CSCI567 Machine Learning (Fall 2021)

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U of Southern California

Sep 16, 2021

Administration

HW1 is being graded. Will discuss solutions today.

HW2 will be released after this lecture. Due on 9/28.

Outline

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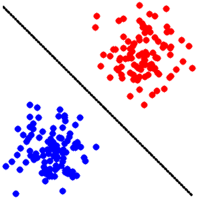
Linear classifiers

Linear models for **binary** classification:

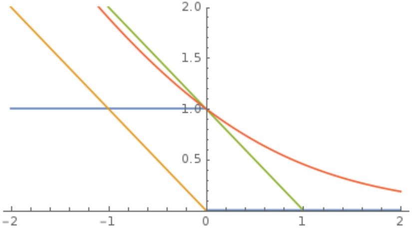
Step 1. Model is the set of **separating hyperplanes**

$$\mathcal{F} = \{f(\boldsymbol{x}) = \mathsf{sgn}(\boldsymbol{w}^{\mathrm{T}}\boldsymbol{x}) \mid \boldsymbol{w} \in \mathbb{R}^{\mathsf{D}}\}$$



Linear classifiers

Step 2. Pick the **surrogate loss**



- perceptron loss $\ell_{\text{perceptron}}(z) = \max\{0, -z\}$ (used in Perceptron)
- hinge loss $\ell_{\text{hinge}}(z) = \max\{0, 1 - z\}$(used in SVM and many others)
- logistic loss $\ell_{\text{logistic}}(z) = \log(1 + \exp(-z))$ (used in logistic regression)

Linear classifiers

Step 3. Find empirical risk minimizer (ERM):

$$\boldsymbol{w}^* = \operatorname*{argmin}_{\boldsymbol{w} \in \mathbb{R}^{\mathsf{D}}} F(\boldsymbol{w}) = \operatorname*{argmin}_{\boldsymbol{w} \in \mathbb{R}^{\mathsf{D}}} \frac{1}{N} \sum_{n=1}^{N} \ell(y_n \boldsymbol{w}^{\mathrm{T}} \boldsymbol{x}_n)$$

using

- **GD:** $\boldsymbol{w} \leftarrow \boldsymbol{w} - \eta \nabla F(\boldsymbol{w})$
- **SGD:** $\boldsymbol{w} \leftarrow \boldsymbol{w} - \eta \tilde{\nabla} F(\boldsymbol{w})$ $\quad (\mathbb{E}[\tilde{\nabla} F(\boldsymbol{w})] = \nabla F(\boldsymbol{w}))$
- **Newton:** $\boldsymbol{w} \leftarrow \boldsymbol{w} - \left(\nabla^2 F(\boldsymbol{w})\right)^{-1} \nabla F(\boldsymbol{w})$

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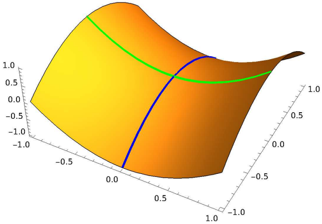
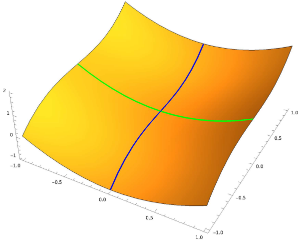
- for nonconvex objectives, can get stuck at local minimizers or "bad" saddle points (random initialization escapes "good" saddle points)



"good" saddle points



"bad" saddle points

Perceptron and logistic regression

Initialize $\boldsymbol{w} = \mathbf{0}$ or randomly.

Repeat:

- pick a data point $\boldsymbol{x}_n$ uniformly at random (**common trick for SGD**)

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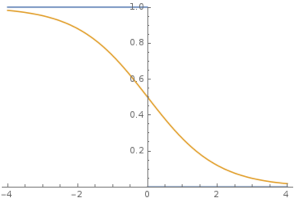
Repeat:

- pick a data point $\boldsymbol{x}_n$ uniformly at random (**common trick for SGD**)
- update parameter:

$$\boldsymbol{w} \leftarrow \boldsymbol{w} + \begin{cases} \mathbb{I}[y_n \boldsymbol{w}^{\mathrm{T}} \boldsymbol{x}_n \leq 0] y_n \boldsymbol{x}_n & \text{(Perceptron)} \\ \eta \sigma(-y_n \boldsymbol{w}^{\mathrm{T}} \boldsymbol{x}_n) y_n \boldsymbol{x}_n & \text{(logistic regression)} \end{cases}$$



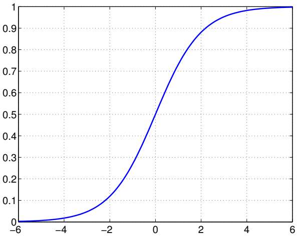
A Probabilistic view of logistic regression

Minimizing logistic loss = MLE for the sigmoid model

$$\boldsymbol{w}^* = \operatorname*{argmin}_{\boldsymbol{w}} \sum_{n=1}^{N} \ell_{\text{logistic}}(y_n \boldsymbol{w}^{\mathrm{T}} \boldsymbol{x}_n) = \operatorname*{argmax}_{\boldsymbol{w}} \prod_{n=1}^{N} \mathbb{P}(y_n \mid \boldsymbol{x_n}; \boldsymbol{w})$$

where

$$\mathbb{P}(y \mid \boldsymbol{x}; \boldsymbol{w}) = \sigma(y\boldsymbol{w}^{\mathrm{T}}\boldsymbol{x}) = \frac{1}{1 + e^{-y\boldsymbol{w}^{\mathrm{T}}\boldsymbol{x}}}$$



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Classification

Recall the setup:

- input (feature vector): $\boldsymbol{x} \in \mathbb{R}^{\mathsf{D}}$
- output (label): $y \in [\mathsf{C}] = \{1, 2, \cdots, \mathsf{C}\}$
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- predicting weather: sunny, cloudy, rainy, etc
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Nearest Neighbor Classifier naturally works for arbitrary C.

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for any $\boldsymbol{w}_1, \boldsymbol{w}_2$ s.t. $\boldsymbol{w} = \boldsymbol{w}_1 - \boldsymbol{w}_2$

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for any $\boldsymbol{w}_1, \boldsymbol{w}_2$ s.t. $\boldsymbol{w} = \boldsymbol{w}_1 - \boldsymbol{w}_2$

Think of $\boldsymbol{w}_k^{\mathrm{T}}\boldsymbol{x}$ as **a score for class k**.

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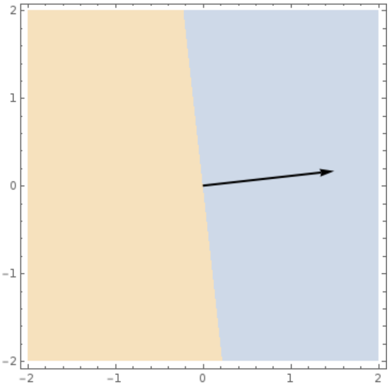

$\boldsymbol{w} = (\frac{3}{2}, \frac{1}{6})$

- Blue class: $\{\boldsymbol{x} : \boldsymbol{w}^{\mathrm{T}}\boldsymbol{x} \geq 0\}$
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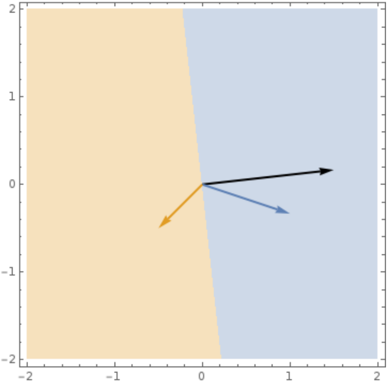

$\boldsymbol{w} = (\frac{3}{2}, \frac{1}{6}) = \boldsymbol{w}_1 - \boldsymbol{w}_2$
$\boldsymbol{w}_1 = (1, -\frac{1}{3})$
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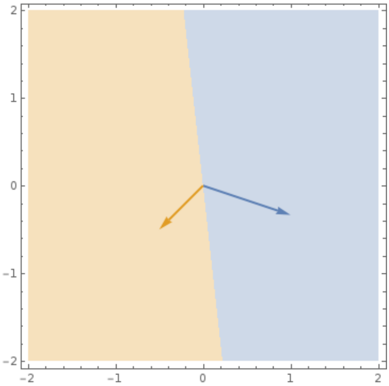

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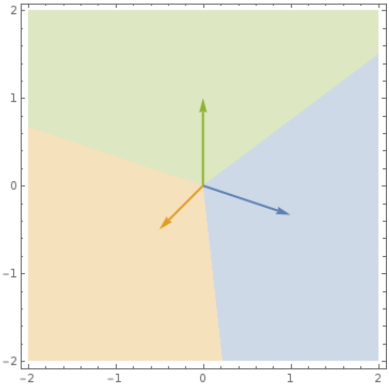

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$\boldsymbol{w}_3 = (0, 1)$

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- Green class: $\{\boldsymbol{x} : 3 = \text{argmax}_k\, \boldsymbol{w}_k^{\mathrm{T}} \boldsymbol{x}\}$

Linear models for multiclass classification

$$\mathcal{F} = \left\{ f(\boldsymbol{x}) = \underset{k \in [\mathsf{C}]}{\operatorname{argmax}} \ \boldsymbol{w}_k^{\mathrm{T}} \boldsymbol{x} \mid \boldsymbol{w}_1, \ldots, \boldsymbol{w}_{\mathsf{C}} \in \mathbb{R}^{\mathsf{D}} \right\}$$

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This lecture: focus on the more popular **logistic loss**

Multinomial logistic regression: a probabilistic view

Observe: for binary logistic regression, with $\boldsymbol{w} = \boldsymbol{w}_1 - \boldsymbol{w}_2$:

$$\mathbb{P}(y = 1 \mid \boldsymbol{x}; \boldsymbol{w}) = \sigma(\boldsymbol{w}^{\mathrm{T}}\boldsymbol{x}) = \frac{1}{1 + e^{-\boldsymbol{w}^{\mathrm{T}}\boldsymbol{x}}} = \frac{e^{\boldsymbol{w}_1^{\mathrm{T}}\boldsymbol{x}}}{e^{\boldsymbol{w}_1^{\mathrm{T}}\boldsymbol{x}} + e^{\boldsymbol{w}_2^{\mathrm{T}}\boldsymbol{x}}} \propto e^{\boldsymbol{w}_1^{\mathrm{T}}\boldsymbol{x}}$$

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This is called the *softmax function*.

Applying MLE again

Maximize probability of seeing labels $y_1, \ldots, y_\mathsf{N}$ given $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_\mathsf{N}$

$$P(\boldsymbol{W}) = \prod_{n=1}^{\mathsf{N}} \mathbb{P}(y_n \mid \boldsymbol{x}_n; \boldsymbol{W}) = \prod_{n=1}^{\mathsf{N}} \frac{e^{\boldsymbol{w}_{y_n}^{\mathrm{T}} \boldsymbol{x}_n}}{\sum_{k \in [\mathsf{C}]} e^{\boldsymbol{w}_k^{\mathrm{T}} \boldsymbol{x}_n}}$$

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By taking **negative log**, this is equivalent to minimizing

$$F(\boldsymbol{W}) = \sum_{n=1}^{\mathsf{N}} \ln \left(\frac{\sum_{k \in [\mathsf{C}]} e^{\boldsymbol{w}_k^{\mathrm{T}} \boldsymbol{x}_n}}{e^{\boldsymbol{w}_{y_n}^{\mathrm{T}} \boldsymbol{x}_n}} \right)$$

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This is the *multiclass logistic loss*, a.k.a. *cross-entropy loss*.

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When $\mathsf{C} = 2$, this is the same as binary logistic loss.

Step 3: Optimization

Apply **SGD**: what is the gradient of

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else:

$$\nabla_{\boldsymbol{w}_k^{\mathrm{T}}} F_n(\boldsymbol{W}) = \frac{-\left(\sum_{k' \neq y_n} e^{(\boldsymbol{w}_{k'} - \boldsymbol{w}_{y_n})^{\mathrm{T}}\boldsymbol{x}_n}\right)}{1 + \sum_{k' \neq y_n} e^{(\boldsymbol{w}_{k'} - \boldsymbol{w}_{y_n})^{\mathrm{T}}\boldsymbol{x}_n}} \boldsymbol{x}_n^{\mathrm{T}}$$

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It's a C $\times$ D matrix. Let's focus on the k-th row:

If $k \neq y_n$:

$$\nabla_{\boldsymbol{w}_k^{\mathrm{T}}} F_n(\boldsymbol{W}) = \frac{e^{(\boldsymbol{w}_k - \boldsymbol{w}_{y_n})^{\mathrm{T}}\boldsymbol{x}_n}}{1 + \sum_{k' \neq y_n} e^{(\boldsymbol{w}_{k'} - \boldsymbol{w}_{y_n})^{\mathrm{T}}\boldsymbol{x}_n}} \boldsymbol{x}_n^{\mathrm{T}} = \mathbb{P}(k \mid \boldsymbol{x}_n; \boldsymbol{W})\boldsymbol{x}_n^{\mathrm{T}}$$

else:

$$\nabla_{\boldsymbol{w}_k^{\mathrm{T}}} F_n(\boldsymbol{W}) = \frac{-\left(\sum_{k' \neq y_n} e^{(\boldsymbol{w}_{k'} - \boldsymbol{w}_{y_n})^{\mathrm{T}}\boldsymbol{x}_n}\right)}{1 + \sum_{k' \neq y_n} e^{(\boldsymbol{w}_{k'} - \boldsymbol{w}_{y_n})^{\mathrm{T}}\boldsymbol{x}_n}} \boldsymbol{x}_n^{\mathrm{T}} = \left(\mathbb{P}(y_n \mid \boldsymbol{x}_n; \boldsymbol{W}) - 1\right) \boldsymbol{x}_n^{\mathrm{T}}$$

SGD for multinomial logistic regression

Initialize $\boldsymbol{W} = \mathbf{0}$ (or randomly). Repeat:

1. pick $n \in [\mathsf{N}]$ uniformly at random
2. update the parameters

$$\boldsymbol{W} \leftarrow \boldsymbol{W} - \eta \begin{pmatrix} \mathbb{P}(y = 1 \mid \boldsymbol{x}_n; \boldsymbol{W}) \\ \vdots \\ \mathbb{P}(y = y_n \mid \boldsymbol{x}_n; \boldsymbol{W}) - 1 \\ \vdots \\ \mathbb{P}(y = \mathsf{C} \mid \boldsymbol{x}_n; \boldsymbol{W}) \end{pmatrix} \boldsymbol{x}_n^{\mathrm{T}}$$

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Think about why the algorithm makes sense intuitively.

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Having learned $\boldsymbol{W}$, we can either

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Reduce multiclass to binary

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Yes, there are in fact many ways to do it.

- **one-versus-all** (one-versus-rest, one-against-all, etc.)
- **one-versus-one** (all-versus-all, etc.)
- **Error-Correcting Output Codes** (ECOC)
- **tree-based reduction**

One-versus-all (OvA)

(picture credit: link)

Idea: train C binary classifiers to learn "**is class k or not?**" for each k.

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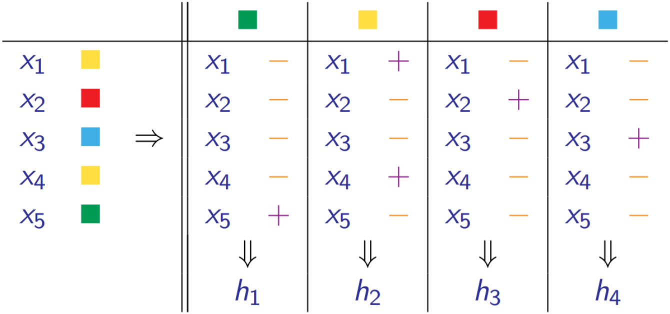

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Issue: will (probably) make a mistake *as long as one of h_k errs*.

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Idea: train $\binom{C}{2}$ binary classifiers to learn "**is class k or k'?**".

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Training: for each pair (k, k'),

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- *discard all other examples*
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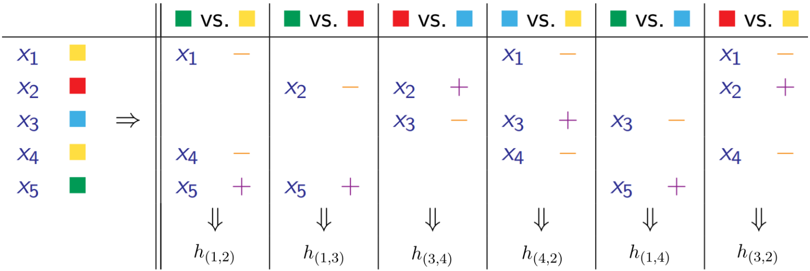

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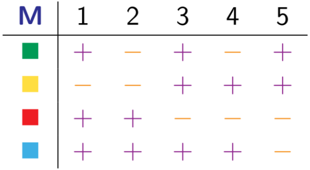
More robust than one-versus-all, but *slower* in prediction.

Error-correcting output codes (ECOC) (picture credit: link)

Idea: based on a code $\boldsymbol{M} \in \{-1, +1\}^{\mathsf{C} \times \mathsf{L}}$, train L binary classifiers to learn "**is bit b on or off**".



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Training: for each bit $b \in [\mathsf{L}]$

- relabel example x_n as $M_{y_n, b}$
- train a binary classifier h_b using this new dataset.

M	1	2	3	4	5
■	+	−	+	−	+
■	−	−	+	+	+
■	+	+	−	−	−
■	+	+	+	+	−

			1		2		3		4		5	
x_1	■		x_1	−	x_1	−	x_1	+	x_1	+	x_1	+
x_2	■		x_2	+	x_2	+	x_2	−	x_2	−	x_2	−
x_3	■	$\Rightarrow$	x_3	+	x_3	+	x_3	+	x_3	+	x_3	−
x_4	■		x_4	−	x_4	−	x_4	+	x_4	+	x_4	+
x_5	■		x_5	+	x_5	−	x_5	+	x_5	−	x_5	+
			$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$	
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Error-correcting output codes (ECOC)

Prediction: for a new example $\boldsymbol{x}$

- compute the **predicted code** $\boldsymbol{c} = (h_1(\boldsymbol{x}), \ldots, h_{\mathsf{L}}(\boldsymbol{x}))^{\mathrm{T}}$

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How to design the code $\boldsymbol{M}$?

- the more *dissimilar* the codes, the more robust
  - if any two codes are d bits away, then prediction can tolerate about $d/2$ errors
- *random code* is often a good choice

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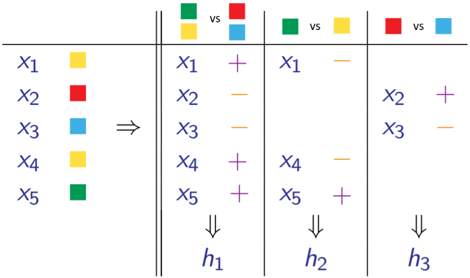
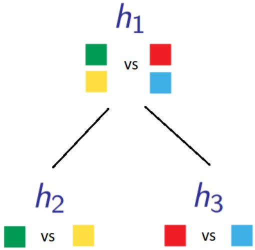
Training: see pictures





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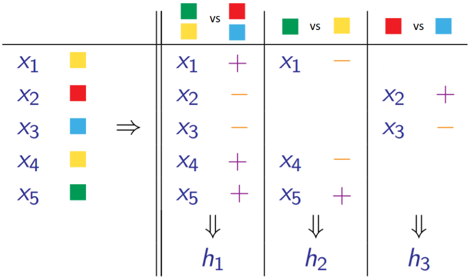

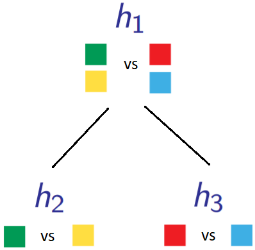

Prediction is also natural,

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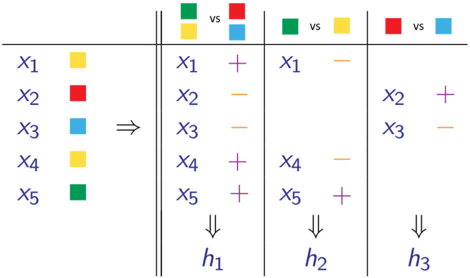

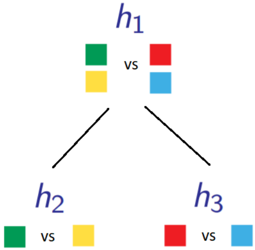

Prediction is also natural, *but is very fast!* (think ImageNet where $C \approx 20K$)

Comparisons

Reduction	**training time**	**prediction time**	**remark**

training time: how many training points are created

prediction time: how many binary predictions are made

Comparisons

Reduction	**training time**	**prediction time**	**remark**
OvA			

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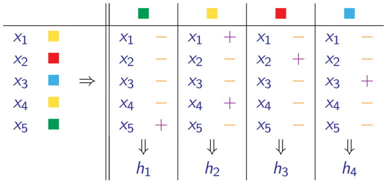

Comparisons

Reduction	**training time**	**prediction time**	**remark**
OvA	CN		

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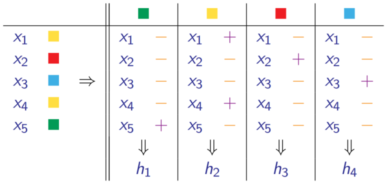

Comparisons

Reduction	**training time**	**prediction time**	**remark**
OvA	CN	C	

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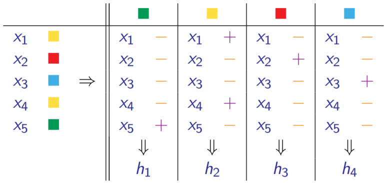

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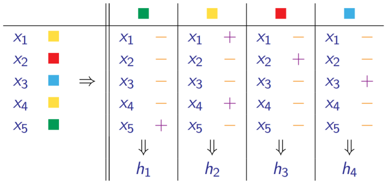

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OvO			

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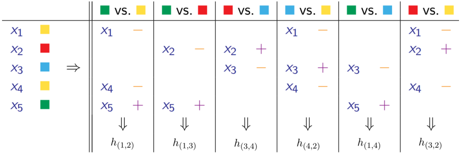

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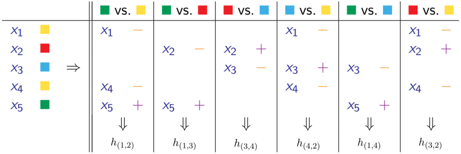

Comparisons

Reduction	**training time**	**prediction time**	**remark**
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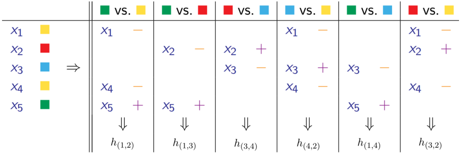

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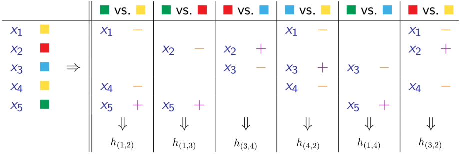

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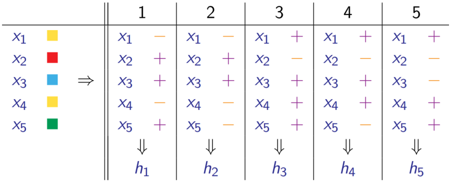

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ECOC	LN		

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			1		2		3		4		5	
x_1			x_1	$-$	x_1	$-$	x_1	$+$	x_1	$+$	x_1	$+$
x_2			x_2	$+$	x_2	$+$	x_2	$-$	x_2	$-$	x_2	$-$
x_3		$\Rightarrow$	x_3	$+$	x_3	$+$	x_3	$+$	x_3	$+$	x_3	$-$
x_4			x_4	$-$	x_4	$-$	x_4	$+$	x_4	$+$	x_4	$+$
x_5			x_5	$+$	x_5	$-$	x_5	$+$	x_5	$-$	x_5	$+$
			$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$	
			h_1		h_2		h_3		h_4		h_5	

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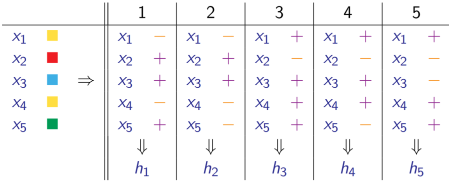

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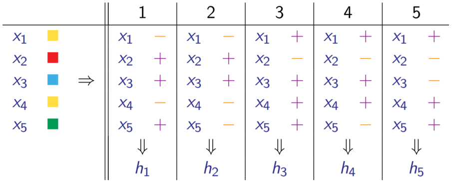

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Tree			

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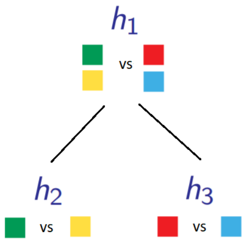

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ECOC	LN	L	need diversity when designing code
Tree	$\mathcal{O}((\log_2 C)N)$		

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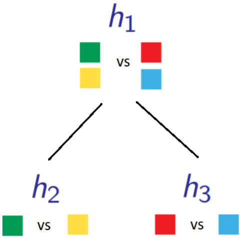

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Reduction	**training time**	**prediction time**	**remark**
OvA	CN	C	not robust
OvO	$(C-1)N$	$\mathcal{O}(C^2)$	can achieve very small training error
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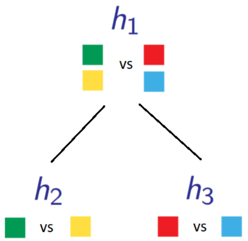

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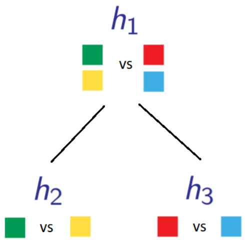
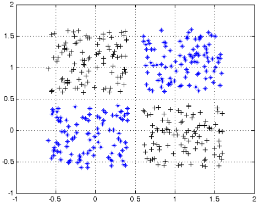
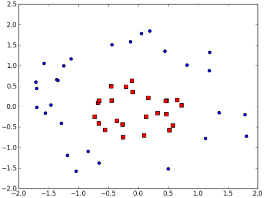


Outline

Linear models are not always adequate





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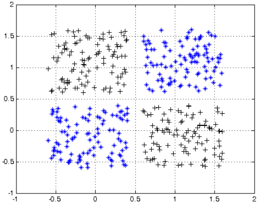

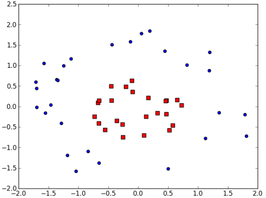

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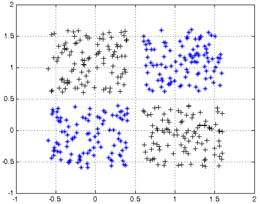

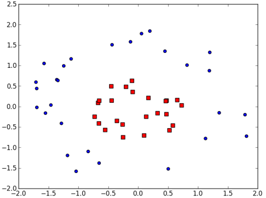

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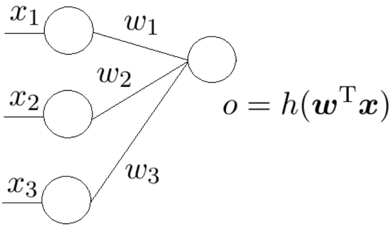
THE most popular nonlinear models nowadays: **neural nets**

Linear model as a one-layer neural net



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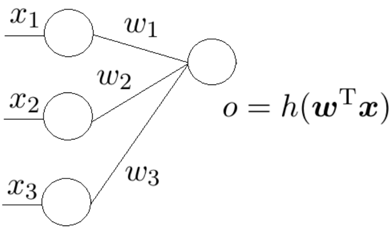
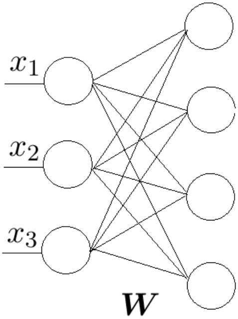


$h(a) = a$ for linear model

To create non-linearity, can use

- Rectified Linear Unit (**ReLU**): $h(a) = \max\{0, a\}$
- sigmoid function: $h(a) = \frac{1}{1+e^{-a}}$
- TanH: $h(a) = \frac{e^{a}-e^{-a}}{e^{a}+e^{-a}}$
- many more

More output nodes



$$\boldsymbol{o} = \boldsymbol{h}(\boldsymbol{W}\boldsymbol{x})$$

$\boldsymbol{W} \in \mathbb{R}^{4\times 3}$, $\boldsymbol{h} : \mathbb{R}^4 \to \mathbb{R}^4$ so $\boldsymbol{h}(\boldsymbol{a}) = (h_1(a_1), h_2(a_2), h_3(a_3), h_4(a_4))$

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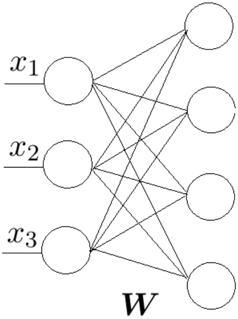

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Can think of this as a nonlinear mapping: $\boldsymbol{\phi}(\boldsymbol{x}) = \boldsymbol{h}(\boldsymbol{W}\boldsymbol{x})$

More layers

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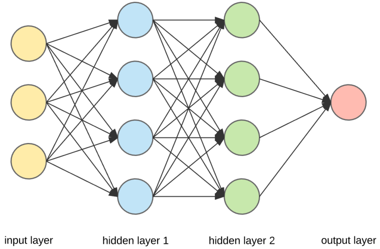

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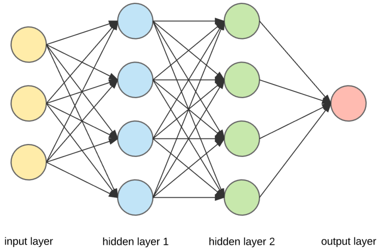

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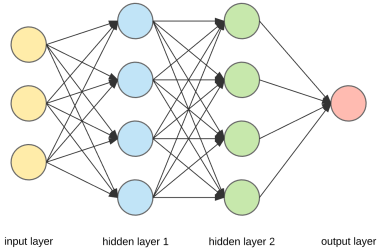

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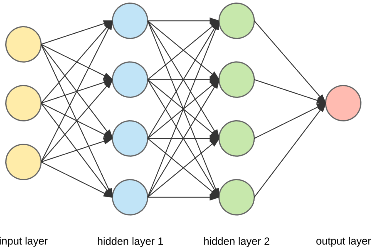

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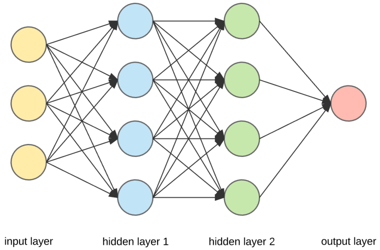

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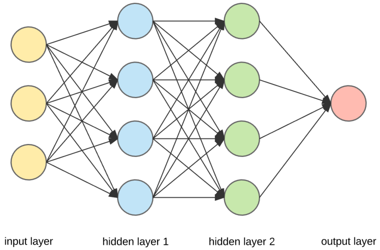

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- **deep** neural nets can have many layers and *millions* of parameters
- this is a **feedforward, fully connected** neural net, there are many variants (convolutional nets, residual nets, recurrent nets, etc.)

How powerful are neural nets?

Universal approximation theorem (Cybenko, 89; Hornik, 91):

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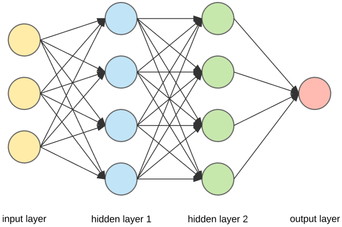
Designing network architecture is important and very complicated

- for feedforward network, need to decide number of hidden layers, number of neurons at each layer, activation functions, etc.

Math formulation

An L-layer neural net can be written as

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{h}_{\mathsf{L}}\left(\boldsymbol{W}_L \boldsymbol{h}_{\mathsf{L}-1}\left(\boldsymbol{W}_{L-1} \cdots \boldsymbol{h}_1\left(\boldsymbol{W}_1 \boldsymbol{x}\right)\right)\right)$$



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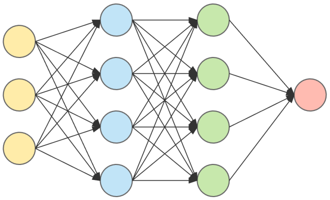

To ease notation, for a given input $\boldsymbol{x}$, define recursively

$$\boldsymbol{o}_0 = \boldsymbol{x}, \qquad \boldsymbol{a}_\ell = \boldsymbol{W}_\ell \boldsymbol{o}_{\ell-1}, \qquad \boldsymbol{o}_\ell = \boldsymbol{h}_\ell(\boldsymbol{a}_\ell) \qquad (\ell = 1, \ldots, \mathsf{L})$$

where

- $\boldsymbol{W}_\ell \in \mathbb{R}^{\mathsf{D}_\ell \times \mathsf{D}_{\ell-1}}$ is the weights between layer $\ell - 1$ and ℓ
- $\mathsf{D}_0 = \mathsf{D}, \mathsf{D}_1, \ldots, \mathsf{D}_\mathsf{L}$ are numbers of neurons at each layer
- $\boldsymbol{a}_\ell \in \mathbb{R}^{\mathsf{D}_\ell}$ is input to layer ℓ
- $\boldsymbol{o}_\ell \in \mathbb{R}^{\mathsf{D}_\ell}$ is output of layer ℓ
- $\boldsymbol{h}_\ell : \mathbb{R}^{\mathsf{D}_\ell} \to \mathbb{R}^{\mathsf{D}_\ell}$ is activation functions at layer ℓ

Learning the model

No matter how complicated the model is, our goal is the same: minimize

$$F(\boldsymbol{W}_1, \ldots, \boldsymbol{W}_{\mathsf{L}}) = \frac{1}{N} \sum_{n=1}^{\mathsf{N}} F_n(\boldsymbol{W}_1, \ldots, \boldsymbol{W}_{\mathsf{L}})$$

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where

$$F_n(\boldsymbol{W}_1, \ldots, \boldsymbol{W}_\mathsf{L}) = \begin{cases} \|\boldsymbol{f}(\boldsymbol{x}_n) - \boldsymbol{y}_n\|_2^2 & \text{for regression} \\ \ln\left(1 + \sum_{k \neq y_n} e^{f(\boldsymbol{x}_n)_k - f(\boldsymbol{x}_n)_{y_n}}\right) & \text{for classification} \end{cases}$$

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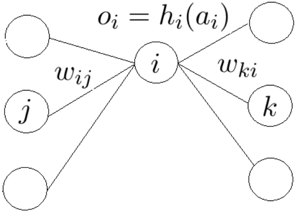
the simplest example $f(g_1(w), g_2(w)) = g_1(w)g_2(w)$

Computing the derivative

Drop the subscript ℓ for layer for simplicity.

Find the **derivative of F_n w.r.t. to w_{ij}**



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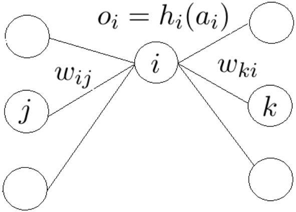

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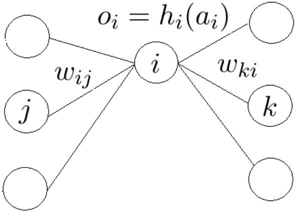

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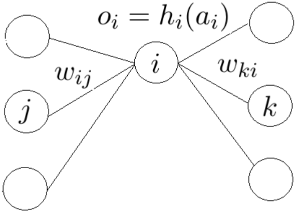

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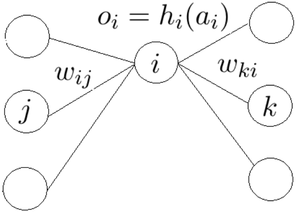

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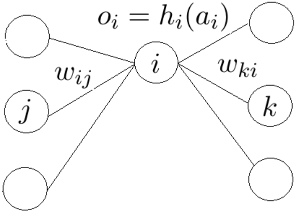

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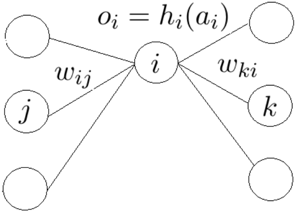

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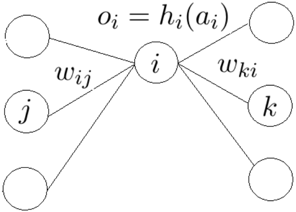

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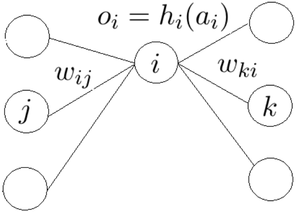

For the last layer, for square loss

$$\frac{\partial F_n}{\partial a_{\mathsf{L},i}} = \frac{\partial (h_{\mathsf{L},i}(a_{\mathsf{L},i}) - y_{n,i})^2}{\partial a_{\mathsf{L},i}}$$

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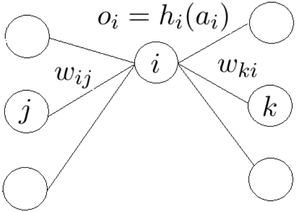

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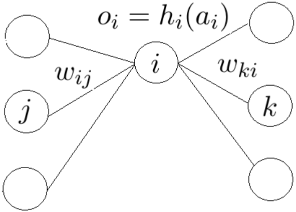

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Exercise: try to do it for logistic loss yourself.

Computing the derivative

Using **matrix notation** greatly simplifies presentation and implementation:

$$\frac{\partial F_n}{\partial \boldsymbol{W}_\ell} = \frac{\partial F_n}{\partial \boldsymbol{a}_\ell} \boldsymbol{o}_{\ell-1}^{\mathrm{T}} \in \mathbb{R}^{\mathsf{D}_\ell \times \mathsf{D}_{\ell-1}}$$

$$\frac{\partial F_n}{\partial \boldsymbol{a}_\ell} = \begin{cases} \left(\boldsymbol{W}_{\ell+1}^{\mathrm{T}} \frac{\partial F_n}{\partial \boldsymbol{a}_{\ell+1}}\right) \circ \boldsymbol{h}_\ell'(\boldsymbol{a}_\ell) & \text{if } \ell < \mathsf{L} \\ 2(\boldsymbol{h}_\mathsf{L}(\boldsymbol{a}_\mathsf{L}) - \boldsymbol{y}_n) \circ \boldsymbol{h}_\mathsf{L}'(\boldsymbol{a}_\mathsf{L}) & \text{else} \end{cases}$$

where $\boldsymbol{v}_1 \circ \boldsymbol{v}_2 = (v_{11}v_{21}, \cdots, v_{1\mathsf{D}}v_{2\mathsf{D}})$ is the element-wise product (a.k.a. Hadamard product).

Verify yourself!

Putting everything into SGD

The **backpropagation** algorithm (**Backprop**)

Initialize $\boldsymbol{W}_1, \ldots, \boldsymbol{W}_\mathsf{L}$ randomly.

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(Important: *should $\boldsymbol{W}_\ell$ be overwritten immediately in the last step?*)

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Many variants based on Backprop

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- **batch normalization**: normalize the inputs of each neuron over the mini-batch (to zero-mean and one-variance; *c.f.* Lec 1)
- **momentum**: make use of previous gradients (taking inspiration from physics)
- $\cdots$

SGD with momentum (a simple version)

Initialize $\boldsymbol{w}_0$ and **velocity** $\boldsymbol{v} = \boldsymbol{0}$

For $t = 1, 2, \ldots$

- form a stochastic gradient $\boldsymbol{g}_t$
- update velocity $\boldsymbol{v} \leftarrow \alpha\boldsymbol{v} + \boldsymbol{g}_t$ for some discount factor $\alpha \in (0, 1)$
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Updates for first few rounds:

- $\boldsymbol{w}_1 = \boldsymbol{w}_0 - \eta\boldsymbol{g}_1$
- $\boldsymbol{w}_2 = \boldsymbol{w}_1 - \alpha\eta\boldsymbol{g}_1 - \eta\boldsymbol{g}_2$
- $\boldsymbol{w}_3 = \boldsymbol{w}_2 - \alpha^2\eta\boldsymbol{g}_1 - \alpha\eta\boldsymbol{g}_2 - \eta\boldsymbol{g}_3$
- $\cdots$

Overfitting

Overfitting is very likely since neural nets are too powerful.

Methods to overcome overfitting:

- data augmentation
- regularization
- dropout
- early stopping
- ...

Data augmentation

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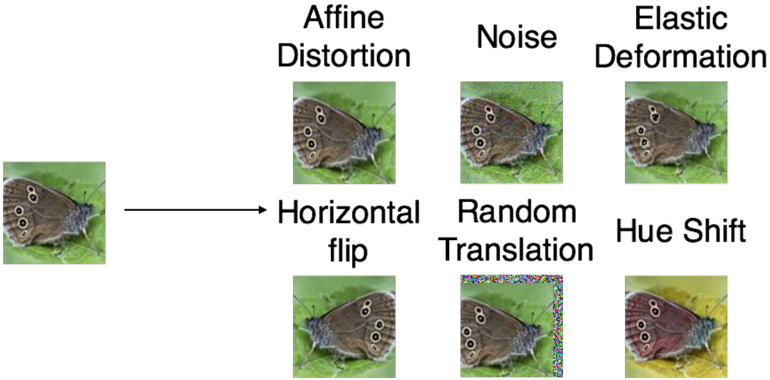
Exploit prior knowledge to add more training data



Regularization

L2 regularization: minimize

$$F'(\boldsymbol{W}_1, \ldots, \boldsymbol{W}_\mathsf{L}) = F(\boldsymbol{W}_1, \ldots, \boldsymbol{W}_\mathsf{L}) + \lambda \sum_{\ell=1}^{\mathsf{L}} \|\boldsymbol{W}_\ell\|_2^2$$

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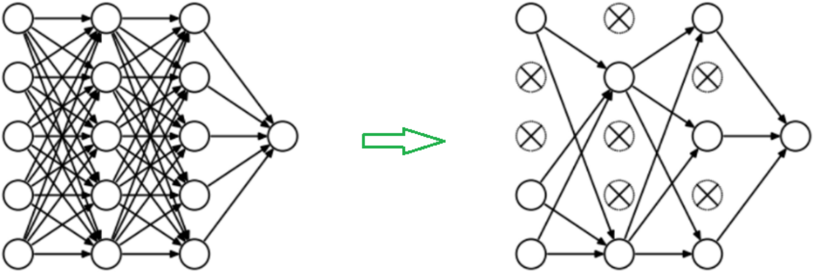
Introduce *weight decaying effect*

Dropout

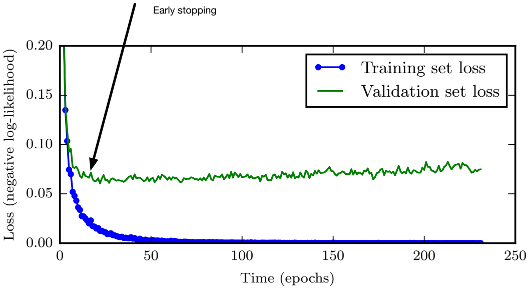
Independently delete each neuron with a fixed probability (say 0.5), during each iteration of Backprop (only for training, not for testing)



Very effective, makes training faster as well

Early stopping

Stop training when the performance on validation set stops improving



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- do need *a lot of data* to work well
- take *a lot of time* to train (need GPUs for massive parallel computing)
- take some work to select architecture and hyperparameters
- are still not well understood in theory