Name ______________________ Class ____________ Date ________

6.5 Dividing Polynomials

Essential Question: **What are some ways to divide polynomials, and how do you know when the divisor is a factor of the dividend?**

Resource Locker

Explore — Evaluating a Polynomial Function Using Synthetic Substitution

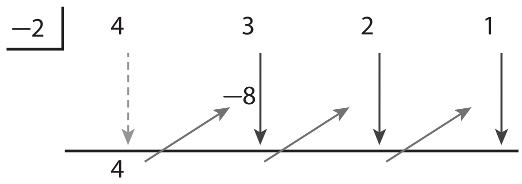
Polynomials can be written in something called nested form. A polynomial in nested form is written in such a way that evaluating it involves an alternating sequence of additions and multiplications. For instance, the nested form of $p(x) = 4x^3 + 3x^2 + 2x + 1$ is $p(x) = x\left(x(4x + 3) + 2\right) + 1$, which you evaluate by starting with 4, multiplying by the value of x, adding 3, multiplying by x, adding 2, multiplying by x, and adding 1.

(A) Given $p(x) = 4x^3 + 3x^2 + 2x + 1$, find $p(-2)$.

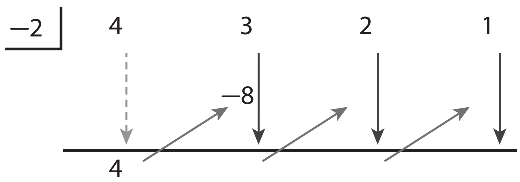
You can set up an array of numbers that captures the sequence of multiplications and additions needed to find $p(a)$. Using this array to find $p(a)$ is called **synthetic substitution**.

Given $p(x) = 4x^3 + 3x^2 + 2x + 1$, find $p(-2)$ by using synthetic substitution. The dashed arrow indicates bringing down, the diagonal arrows represent multiplication by –2, and the solid down arrows indicate adding.

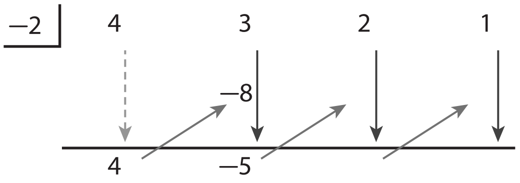
The first two steps are to bring down the leading number, 4, then multiply by the value you are evaluating at, -2.



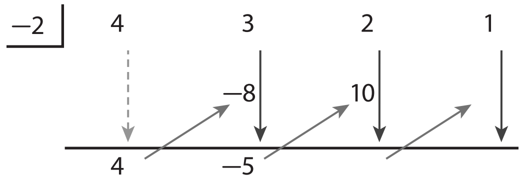
(B) Add 3 and –8.



(C) Multiply the previous answer by –2.



(D) Continue this sequence of steps until you reach the last addition.



(E) $p(-2) = \square$

Reflect

1. Discussion After the final addition, what does this sum correspond to?

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Explain 1 Dividing Polynomials Using Long Division

Recall that arithmetic long division proceeds as follows.

$$\begin{array}{rl} \text{Divisor} \quad 23 & \leftarrow \text{Quotient} \\ 12\overline{)\,277} & \leftarrow \text{Dividend} \\ \underline{24} & \\ 37 & \\ \underline{36} & \\ 1 & \leftarrow \text{Remainder} \end{array}$$

Notice that the long division leads to the result $\frac{dividend}{divisor} = quotient + \frac{remainder}{divisor}$. Using the numbers from above, the arithmetic long division leads to $\frac{277}{12} = 23 + \frac{1}{12}$. Multiplying through by the divisor yields the result $dividend = (divisor)(quotient) + remainder$. (This can be used as a means of checking your work.)

Example 1 **Given a polynomial divisor and dividend, use long division to find the quotient and remainder. Write the result in the form *dividend* = (*divisor*)(*quotient*) + *remainder* and then carry out the multiplication and addition as a check.**

(A) $(4x^3 + 2x^2 + 3x + 5) \div (x^2 + 3x + 1)$

Begin by writing the dividend in standard form, including terms with a coefficient of 0 (if any).

$4x^3 + 2x^2 + 3x + 5$

Write division in the same way as you would when dividing numbers.

$x^2 + 3x + 1\overline{)\,4x^3 + 2x^2 + 3x + 5}$

Find the value you need to multiply the divisor by so that the first term matches with the first term of the dividend. In this case, in order to get $4x^2$, we must multiply x^2 by $4x$. This will be the first term of the quotient.

$$\begin{array}{r|l} & 4x \\ \hline x^2 + 3x + 1 & 4x^3 + 2x^2 + 3x + 5 \end{array}$$

Next, multiply the divisor through by the term of the quotient you just found and subtract that value from the dividend. $(x^2 + 3x + 1)(4x) = 4x^3 + 12x^2 + 4x$, so subtract $4x^3 + 12x^2 + 4x$ from $4x^3 + 2x^2 + 3x + 5$.

$$\begin{array}{r|l} & 4x \\ \hline x^2 + 3x + 1 & 4x^3 + 2x^2 + 3x + 5 \\ & -(4x^3 + 12x^2 + 4x) \\ \hline & \quad -10x^2 - x + 5 \end{array}$$

Taking this difference as the new dividend, continue in this fashion until the largest term of the remaining dividend is of lower degree than the divisor.

$$\begin{array}{r|l} & 4x - 10 \\ \hline x^2 + 3x + 1 & 4x^3 + 2x^2 + 3x + 5 \\ & -(4x^3 + 12x^2 + 4x) \\ \hline & -10x^2 - x + 5 \\ & -(-10x^2 - 30x - 10) \\ \hline & \qquad\qquad 29x + 15 \end{array}$$

Since $29x + 5$ is of lower degree than $x^2 + 3x + 1$, stop. $29x + 15$ is the remainder.

Write the final answer.

$$4x^3 + 2x^2 + 3x + 5 = (x^2 + 3x + 1)(4x - 10) + 29x + 15$$

Check.

$$\begin{aligned} 4x^3 + 2x^2 + 3x + 5 &= (x^2 + 3x + 1)(4x - 10) + 29x + 15 \\ &= 4x^3 + 12x^2 + 4x - 10x^2 - 30x - 10 + 29x + 15 \\ &= 4x^3 + 2x^2 + 3x + 5 \end{aligned}$$

B $(6x^4 + 5x^3 + 2x + 8) \div (x^2 + 2x - 5)$

Write the dividend in standard form, including terms with a coefficient of 0.

Write the division in the same way as you would when dividing numbers.

$$\begin{array}{r|l} & \\ \hline x^2 + 2x - 5 & 6x^4 + 5x^3 + 0x^2 + 2x + 8 \end{array}$$

Divide.

$$
\begin{array}{r}
6x^2 - \square + \square \\
x^2 + 2x - 5 \overline{\big) 6x^4 + 5x^3 + 0x^2 + 2x + 8} \\
-\left(6x^4 + 12x^3 - 30x^2\right) \\
\hline
-7x^3 + 30x^2 + 2x \\
-\left(-7x^3 \quad \square \quad\right) \\
\hline
\square + 8 \\
-\left(\quad \square \quad\right) \\
\hline
\square
\end{array}
$$

Write the final answer.

$6x^4 + 5x^3 + 2x + 8 =$ $\square$

Check.

Reflect

2. How do you include the terms with coefficients of 0?

Your Turn

Use long division to find the quotient and remainder. Write the result in the form *dividend* $=$ (*divisor*)(*quotient*) $+$ *remainder* and then carry out a check.

3. $\left(15x^3 + 8x - 12\right) \div \left(3x^2 + 6x + 1\right)$

4. $(9x^4 + x^3 + 11x^2 - 4) \div (x^2 + 16)$

Explain 2 Dividing $p(x)$ by $x - a$ Using Synthetic Division

Compare long division with synthetic substitution. There are two important things to notice. The first is that $p(a)$ is equal to the remainder when $p(x)$ is divided by $x - a$. The second is that the numbers to the left of $p(a)$ in the bottom row of the synthetic substitution array give the coefficients of the quotient. For this reason, synthetic substitution is also called **synthetic division**.

Long Division	Synthetic Substitution
$\begin{array}{r} 3x^2 + 10x + 20 \\ x - 2 \overline{)\, 3x^3 + 4x^2 + 0x + 10} \\ -(3x^3 - 6x^2) \\ \hline 10x^2 + 0x \\ -(10x^2 - 20x) \\ \hline 20x + 10 \\ -20x - 40 \\ \hline 50 \end{array}$	$\begin{array}{r\|rrrr} 2 & 3 & 4 & 0 & 10 \\ & & 6 & 20 & 40 \\ \hline & 3 & 10 & 20 & 50 \end{array}$

Example 2 **Given a polynomial $p(x)$, use synthetic division to divide by $x - a$ and obtain the quotient and the (nonzero) remainder. Write the result in the form $p(x) = (x - a)(quotient) + p(a)$, then carry out the multiplication and addition as a check.**

Ⓐ $(7x^3 - 6x + 9) \div (x + 5)$

By inspection, $a = -5$. Write the coefficients and a in the synthetic division format.

$$\begin{array}{r|rrrr} -5 & 7 & 0 & -6 & 9 \\ & & & & \\ \hline & & & & \end{array}$$

Bring down the first coefficient. Then multiply and add for each column.

$$\begin{array}{r|rrrr} -5 & 7 & 0 & -6 & 9 \\ & & -35 & 175 & -845 \\ \hline & 7 & -35 & 169 & -836 \end{array}$$

Write the result, using the non-remainder entries of the bottom row as the coefficients.

$$(7x^3 - 6x + 9) = (x + 5)(7x^2 - 35x + 169) - 836$$

Check.

$$\begin{aligned} (7x^3 - 6x + 9) &= (x + 5)(7x^2 - 35x + 169) - 836 \\ &= 7x^3 - 35x^2 - 35x^2 - 175x + 169x + 845 - 836 \\ &= 7x^3 - 6x + 9 \end{aligned}$$

(B) $\left(4x^4 - 3x^2 + 7x + 2\right) \div \left(x - \frac{1}{2}\right)$

Find a. Then write the coefficients and a in the synthetic division format.

Find a = ☐

$$\underline{}\,|\; 4 \quad 0 \quad -3 \quad 7 \quad 2$$

Bring down the first coefficient. Then multiply and add for each column.

$$\underline{}\,|\; 4 \quad 0 \quad -3 \quad 7 \quad 2$$

$$4$$

Write the result.

$\left(4x^4 - 3x^2 + 7x + 2\right) =$ ☐

Check.

Reflect

5. Can you use synthetic division to divide a polynomial by $x^2 + 3$? Explain.

Your Turn

Given a polynomial $p(x)$, use synthetic division to divide by $x - a$ and obtain the quotient and the (nonzero) remainder. Write the result in the form $p(x) = (x - a)(quotient) + p(a)$. You may wish to perform a check.

6. $\left(2x^3 + 5x^2 - x + 7\right) \div (x - 2)$

7. $\left(6x^4 - 25x^3 - 3x + 5\right) \div \left(x + \frac{1}{3}\right)$

Explain 3 Using the Remainder Theorem and Factor Theorem

When $p(x)$ is divided by $x - a$, the result can be written in the form $p(x) = (x - a)q(x) + r$ where $q(x)$ is the quotient and r is a number. Substituting a for x in this equation gives $p(a) = (a - a)q(a) + r$. Since $a - a = 0$, this simplifies to $p(a) = r$. This is known as the **Remainder Theorem**.

If the remainder $p(a)$ in $p(x) = (x - a)q(x) + p(a)$ is 0, then $p(x) = (x - a)q(x)$, which tells you that $x - a$ is a factor of $p(x)$. Conversely, if $x - a$ is a factor of $p(x)$, then you can write $p(x)$ as $p(x) = (x - a)q(x)$, and when you divide $p(x)$ by $x - a$, you get the quotient $q(x)$ with a remainder of 0. These facts are known as the **Factor Theorem**.

Example 3 **Determine whether the given binomial is a factor of the polynomial $p(x)$. If so, find the remaining factors of $p(x)$.**

(A) $p(x) = x^3 + 3x^2 - 4x - 12;\ (x + 3)$

Use synthetic division.

−3	1	3	−4	−12
		−3	0	12
	1	0	−4	0

Since the remainder is 0, $x + 3$ is a factor.

Write $q(x)$ and then factor it.

$q(x) = x^2 - 4 = (x + 2)(x - 2)$

So, $p(x) = x^3 + 3x^2 - 4x - 12 = (x + 2)(x - 2)(x + 3)$.

(B) $p(x) = x^4 - 4x^3 - 6x^2 + 4x + 5;\ (x + 1)$

Use synthetic division.

−1	1	−4	−6	4	5
	1				

Since the remainder is ____, $(x + 1)$ ____ a factor. Write $q(x)$.

$q(x) =$ ____

Now factor $q(x)$ by grouping.

$q(x) =$ ____

$=$ ____

$=$ ____

$=$ ____

So, $p(x) = x^4 - 4x^3 - 6x^2 + 4x + 5 =$ ____.

Your Turn

Determine whether the given binomial is a factor of the polynomial $p(x)$. If it is, find the remaining factors of $p(x)$.

8. $p(x) = 2x^4 + 8x^3 + 2x + 8; (x + 4)$

9. $p(x) = 3x^3 - 2x + 5; (x - 1)$

Elaborate

10. Compare long division and synthetic division of polynomials.

11. How does knowing one linear factor of a polynomial help find the other factors?

12. What conditions must be met in order to use synthetic division?

13. **Essential Question Check-In** How do you know when the divisor is a factor of the dividend?

Evaluate: Homework and Practice

- Online Homework
- Hints and Help
- Extra Practice

Given $p(x)$, find $p(-3)$ by using synthetic substitution.

1. $p(x) = 8x^3 + 7x^2 + 2x + 4$

2. $p(x) = x^3 + 6x^2 + 7x - 25$

3. $p(x) = 2x^3 + 5x^2 - 3x$

4. $p(x) = -x^4 + 5x^3 - 8x + 45$

Given a polynomial divisor and dividend, use long division to find the quotient and remainder. Write the result in the form *dividend* = (*divisor*)(*quotient*) + *remainder*. You may wish to carry out a check.

5. $(18x^3 - 3x^2 + x - 1) \div (x^2 - 4)$

6. $(6x^4 + x^3 - 9x + 13) \div (x^2 + 8)$

7. $\left(x^4 + 6x - 2.5\right) \div \left(x^2 + 3x + 0.5\right)$

8. $\left(x^3 + 250x^2 + 100x\right) \div \left(\frac{1}{2}x^2 + 25x + 9\right)$

Given a polynomial $p(x)$, use synthetic division to divide by $x - a$ and obtain the quotient and the (nonzero) remainder. Write the result in the form $p(x) = (x - a)\left(\text{quotient}\right) + p(a)$. You may wish to carry out a check.

9. $\left(7x^3 - 4x^2 - 400x - 100\right) \div (x - 8)$

10. $\left(8x^4 - 28.5x^2 - 9x + 10\right) \div (x + 0.25)$

11. $\left(2.5x^3 + 6x^2 - 5.5x - 10\right) \div (x + 1)$

Determine whether the given binomial is a factor of the polynomial $p(x)$. If so, find the remaining factors of $p(x)$.

12. $p(x) = x^3 + 2x^2 - x - 2; (x + 2)$

13. $p(x) = 2x^4 + 6x^3 - 5x - 10; (x + 2)$

14. $p(x) = x^3 - 22x^2 + 157x - 360; (x - 8)$

15. $p(x) = 4x^3 - 12x^2 + 2x - 5; (x - 3)$

16. The volume of a rectangular prism whose dimensions are binomials with integer coefficients is modeled by the function $V(x) = x^3 - 8x^2 + 19x - 12$. Given that $x - 1$ and $x - 3$ are two of the dimensions, find the missing dimension of the prism.

17. Given that the height of a rectangular prism is $x + 2$ and the volume is $x^3 - x^2 - 6x$, write an expression that represents the area of the base of the prism.

18. **Physics** A Van de Graaff generator is a machine that produces very high voltages by using small, safe levels of electric current. One machine has a current that can be modeled by $l(t) = t + 2$, where $t > 0$ represents time in seconds. The power of the system can be modeled by $P(t) = 0.5t^3 + 6t^2 + 10t$. Write an expression that represents the voltage of the system. Recall that $V = \frac{P}{l}$.

19. **Geometry** The volume of a hexagonal pyramid is modeled by the function $V(x) = \frac{1}{3}x^3 + \frac{4}{3}x^2 + \frac{2}{3}x - \frac{1}{3}$. Given the height $x + 1$, use polynomial division to find an expression for the area of the base.

(Hint: For a pyramid, $V = \frac{1}{3}Bh$.)

20. **Explain the Error** Two students used synthetic division to divide $3x^3 - 2x - 8$ by $x - 2$. Determine which solution is correct. Find the error in the other solution.

A.	B.
$\begin{array}{r\|rrrr} 2 & 3 & 0 & -2 & -8 \\ & & 6 & 12 & 20 \\ \hline & 3 & 6 & 10 & 12 \end{array}$	$\begin{array}{r\|rrrr} -2 & 3 & 0 & -2 & -8 \\ & & -6 & 12 & -20 \\ \hline & 3 & -6 & 10 & -28 \end{array}$

H.O.T. Focus on Higher Order Thinking

21. Multi-Step Use synthetic division to divide $p(x) = 3x^3 - 11x^2 - 56x - 50$ by $(3x + 4)$. Then check the solution.

22. Critical Thinking The polynomial $ax^3 + bx^2 + cx + d$ is factored as $3(x - 2)(x + 3)(x - 4)$. What are the values of a and d? Explain.

23. Analyze Relationships Investigate whether the set of whole numbers, the set of integers, and the set of rational numbers are closed under each of the four basic operations. Then consider whether the set of polynomials in one variable is closed under the four basic operations, and determine whether polynomials are like whole numbers, integers, or rational numbers with respect to closure. Use the table to organize.

	Whole Numbers	Integers	Rational Numbers	Polynomials
Addition				
Subtraction				
Multiplication				
Division (by nonzero)				

Lesson Performance Task

The table gives the attendance data for all divisions of NCAA Women's Basketball.

NCAA Women's Basketball Attendance			
Season	Years since 2006–2007	Number of teams in all 3 divisions	Attendance (in thousands) for all 3 divisions
2006–2007	0	1003	10,878.3
2007–2008	1	1013	11,120.8
2008–2009	2	1032	11,160.3
2009–2010	3	1037	11,134.7
2010–2011	4	1048	11,160.0
2011–2012	5	1055	11,201.8

Enter the data from the second, third, and fourth columns of the table and perform linear regression on the data pairs (t, T) and cubic regression on the data pairs (t, A) where $t =$ years since the 2006–2007 season, $T =$ number of teams, and $A =$ attendance (in thousands).

Then create a model for the average attendance per team: $A_{avg}(t) = \frac{A(t)}{T(t)}$. Carry out the division to write $A_{avg}(t)$ in the form *quadratic quotient* $+ \frac{remainder}{T(t)}$.

Use an online computer algebra system to carry out the division of $A(t)$ by $T(t)$.