CS 250B: Modern Computer Systems

Cache-Efficient Algorithms

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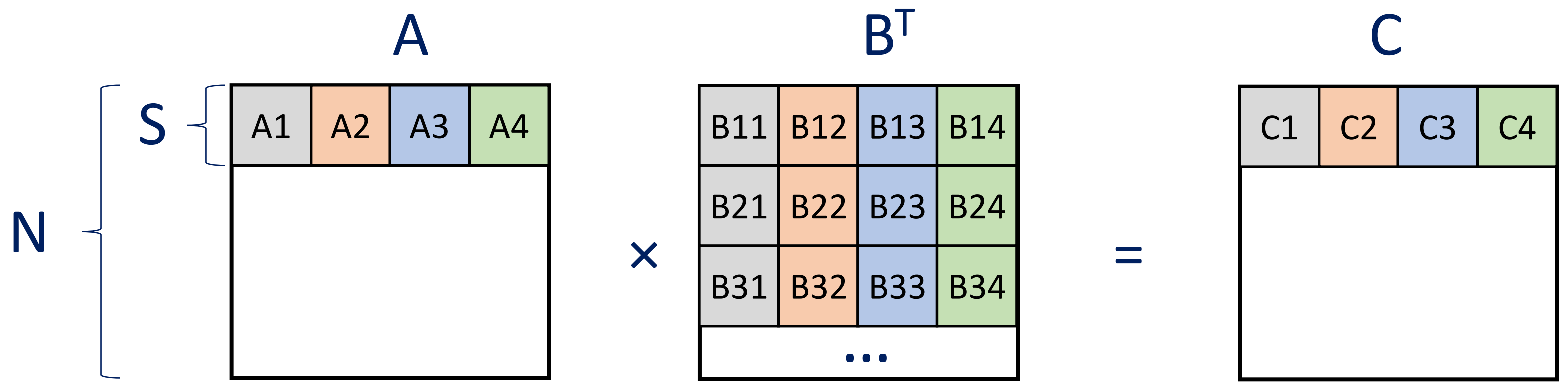
Back To The Matrix Multiplication Example

- Blocked matrix multiplication recap
  - C1 sub-matrix = A1×B11 + A1×B21 + A1×B31 …
  - Intuition: One full read of B^T per S rows in A. Repeated N/S times
- Best performance when S^2 ~= Cache size
  - Machine-dependent magic number!

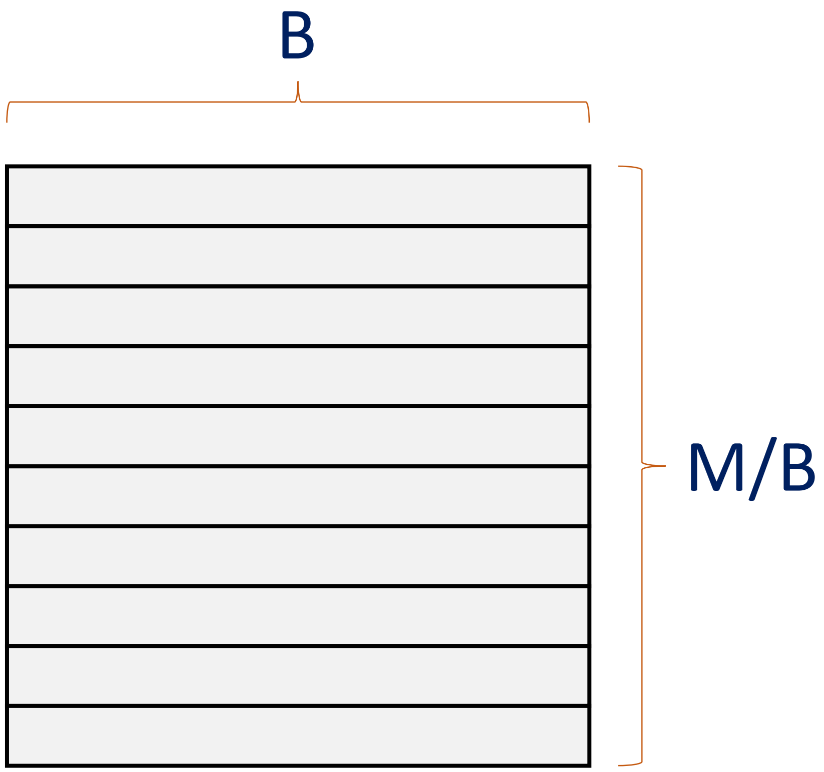


Back To The Matrix Multiplication Example

- For sub-block size S × S -> N * N * (N/S) reads. What S do we use?
  - Optimized for L1? (32 KiB for me, who knows for who else?)
  - If S*S exceeds cache, we lose performance
  - If S*S is too small, we lose performance
- Do we ignore the rest of the cache hierarchy?
  - Say S optimized for L3,
S × S multiplication is further divided into T×T blocks for L2 cache
  - T × T multiplication is further divided into U×U blocks for L1 cache
  - …

Solution: Cache Oblivious Algorithms

- No explicit knowledge of cache architecture/structure
  - Except that one exists, and is hierarchical
  - Also, “tall cache assumption”, which is natural
- Still (mostly) cache optimal
- Typically recursive, divide-and-conquer



Tall cache assumption: $B^2 < cM$ for a small c

ex) Modern Intel L1: M: 64 KiB, B: 16 B

Shorter cache with larger lines can’t efficiently divide data into small blocks

Aside: Even More Important With Storage/Network

- Latency difference becomes even larger
  - Cache: ~5 ns
  - DRAM: 100+ ns
  - Network: 10,000+ ns
  - Storage: 100,000+ ns
- Access granularity also becomes larger
  - Cache/DRAM: Cache lines (64 B)
  - Storage: Pages (4 KB – 16 KB)

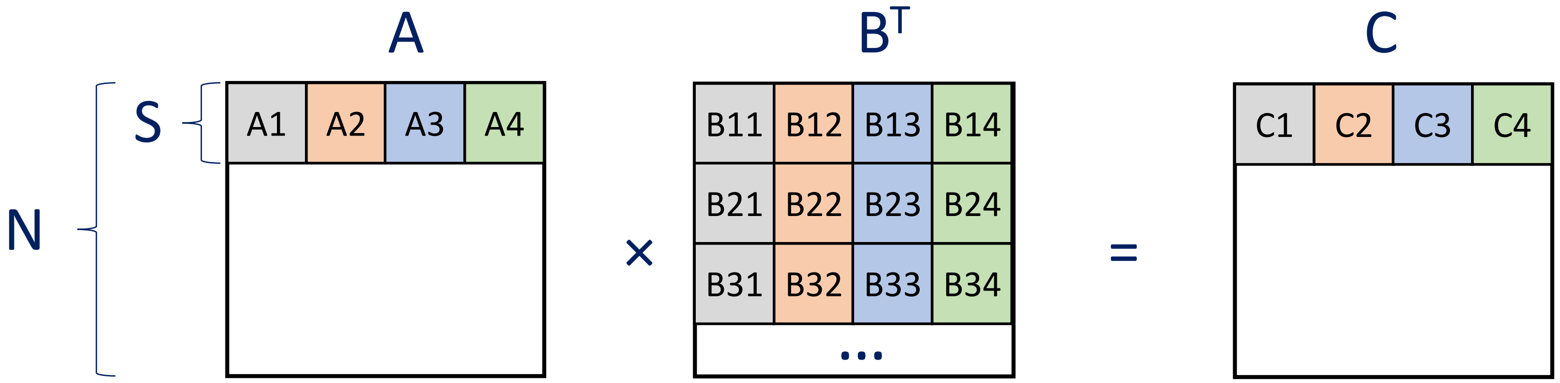
Also see: “Latency Numbers Every Programmer Should Know”
https://people.eecs.berkeley.edu/~rcs/research/interactive_latency.html

Applications of Interest

- Matrix multiplication
- Merge Sort
- Stencil Computation
- Trees And Search

Cache Optimized Matrix Multiplication

- How to make sure we use an optimal S, for all cache levels?



Recursive Matrix Multiplication

$$\overset{C}{\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}} = \overset{A}{\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}} \times \overset{B}{\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}}$$

$$= \begin{bmatrix} A_{11}B_{11} & A_{11}B_{12} \\ A_{21}B_{11} & A_{21}B_{12} \end{bmatrix} + \begin{bmatrix} A_{12}B_{21} & A_{12}B_{22} \\ A_{22}B_{21} & A_{22}B_{22} \end{bmatrix}$$

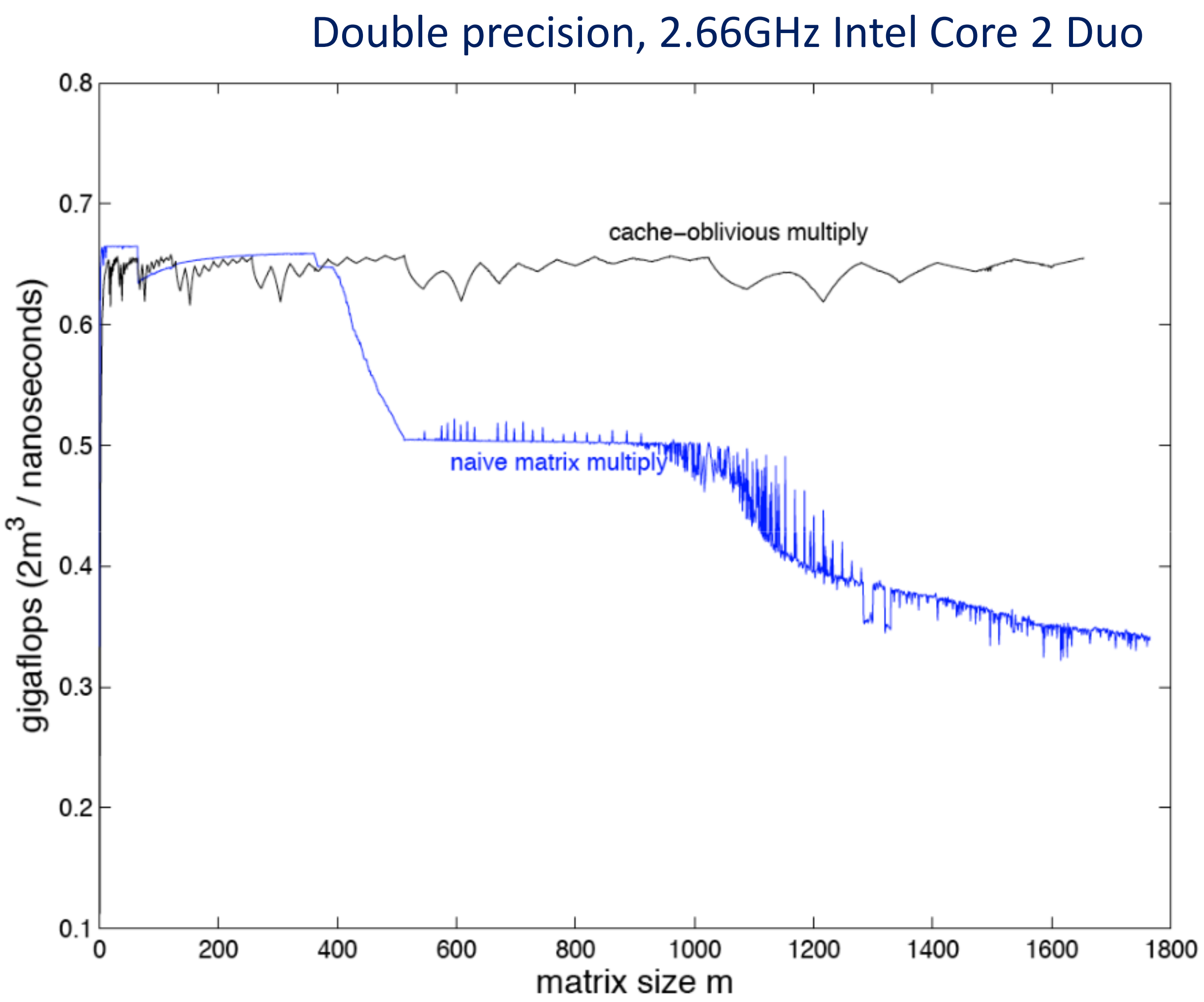
8 multiply-adds of (n/2) × (n/2) matrices
Recurse down until very small

Performance Analysis

- Work:
  - Recursion tree depth is $\log_2(N)$, each node fan-out is 8
  - $8^{\log_2 N} = N^{\log_2 8} = N^3$
  - Same amount of work!
- Cache misses:
  - Recurse tree for cache access has depth log(N)-1/2(log(cM))
    - (Because we stop recursing at $n^2 < cM$ for a small c)
  - So number of leaves = $8^{\log N - 1/2 \log cM} = N^{\log 8} \div cM^{1/2 \log 8} = N^3 / cM^{3/2}$
  - At leaf, we load cM/B cache lines
  - Total cache lines read = $\theta(\frac{n^3}{BM^{1/2}})$ <- Optimal

Also, logN function call overhead is not high

Performance Oblivious to Cache Size



Steven G. Johnson, “Experiments with Cache-Oblivious Matrix Multiplication for 18.335,” MIT Applied Math

Bonus: Cache-Oblivious Matrix Transpose

- Also possible to define recursively

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad A^T = \begin{bmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{bmatrix}$$

Applications of Interest

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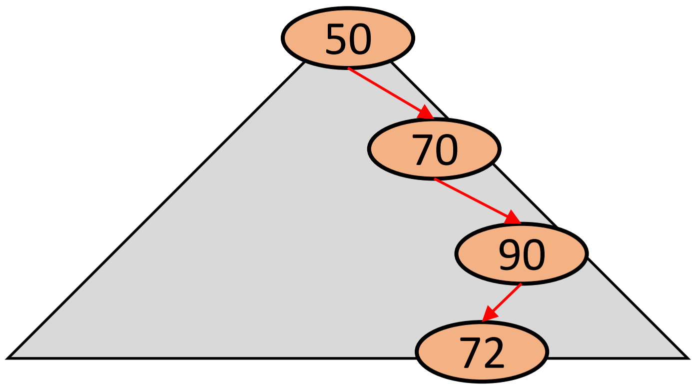
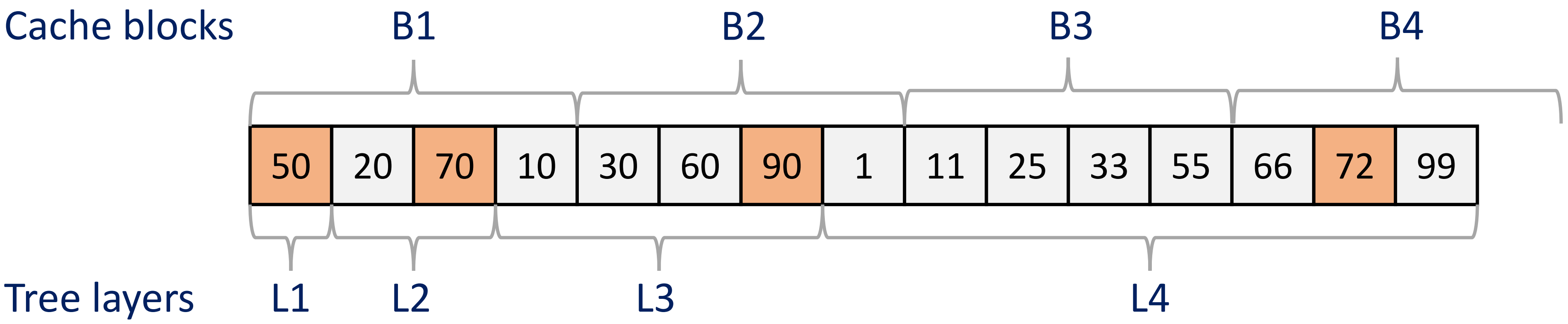
- Trees And Search
- Merge Sort
- Stencil Computation

Trees And Search



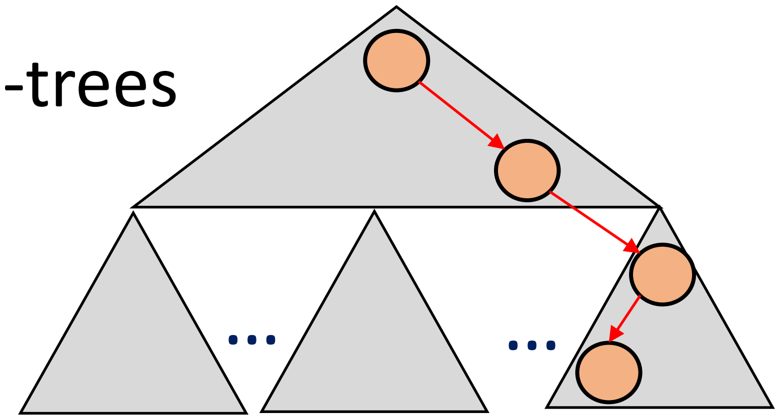
- Binary Search Trees are cache-ineffective
  - e.g., Searching for 72 results in 3 cache line reads
  - Not to mention trees in the heap!

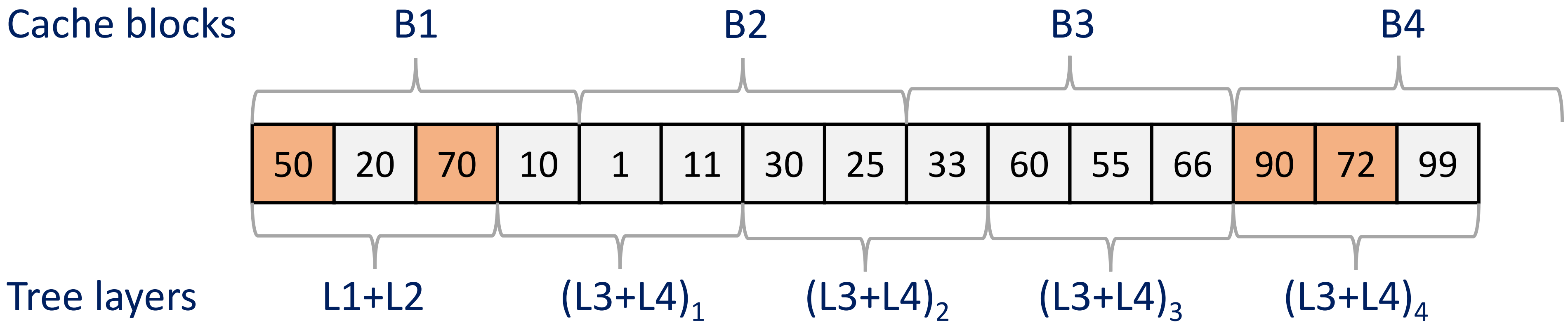


Each traversal pretty much hits new cache line: ϴ(Log(N)) cache lines read

Better Layout For Trees

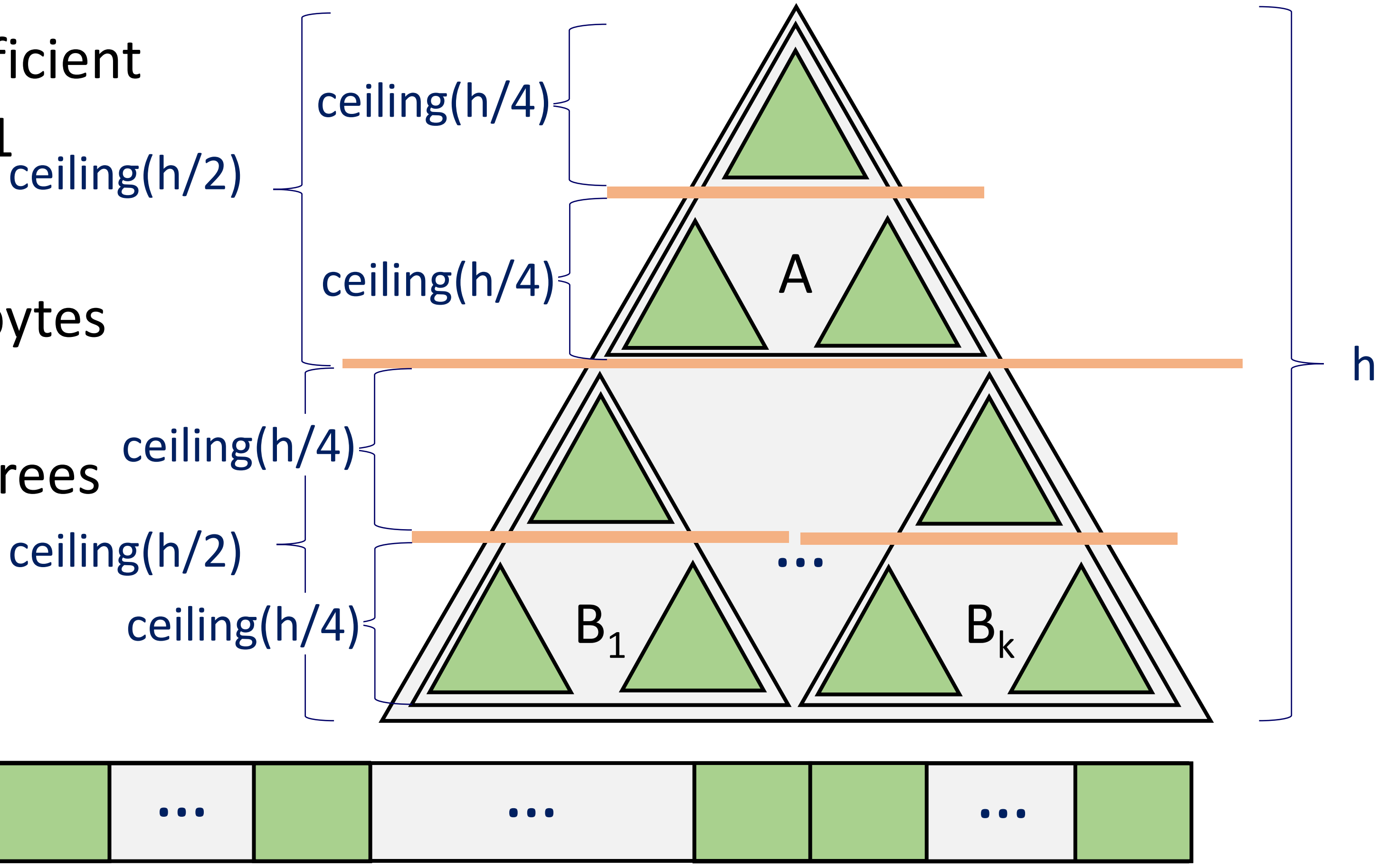
- Tree can be organized into locally encoded sub-trees
  - Much better cache characteristics!
  - We want cache-obliviousness:
    How to choose the size of sub-tree?





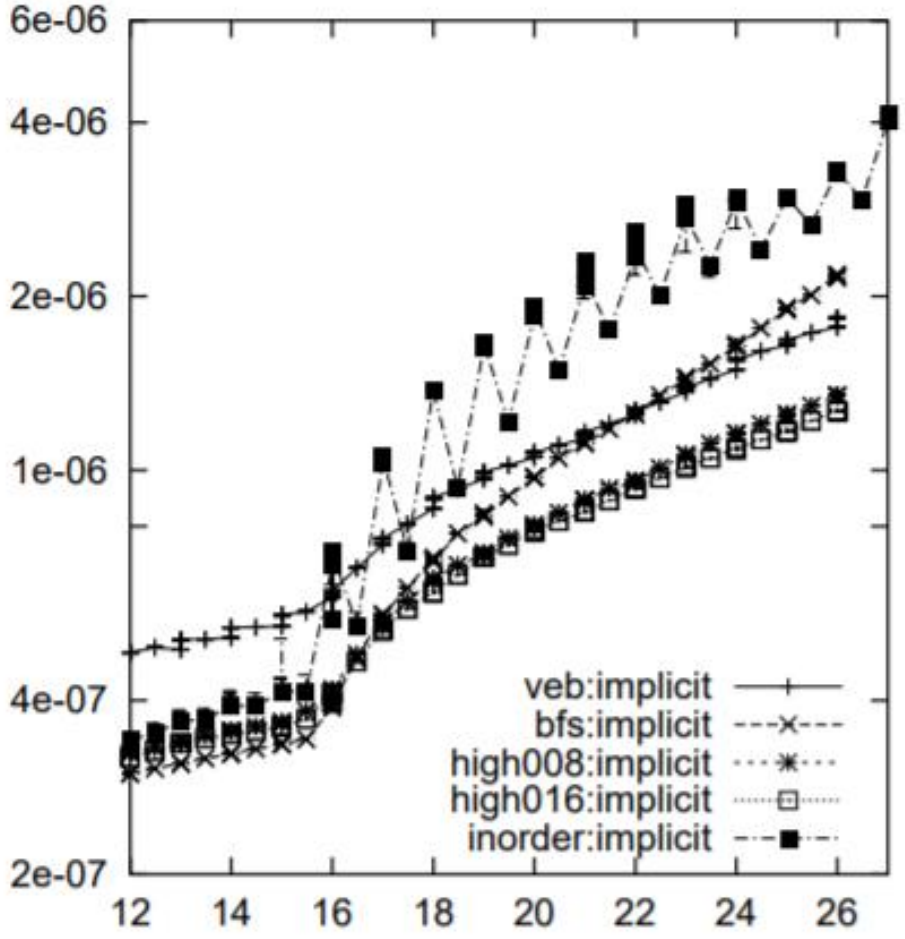
Recursive Tree Layout: van Emde Boas Layout

- Recursively organized binary tree
  - Needs to be balanced to be efficient
  - Recurses until sub-tree is size 1
- In terms of cache access
  - Recursion leaf has **cache line** bytes
  - Sub-tree height: log(B)
  - Traverses $\log_B N$ leaf (green) trees



Performance Evaluations Against Binary Tree

Time (Log)



Tree size

1 GHz Pentium III (Coppermine)
256 KB cache
1 GB DRAM

high8, high16:
8 or 16 children per node

Brodal et.al., "Cache Oblivious Search Trees via Binary Trees of Small Height," SODA 02

Performance Evaluations Against Binary Tree And B-Tree

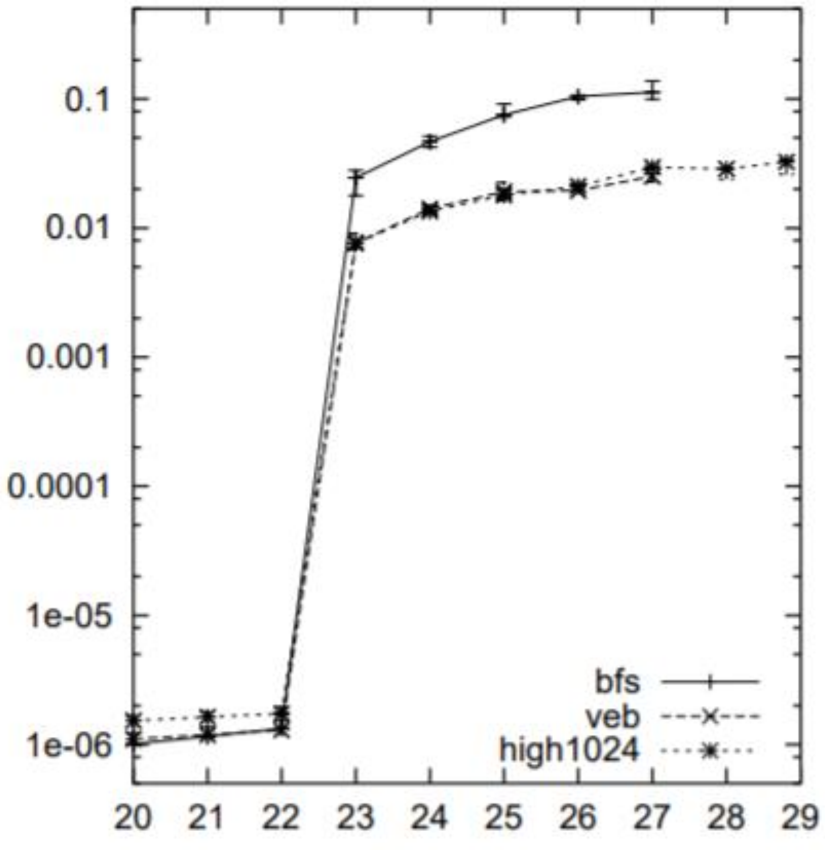


Figure 8: Beyond main memory

**** High1024:*** 1024 elements per node, to make use of the whole cache line (B-Tree)

Question: How do we optimize N in HighN?
Databases use N optimized for storage page

Note: Storage access not explicitly handled!
Letting swap handle storage management

Brodal et.al., “Cache Oblivious Search Trees via Binary Trees of Small Height,” SODA 02

More on the van Emde Boas Tree

- Actually a tricky data structure to do inserts/deletions
  - Tree needs to be balanced to be effective
  - van Emde Boas trees with van Emde Boas trees as leaves?
- Good thing to have, in the back of your head!

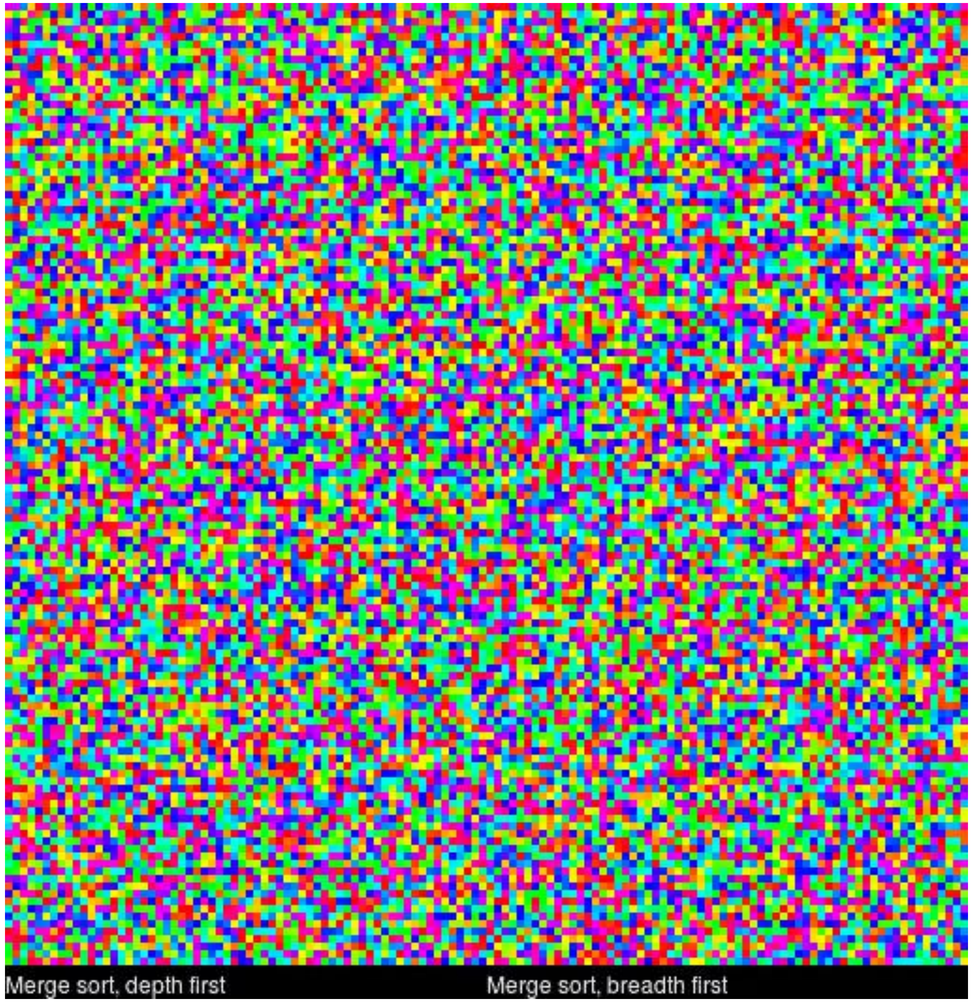
Applications of Interest

- ❑ Matrix multiplication
- ❑ Trees And Search
- ❑ Merge Sort
- ❑ Stencil Computation

Merge Sort

Depth-first

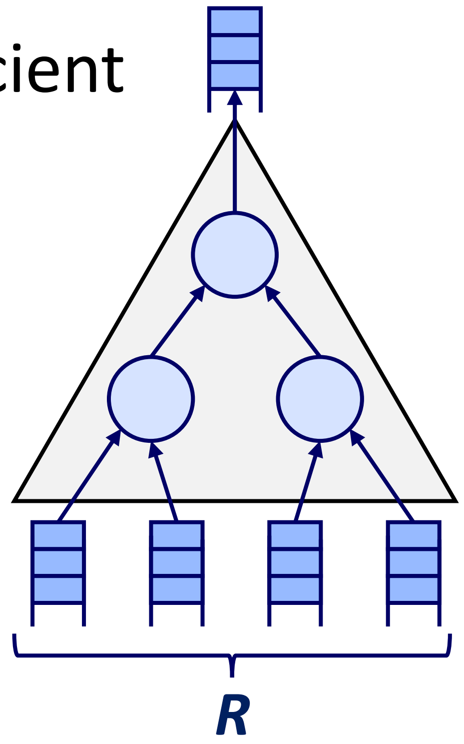
Breadth-first



Source: https://imgur.com/gallery/voutF, created by morolin

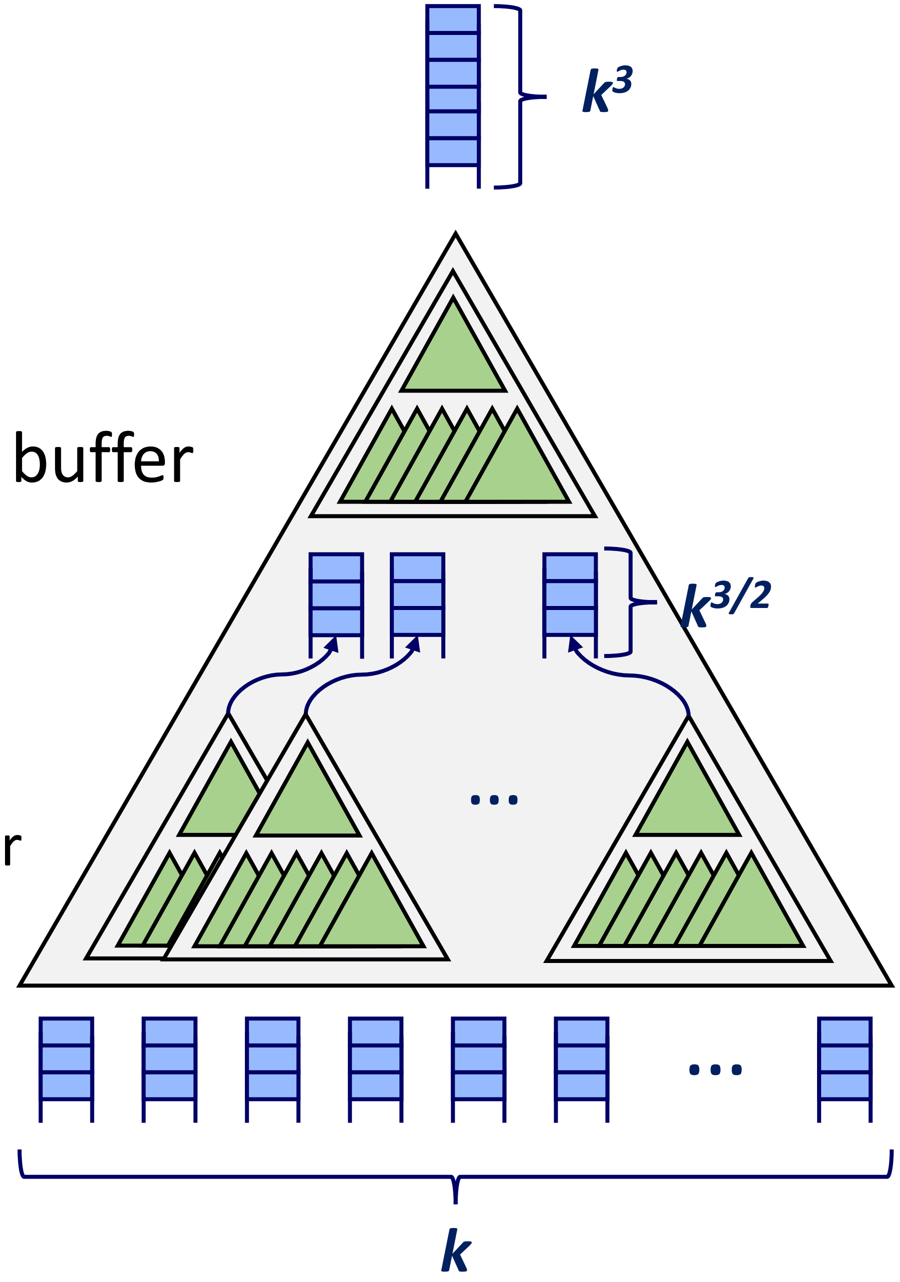
Merge Sort Cache Effects

- Depth-first binary merge sort is relatively cache efficient
  - Log(N) full accesses on data, for blocks larger than M
  - $n \times \log(\frac{n}{M})$
- Binary merge sort of higher fan-in (say, R) is more cache-efficient
  - Using a tournament of mergers!
  - $n \times \log_R(\frac{n}{M})$
- Cache obliviousness: how to choose R?
  - Too large R spills merge out of cache -> Thrash -> Performance loss!



Lazy K-Merger

- Again, recursive definition of mergers!
- Each sub-merger has k^3 element output buffer
- Second level has $\sqrt{k}+1$ sub-mergers
  - $\sqrt{k}$ sub-mergers feeding into 1 sub-merger
  - Each sub-merger has $\sqrt{k}$ inputs
  - $k^{3/2}$-element buffer per bottom sub-merger
  - Recurses until very small fan-in (two?)

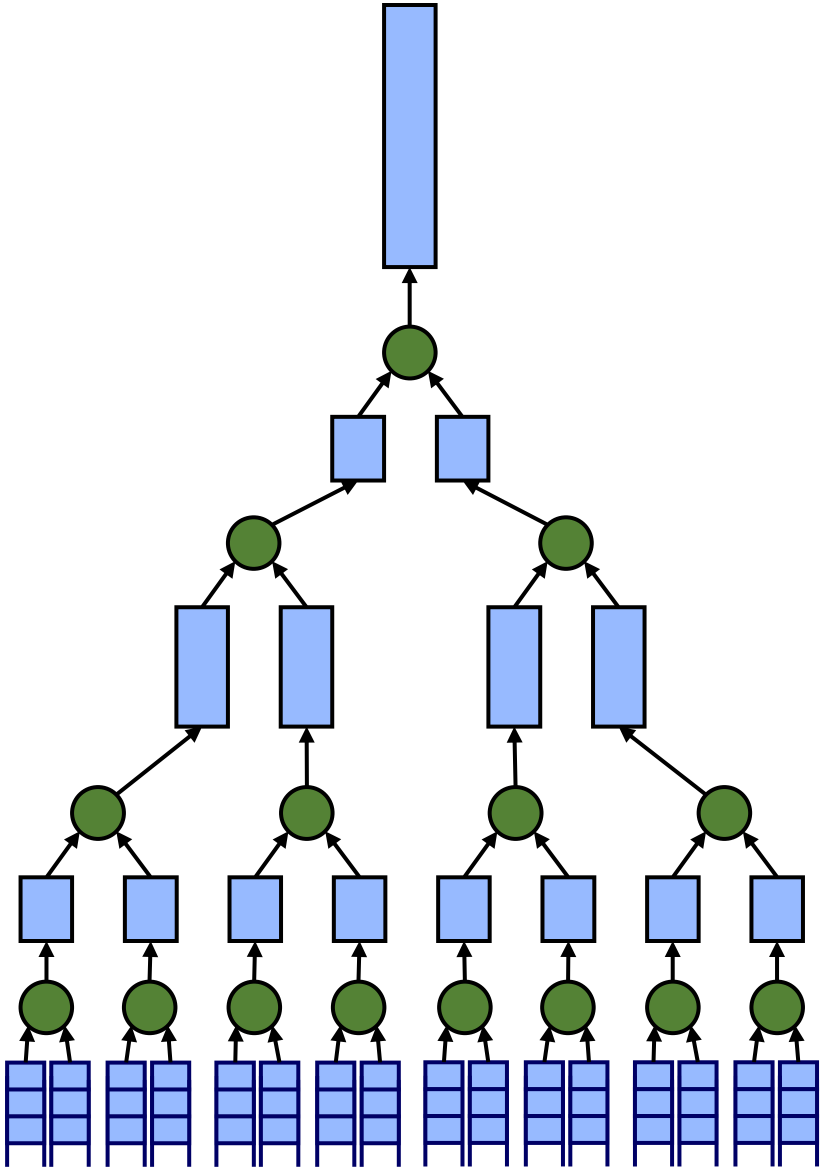


Lazy K-Merger

Procedure **Fill**(v):

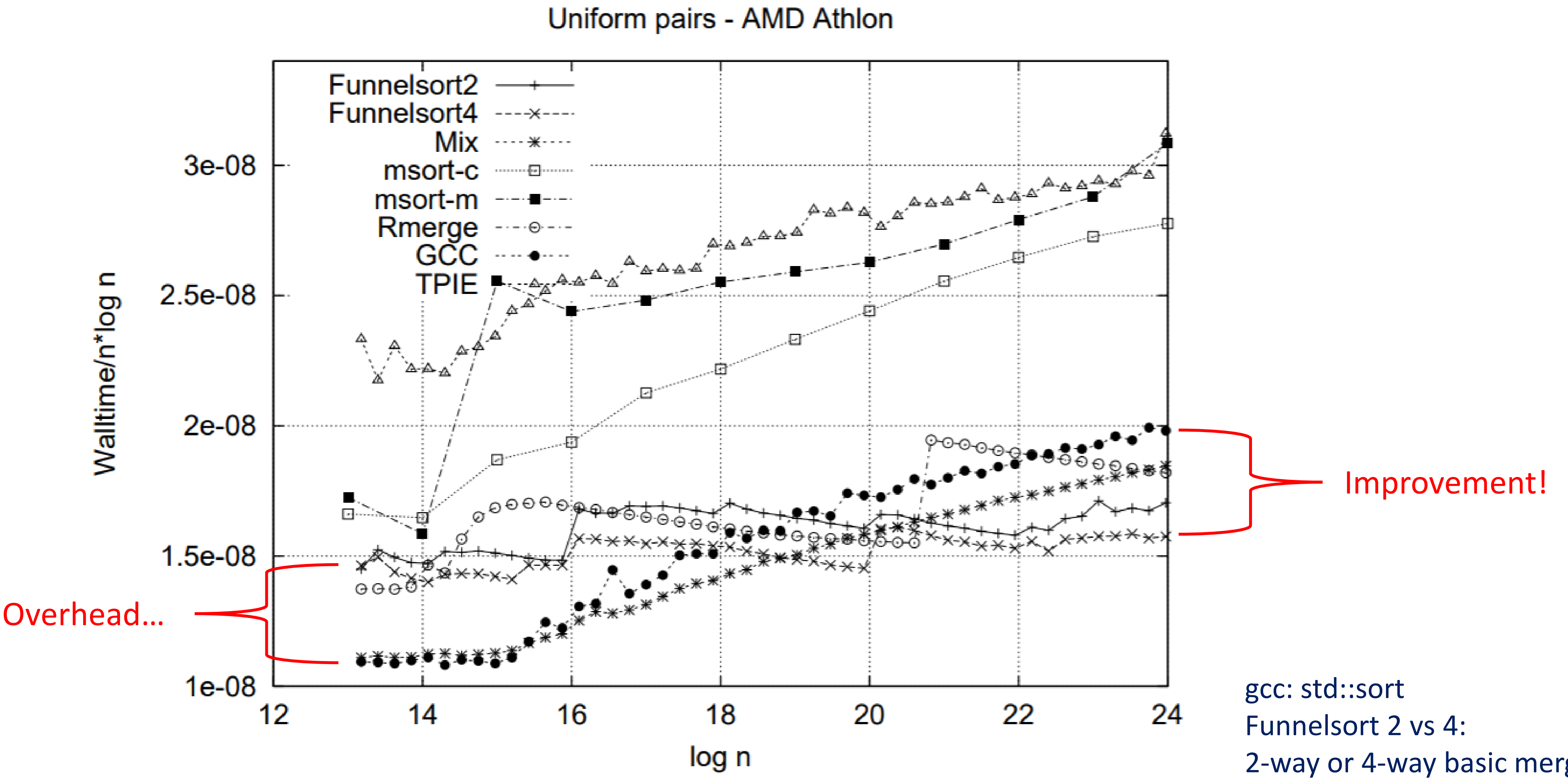
```
while v's output buffer is not full
    if left input buffer empty
        Fill(left child of v)
    if right input buffer empty
        Fill(right child of v)
    perform one merge step
```

- ❑ Each k merger fits in k^2 space
- ❑ Ideal cache effects!
  - Proof too complex to show today…
- ❑ What should k be?
  - Given N elements, $k = N^{(1/3)}$ – "Funnelsort"



Source: Brodal et. al., "Engineering a Cache-Oblivious Sorting Algorithm," 2008

In-Memory Funnelsort Empirical Performance



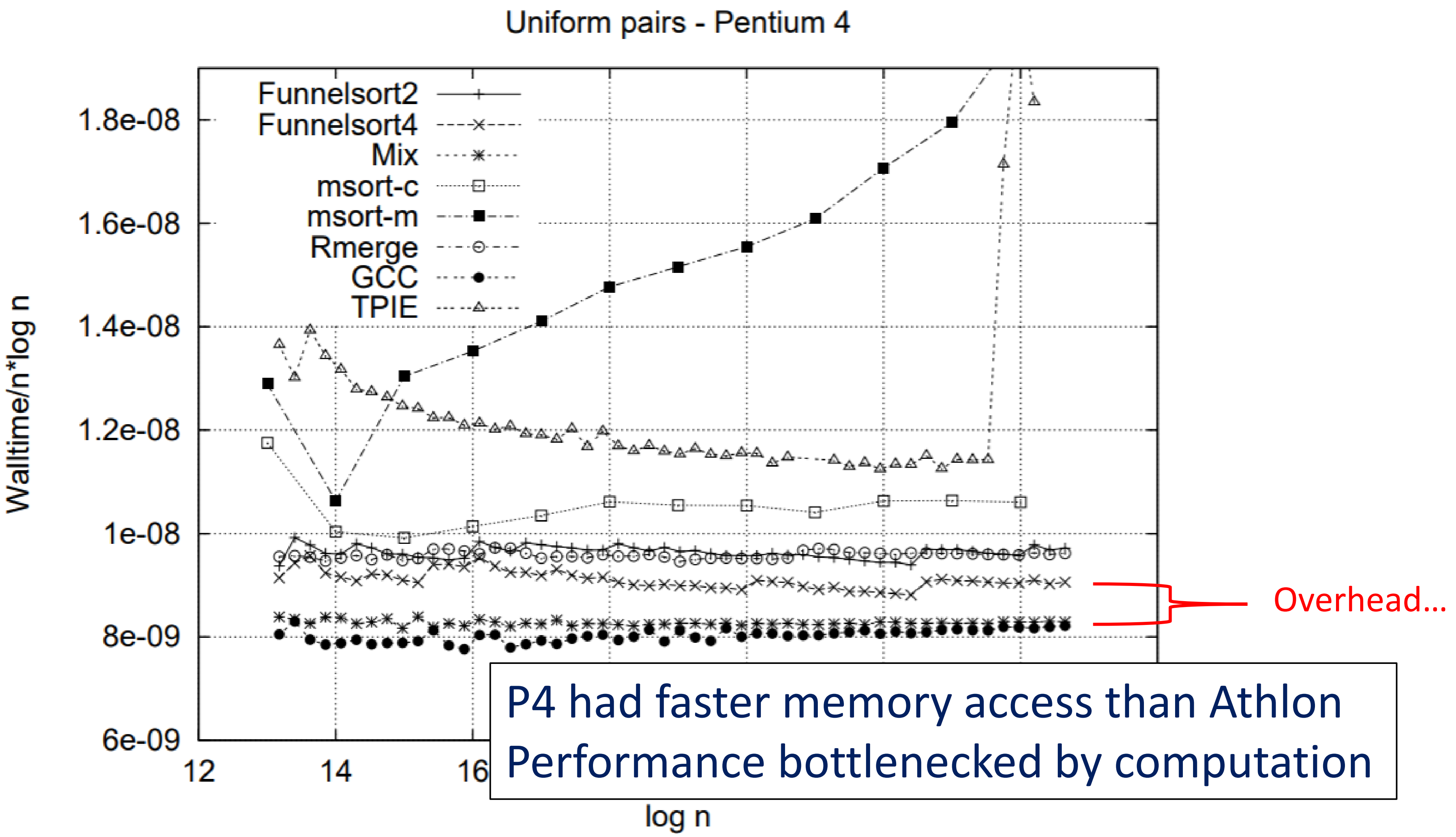
gcc: std::sort

Funnelsort 2 vs 4:

2-way or 4-way basic merger

Source: Brodal et. al., "Engineering a Cache-Oblivious Sorting Algorithm"

In-Memory
Funnelsort Empirical Performance

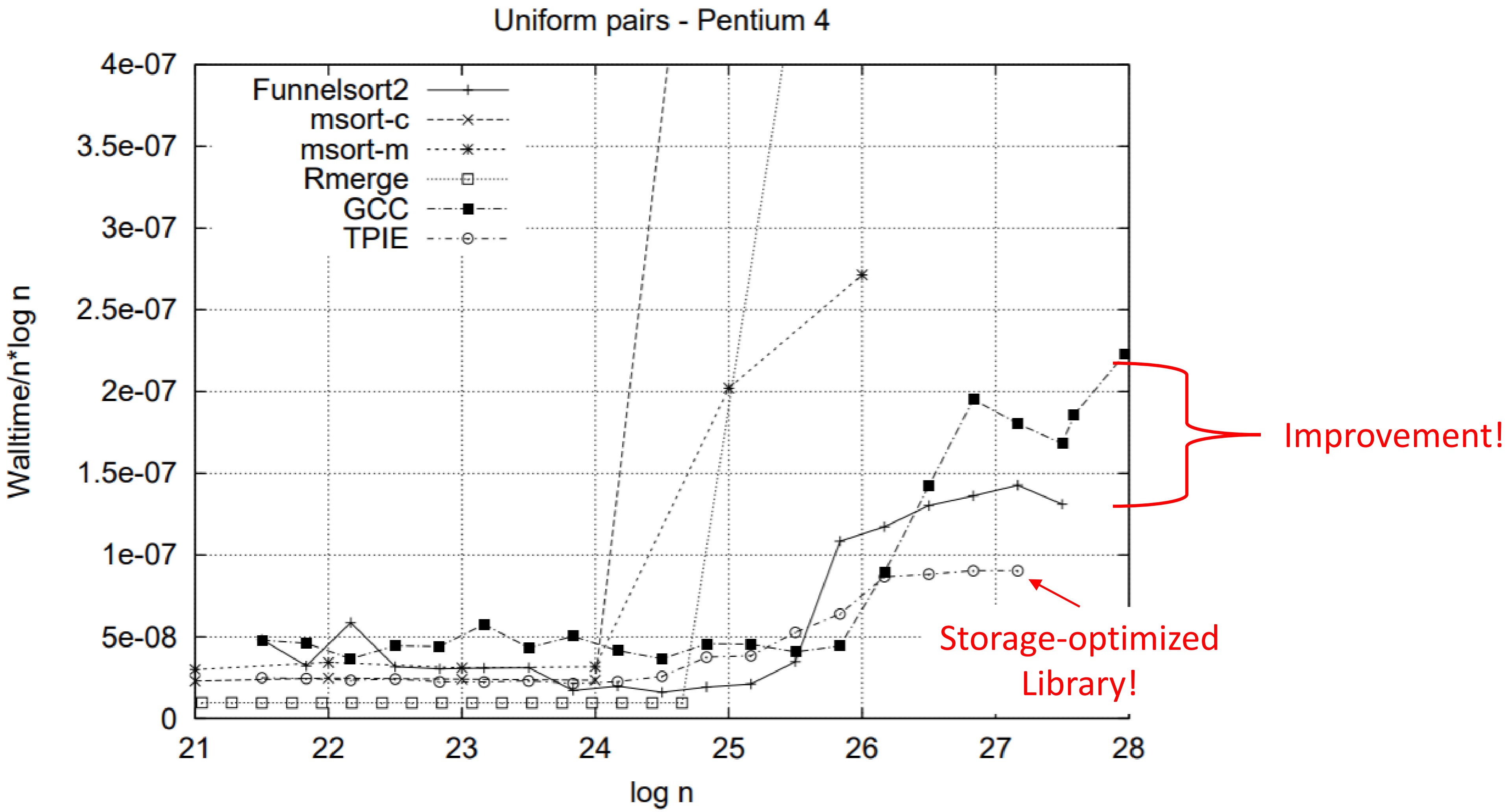


P4 had faster memory access than Athlon

Performance bottlenecked by computation

Source: Brodal et. al., “Engineering a Cache-Oblivious Sorting Algorithm”

In-Storage Funnelsort Empirical Performance



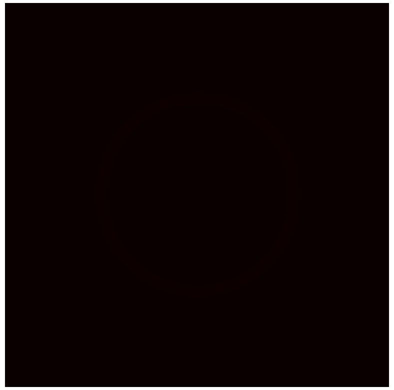
Applications of Interest

- Matrix multiplication
- Trees And Search
- Merge Sort
- Stencil Computation

Stencil Computation

- Example: Heat diffusion
  - Uses parabolic partial differential equation to simulate heat diffusion

$$\frac{\partial u}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$



Heat Equation In Stencil Form

- Simplified model: 1-dimensional heat diffusion $\frac{\partial u}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial x^2} \right)$

$$\frac{\partial u}{\partial t} = \lim_{\Delta t \to 0} \frac{u(x, t+\Delta t) - u(x,t)}{\Delta t}$$

$$\frac{\partial u}{\partial t} \approx \frac{u(x, t+\Delta t) - u(x,t)}{\Delta t}$$

$$u_x(x+\Delta x, t) \approx \frac{u(x+\Delta x, t) - u(x,t)}{\Delta x}$$

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial u_x}{\partial x} \\
&\approx \frac{u_x(x+\Delta x, t) - u_x(x,t)}{\Delta x} \\
&\approx \frac{\frac{u(x+\Delta x,t)-u(x,t)}{\Delta x}}{\Delta x} - \frac{\frac{u(x,t)-u(x-\Delta x,t)}{\Delta x}}{\Delta x} \\
&= \frac{u(x+\Delta x, t) - 2u(x,t) + u(x-\Delta x, t)}{(\Delta x)^2}
\end{aligned}$$

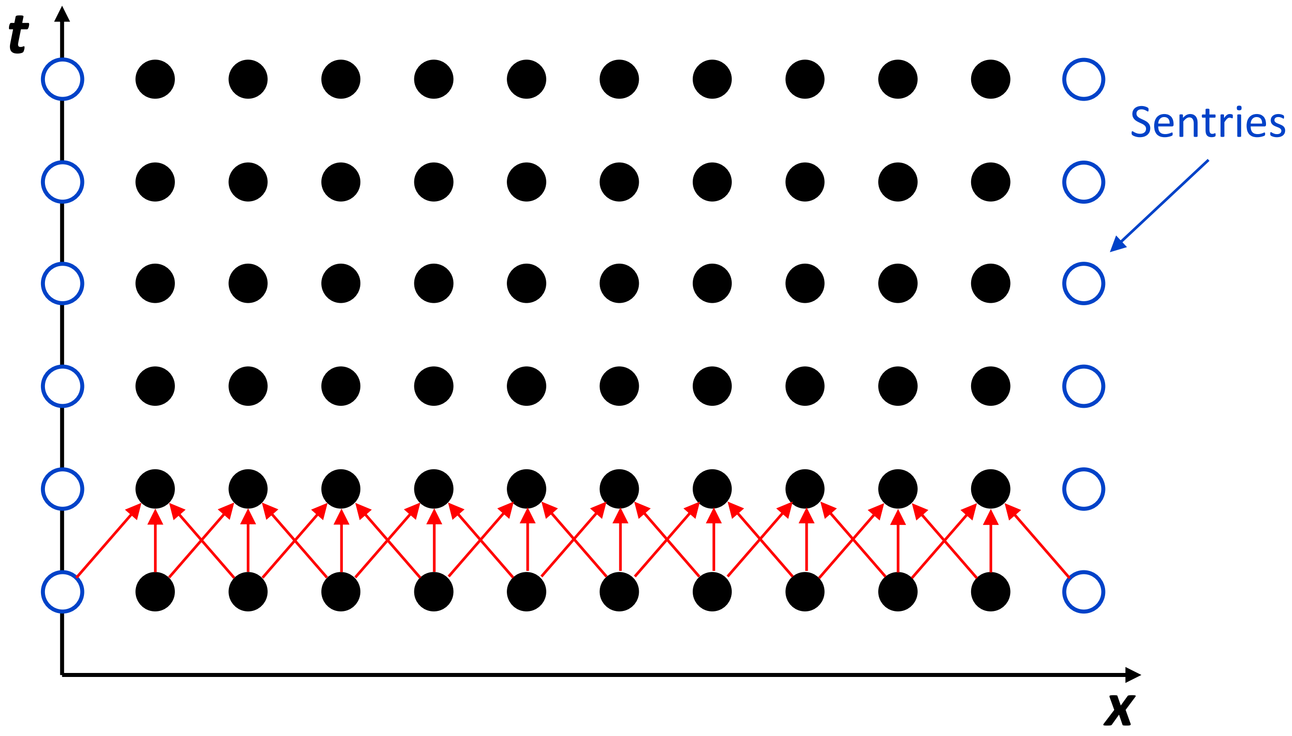
$$\frac{u(x, t+\Delta t) - u(x,t)}{\Delta t} \approx k \frac{u(x+\Delta x, t) - 2u(x,t) + u(x-\Delta x, t)}{(\Delta x)^2}$$

$$u(x, t+\Delta t) \approx u(x,t) + \alpha \left[u(x+\Delta x, t) - 2u(x,t) + u(x-\Delta x, t) \right]$$

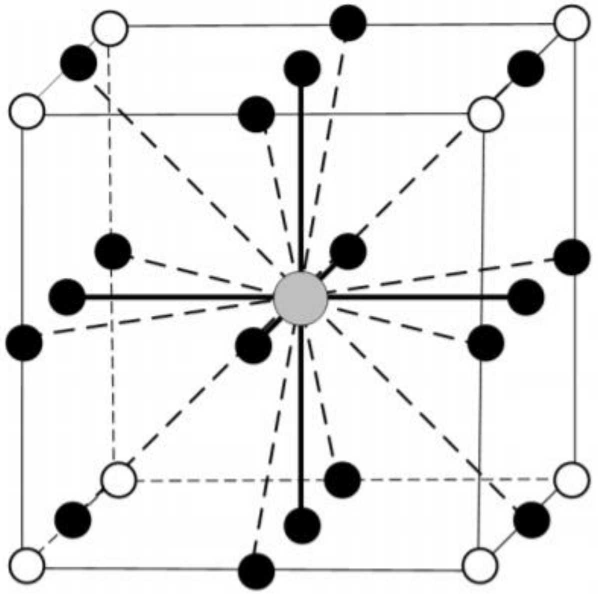
A 3-point Stencil

$$u(x, t + \Delta t) \approx u(x, t) + \alpha \left[u(x + \Delta x, t) - 2u(x, t) + u(x - \Delta x, t) \right]$$

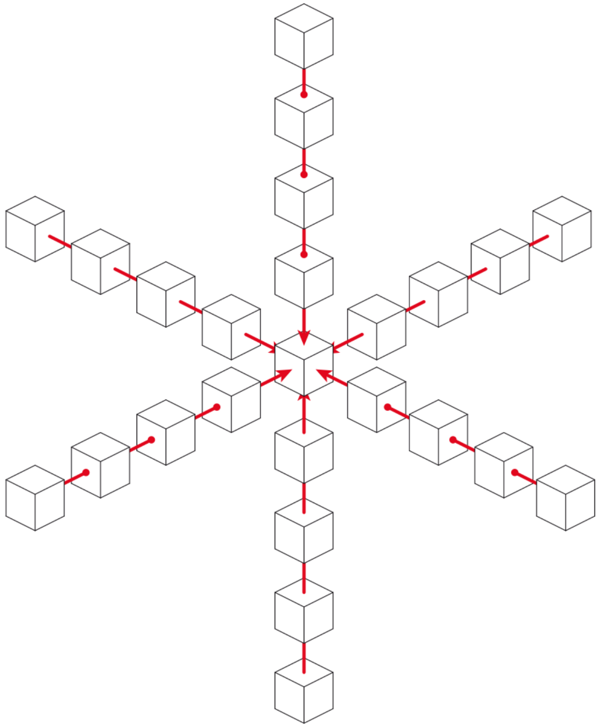
- u(x, t + Δt) can be calculated using u(x, t), u(x + Δx, t), u(x - Δx, t)
- A “stencil” updates each position using surrounding values as input
  - This is a 1D 3-point stencil
  - 2D 5 point, 2D 9 point, 3D 7 point, 3D 25-point stencils popular
  - Popular for simulations, including fluid dynamics, solving linear equations and PDEs



Some Important Stencils



[1] 19-point 3D Stencil for
Lattice Boltzmann Method flow simulation

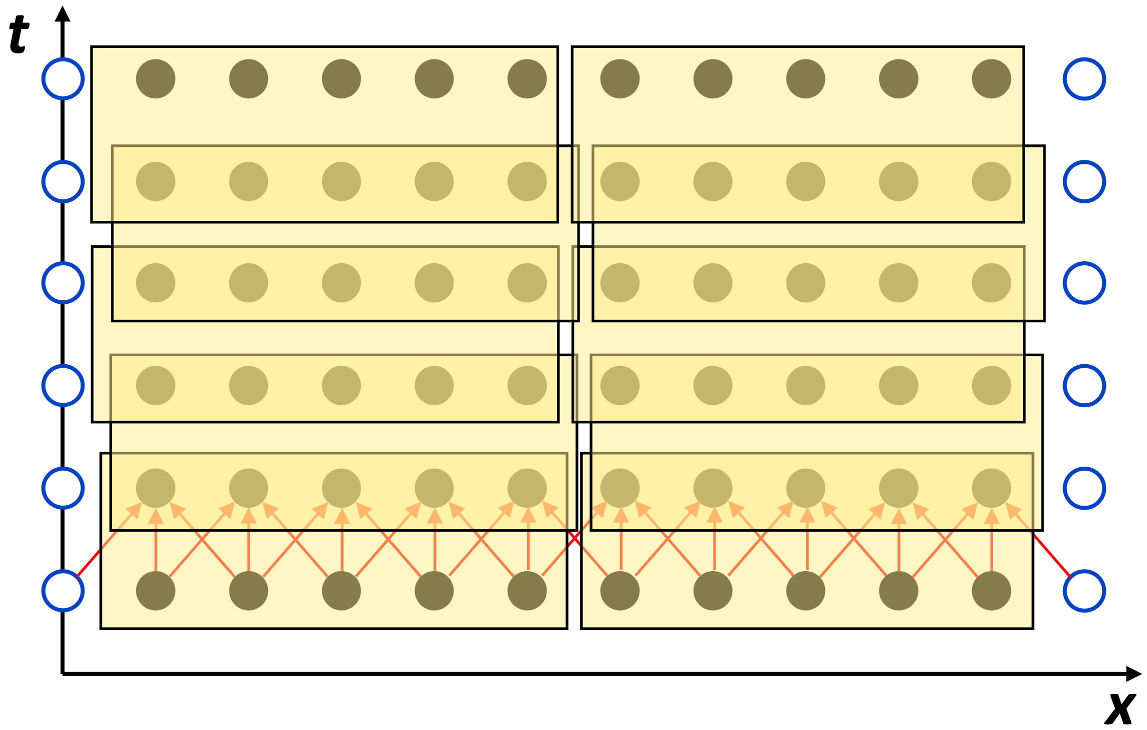


[2] 25-point 3D stencil for
seismic wave propagation applications

[1] Peng, et. al., “High-Order Stencil Computations on Multicore Clusters”
[2] Gentryx, Wikipedia

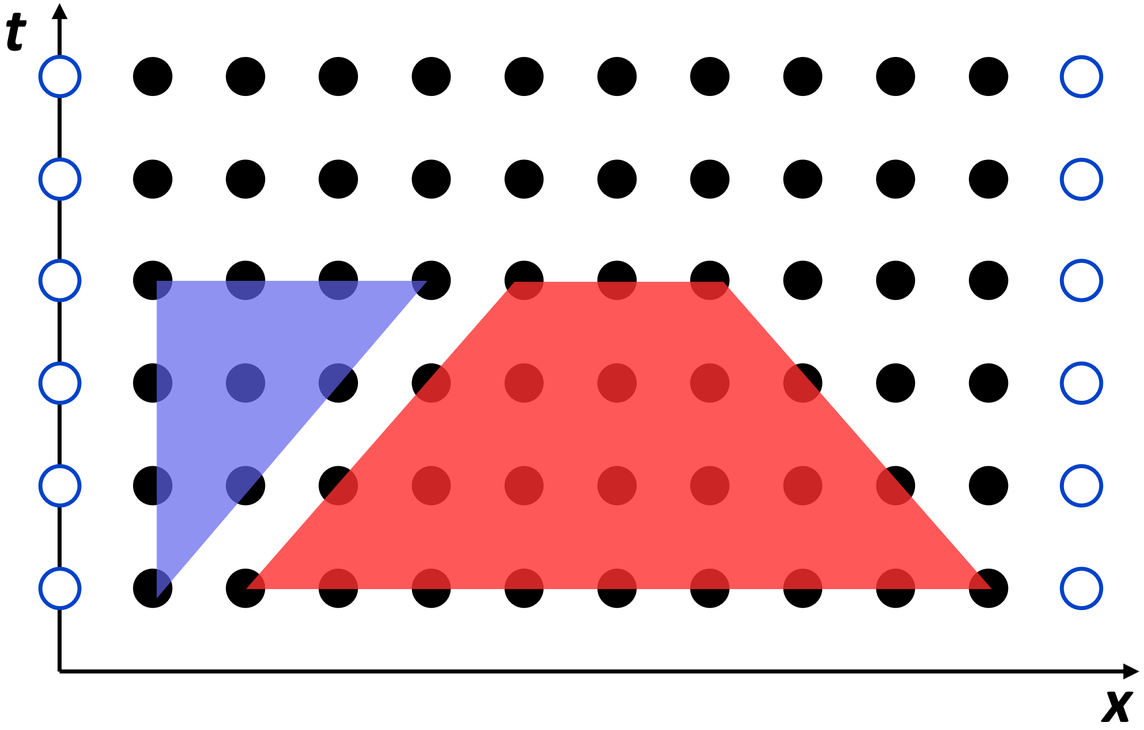
Cache Behavior of Naïve Loops

- Using the 1D 3-point stencil
  - Unless x is small enough, there is no cache reuse
- Continuing the theme, can we recursively process data in a cache-optimal way?



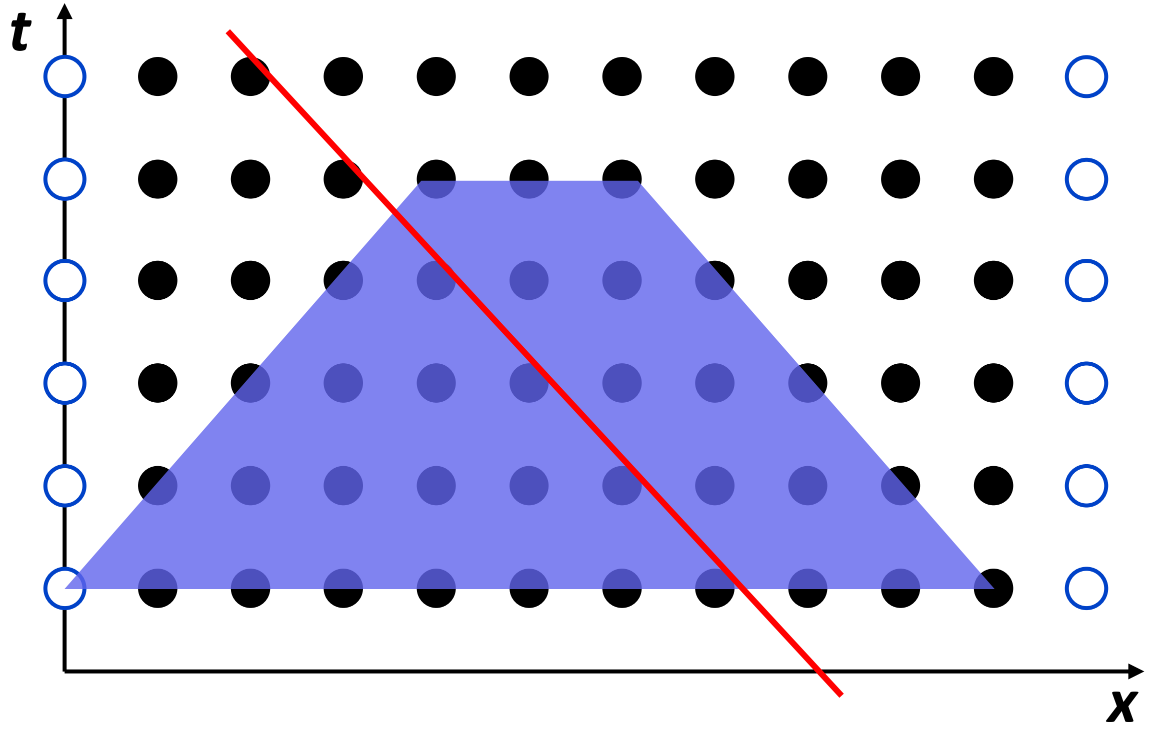
Cache Efficient Processing: Trapezoid Units

- Computation in a trapezoid is either:
  - Self-contained, does not require anything from outside(), or
  - Only uses data that has been computed and ready ( , after )



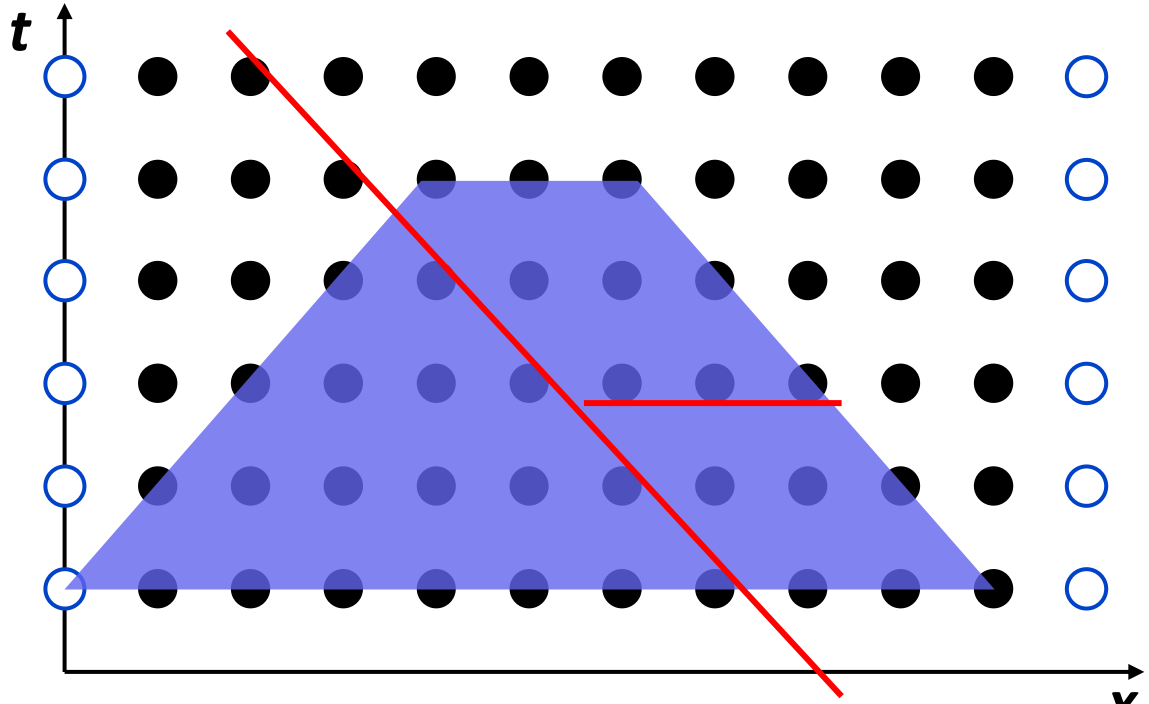
Recursion #1: Space Cut

- If width >= height × 2
  - Cut the trapezoid through the center using a line of slope -1
  - Process left, then right



Recursion #2: Time Cut

- If width < height × 2
  - Cut the trapezoid with a horizontal line through the center
  - Process bottom, then top

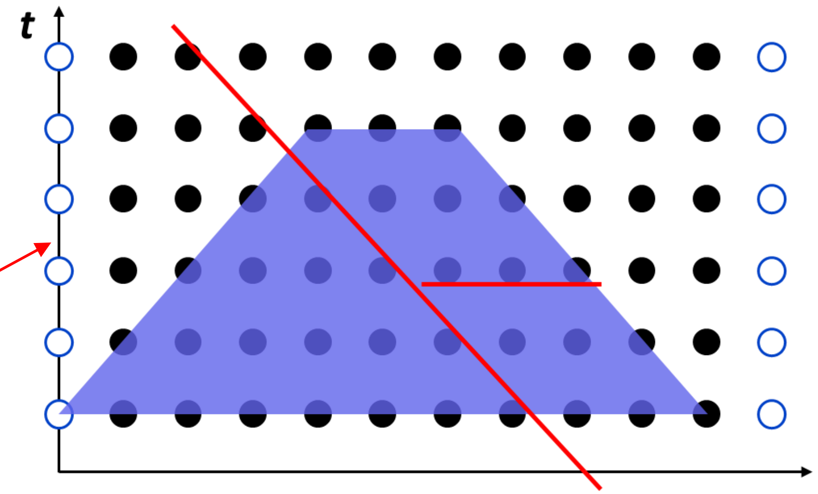


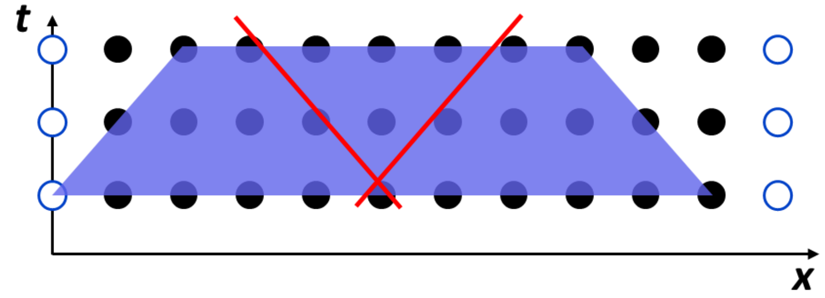
Cache Analysis

- Intuitively, trapezoids are split until they are of size M (cache size)
- Data read = $\Theta(NT/M)$
  - Cache lines read = $\Theta(NT/MB)$
  - Good!

Parallelism-Aware Cutting

- ❑ Vanilla method not good for parallelism
  - Three splits have strict dependencies…
- ❑ Space cuts can be made parallelism-friendly!
  - Bottom two first, top one next





- ❑ Effects on parallel scalability
  - Difference in impact of four cores
  - Why? DRAM bandwidth bottleneck!

Code	Time
Serial looping	128.95s
Parallel looping	66.97s
Serial trapezoidal	66.76s
Parallel trapezoidal	16.86s

} 1.93x

} 3.96x

Performance scaling with four cores
Source: 2008-2018 by the MIT 6.172 Lecturers