CS173
Introduction to Logic, Sets, Graphs, and Functions

Tandy Warnow

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Material (two days)

- Introduction to proofs by contradiction
- Introduction to sets (notation, operations, terminology)
- Introduction to logic (propositions, predicates, quantifiers, operations)
- Introduction to graphs (terminology)
- Introduction to function notation and recursively defined sets and functions
- Satisfiability and Conjunctive Normal Form
- Truth Tables

Proofs

- You want to prove that some statement A is true.
- You can try to prove it directly, or you can prove it indirectly? we will show examples of each type of proof.

Example of Direct Proof

Theorem: Every odd integer is the difference of two perfect squares. (In other words, $\forall$ odd integers x, $\exists y, z$ integers such that $x = y^2 - z^2$.)

Note: $\forall x$ is the same as "for all x" and $\exists x$ is the same as "there exists an x".

Example of Direct Proof

Theorem: Every odd integer is the difference of two perfect squares. (Put more formally $\forall$ odd integers x, $\exists y, z$ integers such that $x = y^2 - z^2$.)

Proof: Since x is odd, there is an integer L such that $x = 2L + 1$.
Note that $(L+1)^2 = L^2 + 2L + 1$
Hence $(L+1)^2 - L^2 = x$
Q.E.D.

Proof by contradiction

Theorem: $\sqrt{7}$ is irrational.

Proof by contradiction

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Proof by contradiction.

If $\sqrt{7}$ is rational, then *by definition* $\exists$ integers a, b such that $\frac{a}{b} = \sqrt{7}$.

Without loss of generality, we will assume a and b are relatively prime (where relatively prime means that they have no common factors greater than 1).

Proof by contradiction that $\sqrt{7}$ is irrational, continued

Recall we assumed that $\sqrt{7}$ is rational and so then $\exists a, b$ relatively prime integers such that $\left(\frac{a}{b}\right)^2 = 7$.

Therefore $a^2 = 7b^2$ (Arithmetic)

Hence 7 divides a^2 (Equality of a^2 and $7b^2$, and 7 divides $7b^2$)

Because 7 is prime, this implies that 7 divides a

But then 7^2 divides a^2 (obvious)

and so 7^2 divides $7b^2$ (because $a^2 = 7b^2$)

And so 7 divides b^2 (obvious)

Because 7 is prime, this implies that 7 divides b

Hence 7 is a common divisor of both a and b, which contradicts our earlier assumption.

Proof by contradiction that $\sqrt{7}$ is irrational, continued

Note that 7 being prime was important in the proof.

We said that if 7 divides a^2 then 7 divides a, and we used that 7 is prime.

This was necessary since it doesn't hold that if 4 divides a^2 then 4 divides a (e.g., let $a = 6$).

Here's a longer justification for why 7 must divide a if 7 divides a^2.

Proof by contradiction that $\sqrt{7}$ is irrational, continued

Here's a longer justification for why 7 must divide a if 7 divides a^2.

Every integer greater than 1 has a unique prime factorization.

So let the prime factorization of a be:

$$a = \prod_{i=1}^{k} p_i^{l_i}$$

where p_i is a prime and l_i is a positive integer.

Hence the unique prime factorization of a^2 must be

$$a^2 = \prod_{i=1}^{k} p_i^{2l_i}$$

If 7 divides a^2 then $7 = p_i$ for some $i, 1 \leq i \leq k$. (the definition of saying that 7 divides a^2)

Hence 7 is one of the prime factors for a, and so 7 divides a.

Class exercise

Prove that $\sqrt{5}$ is irrational.

Introduction to set notation

A set S is just a collection of objects.

Some sets are finite (e.g., $\{1, 2, 3, 5\}$) and some are infinite (e.g., the set $\mathbb{Z}$ of integers).

We can specify a set explicitly, as in $\{1, 2, 3, 5\}$, or implicitly using "set-builder notation?":

  ▶ $\{x \in \mathbb{Z} | 0 < x < 6, x \neq 4\}$

Note that $\{x \in \mathbb{Z} | 0 < x < 6, x \neq 4\} = \{1, 2, 3, 5\}$.

The emptyset is denoted by $\emptyset$ or by $\{\}$, and is the set that has no elements.

Terminology

We write $x \in A$ to indicate that x is an element of set A.
For example, $5 \in \mathbb{Z}$.

We write $x \notin A$ to indicate that x is not an element of A.
For example, $\sqrt{7} \notin \mathbb{Z}$.

We say that a set A is a subset of B if every element of A is an element of B. This is denoted $A \subseteq B$. For example, $\mathbb{Z} \subseteq \mathbb{R}$, where $\mathbb{Z}$ denotes the set of integers and $\mathbb{R}$ denotes the set of real numbers.

The intersection and unions of sets A and B are represented using $A \cap B$ and $A \cup B$, respectively.

The set difference between sets A and B is denoted $A \setminus B$, and is the set $\{x \in A | x \notin B\}$.

The number of elements in a set A (also called its cardinality) is denoted by $|A|$.

Set builder notation

Let $\mathbb{Z}$ denote the integers and $\mathbb{R}$ denote the real numbers. What are these sets?

- $A = \{f : \mathbb{Z} \to \{1, 2, 3\}\}$
- $B = \{x \subseteq \{0, 1, 2, 3\} | 1 \in x\}$
- $C = \{x \subseteq \mathbb{Z} | |x| \leq 2\}$
- $D = \{f : \mathbb{R} \to \mathbb{R} | \forall x(f(x) = f(0))\}$
- $E = \{x \in \mathbb{Z} | x > 0\}$
- $F = \{x \in \mathbb{Z} | x - 1 \in \mathbb{Z}\}$
- $G = \{x \in \mathbb{Z} | x^2 < x\}$

Another proof by contradiction

Theorem: Let $A \subseteq \mathbb{Z}$ be finite and satisfy

- If $x \in A$ then $x + 1 \in A$

Then $A = \emptyset$.

Proving $A = \emptyset$

Recall that we assume A is a finite set of integers and satisfies $x \in A \rightarrow x + 1 \in A$. We first show that A cannot be non-empty, and then we show that $A = \emptyset$ satisfies the constraint.

- If A is finite but non-empty, then it has n elements, and so $A = \{x_1, x_2, \ldots, x_n\}$. Let y be the maximum element of A. Since $y \in A$, it follows that $y + 1 \in A$.
  But $y + 1$ is bigger than every element of A, which is a contradiction. Hence, A cannot be non-empty.
- On the other hand, does $A = \emptyset$ satisfy the required property:
  - If $x \in A$ then $x + 1 \in A$

  Yes, because an "IF-THEN" statement is true whenever the first half is false. And $x \in \emptyset$ is always false.

Introduction to Graphs

Graphs are objects with vertices and edges.

We write this as $G = (V, E)$, so V is the set of vertices and E is the set of edges.
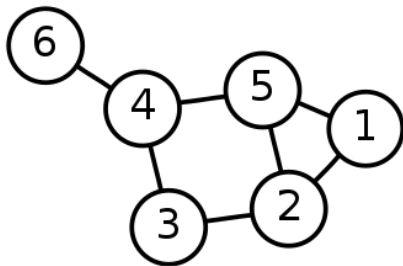
Every edge is an un-ordered pair of vertices.

Figure : Graph with 6 vertices, public domain figure taken from https://en.wikipedia.org/wiki/Graph_(discrete_mathematics),

Properties of graphs
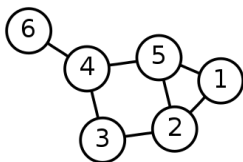

Figure : Graph with 6 vertices, public domain figure taken from
https://en.wikipedia.org/wiki/Graph_(discrete_mathematics)

A graph is simple if it has no self-loops or parallel edges.
The degree of a node v, denoted $deg(v)$, is the number of edges
incident with it.
$deg(1) = 2$
$deg(2) = 3$
$deg(3) = 2$
$deg(4) = 3$
$deg(5) = 3$
$deg(6) = 1$
The number of nodes of odd degree is 4.

Proving theorems about graphs

Theorem: Every finite simple graph has an even number of vertices of odd degree.

Proving theorems about graphs

Theorem: Every finite simple graph has an even number of vertices of odd degree.

Proof: Every edge connects two vertices.
Hence, *SUM* (the sum of the degrees in a graph) is always even (Handshaking Theorem).

Proving theorems about graphs

Theorem: Every finite simple graph has an even number of vertices of odd degree.

Recall *SUM* (sum of degrees) is even.

Let *ODD* denote the set of vertices of odd degree and *EVEN* denote the set of vertices of even degree.

$$SUM = \sum_{v \in ODD} deg(v) + \sum_{v \in EVEN} deg(v)$$

Since *SUM* is even and $\sum_{v \in EVEN} deg(v)$ is even, $\sum_{v \in ODD} deg(v)$ must also be even.

But then $|ODD|$ is even!

Q.E.D.

Introduction to Logic

Logical variables (items that can be true or false)
Propositions (statements that are true or false)
Predicates (statements that are true or false, but depend on the variables)
Quantifiers (for all, there exists)
AND, OR, XOR
If-then
if-and-only-if
negation

Quantifiers: For all (∀) and There Exists (∃)

A proposition is either True or False. Which of these statements are True?

- $\forall x \in \emptyset, x > 0$
- $\exists x \in \emptyset, x > 0$
- $\exists x \in \emptyset$
- All flying elephants eat pizza
- There exists a flying elephant that eats pizza
- There exists a flying elephant
- No flying elephant eats pizza
- For all flying elephants x, x does not eat pizza

Order of quantifiers matters!

We write "s.t." for "such that". Now, think about these two statements:

- $\forall x \in \mathbb{Z} \; \exists y \in \mathbb{Z} \; s.t. \; x > y$
- $\exists y \in \mathbb{Z} \; s.t. \; \forall x \in \mathbb{Z}, \; x > y$

Is either of these statements true?

Predicates

Some logical statements depend on variables. Consider:

- Let $P(x)$ denote the statement "$x \in \mathbb{Z}$". Is $P(3)$ true? Is $P(\sqrt{7})$ true?

- Let $Q(x, y)$ denote the statement "$|x| > |y|$". Is $Q(\{3, 5\}, \mathbb{Z})$ true? Is $Q(\mathbb{Z}, \emptyset)$ true?

- Let $R(x)$ denote the statement "$0 \in x$". Give an example of x for which $R(x)$ is false.

Reading Mathematics

Let $G = (V, E)$ denote a graph.
What do the following statements mean?

1. $\forall v \in V, \exists y \in V s.t. \ (v, y) \in E$
2. $\exists y \in V \ s.t. \ \forall v \in V \setminus \{y\}, \ (v, y) \in E$
3. $\forall \{a, b\} \subseteq V, (a, b) \in E$

Class Exercise

Let $G = (V, E)$ denote a graph.

What do the following statements mean?

1. $\forall v \in V, \exists y \in V s.t. (v, y) \in E$
2. $\exists y \in V \ s.t. \ \forall v \in V \setminus \{y\}, \ (v, y) \in E$
3. $\forall \{a, b\} \subseteq V, (a, b) \in E$

Find an example of a graph that satisfies the first of these statements but not the others.

AND, OR, and XOR

Suppose A and B are propositions (and hence are either true or false).

A AND B (i.e., $A \wedge B$) is True if and only if *both A and B are true.*

A OR B (i.e., $A \vee B$) is True if and only if *at least one of A and B is true.*

A XOR B (i.e., $A \oplus B$) is True if and only if *exactly one of A and B is true.*

Examples:

- All flying elephants eat pizza OR Staten Island is a borough in New York City

- All flying elephants eat pizza AND Staten Island is a borough in New York City

- All flying elephants eat pizza XOR Staten Island is a borough in New York City

If-then

IF A THEN B (sometimes denoted $A \rightarrow B$, and worded as "A implies B") is the same as:

- whenever A is True, B must be True
- It isn't possible for B to be False if A is True

So, how would you show that "IF A THEN B" is False?

If-then statements

Key point: IF A THEN B is only False if A is True and B is False!

In other words,

$$A \rightarrow B \equiv \neg(A \land \neg B) \equiv \neg A \lor B$$

where $\neg X$ is the same as "not X".

Class Exercise

Which of the following statements are true?

1. IF $(0 \in \emptyset)$ THEN (Obama is still president)
2. IF $(0 \notin \emptyset)$ THEN (Obama is still president)
3. IF(all flying elephants eat pizza) THEN (Obama is still president)
4. IF(no flying elephants eat pizza) THEN (Obama is still president)
5. IF (some flying elephant eats pizza) THEN (Obama is still president)

NOT X ($\neg X$)

$\neg X$ is True if and only if X is False.

Can we simplify these?

- $\neg$ (A OR B)
- $\neg$ (A AND B)
- $\neg$ (IF A THEN B)
- $\neg$ (A XOR B)

Logic exercises

- Simplifying logical expressions, and seeing when two logical expressions are equivalent
- Determining if a logical expression can be *satisfied*
- Expressing English statements in logic

Simplifying logical expression

Objectives:

- Remove all unnecessary parentheses
- Remove all $\rightarrow$ or $\leftrightarrow$

Hence, you need to be able to simplify expressions like

$$\neg(a \rightarrow b) \vee (\neg b)$$

Simplifications (warm-up)

When A and B are logical expressions, and you say $A \equiv B$, you mean that they have the same truth values. (You can also write this as $A \leftrightarrow B$.)

For example:

- $\neg\neg x \equiv x$ (obvious)
- $x \vee (x \wedge y) \equiv x$

    Similarly, you can write $x \vee (x \wedge y) \leftrightarrow x$. In other words, $x \vee (x \wedge y)$ is true if and only if x is true.

- $x \vee \neg x \equiv T$

    In other words, $x \vee \neg x$ is always true, no matter what x is.

- $x \wedge \neg x \equiv F$

    In other words, $x \wedge \neg x$ is never true, no matter what x is.

De Morgan's Laws

- Negation of $A \land B$: $\neg A \lor \neg B$
  This is also written as

$$\neg(A \land B) \equiv \neg A \lor \neg B$$

  or as

$$\neg(A \land B) \leftrightarrow \neg A \lor \neg B$$

- Negation of $A \lor B$: $\neg A \land \neg B$
  This is also written as

$$\neg(A \lor B) \equiv \neg A \land \neg B$$

  or as

$$\neg(A \lor B) \leftrightarrow \neg A \land \neg B$$

Negation, warm up with quantifiers

- Negation of $\forall x \in S, P(x)$:

  $\exists x \in S$ s.t. $\neg P(x)$

  Negation of $\exists x \in S$ s.t. $P(x)$

  - $\forall x \in S, \neg P(x)$

Negating Logical Expressions

Consider the expression

$$A \rightarrow B$$

To negate this, we have:

$$\neg(A \rightarrow B)$$
$$\equiv \neg(\neg A \vee B)$$
$$\equiv \neg\neg A \wedge \neg B$$
$$\equiv A \wedge \neg B$$

Negation, again

Negate: $(x \to y) \land \neg x$

First Solution:

$$\neg[(x \to y) \land \neg x]$$

$$\equiv \neg(x \to y) \lor \neg\neg x$$

$$\equiv \neg(\neg x \lor y) \lor x$$

$$\equiv (\neg\neg x \land \neg y) \lor x$$

$$\equiv (x \land \neg y) \lor x$$

$$\equiv x$$

Negation, again

Negate: $(x \to y) \land \neg x$

Second Solution: We begin by simplifying the expression above before negating it. Note that

$$x \to y \equiv \neg x \lor y$$

Hence

$$(x \to y) \land \neg x$$
$$\equiv (\neg x \lor y) \land \neg x$$
$$\equiv (\neg x \land \neg x) \lor (y \land \neg x)$$
$$\equiv \neg x \lor (y \land \neg x)$$
$$\equiv \neg x$$

Therefore,

$$\neg[(x \to y) \land \neg x] \equiv \neg\neg x \equiv x$$

Simplifying a logical expression

Can we simplify these?

- ¬ (A OR B)

Solution:

A OR B is true when at least one of *A* or *B* is true.

Hence it is false if and only if both *A* and *B* are false.

In other words:

$$\neg(A \ OR \ B) \equiv \neg A \ AND \ \neg B$$

Simplifying a logical expression

Can we simplify these?

- ¬ (A AND B)

Solution:

A AND B is true when both *A* or *B* are true.

Hence it is false if and only if at least one of *A* or *B* is false.

In other words:

$$\neg(A \; AND \; B) \equiv \neg A \; OR \; \neg B$$

Note the effect of ¬: AND changes to OR and vice-versa, and *X* changes to ¬*X*.

Class Exercise

Simplify one or both:

- $\neg(A \; OR \neg B)$
- $\neg A \rightarrow A$

Satisfiability

Some logical expressions can never be true, some are always true, and some depend on the values of their variables. **T** and **F** refer to the logical constants True and False, respectively. Examples:

1. $A \vee \neg A$ (always true)

2. $A \wedge \neg A$ (never true)

3. $A \vee B$ (sometimes true and sometimes false, depends on A and B)

4. $A \wedge F$ (never true)

Statements that are always true are called *tautologies*. Statements that can be true (or are always true) are said to be *satisfiable*, and otherwise they are said to be *unsatisfiable*.

Satisfiability

For each of the following expressions, determine if it is satisfiable or not satisfiable. If it is satisfiable, determine if it is a tautology.

1. $(A \land B) \to A$
2. $(A \land B) \to \neg A$
3. $(A \land B) \leftrightarrow A$
4. $(A \to B) \land A \land \neg B$
5. $A \to \neg A$

Satisfiability

For each of the following expressions, determine if it is satisfiable or not satisfiable. If it is satisfiable, determine if it is a tautology.

1. $(A \land B) \to A$
   (Answer: tautology)

2. $(A \land B) \to \neg A$
   (Answer: satisfiable ($A = B = $ **F**) but not a tautology ($A = B = $ **T**))

3. $(A \land B) \leftrightarrow A$
   (Answer: satisfiable ($A = B = $ **T**) but not a tautology ($A = $ **T** and $B = $ **F**))

4. $(A \to B) \land A \land \neg B$
   (Answer: not satisfiable, so never true)

5. $A \to \neg A$
   (Answer: satisfiable ($A = $ **F**) but not a tautology ($A = $ **T**))

Truth Tables

We (sometimes) use truth tables to check our analyses. Here's an example of a very simple truth table for the expression $A \wedge B$:

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

A more complicated truth table

Consider the expression $[(A \to B) \land \neg B] \to A$. Is this always true? Sometimes true and sometimes false? Always false? Let's use a truth table to answer this.

A	B	$(A \to B) \land \neg B$	$[(A \to B) \land \neg B] \to A$
T	T	F	T
T	F	F	T
F	T	F	T
F	F	T	F

So the answer is that it is sometimes true and sometimes false. Note that we also showed $[(A \to B) \land \neg B] \to A \equiv A \lor B$.

Conjunctive Normal Form (CNF)

A logical expression of the form

$$A_1 \lor A_2 \lor A_3 \lor ... \lor A_k$$

where the A_i are literals (statement letters or their negations) is called a *disjunctive clause*.

Then

$$C_1 \land C_2 \land C_3 \land \ldots \land C_p,$$

where each C_i is a disjunctive clause, is said to be in *conjunctive normal form*, or CNF.

CNF is very popular in computer science!

Two-satisfiability

A special case of CNF is where each clause has at most two literals! That is, expression that are written in the form $(A_1 \lor B_1) \land (A_2 \lor B_2) \land \ldots (A_k \lor B_k)$.

Which of the following CNF expressions are satisfiable?

1. $(x \lor y) \land (\neg x \lor \neg y)$
2. $(x \lor y) \land (\neg x \lor \neg y) \land x$
3. $(x \lor y) \land (\neg x \lor \neg y) \land x \land y$
4. $(x \lor y) \land (\neg x \lor \neg z) \land (\neg y \lor z) \land (\neg x \lor z)$
5. $(\neg x \lor y) \land (\neg y \lor z) \land (\neg z \lor x) \land (x \lor z)$

A logic puzzle

In a particular village in the deep valleys in some far-away country, everyone is either a liar (and never tells the truth) or a truth-teller (and never lies).

You are in this village and meet Henry and Allen.

- Henry says "Allen is a truth teller"
- Allen says "Only one of us is a truth teller"

Is either a truth teller? If so, who?