CS672: Approximation ALgorithms
Spring 2017
Crash Course in Linear Programming

Instructor: Shaddin Dughmi

Outline

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1. Linear Programming Basics
2. Duality and Its Interpretations
3. Properties of Duals
4. Weak and Strong Duality
5. Consequences of Duality
6. Uses and Examples of Duality
7. Solvability of LP

A Brief History

- The forefather of convex optimization problems, and the most ubiquitous.
- Developed by Kantorovich during World War II (1939) for planning the Soviet army's expenditures and returns. Kept secret.
- Discovered a few years later by George Dantzig, who in 1947 developed the simplex method for solving linear programs
- John von Neumann developed LP duality in 1947, and applied it to game theory
- Polynomial-time algorithms: Ellipsoid method (Khachiyan 1979), interior point methods (Karmarkar 1984).

LP General Form

$$\begin{array}{lll} \text{minimize (or maximize)} & c^\intercal x & \\ \text{subject to} & a_i^\intercal x \leq b_i, & \text{for } i \in \mathcal{C}^1. \\ & a_i^\intercal x \geq b_i, & \text{for } i \in \mathcal{C}^2. \\ & a_i^\intercal x = b_i, & \text{for } i \in \mathcal{C}^3. \end{array}$$

- Decision variables: $x \in \mathbb{R}^n$
- Parameters:
  - $c \in \mathbb{R}^n$ defines the linear objective function
  - $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ define the i'th constraint.

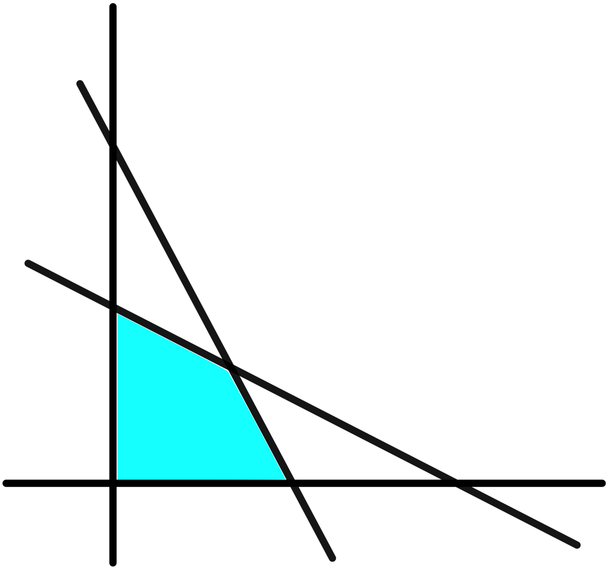
Standard Form

$$\begin{array}{lll} \text{maximize} & c^\mathsf{T} x & \\ \text{subject to} & a_i^\mathsf{T} x \leq b_i, & \text{for } i = 1, \ldots, m. \\ & x_j \geq 0, & \text{for } j = 1, \ldots, n. \end{array}$$

Every LP can be transformed to this form

- minimizing $c^\mathsf{T} x$ is equivalent to maximizing $-c^\mathsf{T} x$
- $\geq$ constraints can be flipped by multiplying by -1
- Each equality constraint can be replaced by two inequalities
- Uconstrained variable x_j can be replaced by $x_j^+ - x_j^-$, where both x_j^+ and x_j^- are constrained to be nonnegative.

Geometric View



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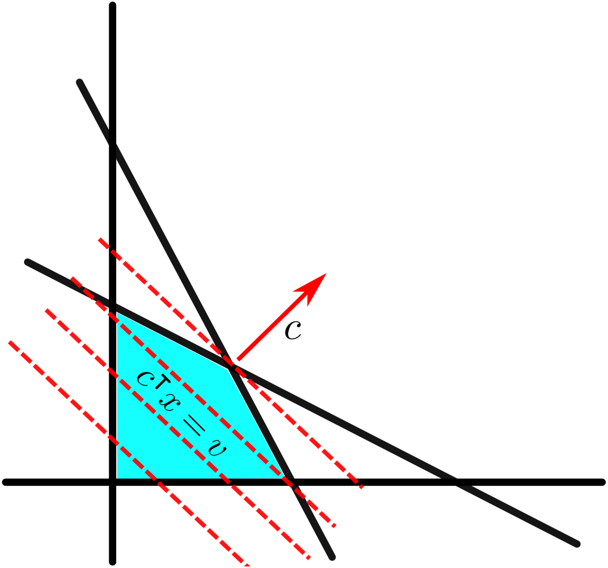
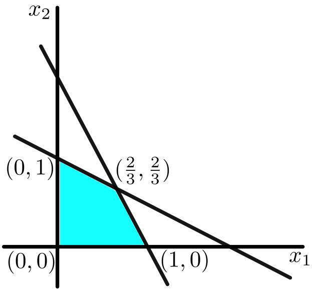


A 2-D example

$$\begin{array}{ll} \text{maximize} & x_1 + x_2 \\ \text{subject to} & x_1 + 2x_2 \leq 2 \\ & 2x_1 + x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{array}$$



Application: Optimal Production

- n products, m raw materials
- Every unit of product j uses a_{ij} units of raw material i
- There are b_i units of material i available
- Product j yields profit c_j per unit
- Facility wants to maximize profit subject to available raw materials

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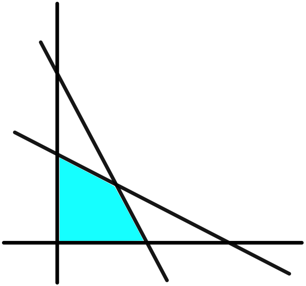
\end{array}
$$

Terminology

- Hyperplane: The region defined by a linear equality
- Halfspace: The region defined by a linear inequality $a_i^\intercal x \leq b_i$.
- Polyhedron: The intersection of a set of linear inequalities
  - Feasible region of an LP is a polyhedron
- Polytope: Bounded polyhedron
  - Equivalently: convex hull of a finite set of points
- Vertex: A point x is a vertex of polyhedron P if $\nexists y \neq 0$ with $x + y \in P$ and $x - y \in P$
- Face of P: The intersection with P of a hyperplane H disjoint from the interior of P



Basic Facts about LPs and Polyhedrons

Fact

Feasible regions of LPs (i.e. polyhedrons) are convex

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Set of optimal solutions of an LP is convex

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- intersection of P with hyperplane $c^\intercal x = OPT$

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- In fact, a face of the polyhedron
- intersection of P with hyperplane $c^{\intercal}x = OPT$

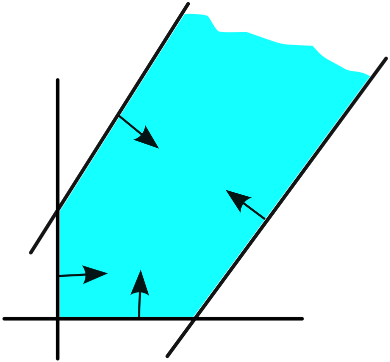
Fact

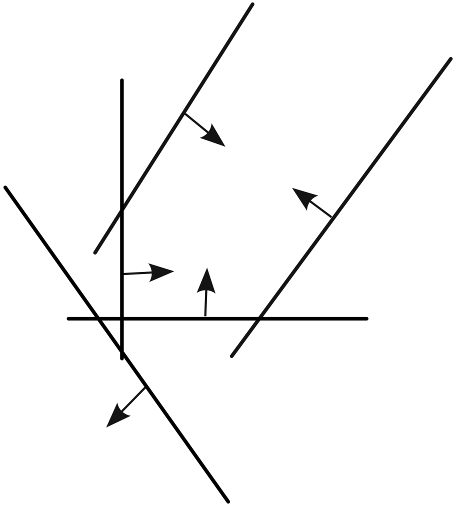
At a vertex, n linearly independent constraints are satisfied with equality (a.k.a. tight)

Basic Facts about LPs and Polyhedrons

Fact

An LP either has an optimal solution, or is unbounded or infeasible





Fundamental Theorem of LP

If an LP in standard form has an optimal solution, then it has a vertex optimal solution.

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- y is perpendicular to the objective function and the tight constraints at x.
  - i.e. $c^\intercal y = 0$, and $a_i^\intercal y = 0$ whenever the i'th constraint is tight for x.

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- Let α be the largest constant such that $x + \alpha y$ is feasible
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- Can choose y s.t. $y_j < 0$ for some j
- Let α be the largest constant such that $x + \alpha y$ is feasible
  - Such an α exists
- An additional constraint becomes tight at $x + \alpha y$, a contradiction.

Counting non-zero Variables

Corollary

If an LP in standard form has an optimal solution, then there is an optimal solution with at most m non-zero variables.

$$\begin{array}{lll} \text{maximize} & c^\intercal x & \\ \text{subject to} & a_i^\intercal x \leq b_i, & \text{for } i = 1, \ldots, m. \\ & x_j \geq 0, & \text{for } j = 1, \ldots, n. \end{array}$$

- e.g. for optimal production with n products and m raw materials, there is an optimal plan with at most m products.

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Linear Programming Duality

Primal LP

$$\begin{aligned} \text{maximize} \quad & c^\intercal x \\ \text{subject to} \quad & Ax \leq b \end{aligned}$$

Dual LP

$$\begin{aligned} \text{minimize} \quad & b^\intercal y \\ \text{subject to} \quad & A^\intercal y = c \\ & y \geq 0 \end{aligned}$$

- $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$
- y_i is the dual variable corresponding to primal constraint $A_i x \leq b_i$
- $A_j^T y \geq c_j$ is the dual constraint corresponding to primal variable x_j

Linear Programming Duality: Standard Form, and Visualization

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y_1	a_{11}	a_{12}	a_{13}	a_{14}	b_1
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- y_i is the dual variable corresponding to primal constraint $A_i x \leq b_i$
- $A_j^T y \geq c_j$ is the dual constraint corresponding to primal variable x_j

Interpretation 1: Economic Interpretation

Recall the Optimal Production problem from last lecture

- n products, m raw materials
- Every unit of product j uses a_{ij} units of raw material i
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- Dual variable y_i is a proposed price per unit of raw material i
- Dual price vector is feasible if facility has incentive to sell materials
- Buyer wants to spend as little as possible to buy materials

Interpretation 2: Finding the Best Upperbound

Consider the simple LP from last lecture

$$
\begin{array}{ll}
\text{maximize} & x_1 + x_2 \\
\text{subject to} & x_1 + 2x_2 \leq 2 \\
& 2x_1 + x_2 \leq 2 \\
& x_1, x_2 \geq 0
\end{array}
$$

- We found that the optimal solution was at $(\frac{2}{3}, \frac{2}{3})$, with an optimal value of $4/3$.

Interpretation 2: Finding the Best Upperbound

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- We found that the optimal solution was at $(\frac{2}{3}, \frac{2}{3})$, with an optimal value of $4/3$.
- What if, instead of finding the optimal solution, we saught to find an upperbound on its value by combining inequalities?
  - Each inequality implies an upper bound of 2
  - Multiplying each by $\frac{1}{3}$ and summing gives $x_1 + x_2 \leq 4/3$.

Interpretation 2: Finding the Best Upperbound

	x_1	x_2	x_3	x_4	
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- Multiplying each row i by y_i and summing gives the inequality

$$y^T Ax \leq y^T b$$

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- Multiplying each row i by y_i and summing gives the inequality

$$y^T Ax \leq y^T b$$

- When $y^T A \geq c^T$, the right hand side of the inequality is an upper bound on $c^T x$ for every feasible x.

$$c^T x \leq y^T Ax \leq y^T b$$

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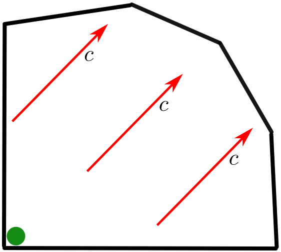
- The dual LP can be thought of as trying to find the best upperbound on the primal that can be achieved this way.

Interpretation 3: Physical Forces



- Apply force field c to a ball inside bounded polytope $Ax \leq b$.

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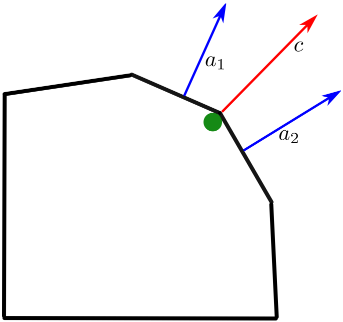

- Apply force field c to a ball inside bounded polytope $Ax \leq b$.
- Eventually, ball will come to rest against the walls of the polytope.

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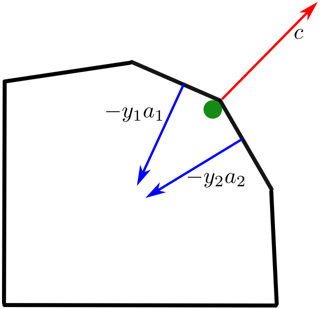

- Apply force field c to a ball inside bounded polytope $Ax \leq b$.
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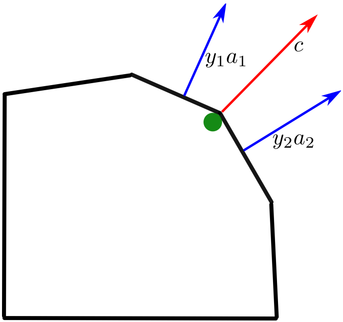

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- Wall $a_i x \leq b_i$ applies some force $-y_i a_i$ to the ball
- Since the ball is still, $c^T = \sum_i y_i a_i = y^T A$.

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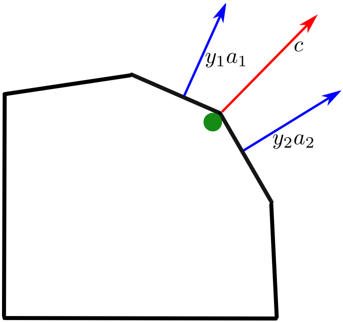

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- Wall $a_i x \leq b_i$ applies some force $-y_i a_i$ to the ball
- Since the ball is still, $c^T = \sum_i y_i a_i = y^T A$.
- Dual can be thought of as trying to minimize "work" $\sum_i y_i b_i$ to bring ball back to origin by moving polytope
- We will see that, at optimality, only the walls adjacent to the ball push (Complementary Slackness)

Outline

Duality is an Inversion

Primal LP

$$\begin{aligned} \text{maximize} \quad & c^\intercal x \\ \text{subject to} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \text{minimize} \quad & b^\intercal y \\ \text{subject to} \quad & A^\intercal y \geq c \\ & y \geq 0 \end{aligned}$$

Duality is an Inversion

Given a primal LP in standard form, the dual of its dual is itself.

Correspondance Between Variables and Constraints

Primal LP

$$
\begin{array}{lll}
\text{max} & \sum_{j=1}^{n} c_j x_j & \\
\text{s.t.} & & \\
& \sum_{j=1}^{n} a_{ij} x_j \le b_i, & \text{for } i \in [m]. \\
& x_j \ge 0, & \text{for } j \in [n].
\end{array}
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Dual LP

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Dual LP

$$\begin{array}{lll} \min & \sum_{i=1}^{m} b_i y_i & \\ \text{s.t.} & & \\ & \sum_{i=1}^{m} a_{ij} y_i \ge c_j, & \text{for } j \in [n]. \\ & y_i \ge 0, & \text{for } i \in [m]. \end{array}$$

- The i'th primal constraint gives rise to the i'th dual variable y_i

Correspondance Between Variables and Constraints

Primal LP

$$\begin{array}{lll} \text{max} & \sum_{j=1}^{n} c_j x_j & \\ \text{s.t.} & & \\ {\color{red} y_i :} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, & \text{for } i \in [m]. \\ & x_j \geq 0, & \text{for } j \in [n]. \end{array}$$

Dual LP

$$\begin{array}{lll} \text{min} & \sum_{i=1}^{m} b_i y_i & \\ \text{s.t.} & & \\ {\color{red} x_j :} & \sum_{i=1}^{m} a_{ij} y_i \geq c_j, & \text{for } j \in [n]. \\ & y_i \geq 0, & \text{for } i \in [m]. \end{array}$$

- The i'th primal constraint gives rise to the i'th dual variable y_i
- The j'th primal variable x_j gives rise to the j'th dual constraint

Syntactic Rules

Primal LP

$$\begin{array}{lll} \max & c^{\intercal}x & \\ \text{s.t.} & & \\ y_i: & a_i x \leq b_i, & \text{for } i \in \mathcal{C}_1. \\ y_i: & a_i x = b_i, & \text{for } i \in \mathcal{C}_2. \\ & x_j \geq 0, & \text{for } j \in \mathcal{D}_1. \\ & x_j \in \mathbb{R}, & \text{for } j \in \mathcal{D}_2. \end{array}$$

Dual LP

$$\begin{array}{lll} \min & b^{\intercal}y & \\ \text{s.t.} & & \\ x_j: & \overline{a}_j^{\intercal} y \geq c_j, & \text{for } j \in \mathcal{D}_1. \\ x_j: & \overline{a}_j^{\intercal} y = c_j, & \text{for } j \in \mathcal{D}_2. \\ & y_i \geq 0, & \text{for } i \in \mathcal{C}_1. \\ & y_i \in \mathbb{R}, & \text{for } i \in \mathcal{C}_2. \end{array}$$

Rules of Thumb

- Loose constraint (i.e. inequality) $\Rightarrow$ tight dual variable (i.e. nonnegative)
- Tight constraint (i.e. equality) $\Rightarrow$ loose dual variable (i.e. unconstrained)

Outline

Weak Duality

Primal LP

$$\begin{aligned} \text{maximize} \quad & c^\intercal x \\ \text{subject to} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \text{minimize} \quad & b^\intercal y \\ \text{subject to} \quad & A^\intercal y \geq c \\ & y \geq 0 \end{aligned}$$

Theorem (Weak Duality)

For every primal feasible x and dual feasible y, we have $c^\intercal x \leq b^\intercal y$.

Corollary

- *If primal and dual both feasible and bounded,* $OPT(Primal) \leq OPT(Dual)$
- *If primal is unbounded, dual is infeasible*
- *If dual is unbounded, primal is infeasible*

Weak Duality

Primal LP

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Dual LP

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Theorem (Weak Duality)

For every primal feasible x and dual feasible y, we have $c^\intercal x \le b^\intercal y$.

Corollary

If x is primal feasible, and y is dual feasible, and $c^\intercal x = b^\intercal y$, then both are optimal.

Interpretation of Weak Duality

Economic Interpretation

If selling the raw materials is more profitable than making any individual product, then total money collected from sale of raw materials would exceed profit from production.

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Upperbound Interpretation

Self explanatory

Interpretation of Weak Duality

Economic Interpretation

If selling the raw materials is more profitable than making any individual product, then total money collected from sale of raw materials would exceed profit from production.

Upperbound Interpretation

Self explanatory

Physical Interpretation

Work required to bring ball back to origin by pulling polytope is at least potential energy difference between origin and primal optimum.

Proof of Weak Duality

Primal LP

maximize	$c^\intercal x$
subject to	$Ax \leq b$
	$x \geq 0$

Dual LP

minimize	$b^\intercal y$
subject to	$A^\intercal y \geq c$
	$y \geq 0$

$$c^\intercal x \leq y^\intercal A x \leq y^\intercal b$$

Strong Duality

Primal LP

$$\begin{aligned} \text{maximize} \quad & c^\intercal x \\ \text{subject to} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \text{minimize} \quad & b^\intercal y \\ \text{subject to} \quad & A^\intercal y \geq c \\ & y \geq 0 \end{aligned}$$

Theorem (Strong Duality)

If either the primal or dual is feasible and bounded, then so is the other and $OPT(Primal) = OPT(Dual)$.

Interpretation of Strong Duality

Economic Interpretation

Buyer can offer prices for raw materials that would make facility indifferent between production and sale.

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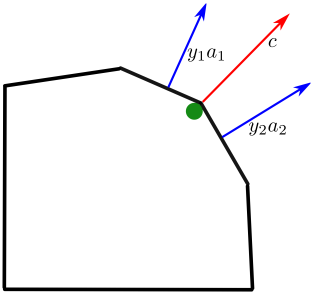
Physical Interpretation

There is an assignment of forces to the walls of the polytope that brings ball back to the origin without wasting energy.

Informal Proof of Strong Duality



- Recall the physical interpretation of duality

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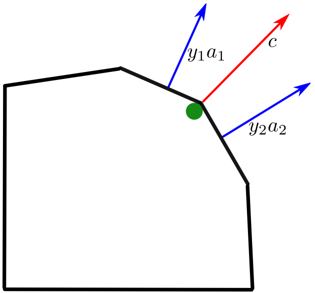

- Recall the physical interpretation of duality
- When ball is stationary at x, we expect force c to be neutralized only by constraints that are tight. i.e. force multipliers $y \geq 0$ s.t.
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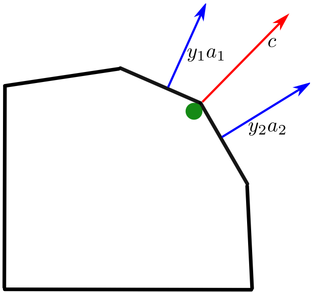

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$$y^\intercal b - c^\intercal x = y^\intercal b - y^\intercal Ax = \sum_i y_i(b_i - a_i x) = 0$$

We found a primal and dual solution that are equal in value!

Outline

Complementary Slackness

Primal LP

$$\begin{array}{ll} \text{maximize} & c^\intercal x \\ \text{subject to} & Ax \le b \\ & x \ge 0 \end{array}$$

Dual LP

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- Let $s_i = (b - Ax)_i$ be the i'th primal slack variable
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Complementary Slackness

x and y are optimal if and only if

- $x_j t_j = 0$ for all $j = 1, \ldots, n$
- $y_i s_i = 0$ for all $i = 1, \ldots, m$

	x_1	x_2	x_3	x_4	
y_1	a_{11}	a_{12}	a_{13}	a_{14}	b_1
y_2	a_{21}	a_{22}	a_{23}	a_{24}	b_2
y_3	a_{31}	a_{32}	a_{33}	a_{34}	b_3
	c_1	c_2	c_3	c_4	

Interpretation of Complementary Slackness

Economic Interpretation

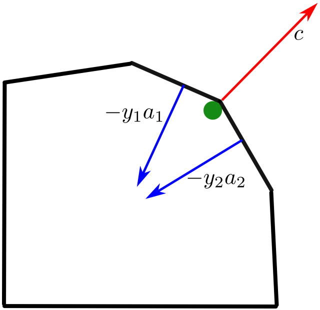
Given an optimal primal production vector x and optimal dual offer prices y,

- Facility produces only products for which it is indifferent between sale and production.
- Only raw materials that are binding constraints on production are priced greater than 0

Interpretation of Complementary Slackness

Physical Interpretation

Only walls adjacent to the balls equilibrium position push back on it.



Proof of Complementary Slackness

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$$\begin{aligned} \text{maximize} \quad & c^\intercal x \\ \text{subject to} \quad & Ax + s = b \\ & x \geq 0 \\ & s \geq 0 \end{aligned}$$

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Gap between primal and dual objectives is 0 if and only if complementary slackness holds.

Recovering Primal from Dual

- Will encounter LPs where the dual is easier to solve than primal
- Complementary slackness allows us to recover the primal optimal from the dual optimal, and vice versa.

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Primal LP
(n variables, $m + n$ constraints)

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  - Exactly n dual constraints are loose
- n loose dual constraints impose n tight primal constraints
  - Assuming non-degeneracy, solving the linear equation yields a unique primal optimum solution x.

Outline

Uses of Duality in Algorithm Design

1. Gain structural insights
   - Dual of a problem gives a “different way of looking at it”.
2. As a benchmark; i.e. to certify (approximate) optimality
   - The primal/dual paradigm
   - A dual may be explicitly constructed by the algorithm, or as part of its analysis

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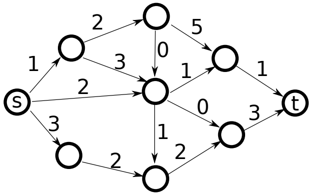
Let’s look at some duals and interpret them.

Shortest Path

Given a directed network $G = (V, E)$ where edge e has length $\ell_e \in \mathbb{R}_+$, find the minimum cost path from s to t.



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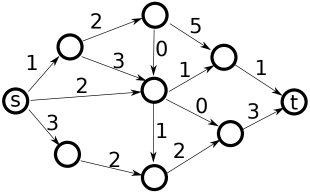

Primal LP

$$
\begin{array}{lll}
\text{min} & \sum_{e \in E} \ell_e x_e & \\
\text{s.t.} & \sum\limits_{e \to v} x_e - \sum\limits_{v \to e} x_e = \delta_v, & \forall v \in \\
 & x_e \geq 0, & \forall e \in
\end{array}
$$

Dual LP

$$
\begin{array}{lll}
\text{max} & y_t - y_s & \\
\text{s.t.} & y_v - y_u \leq \ell_e, & \forall (u,v) \in E.
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Where $\delta_v = -1$ if $v = s$, 1 if $v = t$, and 0 otherwise.

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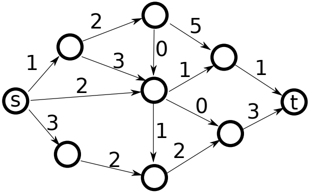

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Where $\delta_v = -1$ if $v = s$, 1 if $v = t$, and 0 otherwise.

Interpretation of Dual

Stretch s and t as far apart as possible, subject to edge lengths.

Vertex Cover

Given an undirected graph $G = (V, E)$, with weights w_i for $i \in V$, find minimum-weight $S \subseteq V$ "covering" all edges.

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Interpretation of Dual

Trying to "sell" coverage to edges at prices y_e.

- Objective: Maximize revenue
- Feasible: charge any neighborhood (of a vertex i) no more than it would cost them if they broke away and bought i themselves

Outline

Solvability of Explicit Linear Programs

$$\begin{aligned} \text{maximize} \quad & c^\intercal x \\ \text{subject to} \quad & Ax \preceq b \\ & x \succeq 0 \end{aligned}$$

- In the examples we have seen so far, the linear program is explicit
- I.e. the constraint matrix A, as well as rhs vector b and objective c, are of polynomial size.

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Theorem (Polynomial Solvability of Explicit LP)

There is a polynomial time algorithm for linear programming, when the linear program is represented explicitly.

Originally using the ellipsoid algorithm, and more recently interior-point algorithms which are more efficient in practice.

Implicit Linear Programs

- These are linear programs in which the number of constraints is exponential (in the natural description of the input)
- These are useful as an analytical tool
- Can be solved in many cases!

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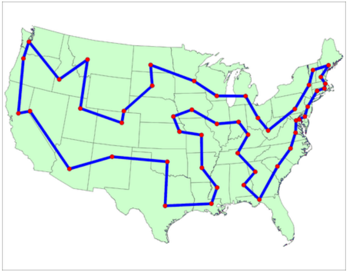
- E.g. Held-Karp relaxation for TSP



$$\begin{array}{lll} \text{min} & \sum_{e \in E} d_e x_e & \\ \text{s.t.} & x(\delta(S)) \geq 2, & \forall \emptyset \subset S \subset \\ & x(\delta(v)) = 2, & \forall v \in V. \\ & 0 \preceq x \preceq 1 & \end{array}$$

Where $\delta(S)$ denotes the edges going out of $S \subseteq V$.

Solvability of Implicit Linear Programs

$$\begin{array}{ll} \text{maximize} & c^\intercal x \\ \text{subject to} & Ax \preceq b \\ & x \succeq 0 \end{array}$$

Theorem (Polynomial Solvability of Implicit LP)

Consider a family Π of linear programming problems $I = (A, b, c)$ admitting the following operations in polynomial time (in $\langle I \rangle$ and n):

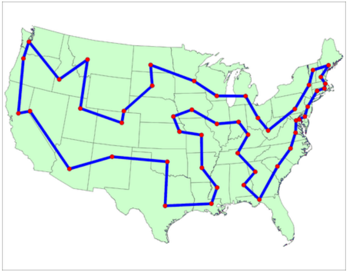
- *A separation oracle for the polyhedron $Ax \preceq b$*
- *Explicit access to c*

Moreover, assume that every $\langle a_{ij} \rangle$, $\langle b_i \rangle$, $\langle c_j \rangle$ are at most $\mathrm{poly}(\langle I \rangle, n)$. Then there is a polynomial time algorithm for Π (both primal and dual).

Separation oracle

An algorithm that takes as input $x \in \mathbb{R}^n$, and either certifies $Ax \preceq b$ or finds a violated constraint $a_i x > b_i$.

E.g. of a Separation Oracle



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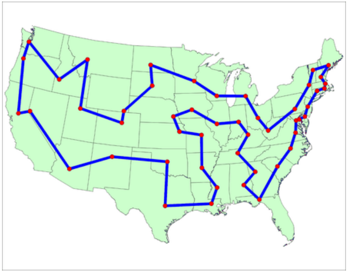

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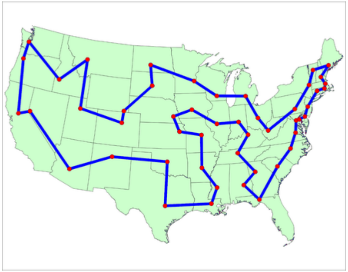

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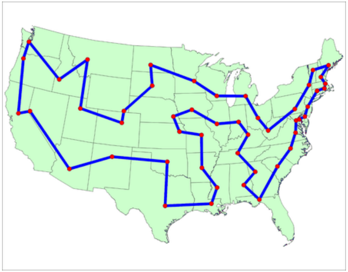

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- This is min-cut in a weighted graph, which we can solve.