FAS-PASS Maths

CSEC MATHEMATICS PAPER 2 JANUARY 2017

SECTION I

1. (a) Using a calculator, or otherwise, calculate the EXACT value of:

(i) $$\frac{3\frac{1}{2}\times 1\frac{2}{3}}{4\frac{1}{5}}$$

SOLUTION:

Required to calculate: The exact value of $\frac{3\frac{1}{2}\times 1\frac{2}{3}}{4\frac{1}{5}}$

Calculation:

$$\frac{3\frac{1}{2}\times 1\frac{2}{3}}{4\frac{1}{5}}=\frac{\frac{7}{2}\times\frac{5}{3}}{\frac{21}{5}}$$

$$=\frac{7}{2}\times\frac{5}{3}\times\frac{5}{21}$$

$$=\frac{5\times 5}{2\times 3\times 3}$$

$$=\frac{25}{18}$$

$$=1\frac{7}{18}\text{ (in exact form)}$$

(ii) $$5.47-\sqrt{\frac{0.1014}{1.5}}$$

SOLUTION:

Required to calculate: The exact value of $5.47-\sqrt{\frac{0.1014}{1.5}}$

Calculation:

$$5.47-\sqrt{\frac{0.1014}{1.5}}=5.47-\sqrt{0.0676}\quad\text{(Using the calculator)}$$

$$=5.47-0.26$$

$$=5.21\text{ (in exact form)}$$

(b) The table below shows the number of tickets sold for a bus tour. Some items in the table are missing.

Tickets Sold for Bus Tour			
Category	**Number of Tickets Sold**	**Cost per Ticket in $**	**Total Cost in $**
Juvenile	5	P	130.50
Youth	14	44.35	Q
Adult	R		2483.60

(i) Calculate the value of P.

SOLUTION:
Data: Table showing the number of tickets sold for a bus tour.
Required to calculate: The value of P
Calculation:
5 Juvenile tickets at \$P each cost \$130.50.
So, 1 Juvenile ticket will cost $\frac{\$130.50}{5} = \$26.10$
So, P = 26.10

(ii) Calculate the value of Q.

SOLUTION:
Required to calculate: The value of Q
Calculation:
14 Youth tickets at \$44.35 will cost \$Q.
$\therefore \$Q = \$44.35 \times 14$
$= \$620.90$
$\therefore Q = 620.90$

(iii) An adult ticket is TWICE the cost of a youth ticket. Calculate the value of R.
SOLUTION:
Data: An adult ticket is twice the cost of a youth ticket.
Required to calculate: The value of R
Calculation:
An adult ticket costs twice as much as the cost of a Youth ticket.
Hence, the cost of an adult ticket $= \$44.35 \times 2$
$= \$88.70$
$= 88.70$
No. of adult tickets sold, $R = \frac{\$2\,483.60}{\$88.70}$
$= 28$

(iv) The bus company pays taxes of 15% on each ticket sold. Calculate the taxes paid by the bus company.

SOLUTION:
Data: The bus company pays 15% taxes on each ticket sold.
Required to calculate: The taxes paid by the bus company.
Calculation:
The amount collected from the sales of tickets is

$$= \$130.50 + \$Q + \$2483.60$$

$$\begin{aligned} = \$\ \ 130.50& \\ \$\ \ 620.90& + \\ \underline{\$2483.60}& \\ \underline{\$3235.00}& \end{aligned}$$

So, the taxes paid

$$\begin{aligned} &= 15\% \text{ of } \$3235 \\ &= \frac{15}{100} \times \$3235 \\ &= \$485.25 \end{aligned}$$

2. (a) Write as a single fraction:

$$\frac{2x+3}{3} + \frac{x-4}{4}$$

SOLUTION:
Required to write: $\frac{2x+3}{3} + \frac{x-4}{4}$ as a single fraction.

Solution:

$$\begin{aligned} \frac{2x+3}{3} + \frac{x-4}{4} &= \frac{4(2x+3)+3(x-4)}{12} \\ &= \frac{8x+12+3x-12}{12} \\ &= \frac{11x}{12} \text{ (as a single fraction in its lowest form)} \end{aligned}$$

(b) Write the following statement as an algebraic expression.

The sum of a number and its multiplicative inverse is five times the number.

SOLUTION:
Data: The sum of a number and its multiplicative inverse is five times the number.
Required to write: The statement as an algebraic expression
Solution:
Let the number be *x*.
Hence, its multiplicative inverse (reciprocal) $= \frac{1}{x}$

$$\underbrace{\text{The sum of a number and its multiplicative inverse}}_{x+\frac{1}{x}} \underbrace{\text{is}}_{=} \underbrace{\text{five times the number.}}_{5\times x}$$

$$x + \frac{1}{x} = 5x$$

(c) Factorise completely:

(i) $x^2 - 36$

SOLUTION:
Required to factorise: $x^2 - 36$
Solution:
$x^2 - 36 = (x)^2 - (6)^2$
This is now in the form of a difference of two squares:
$\therefore x^2 - 36 = (x-6)(x+6)$

(ii) $2x^2 + 5x - 12$

SOLUTION:
Required to factorise: $2x^2 + 5x - 12$

Solution:

$$2x^2 + 5x - 12 = (2x-3)(x+4)$$

$$\begin{array}{r} 2x^2 + 8x \\ -3x - 12 \\ \hline 2x^2 + 5x - 12 \\ \hline \end{array}$$

$$\therefore 2x^2 + 5x - 12 = (2x-3)(x+4)$$

(d) The formula for the volume of a cylinder is given as $V = \pi r^2 h$.

Make *r* the subject of the formula.

SOLUTION:
Data: The formula for the volume of a cylinder is, $V = \pi r^2 h$.

Required to make: r the subject of the formula
Solution:
$V = \pi r^2 h$

$\pi r^2 h = V$

So, $r^2 = \frac{V}{\pi h}$

And $r = \sqrt{\frac{V}{\pi h}}$

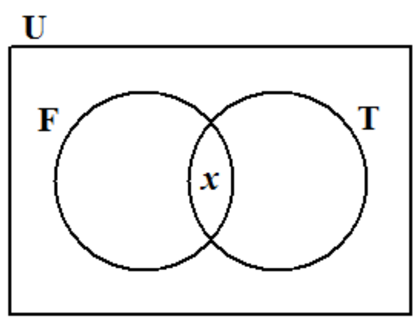
(e) Given that $x^2 + ax + b = (x+2)^2 - 3$, work out the values of a and b.

SOLUTION:
Data: $x^2 + ax + b = (x+2)^2 - 3$
Required to find: The value of a and of b.
Solution:

$$(x+2)^2 - 3 = (x+2)(x+2) - 3$$
$$= x^2 + 2x + 2x + 4 - 3$$
$$= x^2 + 4x + 1$$

Hence, $x^2 + ax + b = x^2 + 4x + 1$.
Equating the coeffcients of the term in x and then the constant term we obtain $a = 4$ and $b = 1$.

3. (a) The incomplete Venn diagram below shows the number of students in a class of 28 who play football and tennis.



U = {all students in the class}
F = {students who play football}
T = {students who play tennis}

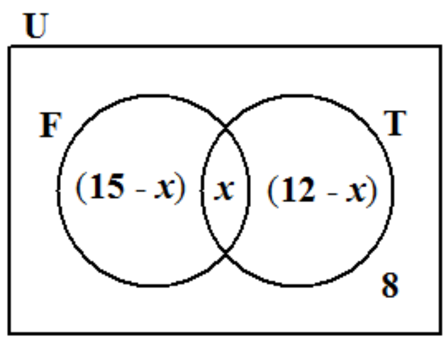
Additional information about the class is that
12 students play tennis

15 students play football
8 students play neither football nor tennis
x students play BOTH football and tennis.

(i) Complete the Venn diagram above to represent the information, showing the number of students in EACH subset.

SOLUTION:
Data: Incomplete Venn diagram showing the numbers of students who play football or tennis in class of 28.

Required to complete: The Venn diagram given
Solution:



(It is grammatically better to say that 8 students do NOT play either football or tennis)

(ii) Calculate the value of x.

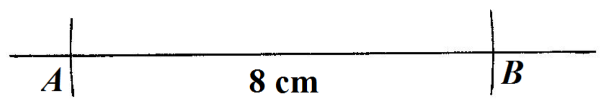
SOLUTION:
Required to calculate: x
Calculation:
The sum of the numbers of students in all the subsets of the Universal set must total 28, which is the number of students in the class.

$$\therefore 15 - x + x + 12 - x + 8 = 28$$
$$35 - x = 28$$
$$\therefore x = 35 - 28$$
$$= 7$$

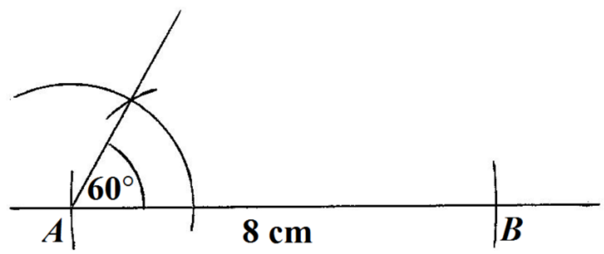
(b) Using **a ruler, a pencil and a pair of compasses**, construct the trapezium ABCD with $AB = 8\text{ cm}$, $B\hat{A}D = 60°$, $AD = 6\text{ cm}$ and AB parallel to CD.
(Credit will be given for clearly drawn construction lines.)

SOLUTION:
Required To Construct: The trapezium ABCD such that $AB = 8 \text{ cm}$, $B\hat{A}D = 60°$, $AD = 6 \text{ cm}$ and AB parallel to CD.

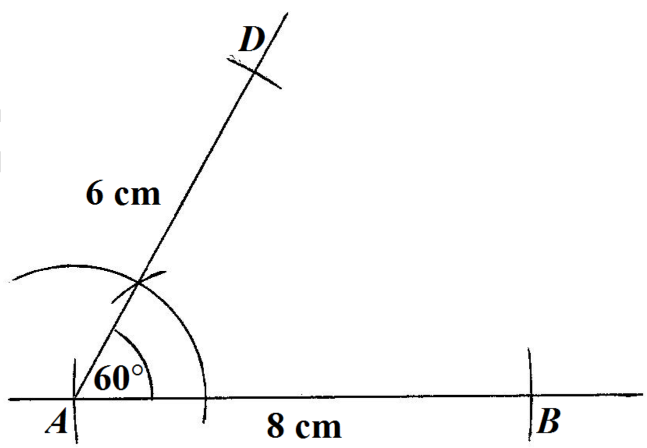
Construction:
We first draw a straight line, longer than 8 cm, and with the pair of compasses, cut off AB = 8 cm



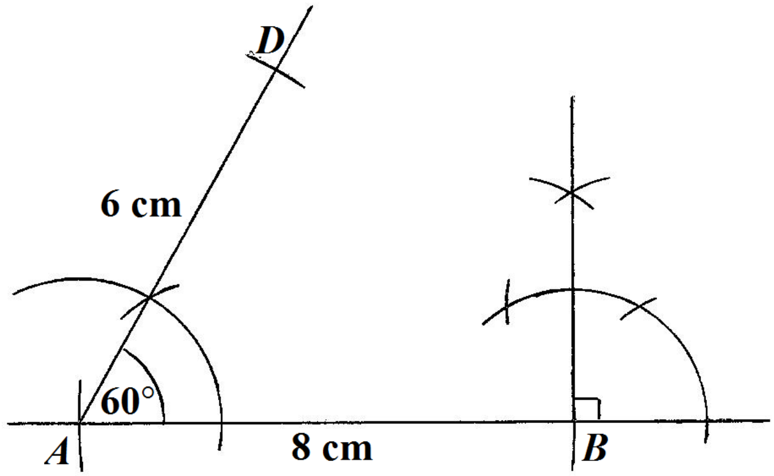
At A, we construct an angle of 60^0

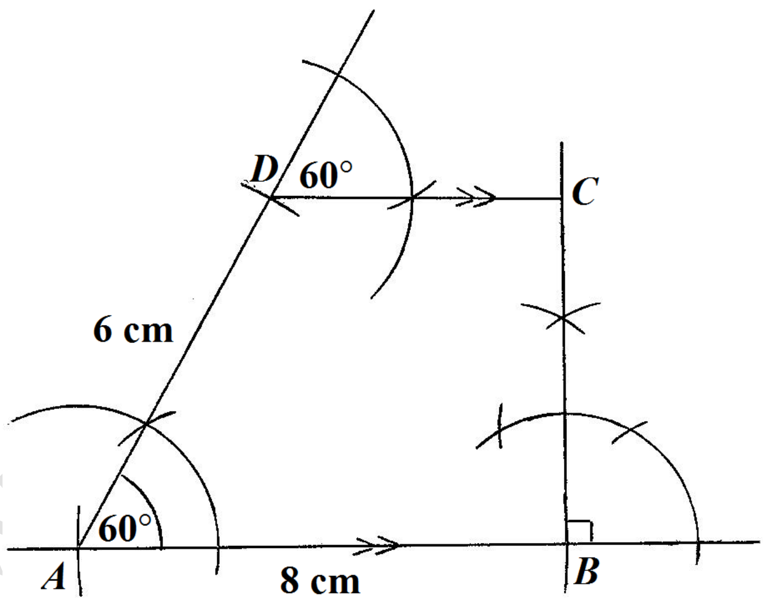


We extend the arm of the angle (if necessary) and cut off AD = 6 cm



At B we construct and angle of 90^0





At D, we construct a 60° angle so that it is corresponding to the 60° angle at A. The arm of the angle at D will then be parallel to AB and meets the perpendicular from B at C. This now completes the trapezium ABCD.

4. (a) Given that $f(x)=4x-7$ and $g(x)=\frac{3x+1}{2}$, determine the values of:

(i) $g(0)+g(5)$

SOLUTION:
Data: $f(x)=4x-7$ and $g(x)=\frac{3x+1}{2}$
Required to calculate: $g(0)+g(5)$
Calculation:

$$g(0)+g(5)=\frac{3(0)+1}{2}+\frac{3(5)+1}{2}$$
$$=\frac{1}{2}+\frac{16}{2}$$
$$=\frac{1}{2}+8$$
$$=8\frac{1}{2}$$

(ii) $fg(5)$

SOLUTION:
Required to calculate: $fg(5)$
Calculation:

$$g(5)=\frac{3(5)+1}{2}$$
$$=\frac{16}{2}$$
$$=8$$

$$\therefore fg(5)=f(8)$$
$$=4(8)-7$$
$$=32-7$$
$$=25$$

Alternative Method:

$$fg(x) = 4g(x) - 7$$
$$= 4\left(\frac{3x+1}{2}\right) - 7$$
$$= 2(3x+1) - 7$$
$$= 6x + 2 - 7$$
$$= 6x - 5$$

$$\therefore fg(5) = 6(5) - 5$$
$$= 30 - 5$$
$$= 25$$

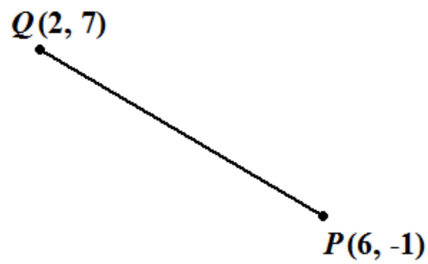
(iii) $f^{-1}(1)$

SOLUTION:
Required to calculate: $f^{-1}(1)$
Calculation:
Let $y = f(x)$
$$\therefore y = 4x - 7$$
$$4x - 7 = y$$
$$4x = y + 7$$
$$x = \frac{y+7}{4}$$
Replace y by x:
$$f^{-1}(x) = \frac{x+7}{4}$$
Hence,
$$f^{-1}(1) = \frac{1+7}{4}$$
$$= \frac{8}{4}$$
$$= 2$$

(b) $P(6, -1)$ and $Q(2, 7)$ are the end points of a line segment PQ. Determine:

(i) the gradient of PQ

SOLUTION:
Data: $P(6, -1)$ and $Q(2, 7)$
Required to calculate: The gradient of PQ
Calculation:

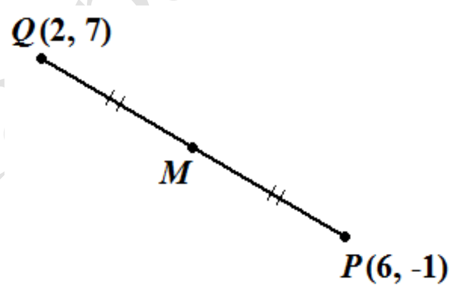


$$\text{Gradient of } PQ = \frac{7-(-1)}{2-6} = \frac{8}{-4} = -2$$

(ii) the coordinates of the midpoint of PQ.

SOLUTION:
Required to calculate: The coordinates of the midpoint of PQ

Calculation:

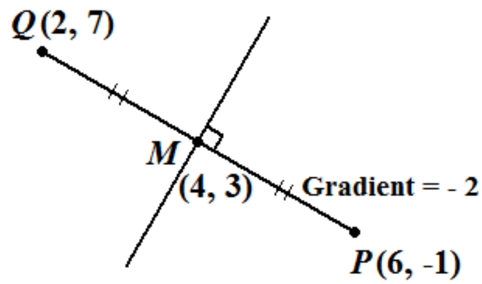


Let M be the midpoint of PQ.

$$\text{Coordinates of } M = \left(\frac{2+6}{2}, \frac{7+(-1)}{2}\right) = \left(\frac{8}{2}, \frac{6}{2}\right) = (4, 3)$$

(iii) the equation of the perpendicular bisector of *PQ*.

SOLUTION:
Required to find: The equation of the perpendicular bisector of *PQ*
Solution:

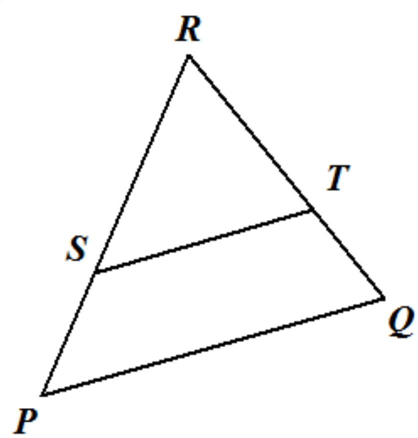


The gradient of any perpendicular line to $PQ = \frac{-1}{-2} = \frac{1}{2}$
(The products of the gradients of perpendicular lines $= -1$)

The equation of the perpendicular bisector of *PQ* is

$$\frac{y-3}{x-4} = \frac{1}{2}$$
$$2(y-3) = 1(x-4)$$
$$2y - 6 = x - 4$$
$$2y = x + 2 \text{ (or any other equivalent form)}$$

5. (a)



Triangles *PQR* and *STR* are similar triangles.

(i) Complete the following statement:

In the diagram above, the corresponding angles of ΔPQR and ΔSTR are ………………….. and the ……………………. of their corresponding sides are the same.

SOLUTION:
Data: Diagram showing similar triangles *PQR* and *STR* and an incomplete statement about these triangles.

Required to complete: The given statement
Solution:
Since the triangles *PQR* and *STR* are similar, then their corresponding angles will be equal. We can deduce the equal angles by simply observing the naming of the triangles, ΔPQR and ΔSTR (a diagram would not have been necessary for this conclusion if the figures are correctly named)
By observing the names of the two triangles, we deduce, $\hat{P} = \hat{S}$, $\hat{Q} = \hat{T}$ and $\hat{R} = \hat{R}$ (common angle).

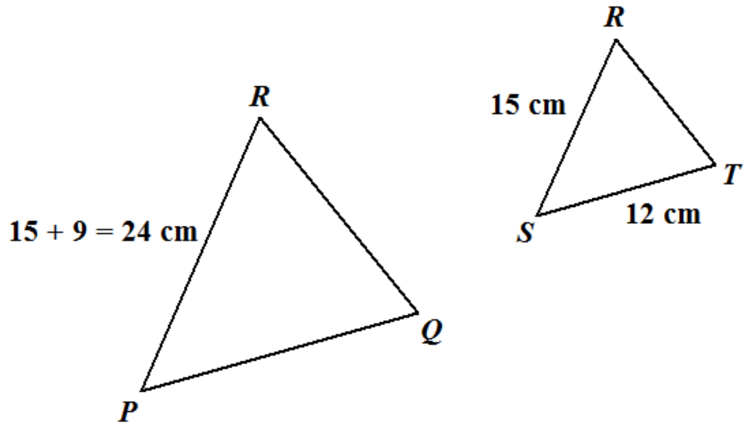
When any two figures are similar, the ratio of their corresponding sides are the equal.

The completed statement is:
In the diagram above, the corresponding angles of ΔPQR and ΔSTR are equal and the ratio of their corresponding sides are the same or equal.

In the diagram above, **not drawn to scale**, $RS = 15$ cm, $SP = 9$ cm and $ST = 12$ cm.

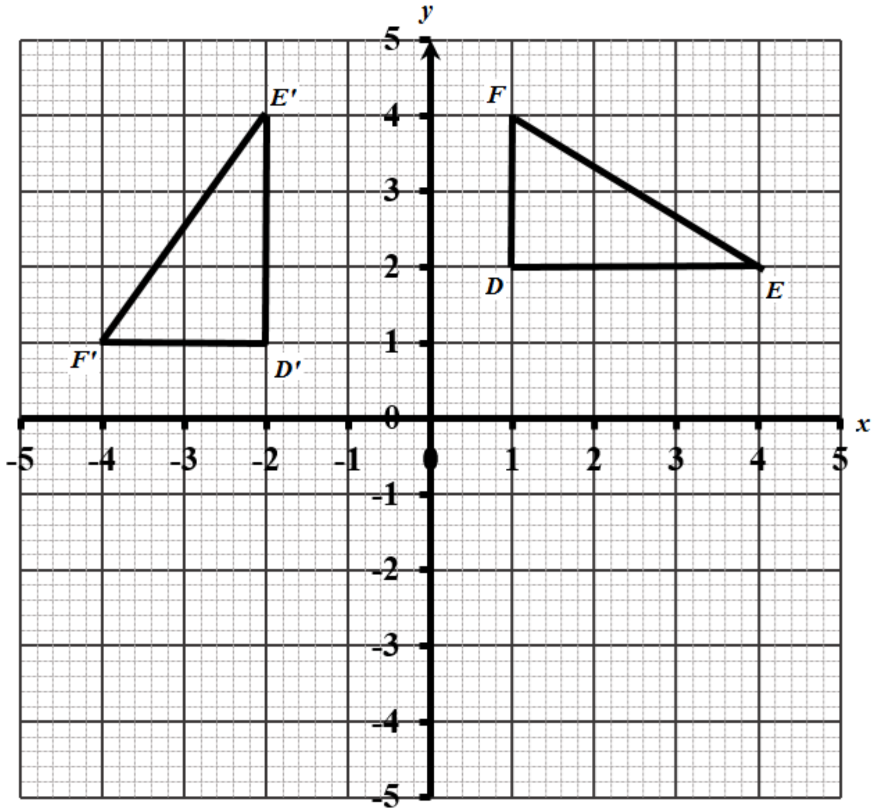
(ii) Determine the length of *PQ*.

SOLUTION:
Data: $RS = 15$ cm, $SP = 9$ cm and $ST = 12$ cm.
Required to calculate: The length of *PQ*
Calculation:
We draw the two triangles separately, for convenience.



$$\frac{RP}{RS} = \frac{PQ}{ST}$$
$$\frac{24}{15} = \frac{PQ}{12}$$
$$PQ = \frac{24 \times 12}{15}$$
$$= 19.2$$
$$\therefore PQ = 19.2 \text{ cm}$$

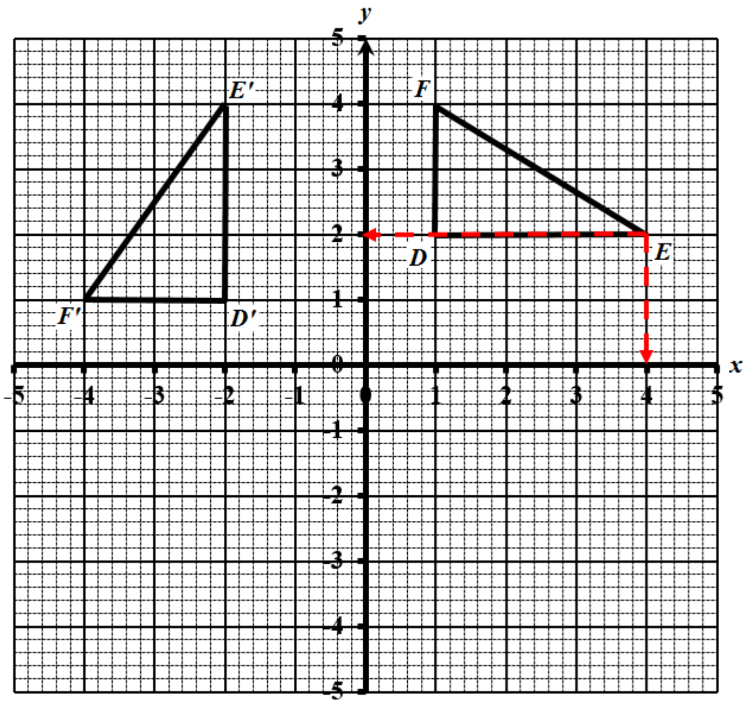
(b) The graph below shows triangle DEF and its image $D'E'F'$ after a transformation.



(i) State the coordinates of the point *E*.

SOLUTION:
Data: Diagram showing triangles *DEF* and $D'E'F'$. Triangle $D'E'F'$ is the image of triangle *DEF* after a transformation.

Required To State: The coordinates of the point *E*
Solution:

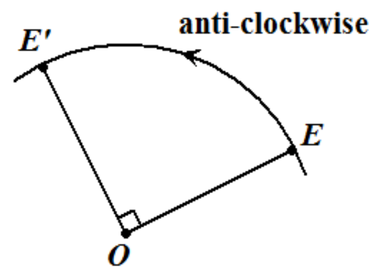


$\therefore$ The coordinates of *E* are $x = 4,\ y = 2$.

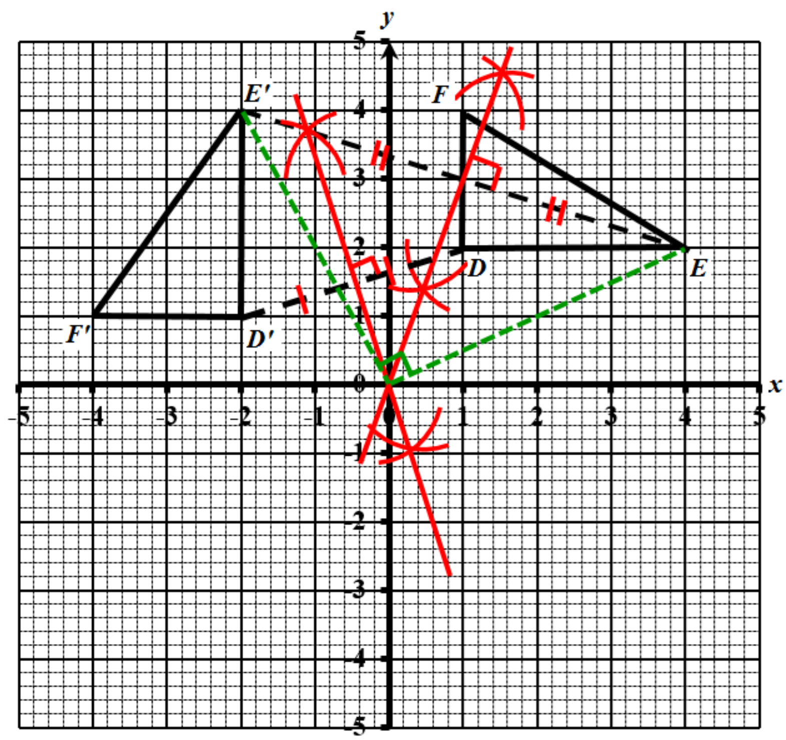
That is, $E = (4,\ 2)$.

(ii) Describe fully the transformation that maps triangle *DEF* to its image, $D'E'F'$.

SOLUTION:
Required to describe: The transformation that maps triangle *DEF* onto triangle $D'E'F'$ fully
Solution:
Triangles *DEF* and $D'E'F'$ are congruent
The image $D'E'F'$ is re-oriented with respect to the object *DEF*. Hence, the transformation is rotation.
We now obtain the center of rotation by the following procedure.
We join *D* to D' and construct the perpendicular bisector.
Next we join *E* to E' and construct the perpendicular bisector. (It is not necessary for this to be done with a third set of points since the three perpendicular bisectors are all concurrent.)
The perpendicular bisectors are produced (if necessary) to meet at the center of rotation, which is *O*, $(0,\ 0)$.
$D\hat{O}D' = E\hat{O}E' = F\hat{O}F' = 90°$
The movement from *E* to E' or *F* to F' or *D* to D' is anti-clockwise.



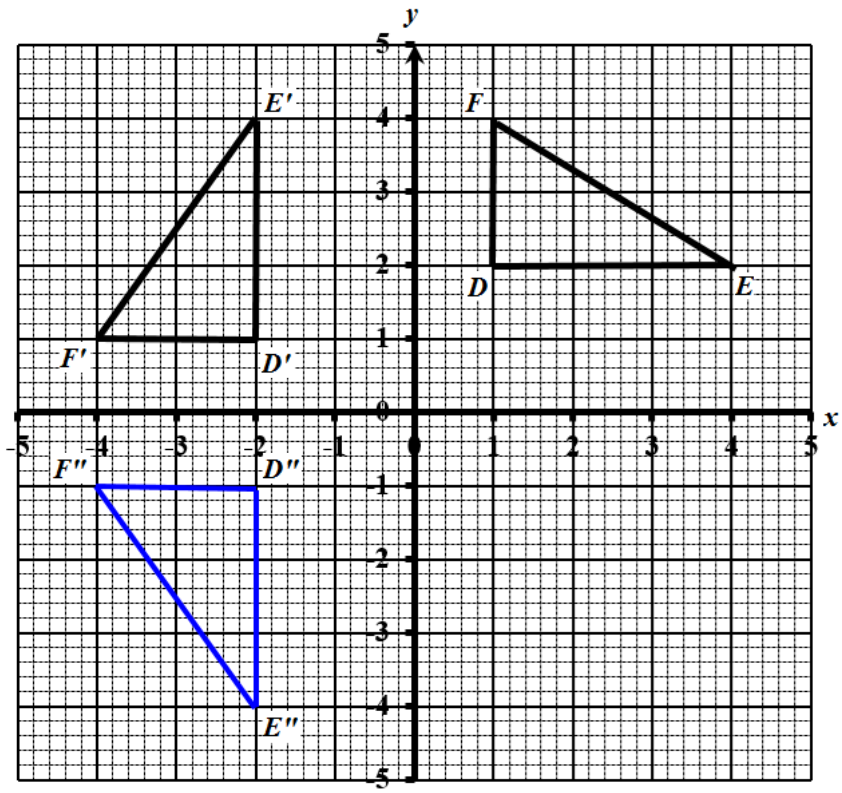
Hence, the transformation is a 90° anti-clockwise rotation about O.



(iii) On the grid above, draw triangle $D''E''F''$, the reflection of triangle $D'E'F'$ in the x – axis.

SOLUTION:
Data: Triangle $D''E''F''$ is the reflection of triangle $D'E'F'$ in the x – axis.
Required to draw: Triangle $D''E''F''$.

Solution: To obtain the image points, we measure the same perpendicular distance as the object points are from the reflection plane, but on the opposite side of the reflection plane.



$$\Delta D'E'F' \xrightarrow[x\text{-axis}]{\text{Reflection in the}} \Delta D''E''F''$$

$D'' = (-2, -1)$

$E'' = (-2, -4)$

$F'' = (-4, -1)$

6. (a) The scale on a map is $1:25\,000$.

(i) Anderlin and Jersey are 31.8 cm apart on the map.

Determine, in km, the actual distance between Anderlin and Jersey.

SOLUTION:
Data: The distance between Anderlin and Jersey is 31.8 cm on a map with a scale of $1:25\,000$.
Required to calculate: The actual distance between Anderlin and Jersey, in km.
Calculation:
Distance on the map $= 31.8$ cm
Scale $= 1:25\,000$
$\therefore$ Actual distance $= 31.8 \times 25\,000$ cm

$1 \text{ km} = 1000 \times 100 \text{ cm}$

$\therefore$ Actual distance, in km $= \dfrac{31.8 \times 25000}{100000}$ km

$= 7.95$ km

(ii) The actual distance between Clifton and James Town is 2.75 km.

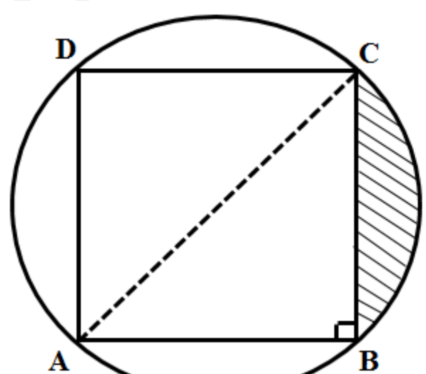
How many units apart are they on the map?

SOLUTION:
Data: The actual distance between Clifton and James Town is 2.75 km.
Required to calculate: The distance between Clifton and James Town on the map
Calculation:
$2.75 \text{ km} = 2.75 \times 100000 \text{ cm}$
Scale $= 1:25000$

$\therefore$ Distance on the map $= \dfrac{2.75 \times 100000}{25000}$ cm

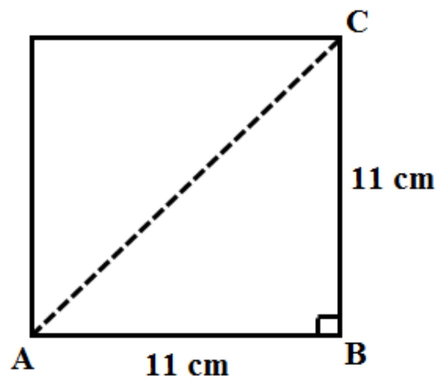
$= 11$ cm

(b) The diagram below shows a square ABCD drawn inside a circle. The vertices of the square lie on the circumference of the circle. The length of a side of the square is 11 cm.



(i) Show that the diameter of the circle is $11\sqrt{2}$ cm.

SOLUTION:
Data: Diagram showing a square with vertices ABCD lying inside a circle, such that the points A, B, C and D lie on the circumference of the circle. The length of a side of the square is 11 cm.
Required to prove: The diameter of the circle is $11\sqrt{2}$ cm
Proof: Let us consider the triangle ABC



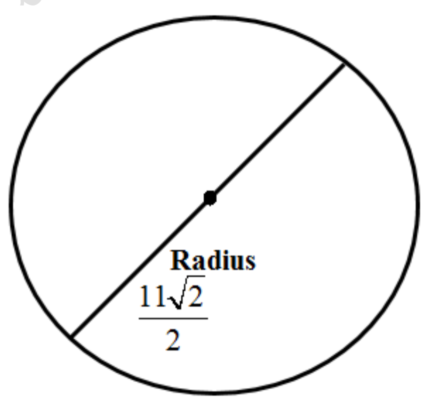
$$AC^2 = (11)^2 + (11)^2 \quad \text{(Pythagoras' Theorem)}$$
$$= 121 + 121$$
$$AC = \sqrt{242}$$
$$= \sqrt{11 \times 11 \times 2}$$
$$= \sqrt{11 \times 11} \times \sqrt{2}$$
$$= 11\sqrt{2}$$

Q.E.D.

Calculate:

(ii) the area of the circle

SOLUTION:
Required to calculate: The area of the circle.
Calculation:



Diameter $= 11\sqrt{2}$ cm

$\therefore$ Radius $= \dfrac{11\sqrt{2}}{2}$ cm

$$\text{Area} = \pi r^2$$
$$= \pi\left(\frac{11\sqrt{2}}{2}\right)^2$$
$$= \pi \times \frac{121 \times 2}{4}$$
$$= \pi \times 60.5 \text{ cm}^2$$
$$= 190.09\underline{\underline{1}} \text{ cm}^2$$
$$= 190.09 \text{ cm}^2 \text{ (correct to 2 decimal places)}$$

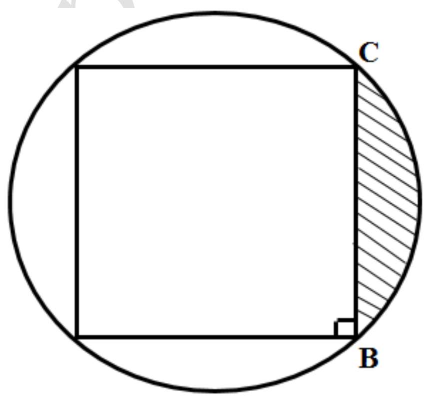
(iii) the area of the square

SOLUTION:
Required to calculate: The area of the square.
Calculation:

$$\text{Area of square} = 11 \times 11 \text{ cm}^2$$
$$= 121 \text{ cm}^2$$

(iv) the area of the shaded section.

SOLUTION:
Required to calculate: The area of the shaded section.
Calculation:



There are 4 segments shown on the diagram (3 of these are un-shaded and 1 shown shaded)

Area of these four equal segments = Area of circle − Area of square ABCD

$$= 190.091 - 121 \text{ cm}^2$$
$$= 69.091 \text{ cm}^2$$

Hence the area of the shaded region

$= \frac{69.091}{4}$ cm^2

$= 17.27\underset{=}{2}$ cm^2

$= 17.27$ cm^2 (correct to 2 decimal places)

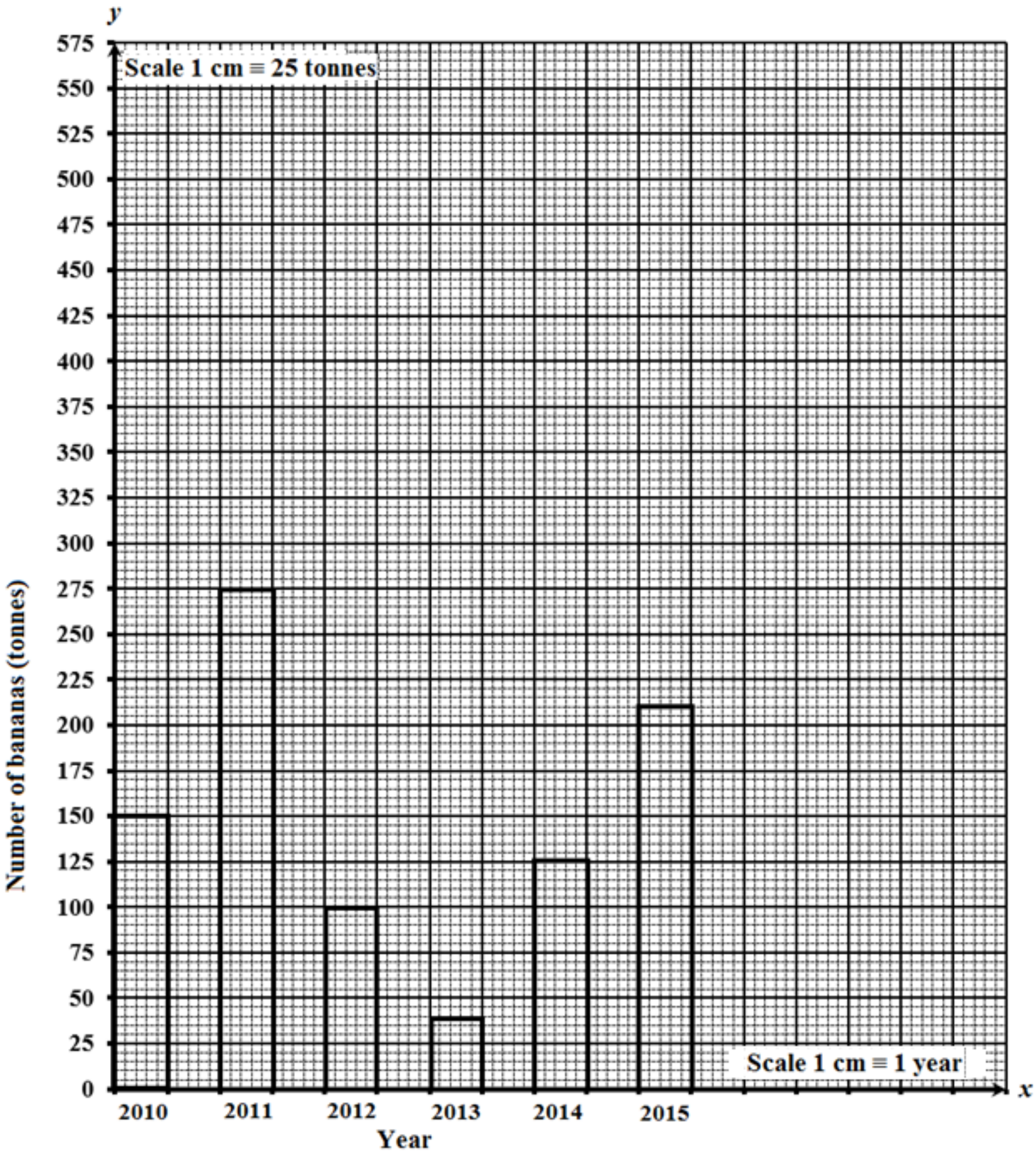
7. The table below shows the number of bananas, to the nearest tonne, produced annually on a farm over a period of 6 years.

Year	2010	2011	2012	2013	2014	2015
Production (tonnes)	150	275	100	40	125	210

(a) **On the graph paper provided**, draw a bar chart to represent the data given in the table above using a scale of 1 cm to represent 1 year on the x – axis and 1 cm to represent 25 tonnes on the y – axis.

SOLUTION:
Data: Table showing the number of bananas, to the nearest tonne, produced annually on a farm from 2010 to 2015.

Required to draw: A bar chart to illustrate the data on the table using a scale of 1 cm to represent 1 year on the x – axis and 1 cm to represent 25 tonnes on the y – axis.



(b) Determine the range of the number of bananas produced between 2010 and 2015.

SOLUTION:
Required to determine: The range of the number of bananas produced between 2010 and 2015
Solution:
Range $=$ Highest value $-$ Lowest value (in the given distribution)

$= (275 - 40)$ tonnes

$= 235$ tonnes

(c) **(i)** During which year was there the greatest production of bananas?

SOLUTION:
Required to state: The year with the greatest production of bananas
Solution:
The year with the greatest production of bananas is 2011 (shown by the highest bar on the bar chart).

(ii) How is this information shown on the bar chart?

SOLUTION:
Required to explain: The way the greatest production of bananas is shown on the bar chart
Solution:
This is shown on the bar chart with the highest bar.

(d) **(i)** Between which two consecutive years was there the greatest change in the production of bananas?

SOLUTION:
Required To State: The two consecutive years between which showed the greatest change in the production of bananas
Solution:

Year	**Change in Production (tonnes)**
2010 – 2011	$275 - 150 = +125$
2011 – 2012	$100 - 275 = -175$
2012 – 2013	$40 - 100 = -60$
2013 – 2014	$125 - 40 = +85$
2014 – 2015	$210 - 125 = +85$

In the above table, a positive value of change in production indicates an increase and a negative value of change in production indicates a decrease from one year to the next year.

The greatest change in production of bananas occurred between the years 2011 and 2012.

(ii) How is this information shown on the bar chart?

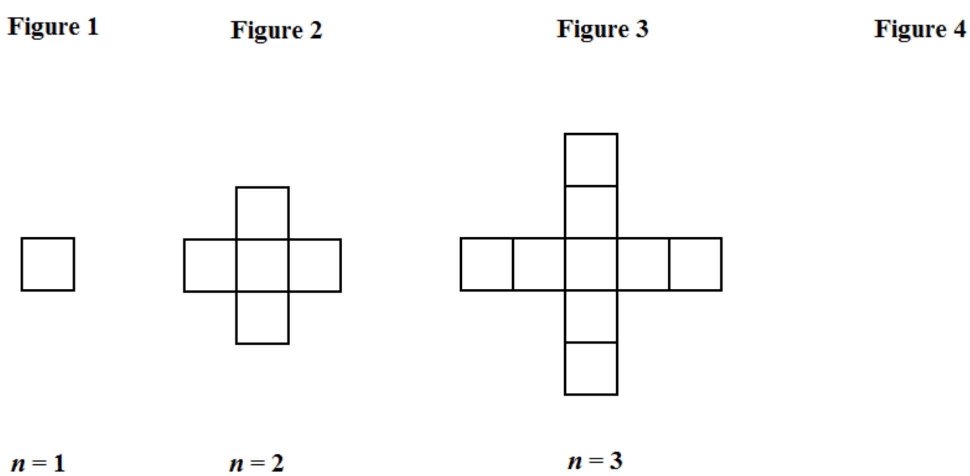
SOLUTION:

Required to explain: The way in which the years between which the greatest change in the production of bananas occurred.
Solution:
This is shown with the difference between the respective heights of any two consecutive bars being the greatest.

(e) Give ONE reason why the bar chart is unsuitable for predicting the number of bananas produced in 2016.

SOLUTION:
Required to state: A reason why the bar chart is not suitable for predicting the number of bananas produced in 2016.
Solution:
The bars do not show a definite pattern with respect to increasing or decreasing over any reasonable period of years. Hence, a prediction for 2016 cannot be made.

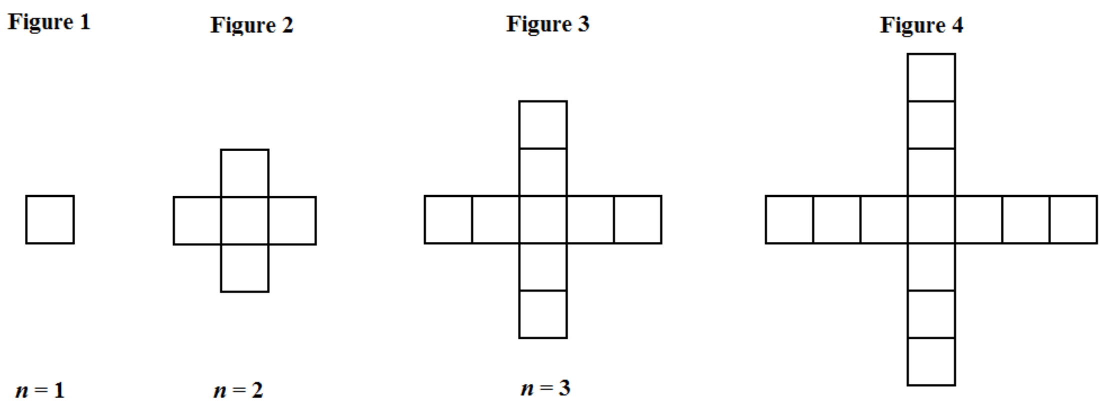
8. A sequence of figures is made up of unit squares with unit sides. The first three figures in the sequence are shown below.



(a) Draw Figure 4 of the sequence in the space provided above.

SOLUTION:
Data: Diagrams showing a sequence of figures made up of unit squares- see Figures 1-4 above.

Required To Draw: The fourth figure in the sequence.
Solution:



(b) Study the pattern of numbers in each row of the table below. Each row relates to one of the figures in the sequence. Some rows have not been included in the table. Complete the rows numbered (i), (ii), (iii) and (iv).

	Figure	Number of Unit Squares	Perimeter of Figure
	1	1	4
	2	5	12
	3	9	20
(i)	4		
(ii)		45	
(iii)	30		
(iv)	n		

SOLUTION:
Data: Incomplete table showing the relationship among the figure number, the number of unit squares and the perimeter of the figures in the given sequence.

Required To Complete: The table given.
Solution:
We observe the figure, n, the number of unit squares which we name (S) and the perimeter, which we name P.

n	S	P
1	1	4
2	5	12
3	9	20

Let us look at the value of n and the corresponding value of S

Figure n	No. of Squares, S
1	1
2	5
3	9

We see that for every increase in n by 1, the number of squares, S increases by 4.
Hence, $S = 4n + a$ where a is an unknown number.
From the values given, we know that when $n = 1, S = 1$
Substituting these values in $S = 4n + a$ we get
$1 = 4(1) + a$, hence $a = -3$
We test again to verify the value of a
When $n = 2, S = 5$
$5 = 4(2) + a$, hence $a = -3$
Therefore, the formula is $S = 4n - 3$.

Let us look at the value of n and the corresponding value of P

Figure n	Perimeter, P
1	4
2	12
3	20

We see that for every increase of n by 1, the perimeter increases by 8.
Hence $P = 8n + b$ where b is an unknown number
From the values given, we know that when $n = 1, P = 4$
Substituting these values in $P = 8n + b$ we get
$4 = 8(1) + p$, hence $a = -4$
We test again to verify the value of b
When $n = 2, P = 12$
$12 = 8(2) + b$, hence $b = -4$
Therefore, the formula is $P = 8n - 4$.

We use the formulae for S and p to complete the table as follows:

Figure	Number of Unit Squares	Perimeter of Figure
1	1	4
2	5	12
3	9	20
n	$4n-3$	$8n-4$
4	$4(4)-3=13$	$8(4)-4=28$
$4n-3=45$ $n=12$	45	$8(12)-4=92$
30	$4(30)-3=117$	$8(30)-4=236$

The completed table looks like:

	Figure	Number of Unit Squares	Perimeter of Figure
	1	1	4
	2	5	12
	3	9	20
(i)	4	13	28
(ii)	12	45	92
(iii)	30	117	236
(iv)	n	$4n-3$	$8n-4$

SECTION II

ALGEBRA AND RELATIONS, FUNCTIONS AND GRAPHS

9. (a) The table below shows pairs of values for x and y, where y is inversely proportional to x.

x	3	4	a	20
y	2	1.5	1.2	b

(i) Express y in terms of x and a constant k.

SOLUTION:
Data: Table showing corresponding values of two variables x and y, where y is inversely proportional to x.

x	3	4	a	20
y	2	1.5	1.2	b

Required to express: y in terms of x and k (a constant).

Solution:

y is inversely proportional to x.

$$y \propto \frac{1}{x}$$

$$y = k \times \frac{1}{x}$$ (where k is the constant of proportionality)

$$\therefore y = \frac{k}{x}$$

(ii) Calculate the value of the constant k.

SOLUTION:
Required to calculate: The value of k.
Calculation:
From the table $y = 2$ when $x = 3$.

$$\therefore 2 = \frac{k}{3}$$

$$k = 2(3)$$

$$k = 6$$

Since $k=6$, then

$y=\frac{6}{x}$

Testing for when $y=1.5$ and $x=4$:

$1.5=\frac{6}{4}$ (True)

(iii) Determine the values of *a* and *b*.

SOLUTION:
Required to calculate: The value of *a* and of *b*.
Calculation:
When $y=1.2$, we get:

$1.2=\frac{6}{x}$

$\therefore x=\frac{6}{1.2}$

$=5$

Hence, $a=5$.

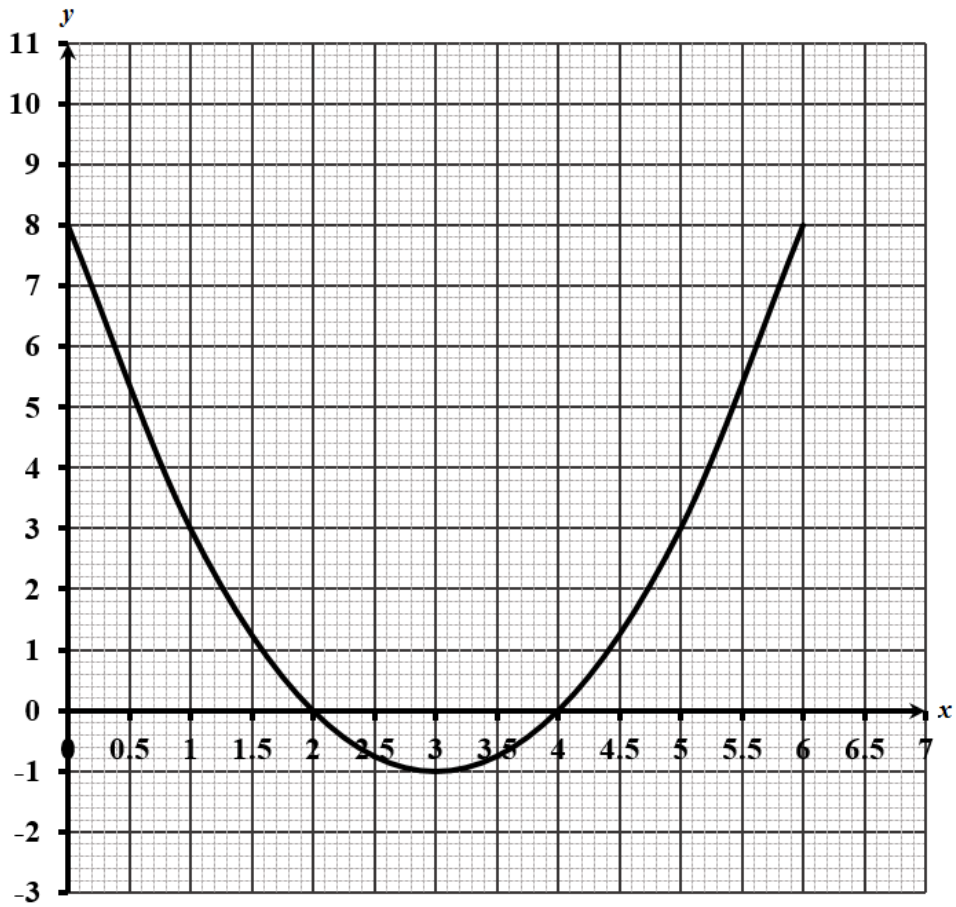
When $x=20$

$y=\frac{6}{20}$

$=0.3$

Hence, $b=0.3$.

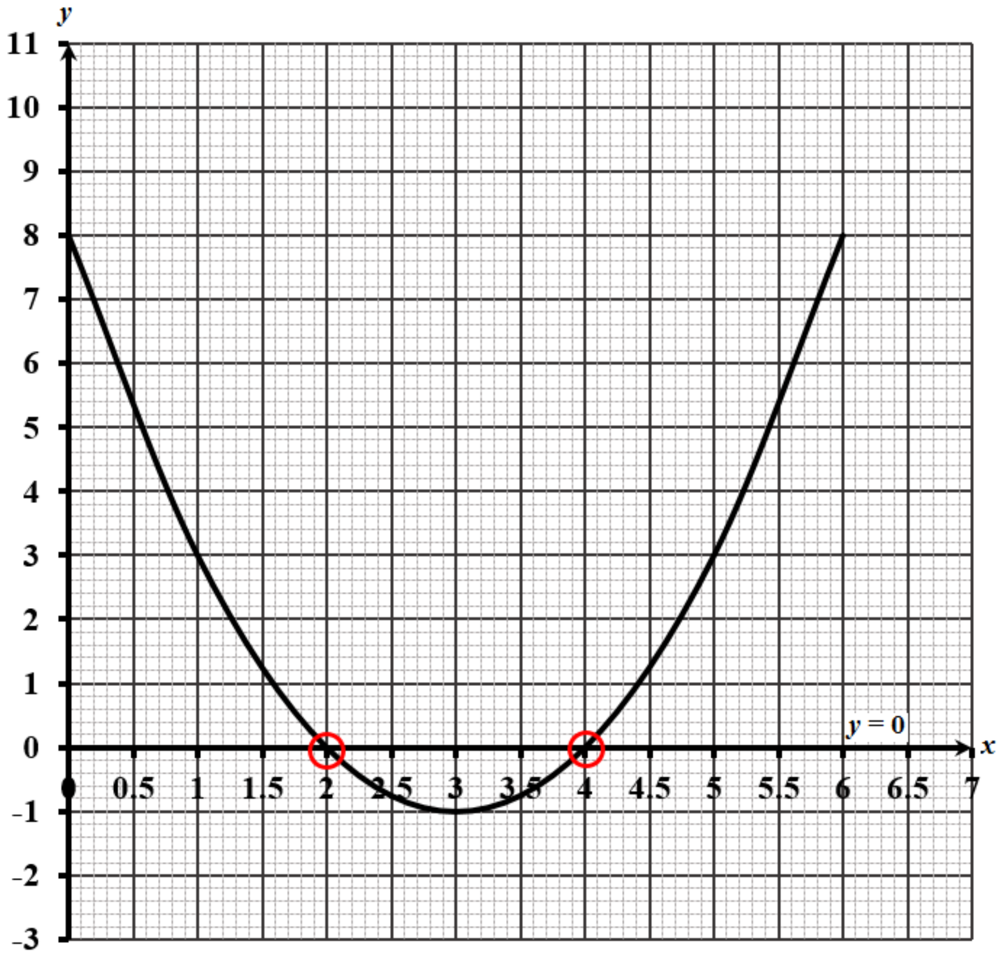
(b) The diagram below shows the graph of the function $f(x)=x^2-6x+8$ for values of *x* from 0 to 6.



(i) Use the graph to solve the equation $x^2 - 6x + 8 = 0$.

SOLUTION:
Data: Diagram of the graph of the function $f(x) = x^2 - 6x + 8$ for $0 \le x \le 6$.

Required to solve: $x^2 - 6x + 8 = 0$ using the graph.
Solution: The curve cuts the x-axis($y = 0$) at 2 and at 4

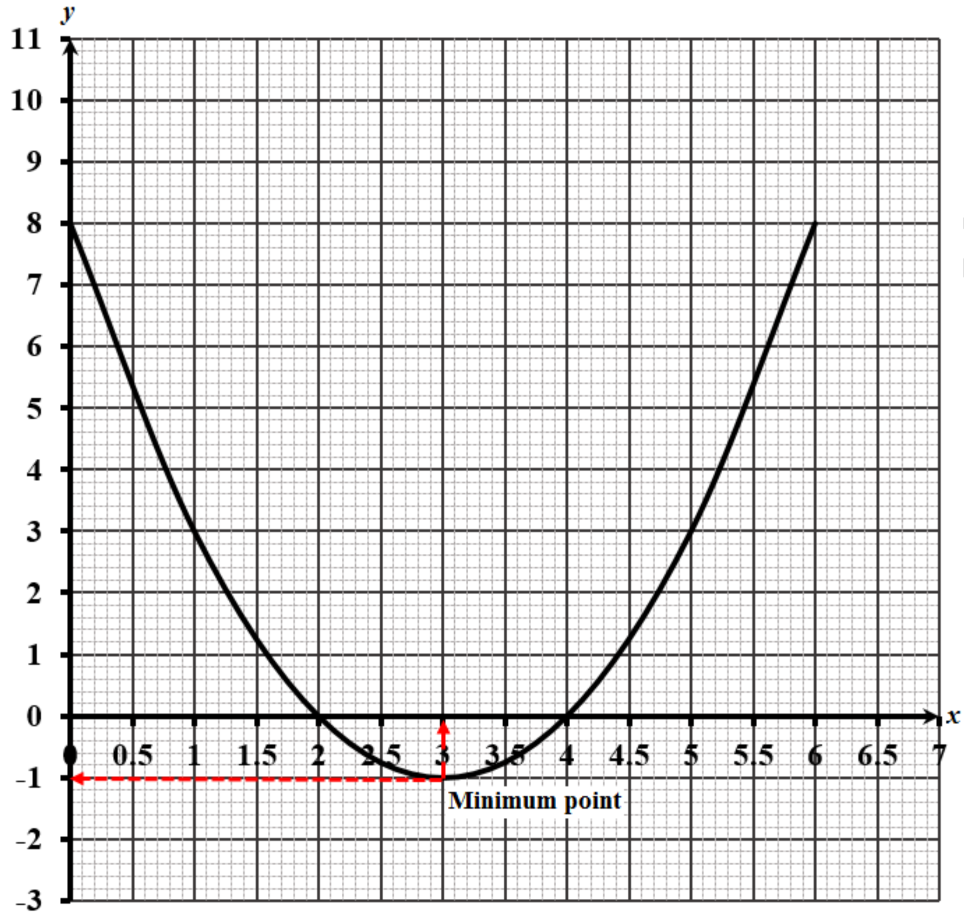


Hence, $x = 2$ and $x = 4$ are the solutions by using the graph.

(ii) Write down the coordinates of the minimum point in the form $(x,\ y)$.

SOLUTION:
Required to write: The coordinates of the minimum point of the graph in the form $(x,\ y)$.

Solution:



The minimum point occurs at $x = 3$ and $y = -1$ and is therefore $(3, -1)$ and is of the form (x, y), where $x = 3$ and $y = -1$.

(iii) Write $x^2 - 6x + 8$ in the form $a(x+h)^2 + k$ where *a*, *h* and *k* are constants.

SOLUTION:
Required to write: $x^2 - 6x + 8$ in the form $a(x+h)^2 + k$ where *a*, *h* and *k* are constants.
Solution:
$\frac{1}{2}$ the coefficient of *x* is $\frac{1}{2}(-6) = -3$

So, $\underbrace{x^2 - 6x}_{(x-3)^2+?} + 8$

$(x-3)(x-3)=x^2-6x+9$

$$\frac{-1=?}{8}$$

Hence, $x^2-6x+8=(x-3)^2-1$ is of the form $a(x+h)^2+k$, where $a=1$, $h=-3$ and $k=-1$.

Alternative Method:

$$\begin{aligned} a(x+h)^2+k &= a(x+h)(x+h)+k \\ &= a\left(x^2+2hx+h^2\right)+k \\ &= ax^2+2ahx+ah^2+k \\ x^2-6x+8 &= ax^2+2ahx+ah^2+k \end{aligned}$$

Equating coefficients:
For x^2:
$a=1$

For x:
$2(1)h=-6$
$\therefore h=-3$

For the constant:
$1(-3)^2+k=8$
$k=-1$

So, $x^2-6x+8=(x-3)^2-1$ is of the form $a(x+h)^2+k$, where $a=1$, $h=-3$ and $k=-1$.

(iv) On the same axes, draw the graph of the straight line $g(x)=x-2$.

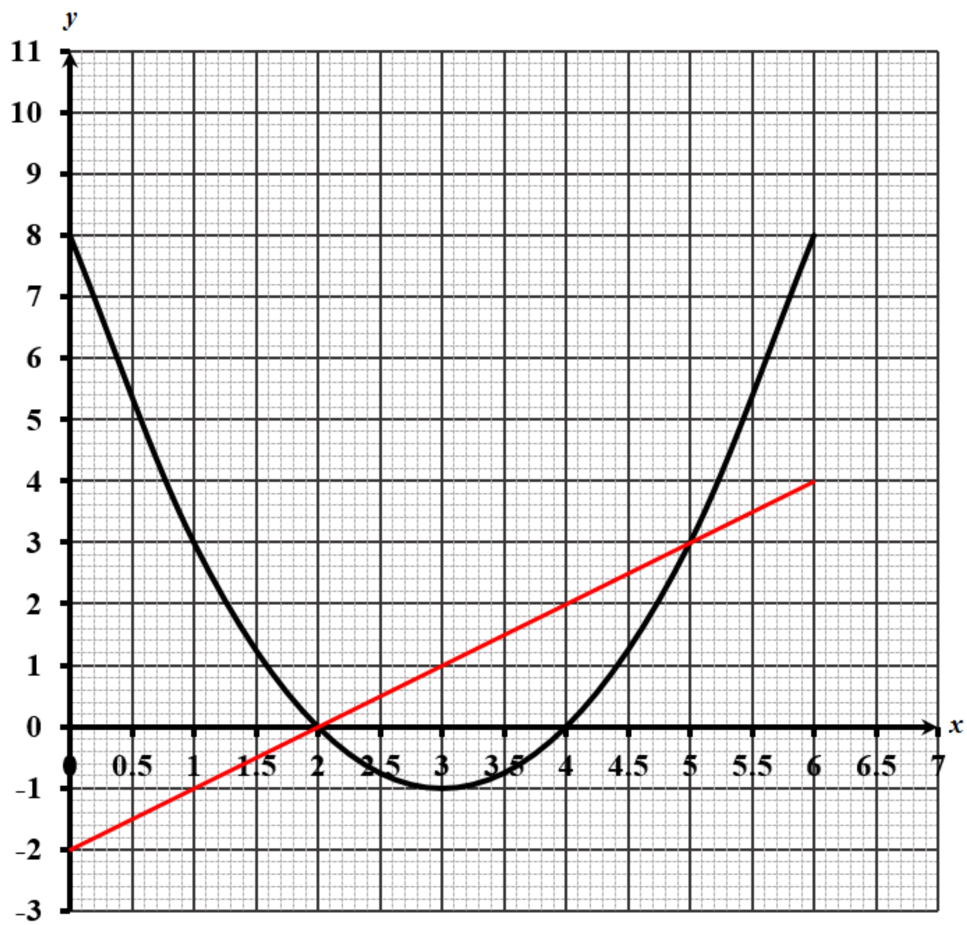
SOLUTION:
Required to draw: The graph of the straight line $g(x)=x-2$ on the same axes
Solution:
To draw a straight line, we need the coordinates of only two points
$g(x)=x-2$

When $x=0$
$g(0)=0-2$
$=-2$

When $x=6$
$g(6)=6-2$
$=4$

x	y
0	–2
6	4

We plot the points $(0, -2)$ and $(6, 4)$, shown in red



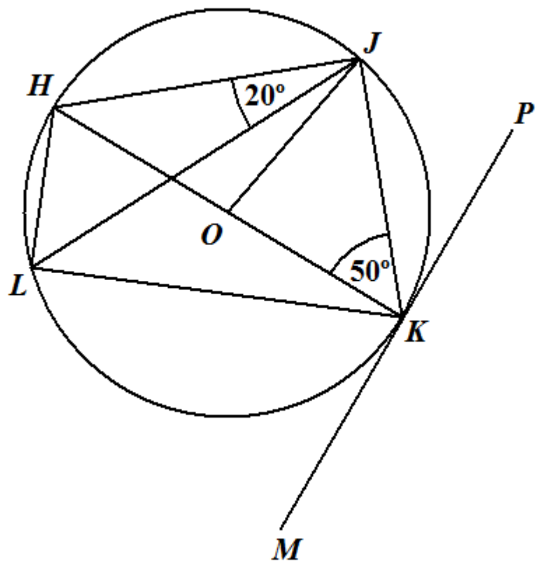
(v) Hence, solve the equation $x^2 - 6x + 8 = x - 2$.

SOLUTION:
Required to solve: $x^2 - 6x + 8 = x - 2$
Solution:

The graphs of $y = x^2 - 6x + 8$ and $y = x - 2$ meet at $(2, 0)$ and $(5, 3)$.
Hence, the solutions of $x^2 - 6x + 8 = x - 2$ are $x = 2$ and $x = 5$.

MEASUREMENT, GEOMETRY AND TRIGONOMETRY

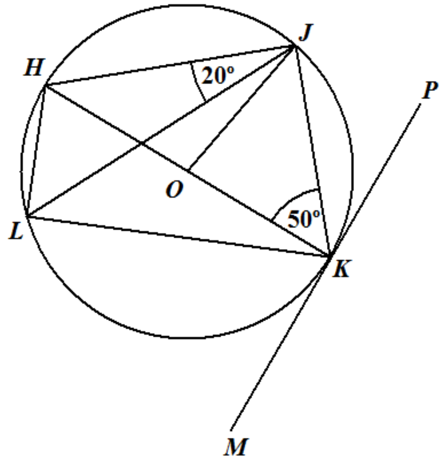
10. (a) The diagram below, **not drawn to scale**, shows a circle with center *O*. The vertices *H*, *K* and *L* of a quadrilateral lie on the circumference of the circle and *PKM* is a tangent to the circle at *K*. The measure of angle $H\hat{J}L = 20°$ and $J\hat{K}H = 50°$.



Calculate, giving reasons for each step of your answer, the measure of:

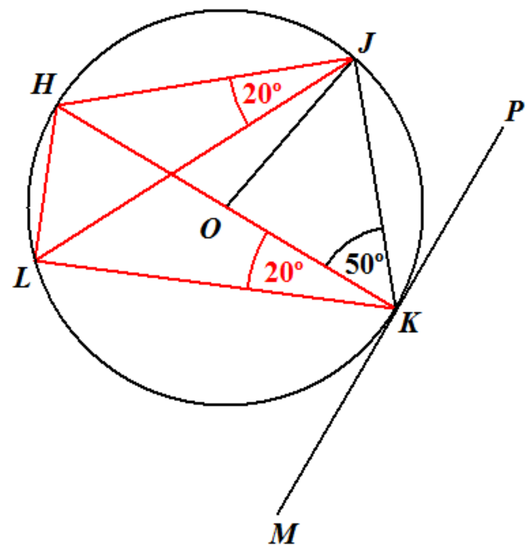
(i) $H\hat{K}L$

SOLUTION:
Data: Diagram showing a circle with center *O*. The vertices *H*, *K* and *L* of a quadrilateral lie on the circumference of the circle and *PKM* is a tangent to the circle at *K*. The measure of angle $H\hat{J}L = 20°$ and $J\hat{K}H = 50°$.



Required to calculate: The measure of $H\hat{K}L$.

Calculation:

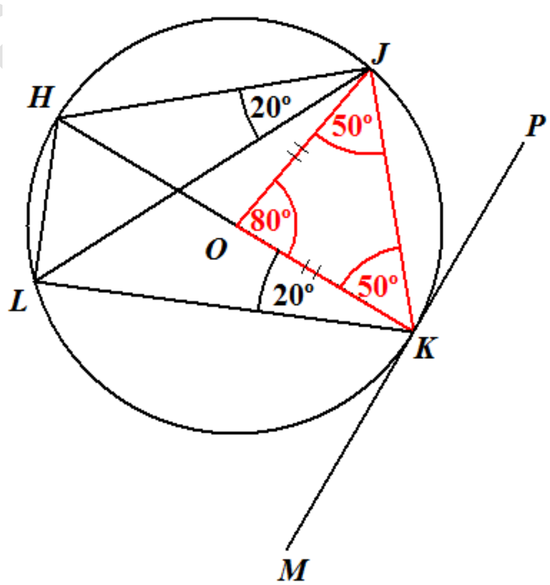


$H\hat{K}L = 20°$
(Angles subtended by a chord, *HL*, at the circumference of the circle and standing on the same arc are equal.)

(ii) $J\hat{O}K$

SOLUTION:
Required To Calculate: The measure of $J\hat{O}K$.

Calculation:



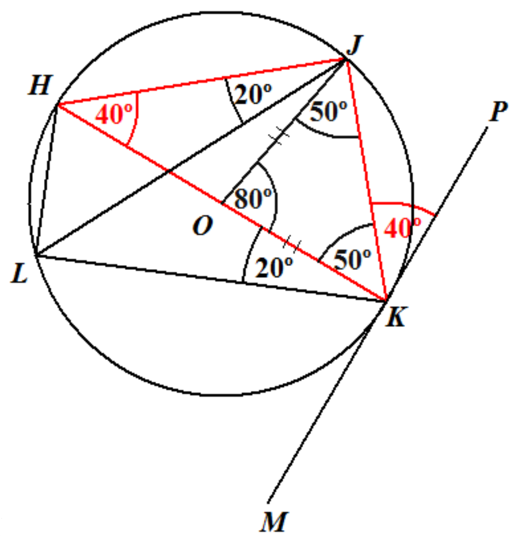
$OJ = OK$ (radii) and which makes OJK isosceles.

Hence, $O\hat{J}K = 50°$
(The base angles in an isosceles triangle are equal)

$J\hat{O}K = 180° - (50° + 50°)$
$= 80°$
(Sum of the interior angles in a triangle is equal to 180°)

(iii) $J\hat{H}K$

SOLUTION:
Required to calculate: The measure of $J\hat{H}K$.
Calculation:



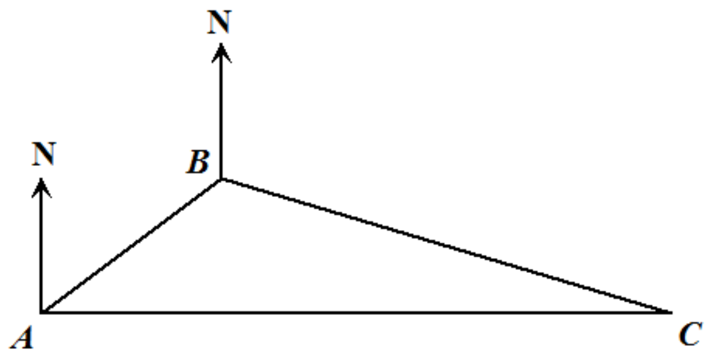
$O\hat{K}P = 90°$
(The angle made by a tangent,(*MKP*,) to a circle and a chord (*JK*) at the point of contact is 90°)

$J\hat{K}P = 90° - 50°$
$= 40°$
$J\hat{H}K = 40^0$
(angle between tangent, *MKP* and chord *JK* is equal to the angle subtended by the chord *JK* in the alternate segment).

OR

$J\hat{H}K$ can be calculated by considering the triangle *HJK*
$H\hat{J}K = 90^0$ (angle in a semicircle is a right angle)
$H\hat{K}J = 50^0$
$J\hat{H}K = 180^0 - (90^0 + 50^0) = 40^0$
(sum of the angles in a triangle is 180^0)

(b) A ship travels from Akron (A) on a bearing or 030° to Bellville (B), 90 km away. It then travels to Comptin (C) which is 310 km east of Akron (A), as shown in the diagram below.

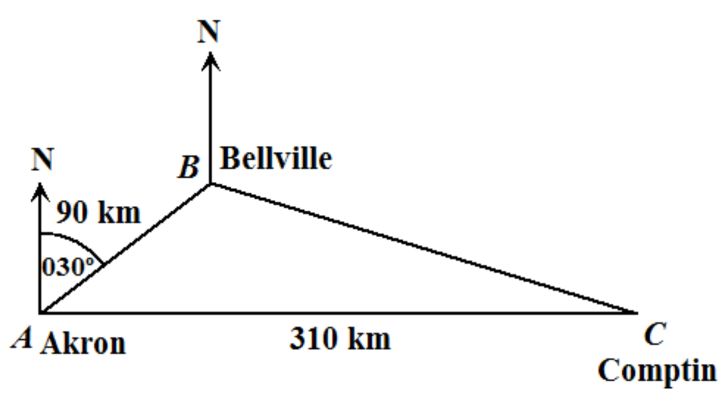


(i) Indicate on the diagram the bearing 030° and the distances 90 km and 310 km.

SOLUTION:
Data: Diagram showing the movement of a ship from Akron (A) on a bearing or 030° to Bellville (B), 90 km away. It then travels to Comptin (C) which is 310 km east of Akron (A), as shown in the diagram below.

Required to show: The bearing 030° and the distances 90 km and 310 km on the diagram.

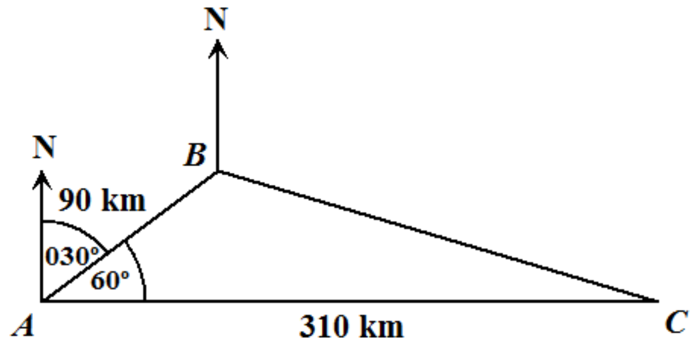
Solution:



(ii) Calculate, to the nearest km, the distance between Bellville (B) and Comptin (C).

SOLUTION:

Required to calculate: The distance between Bellville (B) and Comptin (C).
Calculation:

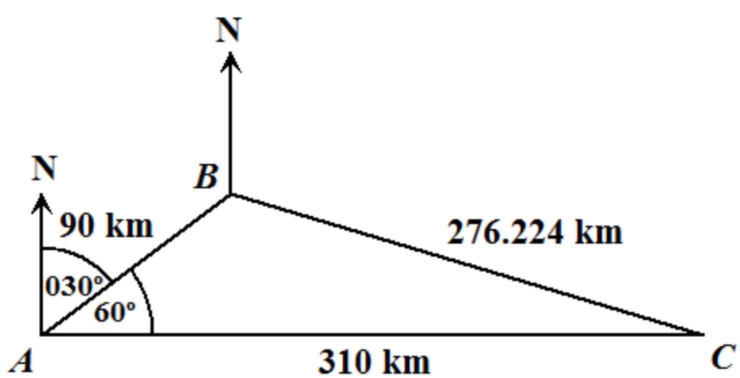


By the cosine rule:

$$BC^2 = (90)^2 + (310)^2 - 2(90)(310)\cos 60°$$
$$= 8100 + 96100 - 55800(0.5)$$
$$= 104200 - 27900$$
$$= 76300$$
$$\therefore BC = \sqrt{76300}$$
$$= 276.\underline{\underline{2}}24 \text{ km}$$
$$= 276 \text{ km (correct to the nearest km)}$$

(iii) Calculate, to the nearest degree, the measure of $A\hat{B}C$.

SOLUTION:
Required to calculate: The measure of $A\hat{B}C$.
Calculation:



By the sine rule:

$$\frac{310}{\sin A\hat{B}C} = \frac{276.224}{\sin 60^0}$$

$$\sin A\hat{B}C = \frac{310 \times \sin 60^0}{276.224}$$

$$= 0.9719$$

$$A\hat{B}C = sin^{-1}(0.9719)$$
$$= 76.4^0$$

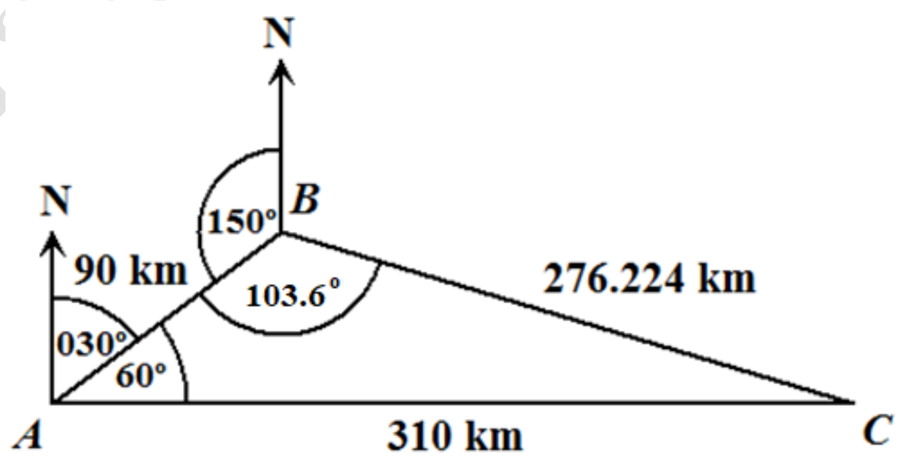
However, according to the diagram, $A\hat{B}C$ is obtuse.
Recall
$\sin(180^0 - \theta) = \sin\theta$

$\therefore A\hat{B}C = 180^0 - 76.4^0 = 103.6^0$

$= 104^0$(correct to the nearest degree)

(iv) Determine the bearing of Comptin (C) from Bellville (B).

SOLUTION:
Required to determine: The bearing of Comptin (C) from Bellville (B).
Solution:



The bearing of C from B is shown as $N\hat{B}C$.
Bearing of C from $B = 360^0 - (150^0 + 103.6^0)$
$= 106.4^0$
$= 106^0$ (correct to the nearest degree)

VECTORS AND MATRICES

11. (a) The matrix $T=\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$ maps the point $P\,(2,\,3)$ onto the point $Q\,(2,\,-3)$.

(i) Determine the values of *c* and *d*.

SOLUTION:
Data: $T=\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$ maps the point $P\,(2,\,3)$ onto the point $Q\,(2,\,-3)$.
Required to determine: The value of *c* and of *d*.
Solution:

$$P \xrightarrow{T} Q$$

$$\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}=\begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$2\times 2\times 2\times 1=2\times 1$$

$$=\begin{pmatrix} e_{11} \\ e_{21} \end{pmatrix}$$

$$e_{11}=(c\times 2)+(0\times 3)$$
$$=2c$$
$$e_{21}=(0\times 2)+(d\times 3)$$
$$=3d$$

$$\begin{pmatrix} 2c \\ 3d \end{pmatrix}=\begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

Equating corresponding entries we get:

$2c=2$ $\quad\quad$ $3d=-3$

$c=1$ $\quad\quad$ $d=-1$

Hence, $c=1$ and $d=-1$.

(ii) Determine the image of $(-5,\,4)$, under the transformation *T*.

SOLUTION:
Required to find: The image of $(-5,\,4)$, under the transformation *T*.
Solution:

$$T=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -5 \\ 4 \end{pmatrix} = \begin{pmatrix} e_{11} \\ e_{21} \end{pmatrix}$$

$2 \times 2 \times 2 \times 1 = 2 \times 1$

$$e_{11} = (1 \times -5) + (0 \times 4)$$
$$= -5$$
$$e_{21} = (0 \times -5) + (-1 \times 4)$$
$$= -4$$

$$\begin{pmatrix} e_{11} \\ e_{21} \end{pmatrix} = \begin{pmatrix} -5 \\ -4 \end{pmatrix}$$

$\therefore$ The image is $(-5, -4)$.

(iii) Describe fully the transformation T.

SOLUTION:
Required to describe: The transformation T fully.
Solution:

$T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ describes a reflection in the x-axis.

(iv) Find the matrix that maps the point Q back onto the point P
SOLUTION:
Required to find: The matrix that maps Q onto P.
Solution:

$$P \underset{T^{-1}}{\overset{T}{\rightleftarrows}} Q$$

The inverse of T, written as T^{-1} will map Q back onto P.

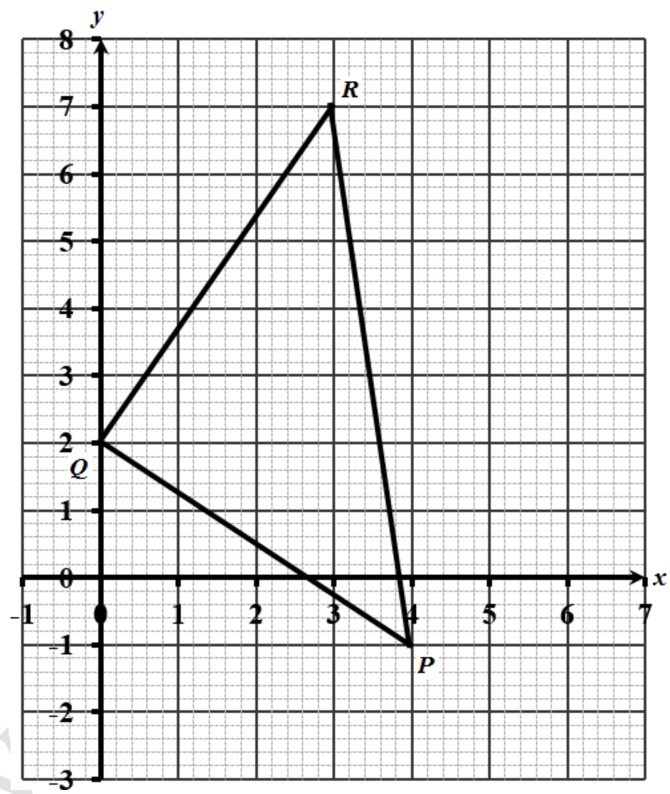
$$T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$|T| = (1 \times -1) - (0 \times 0)$$
$$= -1$$

$$\therefore T^{-1} = \frac{1}{-1}\begin{pmatrix} -1 & -(0) \\ -(0) & 1 \end{pmatrix}$$

$$= -1\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(b) The graph below shows three points P, Q and R, relative to the origin O.



(i) Write a column vector in the form $\begin{pmatrix} x \\ y \end{pmatrix}$:

- the vector $\overrightarrow{OP}$.

SOLUTION:
Data: Graph showing three points P, Q and R, relative to the origin O.

Required to write: $\overrightarrow{OP}$ in the form $\begin{pmatrix} x \\ y \end{pmatrix}$.

Solution:

$P=(4,\,-1)$

Hence, $\overrightarrow{OP}=\begin{pmatrix}4\\-1\end{pmatrix}$ is of the form $\begin{pmatrix}x\\y\end{pmatrix}$, where $x=4$ and $y=-1$.

- the vector $\overrightarrow{QR}$.

SOLUTION:

Required to write: $\overrightarrow{OP}$ in the form $\begin{pmatrix}x\\y\end{pmatrix}$.

Solution:

$\overrightarrow{QR}=\overrightarrow{QO}+\overrightarrow{OR}$

$Q=(0,\,2)\Rightarrow\overrightarrow{OQ}=\begin{pmatrix}0\\2\end{pmatrix}$ and $\overrightarrow{QO}=-\begin{pmatrix}0\\2\end{pmatrix}$

$R=(3,\,7)\Rightarrow\overrightarrow{OR}=\begin{pmatrix}3\\7\end{pmatrix}$

$\therefore\overrightarrow{QR}=-\begin{pmatrix}0\\2\end{pmatrix}+\begin{pmatrix}3\\7\end{pmatrix}$

$=\begin{pmatrix}3\\5\end{pmatrix}$ which is of the form $\begin{pmatrix}x\\y\end{pmatrix}$, where $x=3$ and $y=5$.

(ii) Determine the magnitude of the vector $\overrightarrow{QR}$.

SOLUTION:

Required to find: The magnitude of $\overrightarrow{QR}$.

Solution:

$\overrightarrow{QR}=\begin{pmatrix}3\\5\end{pmatrix}$

$\therefore\left|\overrightarrow{QR}\right|=\sqrt{(3)^2+(5)^2}$

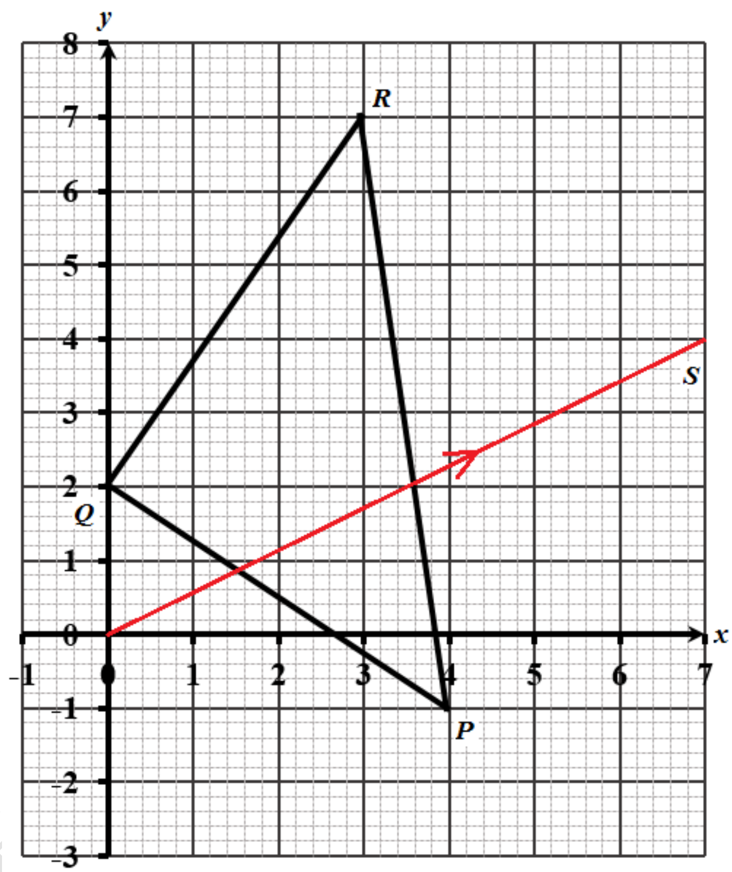
$=\sqrt{34}$ units

(iii) On the graph provided, draw the vector $\overrightarrow{OS}=\begin{pmatrix}7\\4\end{pmatrix}$. Show that $PQRS$ is a parallelogram.

SOLUTION:

Required to draw: $\overrightarrow{OS} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$ on the diagram and show that *PQRS* is a parallelogram.

Solution:



By the vector triangle law

$$\overrightarrow{QP} = \overrightarrow{QO} + \overrightarrow{OP}$$
$$= -\begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$
$$= \begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

By the vector triangle law

$$\overrightarrow{RS} = \overrightarrow{RO} + \overrightarrow{OS}$$
$$= -\begin{pmatrix} 3 \\ 7 \end{pmatrix} + \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$
$$= \begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

Hence, $\overrightarrow{QP} = \overrightarrow{RS}$, that is, $\left|\overrightarrow{QP}\right| = \left|\overrightarrow{RS}\right|$ and $\overrightarrow{QP}$ is parallel to $\overrightarrow{RS}$.

If the opposite sides of a quadrilateral are parallel and equal, then the quadrilateral is a parallelogram. Hence, *PQRS* is a parallelogram.
We could have used the sides *QR* and *PS* and done the same.

Alternative Method:
$Q = (0, 2)$ and $S = (7, 4)$

Midpoint of $QS = \left(\frac{0+7}{2}, \frac{2+4}{2}\right)$ Type equation here.
$$= \left(3\frac{1}{2}, 3\right)$$

$P = (4, -1)$ and $R = (3, 7)$

Midpoint of $PR = \left(\frac{4+3}{2}, \frac{-1+7}{2}\right)$
$$= \left(3\frac{1}{2}, 3\right)$$

Both diagonals of *PQRS* bisect at the same point. If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram. Hence, *PQRS* is a parallelogram.

Alternative Method:
We could prove that $\overrightarrow{PS}$ and $\overrightarrow{QR}$ are parallel and $\overrightarrow{PQ}$ and $\overrightarrow{SR}$ are parallel.

If the opposite sides of a quadrilateral are parallel, then the quadrilateral is a parallelogram.

Alternative Method:
We could also prove that $\left|\overrightarrow{PQ}\right| = \left|\overrightarrow{SR}\right|$ and $\left|\overrightarrow{QR}\right| = \left|\overrightarrow{PS}\right|$.

If the opposite sides of a quadrilateral are equal, then the quadrilateral is a parallelogram.

Alternative Method:
We could have proven that the angles at the opposite vertices are equal, that is, $\hat{Q} = \hat{S}$ and $\hat{R} = \hat{P}$ and concluded that *PQRS* is a parallelogram. This method, though, is long and not very practical and involves a higher level of mathematics than is required at CSEC mathematics.