the description of the situation carefully and identify the initial velocity, the initial height, and the height at time t.

Example 3 **Read the real-world situation and substitute in values for the projectile motion formula. Then solve the resulting quadratic equation by factoring to answer the question.**

(A) When a baseball player hits a baseball into the air, the height of the ball at t seconds after the ball is hit can be modeled with the projectile motion formula. If the ball is hit at 3 feet off the ground with an upward velocity of 47 feet per second, how long will it take for the ball to hit the ground, assuming it is not caught?

Use the equation $h = -16t^2 + vt + s$. Find the parameters v and s from the description of the problem.

$v = 47$ $\qquad$ $s = 3$ $\qquad$ $h = 0$

Substitute parameter values. $\qquad -16t^2 + 47t + 3 = 0$

Divide both sides by -1. $\qquad 16t^2 - 47t - 3 = 0$

Use the combination pair that results in a sum of -47.

$$(t - 3)(16t + 1) = 0$$

$$t - 3 = 0 \quad \text{or} \quad 16t + 1 = 0$$

$$t = 3 \qquad\qquad 16t = -1$$

$$t = -\frac{1}{16}$$

The solutions are 3 and $-\frac{1}{16}$.

The negative time answer can be rejected because it is not a reasonable value for time in this situation. The correct answer is 3 seconds.

(B) A child standing on a river bank ten feet above the river throws a rock toward the river at a speed of 12 feet per second. How long does it take before the rock splashes into the river?

Find the parameters v and s from the description of the problem.

$\square = -12$ $\quad s = \square$ $\quad h = \square$

Substitute parameter values. $\quad \square t^2 + \square t + \square = 0$

Divide both sides by $\square$. $\quad 8t^2 + \square t + \square = 0$

Use the combination pair that results in a sum of 6.

$$\left(\square t - 1\right)\left(\square t + 5\right) = 0$$

$$2t - 1 = 0 \quad \text{or} \quad 4t + 5 = 0$$

$$2t = \square \qquad 4t = \square$$

$$t = \square \qquad t = \square$$

The solutions are $\square$ and $\square$.

The only correct solution to the time it takes the rock to hit the water is $\square$ second.

Your Turn

8. How long does it take a rock to hit the ground if thrown off the edge of a 72-foot tall building roof with an upward velocity of 24 feet per second?

Elaborate

9. **Discussion** What happens if you do not remove the common factor from the coefficients before trying to factor the quadratic equation?

10. Explain how you can know there are never more than two solutions to a quadratic equation, based on what you know about the graph of a quadratic function.

11. **Essential Question Check-In** Describe the steps it takes to solve a quadratic equation by factoring.

Evaluate: Homework and Practice

Personal Math Trainer

- Online Homework
- Hints and Help
- Extra Practice

Factor the following quadratic expressions.

1. $6x^2 + 5x + 1$

2. $9x^2 + 33x + 30$

3. $4x^2 - 8x + 3$

4. $24x^2 - 44x + 12$

5. $3x^2 - 2x - 5$

6. $-10x^2 + 3x + 4$

7. $12x^2 + 22x - 14$

8. $-15x^2 + 21x + 18$

Solve the following quadratic equations.

9. $5x^2 + 18x + 9 = 0$

10. $12x^2 - 36x + 15 = 0$

11. $6x^2 + 28x - 2 = 2x - 10$

12. $-100x^2 + 55x + 3 = 50x^2 - 55x + 23$

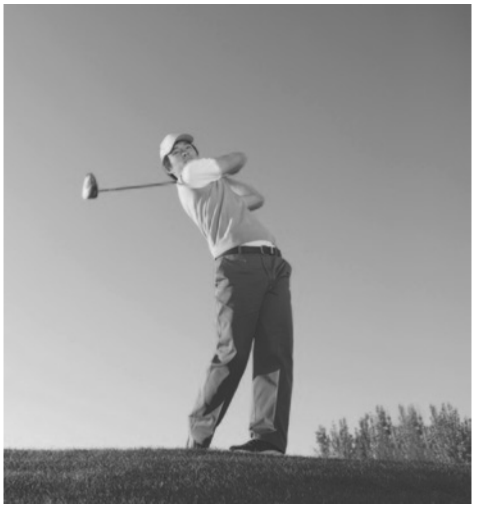
13. $8x^2 - 10x - 3 = 0$

14. $-12x^2 = 34x - 28$

15. $(8x + 7)(x + 1) = 9$

16. $3(4x - 1)(4x + 3) = 48x$

Read the real-world situation and substitute in values for the projectile motion formula. Then solve the resulting quadratic equation by factoring to answer the question.



17. A golfer takes a swing from a hill twenty feet above the cup with an initial upwards velocity of 32 feet per second. How long does it take the ball to land on the ground near the cup?

18. An airplane pilot jumps out of an airplane and has an initial velocity of 60 feet per second downwards. How long does it take to fall from 1000 feet to 900 feet before the parachute opens?

A race car driving under the caution flag at 80 feet per second begins to accelerate at a constant rate after the warning flag. The distance traveled since the warning flag in feet is characterized by $30t^2 + 80t$, where t is the time in seconds after the car starts accelerating again.

19. How long does it take the car to travel 30 feet after it begins accelerating?

20. How long will the car take to travel 160 feet?

Geometry **For each rectangle with area given, determine the binomial factors that describe the dimensions.**

21.

area $= 6x^2 + 17x - 3$

22.

area $= 21x^2 + 13x + 2$

23. **Multiple Response** Which of the following expressions in the list describes the complete factorization of the quadratic expression $15x^2 - 25x - 10$? Circle all that apply.

a. $(3x + 1)(5x - 10)$ **b.** $5(3x + 1)(x - 2)$ **c.** $5(x + 2)(3x - 1)$

d. $5(x - 2)(3x + 1)$ **e.** $5(3x - 1)(x + 2)$ **f.** $(5x - 10)(3x + 1)$

H.O.T. Focus on Higher Order Thinking

24. **Multi-Part Response** A basketball player shoots at the basket from a starting height of 6 feet and an upwards velocity of 20 feet per second. Determine how long it takes for the shot to drop through the basket, which is mounted at a height of 10 feet.

a. Set up the equation for projectile motion to solve for time and convert it to standard form.

b. Solve the equation by factoring.

c. Explain why you got two positive solutions to the equation, and determine how you can rule one of them out to find the answer to the question. Hint: Solving the equation graphically may give you a hint.

25. **Critical Thinking** Find the binomial factors of $4x^2 - 25$.

26. **Communicate Mathematical Ideas** Find all the values of b that make the expression $3x^2 + bx - 4$ factorable.

Lesson Performance Task

The equation for the motion of an object with constant acceleration is $d = d_0 + vt + \frac{1}{2}at^2$, where d is distance from a given point in meters, d_0 is the initial distance from the starting point in meters, v is the starting velocity in meters per second, a is acceleration in meters per second squared, and t is time in seconds.

A car is stopped at a traffic light. When the light turns green, the driver begins to drive, accelerating at a constant rate of 4 meters per second squared. A bus is traveling at a speed of 15 meters per second in another lane. The bus is 7 meters behind the car as it begins to accelerate.

Find when the bus passes the car, when the car passes the bus, and how far each has traveled each time they pass one another.