Costly entry and the optimality of asymmetric auction designs

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Abstract

We investigate optimal auction mechanisms when bidders base costly entry decisions on their valuations, and payments depend on both the bids and asset payoffs generated by the winning bidder. We show the optimal mechanism can feature asymmetry, where the seller sets differential reserve prices so that bidders enter with strictly positive but different (ex-ante) probabilities, even when bidders are ex-ante identical. The optimality of asymmetric mechanisms extends to cash auctions when there is sufficient valuation uncertainty relative to entry costs. When bidders pay with a fixed royalty rate plus cash, the optimal degree of asymmetry rises with the royalty rate.

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1 Introduction

This paper investigates optimal selling mechanisms in auctions with entry costs. We show that the optimal mechanism can feature asymmetry, where the seller sets differential reserve prices so that bidders enter with strictly positive but different (ex-ante) probabilities, even when bidders are ex-ante identical.

To make the driving forces transparent, we analyze settings with ex-ante identical bidders who have independent and private valuations for the asset, and know their valuations before incurring entry costs, as in Samuelson (1985) or Sogo, Bernhardt, and Liu (2016).[1] The qualitative nature of the optimality of asymmetric designs is robust to the payment method, holding both when bidders pay with cash or with securities whose values depend on their bids as well as the contractible ultimate (stochastic) payoff of the asset won by the winning bidder.[2]

To highlight the basic insights and tradeoffs, we first consider a scenario in which a seller can ask bidders to pay with any security. This simplifies analysis because a seller can extract all bidder rents with cash and a large fixed equity share, as in Cremer (1987). The optimal mechanism trades off between the increased rents that more entrants can bring versus the higher total entry costs incurred by more bidders.

We start by considering modest valuation uncertainty for bidders. Then, the welfare gains from greater selection are small, but the probability that the asset is not sold falls with the number of potential bidders in a symmetric auction setting in which all bidders have the same entry cutoff.[3] We provide sufficient conditions under which the seller does best to restrict entry to a single bidder, setting a take-it-or-leave-it price.

More typically, uncertainty over bidder valuations is more extensive, and one must extend the analysis to allow for asymmetric mechanisms in which entering bidders may face

[1] In the appendix we investigate auction designs in which bidders do not know their valuations before incurring entry costs.

[2] For example, Andrade, Mitchell, and Stafford (2001) report that 58% of mergers and acquisitions are paid entirely in equity, and 70% involve at least some equity; Skrzypacz (2013) reports oil and gas lease auctions typically feature equity payments in the form of royalties; and venture capital financing, procurement auctions, and lead-plaintiff auctions also often use security payments.

[3] For symmetric mechanisms of cash auctions in which entering bidders have the same cutoff valuation, Samuelson (1985) was the first to note that a seller can gain by restricting entry.

different reserves. Given higher—and more plausible—levels of bidder valuation uncertainty relative to entry costs, we identify mild conditions in a two-bidder setting under which it is optimal to use an asymmetric auction that favors one bidder over another, so that bidders enter with strictly positive but different ex-ante probabilities. With enough uncertainty over bidder valuations, a seller wants to encourage entry of multiple bidders when their valuations are high. At the same time, to reduce the risk of a no-trade outcome, the seller does better with an asymmetric auction that favors one bidder over another.

The optimal cutoffs balance the gains from selling to a higher valuation bidder with the costs of a no-trade outcome. To encourage entry by a bidder with a low valuation, the seller sets a higher reserve for the other bidder. This preserves a high probability of trade, while obtaining efficient allocations when the handicapped bidder has a high valuation. Due to the rival's handicap, the bidder facing a low reserve remains willing to enter even with a low valuation, understanding that the probability of competition is not that high. Specializing to uniformly-distributed valuations we identify conditions on the extent of bidder valuation uncertainty under which in the optimal auction design, one bidder enters even with the lowest valuation possible, eliminating the possibility of a no-trade outcome, and the other bidder is not excluded but enters only when its valuation is high enough.

We then consider auctions featuring any fixed royalty rate plus cash,[4] including pure cash auctions. Challenges arise in evaluating the relative merits of symmetric and asymmetric mechanisms because bidders earn strictly positive rents, and these rents differ between these two types of mechanisms. Nonetheless, the optimality of asymmetric auction designs extends under mild additional structure, and we show that the attraction of asymmetric auction designs rises with the royalty rate. The logic is as follows: spreading entry thresholds raises total bidder profits at a seller's expense, and greater tying via a higher royalty rate reduces bidder profit in the optimal design, reflecting the intuition from Demarzo, Kremer and Skrzypacz (2005) that steeper securities extract more rents from bidders with high valuations. An application of the envelope theorem reveals that higher royalty rates differentially reduce the profit of a bidder who faces a lower reserve when its valuation is high. It follows that higher

[4]Gorbenko and Malenko (2011) and Skrzypacz (2013) highlight the extensive use of such payment schemes.

royalty rates reduce the value attached by a seller to competition by another bidder, making it optimal to set a higher reserve for that bidder, i.e., to increase the degree of asymmetry.[5]

Our analysis provides theoretical foundations for 'poison pills' or shareholder rights plans that favor one bidder over another, and for hostile takeover offers that seek to circumvent a board's resistance to an offer that may exceed other offers, but is not sufficiently high. Such resistance is often interpreted as indicative of managerial entrenchment. We show that such asymmetric auction designs can be optimal, even when bidders are ex-ante symmetric and management only cares about maximizing expected revenues from being sold.

Our analysis also provides grounding for the favoring of one supplier in procurement over another. For example, the US government provides preferential treatment to particular bidders (e.g., women, minorities, disabled veterans, domestic firms, small businesses). In a setting with ex-ante *asymmetric* bidders, McAfee and McMillan (1989) show that favoring ex-ante weaker bidders can enhance auction revenues. Ayres and Cramton (1996) provide evidence of this in auctions for paging licenses by the FCC, in which bid preference is given by subsidizing the winning bids of favored bidders by a fixed rate.[6] Krasnokutskaya and Seim (2011) estimate a costly auction entry model in which bidders are ex-ante uninformed of their valuations. Their estimates indicate that incorporating bidders' endogenous entry decisions in response to bid preference policies significantly changes procurement costs, highlighting the importance of entry decisions in auctions. We take on this theoretical challenge: our endogenous entry model reveals the revenue-enhancing effect of bid preference policies even when bidders are ex-ante *symmetric*, but know their private valuations.

We contribute to research on the optimal design of security auctions without entry costs, and research on standard (symmetric) auction designs with entry costs. Absent entry costs, Cremer (1987) shows that optimal securities auctions extract almost all surplus; and De-

[5]The asymmetry refers to direct-mechanisms, where, in the analogous costless entry setting, Myerson (1981) shows the optimal direct-mechanism is necessarily symmetric when bidders are ex-ante identical. Deb and Pai (2017) show in a no entry cost setting that if one maintains interim individual rationality, but relaxes ex-post individual rationality so a winning bidder can pay more than the asset's expected value, then an asymmetric direct-mechanism can almost always be implemented via a symmetric indirect-mechanism in which payments hinge on the bids of all bidders.

[6]In the unknown valuations setting, Athey, Coey, and Levin (2013) and Nakabayashi (2013) empirically investigate another common method of preferential treatment—setting aside some contracts for certain bidders.

Marzo, Kremer and Skrzypacz (2005) show that if a seller restricts bids to an ordered set of securities and uses a standard auction format, then steeper securities yield higher revenues. Liu (2016) identifies the optimal mechanism when bidders are heterogeneous and pay with equities, generalizing Myerson (1981) from cash auctions to equity auctions. Skrzypacz (2013) reviews the security-bid auction literature. Our contribution is to identify optimal mechanisms when bidders bid with securities and incur entry costs, so that entry is endogenous.

Fishman (1988) studies takeover contests in which an acquirer faces a potential rival that must incur a cost to learn its target firm valuation, showing that a high-valuation acquirer may offer the target a high price to pre-emptively discourage a rival from becoming informed. Marquez and Singh (2013) investigate club bidding in private equities, and how entry costs affect club formation and seller profits. Gorbenko and Malenko (2011) endogenize competition between sellers in the design of security-bid auctions when bidders learn valuations after incurring entry costs. Sogo, Bernhardt, and Liu (2016) examine entry decisions in security-bid auctions when bidders know their valuations prior to entering. These papers study standard auction formats with entry costs. By contrast we analyze optimal auction mechanisms, optimizing over the entire space of symmetric and asymmetric mechanisms.

2 Model

There is a risk-neutral seller and $n \geq 1$ ex-ante identical risk-neutral potential bidders. The indivisible asset being auctioned has a value of zero if the seller retains it. A bidder incurs cost $\phi > 0$ from entering the auction. For the asset to pay off, the auction winner must invest $X > 0$. If bidder i acquires the asset and invests X, then it will yield a stochastic payoff of Z_i.

At date 0, each potential bidder i receives a private signal Θ_i of the incremental value of the asset if he wins it. If acquired by bidder i with type $\Theta_i = \theta$, the asset is expected to pay

$$E(Z_i|\Theta_i = \theta) = X + \theta.$$

The expected value added by this bidder type is $E(Z_i|\Theta_i = \theta) - X = \theta$. Signals are distributed i.i.d. according to a distribution with cdf $F(\theta)$ (and pdf $f(\theta)$), where $f(\theta) > 0$ is differentiable for $\theta \in [\underline{\theta}, \overline{\theta}]$. For simplicity, we assume $0 < \phi < \underline{\theta}$.

At date 1, after receiving signals, potential bidders simultaneously decide whether to enter the auction. We consider a general, possibly asymmetric, mechanism that involves security payments. Given the bids and identities of entering bidders, the auction specifies the winning bidder i and its payment, which can depend on the ultimate cash flow Z_i. In equilibrium, for each bidder i, we assume that only $\theta_i \geq \hat{\theta}_i$ participate for some $\hat{\theta}_i \in [\underline{\theta}, \overline{\theta}]$. Thus, bidder i participates with (ex-ante) probability $1 - F(\hat{\theta}_i)$, where $\hat{\theta}_i$ can differ across bidders. Focusing on mechanisms in which a bidder enters only if its valuation exceeds a threshold is without loss of generality, because for a given probability of entry by a potential bidder (and hence given expected entry costs), social welfare is maximized when a bidder enters if and only if its private valuation is sufficiently high; and this is a property of any optimal mechanism.

To highlight tradeoffs associated with greater entry, in Section 3, we first identify the optimal symmetric mechanism when the seller can require bidders to pay with any form of securities. We show that with limited uncertainty over valuations, a seller does best to restrict entry to a single bidder. In Section 4, we establish that with more substantial valuation uncertainty, the optimal mechanism is asymmetric: bidders enter with positive, but heterogeneous, ex-ante probabilities. Section 5 shows that the optimality of asymmetric mechanisms extends to auctions featuring any fixed royalty rate plus cash, including the special case of pure cash auctions.

3 Optimal Symmetric Mechanism

In a symmetric mechanism, $\hat{\theta}_i = \hat{\theta}$ for all i. With entry costs, paying for entry is costly and duplicative: only the winner incurs investment costs, but the entry costs of all bidders save the winner are wasted. Thus, the optimal mechanism trades off between the increased rents that more entrants can bring versus the higher total entry costs incurred by more bidders that a seller indirectly bears via the entry choices of bidders.

Let θ_n^1 be the highest type among n potential bidders and $Q_n(\theta_n^1) \equiv F^n(\theta_n^1)$ be its distribution. Then the seller's expected payoff can be written as

$$\Pi(\hat{\theta}) = \int_{\hat{\theta}}^{\bar{\theta}} \theta_n^1 dQ_n \left(\theta_n^1\right) - n\phi(1 - F(\hat{\theta})) - n\pi_b, \tag{1}$$

where the first term equals social welfare, the second term represents expected total entry costs, and π_b denotes the ex-ante expected profit of each bidder. Because expected bidder profit is nonnegative, (1) readily yields:

Lemma 1 *In any symmetric mechanism in which only potential bidders with types exceeding $\hat{\theta}$ enter, a seller's expected profit cannot exceed*

$$\Pi^*(\hat{\theta}) \equiv \int_{\hat{\theta}}^{\bar{\theta}} \theta_n^1 dQ_n \left(\theta_n^1\right) - n\phi(1 - F(\hat{\theta})). \tag{2}$$

Equation (2) yields via first-order condition that the upper bound on a seller's expected payoff $\Pi^*(\hat{\theta})$ is maximized when the threshold bidder type $\hat{\theta}^{opt}$ solves

$$\begin{cases} \hat{\theta}^{opt} F^{n-1}(\hat{\theta}^{opt}) = \phi & \text{if} \quad n \geq 2 \\ \hat{\theta}^{opt} = \underline{\theta} & \text{if} \quad n = 1. \end{cases} \tag{3}$$

This $\hat{\theta}^{opt}$ and the associated upper bound on a seller's expected profit $\Pi^*(\hat{\theta}^{opt})$ are attainable:

Lemma 2 *Suppose the auction winner pays a fixed royalty rate $\alpha \in [0, 1)$ plus cash, where the highest cash bid wins and pays the second highest bid. Let the cash reserve price be*

$$(1 - \alpha)\left(X + \hat{\theta}^{opt}\right) - X - \frac{\phi}{F^{n-1}(\hat{\theta}^{opt})}, \tag{4}$$

where $\hat{\theta}^{opt}$ is given by (3). Then as $\alpha \to 1$, expected revenue approaches $\Pi^*(\hat{\theta}^{opt})$.

Proof: We first show that in the equilibrium, a bidder enters if and only if its valuation exceeds $\hat{\theta}^{opt}$. Observe that for any $\alpha \in [0, 1)$, and both $n = 1$ and $n > 1$, with the reserve in (4), a bidder with $\hat{\theta}^{opt}$ earns zero expected profit if it enters. Types above $\hat{\theta}^{opt}$ strictly prefer to enter, while those below $\hat{\theta}^{opt}$ strictly prefer to not enter. Furthermore, as $\alpha \to 1$, expected profits of entering bidders approach zero. To see this, let $q_i\left(\theta_i\right)$ and $\pi_i\left(\theta_i\right)$ be the equilibrium winning probability and profit of bidder i with $\theta_i \geq \hat{\theta}_i^{opt} \equiv \hat{\theta}^{opt}$. The envelope theorem yields $\frac{d}{d\theta_i}\pi_i\left(\theta_i\right) = (1 - \alpha)\, q_i\left(\theta_i\right)$. Hence, $\pi_i\left(\theta_i\right) = 0$. The lemma follows. □

By (3), as entry cost ϕ decreases, so does the optimal entry threshold type $\hat{\theta}^{opt}$. In the limit as $\phi \to 0$, (3) yields $\hat{\theta}^{opt}_{\phi=0} = \underline{\theta}$. Intuitively, with costless entry, a seller's profit equals the social welfare gain created by the allocation of the asset (as the optimal mechanism leaves no rents to bidders). Then, because $\underline{\theta} > 0$, it is optimal to always award the asset (i.e., $\hat{\theta}^{opt}_{\phi=0} = \underline{\theta}$).

Comparing a seller's maximum rents for the costless entry benchmark and our costly entry setting where $\phi > 0$, (2) reveals that

$$\Pi^* \left(\hat{\theta}^{opt}_{\phi=0}\right) = \int_{\underline{\theta}}^{\bar{\theta}} \theta^1_n dQ_n \left(\theta^1_n\right)$$

and

$$\Pi^* \left(\hat{\theta}^{opt}_{\phi=0}\right) - \Pi^* \left(\hat{\theta}^{opt}\right) = n\phi \left(1 - F\left(\hat{\theta}^{opt}\right)\right) + \int_{\underline{\theta}}^{\hat{\theta}^{opt}} \theta^1_n dQ_n \left(\theta^1_n\right).$$

Costly entry reduces a seller's maximum rents in two ways. The first term captures direct cost of entry. The second term reflects an indirect efficiency loss: as reasoned above, costly entry impairs the efficiency of allocations as it is no longer optimal to always award the asset—the threshold $\hat{\theta}^{opt}$ is set higher than $\underline{\theta}$, so the seller foregoes some socially optimal trades.

Restricting the number of potential bidders. We first show that with limited uncertainty about bidder valuations, restricting entry to a single bidder may be optimal to conserve on entry costs.[7] We then show that with more extensive uncertainty, asymmetric mechanisms that handicap some bidders are optimal.

To start, we motivate the tradeoffs associated with exclusion faced by a seller. Fixing the sum of bidders' entry probabilities, and hence fixing the total expected direct costs of entry (the second term in equation (1)), but increasing the number of potential bidders n, has two opposing effects on social welfare. Conditional on some bidder entering, welfare gains typically rise with n, as the expected winner's valuation is higher due to a *greater-selection effect*. However, the probability that no bidder enters and no trade occurs also rises with n, which harms welfare. Intuition for this *deterred-entry effect* derives from a simple inequality:

[7] In a cash auction, Samuelson (1985) is the first to recognize that with symmetric mechanisms, a seller can gain by restricting entry when entry is costly. He notes that with limited uncertainty, increasing the number of potential bidders conveys little benefit, but the equilibrium probability that the asset is not sold (of $F^{n-1}(\hat{\theta}^{opt})$, where $\hat{\theta}^{opt}$ solves (3) in our setting) can fall with the number of potential bidders.

Lemma 3 *Given $n > 1$ numbers $p_1, ..., p_n \in (0,1)$, $1 - \min\{\sum_{j=1}^{n} p_j, 1\} < \Pi_{j=1}^{n}(1 - p_j)$.*

Corollary 1 *Suppose that rather than $n > 1$ potential bidders entering with (ex-ante) probabilities $p_1, ..., p_n \in (0,1)$, a single potential bidder enters with probability $min\{\sum_{j=1}^{n} p_j, 1\}$. Then, the asset is strictly more likely to be sold, and total expected entry costs are weakly less.*

Proofs: We first prove the lemma. Suppose $\Sigma_{j=1}^{n} p_j < 1$ (else the proof is trivial). For $n = 2$, the result follows by $(1 - p_1)(1 - p_2) = 1 - p_1 - p_2 + p_1 p_2 > 1 - p_1 - p_2 = 1 - min\{\Sigma_{j=1}^{n} p_j, 1\}$, and for $n > 2$, the result follows by induction. The corollary follows directly. □

Because expected total entry costs equal the sum of participation probabilities times ϕ, a single potential bidder can achieve a higher probability of sale while paying an entry cost that does not exceed the total entry costs of n potential bidders, i.e., $\min\{\Sigma_{j=1}^{n} p_j, 1\} \leq \Sigma_{j=1}^{n} p_j$. This is the deterred-entry effect of having more potential bidders. The greater-selection effect—the value of sampling more bidders to improve the draw of the entrant with the highest valuation (and possibly the second highest)—scales with the extent of variation in bidder valuations. When the variation in valuations is small enough relative to the entry cost ϕ, the deterred-entry effect dominates the greater-selection effect in an extreme form:

Proposition 1 *Let bidder valuations be distributed on $[\bar{\theta} - \epsilon, \bar{\theta}]$. If $\epsilon < \phi\left(1 - \frac{\phi}{\bar{\theta}}\right)$, then expected seller profits are strictly higher in the optimal mechanism with one potential bidder than in any symmetric mechanism with $n \geq 2$ potential bidders.*

Proof: See Appendix.

To illustrate the result, consider the optimal symmetric mechanism (Lemma 2) when $\epsilon \to 0$. Indexing $\hat{\theta}^{opt}$ and profits by the number of potential bidders n, (3) yields, for $n \geq 1$,

$$\lim_{\epsilon \to 0} F\left(\hat{\theta}^{opt}(n)\right) = \left(\frac{\phi}{\bar{\theta}}\right)^{\frac{1}{n-1}}.$$

Thus, by (1),

$$\lim_{\epsilon \to 0} \Pi\big(\hat{\theta}^{opt}(n), n\big) = \bar{\theta}\left(1 - \left(\frac{\phi}{\bar{\theta}}\right)^{\frac{n}{n-1}}\right) - n\phi\left(1 - \left(\frac{\phi}{\bar{\theta}}\right)^{\frac{1}{n-1}}\right) = \bar{\theta} - \phi - (n-1)\phi\left(1 - \left(\frac{\phi}{\bar{\theta}}\right)^{\frac{1}{n-1}}\right)$$

for $n > 1$. Similarly, $\lim_{\epsilon \to 0} \Pi(\hat{\theta}^{opt}(1), 1) = \bar{\theta} - \phi$ for $n = 1$. It is straightforward to show that expected revenue strictly decreases in n; that is, expected revenue is maximized at $n = 1$.[8]

4 Optimality of Asymmetric Mechanisms

We now show that symmetric mechanisms are not typically optimal, even with exclusion: when the possible valuations of potential bidders differ by enough then while excluding bidders is not optimal, neither is a symmetric mechanism. For simplicity, we consider $n = 2$. For asymmetric mechanisms, Lemma 1 generalizes to:

Lemma 4 *In any (potentially asymmetric) mechanism in which bidder* $i \in \{1, 2\}$ *enters if and only if its type exceeds* $\hat{\theta}_i \in [\underline{\theta}, \bar{\theta}]$, *a seller's expected profit cannot exceed* $\Pi^*(\hat{\theta}_1, \hat{\theta}_2) \equiv$

$$\int_{\hat{\theta}_2}^{\bar{\theta}} \int_{\hat{\theta}_1}^{\bar{\theta}} \max\{\theta_1, \theta_2\} \, dF(\theta_1) \, dF(\theta_2) + F(\hat{\theta}_2) \int_{\hat{\theta}_1}^{\bar{\theta}} \theta_1 dF(\theta_1) + F(\hat{\theta}_1) \int_{\hat{\theta}_2}^{\bar{\theta}} \theta_2 dF(\theta_2) - \phi(2 - F(\hat{\theta}_1) - F(\hat{\theta}_2)). \tag{5}$$

The first term in (5) is the social welfare gain when both bidders enter, and the second and third are the social welfare gains when only one of the bidders enters. Explicitly writing out the max term in (5) yields that, for $\hat{\theta}_1 \leq \hat{\theta}_2$,

$$\begin{aligned} \Pi^*(\hat{\theta}_1, \hat{\theta}_2) = & \int_{\hat{\theta}_2}^{\bar{\theta}} \left(\int_{\hat{\theta}_1}^{\theta_2} \theta_2 dF(\theta_1) + \int_{\theta_2}^{\bar{\theta}} \theta_1 dF(\theta_1) \right) dF(\theta_2) \\ & + F(\hat{\theta}_2) \int_{\hat{\theta}_1}^{\bar{\theta}} \theta_1 dF(\theta_1) + F(\hat{\theta}_1) \int_{\hat{\theta}_2}^{\bar{\theta}} \theta_2 dF(\theta_2) - \phi(2 - F(\hat{\theta}_1) - F(\hat{\theta}_2)). \end{aligned} \tag{6}$$

Let $(\hat{\theta}_1^{opt}, \hat{\theta}_2^{opt})$ maximize $\Pi^*(\hat{\theta}_1, \hat{\theta}_2)$, the upper bound on the seller's expected payoff:

$$(\hat{\theta}_1^{opt}, \hat{\theta}_2^{opt}) \in \arg\max \Pi^*(\hat{\theta}_1, \hat{\theta}_2). \tag{7}$$

$\Pi^*(\hat{\theta}_1^{opt}, \hat{\theta}_2^{opt})$ is attainable:

Lemma 5 *Let the auction winner pay fixed royalty rate* $\alpha \in [0, 1)$ *plus cash, where the highest cash bid wins and pays the second highest bid. Then a mechanism with bidder-specific cash*

[8] This follows because $m(1 - a^{1/m})$ increases in $m = n - 1$ for $a < 1$. The derivative with respect to m is $1 - a^b(1 - \ln(a^b))$, where $b = 1/m$. This derivative is positive since $\frac{d}{dx} x(1 - \ln(x)) = -\ln(x) > 0$ for $x < 1$, and the expression goes to zero as $a^b \to 1$.

reserves and cash reimbursements exists for which revenues approach $\Pi^*(\hat{\theta}_1^{opt}, \hat{\theta}_2^{opt})$ *as* $\alpha \to 1$.

Proof: See Appendix.

The proof mirrors that for Lemma 2, save that the cash reserves that leave bidders with signals $\hat{\theta}_1^{opt}$ and $\hat{\theta}_2^{opt}$ indifferent to entry are slightly more complicated due to the asymmetries.

We next establish our central result that asymmetric mechanisms that handicap one bidder are optimal when valuations can differ sufficiently:

Theorem 1 *Suppose (i)* $\frac{df(\theta)}{f(\theta)} > -\frac{1}{(\bar{\theta}-\underline{\theta})} \ln \frac{\bar{\theta}}{(\bar{\theta}-\underline{\theta})}$ *for all* $\theta \in [\underline{\theta}, \bar{\theta}]$*, i.e., the pdf does not decrease too quickly, and (ii) there is enough valuation uncertainty that* $\bar{\theta} - E[\theta] > \phi$*. Then optimal mechanisms are necessarily asymmetric, with both bidders entering with strictly positive but different (ex-ante) probabilities. That is,* $\hat{\theta}_1^{opt} \neq \hat{\theta}_2^{opt}$ *and* $\max\{\hat{\theta}_1^{opt}, \hat{\theta}_2^{opt}\} < \bar{\theta}$.

Proof: See Appendix.

Theorem 1 reveals the counter-intuitive result that the optimal mechanism with ex-ante identical bidders can be asymmetric, and this result does not hinge on large entry costs (see condition (ii)). To see the logic, consider a small spread away from a symmetric mechanism with entry thresholds $\hat{\theta}_1 = \hat{\theta}_2 = \hat{\theta}$, where each of the bidders enters with probability $p \in (0,1)$, to an asymmetric mechanism with entry thresholds $\hat{\theta}_1 = \hat{\theta} - \epsilon$ and $\hat{\theta}_2 = \hat{\theta} + \epsilon^*$, where ϵ^* is chosen so that one enters with probability $p + \Delta p$ and the other with $p - \Delta p$.

The introduction of asymmetry reduces the probability of no sale from $(1-p)^2$ to $(1-p)^2 - (\Delta p)^2$, while leaving total expected entry costs unchanged.[9] However, it forsakes some choice when the higher valuation bidder is excluded. Condition (i) ensures that the density does not decline so quickly that the potential (and hence expected) value of that foregone choice is too high, making it always optimal to spread the cutoffs of a symmetric mechanism. Condition (ii) ensures that always excluding a bidder (in which case, it is optimal for the other bidder always to enter since $\phi < \underline{\theta}$) is not optimal: the left-hand side of $\bar{\theta} - E[\theta] > \phi$ is the expected benefit of entry by a bidder with valuation $\bar{\theta}$ when the other bidder always enters, while the right-hand side is the cost.

[9]This intuition leads to optimality of asymmetric mechanism only if $\phi > 0$. With no entry costs, the logic breaks down because the optimal mechanism features $p = 1$, so it is impossible to have a probability of $p + \Delta p$.

To strengthen Theorem 1, we specialize to a uniform distribution over valuations. With a uniform distribution, condition (i) always holds, and condition (ii) reduces to

$$\frac{1}{2}\left(\bar{\theta}-\underline{\theta}\right)>\phi.$$

Corollary 2 *When valuations are uniform distributed, the optimal mechanism is asymmetric as long as* $\frac{1}{2}\left(\bar{\theta}-\underline{\theta}\right)>\phi$.

With an additional condition, we now pin down the optimal degree of asymmetry: the optimal mechanism is asymmetric with one bidder always entering regardless of its valuation, and the other bidder entering with an ex-ante probability strictly between zero and one, entering only when its valuation is high enough.

Proposition 2 *Let valuations be uniformly distributed with* $3\underline{\theta}>\bar{\theta}$ *and* $\frac{1}{2}\left(\bar{\theta}-\underline{\theta}\right)>\phi$. *Then in the optimal mechanism, one bidder always enters, while the other bidder enters with an ex-ante probability strictly between zero and one: the cutoff valuation is* $\underline{\theta}+\sqrt{2(\bar{\theta}-\underline{\theta})\phi}\in\left(\underline{\theta},\bar{\theta}\right)$.

Proof: See Appendix.

When $3\underline{\theta}>\bar{\theta}$, a slight ϵ increase in the spread between $\hat{\theta}_1$ and $\hat{\theta}_2$ raises seller revenues—the benefit of increased probability of including one bidder swamps the cost of increased probability of excluding the other bidder when its valuation is higher. But then the optimum features a boundary solution. However, as explained below Theorem 1, the condition $\bar{\theta}-E[\theta]=\frac{1}{2}\left(\bar{\theta}-\underline{\theta}\right)>\phi$ ensures that it is not optimal to always exclude one bidder, implying that it is optimal for one bidder to always enter, and for the other to enter with an ex-ante probability between zero and one. In this case, first-order conditions yield that the other potential bidder should enter when its valuation is at least $\underline{\theta}+\sqrt{2(\bar{\theta}-\underline{\theta})\phi}<\bar{\theta}$.

This analysis underscores the optimality of treating bidders asymmetrically, even in a symmetric setting, where either bidder can be favored. Of course, if there is a natural heterogeneity among bidders, no matter how tiny, then the optimal way to select the favored bidder is unique. This means that there is a *discontinuity* in the optimal degree of asymmetry with respect to the underlying bidder heterogeneity—in contrast to a costless entry setting, as in Myerson (1981), where the relationship is continuous.

It is insightful to identify ways in which the optimal symmetric mechanism is and is not robust. Fixing $\hat{\theta}_1$ at the optimal symmetric cutoff $\hat{\theta}_1^{opt}$ (given by (3)) and varying $\hat{\theta}_2 \in \left[\underline{\theta}, \bar{\theta}\right]$ in equation (6) yields

$$\frac{\partial}{\partial \hat{\theta}_2} \Pi^* \left(\hat{\theta}_1^{opt}, \hat{\theta}_2\right) = \frac{\phi}{\bar{\theta} - \underline{\theta}} - \frac{\hat{\theta}_2}{\bar{\theta} - \underline{\theta}} \frac{\hat{\theta}_1^{opt} - \underline{\theta}}{\bar{\theta} - \underline{\theta}} - \frac{1}{2} \left(\frac{\max\{\hat{\theta}_1^{opt}, \hat{\theta}_2\} - \hat{\theta}_1^{opt}}{\bar{\theta} - \underline{\theta}} \right)^2.$$

Algebra reveals that $\frac{\partial}{\partial \hat{\theta}_2} \Pi^*(\hat{\theta}_1^{opt}, \hat{\theta}_2)$ decreases in $\hat{\theta}_2$, and is zero at $\hat{\theta}_2 = \hat{\theta}_1^{opt}$. Thus, $\Pi^*(\hat{\theta}_1^{opt}, \hat{\theta}_1^{opt}) \geq \Pi^*(\hat{\theta}_1^{opt}, \hat{\theta}_2)$ for all $\hat{\theta}_2 \in [\underline{\theta}, \bar{\theta}]$. That is, the symmetric mechanism is "optimal" as long as we only vary one cutoff while fixing the other at $\hat{\theta}_1^{opt}$, or if we vary both cutoffs up and down together (as $\hat{\theta}_i^{opt}$ satisfies (3)). However, as Theorem 1 shows, a symmetric mechanism is not optimal: it is better to vary the two cutoffs by different amounts in opposite directions.

5 Generality of Optimality of Asymmetric Mechanisms

The optimality of asymmetric mechanisms extends beyond optimal securities auctions that use steep securities to extract full rents, to hold for optimal mechanisms featuring any fixed royalty rate $\alpha \in [0, 1)$ plus cash. To highlight this, we consider two bidders and uniform uncertainty over valuations on $[\underline{\theta}, \bar{\theta}]$, where

$$2\underline{\theta} - \bar{\theta} > \phi. \tag{8}$$

This assumption ensures that with a single potential entrant, the optimal cutoff is $\underline{\theta}$.[10] We now show that the qualitative content of Theorem 1 holds for all fixed royalty rates given only slightly stronger sufficient conditions:

Proposition 3 *Let bidders pay with a fixed royalty rate $\alpha \in [0, 1)$ plus cash, where the highest cash bid wins but the winner pays the second highest bid. Let valuations be uniformly distributed with $2\underline{\theta} - \bar{\theta} > \phi$ and $\frac{(\bar{\theta} - \underline{\theta})}{2} > \phi$. Then*

(*i*) *The optimal mechanism is asymmetric, with both bidders entering with strictly positive but different probabilities.*

[10] This condition (equation (8)) ensures that the virtual valuation for the fixed royalty plus cash (see Myerson 1981 and Liu 2016), calculated at $\underline{\theta}$, exceeds ϕ for any α.

(ii) If, in addition, $\alpha > \frac{2(2\bar{\theta}-3\underline{\theta})}{3(\bar{\theta}-\underline{\theta})}$*, then in the optimal mechanism, one bidder always enters and the other bidder enters with an ex-ante probability strictly between zero and one, where the cutoff valuation is* $\underline{\theta} + \sqrt{\frac{2\phi(\bar{\theta}-\underline{\theta})}{2-\alpha}} \in (\underline{\theta}, \bar{\theta})$*. This cutoff strictly increases in* α*.*

Proof: See Appendix.

As $\alpha \to 1$ (which corresponds to the steepest security that leaves bidders with no rents), the third condition reduces to $3\underline{\theta} > \bar{\theta}$, as in Proposition 2. When $\alpha = 0$, the third condition is more stringent, $3\underline{\theta} > 2\bar{\theta}$; under this condition, the characterization applies to cash auctions.

When bidders pay with fixed royalty rate plus cash, challenges arise (vis à vis when $\alpha \to 1$) in evaluating the relative merits of symmetric and asymmetric mechanisms, because bidders earn strictly positive rents, and the rents differ between these two types of mechanisms. Proposition 3 reveals that the benefit of the asymmetric mechanism increases in α. The fixed royalty rate α does not affect the social welfare benefits of spreading. However, the proof reveals that spreading the entry thresholds increases total bidder payoffs at a seller's expense. Decreased tying (reduced α) raises bidder profits, magnifying this effect, especially when the entry threshold is low.[11] The conditions of the proposition ensure that the positive effect of spreading on social welfare outweighs the negative effect of increasing total bidder payoffs. Finally, increasing the royalty rate α reduces the profit of the bidder who always enters. In turn, this reduces the value that the seller attaches to entry by the other bidder, making it optimal to set a higher entry threshold for that bidder.

6 Conclusion

With entry costs, optimal selling mechanisms trade off between the increased rents that more entrants can bring versus the higher total entry costs incurred by more bidders that a seller indirectly bears via the endogenous entry choices of bidders. With very limited dispersion in bidder valuations, it is optimal to restrict entry to a single bidder to avoid a no-trade outcome.

Given the extensive uncertainty over bidder valuations relative to entry costs that is com-

[11] Applying the envelope condition, bidder i's expected profit given signal θ_i is $(1-\alpha)\int_{\hat{\theta}_i}^{\theta_i} W_i(\theta)d\theta$, where $W_i(\theta)$ is bidder i's probability of winning given signal θ. The expected profit decreases in both α and $\hat{\theta}_i$.

mon in practice, we then provide theoretical foundations for the asymmetric auction designs found in shareholder rights plans that favor one bidder over another, or in the favoring of certain designated suppliers in procurement auctions. When valuation uncertainty is extensive, we show that it is optimal to handicap some bidders in order to encourage other bidders to enter even when their valuations are low. Thus, asymmetric designs can be optimal even when bidders are ex-ante symmetric and management (or the procurer) seeks to maximize expected revenues. Steeper securities raise the attraction of asymmetric designs, differentially reducing the profit of a bidder who enters more frequently, making it optimal to raise the reserve for another bidder, thereby increasing the degree of asymmetry.

It is useful to contrast our setting with one in which potential bidders only learn valuations *after* entering. The Appendix establishes that when potential bidders do not have an information advantage over the seller when making entry decisions, it is unnecessary to tie payments to valuations as a seller can extract all bidder rents by using lump-sum transfers. Moreover, the seller need not handicap particular bidders when bidders are ex-ante identical—the optimal amount of entry will endogenously arise in equilibrium.

Appendix

Proof of Proposition 1: With a single potential bidder and any security (with optimal reserve), the seller can always extract a surplus of at least $\bar{\theta}-\epsilon-\phi$ via a naive reserve that leaves type $\bar{\theta}-\epsilon$ indifferent to entry: $\Pi\left(p, n=1\right) \geq \bar{\theta}-\epsilon-\phi$. Now consider $n>1$ potential bidders. If the probability that each bidder enters is p, the seller's payoff is bounded as follows:

$$\Pi\left(p, n>1\right) < \left(1-\left(1-p\right)^{n}\right)\bar{\theta}-np\phi, \tag{9}$$

where $1-\left(1-p\right)^{n}$ is the probability that at least one potential bidder enters. Thus, $\left(1-\left(1-p\right)^{n}\right)\bar{\theta}$ is an upper bound on welfare gains and $np\phi$ is the expected entry cost (the inequality is slack because the winner's type is typically below $\bar{\theta}$ and bidders may earn positive rents). Maximizing the right-hand side of (9) with respect to p yields $p^{*}=1-(\frac{\phi}{\bar{\theta}})^{\frac{1}{n-1}}$.

Substituting this into the right-hand side of (9) yields

$$\Pi\left(p,n>1\right)<\bar{\theta}-n\phi+(n-1)\left(\frac{\phi}{\bar{\theta}}\right)^{\frac{1}{n-1}}\phi.$$

One can show that the right-hand side decreases in n for $n \geq 2$. Thus,

$$\Pi\left(p,n>1\right)<\bar{\theta}-2\phi+\frac{\phi^2}{\bar{\theta}}<\bar{\theta}-\varepsilon-\phi\leq\Pi\left(p,n=1\right),$$

where the second inequality follows from $\epsilon<\phi\left(1-\frac{\phi}{\bar{\theta}}\right)$. $\square$

Proof of Lemma 5: Without loss of generality let $\hat{\theta}_1^{opt}<\hat{\theta}_2^{opt}$ (the symmetric case with $\hat{\theta}_1^{opt}=\hat{\theta}_2^{opt}$ is proved in Lemma 2). If $\hat{\theta}_1^{opt}>\underline{\theta}$, then the cash reserve prices for each bidder are

$$C_1=(1-\alpha)\left(X+\hat{\theta}_1^{opt}\right)-X-\frac{\phi}{F(\hat{\theta}_2^{opt})} \tag{10}$$

and

$$C_2=(1-\alpha)\left(X+\hat{\theta}_2^{opt}\right)-X-\frac{\phi-(1-\alpha)\int_{\hat{\theta}_1^{opt}}^{\hat{\theta}_2^{opt}}(\hat{\theta}_2^{opt}-\theta)dF\left(\theta\right)}{F(\hat{\theta}_1^{opt})},$$

where $\hat{\theta}_i^{opt}$ $(i=1,2)$ is given by (7).

If $\hat{\theta}_1^{opt}=\underline{\theta}$, then the cash reserve price for bidder 1 is $C_1=(1-\alpha)\left(X+\underline{\theta}\right)-X-\frac{\phi}{F(\hat{\theta}_2^{opt})}$ (i.e. (10) with $\hat{\theta}_1^{opt}=\underline{\theta}$), and reserve for bidder 2 is irrelevant. If bidder 2 enters, the seller reimburses bidder 2 with $\phi-(1-\alpha)\int_{\underline{\theta}}^{\hat{\theta}_2^{opt}}(\hat{\theta}_2^{opt}-\theta)dF\left(\theta\right)$ regardless of whether he wins.

Bidder 1 with $\hat{\theta}_1^{opt}$ wins with probability $F(\hat{\theta}_2^{opt})$, and whenever it happens, bidder 1 pays its reserve because bidder 2 does not enter. Bidder 2 with $\hat{\theta}_2^{opt}$ also wins with probability $F(\hat{\theta}_2^{opt})$, but its payment depends on whether bidder 1 enters: with probability $F(\hat{\theta}_1^{opt})$ bidder 1 does not enter and hence bidder 2 pays its reserve; with probability $F(\hat{\theta}_2^{opt})-F(\hat{\theta}_1^{opt})$ bidder 1 enters and hence bidder 2 pays bidder 1's bid. In this latter situation, bidder 2 earns an expected profit of $(1-\alpha)\int_{\hat{\theta}_1^{opt}}^{\hat{\theta}_2^{opt}}(\hat{\theta}_2^{opt}-\theta)dF\left(\theta\right)$. Thus, for example, if $\hat{\theta}_1^{opt}>\underline{\theta}$, then C_2 must equate the expected profit to $\hat{\theta}_2^{opt}$ from entering with the cost ϕ:

$$F(\hat{\theta}_1^{opt})[(1-\alpha)(X+\hat{\theta}_2^{opt})-X-C_2]+(1-\alpha)\int_{\hat{\theta}_1^{opt}}^{\hat{\theta}_2^{opt}}(\hat{\theta}_2^{opt}-\theta)dF(\theta)=\phi.$$

So, too, if $\hat{\theta}_1=\underline{\theta}$ and bidder 2 has valuation $\hat{\theta}_2$, its expected revenues from entering of

$(1-\alpha)\int_{\underline{\theta}}^{\hat{\theta}_2}(\hat{\theta}_2-\theta_1)dF(\theta_1)$ must equal ϕ minus the cash reimbursement for entering. The solution for C_1 is obtained analogously; and the rest of the proof follows that of Lemma 2. $\square$

Proof of Theorem 1: We first show that if $\phi < \bar{\theta} - E[\theta]$, then always excluding a bidder is not optimal. By way of contradiction, suppose that always excluding bidder 2 is optimal. Because $\phi < \underline{\theta}$, if only bidder 1 enters, then setting $\hat{\theta}_1 = \underline{\theta}$ is optimal. Differentiating (6) at $(\hat{\theta}_1 = \underline{\theta}, \hat{\theta}_2 = \bar{\theta})$ with respect to $\hat{\theta}_2$ yields

$$\left.\frac{\partial \Pi^*(\hat{\theta}_1, \hat{\theta}_2)}{\partial \hat{\theta}_2}\right|_{\hat{\theta}_1=\underline{\theta},\, \hat{\theta}_2=\bar{\theta}} = -f\left(\bar{\theta}\right)\left(\bar{\theta} - E[\theta] - \phi\right) < 0, \tag{11}$$

where the inequality holds by $\bar{\theta} - E[\theta] > \phi$. Thus, always excluding a bidder is not optimal.

We now show that symmetric mechanisms with $\hat{\theta}_1 = \hat{\theta}_2$ are never optimal. Because always excluding a bidder and always having two bidders (i.e., setting $\hat{\theta}_1 = \hat{\theta}_2 = \underline{\theta}$) are never optimal, without loss of generality suppose that $\hat{\theta}_1 \leq \hat{\theta}_2 \in \left(\underline{\theta}, \bar{\theta}\right)$. Consider the asymmetric mechanism: $\hat{\theta}_2 = \theta^* + \epsilon$ and $\hat{\theta}_1 = \theta^* - \epsilon^*$, where ϵ is small and ϵ^* solves $F(\theta^*) - F(\theta^* - \epsilon^*) = F(\theta^* + \epsilon) - F(\theta^*) \equiv \Delta p$. Using "$o$" to stand for "terms of order," we have

$$\Delta p = f(\theta^*)\,\epsilon + o\left(\epsilon^2\right) \quad \text{and} \quad \epsilon^* = \epsilon + o\left(\epsilon^2\right).$$

We show that $\Delta\Pi^* = \Pi^*(\theta^* - \epsilon^*, \theta^* + \epsilon) - \Pi^*(\theta^*, \theta^*) > 0$ (see equation (6)). Because ϵ^* is set so that the terms with ϕ are the same for both mechanisms, we need only compare terms concerning social welfare. We retain terms up to order ϵ^2. There exist contributions to $\Delta\Pi^*$ only in 3 cases:

Case 1: $\theta_2 \in (\theta^*, \theta^* + \epsilon)$ and $\theta_1 \in (\theta^* - \epsilon^*, \theta^*)$. The contribution to $\Delta\Pi^*$ is

$$-\left(\Delta p\right)^2\left(\epsilon + o\left(\epsilon^2\right)\right) = 0 + o\left(\epsilon^3\right).$$

Case 2: $\theta_2 \in (\theta^*, \theta^* + \epsilon)$ and $\theta_1 \notin (\theta^* - \epsilon^*, \theta^*)$. The contribution exists only when $\theta_1 \in (\underline{\theta}, \theta^* - \epsilon^*)$ and it is

$$\begin{aligned}-\Delta p\left(F(\theta^*) - \Delta p\right)\left(\theta^* + \frac{\epsilon}{2} + o\left(\epsilon^2\right)\right) &= -\Delta p\left(F(\theta^*) - \Delta p\right)\left(\theta^* + \frac{\epsilon}{2}\right) + o\left(\epsilon^3\right) \\ &= -\Delta p F(\theta^*)\left(\theta^* + \frac{\epsilon}{2}\right) + \left(\Delta p\right)^2\theta^* + o\left(\epsilon^3\right).\end{aligned}$$

Case 3: $\theta_2 \notin (\theta^*, \theta^* + \epsilon)$ and $\theta_1 \in (\theta^* - \epsilon^*, \theta^*)$. The contribution exists only when $\theta_2 \in (\underline{\theta}, \theta^*)$ and it is

$$\Delta p F(\theta^*) \left(\theta^* - \frac{\epsilon}{2} + o\left(\epsilon^2\right)\right) = \Delta p F(\theta^*) \left(\theta^* - \frac{\epsilon}{2}\right) + o\left(\epsilon^3\right).$$

Adding up all contributions from the 3 cases yields

$$\begin{aligned}\Delta\Pi^* =& \Delta p F(\theta^*) \left(\theta^* - \frac{\epsilon}{2}\right) - \Delta p F(\theta^*) \left(\theta^* + \frac{\epsilon}{2}\right) + (\Delta p)^2 \theta^* + o\left(\epsilon^3\right) \\ =& \Delta p \left(\Delta p \theta^* - F(\theta^*)\epsilon\right) + o\left(\epsilon^3\right) \\ =& \Delta p \epsilon \left(f(\theta^*)\theta^* - F(\theta^*)\right) + o\left(\epsilon^3\right).\end{aligned}$$

Thus, symmetric mechanisms are never optimal if

$$f(\theta^*)\theta^* - F(\theta^*) > 0, \text{ for all } \theta^* \in (\underline{\theta}, \bar{\theta}), \tag{12}$$

or equivalently if $\frac{F(\theta^*)}{f(\theta^*)\theta^*} < 1$. We now show that (12) holds. Define $k \equiv \frac{1}{(\bar{\theta}-\underline{\theta})} \ln \frac{\bar{\theta}}{(\bar{\theta}-\underline{\theta})}$. The premise $\frac{df}{f} > -k$ implies that, for all $\theta \in [\underline{\theta}, \theta^*]$, $\frac{\ln f(\theta^*) - \ln f(\theta)}{\theta^* - \theta} > -k$, or $\ln \frac{f(\theta)}{f(\theta^*)} < k(\theta^* - \theta)$, which yields $f(\theta) < f(\theta^*) \exp(k(\theta^* - \theta))$. Thus,

$$\frac{F(\theta^*)}{f(\theta^*)\theta^*} < \frac{1}{\theta^*} \int_{\underline{\theta}}^{\theta^*} \exp(k(\theta^* - \theta))\, d\theta = \frac{\exp(k(\theta^* - \underline{\theta})) - 1}{k\theta^*}.$$

By the mean value theorem, there exists a $\theta^{**} \in [\underline{\theta}, \theta^*]$ such that $\exp(k(\theta^* - \underline{\theta})) - 1 = k \exp(k(\theta^{**} - \underline{\theta}))(\theta^* - \underline{\theta})$. Thus,

$$\begin{aligned}\frac{\exp(k(\theta^* - \underline{\theta})) - 1}{k\theta^*} &\le \frac{k \exp(k(\theta^* - \underline{\theta}))(\theta^* - \underline{\theta})}{k\theta^*} \\ &= \exp(k(\theta^* - \underline{\theta})) \frac{(\theta^* - \underline{\theta})}{\theta^*} \\ &\le \exp\left(k\left(\bar{\theta} - \underline{\theta}\right)\right) \frac{\left(\bar{\theta} - \underline{\theta}\right)}{\bar{\theta}} = 1.\end{aligned}$$

Thus, $\frac{F(\theta^*)}{f(\theta^*)\theta^*} < 1$, proving (12). □

Proof of Proposition 2: To proceed, we compare mechanism 1 in which $\underline{\theta} < \hat{\theta}_1 \le \hat{\theta}_2 < \bar{\theta}$, with mechanism 2 where $\hat{\theta}_1$ is replaced by $\hat{\theta}_1 - \epsilon$ and $\hat{\theta}_2$ replaced by $\hat{\theta}_2 + \epsilon$. We show that $\Delta\Pi^* = \Pi^*(\hat{\theta}_1 - \epsilon, \hat{\theta}_2 + \epsilon) - \Pi^*(\hat{\theta}_1, \hat{\theta}_2) > 0$ (see equation (6)).

Observe that there is a contribution to $\Delta\Pi^*$ only when both θ_1 and θ_2 lie in $[\underline{\theta}, \hat{\theta}_2 + \epsilon]$, which occurs with probability $\left(\frac{\hat{\theta}_2+\epsilon-\underline{\theta}}{\bar{\theta}-\underline{\theta}}\right)^2$. Thus, with an abuse of notation, we compute contributions to $\Pi^*(\hat{\theta}_1 - \epsilon, \hat{\theta}_2 + \epsilon)$ and $\Pi^*(\hat{\theta}_1, \hat{\theta}_2)$ only when θ_1 and θ_2 are in $[\underline{\theta}, \hat{\theta}_2 + \epsilon]$. That is, replacing $\bar{\theta}$ with $\hat{\theta}_2 + \epsilon$ in equation (6), we have for mechanism 1

$$\Pi^*\left(\hat{\theta}_1, \hat{\theta}_2\right) = \left(\frac{\hat{\theta}_2 + \epsilon - \underline{\theta}}{\bar{\theta} - \underline{\theta}}\right)^2 \left\{ \left(\frac{\epsilon}{\hat{\theta}_2 + \epsilon - \underline{\theta}}\right)^2 \left(\hat{\theta}_2 + \frac{2}{3}\epsilon\right) + \frac{\epsilon}{\hat{\theta}_2 + \epsilon - \underline{\theta}} \frac{\hat{\theta}_2 - \underline{\theta}}{\hat{\theta}_2 + \epsilon - \underline{\theta}} \left(\hat{\theta}_2 + \frac{1}{2}\epsilon\right) \right.$$
$$\left. + \left(1 - \frac{\epsilon}{\hat{\theta}_2 + \epsilon - \underline{\theta}}\right) \frac{\hat{\theta}_2 - \hat{\theta}_1 + \epsilon}{\hat{\theta}_2 + \epsilon - \underline{\theta}} \frac{\hat{\theta}_2 + \hat{\theta}_1 + \epsilon}{2} \right\} - \phi \left(\frac{2\bar{\theta} - \hat{\theta}_2 - \hat{\theta}_1}{\bar{\theta} - \underline{\theta}}\right).$$

The first term inside the braces corresponds to $\theta_2, \theta_1 \in [\hat{\theta}_2, \hat{\theta}_2 + \epsilon]$; the second term corresponds to $\theta_2 \in [\hat{\theta}_2, \hat{\theta}_2 + \epsilon]$ and $\theta_1 \in [\underline{\theta}, \hat{\theta}_2]$, and the third term corresponds to $\theta_2 \in [\underline{\theta}, \hat{\theta}_2]$ and $\theta_1 \in [\hat{\theta}_1, \hat{\theta}_2 + \epsilon]$. For mechanism 2,

$$\Pi^*(\hat{\theta}_1 - \epsilon, \hat{\theta}_2 + \epsilon) = \left(\frac{\hat{\theta}_2 + \epsilon - \underline{\theta}}{\bar{\theta} - \underline{\theta}}\right)^2 \left\{ \frac{\hat{\theta}_2 - \hat{\theta}_1 + 2\epsilon}{\hat{\theta}_2 + \epsilon - \underline{\theta}} \frac{\hat{\theta}_2 + \hat{\theta}_1}{2} \right\} - \phi \left(\frac{2\bar{\theta} - \hat{\theta}_2 - \hat{\theta}_1}{\bar{\theta} - \underline{\theta}}\right).$$

Thus,

$$\begin{aligned}
\Delta\Pi^* = &\left(\frac{1}{\bar{\theta} - \underline{\theta}}\right)^2 \left\{ \frac{1}{2}\left(\hat{\theta}_2 + \epsilon - \underline{\theta}\right)\left(\hat{\theta}_2 - \hat{\theta}_1 + 2\epsilon\right)\left(\hat{\theta}_2 + \hat{\theta}_1\right) - \epsilon^2 \left(\hat{\theta}_2 + \frac{2}{3}\epsilon\right) \right. \\
&\left. - \epsilon\left(\hat{\theta}_2 - \underline{\theta}\right)\left(\hat{\theta}_2 + \frac{1}{2}\epsilon\right) - \frac{1}{2}\left(\hat{\theta}_2 - \underline{\theta}\right)\left(\hat{\theta}_2 - \hat{\theta}_1 + \epsilon\right)\left(\hat{\theta}_2 + \hat{\theta}_1 + \epsilon\right) \right\} \\
= &\left(\frac{1}{\bar{\theta} - \underline{\theta}}\right)^2 \left\{ -\frac{2}{3}\epsilon^3 + \left(\hat{\theta}_1 - \hat{\theta}_2 + \underline{\theta}\right)\epsilon^2 + \left(\hat{\theta}_2 - \hat{\theta}_1\right)\left(\underline{\theta} - \frac{1}{2}\left(\hat{\theta}_2 - \hat{\theta}_1\right)\right)\epsilon \right\}. \qquad (13)
\end{aligned}$$

The leading term in (13) is ϵ whose coefficient is $(\hat{\theta}_2 - \hat{\theta}_1)(\underline{\theta} - \frac{1}{2}(\hat{\theta}_2 - \hat{\theta}_1))$. As long as $\underline{\theta} > \frac{1}{3}\bar{\theta}$, this coefficient is positive, meaning that it is better to widen the gap (slightly) between the two cutoffs. We can repeat this operation until either $\hat{\theta}_1 = \underline{\theta}$ or $\hat{\theta}_2 = \bar{\theta}$.

Suppose $\hat{\theta}_1 = \underline{\theta}$ (i.e., one potential bidder always enters). Then, setting $\partial\Pi^*(\underline{\theta}, \hat{\theta}_2)/\partial\hat{\theta}_2 = 0$ yields $\hat{\theta}_2 = \underline{\theta} + \sqrt{2(\bar{\theta} - \underline{\theta})\phi} < \bar{\theta}$, where the inequality holds by $\phi < \frac{1}{2}\left(\bar{\theta} - \underline{\theta}\right)$.

Suppose now that $\hat{\theta}_2 = \bar{\theta}$ (i.e, one potential bidder never enters). Then since $\underline{\theta} > \phi$, it is optimal to set $\hat{\theta}_1 = \underline{\theta}$. But this implies $\hat{\theta}_2 < \bar{\theta}$, a contradiction. The proposition follows. □

Proof of Proposition 3: We first calculate expected revenue in a mechanism characterized by cutoffs $\hat{\theta}_1 \leq \hat{\theta}_2$. One can show via standard mechanism design arguments that when

$2\underline{\theta} > \bar{\theta}$, which is implied by (8), the expected-revenue-maximizing mechanism among all such mechanisms (as characterized by $\hat{\theta}_1$ and $\hat{\theta}_2$) is such that bidder i with $\hat{\theta}_i$ earns zero rent, the higher valuation bidder wins if both bidders enter, and a bidder wins if he is the sole entrant. Applying the envelope condition, expected profit of bidder i with valuation θ_i is $(1-\alpha)\int_{\hat{\theta}_i}^{\theta_i} W_i(\theta)\, d\theta$, where $W_i(\theta)$ is the winning probability of bidder i with valuation θ. The unconditional expected profit of bidder i is

$$\begin{aligned}
\pi_i &= (1-\alpha)\int_{\hat{\theta}_i}^{\bar{\theta}}\int_{\hat{\theta}_i}^{\theta_i} W_i(\theta)\, d\theta dF(\theta_i) \\
&= -(1-\alpha)\int_{\hat{\theta}_i}^{\bar{\theta}}\int_{\hat{\theta}_i}^{\theta_i} W_i(\theta)\, d\theta d(1-F(\theta_i)) \\
&= (1-\alpha)\int_{\hat{\theta}_i}^{\bar{\theta}} (1-F(\theta_i))\, W_i(\theta_i)\, d\theta_i.
\end{aligned}$$

Because $\hat{\theta}_1 \leq \hat{\theta}_2$, we have $W_2(\theta_2) = F(\theta_2)$ and

$$W_1(\theta_1) = \begin{cases} F(\theta_1) & \text{if} \quad \theta_1 > \hat{\theta}_2 \\ F(\hat{\theta}_2) & \text{if} \quad \theta_1 \in [\hat{\theta}_1, \hat{\theta}_2]. \end{cases}$$

Thus,

$$\begin{aligned}
\pi_1 &= (1-\alpha)\, F(\hat{\theta}_2)\int_{\hat{\theta}_1}^{\hat{\theta}_2} (1-F(\theta_1))\, d\theta_1 + (1-\alpha)\int_{\hat{\theta}_2}^{\bar{\theta}} (1-F(\theta_1))\, F(\theta_1)\, d\theta_1 \\
\pi_2 &= (1-\alpha)\int_{\hat{\theta}_2}^{\bar{\theta}} (1-F(\theta_2))\, F(\theta_2)\, d\theta_2.
\end{aligned}$$

Letting $\pi^*(\hat{\theta}_1, \hat{\theta}_2)$ be the sum of both bidders' equilibrium payoffs,

$$\begin{aligned}
\pi^*(\hat{\theta}_1, \hat{\theta}_2) &= \pi_1 + \pi_2 \\
&= (1-\alpha)\, F\left(\hat{\theta}_2\right)\int_{\hat{\theta}_1}^{\hat{\theta}_2} (1-F(\theta_1))\, d\theta_1 + 2(1-\alpha)\int_{\hat{\theta}_2}^{\bar{\theta}} (1-F(\theta))\, F(\theta)\, d\theta.
\end{aligned}$$

With the uniform distribution, $F(\theta) = \frac{\theta - \underline{\theta}}{\bar{\theta} - \underline{\theta}}$ and $1 - F(\theta) = \frac{\bar{\theta} - \theta}{\bar{\theta} - \underline{\theta}}$. Thus,

$$\begin{aligned}\frac{\left(\bar{\theta} - \underline{\theta}\right)^2}{1-\alpha}\pi^*(\hat{\theta}_1, \hat{\theta}_2) &= (\hat{\theta}_2 - \underline{\theta})\left[\bar{\theta}(\hat{\theta}_2 - \hat{\theta}_1) - \frac{1}{2}(\hat{\theta}_2^2 - \hat{\theta}_1^2)\right] + 2\int_{\hat{\theta}_2}^{\bar{\theta}}\left(-\theta^2 + \theta\left(\bar{\theta} + \underline{\theta}\right) - \bar{\theta}\underline{\theta}\right)d\theta \\ &= (\hat{\theta}_2 - \underline{\theta})\left[\bar{\theta}(\hat{\theta}_2 - \hat{\theta}_1) - \frac{1}{2}(\hat{\theta}_2^2 - \hat{\theta}_1^2)\right] \\ &\quad + 2\left[\frac{1}{3}\hat{\theta}_2^3 - \frac{1}{3}\bar{\theta}^3 + \frac{1}{2}\left(\bar{\theta} + \underline{\theta}\right)(\bar{\theta}^2 - \hat{\theta}_2^2) - \bar{\theta}\underline{\theta}(\bar{\theta} - \hat{\theta}_2)\right]. \end{aligned} \tag{14}$$

We next compare the two mechanisms considered in the proof of Proposition 2 (where in mechanism 2, $\hat{\theta}_1$ is replaced by $\hat{\theta}_1 - \epsilon$ and $\hat{\theta}_2$ replaced by $\hat{\theta}_2 + \epsilon$). Define $\Delta\pi^* = \pi^*(\hat{\theta}_1 - \epsilon, \hat{\theta}_2 + \epsilon) - \pi^*(\hat{\theta}_1, \hat{\theta}_2)$ to be the difference in bidders' total payoffs from the two mechanisms:

$$\begin{aligned}\frac{\left(\bar{\theta} - \underline{\theta}\right)^2}{1-\alpha}\Delta\pi^* &= (\hat{\theta}_2 - \underline{\theta} + \epsilon)\left[\bar{\theta}(\hat{\theta}_2 - \hat{\theta}_1 + 2\epsilon) - \frac{1}{2}\left((\hat{\theta}_2 + \epsilon)^2 - (\hat{\theta}_1 - \epsilon)^2\right)\right] \\ &\quad + 2\left[\frac{1}{3}(\hat{\theta}_2 + \epsilon)^3 - \frac{1}{3}\bar{\theta}^3 + \frac{1}{2}\left(\bar{\theta} + \underline{\theta}\right)\left(\bar{\theta}^2 - (\hat{\theta}_2 + \epsilon)^2\right) - \bar{\theta}\underline{\theta}(\bar{\theta} - \hat{\theta}_2 - \epsilon)\right] \\ &\quad - (\hat{\theta}_2 - \underline{\theta})\left[\bar{\theta}(\hat{\theta}_2 - \hat{\theta}_1) - \frac{1}{2}(\hat{\theta}_2^2 - \hat{\theta}_1^2)\right] \\ &\quad - 2\left[\frac{1}{3}\hat{\theta}_2^3 - \frac{1}{3}\bar{\theta}^3 + \frac{1}{2}\left(\bar{\theta} + \underline{\theta}\right)(\bar{\theta}^2 - \hat{\theta}_2^2) - \bar{\theta}\underline{\theta}(\bar{\theta} - \hat{\theta}_2)\right]. \end{aligned} \tag{15}$$

On the right-hand side of (15), the terms proportional to ϵ sum up to

$$\begin{aligned}&\left[\left(\bar{\theta}(\hat{\theta}_2 - \hat{\theta}_1) - \frac{1}{2}\hat{\theta}_2^2 + \frac{1}{2}\hat{\theta}_1^2\right) + (\hat{\theta}_2 - \underline{\theta})\left(2\bar{\theta} - \hat{\theta}_2 - \hat{\theta}_1\right) + 2\left(\hat{\theta}_2^2 - \left(\bar{\theta} + \underline{\theta}\right)\hat{\theta}_2 + \bar{\theta}\underline{\theta}\right)\right]\epsilon \\ &= (\hat{\theta}_2 - \hat{\theta}_1)\left(\bar{\theta} - \underline{\theta} + \frac{1}{2}(\hat{\theta}_2 - \hat{\theta}_1)\right)\epsilon,\end{aligned}$$

and the terms proportional to ϵ^2 sum up to

$$\left[2\bar{\theta} - \hat{\theta}_2 - \hat{\theta}_1 + 2\hat{\theta}_2 - \left(\bar{\theta} + \underline{\theta}\right)\right]\epsilon^2 = (\bar{\theta} - \underline{\theta} + \hat{\theta}_2 - \hat{\theta}_1)\epsilon^2.$$

Including all terms yields

$$\frac{\left(\bar{\theta} - \underline{\theta}\right)^2}{1-\alpha}\Delta\pi^* = (\hat{\theta}_2 - \hat{\theta}_1)\left(\bar{\theta} - \underline{\theta} + \frac{1}{2}(\hat{\theta}_2 - \hat{\theta}_1)\right)\epsilon + (\bar{\theta} - \underline{\theta} + \hat{\theta}_2 - \hat{\theta}_1)\epsilon^2 + \frac{2}{3}\epsilon^3.$$

Because the right-hand side of the above is strictly positive for $\epsilon > 0$, and $1 - \alpha > 0$, we have

$\Delta\pi^* > 0$, implying that spreading the cutoffs raises bidders' payoffs at the seller's expense.

Together with $\Delta\Pi^*$ in (13), we have

$$\begin{aligned}
\left(\bar{\theta}-\underline{\theta}\right)^2\left(\Delta\Pi^*-\Delta\pi^*\right) &= -\frac{2}{3}\epsilon^3+\left(\hat{\theta}_1-\hat{\theta}_2+\underline{\theta}\right)\epsilon^2+\left(\hat{\theta}_2-\hat{\theta}_1\right)\left(\underline{\theta}-\frac{1}{2}\left(\hat{\theta}_2-\hat{\theta}_1\right)\right)\epsilon \\
&\quad -(1-\alpha)\left\{(\hat{\theta}_2-\hat{\theta}_1)\left(\bar{\theta}-\underline{\theta}+\frac{1}{2}(\hat{\theta}_2-\hat{\theta}_1)\right)\epsilon+(\bar{\theta}-\underline{\theta}+\hat{\theta}_2-\hat{\theta}_1)\epsilon^2+\frac{2}{3}\epsilon^3\right\} \\
&= \left(\hat{\theta}_2-\hat{\theta}_1\right)\left[2\underline{\theta}-\bar{\theta}-\left(\hat{\theta}_2-\hat{\theta}_1\right)+\alpha\left(\bar{\theta}-\underline{\theta}+\frac{1}{2}\left(\hat{\theta}_2-\hat{\theta}_1\right)\right)\right]\epsilon \\
&\quad +\left[(2-\alpha)\left(\underline{\theta}-\hat{\theta}_2+\hat{\theta}_1\right)-(1-\alpha)\,\bar{\theta}\right]\epsilon^2+o\left(\epsilon^3\right), \qquad (16)
\end{aligned}$$

which is positive for small ϵ at $\hat{\theta}_2=\hat{\theta}_1$, for all $\alpha \geq 0$, given the assumption that $2\underline{\theta} > \bar{\theta}$. Thus, interior symmetric cutoffs cannot be optimal.

Next, it follows from (6) and (14) that

$$\begin{aligned}
&\left(\bar{\theta}-\underline{\theta}\right)^2\frac{\partial}{\partial\hat{\theta}_2}\left(\Pi^*(\hat{\theta}_1,\hat{\theta}_2)-\pi^*(\hat{\theta}_1,\hat{\theta}_2)\right)\Big|_{\hat{\theta}_1=\underline{\theta}} \\
&= -\left(\frac{1}{2}\bar{\theta}^2-\hat{\theta}_2\underline{\theta}+\frac{1}{2}\hat{\theta}_2^2\right)+\frac{1}{2}\bar{\theta}^2-\frac{1}{2}\underline{\theta}^2+\left(\bar{\theta}-\underline{\theta}\right)\phi \\
&\quad -(1-\alpha)\left\{(\hat{\theta}_2-\underline{\theta})\left[\bar{\theta}-\frac{1}{2}(\hat{\theta}_2+\underline{\theta})\right]+(\hat{\theta}_2-\underline{\theta})(\bar{\theta}-\hat{\theta}_2)+2\hat{\theta}_2^2-2\left(\bar{\theta}+\underline{\theta}\right)\hat{\theta}_2+2\bar{\theta}\underline{\theta}\right\} \\
&= -\frac{1}{2}(2-\alpha)(\hat{\theta}_2-\underline{\theta})^2+\left(\bar{\theta}-\underline{\theta}\right)\phi. \qquad (17)
\end{aligned}$$

The right-hand side of (17) is strictly positive at $\hat{\theta}_2=\underline{\theta}$, implying that $\hat{\theta}_2=\hat{\theta}_1=\underline{\theta}$ cannot be optimal, and it is immediate that $\hat{\theta}_2=\hat{\theta}_1=\bar{\theta}$ cannot be optimal. Thus, no symmetric mechanism is optimal.

Next, to show that always excluding a bidder is not optimal, suppose by way of contradiction that it is. Then under $2\underline{\theta}-\bar{\theta} > \phi$, setting $\hat{\theta}_1=\underline{\theta}$ is optimal (given the premise that bidder 1 is the sole entrant). Differentiating (14) at $(\hat{\theta}_1=\underline{\theta},\hat{\theta}_2=\bar{\theta})$ with respect to $\hat{\theta}_2$ yields

$$\begin{aligned}
\frac{(\bar{\theta}-\underline{\theta})^2}{1-\alpha}\frac{\partial\pi^*(\hat{\theta}_1,\hat{\theta}_2)}{\partial\hat{\theta}_2}\Big|_{\hat{\theta}_1=\underline{\theta},\hat{\theta}_2=\bar{\theta}} &= \bar{\theta}(\bar{\theta}-\underline{\theta})-\frac{1}{2}(\bar{\theta}^2-\underline{\theta}^2)+(\bar{\theta}-\underline{\theta})\bar{\theta}-\bar{\theta}(\bar{\theta}-\underline{\theta}) \\
&\quad +2(\bar{\theta}^2-\bar{\theta}(\bar{\theta}+\underline{\theta})+\bar{\theta}\underline{\theta}) \\
&= \frac{(\bar{\theta}-\underline{\theta})^2}{2} > 0. \qquad (18)
\end{aligned}$$

Moreover, by (11), $\left.\frac{\partial \Pi^*(\hat{\theta}_1,\hat{\theta}_2)}{\partial \hat{\theta}_2}\right|_{\hat{\theta}_1=\underline{\theta},\,\hat{\theta}_2=\bar{\theta}} < 0$, implying that

$$\left.\frac{\partial}{\partial \hat{\theta}_2}\left(\Pi^*(\hat{\theta}_1,\hat{\theta}_2) - \pi^*(\hat{\theta}_1,\hat{\theta}_2)\right)\right|_{\hat{\theta}_1=\underline{\theta},\,\hat{\theta}_2=\bar{\theta}} < 0.$$

Thus, always excluding one bidder is not optimal; that is, $\hat{\theta}_2 \neq \bar{\theta}$. This completes the proof of part (i) of the proposition.

Because symmetric cutoffs are not optimal, assume that $\hat{\theta}_2 > \hat{\theta}_1$. To establish when $\hat{\theta}_1 = \underline{\theta}$ is optimal, observe that the leading term in (16) is ϵ, which has coefficient

$$\begin{aligned}
&\left(\hat{\theta}_2 - \hat{\theta}_1\right)\left[2\underline{\theta} - \bar{\theta} - \left(\hat{\theta}_2 - \hat{\theta}_1\right) + \alpha\left(\bar{\theta} - \underline{\theta} + \frac{1}{2}\left(\hat{\theta}_2 - \hat{\theta}_1\right)\right)\right] \\
&\geq \left(\hat{\theta}_2 - \hat{\theta}_1\right)\left[2\underline{\theta} - \bar{\theta} - \left(\bar{\theta} - \underline{\theta}\right) + \alpha\left(\bar{\theta} - \underline{\theta} + \frac{1}{2}\left(\bar{\theta} - \underline{\theta}\right)\right)\right] \\
&= \left(\hat{\theta}_2 - \hat{\theta}_1\right)\left[3\underline{\theta} - 2\bar{\theta} + \frac{3}{2}\alpha\left(\bar{\theta} - \underline{\theta}\right)\right].
\end{aligned}$$

This leading coefficient is positive (meaning that it is optimal to spread entry cutoffs until either $\hat{\theta}_1 = \underline{\theta}$ or $\hat{\theta}_2 = \bar{\theta}$) if and only if $\alpha > \frac{2(2\bar{\theta}-3\underline{\theta})}{3(\bar{\theta}-\underline{\theta})}$. But, since $\hat{\theta}_2 \neq \bar{\theta}$, we must have $\hat{\theta}_1 = \underline{\theta}$. Then, setting the right-hand side of the first-order condition (17) for $\hat{\theta}_2$ to zero yields

$$\hat{\theta}_2 = \underline{\theta} + \sqrt{\frac{2\phi\left(\bar{\theta} - \underline{\theta}\right)}{2-\alpha}} < \underline{\theta} + \sqrt{2\phi\left(\bar{\theta} - \underline{\theta}\right)} < \bar{\theta},$$

where the last inequality holds by $\bar{\theta} - \underline{\theta} > 2\phi$. Therefore, the optimal cutoffs are $\hat{\theta}_1 = \underline{\theta}$ and $\hat{\theta}_2 \in (\underline{\theta}, \bar{\theta})$, establishing part (ii) of the proposition. □

Unknown Valuations. Suppose that bidders do not know their valuations before making entry decisions. Then, if m potential bidders enter, defining $Q_m(\theta^1)$ to be the distribution over the highest valuation, a seller's expected payoff cannot exceed

$$\bar{\Pi}_m \equiv \int_{\underline{\theta}}^{\bar{\theta}} \theta^1 dQ_m\left(\theta^1\right) - m\phi.$$

This reflects that social welfare cannot exceed $\int_{\underline{\theta}}^{\bar{\theta}} \theta^1 dQ_m\left(\theta^1\right)$, and expected bidder payoffs (net of entry costs) must be nonnegative. Thus, an upper bound on a seller's expected payoff is $\bar{\Pi}_{m^*}$, where $m^* = \arg\max_{m \leq n} \bar{\Pi}_m$. It follows that $m^* \geq 1$ when $\phi < \underline{\theta}$. We now show that $\bar{\Pi}_{m^*}$ is attainable simply by using lump-sum transfers (the same fee for each bidder, where a negative entry fee corresponds to a cash reimbursement):

Proposition 4 $\bar{\Pi}_{m^*}$ *is implementable in the pure-strategy equilibrium of any standard format in which bidders bid with a fixed royalty rate* $\alpha \in [0, 1)$ *plus cash, face a reserve that does not exceed the break-even bid of a bidder with valuation* $\underline{\theta}$*, and pay an entry fee of* $\pi^* - \phi$*, where* π^* *is the expected payoff of an entering bidder (excluding entry costs) given* m^* *entrants.*

Proof: We show that m^* potential bidders' entering constitutes an equilibrium. Any entering bidder receives expected payoff (gross of entry cost) of ϕ. Thus, entering is a best response. Further, if $n > m^*$, then each potential bidder who did not enter strictly prefers not to enter: the expected payoff (gross of entry cost) from entry would be strictly less than ϕ due to the heightened competition, making entering unprofitable. Thus, the equilibrium holds. In equilibrium, each bidder's ex-ante expected payoff (including entry costs) is zero, and social welfare is maximized for the given m^* entrants, establishing the proposition. □

One way to implement this mechanism is to use $\alpha = 0$, i.e., pure cash auctions (hence, no tying) and an entry fee.[12] By contrast, if bidders know their valuations before entry, potential bidders have an informational advantage that a seller must offset by tying payments to their private information, as in Lemma 1. Further, with unknown-valuations, efficiency is not impaired by having no trade—a seller always awards the asset, as the profit equals the welfare gain from trade. In contrast, with known valuations, a seller raises entry thresholds, screening out low-valuation bidders.

When bidders do not know their valuations prior to making entry decisions, two types of equilibria exist: a pure strategy equilibrium (McAfee and McMillan, 1987) in which entrants expect non-negative profits, but with greater entry, expected profits would become negative; and a mixed strategy equilibrium (Levin and Smith, 1994) in which potential bidders enter with a common probability p. The equilibrium in Proposition 4 delivers the optimal number of entrants: full surplus extraction is obtained via the pure-strategy equilibrium,

[12] Alternatively, if there exists $\alpha \in [0, 1)$ such that $\pi^* = \phi$, then the mechanism in Proposition 4 can be implemented without charging an entry fee.

as in McAfee and McMillan (1987), in which the right (deterministic) number of bidders endogenously choose to enter, making it unnecessary to restrict entry.

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