Cost Functions

[See Chap 10]

Definitions of Costs

- Economic costs include both implicit and explicit costs.
- Explicit costs include wages paid to employees and the costs of raw materials.
- Implicit costs include the opportunity cost of the entrepreneur and the capital used for production.

Economic Cost

- The economic cost of any input is its opportunity cost:
  - the remuneration the input would receive in its best alternative employment

Model

- Firm produces single output, q
- Firm has N inputs $\{z_1, \ldots z_N\}$.
- Production function $q = f(z_1, \ldots z_N)$
  - Monotone and quasi-concave.
- Prices of inputs $\{r_1, \ldots r_N\}$.
- Price of output p.

Firm's Payoffs

- Total costs for the firm are given by

$$\text{total costs} = C = r_1 z_1 + r_2 z_2$$

- Total revenue for the firm is given by

$$\text{total revenue} = pq = pf(z_1, z_2)$$

- Economic profits (π) are equal to

$$\pi = \text{total revenue - total cost}$$

$$\pi = pq - r_1 z_1 - r_2 z_2$$

$$\pi = pf(z_1, z_2) - r_1 z_1 - r_2 z_2$$

Firm's Problem

- We suppose the firm maximizes profits.
- One-step solution
  - Choose (q,z_1,z_2) to maximize π
- Two-step solution
  - Minimize costs for given output level.
  - Choose output to maximize revenue minus costs.
- We first analyze two-step method
  - Where do cost functions come from?

Cost Minimization Problem

Cost-Minimization Problem (CMP)

- The cost minimization problem is

$$\min\ r_1 z_1 + r_2 z_2 \qquad \text{s.t.} \qquad f(z_1, z_2) \geq q \ \ \text{and} \ \ z_1, z_2 \geq 0$$

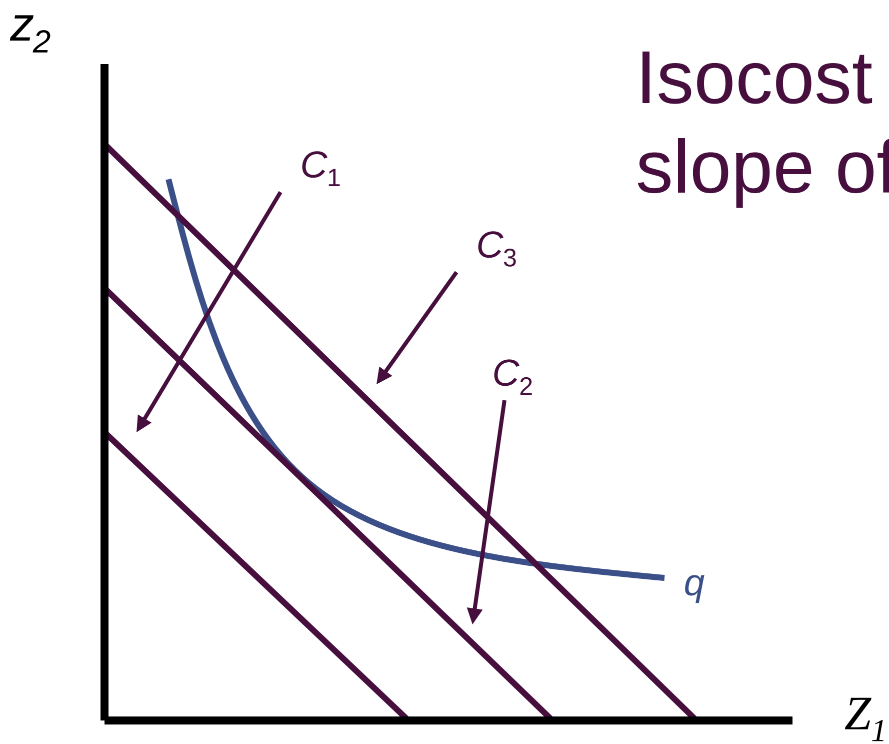
- Denote the optimal demands by $z_i^*(r_1,r_2,q)$
- Denote cost function by

$$C(r_1,r_2,q) = r_1 z_1^*(r_1,r_2,q) + r_2 z_2^*(r_1,r_2,q)$$

- Problem very similar to EMP.
- Output constraint binds if f(.) is monotone.

CMP: Graphical Solution

Given output q, we wish to find the lowest cost point on the isoquant

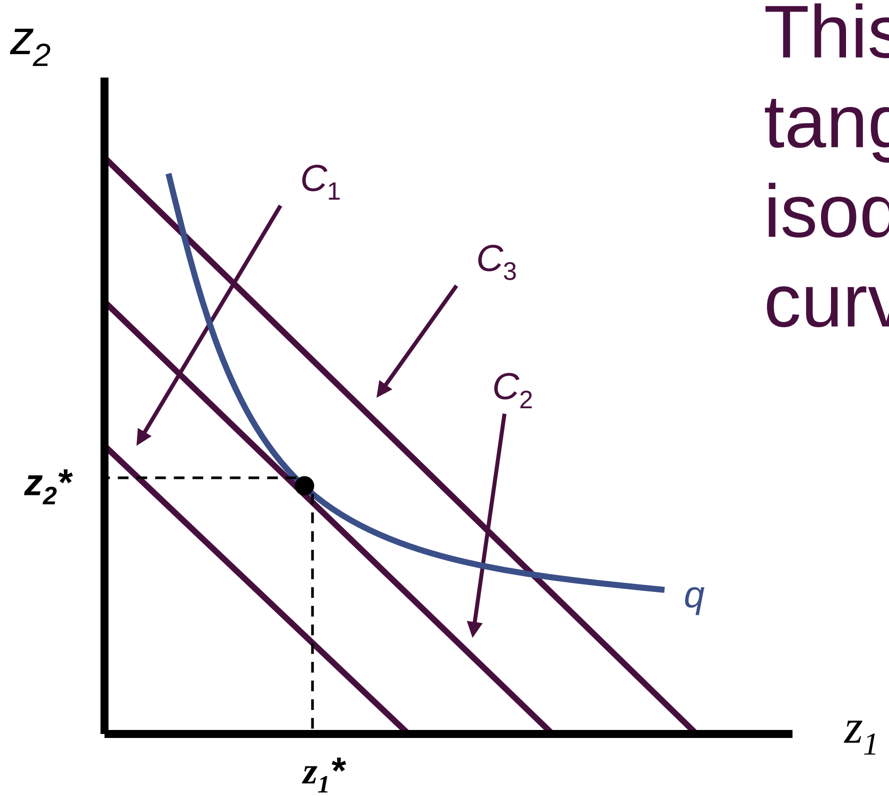


Isocost line are parallel with a slope of $-r_1/r_2$:

$C_1 < C_2 < C_3$

CMP: Graphical Solution

The minimum cost of producing q is C_2



This occurs at the tangency between the isoquant and the total cost curve

The optimal choice is (z_1*,z_2*)

CMP: Lagrangian Method

- Set up the Lagrangian:

$$\mathbf{L} = r_1 z_1 + r_2 z_1 + \lambda[q - f(z_1, z_2)]$$

- Find the first order conditions:

$$\partial \mathbf{L}/\partial z_1 = r_1 - \lambda(\partial f/\partial z_1) = 0$$

$$\partial \mathbf{L}/\partial z_2 = r_2 - \lambda(\partial f/\partial z_2) = 0$$

$$\partial \mathbf{L}/\partial \lambda = q - f(z_1, z_2) = 0$$

Cost-Minimizing Input Choices

- Dividing the first two conditions we get:

$$\frac{r_1}{r_2} = \frac{\partial f / \partial z_1}{\partial f / \partial z_2} = MRTS$$

- The cost-minimizing firm equates the *MRTS* for the two inputs to the ratio of their prices.
- Equivalently, the firm equates the bang-per-buck from each input

$$\frac{\partial f / \partial z_1}{r_1} = \frac{\partial f / \partial z_2}{r_2}$$

Interpretation of Multiplier

- Note that the first order conditions imply the following:

$$\frac{r_1}{f_1} = \frac{r_2}{f_2} = \lambda$$

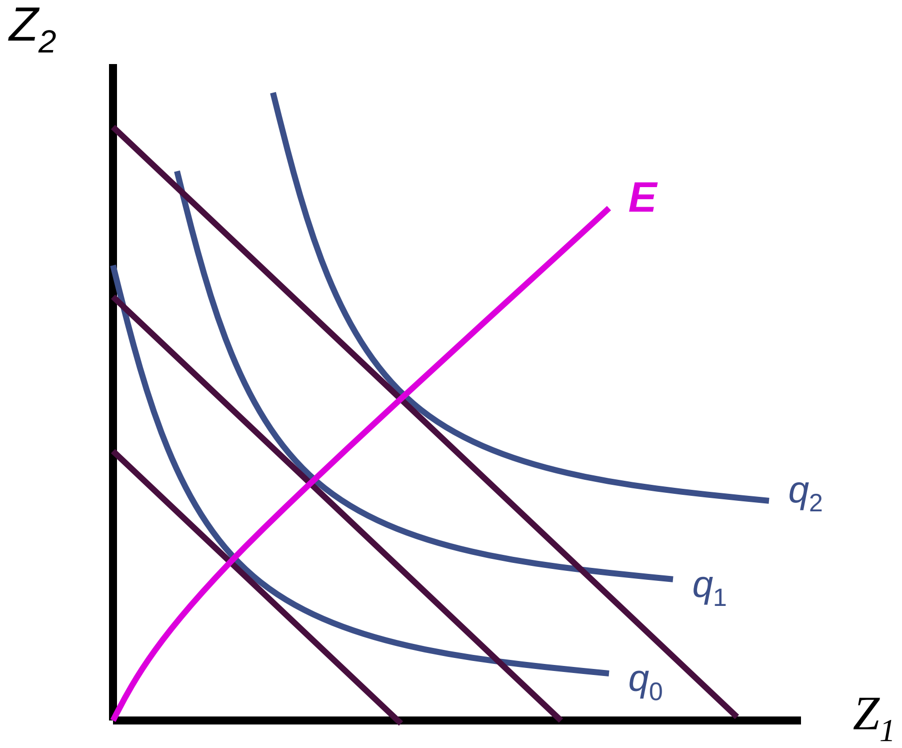
- The Lagrange multiplier describes how much total costs would increase if output q would increase by a small amount.

The Firm's Expansion Path

- The firm can determine the cost-minimizing combinations of z_1 and z_2 for every level of output
- The set of combinations of optimal amount of z_1 and z_2 is called the firm's <u>expansion path</u>.

The Firm’s Expansion Path

The expansion path is the locus of cost-minimizing tangencies



The curve shows how inputs increase as output increases

The Firm’s Expansion Path

- The expansion path does not have to be a straight line
  - the use of some inputs may increase faster than others as output expands
  - depends on the shape of the isoquants
- The expansion path does not have to be upward sloping.

Example: Symmetric CD

- Production function is symmetric cobb-douglas:

$$q = z_1^{\gamma} z_2^{\gamma}$$

- The Lagrangian for the CMP is

$$\mathbf{L} = r_1 z_1 + r_2 z_2 + \lambda[q - z_1^{\gamma} z_2^{\gamma}]$$

Example: Symmetric CD

- FOCs for a minimum:

$$\partial \mathbf{L}/\partial z_1 = r_1 - \lambda z_1^{(\gamma-1)} z_2^{\gamma} = 0$$

$$\partial \mathbf{L}/\partial z_2 = r_2 - \lambda z_1^{\gamma} z_2^{(\gamma-1)} = 0$$

- Rearranging yields $r_1 z_1 = r_2 z_2$.
- Using the constraint $q = z_1^{\gamma} z_2^{\gamma}$,

$$z_1^*(r_1, r_2, q) = \left(\frac{r_2}{r_1}\right)^{1/2} q^{1/2\gamma} \quad \text{and} \quad z_2^*(r_1, r_2, q) = \left(\frac{r_1}{r_2}\right)^{1/2} q^{1/2\gamma}$$

- Substituting, the cost is

$$c(r_1, r_2, q) = r_1 z_1^* + r_2 z_2^* = 2(r_1 r_2)^{1/2} q^{1/2\gamma}$$

Example: Perfect Complements

- Suppose

$$q = f(z_1, z_2) = \min(z_1,z_2)$$

- Production will occur at the vertex of the L-shaped isoquants, $z_1 = z_2$.
- Using constraint, $z_1 = z_2 = q$
- Hence cost function is

$$C(r_1,r_2,q) = r_1 z_1 + r_2 z_2 = (\mathrm{r}_1+\mathrm{r}_2)\mathrm{q}$$

COST FUNCTIONS

Total Cost Function

- The cost function shows the minimum cost incurred by the firm is

$$C(r_1,r_2,q) = r_1 z_1^*(r_1,r_2,q) + r_2 z_2^*(r_1,r_2,q)$$

- Cost is a function of output and input prices.
- When prices fixed, sometimes write C(q)

Average Cost Function

- The average cost function (*AC*) is found by computing total costs per unit of output

$$\text{average cost} = AC(r_1, r_2, q) = \frac{C(r_1, r_2, q)}{q}$$

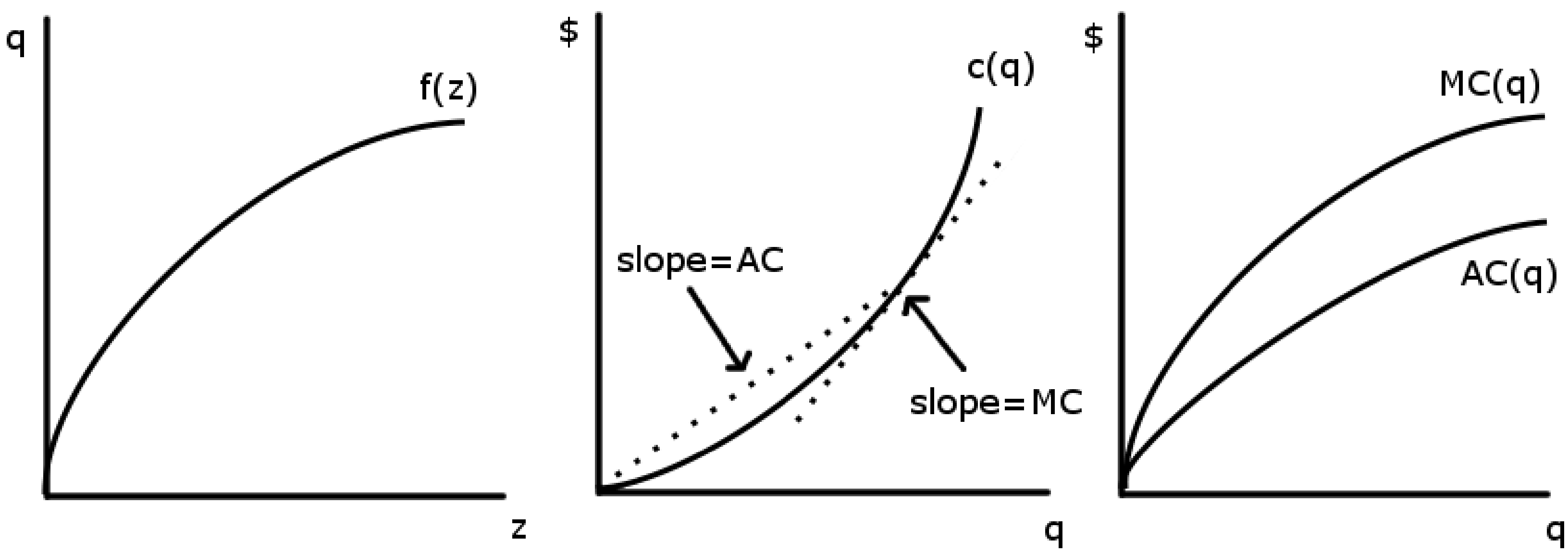
Marginal Cost Function

- The <u>marginal cost function</u> (*MC*) equals the extra cost from one extra unit of output.

$$\text{marginal cost} = MC(r_1, r_2, q) = \frac{\partial C(r_1, r_2, q)}{\partial q}$$

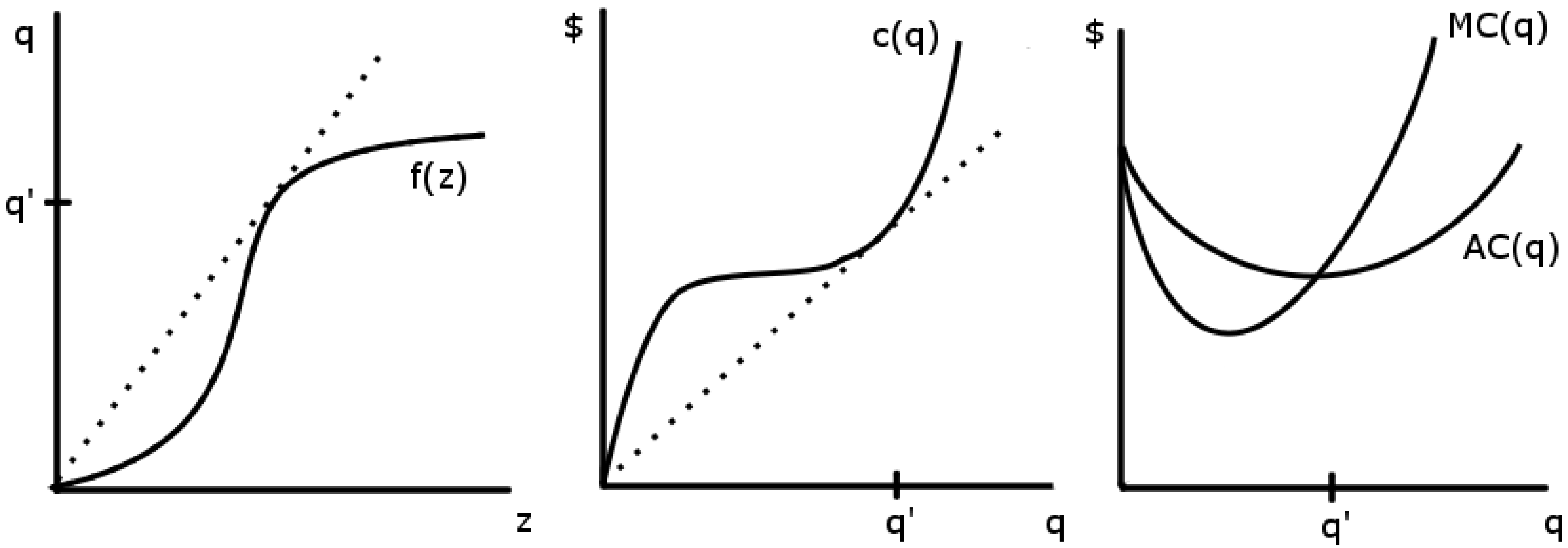
Picture #1

- Concave production function.



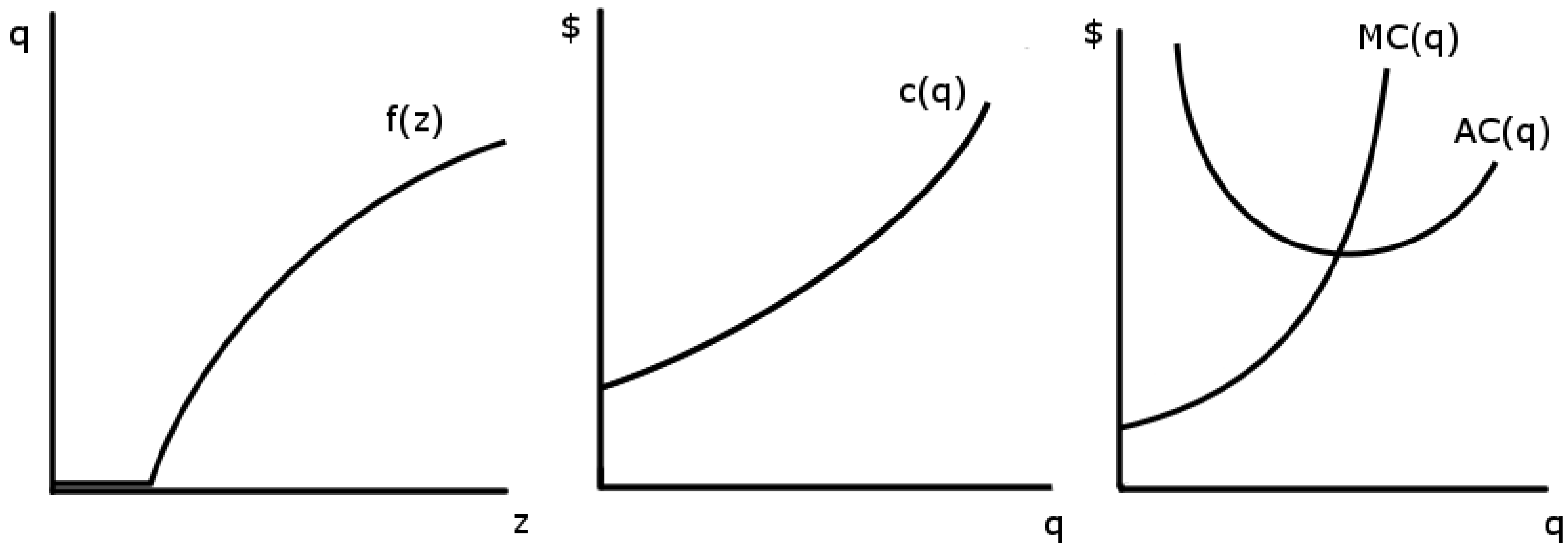
Picture #2

- Non-concave production function



Picture #3

- Non-concave production function.
- Fixed cost of production.



Cost Function: Properties

1. $c(r_1,r_2,\underline{q})$ is homogenous of degree 1 in (r_1,r_2)
   – If prices double constraint unchanged, so cost doubles.
2. $c(r_1,r_2,q)$ is increasing in (r_1,r_2,q)
3. Shepard's Lemma:

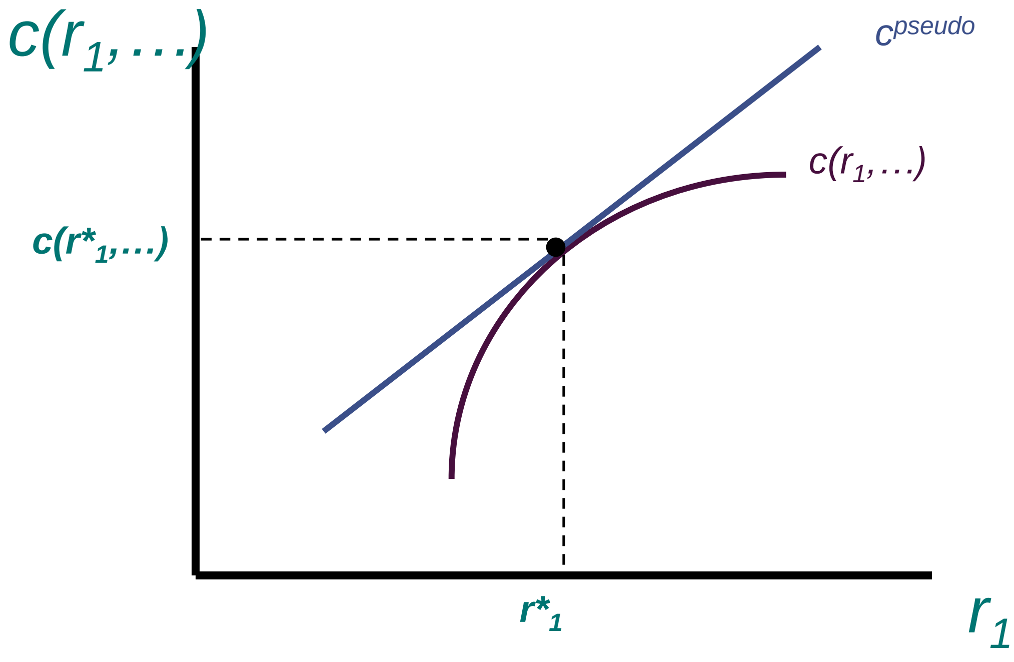
$$\frac{\partial}{\partial r_i} c(r_1, r_2, q) = z_i^*(r_1, r_2, q)$$

   – If r_1 rises by Δr, then c(.) rises by $\Delta r \times z^*_1(.)$
   – Input demand also changes, but effect second order.
4. $c(r_1,r_2,q)$ is concave in (r_1,r_2)

Cost Function: Concavity and Shepard’s Lemma

At r^*_1, the cost is

$c(r^*_1, \ldots) = r^*_1 z^*_1 + r^*_2 z^*_2$



If the firm continues to buy the same input mix as r_1 changes, its cost function would be C^{pseudo}

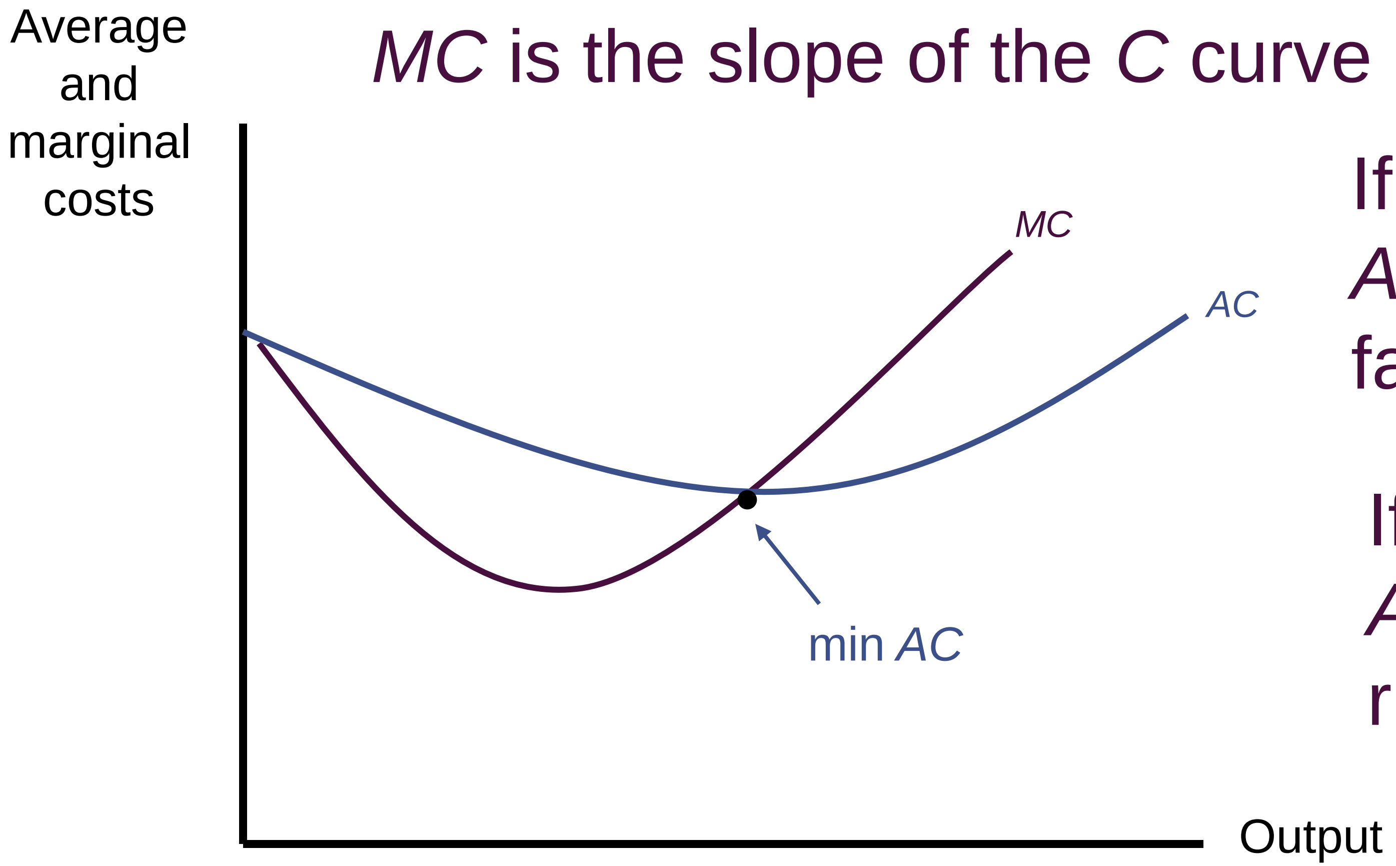
Since the firm’s input mix will likely change, actual costs will be less than C^{pseudo} such as $C(r_1, r_2, q)$

Cost Function: Properties

5. If $f(z_1,z_2)$ is concave then $c(r_1,r_2,q)$ is convex in q. Hence MC(q) increases in q.
   – Concavity implies decreasing returns.
   – More inputs needed for each unit of q, raising cost.
6. If $f(z_1,z_2)$ is exhibits decreasing (increasing) returns then AC(q) increases (decreases) in q.
   – Under DRS, doubling inputs produces less than double output. Hence average cost rises.
7. AC(q) is increasing when MC(q)≥AC(q), and decreasing when MC(q)≤AC(q).
   – If MC(q) ≥ AC(q) then cost being dragged up.
   – When AC(q) minimized, MC(q)=AC(q).

Average and Marginal Costs

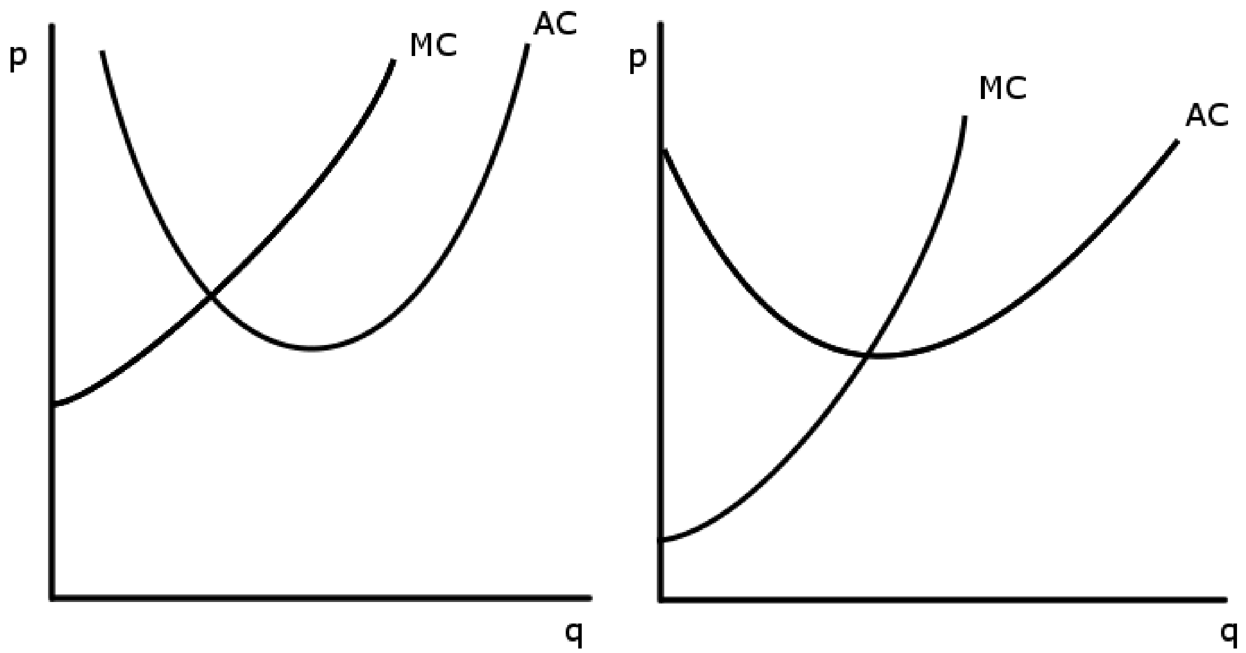
MC is the slope of the *C* curve



If *AC* > *MC*, *AC* must be falling

If *AC* < *MC*, *AC* must be rising

Can Costs Look Like This?



- Left: When AC minimized, MC=AC.
- Right: If no fixed costs AC=MC for first unit. If fixed costs, AC=∞ for first unit.

Input Demand: Properties

1. $z^*_i(r_1,r_2,q)$ is homogenous of degree 0 in (r_1,r_2)
   – If prices double constraint unchanged, so demand unchanged.
2. Symmetry of cross derivatives

$$\frac{\partial}{\partial r_2} z_1^* = \frac{\partial}{\partial r_2}\left[\frac{\partial}{\partial r_1} c\right] = \frac{\partial}{\partial r_1}\left[\frac{\partial}{\partial r_2} c\right] = \frac{\partial}{\partial r_1} z_2^*$$

   – Uses Shepard's Lemma
3. Law of demand

$$\frac{\partial}{\partial r_1} z_1^* = \frac{\partial}{\partial r_1}\left[\frac{\partial}{\partial r_1} c\right] \leq 0$$

   – Uses Shepard's Lemma and concavity of c(.)

SHORT-RUN VS. LONG-RUN

Short-Run, Long-Run Distinction

- Costs may differ in the short and long run.
- In the short run it is (relatively) easy to hire and fire workers but relatively difficult to change the level of the capital stock.
- Suppose firm wishes to raise production
  - Can't change capital stock
  - Hires more workers.
  - Capital/Labor balance no longer optimal.
  - High production costs.

Time Frames

- In *very short run*, all inputs are fixed.
- In *short run*, some inputs fixed with others are flexible.
- In *medium run*, all inputs are flexible but firm cannot enter/exit.
  - Fixed costs are sunk.
- In *long run*, all factor are flexible and firm can exit without cost.

Example: $f(z_1,z_2)=(z_1-1)^{1/3}(z_2-1)^{1/3}$

- Cobb-Douglas production but first unit of each input is useless.
- In long run,

$$L = r_1 z_1 + r_2 z_2 + \lambda[q - (z_1-1)^{1/3}(z_2-1)^{1/3}]$$

- FOC becomes $r_1(z_1-1)=r_2(z_2-1)$.
- Using constraint, demands are

$$z_1^*(r_1,r_2,q)=\left(\frac{r_2}{r_1}\right)^{1/2} q^{3/2}+1 \quad \text{and} \quad z_2^*(r_1,r_2,q)=\left(\frac{r_1}{r_2}\right)^{1/2} q^{3/2}+1$$

- Long-run cost function

$$c(r_1,r_2,q)=r_1 z_1^* + r_2 z_2^* = 2(r_1 r_2)^{1/2} q^{1/2\gamma} + (r_1+r_2)$$

with $c(r_1,r_2,0)=0$.

Example: $f(z_1,z_2)=(z_1-1)^{1/3}(z_2-1)^{1/3}$

- In medium run, startup cost of (r_1+r_2) is sunk.

- Cost function is thus

$$c(r_1,r_2,q)=r_1z_1^*+r_2z_2^*=2(r_1r_2)^{1/2}q^{1/2\gamma}+(r_1+r_2)$$

with $c(r_1,r_2,0) = r_1+r_2$.

Example: $f(z_1,z_2)=(z_1-1)^{1/3}(z_2-1)^{1/3}$

- In short run, z_2 is fixed at z_2'.
- The constraint in the CMP becomes

$$q = (z_1-1)^{1/3}(z_2'-1)^{1/3}$$

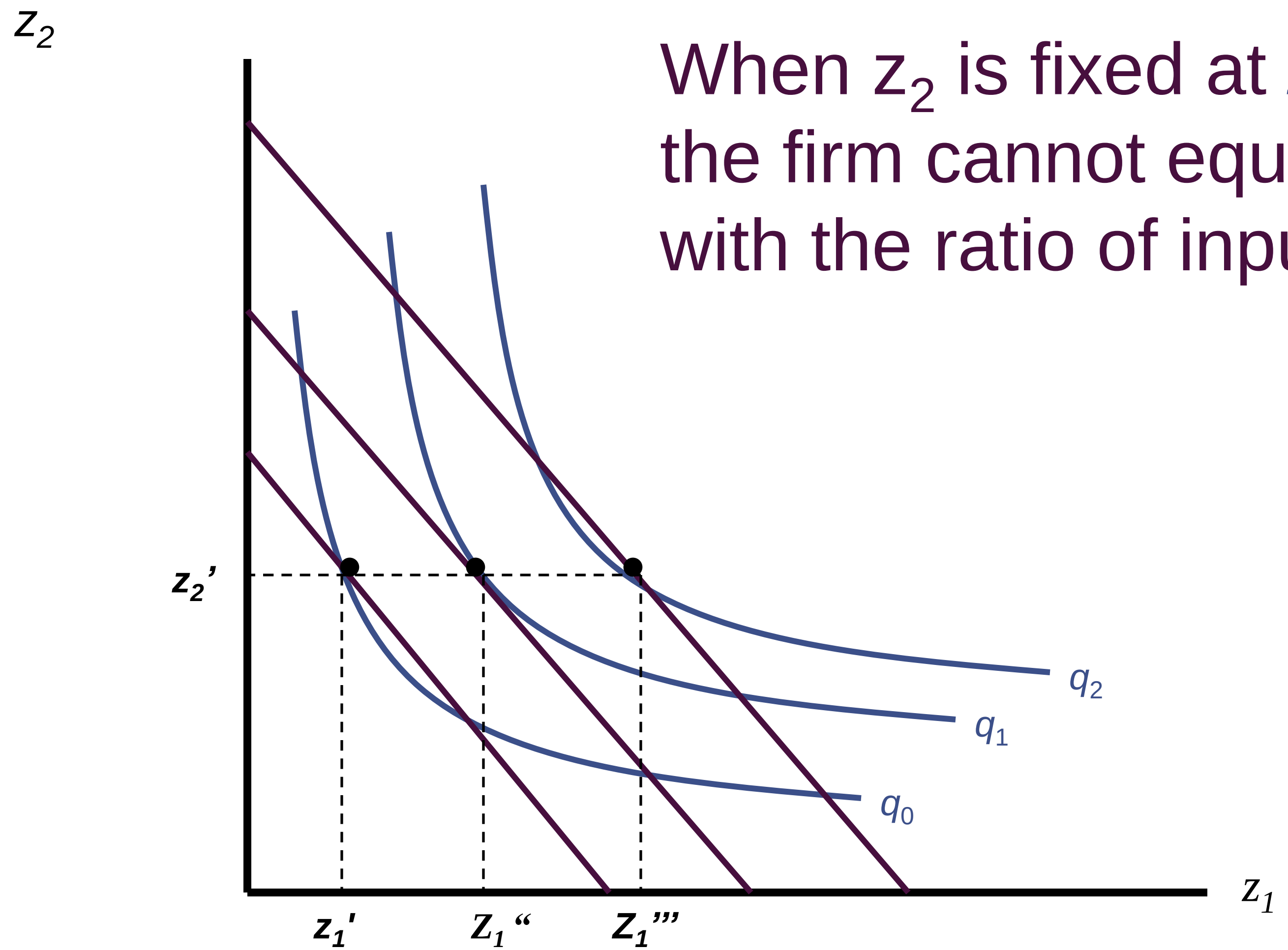
- Rearranging, $$z_1^* = \frac{q^3}{z_2'-1} + 1$$
- Cost function is

$$c(r_1, r_2, q) = r_1 z_1^* + r_2 z_2' = r_1 \frac{q^3}{z_2'-1} + r_1 + r_2 z_2'$$

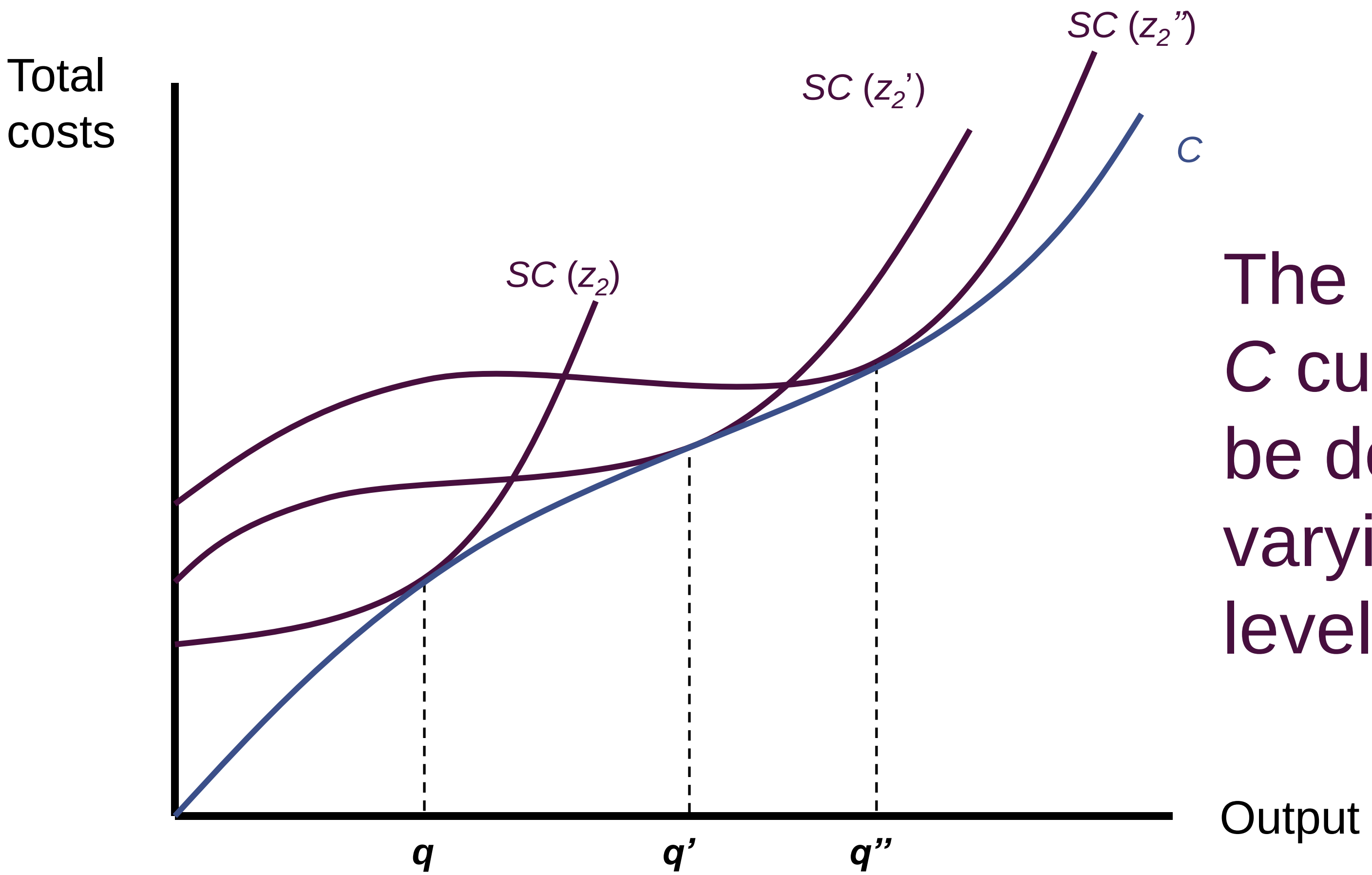
- In very short run, (z_1,z_2) fixed so output fixed.

Short-Run Total Costs

When z_2 is fixed at z_2', the firm cannot equate M*RTS* with the ratio of input prices



Relationship between Short-Run and Long-Run Costs



The long-run C curve can be derived by varying the level of z_2

Short-Run Marginal and Average Costs

- The short-run average total cost (*SAC*) function is

$$SAC = \text{total costs/total output} = SC/q$$

- The short-run marginal cost (*SMC*) function is

$$SMC = \text{change in } SC\text{/change in output} = \partial SC/\partial q$$