SLAC – PUB – 3623
March 1985
T/E/AS

COVARIANT AMPLITUDES IN POLYAKOV'S STRING THEORY*

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ABSTRACT

A manifestly Lorentz-covariant and reparametrization-invariant procedure for computing string amplitudes using Polyakov's formulation is described. Both bosonic and superstring theories are dealt with. The computation of string amplitudes is greatly facilitated by this formalism.

Submitted to *Nuclear Physics B*

* Work supported by the Department of Energy, contract DE – AC03 – 76SF00515.
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1. Introduction

Less than a year ago Green and Schwarz[1] made the remarkable discovery of anomaly cancellations in the 10-dimensional SO(32) superstring theory. Subsequently they also showed[2] that this theory is one-loop (and presumably to all orders) finite. Since then two more superstring theories, the SO(32) and $E_8 \times E_8$ heterotic strings,[3] have been added to this list. Given the present framework and the level of our understanding, these theories appear to be quite possibly the only ones capable of consistently unifying quantum gravity with other interactions. From recent investigations[4] they also appear to have bright phenomenological prospects, which has sparked off an upsurge of renewed interest in string theories.

Despite these successes, so far there does not exist in the literature a systematic procedure for computing string amplitudes which is manifestly Lorentz covariant and reparametrization invariant and which preserves the geometrical properties of strings and their interactions. In the old first quantized light-cone gauge operator formalism,[5] the geometrical picture of strings and their interactions is obscure. Mandelstam's path integral approach[6] preserves the geometrical features but his approach is not manifestly covariant. Moreover, his procedure for establishing the Möbius invariant Koba-Nielsen volume element is not straightforward. Second quantized Lagrangians including interactions have also been written down in the light-cone gauge for the bosonic string[7] as well as for superstrings[8] in terms of string field functionals. These too, however, do not appear to have any particular geometrical interpretation and their computational utility is doubtful.

Some years ago Polyakov[9] suggested a novel geometrical approach to string theories. His manifestly Lorentz covariant sum over random surfaces is a natural generalization to strings of Feynman's sum over random paths in particle quantum mechanics. Polyakov's framework is also a natural setting for computation of string amplitudes which have all the desired properties mentioned earlier. In this paper we develop a systematic procedure for computing arbitrary n-point amplitudes within Polyakov's framework. There is an essentially unique prescription, based on maintaining explicit 2-dimensional reparametrization invariance of the string world-sheet and D-dimensional Lorentz covariance, for the construction of such amplitudes. This procedure carries over to the old fermionic strings of Neveu-Schwarz[10] and Ramond[11] by simply requiring 2-dimensional local supersymmetry in addition to general coordinate invariance. Amplitudes of the Green and Schwarz superstring theory[12] arise naturally in this formulation by projecting onto a certain even "parity" or "fermion number" sector;[13] otherwise amplitudes of the old Neveu-Schwarz and Ramond spinning strings are obtained. The present procedure is, however, sufficient for computing amplitudes with external bosonic lines only. To obtain amplitudes with external fermionic lines one needs to construct objects which transform as D-dimensional fermions (since among the basic degrees of freedom in the old formulation of string theories no such objects are present). This nontrivial task remains an open problem.

In the next section we first briefly review Polyakov's formulation and then discuss the construction of amplitudes for bosonic closed and open strings in general. The straightforward emergence of open string N-tachyon and 4-point Yang-Mills amplitudes in critical dimensions is then demonstrated by way of examples. In this section we also present a generalization of the Fradkin-Tseytlin prescription[14] of introducing sources for generating amplitudes to open strings. We shall see that this procedure correctly reproduces Chan-Paton group theory factors in the amplitudes.

In Section 3 the question of off-shell amplitudes in this formalism is entered into in some detail. For on-shell values of the external momenta, the path integral over the conformal factor is trivial in critical dimensions and it decouples from integration over Koba-Nielsen variables. However, for off-shell values of the external momenta a nontrivial, divergent integral over the conformal factor remains. Moreover, for these values of the external momenta the amplitudes also contain certain ill-defined factors (which involve derivatives of coincident propagators). We shall give prescriptions for regularizing these divergences. The resulting off-shell amplitudes are well-defined and seem to satisfy certain minimum restrictions.

Extension of the above formalism to the Neveu-Schwarz and Ramond spinning strings is undertaken in Section 4. To retain manifest 2-dimensional reparametrization and local supersymmetry we use the superfield formalism developed in Ref. 15 and used in Polyakov's framework in Ref. 16. Section 5 contains a summary of the foregoing results and concluding remarks. In the Appendix we collect some formulae useful for computation of string amplitudes and discuss a problem which one faces in the case of open fermionic strings in this context.

After the completion of most of this work, we came to know that essentially the same formalism has been independently developed by Friedan and Shenker. These authors have also reported some progress in constructing vertices for external D-dimensional fermions.[17] For a different approach to covariantly second-quantized strings see Ref. 18.

2. Bosonic Strings

The starting point of Polyakov's formulation is the partition function

$$\int \mathcal{D}g^{ab}\mathcal{D}x\, e^{-S-\mu_0^2\int_M d^2z\sqrt{g}(z)} \quad , \tag{2.1}$$

where S is the free string action

$$S = \frac{1}{2\pi\alpha'}\int\limits_M d^2z\sqrt{g}\,\frac{1}{2}\,g^{ab}\partial_a x\cdot\partial_b x \quad , \tag{2.2}$$

and μ_0^2 term is a cosmological-constant term. In (2.1), M is a compact 2-manifold, $g_{ab}(z)$ is a metric on it ($g = det(g_{ab})$), and $x(z)$ is the string variable. A D-dimensional Lorentz index on x is suppressed. Both the 2- and D-dimensional metrics are taken to be of Euclidean signature. At the tree level M is simply connected. Loop corrections are taken into account by summing (2.2) over different topologies (holes and handles) with suitable factor of a coupling constant multiplied. We shall restrict ourselves to the tree level throughout this paper.

The action in (2.2) has 2-dimensional general coordinate invariance. Therefore, in order to evaluate the functional integral (2.1) one needs to fix a gauge. In the conformal gauge

$$g_{ab}(z) = e^{\phi(z)}\delta_{ab} \quad , \tag{2.3}$$

the integration $\int \mathcal{D}\, g^{ab}$ can be replaced by $V \int \mathcal{D}\phi \Delta_{FP}[\phi]$, where V is the volume of reparametrizations and Δ_{FP} is the Faddeev-Popov determinant associated with the gauge fixing (2.3). Polyakov showed that the conformal anomaly (associated with the x-integral in (2.1)) and the Faddeev-Popov determinant cancel each other at the critical dimensions, $D = 26$. For $D < 26$, one must solve the Liouville field theory for the conformal mode $\phi(z)$.

2.1 VERTICES

Polyakov introduced an expression for obtaining physical quantities in his approach,

$$A(p_1, ...p_N) = \langle \prod_{i=1}^{N} \int d^2 z_i \sqrt{g}(z_i)\, e^{ip_i x(z_i)} \rangle \quad . \tag{2.4}$$

The average $\langle ... \rangle$ in the above is defined by the functional integral (2.1). It turns out that $A(p_1, ...p_N)$ only describes the amplitude for the scattering of N-tachyons of the closed strings, carrying external momenta $p_1, ...p_N$.[10]

Using the definition

$$v_T(p) = \int d^2 z \sqrt{g}\, e^{ipx(z)} \quad , \tag{2.5}$$

we can rewrite (2.4) as

$$A(p_1, ...p_N) = \langle \prod_{i=1}^{N} v_T(p_i) \rangle \quad . \tag{2.6}$$

Thus the basic ingredient for obtaining the N-tachyon amplitude is $v_T(p)$, which we call "vertex". We notice that this vertex is manifestly reparametrization and Lorentz-invariant. Clearly, to obtain amplitudes for the excited states of the closed string we need vertices for them. Now, the only nontrivial factors involving the string variable $x(z)$ that can be inserted in the integrand in (2.5) are of the the general form*

$$\prod_{i=1}^{N} \nabla_{a_i} x^{I_i} \prod_{j=1}^{M} \nabla_{b_j} \nabla_{c_j} x^{J_j} \ldots , \tag{2.7}$$

where ∇_a is the 2-dimensional covariant derivative. Note that x must always appear with a derivative to maintain translation invariance in D-dimensions. Thus the only vertices which are manifestly reparametrization invariant and Lorentz covariant that can be written down for closed strings are of the generic form

$$\int d^2 z \sqrt{g}\, e^{ipx(z)} \times \begin{bmatrix} \text{(2.7) with all the 2-dimensional indices} \\ \text{appropriately contracted with } g^{ab} \text{ and } \dfrac{\epsilon^{ab}}{\sqrt{g}} \end{bmatrix} . \tag{2.8}$$

The graviton vertex is, therefore, uniquely fixed to be

$$V_G^{IJ}(p) = \int d^2 z \sqrt{g}\, \frac{1}{2} g^{ab} \partial_a x^I \partial_b x^J e^{ipx(z)} \quad , \tag{2.9}$$

which is natural in the sense that the graviton couples to the energy-momentum tensor. The vertex for an antisymmetric tensor is

$$V_B^{IJ}(p) = \int d^2 z \sqrt{g}\, \frac{1}{2} \epsilon^{ab} \partial_a x^I \partial_b x^J e^{ipx(z)} \quad , \tag{2.10}$$

We can also introduce the dilaton vertex

$$V_D(p) = \int d^2 z \sqrt{g}\, \frac{1}{2} g^{ab} \partial_a x^I \partial_b x^I e^{ipx(z)} \quad . \tag{2.11}$$

As we shall see, an important feature of these vertices is that when the external lines are on-shell, the dependence on the conformal factor $\phi(z)$ vanishes, and

* For the sake of uniformity of notation, we shall use $I, J, \ldots$ as Lorentz indices throughout this paper. The Greek indices $\mu, \nu, \ldots$ are reserved for spinor components (Sec. 4).

therefore the S-matrix elements are obtained rather trivially. For this to occur, the vertex for the m-th excited state must have m g^{ab}'s or $\epsilon^{ab}/\sqrt{g}$'s. Moreover, their appropriate "antisymmetrized" (with respect to Lorentz indices) combinations must be formed to cancel the dependence on derivatives of ϕ coming from multiple derivatives on x. By requiring these properties to hold, we can *completely* fix the vertices for arbitrary higher excited states. This point is further discussed in the next subsection.

These vertices attach external strings to a point on the world sheet. This is due to the fact that external strings represent physical point particle states. This is clearer in the coordinate representation in which the tachyon vertex, for example, is given by

$$v_T(x) = \int d^2z\sqrt{g}\,\delta^D(x - x(z)) \quad . \tag{2.12}$$

This form for the vertex suggests that there exists a simple generating functional of the amplitudes*

$$\Gamma\left[T, G_{IJ}, B_{IJ}, D \ldots\right] = \left\langle \exp\left[\int d^2z\sqrt{g}\left(T(x(z)) + \frac{1}{2} g^{ab}\, \partial_a x^I \partial_b x^J G_{IJ}(x(z)) \right.\right.\right.$$
$$\left.\left.\left. + \frac{1}{2}\frac{\epsilon^{ab}}{\sqrt{g}} \partial_a x^I \partial_b x^J B_{IJ}(x(z)) + \frac{1}{2}\, \partial_a x^I \partial_b x^I D(x(z)) + \ldots \right)\right]\right\rangle . \tag{2.13}$$

This object has recently been considered by Fradkin and Tseytlin,[14] who have interpreted it as an effective action. A little later we shall introduce a generalization of this to open strings which reproduces the Chan-Paton group-theory factors in the amplitudes.

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* Note that the derivatives are to be taken with respect to sources which are ordinary functions in D-dimensional spacetime, and not functionals of strings, e.g. $T(x)$, not $T(x(z))$. For example,

$$\langle v_T(x)\rangle = \left\langle \frac{\delta\Gamma\left[T, 0, \ldots\right]}{\delta T(x)} \right\rangle_{T=0} .$$

Construction of vertices in the open string case is a straightforward generalization of the procedure used above. Since open strings couple only to the boundary of the manifold M, integrals over the surface $\int_M d^2z\sqrt{g}$ must now be replaced by integrals over the boundary $\int_{\partial M} ds$, where $ds^2 = g_{ab}dz^a dz^b$ is the invariant line element on M. Moreover, for translation invariance the string variable x must now appear as the derivative $dx(z(s))/ds$. Thus the generic vertex in the open string case is of the form

$$\int\limits_{\partial M} ds e^{ipx(z(s))} \prod_{i=1}^{N} \frac{dx^{I_i}(z(s))}{ds} \prod_{j=1}^{M} \frac{d^2x^{J_j}(z(s))}{ds^2} \ldots \quad . \tag{2.14}$$

The tachyon and vector vertices are, therefore, uniquely determined to be

$$v_{T_o}(p) = \int\limits_{\partial M} ds\, e^{ipx(z(s))} \quad , \tag{2.15}$$

and

$$v_A^I(p) = \int\limits_{\partial M} ds \frac{dx^I(z(s))}{ds} e^{ipx(z(s))} \quad . \tag{2.16}$$

We introduce the following generating functional for amplitudes in the present case

$$\Gamma[T_o, A_I, \ldots] =$$

$$\left\langle \mathrm{Tr}\,\mathrm{P}\exp\left[\int\limits_{\partial M} ds \left(T_o(x(z(s))) + \frac{dx^I(z(s))}{ds} A_I(x(z(s))) + \ldots\right)\right]\right\rangle . \tag{2.17}$$

The new feature here is that the sources can carry representations of an allowed symmetry group (U, SO, Sp). Hence the trace 'Tr' and the path-ordering 'P'. It is straightforward to check that (2.17) reproduces the usual Chan-Paton rules.

For example, the amplitude for four vectors which follows from (2.17) is

$$A^{(I_i)}_{(\alpha_i)}(p_1, \ldots p_4) \sim \mathrm{Tr}\langle \prod_{i=1}^{4} \lambda_{\alpha_i} \int_{\partial M} ds_i \, \frac{dx^{I_i}(z(s_i))}{ds_i} \theta(s_{i+1} - s_i) \, e^{ip_i x(z(s_i))} \rangle$$
$$+ \, (\text{simultaneous permutations of} \, (\alpha_i, I_i, p_i)) \quad , \tag{2.18}$$

which contains the correct Chan-Paton factors.

2.2 S-MATRIX ELEMENTS

We have checked in several simple cases that the vertices given above reproduce the usual string (on-shell) amplitudes with the correct Koba-Nielsen measure. We give below two examples - the open-string N-tachyon and 4-vector amplitudes. The closed-string N-tachyon amplitude was considered in Ref. 19 and briefly in Polyakov's work.

At the tree level, the relevant manifold for open-string amplitudes is a simply connected one (topologically equivalent to a disc). We map this manifold to the upper-half of the complex z-plane ($z = z^1 + iz^2$). In the conformal gauge (2.3), the boundary of this manifold, the real axis, has $ds = e^{\frac{1}{2}\phi(z)} dz^1$. Then the result of the x-integral in critical dimensions is*

$$A(p_1, \ldots p_N) = (2\pi)^{26} \delta^{26}(\sum_{i=1}^{N} p_i) \int \mathcal{D}\phi$$
$$\times \int_{-\infty}^{\infty} \prod_{i=1}^{N} dz_i^1 \, e^{\frac{1}{2}\phi(z_i)} \, \theta(z_{i+1}^1 - z_i^1) \exp\left[-\pi\alpha' \sum_{ij} p_i p_j K(z_i, z_j|\phi)\right] \, , \tag{2.19}$$

The momentum conserving δ-function comes from integration over the zero mode of x. In the following, we shall not display it explicitly. We shall also drop the

* The total amplitude is obtained by multiplying appropriate Chan-Paton factors and summing over different orderings (see (2.18)).

superscript 1 of z_i^1. The propagator $K(z_i, z_j|\phi)$ is given in the Appendix. Using equation (A8), we get

$$A(p_1, ...p_N) = \prod_{i=1}^{N} e^{-\frac{\alpha'}{2}p_i^2 \ln \epsilon} \int D\phi \int_{-\infty}^{\infty} \prod_{i=1}^{N} dz\, e^{\frac{1}{2}\phi(z_i)(1-\alpha' p_i^2)}\, \theta(z_{i+1} - z_i) \times \prod_{i>j} |z_i - z_j|^{2\alpha' p_i p_j} . \tag{2.20}$$

On-shell, $\alpha' p_i^2 = 1$, so the dependence on ϕ drops out and the ϕ functional integral is a trivial infinite factor which cancels in the normalized amplitude. In fact, this is actually the case for all on-shell amplitudes. The reason is that for every external line there is always a factor $\frac{1}{2}\alpha' p^2 \phi$ in the exponent coming from the coincident point singularity of the propagator. The other appropriate factor ($\frac{1}{2}\phi$ for a tachyon, zero for a vector, etc.) is supplied by the integration measure 'ds' and the derivatives 'd/ds' in the vertex associated with the corresponding external line. We could actually turn this argument around and set the mass-shell condition by insisting that the ϕ-dependence drop out of the S-matrix elements. This criterion reproduces the correct spectrum in both the open and closed string sectors. In the latter case, the coincident point singularity of the propagator gives a factor $\frac{1}{4}\alpha' p^2 \phi$ in the exponent. (The additional factor of $\frac{1}{2}$ compared to the open string case is due to the absence of the image term in the closed string propagator.) The other appropriate factor (ϕ for a tachyon, zero for graviton, etc.) is supplied by the corresponding vertex. The remaining dependence of the on-shell amplitudes on (the derivatives of) ϕ coming from the multiple derivatives on x can be made to cancel by taking appropriate "antisymmetrized" (in Lorentz indices) combination of the vertices.† This point is, however, irrelevant for the massless sector.

† For example, at the 2nd mass level in the open string sector, the appropriate vertices are $\int ds\, e^{ipx} \frac{dx^I}{ds}\frac{dx^J}{ds}\frac{dx^K}{ds}$ and $\int ds\, e^{ipx} \left(\frac{d^2x^I}{ds^2}\frac{dx^J}{ds} - \frac{d^2x^J}{ds^2}\frac{dx^I}{ds} \right)$.

Thus, for the on-shell N-tachyon amplitude we get the well known expression,

$$A(p_1,...p_N) = V_\phi \int\limits_{-\infty}^{\infty} \prod_{i=1}^{N} dz_i\, \theta(z_{i+1}-z_i) \prod_{i>j} |z_i - z_j|^{2\alpha' p_i p_j}$$

$$= V_\phi V_M |(z_a - z_b)(z_b - z_c)(z_c - z_a)| \int\limits_{-\infty}^{\infty} \prod_{i\neq a,b,c} dz_i \prod_{i=1}^{N} \theta(z_{i+1}-z_i) \prod_{i>j} |z_i - z_j|^{2\alpha' p_i p_j} , \tag{2.21}$$

where $V_\phi \equiv \int \mathcal{D}\phi$, and the V_M is the group volume of the Mōbius transformations. The singular factors of ϵ can be consistently absorbed in the definition of the vertex and so we have omitted them from (2.21).

Our second example is the four-point Yang-Mills amplitude, (2.18). Doing the x-integral in the conformal gauge as in the previous example, we find

$$A^{(I_i)}(p_1,...p_4) \propto \int \mathcal{D}\phi \int\limits_{-\infty}^{\infty} \prod_{i=1}^{N} dz_i\, e^{-\frac{1}{2}\phi(z_i)\alpha' p_i^2}\, \theta(z_{i+1}-z_i) \prod_{i>j} |z_i - z_j|^{2\alpha' p_i p_j}$$

$$\times \left[\frac{1}{4}\left(Q^{I_1 I_2} Q^{I_3 I_4} + ... \right) + \frac{\alpha'}{2}\left(Q^{I_1 I_2} P_3^{I_3} P_4^{I_4} + ... \right) + \alpha'^2 \prod_{i=1}^{4} P_i^{I_i} \right] . \tag{2.22}$$

where the dots represent appropriate permutation terms and

$$Q^{I_i I_j} \equiv -\frac{1}{\pi} \frac{\delta^{I_i I_j}}{(z_i - z_j)^2} \quad , \tag{2.23}$$

$$P_j^{I_j} \equiv \frac{1}{\pi} \sum_{i\neq j} \frac{p_i^{I_j}}{z_i - z_j} + \frac{1}{2} p_j^{I_j} \frac{d}{dz_j} K(z_j, z_j|\phi) \quad . \tag{2.24}$$

The last term of $P_j^{I_j}$ is singular and needs careful treatment. On shell, however, it does not contribute to (2.22) because the S-matrix elements are obtained by contracting (2.22) with polarization vectors $\xi_j^{I_j}$ and $\xi_j p_j = 0$. The resulting expression agrees with the amplitude calculated in the first-quantized operator formalism.[20] We shall give a regularized definition of (2.24) in the next section where we discuss off-shell amplitudes.

3. Off-shell Amplitudes

In the previous section, we have obtained expressions for amplitudes which have the generic form*

$$A(p_1,...p_N) \propto \int \mathcal{D}\phi \int_{-\infty}^{\infty} \prod_{i=1}^{N} dz_i \, e^{-\frac{1}{2}\phi(z_i)\Gamma_i} \, \theta(z_{i+1}-z_i) \prod_{i>j} |z_i - z_j|^{2\alpha' p_i p_j} T(p,z,\phi) \, . \tag{3.1}$$

where $T(p,z,\phi)$ denotes appropriate tensor structure and Γ_i is the "inverse propagator" for the i-th external line ($1-\alpha' p_i^2$ for tachyons, $-\alpha' p_i^2$ for massless vectors, etc). This expression is not well-defined because of the divergent functional integral over ϕ and expressions of the form given in (2.24) which are involved in $T(p,z,\phi)$. As remarked earlier, these problems disappear on-shell and one recovers the usual Koba-Nielsen amplitudes. For off-shell values of external momenta, however, a careful treatment of both these problems is required in order to give a sensible meaning to (3.1). In the following we shall give prescriptions for handling the ϕ functional integral as well as expressions of the form (2.24). We shall see that amplitudes so obtained satisfy certain minimum restrictions, namely, that they are well-defined off-shell, have a smooth on-shell limit and reproduce the known S-matrix elements.

We shall first deal with the regularization of expressions of the form (2.24). Such expression result from sums of the form‡

$$P_j^{I_j} = \sum_{i=1}^{N} p_i^{I_j} \sum_n \lambda_n^{-1} f_n(z_i) \frac{df_n(z_j)}{dz_j} \quad . \tag{3.2}$$

The $i=j$ term in the above is the singular term that needs a careful treatment.

* For simplicity we shall restrict ourselves to open strings in this section. The treatment for closed strings is basically no different.

‡ These are obtained if one does the Gaussian integration over $x(z)$ by expanding it in terms of the covariant eigenfunctions defined in the appendix.

A covariantly regularized expression for (3.2) is

$$P_j^{I_j} = \sum_{i=1}^{N} p_i^{I_j} \sum_n \lambda_n^{-1} f_n(z_i) \frac{df_n(z_j)}{dz_j} e^{-\epsilon\lambda_n} \quad . \tag{3.3}$$

Using equation (A6) from the appendix we can rewrite (3.3) as

$$P_j^{I_j} = \sum_{i\neq j} p_i^{I_j} \frac{d}{dz_j} K_\epsilon(z_j, z_i|\phi) + p_j^{I_j} \frac{1}{2} \frac{d}{dz_j} K_\epsilon(z_j, z_j|\phi) \quad . \tag{3.4}$$

Expressions for $K_\epsilon(z, z'|\phi)$ have been given in the appendix using a different regulator. It turns out, however, that they can also be used in (3.4). The reason for this is the following. A crucial requirement that a regularized expression for (3.1) must satisfy is that it should be Mōbius invariant since that would allow us to extract the infinite group volume V_M even off-shell. Otherwise the on-shell limit of these amplitudes would be singular since Mōbius invariance would be recovered on-shell. This requirement implies that under the Mōbius transformation

$$z_i \to \omega_i \,, \quad z_i = \frac{\omega_i + B}{C\omega_i + D} \tag{3.5}$$

$P_j^{I_j}$ must transform as

$$P_j^{I_j}(z) \to \left[\frac{(C\omega_j + D)^2}{D - CB}\right] \tilde{P}_j^{I_j}(\omega) \,, \tag{3.6}$$

where the 'tilde' on $P_j^{I_j}$ signifies the fact that ϕ also transform under (3.5):

$$\phi(z) \to \tilde{\phi}(\omega) = \phi(z(\omega)) - \ln\left|\frac{(C\omega + D)^2}{D - CB}\right|^2 \quad . \tag{3.7}$$

That (3.6) is the appropriate transformation property for $P_j^{I_j}$ can be seen by considering, for example, the 4-point Yang-Mills amplitude given in (2.22). Using

(3.5)-(3.7) and the fact that under (3.5) we also have

$$\frac{1}{z_i - z_j} \rightarrow \frac{(C\omega_j + D)^2}{D - CB} \frac{1}{\omega_i - \omega_j} + \frac{C(C\omega_j + D)}{D - CB} \quad , \tag{3.8}$$

$$\frac{d\phi(z_j)}{dz_j} \rightarrow \left[\frac{(C\omega_j + D)^2}{D - CB}\right] \frac{d\tilde{\phi}(\omega_j)}{d\omega_j} + \frac{4C(C\omega_j + D)}{D - CB} \quad . \tag{3.9}$$

We can see that all the terms in the square bracket [...] in (2.22) transform in precisely the same way so that the entire square bracket transforms as

$$[...]_z \rightarrow \prod_{i=1}^{4} \left[\frac{(C\omega_i + D)^2}{D - CB}\right] [...]_\omega \quad . \tag{3.10}$$

This exactly cancels the change in the integration measure $\prod_{i=1}^{4} dz_i$, i. e. $\prod_{i=1}^{4} dz_i[...]_z \rightarrow \prod_{i=1}^{4} d\omega_i[...]_\omega$. The remaining factors also compensate for changes in each other, as can be easily verified. So, with the transformation property (3.6) for $P_j^{I_j}$, the amplitude in (2.22) is Mõbius invariant even for off-shell values of the external momenta. Now, it can be easily seen that the expression

$$P_j^{I_j} = \frac{1}{\pi} \left[\sum_{i \neq j} \frac{p_i^{I_j}}{z_i - z_j} + \frac{1}{4} p_j^{I_j} \frac{d\phi(z_j)}{dz_j}\right] \quad , \tag{3.11}$$

which can be obtained by using equation (A8) from the appendix in (3.4), satisfies the transformation property given in (3.6). Hence, as stated above, we can use the regularized expression for K_ϵ given in the appendix also in (3.4).

The next step of our prescription is to fix the above Mõbius invariance and extract the infinite group volume V_M. This is done by the well-known procedure of using the Faddeev-Popov trick to trade three of the z-integrations with the three group parameters B, C and D. Then (3.1) takes the form

$$A(p_1,\dots p_N) = V_M |(\omega_a-\omega_b)(\omega_b-\omega_c)(\omega_c-\omega_a)| \int D\tilde{\phi} \int \prod_{i\neq a,b,c} d\omega_i \, e^{\frac{1}{2}\tilde{\phi}(\omega_i)\Gamma_i}$$
$$\times \prod_{i=1}^{N} \theta(\omega_{i+1}-\omega_i) \prod_{i>j} |\omega_i-\omega_j|^{2\alpha' p_i p_j} T(p,\omega,\tilde{\phi}), \tag{3.12}$$

where $V_M = \int dBdCdD/(D-BC)^2$. On-shell, $\tilde{\phi}$-integral decouples as before and the known S-matrix elements are reproduced.

What we propose to do off-shell is to choose a particular $\tilde{\phi}$ and drop the functional-integral over it. Thus, we define off-shell amplitudes by the following expression

$$A(p_1,\dots p_N) \sim |(\omega_a-\omega_b)(\omega_b-\omega_c)(\omega_c-\omega_a)| \int \prod_{i\neq a,b,c} d\omega_i \, e^{\frac{1}{2}\tilde{\phi}(\omega_i)\Gamma_i}$$
$$\times \prod_{i=1}^{N} \theta(\omega_{i+1}-\omega_i) \prod_{i>j} |\omega_i-\omega_j|^{2\alpha' p_i p_j} T(p,\omega,\tilde{\phi}). \tag{3.13}$$

Note that in order to obtain sensible off-shell amplitudes it is important to fix a "gauge" for Möbius transformations *before* dropping the $\tilde{\phi}$ functional integral. Also this prescription for N-point amplitudes seems to make sense only for N$\geq$ 3. It is not clear how 2-point amplitudes can be defined in the present approach.

According to our prescription, off-shell amplitudes will in general depend on the choice of the metric $\tilde{\phi}$ and ω_a, ω_b and ω_c. However, this may not cause any problem, and, in fact, may be similar to the gauge dependence of off-shell quantities in Yang-Mills theories. Of course, these amplitudes give the usual $(\tilde{\phi}, \omega_a, \omega_b, \omega_c)$-independent results on-shell. We also note here that it is important to choose $\tilde{\phi}$ in (3.13) appropriate for the topology of a compact manifold. This is because the propagator $K(\omega,\omega'|\tilde{\phi})$ given in (A8) is singular in the limit $|\omega| \to \infty$. But as long as $\tilde{\phi}$ has the right asymptotic behavior for the manifold to be compact, i.e.,

$$\tilde{\phi} \sim -4 \ln |\omega| \quad (\text{as } |\omega| \to \infty) . \tag{3.14}$$

this singularity does not contribute to (3.13) and one gets finite answers.

We shall now explicitly evaluate (3.13) in some simple cases. We take $\tilde{\phi}$ appropriate for a flat disc or a hemisphere. The former is given by

$$\tilde{\phi}(\omega) = -4 \ln \left| 1 + i\frac{\omega}{R} \right| \quad , \tag{3.15}$$

where ω lies in the complex upper-half plane and R is the diameter of the disc.*
The metric of the latter is obtained by a projection from a pole,

$$\tilde{\phi}(\omega) = -2 \ln \left(1 + \frac{|\omega|^2}{R^2} \right) \quad . \tag{3.16}$$

We actually need the metric on the boundary, the real axis, which is the same for both cases

$$\tilde{\phi}(\omega) = -2 \ln \left(1 + \frac{\omega^2}{R^2} \right) \quad (\omega \quad \text{real}) \quad . \tag{3.17}$$

It is convenient to rewrite the amplitudes using the angle variable θ on the boundary, defined by $\omega = R \cot(\theta/2)$. Using the relation

$$|\omega_i - \omega_j| = \frac{R}{2} \frac{|\vec{e}_i - \vec{e}_j|}{\sin\frac{\theta_i}{2} \sin\frac{\theta_j}{2}} \quad , \tag{3.18}$$

where $\vec{e}_i$ is a unit vector from the center of the disc to a point i on the boundary, we can recast (3.13) in the form

$$\begin{aligned} A(p_1, \ldots p_N) \sim \prod_{i=1}^{N} \left(\frac{R}{2} \right)^{\Gamma_i} & |\vec{e}_a - \vec{e}_b||\vec{e}_b - \vec{e}_c||\vec{e}_c - \vec{e}_a| \\ & \times \int \prod_{i \neq a,b,c} d\theta_i \prod_{i=1}^{N} \theta(\theta_{i+1} - \theta_i) \prod_{i>j} |\vec{e}_i - \vec{e}_j|^{2\alpha' p_i p_j} T(p, \theta, \phi) . \end{aligned} \tag{3.19}$$

Note that the $\omega \to \infty$ ($\theta \to 0$) singularity has disappeared from (3.19). Also this expression depends on four "gauge" parameters R, θ_a, θ_b and θ_c. It is convenient

* The flat coordinate ω' is given by a conformal transformation, $\omega' = iR\omega/(\omega + iR)$.

to make the following symmetrical choices for the last three:

$$\theta_a = \theta_1 = 0\,, \quad \theta_b = \theta_2 = \frac{2\pi}{3}\,, \quad \theta_c = \theta_N = \frac{4\pi}{3} \quad . \tag{3.20}$$

We can now compute (3.19) for different cases. We quote the results for 3-tachyon, tachyon-tachyon-vector, and 3-vector amplitudes:

$$A(p_1, p_2, p_3) \sim \prod_{i=1}^{3} \left(\frac{\sqrt{3}}{2} R \right)^{\Gamma_i} \quad , \tag{3.21}$$

$$A^I(p_1, p_2, p_3) \sim \prod_{i=1}^{3} \left(\frac{\sqrt{3}}{2} R \right)^{\Gamma_i} (p_1 - p_2)^I \quad , \tag{3.22}$$

$$\begin{aligned} A^{I_1 I_2 I_3}(p_1, p_2, p_3) \sim & \prod_{i=1}^{4} \left(\frac{\sqrt{3}}{2} R \right)^{\Gamma_i} \\ & \times \left[\delta^{I_1 I_2} (p_1 - p_2)^{I_3} + \delta^{I_2 I_3} (p_2 - p_3)^{I_1} + \delta^{I_3 I_1} (p_3 - p_1)^{I_2} \right] \quad . \end{aligned} \tag{3.23}$$

In (3.21) - (3.23) we have quoted only the momentum dependent factors and omitted overall constant factors. All these amplitudes contain an appropriate factor of $\left(\frac{\sqrt{3}}{2}R\right)^{\Gamma_i}$ for each external line. In fact, it can be easily verified that this is true for an arbitrary amplitude. One can, therefore, consistently ignore such factors. (Alternatively one can set $R = 2/\sqrt{3}$.) We are then left with 3-point amplitudes which are just the right vertices for triple scalar, scalar-scalar-Yang-Mills and triple Yang-Mills field theory interactions. Of course, for a different choice of the values of θ_a, θ_b and θ_c the amplitudes in (3.13)-(3.15) would look different. In particular, the 3-point Yang-Mills amplitude would also contain terms of the form $\delta^{I_1 I_2} p_3^{I_3}$. However, such terms can be induced in field theory by choosing a nonlinear gauge fixing function (e.g. $F(A) = \partial A + A^2$). It is, therefore, not unreasonable to expect that different choices of θ's would correspond to different gauges, although the exact connection is not clear at the moment. The symmetrical choice in (3.12) is the simplest one and it reproduces the more familiar field theory vertices.

Amplitudes with more than three external lines can be computed similarly. They do not involve any new point of principle, only more labour is involved.

4. Fermionic Strings

Fermions were first introduced in string theories by Neveu and Schwarz[10] and by Ramond.[11] For our purposes, the most convenient starting point is the supersymmetrized version of the bosonic string action.[21] The procedure used in the bosonic case has then a straightforward extension to the 2-dimensional supergravity coupled to matter obtained in this way. The vertices have now to be locally supersymmetric (in the 2-dimensional sense) as well as general coordinate invariant. The simplest way to implement these twin requirements is to use superfields; the necessary formalism has been presented in Refs. 15 and 16. We therefore briefly summarize those aspects relevant to our discussion.

The 2-dimensional world sheet is extended to a curved, graded manifold with coordinates* $Z_M \equiv (z_m, \theta_\mu)$, where θ_μ is a real 2-component spinor. Diffeomorphisms, $Z_M \to Z_M + \xi_M(z,\theta)$, on the graded manifold can be defined as well as local frame rotations with parameter $\Lambda(z,\theta)$. The independent component fields that remain after the imposition of torsion constraints, solving the Bianchi identities and gauging away redundant components are accommodated in the $E_m{}^a(z,\theta)$ components of the supervierbein $E_M{}^A(z,\theta)$ ($A \equiv (a,\alpha)$ is the tangent space index); viz.

$$E_m{}^a = e_m{}^a(z) + \bar\theta\gamma^a\lambda_m(z) + \frac{i}{4}\,\bar\theta\,\theta\, e_m{}^a(z)\,A(z) \tag{4.1}$$

where $e_m{}^a$ is the 2-dimensional vierbein field, $\lambda_{m,\mu}(z)$ is the gravitino field and $A(z)$ is an auxiliary field. The string variable $x(z)$ is likewise generalized to a

* Except for the fact that $\bar\theta^\alpha = \epsilon^{\alpha\beta}\theta_\beta$ and $\theta_\alpha = \epsilon_{\alpha\beta}\bar\theta^\beta$, the metric and conventions used in this section are those of Ref. 15. A continuation to Euclidean metric in the function integral is understood.

scalar superfield[†]

$$V(z,\theta) = x(z) + \bar{\theta}\psi(z) + \frac{1}{2}\bar{\theta}\theta\, F(z) \tag{4.2}$$

where $F(z)$ is again an auxiliary field which vanishes by virtue of its equations of motion; ψ_μ is the 2-dimensional spinor and D-dimensional vector field of the Neveu-Schwarz-Ramond spinning strings.

The superspace string action is given by

$$S = \frac{1}{2\pi\,\alpha'}\int_M d^2z \int d^2\theta\, \frac{1}{4}\, E\overline{\nabla}^\alpha V \cdot \nabla_\alpha V \ , \tag{4.3}$$

where E is the superdeterminant of the supervierbein,

$$E = \mathrm{sdet} E_M{}^A \equiv \det\left(E_m{}^a - E_m{}^\alpha E_\alpha{}^\nu E_\nu{}^a\right) \det{}^{-1}(E_\mu{}^\alpha) \ , \tag{4.4}$$

$E_A{}^M$ being the inverse of $E_M{}^A$. Also $\nabla_\alpha = E_\alpha{}^M \nabla_M$, where ∇_M is the covariant derivative in superspace.

Manifestly reparametrization invariant (in superspace) and Lorentz covariant vertices for external boson lines can now be written down directly for closed fermionic strings. An obvious analogue of the tachyon vertex of the bosonic theory, Eq. (2.6), is the vertex*

$$V_T(p) = \int d^2z \int d^2\theta\ E(z,\theta)\, e^{ip\cdot V(z,\theta)} \tag{4.5}$$

and the N-tachyon amplitude is given, as before, by

$$A(p_1,\ldots,p_N) = \left\langle \prod_{i=1}^{N} V_T(p_i) \right\rangle \ . \tag{4.6}$$

It is also not difficult to see that an external graviton line should be associated

† A D-dimensional Lorentz index on V, x, ψ and F is understood; it will not be shown explicitly unless necessary.

* We use the same notation for vertices as in the bosonic case. Obviously, there is no scope for confusion.

with the vertex

$$V_G^{IJ}(p) = \int d^2z \int d^2\theta \, \frac{1}{4} E \, \overline{\nabla}^\alpha V^I \nabla_\alpha V^J \, e^{ip\cdot V} \ , \tag{4.7}$$

while for an external antisymmetric tensor field the appropriate vertex is

$$V_B^{IJ}(p) = -\int d^2z \int d^2\theta \, \frac{1}{4} E \, \overline{\nabla}^\alpha V^I \gamma_{5\alpha}{}^\beta \nabla_\beta V^J e^{ip\cdot V} \ . \tag{4.8}$$

In (4.8) $\gamma_5 = \gamma^0\gamma^1$ is the 2-dimensional γ_5-matrix. The case of open fermionic strings is a little more delicate and we shall discuss this separately below.

To analyze the component field content of these vertices it is convenient to work in the superconformal gauge in which the bosonic and fermionic parts of the string action decouple. In terms of component fields this gauge is specified by

$$e_m{}^a(z) = e^{\phi(z)} \delta_m^a \quad \text{and} \quad \lambda_{m,\alpha}(z) = \gamma_m \lambda_\alpha(z) \tag{4.9}$$

or, equivalently, in terms of superfields,

$$E_M{}^a = e^{\Phi} \widehat{E}_M{}^a \quad \text{and} \quad E_M{}^\alpha = e^{\frac{1}{2}\Phi} \left(\widehat{E}_M{}^\alpha + \frac{i}{2} \widehat{E}_M{}^a \overline{D}^\beta \Phi \gamma_{a,\beta}{}^\alpha \right) \ , \tag{4.10}$$

where the flat-space supervierbein, $\widehat{E}_M{}^A$, has the form

$$\widehat{E}_M{}^A = \left(\begin{array}{c|c} \widehat{E}_m{}^a = \delta_m^a & \widehat{E}_m{}^\alpha = 0 \\ \hline \widehat{E}_\mu{}^a = i(\gamma^a\theta)_\mu & \widehat{E}_\mu{}^\alpha = \delta_\mu^\alpha \end{array} \right) \ , \tag{4.11}$$

D_α is the usual spinor covariant derivative,

$$D_\alpha = \frac{\partial}{\partial \overline{\theta}^\alpha} - i(\partial\!\!\!/ \theta)_\alpha \ , \tag{4.12}$$

and $\Phi(z,\theta)$ is the scalar superfield

$$\Phi(z,\theta) = \phi(z) + \overline{\theta}\lambda(z) + \frac{1}{2} \overline{\theta}\,\theta\, f(z) \ . \tag{4.13}$$

In this gauge the superdeterminant of the supervierbein has the simple form

$$E = e^{\Phi(z,\theta)} . \tag{4.14}$$

Moreover, the derivative in (4.3) becomes $\nabla_\alpha V = e^{-\frac{1}{2}\Phi} D_\alpha V$. Thus the action becomes

$$\begin{aligned} S &= \frac{1}{2\pi\,\alpha'} \int\limits_M d^2 z \int d^2\theta\, \frac{1}{4} \overline{D}^\alpha V \cdot D_\alpha V \\ &= \frac{1}{2\pi\,\alpha'} \int\limits_M d^2 z\, \frac{1}{2} \left[(\partial_a x)^2 + i\overline{\psi}\partial\!\!\!/\psi + F^2 \right] . \end{aligned} \tag{4.15}$$

Similarly, one can work out the component content of vertices in this gauge. The tachyon vertex, for example, has the component structure

$$V_T(p) = \int d^2 z\, e^{\phi(z)} \left(\frac{1}{2} p \cdot \overline{\psi}\,\psi \cdot p + \cdots \right) e^{ip\cdot x(z)} . \tag{4.16}$$

while the graviton and the antisymmetric tensor vertices are respectively given by

$$\begin{aligned} V_G^{IJ}(p) = \frac{1}{2} \int d^2 z \Big\{ & \partial_a x^I \partial_a x^J + \frac{1}{2} (\overline{\psi}^I i\partial\!\!\!/\psi^J + \overline{\psi}^J i\partial\!\!\!/\psi^I) \\ & - \frac{1}{2} (\overline{\psi}^I \partial\!\!\!/ x^J \psi \cdot p + \overline{\psi}^J \partial\!\!\!/ x^I \psi \cdot p) \\ & + \frac{1}{4} p \cdot \overline{\psi}\,\psi \cdot p \overline{\psi}^I \psi^J + \cdots \Big\} e^{ip\cdot x(z)} \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} V_B^{IJ}(p) = \frac{1}{2} \int d^2 z \Big\{ & \epsilon^{ab} \partial_a x^I \partial_b x^J - \frac{i}{2} (\overline{\psi}^I \gamma_5 \partial\!\!\!/\psi^J - \overline{\psi}^J \gamma_5 \partial\!\!\!/\psi^I) \\ & + \frac{1}{2} (\psi^I \gamma_5 \partial\!\!\!/ x^J \psi \cdot p - \overline{\psi}^J \gamma_5 \partial\!\!\!/ x^I \psi \cdot p) \\ & - \frac{1}{4} \overline{\psi}^I \gamma_5 \psi^J p \cdot \overline{\psi}\,\psi \cdot p + \cdots \Big\} e^{ip\cdot x(z)} . \end{aligned} \tag{4.18}$$

The dots represent λ-dependent terms (λ has been defined in (4.9)).

As in the purely bosonic problem, amplitudes can be generated by coupling source superfield functions to the vertices and taking appropriate functional derivatives. For example, for the three vertices discussed above the source terms are given by

$$\exp\left[\int d^2z \int d^2\theta\, E\Big\{T(V) + \frac{1}{4}G_{IJ}(V)\overline{\nabla}^{\alpha}V^I\nabla_{\alpha}V^J \right.$$
$$\left. - \frac{1}{4}B_{IJ}(V)\overline{\nabla}^{\alpha}V^I\gamma_{5\alpha}{}^{\beta}\nabla_{\beta}V^J\Big\}\right] . \tag{4.19}$$

The action in (4.3) possesses a discrete symmetry which we shall discuss now. One can define the transformations $\theta \to \pm\gamma_5\theta$ in superspace under which $V(z,\theta)$ remains invariant if we assign to the component fields the transformations $x \to x$, $\psi \to \mp\gamma_5\psi$ and $F \to -F$. The action is invariant under these transformations provided the component fields of the supervierbein $E_m{}^a$, Eq. (4.1), transform as follows: $e_m{}^a \to e_m{}^a$, $\lambda_m \to \pm\gamma_5\lambda_m$ and $A \to -A$. These symmetries count (modulo two) separately the number of positive and negative helicity (in the 2-dimensional sense) excitations of the 2-component fermion field ψ in any physical state. They are essentially a generalization to closed strings of the G-parity encountered in the operator formulation of the fermionic open string theory. Since $d^2\theta \to -d^2\theta$ under $\theta \to \pm\gamma_5\theta$, it is clear that the tachyon vertex is odd while the graviton and the antisymmetric tensor vertices are even under these "parity" operations. Therefore, amplitudes with an odd number of external tachyon lines vanish. Moreover, if we impose the condition that the action including the source terms be invariant, then the tachyon source term in (4.19) is in fact entirely excluded. Such a condition can be reasonably enforced here because the sources can be assigned definite transformation properties under the above operations. (They are in fact trivially invariant since they are required to be functions of V only.) Along with the exclusion of tachyons, the above argument eliminates all source terms of the form

$$\int d^2z \int d^2\theta\, E\, A_{I_1J_1\cdots I_NJ_N,K_1L_1\cdots K_ML_M}(V)$$

$$\times \prod_{i=1}^{N} \overline{\nabla}^{\alpha_i} V^{I_i} \cdot \nabla_{\alpha_i} V^{J_i} \prod_{j=1}^{M} \overline{\nabla}^{\alpha_j} V^{K_j} \cdot \gamma_{5\alpha_j}{}^{\beta_j} \nabla_{\beta_j} V^{L_j}$$

for $N + M = 2k$, $k = 1, 2, \ldots$.

We now consider the case of open strings; here a rather careful treatment of boundary conditions is necessary. A detailed discussion is given in Ref. 16. The vanishing of boundary contributions to the variational problem for the action (4.3) requires, in terms of component fields, that

$$n \cdot \partial x = 0 \quad \text{and} \quad \psi_1 = \pm\sqrt{\frac{n_+}{n_-}}\,\psi_2 \quad (n_\pm \equiv n_1 \pm n_0)\ , \tag{4.20}$$

where n^a is a unit vector normal to the boundary and $\psi_{1,2}$ are the 2 components of ψ. On the upper half-plane with x-axis as the boundary, the last condition reads

$$\psi_1 = \pm\,\psi_2\ . \tag{4.21}$$

The two different possibilities in (4.21) give rise to the well known bosonic (Neveu-Schwarz) and fermionic (Ramond) sectors of the spinning string. By considering supersymmetry transformations of the component fields with the spinor parameter ϵ_α $(\alpha = 1, 2)$ it can be seen that these boundary conditions are compatible with only one supersymmetry on the boundary, namely those transformations that satisfy

$$\epsilon_1 = \mp\,\epsilon_2\ , \tag{4.22}$$

provided one also imposes the further boundary conditions

$$F = 0 \quad \text{and} \quad (\partial_0 + \partial_1)\psi_2 = \pm\,(\partial_0 - \partial_1)\psi_1\ . \tag{4.23}$$

These additional conditions lead to a problem discussed in the appendix.

The boundary condition (4.21) on ψ also implies that the action no longer has the symmetries discussed earlier. However, $V(z,\theta)$ is still invariant under the transformations $\theta \to -\theta$, $\psi \to -\psi$, $x \to x$, $F \to F$. If we assign similar transformations to the components of $E_m{}^a$ then the action is also invariant under this "parity" operation. This symmetry simply counts (modulo two) the number of ψ-excitations present in any physical state and is just the G-parity of the operator formulation in the Neveu-Schwarz sector and fermion number "parity" in the Ramond sector. Since the tachyon is odd under G-parity, it is clear that the vertex for it must have a "loose" fermionic index. On the other hand, the vector is even under G-parity and so the vector vertex must have only contracted fermionic indices. The vector vertex also has to carry a Lorentz index, so it must involve the derivative $\nabla_\alpha V^I$. These requirements uniquely fix the vector vertex to be

$$V_A^I(p) = \frac{1}{2}\int d\sigma_m \int d^2\theta\, E\, \overline{\nabla}^\alpha V^I \gamma_{5\alpha}{}^\beta E_\beta{}^m\, e^{ip\cdot V} \tag{4.24}$$

where $d\sigma_m = dz^n \epsilon_{nm}$. The γ_5 is essential, otherwise the vertex vanishes when boundary conditions are used. The component content of this vertex is easily worked out in the superconformal gauge:

$$V_A^I(p) = \int ds \left(\frac{d}{ds}x^I - \frac{1}{2}e^{-\phi}\overline{\psi}^I\gamma_5\, \not{n}\psi\cdot p\right) e^{ip\cdot x}\ . \tag{4.25}$$

This may be compared with a very similar expression for the vector vertex in the operator approach.[22] For the tachyon vertex we try the expressions

$$V_{T\pm}(p) = \int d\sigma_m \int d^2\theta\, E\,(E_1{}^m \pm E_2{}^m)\, e^{ip\cdot V} \tag{4.26}$$

obtained form the quantity

$$\int d\sigma_m \int d^2\theta\, E\, E_\alpha{}^m\, e^{ip\cdot V}$$

which has a "loose" fermionic index and is suggested by the form of the vector vertex in (4.24). The component content of (4.26) in the superconformal gauge (on the upper half-plane with x-axis as the boundary) is

$$V_{T_\pm}(p) = \int dx \left\{ (\psi_2 \pm \psi_1) \cdot p + \cdots \right\} e^{ip\cdot x} \; , \tag{4.27}$$

where the dots represent λ-dependent terms. Clearly, V_{T_+} vanishes for the '-' boundary condition in (4.21) while V_{T_-} vanishes for the '+' boundary condition. The vertex, however, looks essentially the same in both cases and may be compared with a similar expression for the tachyon vertex in the operator approach.[22]

Vertices for higher excited states can be obtained simply by inserting factors of $\overline{\nabla}^\alpha V \nabla_\alpha V$ and $\overline{\nabla}^\alpha V \gamma_{5\alpha}{}^\beta \nabla_\beta V$ in the tachyon and vector vertices described above. In this way one gets two sets of vertices—those odd under G-parity and the even ones. As in the previous cases, amplitudes can be generated by adding source terms to the action. We can also eliminate the odd vertices (including the tachyon) from external lines by demanding that the generating functional for amplitudes be invariant under the "parity" symmetry discussed earlier.

5. Concluding Remarks

In this paper, we have described a manifestly covariant procedure for computing string amplitudes within Polyakov's geometrical formulation of string theories. In the present approach calculations are simpler and more transparent than in previous approaches. For bosonic strings we have also given a definition of off-shell amplitudes which seems to give sensible results. We expect a similar definition to carry over to fermionic strings. In the case of fermionic strings we have succeeded in constructing amplitudes with external bosonic lines only. It seems much more nontrivial to incorporate external fermions in this framework. Needless to say that a solution of this problem is urgently required in view of the emergence of the 10-dimensional superstring theory as a potential unified theory of all known interactions. The formalism presented here is sufficient to compute tree level amplitudes in both the bosonic and the superstring theory. We hope to return to loop calculations in a future publication.

ACKNOWLEDGEMENTS

It is a pleasure to thank M. Peskin for useful discussions and critical comments on the manuscript. We would also like to thank I. Antoniadis for discussions and to B. Ratra and P. Franzini for help in obtaining equation (A18). One of us (M. A. N.) would like to thank V. P. Nair and B. F. L. Ward for discussions. He is also indebted to Prof. S. D. Drell for hospitality at the SLAC theory group.

APPENDIX

To calculate amplitudes for bosonic strings one needs the 2-dimensional Green's function $K(z,z'|g)$ which satisfies the equation

$$\Delta K(z,z'|g) = \delta(z,z'|g) \tag{A1}$$

where the covariant Laplacian Δ and the δ-function $\delta(z,z'|g)$ are given by

$$\Delta = -\frac{1}{\sqrt{g}}\,\partial_a\left(\sqrt{g}\,g^{ab}\partial_b\right) \tag{A2}$$

$$\delta(z,z'|g) = \frac{1}{\sqrt{g}}\,\delta^{(2)}(z-z') \; . \tag{A3}$$

In the open string case K also satisfies the boundary condition $n\cdot\partial k(z,z'|g) = 0$. One can write down an expression for K in terms of the eigenfunctions of Δ:

$$K(z,z'|g) = \sum_n \lambda_n^{-1} f_n(z)\, f_n(z') \; , \tag{A4}$$

$$\Delta f_n(z) = \lambda_n f_n(z) \ , \tag{A5}$$

where for simplicity we have assumed a discrete spectrum. The zero mode is excluded from the sum in (A4). This expression for K is singular at coincident points. A covariant prescription for regularizing this singularity is

$$K_\epsilon(z, z'|g) = \sum_n \lambda_n^{-1} f_n(z)\, f_n(z')\, e^{-\epsilon\lambda_n} \ . \tag{A6}$$

In practice it is hard to obtain a closed form expression for K_ϵ using (A6). A simpler prescription is to use a sharp cut-off on the invariant distance between z and z' when it is smaller than ϵ. This leads to the following expressions for K_ϵ in the conformal gauge, $g_{ab} = e^\phi \delta_{ab}$:

$$K_\epsilon(z, z'|\phi) = \begin{cases} -\frac{1}{4\pi} \ln|z - z'|^2 & (z \neq z') \\ \frac{1}{2\pi} \ln \frac{1}{\epsilon} + \frac{1}{4\pi}\phi(z) & (z = z') \ , \end{cases} \tag{A7}$$

on the entire complex plane and

$$K_\epsilon(z, z'|\phi) = \begin{cases} -\dfrac{1}{4\pi}(\ln|z - z'|^2 + \ln|z - \bar{z}'|^2) & (z \neq z') \\ \dfrac{1}{2\pi}\ln\dfrac{1}{\epsilon} + \dfrac{1}{4\pi}\phi(z) - \dfrac{1}{4\pi}\ln|z - \bar{z}'|^2 & (z = z',\ \mathrm{Im} z \neq 0) \\ \dfrac{1}{\pi}\ln\dfrac{1}{\epsilon} + \dfrac{1}{2\pi}\phi(z) & (z = z',\ \mathrm{Im} z = 0) \ , \end{cases} \tag{A8}$$

on the upper half-plane with x-axis as the boundary.

In computing off-shell amplitudes involving external particles other than tachyons, one also encounters derivatives of K at coincident points. It turns out that a proper regularization for these singular objects is to simply take derivatives of appropriate expressions in (A7) and (A8). As explained in the main text, this regularization preserves the Mōbius invariance of the amplitudes.

The above procedure can be used for obtaining Green's functions in the case of fermionic strings. Here the analogues of Eqs. (A1)-(A3) are

$$\Delta_s K_s(z\theta, z'\theta'|E) = \delta(z\theta, z'\theta'|E) \tag{A9}$$

where the superspace Laplacian Δ_s is

$$\Delta_s = -\frac{1}{E}\,\overline{\nabla}^\alpha E \nabla_\alpha \,, \tag{A10}$$

and

$$\delta(z\theta, z'\theta'|E) = \frac{1}{E}\,\delta^{(2)}(z-z')\,\delta^{(2)}(\theta-\theta')\,,\ \delta^{(2)}(\theta-\theta') = \frac{1}{2}\,(\bar\theta-\bar\theta')\,(\theta-\theta')\,. \tag{A11}$$

The eigenfunctions of Δ_s are now superfields:

$$\Delta_s V_n(z,\theta) = \lambda_n V_n(z,\theta) \tag{A12}$$

and a regularized expression, analogous to (A6), can be written down for K_s. In the superconformal gauge, however, (A9) can be easily solved for the case of closed strings. In this gauge (A9) becomes

$$-\overline{D}D\,K_s(z\theta, z'\theta'|\Phi) = \delta^{(2)}(z-z')\,\delta^{(2)}(\theta-\theta')\ . \tag{A13}$$

Then using the identify

$$(\overline{D}D)^2 = -4\partial^2 \tag{A14}$$

we can solve (A13) for K_s in terms of the bosonic Green's function given in (A7):

$$\begin{aligned} K_s(z\theta, z'\theta'|\Phi) &= -\frac{1}{4}\,\overline{D}D\,K_\epsilon(z,z'|\phi)\,\delta^2(\theta-\theta') \\ &= \frac{1}{2}\left\{1 + i\bar\theta\not{\partial}\theta' - \frac{1}{4}(\bar\theta\theta)(\bar\theta'\theta')\partial^2\right\} K_\epsilon(z,z'|\phi)\ . \end{aligned} \tag{A15}$$

Green's function for the open string case is not obtained so readily because of the complicated boundary conditions (4.20)-(4.23). In fact, it is impossible to obtain a K_s for these boundary conditions. To understand the reason for this, consider the eigenvalue problem

$$\not{\partial}\psi_n = \lambda_n\psi_n \tag{A16}$$

which must be solved to obtain the fermion propagator on the world sheet. The boundary conditions $\psi_{1n} = \pm\psi_{2n}$ and (A16) then imply that we must also have $(\partial_0 + \partial_1)\psi_2 = \mp(\partial_0 - \partial_1)\psi_1$ on the boundary, otherwise no solutions to (A16) exist (except for $\lambda_n = 0$). Also, by looking at the component equations of (A12) one can see that f_n and F_n must satisfy identical boundary conditions. Thus the appropriate set of boundary conditions for which we can obtain Green's function is

$$n \cdot \partial x = 0 = n \cdot \partial F\,, \quad \psi_1 = \pm\psi_2 \quad \text{and} \quad (\partial_0 + \partial_1)\psi_2 = \mp(\partial_0 - \partial_1)\psi_1\ . \tag{A17}$$

These boundary conditions spontaneously break the remaining supersymmetry on the boundary. This situation can be avoided if one is interested only in the determinant of the Laplacian Δ_s,[16] since then one can define the determinant to be the square root of the determinant of the square of the Laplacian, for which the above problem does not exist. For computing amplitudes, however, we need the Green's function and so we must use (A17). For these boundary conditions the bosonic part of the Green's function is the same as (A8). The fermionic part is given (in Euclidean space) by

$$S(z,z') = i\ \not{\partial}K(z,z') + 2\begin{pmatrix} \mp i\partial_{\bar{z}} & -\partial_{\bar{z}} \\ \partial_z & \mp i\partial_z \end{pmatrix}\tilde{K}(z,z'), \tag{A18}$$

where $i\ \not{\partial} = \begin{pmatrix} 0 & 2\partial_{\bar{z}} \\ -2\partial_z & 0 \end{pmatrix}$, $K(z,z')$ is the bosonic propagator on the upper half-plane and $\tilde{K}(z,z') \equiv -\frac{1}{4\pi}\ln|z-\bar{z}'|^2$. One can regularize the short distance behavior of $S(z,z')$ by using regularized expression for K and $\tilde{K}$ as in (A8).

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