CS 374: Algorithms & Models of Computation

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Strings and Languages

Lecture 1

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Part I

Strings

String Definitions

Definition

1. An alphabet is a **finite** set of symbols. For example $\Sigma = \{0, 1\}$, $\Sigma = \{a, b, c, \ldots, z\}$, $\Sigma = \{\langle\text{moveforward}\rangle, \langle\text{moveback}\rangle\}$ are alphabets.
2. A string/word over Σ is a **finite sequence** of symbols over Σ. For example, '0101001', 'string', '$\langle\text{moveback}\rangle\langle\text{rotate90}\rangle$'
3. ϵ is the empty string.
4. The length of a string w (denoted by $|w|$) is the number of symbols in w. For example, $|101| = 3$, $|\epsilon| = 0$
5. For integer $n \geq 0$, Σ^n is set of all strings over Σ of length n. Σ^* is th set of all strings over Σ.

Formally

Formally strings are defined recursively/inductively:

- ϵ is a string of length $\mathbf{0}$
- ax is a string if $a \in \Sigma$ and x is a string. The length of ax is $1 + |x|$

The above definition helps prove statements rigorously via induction.

- Alternative recursive defintion useful in some proofs: xa is a string if $a \in \Sigma$ and x is a string. The length of xa is $1 + |x|$

Convention

- $a, b, c, \ldots$ denote elements of Σ
- $w, x, y, z, \ldots$ denote strings
- $A, B, C, \ldots$ denote sets of strings

Much ado about nothing

- ϵ is a string containing no symbols. It is not a set
- $\{\epsilon\}$ is a set containing one string: the empty string. It is a set, not a string.
- $\emptyset$ is the empty set. It contains no strings.
- $\{\emptyset\}$ is a set containing one element, which itself is a set that contains no elements.

Concatenation and properties

- If x and y are strings then xy denotes their concatenation. Formally we define concatenation recursively based on definition of strings:
  - $xy = y$ if $x = \epsilon$
  - $xy = a(wy)$ if $x = aw$

  Sometimes xy is written as $x \bullet y$ to explicitly note that $\bullet$ is a binary operator that takes two strings and produces another string.
- concatenation is associative: $(uv)w = u(vw)$ and hence we write uvw
- **not** commutative: uv not necessarily equal to vu
- identity element: $\epsilon u = u\epsilon = u$

Substrings, prefix, suffix, exponents

Definition

1. v is substring of w iff there exist strings x, y such that $w = xvy$.
   - If $x = \epsilon$ then v is a prefix of w
   - If $y = \epsilon$ then v is a suffix of w
2. If w is a string then w^n is defined inductively as follows:
   $w^n = \epsilon$ if $n = 0$
   $w^n = ww^{n-1}$ if $n > 0$

   Example: $(blah)^4 = blahblahblahblah$.

Set Concatenation

Definition

Given two sets A and B of strings (over some common alphabet Σ) the concatenation of A and B is defined as:

$$AB = \{xy \mid x \in A, y \in B\}$$

Example: $A = \{fido, rover, spot\}$, $B = \{fluffy, tabby\}$ then $AB = \{fidofluffy, fidotabby, roverfluffy, \ldots\}$.

Σ^* and languages

Definition

1. Σ^n is the set of all strings of length n. Defined inductively as follows:
   $\Sigma^n = \{\epsilon\}$ if $n = 0$
   $\Sigma^n = \Sigma\Sigma^{n-1}$ if $n > 0$
2. $\Sigma^* = \cup_{n \geq 0} \Sigma^n$ is the set of all finite length strings
3. $\Sigma^+ = \cup_{n \geq 1} \Sigma^n$ is the set of non-empty strings.

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Definition

A language L is a set of strings over Σ. In other words $L \subseteq \Sigma^*$.

Exercise

Answer the following questions taking $\Sigma = \{0, 1\}$.

1. What is Σ^0? $= \{\epsilon\}$
2. How many elements are there in Σ^3? $= |\Sigma|^3 = 8$
3. How many elements are there in Σ^n? $= 2^n$
4. What is the length of the longest string in Σ^*? Does Σ^* have strings of infinite length?
5. If $|u| = 2$ and $|v| = 3$ then what is $|u \bullet v|$? 5
6. Let u be an arbitrary string in Σ^*. What is ϵu? What is $u\epsilon$? u
7. Is $uv = vu$ for every $u, v \in \Sigma^*$? No
8. Is $(uv)w = u(vw)$ for every $u, v, w \in \Sigma^*$? Yes

Canonical order and countability of strings

Definition

An set A is countably infinite if there is a bijection f between the natural numbers and A.

Alternatively: A is countably infinite if A is an infinite set and there enumeration of elements of A

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Theorem

Σ^* is countably infinite for every finite Σ.

Enumerate strings in order of increasing length and for each given length enumerate strings in dictionary order (based on some fixed ordering of Σ).
Example: $\{0, 1\}^* = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, \ldots\}$.
$\{a, b, c\}^* = \{\epsilon, a, b, c, aa, ab, ac, ba, bb, bc, \ldots\}$

Exercise

Question: Is $\Sigma^* \times \Sigma^* = \{(x, y) \mid x, y \in \Sigma^*\}$ countably infinite?

$N \times N$ is countably infinite

$= \{ (a,b) \mid a \geq 0, b \geq 0 \}$

Exercise

Question: Is $\boldsymbol{\Sigma}^* \times \boldsymbol{\Sigma}^* = \{(x, y) \mid x, y \in \boldsymbol{\Sigma}^*\}$ countably infinite?

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Inductive proofs on strings

Inductive proofs on strings and related problems follow inductive definitions.

Definition

The reverse w^R of a string w is defined as follows:

- $w^R = \epsilon$ if $w = \epsilon$
- $w^R = x^R a$ if $w = ax$ for some $a \in \Sigma$ and string x

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Theorem

Prove that for any strings $u, v \in \Sigma^*$, $(uv)^R = v^R u^R$.

Example: $(dog \bullet cat)^R = (cat)^R \bullet (dog)^R = tacgod$.

Principle of mathematical induction

Induction is a way to prove statements of the form $\forall n \geq 0, P(n)$ where $P(n)$ is a statement that holds for integer n.

Example: Prove that $\sum_{i=0}^{n} i = n(n+1)/2$ for all n.

Induction template:

- **Base case:** Prove $P(0)$
- **Induction Step:** Let $n > 0$ be **arbitrary** integer. Assuming that $P(k)$ holds for $0 \leq k < n$, prove that $P(n)$ holds.

Unlike the simple cases we will be working with various more complicated "structures" such as strings, tuples of strings, graphs etc. We need to translate a statement "Q" into a (stronger or equivalent) statement that looks like "$\forall n \geq 0, P(n)$ and then apply induction. We call $\forall n \geq 0, P(n)$ the induction hypothesis.

Proving the theorem

Theorem

Prove that for any strings $u, v \in \Sigma^*$, $(uv)^R = v^R u^R$.

Proof: by induction.
On what?? $|uv| = |u| + |v|$?
$|u|$?
$|v|$?

What does it mean to say "induction on $|u|$"?

By induction on $|\mathbf{u}|$

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Prove that for any strings $u, v \in \Sigma^*$, $(uv)^R = v^R u^R$.

Proof by induction on $|u|$ means that we are proving the following.
Induction hypothesis: $\forall n \geq 0$, for any string u of length n (for all strings $v \in \Sigma^*$, $(uv)^R = v^R u^R$).

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Base case: Let u be an arbitrary stirng of length 0. $u = \epsilon$ since there is only one such string. Then

$$(uv)^R = (\epsilon v)^R = v^R = v^R \epsilon = v^R \epsilon^R = v^R u^R$$

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Note that we did not assume anything about v, hence the statement holds for all $v \in \Sigma^*$.

Inductive step

- Let u be an arbitrary string of length $n > 0$. Assume inductive hypothesis holds for all strings w of length $< n$.
- Since $|u| = n > 0$ we have $u = ay$ for some string y with $|y| < n$ and $a \in \Sigma$.
- Then

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$$
\begin{aligned}
(uv)^R &= (ayv)^R = (a(yv))^R \\
&= (yv)^R a^R \\
&= v^R \underline{y^R a^R} \\
&= v^R (ay)^R \\
&= v^R u^R
\end{aligned}
$$

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$$\begin{aligned}
(uv)^R &= ((ay)v)^R \\
&= (a(yv))^R && \text{- associativity of concat} \\
&= (yv)^R a^R && \text{- definition of reverse} \\
&= (v^R y^R) a^R && \text{- induction since } |y| < |u| \\
&= v^R (y^R a^R) && \text{- associativity concat} \\
&= v^R (ay)^R && \text{- defn of reverse} \\
&= v^R u^R && \text{- Since } u = ay.
\end{aligned}$$

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$$\begin{aligned}(uv)^R &= (u(ay))^R \\ &= ((ua)y)^R \\ &= y^R(ua)^R \\ &= ?? \quad y^R a^R u^R \quad \text{by ind}\end{aligned}$$

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$$\begin{aligned} (uv)^R &= (u(ay))^R \\ &= ((ua)y)^R \\ &= y^R(ua)^R \\ &= ?? \end{aligned}$$

Cannot simplify $(ua)^R$ using inductive hypotheis. Can simplify if we extend base case to include $n = 0$ **and** $n = 1$. However, $n = 1$ itself requires induction on $|u|$!

Induction on $|\mathbf{u}| + |\mathbf{v}|$

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Proof by induction on $|u| + |v|$ means that we are proving the following.

Induction hypothesis: $\forall n \geq 0$, for any $u, v \in \Sigma^*$ with $|u| + |v| \leq n$, $(uv)^R = v^R u^R$.

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Base case: $n = 0$. Let u, v be an arbitrary stirngs such that $|u| + |v| = 0$. Implies $u, v = \epsilon$.

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Inductive step~~e~~: $n > 0$. Let u, v be arbitrary strings such that $|u| + |v| = n$.

Part II

Languages

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Standard set operations apply to languages.

- For languages A, B the concatenation of A, B is $AB = \{xy \mid x \in A, y \in B\}$.
- For languages A, B, their union is $A \cup B$, intersection is $A \cap B$, and difference is $A \setminus B$ (also written as $A - B$).
- For language $A \subseteq \Sigma^*$ the complement of A is $\bar{A} = \Sigma^* \setminus A$.

Exponentiation, Kleene star etc

Definition

For a language $L \subseteq \Sigma^*$ and $n \in \mathbb{N}$, define L^n inductively as follows.

$$L^n = \begin{cases} \{\epsilon\} & \text{if } n = 0 \\ L \bullet (L^{n-1}) & \text{if } n > 0 \end{cases}$$

And define $L^* = \cup_{n \geq 0} L^n$, and $L^+ = \cup_{n \geq 1} L^n$

Exercise

Problem

Answer the following questions taking $A, B \subseteq \{0,1\}^*$.

1. *Is* $\epsilon = \{\epsilon\}$*? Is* $\emptyset = \{\epsilon\}$*?*
2. *What is* $\emptyset \bullet A$*? What is* $A \bullet \emptyset$*?*
3. *What is* $\{\epsilon\} \bullet A$*? And* $A \bullet \{\epsilon\}$*?*
4. *If* $|A| = 2$ *and* $|B| = 3$*, what is* $|A \bullet B|$*?*

Exercise

Problem

Consider languages over $\Sigma = \{0, 1\}$.

1. *What is* $\emptyset^0$*?*
2. *If* $|L| = 2$, *then what is* $|L^4|$*?*
3. *What is* $\emptyset^*$, $\{\epsilon\}^*$, ϵ^**?*
4. *For what* L *is* L^* *finite?*
5. *What is* $\emptyset^+$, $\{\epsilon\}^+$, ϵ^+*?*

Languages and Computation

What are we interested in computing? Mostly functions.

Informal defintion: An algorithm $\mathcal{A}$ computes a function $f : \Sigma^* \rightarrow \Sigma^*$ if for all $w \in \Sigma^*$ the algorithm $\mathcal{A}$ on input w terminates in a finite number of steps and outputs $f(w)$.

Examples of functions:

- Numerical functions: length, addition, multiplication, division etc
- Given graph G and s, t find shortest paths from s to t
- Given program M check if M halts on empty input
- Posts Correspondence problem

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- Given boolean function $f : \Sigma^* \rightarrow \{0, 1\}$ define language $L_f = \{w \in \Sigma^* \mid f(w) = 1\}$

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- Given boolean function $f : \Sigma^* \rightarrow \{0,1\}$ define language $L_f = \{w \in \Sigma^* \mid f(w) = 1\}$
- Given language $L \subseteq \Sigma^*$ define boolean function $f : \Sigma^* \rightarrow \{0,1\}$ as follows: $f(w) = 1$ if $w \in L$ and $f(w) = 0$ otherwise.

Language recognition problem

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- Language recognition is same as boolean function computation
- How difficult is a function f to compute? How difficult is the recognizing L_f?

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Why two different views? Helpful in understanding different aspects?

How many languages are there?

Recall:

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Theorem

Σ^* is countably infinite for every finite Σ.

The set of all languages is $\mathbb{P}(\Sigma^*)$ the power set of Σ^*

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Theorem (Cantor)

*$\mathbb{P}(\Sigma^*)$ is **not** countably infinite if $|\Sigma| \geq 2$.*

Cantor's diagonalization argument

Theorem (Cantor)

$\mathbb{P}(\mathbb{N})$ *is not countably infinite.*

- Suppose $\mathbb{P}(\mathbb{N})$ is countable infinite. Let $S_1, S_2, \ldots,$ be an enumeration of all subsets of numbers.
- Let D be the following diagonal subset of numbers.

$$D = \{i \mid i \not\in L_i\}$$

- Since D is a set of numbers, by assumption, $D = S_j$ for some j.
- **Question:** Is $j \in D$**?**

Consequences for Computation

- How many C programs are there? The set of C programs is countably infinite since each of them can be represented as a string over a finite alphabet.
- How many languages are there? Uncountably many!
- Hence some (in fact almost all!) languages/boolean functions do not have any C program to recognize them.

Questions:

Consequences for Computation

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- Hence some (in fact almost all!) languages/boolean functions do not have any *C* program to recognize them.

Questions:

- Maybe interesting languages/functions have *C* programs and hence computable. Only uninteresting langues uncomputable?
- Why should *C* programs be the definition of computability?
- Ok, there are difficult problems/languages. what lanauges are computable and which have efficient algorithms?

Easy languages

Definition

A language $L \subseteq \Sigma^*$ is finite if $|L| = n$ for some integer n.

Exercise: Prove the following.

Theorem

The set of all finite languages is countably infinite.