**Chapter 5**

## *Discrete Fourier Transform, DFT and FFT*

In the previous chapters we learned about Fourier series and the Fourier transform. These representations can be used to both synthesize a variety of continuous and discrete-time signals as well as understanding the frequency spectrum of the given signal whether periodic or otherwise.

A majority of real life signals are discrete. Most are also aperiodic. They are also limited in time. We never have an infinitely long signal, nor are most to the signals we deal with perfectly periodic. Examples are stock market quotations, temperature measurement, communications signals that carry voice, TV signals etc.. These are signals with a large random component. Fourier opened the door to the analysis of these signals but at every step some mathematical incongruity surfaced making necessary the development of the tool into many different types.

So let's look at our primary desire, which is that the input signal be discrete. For discrete signals we can use the discrete Fourier series (DFS), but it is applicable only to periodic signals. Or we can use the discrete-time Fourier Transform (DTFT), which is applicable to non-periodic signals. The spectrum of the DFS is discrete whereas the spectrum of DTFT is continuous. As a consequence of sampling, the output spectrums for both of these constructs are periodic.

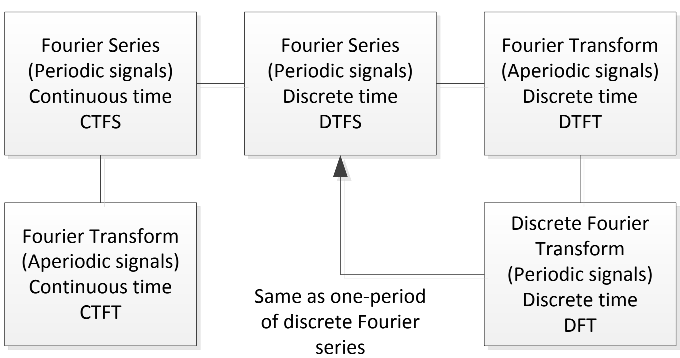



**Figure 5.1 – Relationship of various Fourier transforms**

In the last chapter we arrived at DTFT, but DTFT falls short of an ideal analysis tool. Yes, it applies to discrete time signals and to aperiodic signals. But the rub is that the DTFT in frequency domain is continuous. Computer storage and calculation mode is inherently discrete and DTFT is not an algorithm that can be computed using discrete (which means essentially by computer) math. What we need is a combination of DFS which can be discrete in frequency domain and the DTFT which is discrete in time domain. We saw that when we compute a DTFT of a periodic signal, it is also discrete because this form of the DTFT is same as sampled DFS coefficients. The results from the DTFT of periodic signals in Chapter 4 leads directly to the development of the Discrete Fourier Transform (DFT).

So we now move a new transform called the **Discrete Fourier Transform (DFT)**. It borrows elements from both the Fourier series and the Fourier transform. DFT was developed after it became clear that our previous transforms fell a little short of what was needed.

Let's go through the path we took to get to the DFT. In the beginning there was continuous time Fourier series. The input signal is periodic with period T and continuous in time, which we don't like. But CTFS gave us a discrete frequency spectrum which we do like.

Then we let the period go to infinity. This made the signal aperiodic which is something that we also want, this we called the Continuous-time Fourier transform (CTFT). CTFT gave us a discrete frequency spectrum, also good. But continuous time is still a problem and we want to avoid dealing with it.

Then we sample the signal, get a discrete version of the signal and calculate the discrete-time Fourier series (DTFS). The DTFS gives discrete frequency spectrum but unfortunately the spectrum replicates.

Again by letting the period go to infinity we get the discrete-time Fourier Transform (DTFT). This gives us a continuous frequency spectrum, which is not desirable. But we notice that if take the DTFT of a periodic signal, which has the effect of limiting the signal to a time window, the spectrum becomes discrete. This gives us the idea for the next development in the series, the Discrete Fourier Transform (DFT), the topic of this chapter.

| | Input is | | Output Spectrum is | |
|---|---|---|---|---|
| | **Periodic** | **Time Resolution** | **Periodic** | **Frequency Resolution** |
| Fourier Series Continuous CT-DFS | Periodic | Continuous | No | Discrete |
| Fourier Series Discrete DT-DFS | Periodic | Discrete | Periodic | Discrete |
| Fourier Transform CTFT | Non-periodic | Continuous | Non-periodic | Continuous |
| Fourier Transform DTFT | Non-periodic | Discrete | Periodic | Continuous |
| Discrete Fourier Transform DFT | Non-periodic | Discrete | Periodic | Discrete |

**Table I – Properties of Fourier transform input and output signals**

Taking this further we present now the Discrete Fourier transform (DFT) which has all three desired properties. It applies to discrete signals which may be

(a) Periodic or non-periodic
(b) Of finite duration
(c) Have a discrete frequency spectrum

DFT is similar to both DTFS and DTFT. The differences between the DFT and DTFT and DFS are subtle. So first let's look at the definition for the DTFT.

$$X(\Omega) = \sum_{k=-\infty}^{\infty} x[k] e^{-j\Omega k} \quad DTFT \tag{1.1}$$

Notable here are an infinite number of harmonics used in the calculation of the DTFT. And because there are an infinite number of harmonics, resolution is infinitesimally small and hence the spectrum of the DTFT is continuous.

The DFT is essentially a discrete version of the DTFT. It is a function of the frequency index n, where the nth harmonic of the DTFT $\Omega_n = 2\pi \, n/N$ for $0 \le n \le N-1$. Here N is the period of the signal. To derive the expression of the DFT, we substitute this value of the harmonic into the DTFT equation (1.1). This is given by the following expression.

$$X(n) = \sum_{k=0}^{N-1} x[k] e^{-j(2\pi kn/N)} \quad DFT \tag{1.2}$$

The inverse DFT is given by

$$x[k] = \frac{1}{N} \sum_{k=0}^{N-1} X[n] e^{j(2\pi kn/N)} \quad IDFT \tag{1.3}$$

Here k is the time index, N the number of harmonics and r is the index of these, so that n/N is the nth harmonic. We note that the only thing different between the DTFT expression (1.1) and the DFT (1.2) is the range of summation. The DFT takes place over N points. The DFT uses only a finite number of harmonics, instead of the infinite for the DTFT, namely N, so its spectrum resolution as measured by the fundamental frequency $\Omega_0 = \frac{2\pi}{N}$ is discrete and finite. Hence this expression gives a spectrum that is discrete. Of course we can pick a very large N, but that does not change the fact that it is still a finite number and so the DFT output is discrete. Conceptually speaking, it may be easier to think of the DFT as sampled DTFT at specific frequency points.

In previous chapter, we calculated DTFTs by closed form analysis. DTFT of arbitrary signals are often hard to calculate. DFT on the other hand lends itself to calculation by digital devices. It has an easy to understand architecture as shown in Figure 1. It can be implemented as a linear device with N inputs and N outputs. It is completely lossless and bidirectional. N numbers go in and N numbers come out.

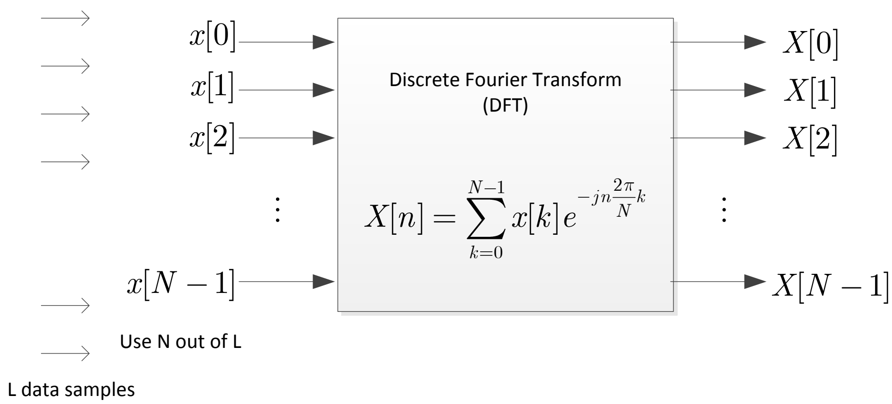



**Figure 5.2 – Architecture of a DFT**

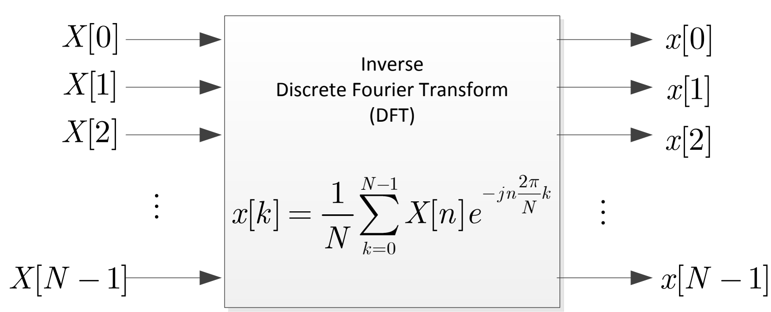



**Figure 5.3 – Inverse Discrete Fourier Transform**

The input to the inverse DFT are the N frequency domain samples. Let's compare this to the inverse expression for the DFS (Eq. 1.17, Chapter 3)

$$x[k] = \sum_{k=0}^{K_0-1} D_n \; e^{jn\Omega_0 k} \quad IDFS \tag{1.4}$$

The two equations (1.4) and (1.3) are very similar, with only a scaling difference. The coefficients $D_n$ from the DFS are same as the transform X[n] in (1.4) The range is also same as N samples. So we see how the DFT is a hybrid transform that is like the both the DTFT and the DFS. It is very nearly identical to the DFS if we think of it as looking through an observation window one period long and choosing to ignore what is outside that window. DFT spectrum also repeats but because we limit the range to just $2\pi$, calculations are limited to just one main part which is sometimes also called the principal alias. But the true spectrum does have all the other aliases besides the principal one located at the zero, as well all the others we have to account for.

The difference between the DFT and its inverse IDFT is just a scaling term in the front of the IDFT and a change of the sign of the exponent. The two expressions are very similar and this makes the hardware/software implementation of DFT algorithm easy. You can easily do it in Matlab with just one or two lines.

## Matrix method for computing DFT

We can also view the DFT as a linear input/output processor consisting of N equations. Assume we have collected L data symbols, then we need to make a decision about how long the N as specified for the DFT processor should be. If N is less than L, then we select N samples out of the L and perform the DFT calculations on these N points. If we choose N that is larger than the L collected samples, then we have to perform some sort interpolation to increase the number of samples from L to N. Some algorithms of the DFT will automatically add a bunch of zeros to make the length N you have specified.

The exponential is kind of hard to type, so let's make this substitution.

$$e^{j2\pi/N} = W_N$$

And also this

$$e^{jn\frac{2\pi}{N}k} = \left(e^{\,j\frac{2\pi}{N}}\right)^{nk} = W_N^{nk} \qquad (1.5)$$

Remember to think of $W_N$ as a single complex exponential of frequency $\frac{2\pi}{N}$. Hence $W_N^{nk}$ is a matrix of several exponentials with both n and k ranging from 0 to N. Now rewrite the DFT using this form.

$$X[n] = \sum_{k=0}^{N} x[k]\, W_N^{nk} \qquad (1.6)$$

The index N indicates that this complex number is based on period N. Now we write Eq.(1.6) in matrix form as

$$\mathbf{X} = W_N^{nk}\mathbf{x} \qquad (1.7)$$

X is the output Fourier transform vector of size N and x is the input vector of same size.

$$\mathbf{X} = \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} \tag{1.8}$$

The $W_N^{nk}$ term in Eq. (1.6) is called the DFT matrix and is given by

$$W_N^{nk} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(n-1)} & \cdots & W_N^{(n1)(n-1)} \end{bmatrix}_{n \times k}$$

Note that the exponent of $W_N$ is the product of n and k. The matrix size is n by k, but since both n and k are equal to, the size is also equal to N x N. The rows represent the index n of the harmonics and columns the index k, time. The first row has index n = 0, so all the first row terms are equal to 1, because

$$W_N^{(n=0)k} = e^{j0\Omega_0 k} = 1\,.$$

The first column has index k = 0, hence all $W_N^{nk}$ terms in the first column have a zero exponent and so all are also equal to 1. The second row, second term is equal to $W_N^{(n=1)(k=1)} = W_N$ and the rest can be understood the same way. Note that the matrix is symmetrical about the right diagonal such that

$$W_N = W_N^T$$

We can also show that the inverse of this matrix is just the inverse of the individual terms.

$$W_N^{-1} = \frac{1}{N} W_N^* \tag{1.9}$$

We will call $W_N^{nk}$ the **DFT Matrix**.

Let's write out the DFT matrix for N = 4. If N = 4, then the product of n and k is also equal to 4 x 4 = 16. To calculate the individual terms of the DFT Matrix $W_N^{nk}$, we need to recall the following property of the complex exponential from Chapter 2.

N is equal to 4, so the fundamental frequency is equal to $\Omega_0 = \frac{2\pi}{4}$.

$$
\begin{aligned}
W_4^{nk} &= e^{-j\left(\frac{2\pi}{4}\right)nk} \\
&= \cos\left(\tfrac{\pi}{2}nk\right) - j\sin\left(\tfrac{\pi}{2}nk\right) \qquad (1.10) \\
&= (-j)^{nk}
\end{aligned}
$$

For all nk, a product which is always an integer, the cosine of $\frac{\pi}{2}nk$ is always zero. Sine however is always equal to 1. So we get the various values of the matrix

$$
\begin{aligned}
&W_4^{nk} = -j^{\;nk} \\
&W_4^{n=1,k=0} = 1 \\
&W_4^{n=1,k=1} = -j \\
&W_4^{n=1,k=2} = -1 \\
&W_4^{n=1,k=3} = j
\end{aligned}
$$

From this we can develop the $W_4^{nk}$ matrix as

$$
W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \qquad (1.11)
$$

The inverse comes easily from Eq. (1.9)

$$
Inverse\ W_N = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}
$$

Notice that only the imaginary components changed sign. For this you have to remember what matrix W is. Each term is a complex exponential. For the forward transform we need $e^{-j\ n\Omega_0\ k}$ and for the reverse we need $e^{j\ n\Omega_0\ k}$. The difference between these is that the imaginary part changes sign. $\cos\theta + j\sin\theta$ vs. $\cos\theta - j\sin\theta$. So here you are seeing the imaginary part changing sign for the inverse but not the real part.

We can now use this DFT matrix to calculate the DFT of any arbitrary signal of 4 input points using the equation of (1.7). Larger DFT matrices would be needed to compute larger size DFT.

Now we will revisit ideas some elementary signals and their DTFT and the DFTs.

## DFT of Important functions

### DFT of a delta function at the origin

Here we have a sequence of just one impulse at the origin. We already know that the DTFT of an impulse is a constant 1. The DFT however is a train of impulses of amplitude 1.

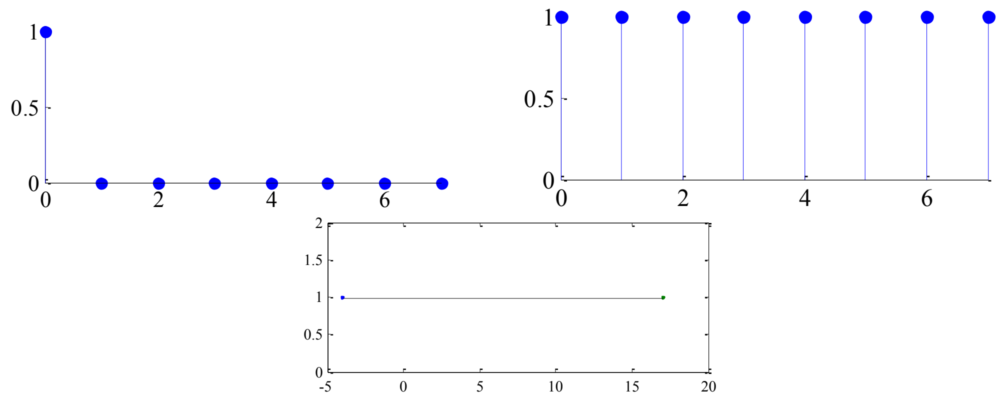



**Figure 5.4 – (a) Single impulse function at origin, (b) its Discrete Fourier Transform (DFT) (c) its discrete-time Fourier Transform (DTFT)**

If the DTFT of an impulse is a constant, then for the DFT, how did we go from a constant of 1 to a train of impulses? The reason is this. DFT is essentially the sampled version of the DTFT. If the DTFT is a constant 1, then for DFT, we sample the DTFT at each of the DFT harmonics which are every $\frac{\pi}{4}$. From the sifting property of delta function, we get 8 samples which are called the **principal alias**, as we see in Fig. 5.4(b). Thereafter they repeat again and hence we get an infinite pulse train.

This idea that the spectrum repeats every $2\pi$ is a key point which you will see repeated often in this chapter. The inverse DFT or the IDFT of an impulse train will of course then be a single impulse.

## DFT/DTFT of a constant

Conversely if we have a time domain signal that is constant of amplitude 1, its DTFT is given as

$$X\left[\Omega\right] = 2\pi \sum_{m=-\infty}^{\infty} \delta\left[\Omega - 2\pi m\right] \qquad (1.12)$$

This is a pulse train of magnitude $2\pi$. The DFT of this case is exactly the same as the DTFT.

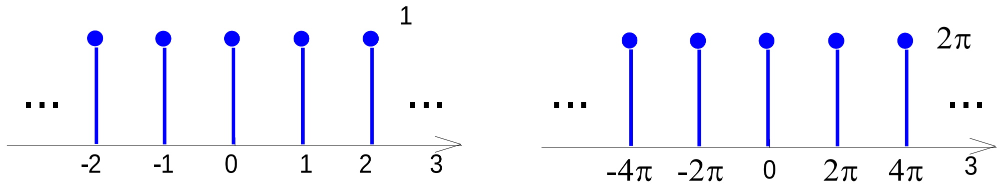



**Figure 5.5 – DFT of a constant function is same an impulse train.**

## DFT of a sinusoid

Here is a cosine of frequency 4 Hz of amplitude 0.7. The sampling frequency is 20 Hz. In (b) we plot its spectrum which gives us two impulses of half the amplitude located at 4 Hz from the origin, just as we would expect. The third figure shows the full DFT spectrum. The center part is called the **principal alias**, thereafter the same repeats with the sampling frequency of 20 Hz. Note that the second set is centered at 20 Hz on the high side at -20 Hz on the low side. These repeat at -40, -60 etc. the integer multiple of the sampling frequency.

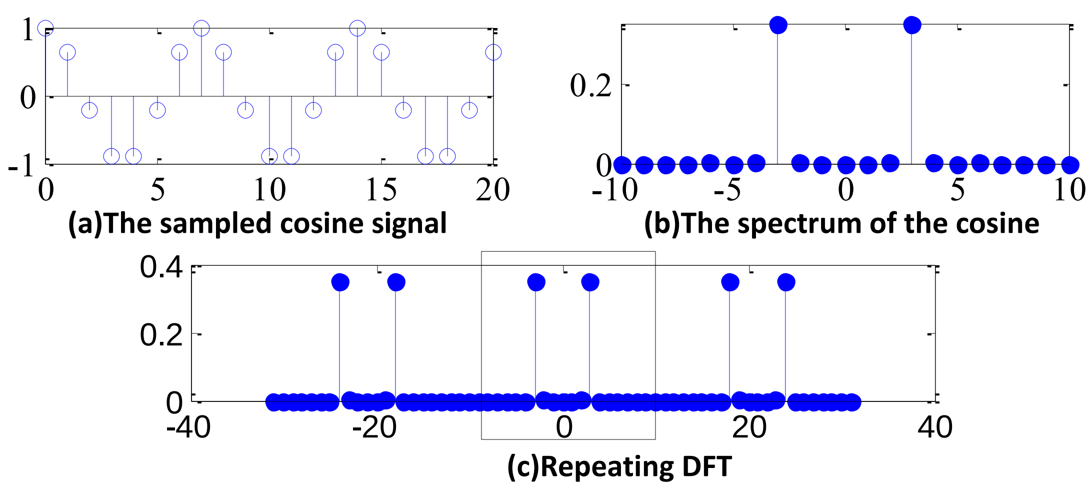



**Figure 5.6 – The DFT of a cosine. (a) Time domain signal, (b) The DFT showing only the principal alias, (c) the repeating DFT**

The DTFT of this signal is given as

$$X\left[\Omega\right] = \pi \sum_{k=-\infty}^{\infty} \delta \; \Omega + \Omega_0 - 2\pi m \; + \pi \sum_{k=-\infty}^{\infty} \delta \; \Omega - \Omega_0 - 2\pi m \qquad (1.13)$$

The DTFT of a cosine is discrete and so is the DFT.

## DFT of a complex exponential

This example shows the DFT of the sum of two complex exponentials of frequencies 4 and 6 Hz. We sample the signal with a sampling frequency of 32 Hz, which is of course much bigger than minimum needed (12 Hz.)

$$x[k] = 0.5\exp(-j2\pi f_1 k \,/\, F_s) + 0.3\exp(j2\pi f_2 k \,/\, F_s)$$
$$f_1 = 6, \quad f_2 = 4, \quad F_s = 32$$

The DFT of this signal are two impulses one at -6 Hz of amplitude .5 and the other at 4 Hz of amplitude .3, which you already know. The set around the origin is the principal alias and the rest are repetitions centered at 32 Hz from the origin.

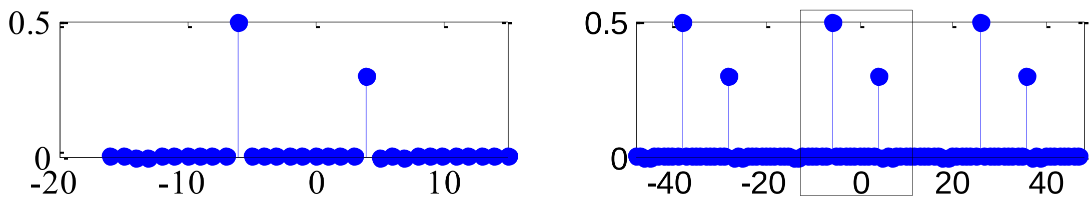



**Figure 5.7 – The spectrum of a complex exponential is just a single impulse at the frequency of the exponential (a) The DFT showing only the principal alias, (c) the repeating DFT**

The DTFT of a complex exponential is given by

$$X\left[\Omega\right] = 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi m) \qquad \textbf{(1.14)}$$

This is quite similar to the DTFT/DFT of a cosine, but of course, there is only one impulse instead of two and it also repeats every $2\pi$.

## Properties of DTFT/DFT

We will go over properties of both DTFT and DFT in this section. DFT derives its properties from both DFS and DTFT.

### Periodicity: Repeating spectrum

The DTFT of a signal is periodic with $2\pi$. The same is true for DFT.

$$X[\Omega] = X[\Omega + 2\pi] \qquad (1.15)$$

Most calculations of the DFT, be they done in closed form, software, hardware or Matlab etc. show only the principal alias, the DFT actually consists of an infinity of the repeating spectrum. Hence, in frequency domain, for signal processing purposes, we often have to filter out the principal alias from all the others.

## Time shifting: DFT of a delayed delta function

In this example, we show the effect of time shift in time domain on the DFT. Take a single impulse which is displaced or delayed from the origin by one sample. The signal length is 16 samples but for this example, the length does not matter.

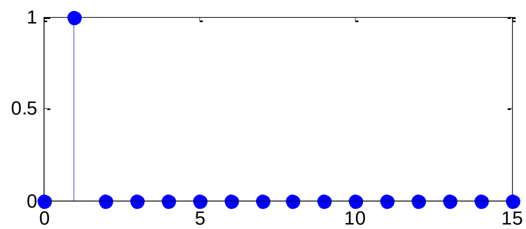



**Figure 5.8 - A delayed impulse function**

We calculate the DFT of this signal by

$$X(n) = \sum_{k=0}^{N-1} x[k] e^{-jn\Omega_0 k}$$

What is $\Omega_0$ in the DFT equation? It is $\frac{2\pi}{16} = \frac{\pi}{8}$ because we have N = 16 samples in the signal, although only one is non-zero.

What is x[k]?

It is [0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0] or we can say it is $x[k] = \delta[k-1]$.

$$X(\Omega) = \sum_{k=0}^{N-1} x[k] e^{-jn\Omega_0 k}$$

$$= \sum_{k=0}^{15} \delta[k-1] e^{-jn\Omega_0 k}$$

The index k has disappeared in the last step because its value is equal to 1 from the delta function. The DFT of an un-delayed delta function was a train of constant impulses but here we are getting a complex exponential. The frequency of this exponential is 1 Hz, because the signal goes through one cycle or $2\pi$ in 16 samples. This is not very obvious if we look at the DFT. We don't see the 1 Hz. But note that n is the sample number. There are 16 samples. The fundamental frequency which is not in Hz, but in radians per sample is $\pi / 8$.

$$X(n) = e^{-jn\frac{\pi}{8}}$$

$$X(0) = -1$$
$$X(1) = -.92 + .38j$$
$$X(2) = -.70 + .70j$$
$$X(3) = -.38 + .92j$$

Here we plot the real and imaginary parts of the DFT.

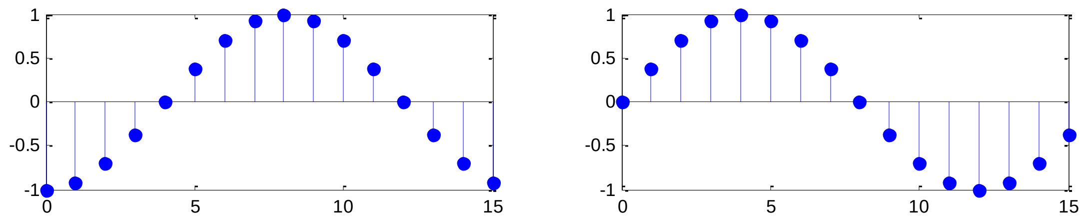



**Figure 5.9 - (a) the Real part of the DFT, a cosine of 1 Hz, (b) the Imaginary part of the DFT of the delayed delta function.**

It may help instead to look at the magnitude and the phase, the most common way of looking at the spectrum of a signal. The delay does not effect the magnitude at all. Only the phase changes. In (b), we see that the phase is changing. It progresses $\pi / 8$ radians per sample, and we get $2\pi$ in 16 samples.

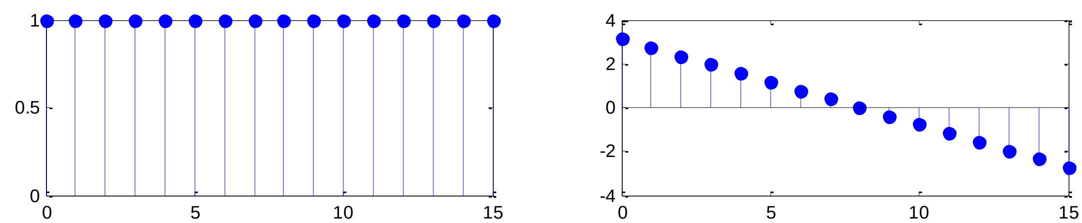
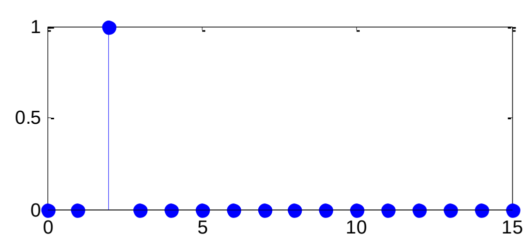
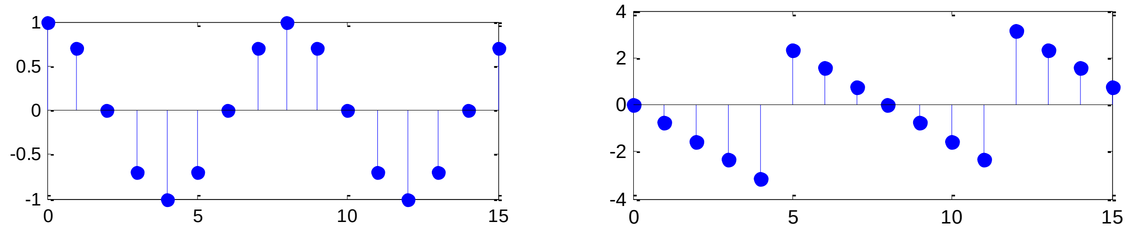



**Figure 5.10 - (a) The magnitude of the DFT, (b) Phase of the DFT.**

Now try to guess what happens if the impulse is delayed by two samples instead of one. The magnitude stays the same but the phase changes twice as much. This gives us a cosine of 2 Hz as the real part of the spectrum. In Fig. 5.10 (b), the phase change over 16 samples is $2\pi$ but in Fig. 5.11 (c), it is $4\pi$.




(a) Impulse delayed by k = 2 units.




(b) The real part of the signal (a cosine of frequency 2 Hz.) (c) The phase

**Figure 5.11 – (a) Impulse function delayed by $k_0$ units, (b) the Real part of its DFT shows increasing frequency (c) Phase similarly also increases at twice the speed as case with one time unit delay.**

These results show the shifting property of the DFT same as DTFT and other Fourier transforms. The DTFT of this delayed signal is given by

$$X[\Omega] = e^{j\Omega k_0} \qquad (1.16)$$

And equivalently we can write that if a signal is delayed by k0 units, then its DTFT (and also DFT) is multiplied by a complex exponential of $\Omega k_0$. For the DFT, this is equal to $\frac{2\pi}{N} k_0$.

$$x[k - k_0] \Leftrightarrow X[n] e^{-j\Omega k_0} \qquad (1.17)$$

How does the delay turn into a frequency/phase shift? To understand what the delay means, look carefully at the complex exponential multiplier. We are multiplying frequency $\Omega$ by an integer, k0. What is this frequency? Okay, this gets confusing here. Think of this term as an exponential of frequency k0, and not $\Omega$. The term $\Omega$ is not actually frequency here. As we said earlier, it is measured as radians per sample, so it is not frequency per say in Hz. The only way you know it is 1 Hz frequency is by noting the phase changes $2\pi$ over the period and that makes the frequency 1 Hz. We know this indirectly. So it is a frequency of 1 Hz (although the digital frequency is equal to $\pi / 8$) for k0 = 1 and as you step through n, you just trace out the sinusoid as shown below. This is why this phase shift is not a function of the N, the size of the DFT.

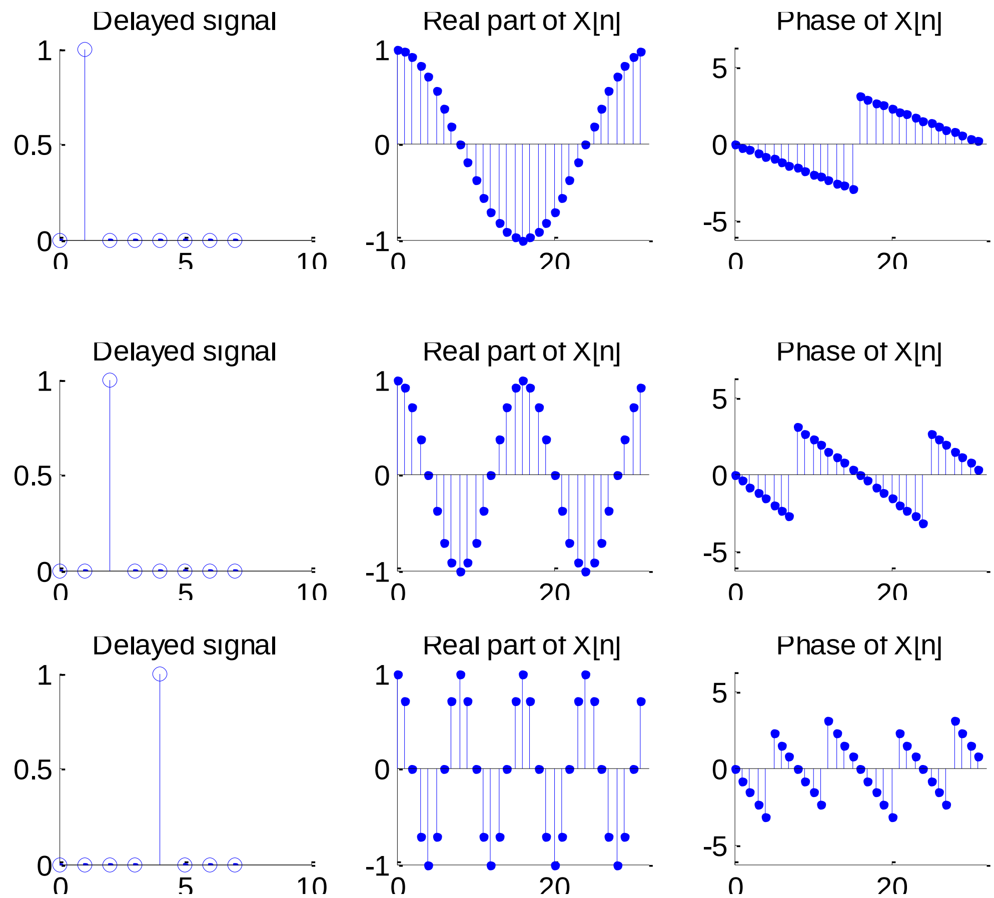



**Fig. 5.12 – The effect of delay is to introduce a phase shift in the frequency domain. The phase shift is always $2\pi$ per sample. The amplitude is not effected. (a) Delay of one sample results in the real part of cosine of 1 Hz and a total phase shift of 2$\pi$, (b) Delay of 2 samples results in the real part of cosine of 2 Hz and a total phase shift of 4$\pi$, (c) Delay of 4 samples results in the real part of cosine of 4 Hz and a total phase shift of 8$\pi$.**

## Modulation property: shifting the spectrum of a signal to a carrier frequency

If we multiply a signal by a signal of certain frequency, the resulting DFT shows the original spectrum has shifted to the multiplying signals' frequency. This is also called the modulation property. It is this property of signals that allows us to take a baseband signal, multiply it by a carrier frequency and create a rf signal, or one called the modulated signal. The multiplying signal, usually a sinusoid is called a carrier.

$$x[k]\, e^{-j\Omega_0 k} \Leftrightarrow X\left[\Omega - \Omega_0\right] \tag{1.18}$$

Let's take a signal given by

$$x[k] = \cos(2\pi f_1 k \ / \ F_s) - 0.7\cos(2\pi f_2 k \ / \ F_s)$$
$$f_1 = 1, \quad f_1 = 4, \quad F_s = 30 \ or \ 50$$

First we compute the DFT of this signal. It has two impulses, one of magnitude 0.5 and the second of 0.35. These are shown in (c). This signal is then multiplied by a carrier signal of frequency 10 Hz to modulate it. The DFT of the modulating signal which is a single frequency sinusoid, is given by two impulses located the frequency of the sinusoid as in (d).

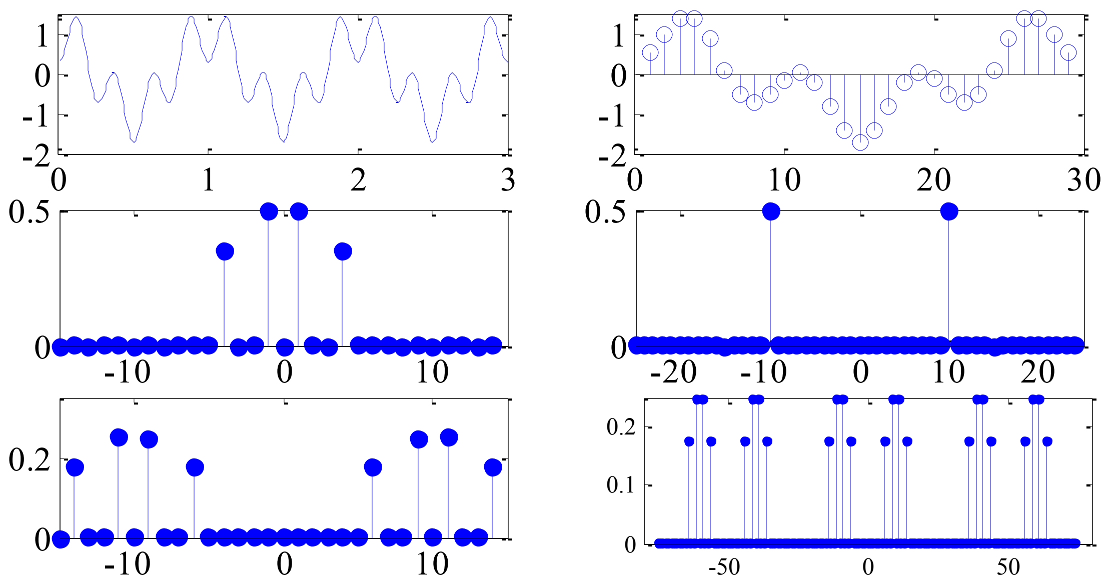



**Fig 5.13 – (a) the continuous signal, (b) Sampled signal with Fs = 30, (c) the spectrum of the baseband signal of (b). It spectrum is located at the origin, (d) the DFT of the carrier signal of frequency 10 Hz. (e) The spectrum of same signal after it is multiplied by a carrier signal of frequency 10 Hz. (f) Same signal sampled with Fs = 50, which increases the distance between the copies.**

In Fig. 5.13 (e) we see just the principal alias of the spectrum when the signal is sampled at 30 Hz. The edges are samples 15, -16. The spectrum which was centered at origin, is now replicated on each carrier frequency and is now located at $\pm 10$ Hz. Figure (f) shows the DFT with additional copies of the spectrum. When the signal is sampled with Fs $= 50$ Hz. The principal alias did not change, but the distance between the principal alias and the copies has increased. Each copy is centered at integer multiple of 50 Hz.

Note when we sampled the signal at 30 hz, due to the 10 Hz shift, the 4 Hz component on left is now at 14 Hz on the right, the edge of the principal alias. When we see the replicated spectrum, note that next copy of the spectrum is very close to the principal alias because its right edge is at 16 Hz. The same signal if it is sampled with a sampling frequency of 50 Hz, moves the next copy of the spectrum to 50 Hz and hence we see larger distances between the copies. This makes filtering less of a problem, we won't need such a sharp filter.

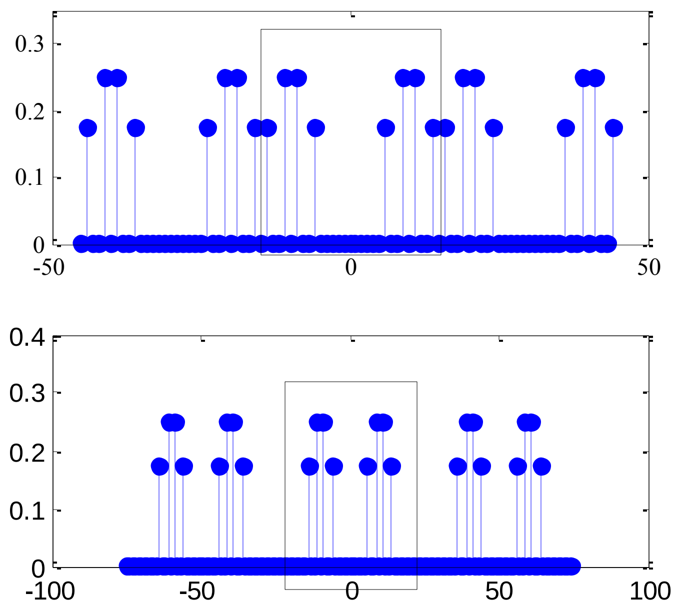



**Figure 5.14 – Increasing the sampling frequency moves the copies of the spectrum further apart**

## DFT of square-pattern pulses

Now let's look at another class of signals, ones that are a discrete version of a square pulse. The first pulse consists of four samples and the rest 28 are zeros for a total of 32 samples. We write the DFT equation as

$$X[\Omega] = \tfrac{1}{N}\sum_{k=0}^{N}\left(\delta[k]+\delta[k-1]+\delta[k-2]+\delta[k-3]\right)e^{-j(n\Omega_0)k}$$

$$\Omega_0 = \tfrac{2\pi}{32}$$

$$= \tfrac{1}{N}\left(1+e^{-j(n\pi/16)}+e^{-j(n\pi/16)2}+e^{-j(n\pi/16)3}\right)$$

$$= \tfrac{1}{N}\left(1+e^{-jn\frac{\pi}{16}}+e^{-jn\frac{\pi}{8}}+e^{-jn\frac{3\pi}{16}}\right)$$

$X[0] = 0.0313 + 0.0129j$

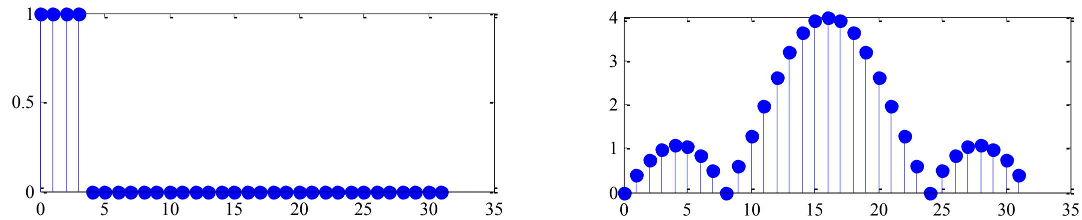
$X[1] = 0.0326 + 0.0032j$

$X[2] = 0.0261 - 0.0052j$

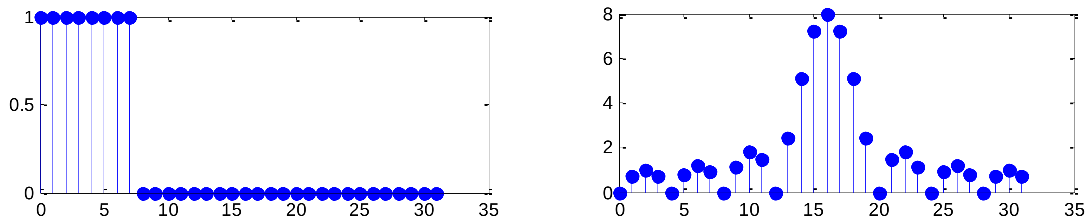
$X[3] = 0.0136 - 0.0073j$

etc.

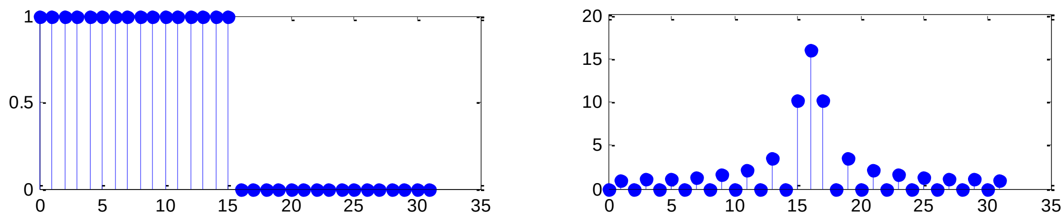
We see the magnitude of the spectrum plotted in Figure 5.15. Note that magnitude has a Diric function pattern and not sinc. The width of the main lobe is equal to 2N/K, if N is the total length and K the length of the non-zero pulses. In the first case below, N = 32, so we see that the width of the main lobe is equal to 64/4 = 16 samples.



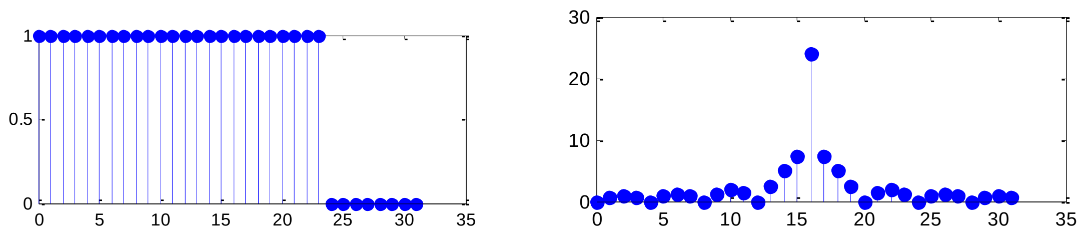

**(a) Four impulses representing a square**



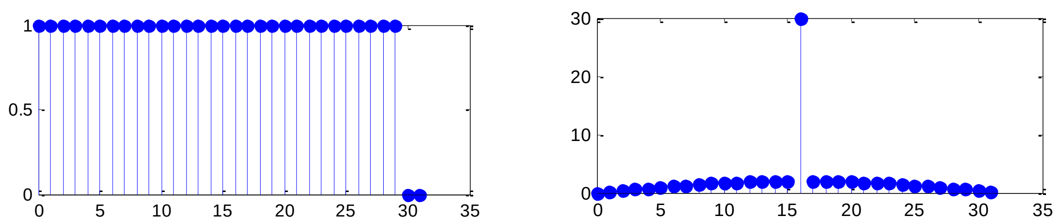

**(b) Square pulse length = 8 samples**



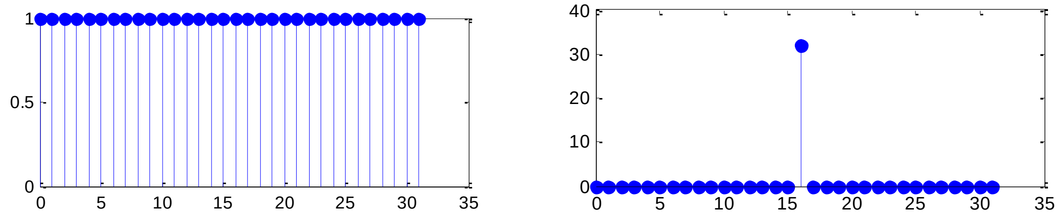

(c) **Square pulse length = 16 samples**




(d) **Square pulse length = 24 samples**




(e) **Square pulse length = 30 samples**




**(f) Square pulse length = 32 samples (Isn't that just an impulse train?)**

**Figure 5.15 – Spectrum of sampled square pulses.**

One thing to recognize is that here we have two boundary cases, an impulse and an impulse train. In each case, the spectrum moves either to one type or the other as we compress the non-zero part of the signal.

The DTFT of this type of signal is given by

$$X\left[\Omega\right] = e^{-j\Omega(N-1)/2}\frac{\sin\ N\Omega / 2}{\sin(\Omega / 2)} \qquad (1.19)$$

Note the exponential term in the front. What is that about? First of all, it is the imaginary term and hence effects the phase. This term also is an indication of a phase/frequency shift. It is there because the pulses are not centered at 0. They were centered at (N-1)/2 which is same as a delay. So we see that the delay has introduced a frequency shift.

In the next example, we show the DTFT/DFT of square pulses located at 0. In this case, there is no delay and all we have is the Diric function without a frequency shift.

$$X\left[\Omega\right] = \frac{\sin\ N\Omega / 2}{\sin(\Omega / 2)} \qquad (1.20)$$

## Square pulses centered at origin

Here we will look at the square pulses centered at zero. These will have a different spectrum owing to the fact because they not displaced from the center, so have no frequency shift. The method of calculation is the same as in previous case but since the range of N goes from – to +, we get a slightly different result. Here N is an odd number, the number of impulses on each side of center impulse.

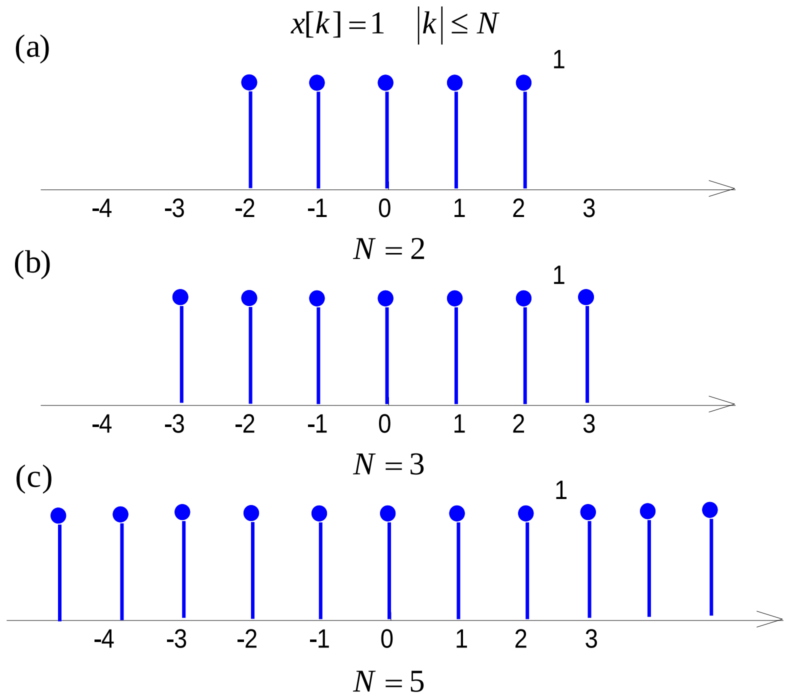



**Figure 5.16 - Sampled square pulse centered at zero.**

Now let's plot the DFT of this signal as given by the following formula.

$$x[k] = \begin{cases} 1 & |k| \le N \\ 0 & N < k < K_0 \end{cases}$$

Also

$x[k] = x[k + K_0]$ the signal is periodic.

$$C_n = \begin{cases} \dfrac{(2N+1)}{K_0} & k = lK_0 \\ \dfrac{1}{K_0}\left(\dfrac{\sin\dfrac{(2N+1)}{K_0}n\pi}{\sin\left(\dfrac{1}{K_0}n\pi\right)}\right) & elsewhere \end{cases} \tag{1.21}$$

DFT is the repeating version of these coefficients given by

$$X[\Omega] = 2\pi \sum_{k=-\infty}^{\infty} C_n \delta\left(\Omega - \frac{2n\pi}{K_0}\right) \tag{1.22}$$

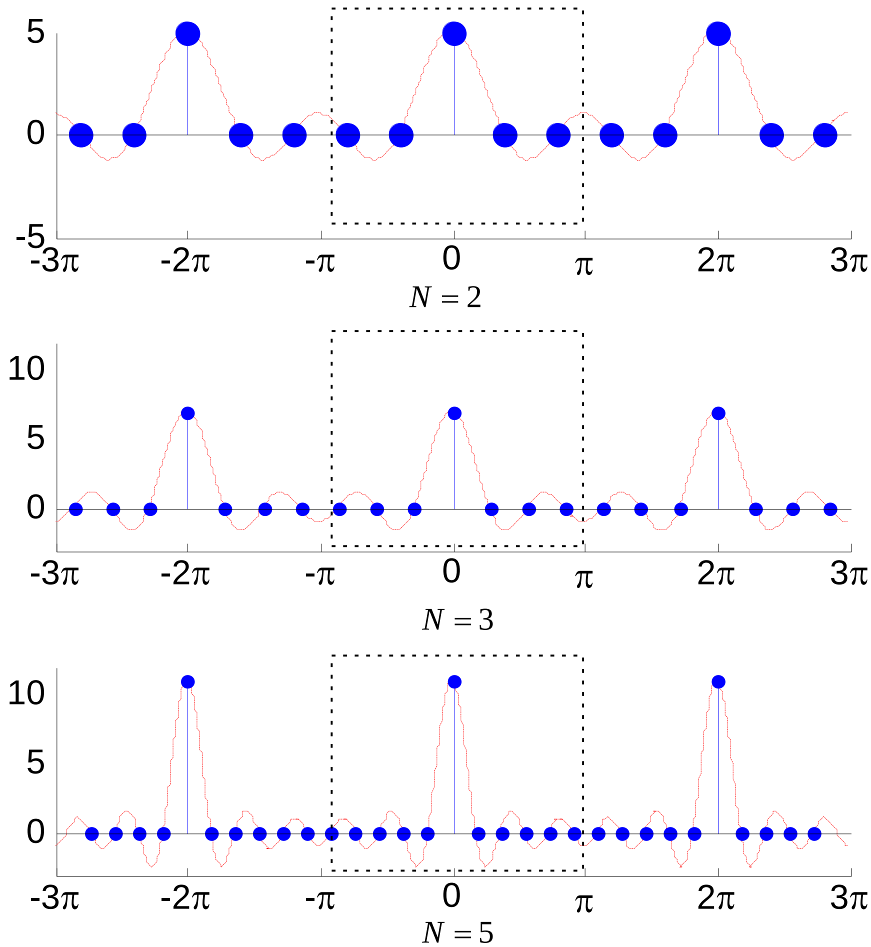



**Figure 5.17 – Spectrums of square pulse centered at zero.**

Here we have plotted the DTFT along with DFT. As we increase N, the size of the square pulse gets bigger so are we not approaching an impulse train again? The spectrum also approaches impulses since the lobes are getting narrower which is the spectrum of an impulse train.

## Sinc function as an input signal

When the input is a sinc function, such as

$$x[k] = \frac{W}{\pi}\operatorname{sinc}\left(\frac{Wk}{\pi}\right) \tag{1.23}$$

Then its DTFT is given as

$$X[\Omega] = \begin{cases} 1 & |\Omega| \le W \\ 0 & W \le |\Omega| \le 2\pi \end{cases} \tag{1.24}$$

In theory, the sinc signal would give a perfectly flat top-hat like spectrum as shown in Fg. 5-.

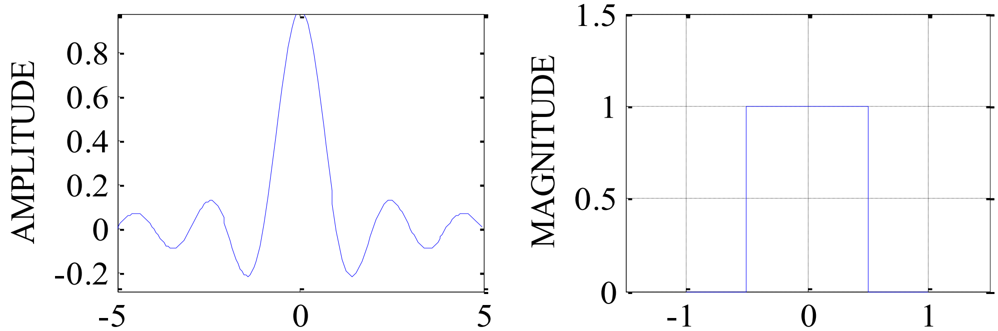



**Figure 5.18 – A sinc pulse of infinite length has a top-hat like DTFT.**

But this does not happen if you employ the DFT. The reason is that the sinc function is infinitely long and when we truncate it to compute the DFT, we get what is called ringing (also called leakage). Since we can never get an infinitely long signal, the spectrum of a truncated sinc function will always show this pattern, This is shown in Fig. 5.19. And in (b) and (d), same spectrum plotted in stem and the in classical fashion by connecting the values.

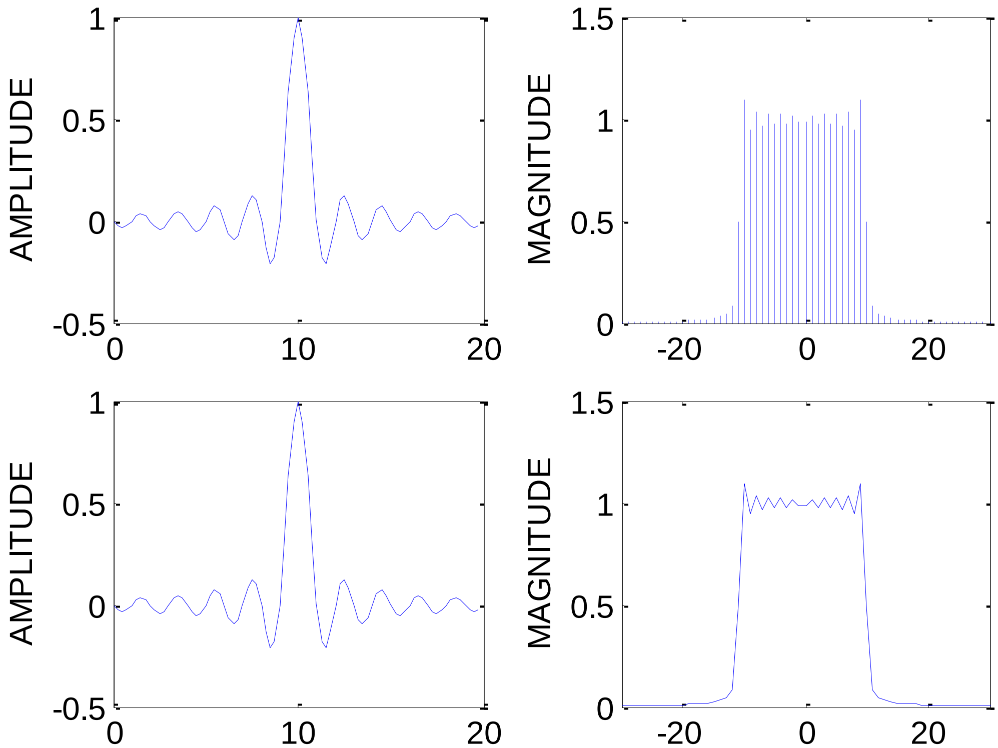



**Figure 5.19 – The DFT of a since function that has been truncated to four lobes on each side.**

The actual spectrum of course repeats with sampling frequency. Here we see the square pulses located at -64 and +64 Hz which is the sampling frequency of this signal. Notice that each “square” pulse is centered at integer multiple of the sampling frequency.

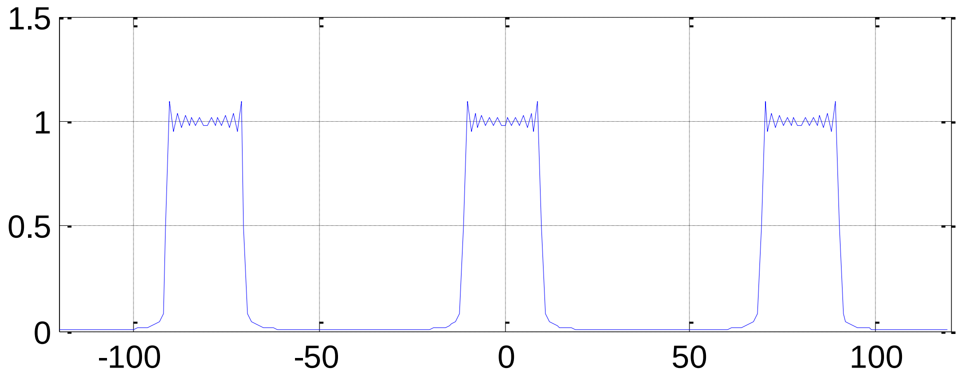



**Figure 5.20 – When sampled with Fs = 64, the spectrum repeats with same frequency.**

### Time scaling: Effect of oversampling and downsampling

In 5.21 (a) we see a signal of 4 samples. Its DFT is shown in (b). Now we stretch this signal by inserting a zero in between each sample. The effect of this oversampling is actually surprising. The oversampling by a factor k, results in the spectrum replicating k times. We see this in (e) and (f).

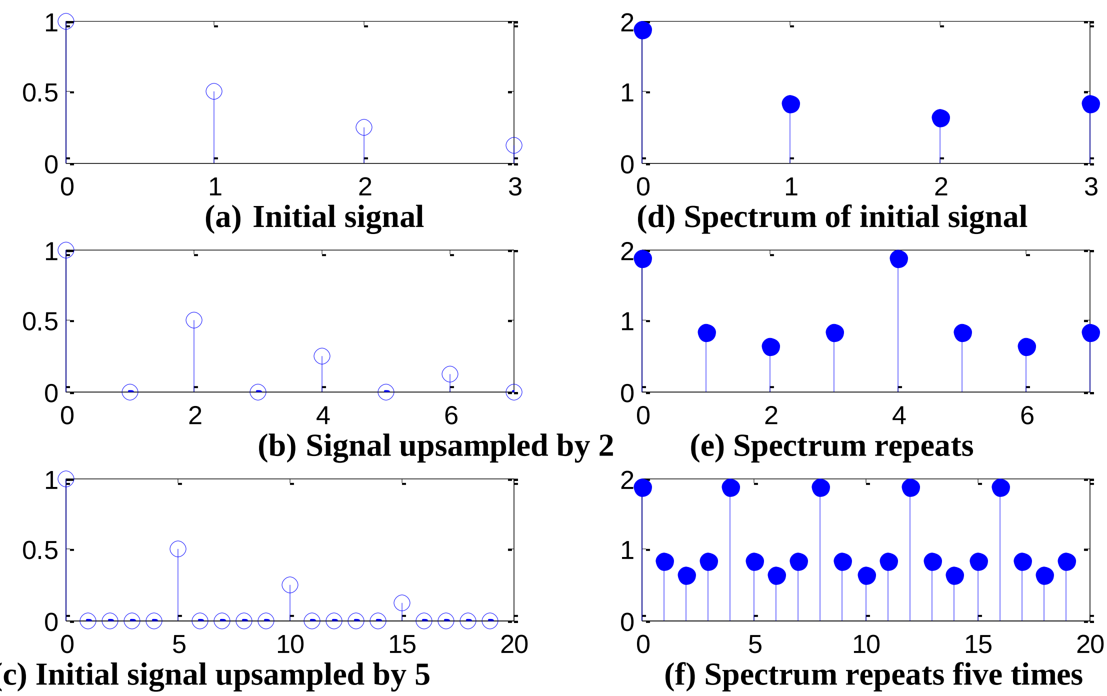



**Figure 5.21 – Effect of oversampling**

Here is an another example. Let's take the square pulse centered at origin with N = 3. We get the following signal and its DFT. When we oversample this signal and insert 2 zeros between each sample, we see the following spectrum. The spectrum repeats by the factor of oversampling.

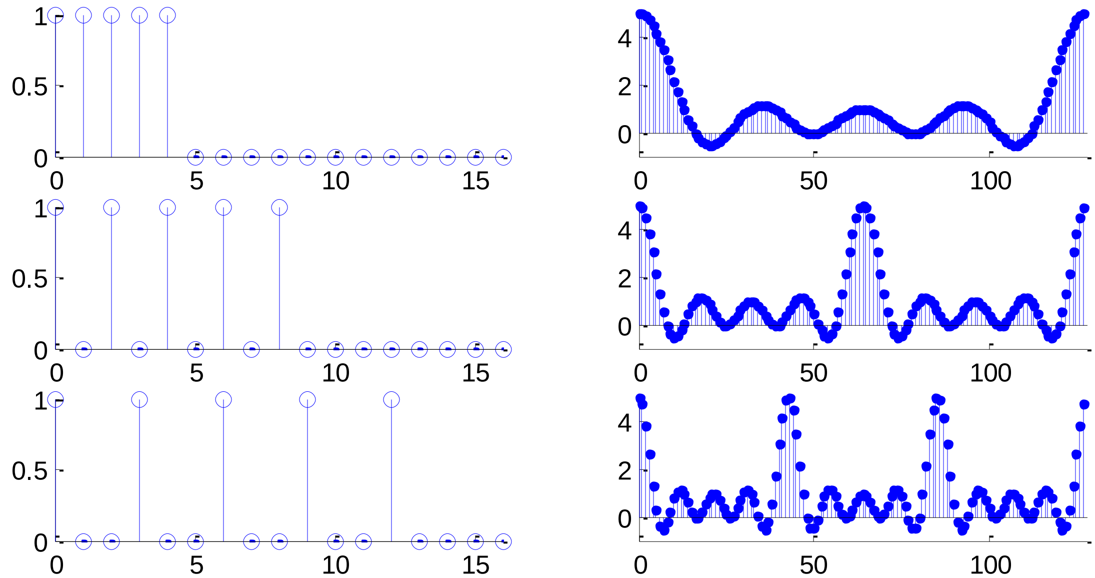



**Figure 5.22 – Effect of oversampling on a square pulse**

Down-sampling the oversampled signal by the same amount, will return the signal to its un-stretched configuration, both in time and frequency domain. Try it with the Matlab code.

## Zero-padding

In figure 5.22, we show that the signal size is 16 points, however the DFT size is 128 points. The extras points come as additional zeros appended to the signal. The appended points reduce the frequency resolution by the relationship $\frac{2\pi}{N}$. As N gets larger, the resolution get smaller and we get a better defined spectrum such as we can see. Zero padding can only be done at the end or at the start of the signal. It will not provide any additional information as information is a function of the sampling frequency and not the size of the DFT.

## Example 5-1

To compute the DFT of an arbitrary signal, we will now use both the closed solution first and then use the metric method. The signal in Fig. 5.16 is given as

$$x[k] = \delta[k] + .5\delta[k-6]$$

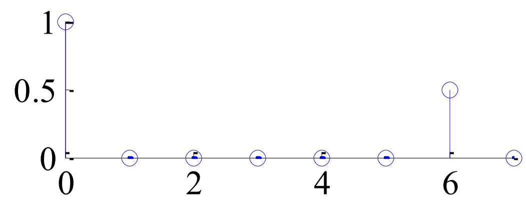



**Figure 5.23 – Time domain signal of example problem 5-2**

Signal length is: L = 8. Taking DFT of a series of impulses can be pretty easy. First we determine the fundamental frequency of this signal. N = 8 and so $\Omega_0 = \frac{2\pi}{N} = \frac{2\pi}{8} = \frac{\pi}{4}$. This is a good resolution because of both the non-zero impulses fall on one of the bins which are separated by

$$\frac{\pi}{4}k$$

Let's compute the DFT of the signal as

$$\begin{aligned} X[n] &= \sum_{k=0}^{N-1} x[k] \; e^{-j \; n\frac{2\pi}{N} \; k} \\ &= \sum_{k=0}^{7} x[k] \; e^{-j \; n\frac{2\pi}{8} \; k} \\ &= \sum_{k=0}^{7} \; \delta[k] + .5\delta[k-6] \quad e^{-j \; n\frac{2\pi}{8} \; k} \end{aligned}$$

In the last step, the multiplication of the complex exponential by a delta function just means that we grab the value of the exponential at the location of the delta function which we do here for k = 0 and k = 6.

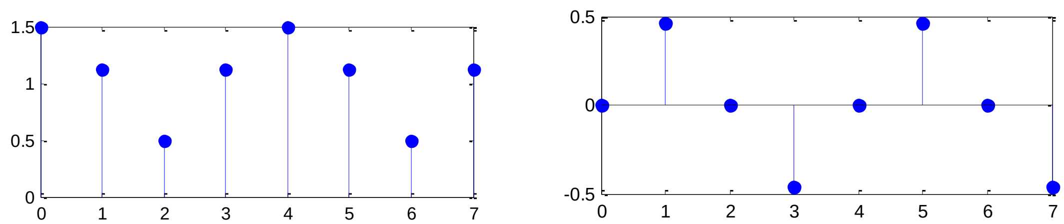
$$\begin{aligned} &\delta[k]e^{-j \; n\frac{2\pi}{8} \; k} + .5\delta[k-6] \, e^{-j \; n\frac{2\pi}{8} \; k} \\ &= e^{-j \; n\frac{2\pi}{8} \; k}\Big|_{k=0} + .5e^{-j \; n\frac{2\pi}{8} \; k}\Big|_{k=6} \\ &= 1 + .5e^{-j \; n\frac{2\pi}{8} \; 6} \end{aligned}$$

Now we finish the job, noting that index k is gone.

$$\begin{aligned} X[n] &= \sum_{k=0}^{7}\left(1 + .5e^{-j\left(n\frac{2\pi}{8}\right)6}\right) \\ &= 1 + .5e^{-j\left(n\frac{3\pi}{2}\right)} \end{aligned} \qquad (1.25)$$

Now take a careful look at this. Is this not just the amplitudes of the two impulse functions, one at 0 and the other at n = 6. We have to actually go through each k and determine the values of the DFT for each n. These are given by

$$X[\Omega] = 1 + .5e^{-j\ n\frac{3\pi}{2}}$$
$$X[0] = 1.5$$
$$X[1] = 1 + .5e^{-j\ \frac{3\pi}{2}} = 1.0 + 0.5\text{j}$$
$$X[2] = 1 + .5e^{-j\ 3\pi} = 0.5$$
$$X[3] = 1 + .5e^{-j\ 4.5\pi} = 1.0 - 0.5\text{j}$$
$$X[4] = 1 + .5e^{-j\ 6\pi} = 1.5$$
$$X[5] = 1 + .5e^{-j\ 7.5\pi} = 1.0 + 0.5\text{j}$$
$$X[6] = 1 + .5e^{-j\ 9\pi} = 0.5$$
$$X[7] = 1 + .5e^{-j\ 11.5\pi} = 1.0 - 0.5\text{j}$$




**(a) The magnitude of the spectrum**

**(b) The phase of the spectrum**

**Figure 5.24 – Spectrum of signal of example 5-1**

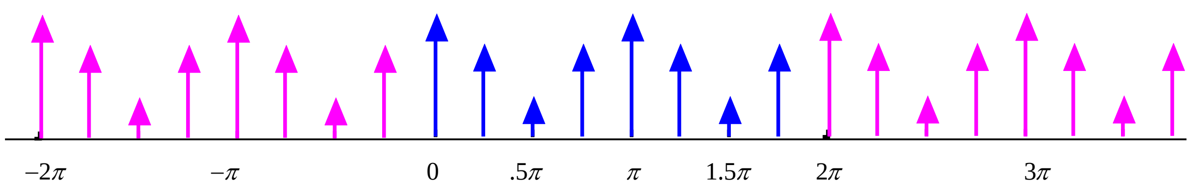
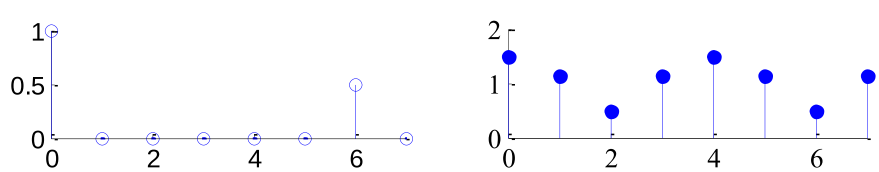



**Figure 5.25 – Magnitude spectrum of signal of example 5-1. Note it repeats what is shown in Fig. 5.18 (a)**

We did the analysis for 8 points. Now we will do exactly the same thing first for 7 points and then for 128 points by padding the end of the signal with a lot of zeros.



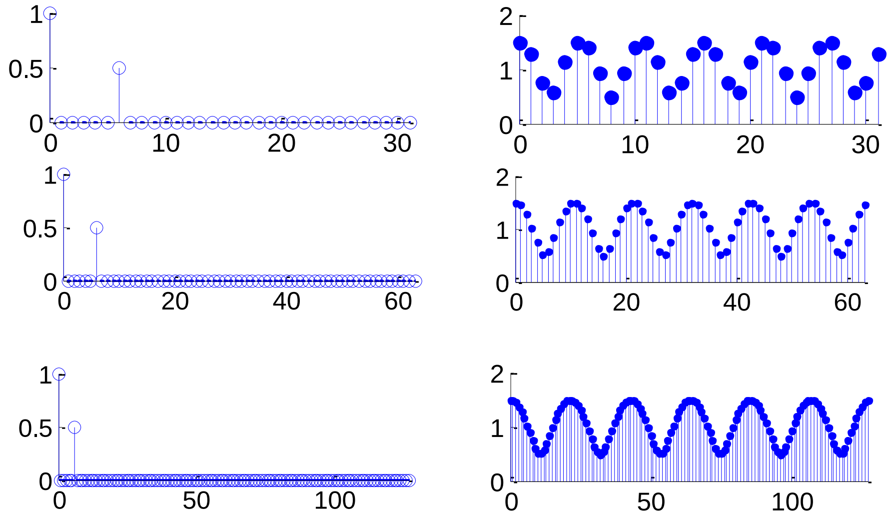



**Figure 5.26 DFT of a signal with different amounts of zero padding, size = 7, size = 8, size = 128**

So zero padding gives more points of interpolation. However, is the signal contained a higher frequency signal beyond the twice sampling frequency, zero-padding would do nothing to help discover it.

## Example 5-2

Find the DFT of sequence

$$x[k] = 1 + \cos^2\left(\frac{2\pi k}{8}\right) \quad k = 0,1,\cdots 7$$

Its easier to convert the sinusoid into complex exponentials. So we write

$$x[k] = 1 + \frac{1}{4}\left[e^{j\left(\frac{2\pi k}{8}\right)} + e^{-j\left(\frac{2\pi k}{8}\right)}\right]^2 = \frac{3}{2} + \frac{1}{4}e^{j\left(\frac{2\pi k}{4}\right)} + \frac{1}{4}e^{-j\left(\frac{2\pi k}{4}\right)}$$

The DFT coefficients are;

$$X[0] = \begin{cases} \frac{3}{2} & n = 0 \\ 2 & n = 2,6 \\ 0 & else \end{cases}$$

## Example 5-3

Find the sequence $y[k]$ that has for its DFT $Y[n] = e^{j2n\frac{2\pi}{12}} X[n]$, given that $x[k] = \delta(k) + 2\delta(n-6)$ and $X[k]$ is the 12 point DFT of $x[k]$.

The DFT of $x[k]$ is equal to

$$X[n] = 1 + 2\,e^{-j\left(6n\frac{2\pi}{12}\right)} = 1 + 2(-1)^n$$

Multiplication of this DFT by a complex exponential has the effect of delaying the signal in time domain. So $y[k]$ is a delayed form of $x[k]$. The delay operator is

$$e^{j\left(\frac{2\pi}{12}\right)6}$$

The delay is $k_0 = -6$, hence $y[k]$ is given by

$$y[k] = x[k+2]_{12} = 2\ \ \delta - 4\ \ +\ \ \delta - 10$$

## Example 5-4

Compute the DFT of the aperiodic signal given by the following four samples.

X[k] = [ 2 1 -3 4]

We write its DFT by observation like this.

$$\sum_{n=0}^{3} 2 + 1e^{-j\pi n/2} - 3e^{-j\pi n} + 4e^{-j3\pi n/2}$$

From here we can compute the first two terms as

$$\begin{aligned} X[0] &= 2 + 1 - 3 + 4 = 4 \\ X[1] &= 2 + e^{-\pi j/2} - 3e^{-j\pi} + 4e^{-j3\pi/2} \\ &= 2 + j + 3 - 4j \\ &= 5 - 3j \end{aligned}$$

Now we will use the matrix method for the solution to see if we get the same thing.

$$x[k] = [2\ 1 - 3\ 4]$$

We write the matrix equation using the DFT matrix for N = 4.

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

$$= \left[ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \right] \begin{bmatrix} 2 \\ 1 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5-3j \\ -6 \\ 5+3j \end{bmatrix}$$

Okay, we got the same thing!

Now let's compute the inverse DFT.

$$x[k] = W_N^{-1} X[n]$$

Here we will use the matrix we developed for the inverse DFT. Note that various sizes of the DFT can be pre-computed and saved in the computer hence making computations easy for any size B.

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 4 \\ 5+3j \\ -6 \\ 5-3j \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 4 \end{bmatrix}$$

## Example 5-5

A band-limited signal DFT is sampled at the rate of 100 Hz and is band-limited to 25 Hz. The number of samples is N = 50.

(a) What is spectral spacing?
(b) What frequency is represented by n = 20 and 40?

The DFT is basically a set of sampled points at N frequencies. So each bin of 50 samples (the size of the DFT) represents 2 Hz which is the spectral spacing. Bin 20 represents 40 Hz and bin 40 represents 80 Hz.

What if this signal was sampled at the rate of 25 Hz instead of 100? What would be the value of these bins then.

Sampling frequency of 25 Hz is less than the minimum rate (needs to be 50 Hz) and so we will get aliasing. The spectral resolution is 0.5 Hz. For bin 20, we get 10 Hz, and 20 Hz for bin 40. This is beyond the $\pm 12.5$ Hz range of the DFT, so at bin 20 we will get aliased adjacent spectrum value and the answer would not be correct.

## Truncation effects

Let's take this sampled cosine wave with period equal to N = 8.

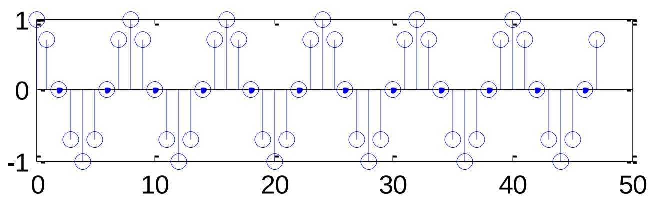



**Figure 5.27 – A sampled cosine wave with N = 8.**

The DFT of the signal should be pretty clear to you by now. It consists of just two impulses located at the positive and negative frequency of the signal centered at the origin.

To do the DFT of this signal, how many points do we need? This is a crucial point. The signal repeats every 8 samples. So we need at least 8, anything more is extra and may or may not be helpful, as we will see.

Assume that we are going to use some part of the above sampled cosine signal, which we obtain by multiplying this signal with a "window", a constant signal of a certain length. For example, if we wanted just one period, the window would be 8 samples long and would have an amplitude of 1.0 . So the N-sample segment of the signal is equivalent to the input signal times a window function of length N. We call the resulting "truncated" signal a windowed signal.

$$x_w(m) = w(m)x(m)$$

We define the window function as a rectangular function given by

$$w(m) = \begin{cases} 1 & 0 \le m \le N-1 \\ 0 & otherwise \end{cases} \qquad (1.26)$$

Where $-M \le n \le M$ is the time length of the window. This window function is all ones. So it is just a rectangular function over a certain number of samples, N. The truncated function $x_w(m)$ now becomes a signal of finite duration. We can also think of this signal as

$$x_m(m) = \begin{cases} x[n] & -M \le n \le M \\ 0 & otherwise \end{cases} \quad (1.27)$$

Multiplying these two signals together is same as doing convolution in frequency domain, so we can write the relationship of the spectrums as a convolution.

$$X_w(k) = W(k) * X(k) \qquad (1.28)$$

Our signal is a sinusoid and we know that its transforms looks something like this

$$X(\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0) \qquad (1.29)$$

The DTFT of the window function, which is a square pulse like function is a Diric (repeating sinc) function.

$$W(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)} \frac{\sin N\frac{\omega}{2}}{\sin\frac{\omega}{2}} \qquad (1.30)$$

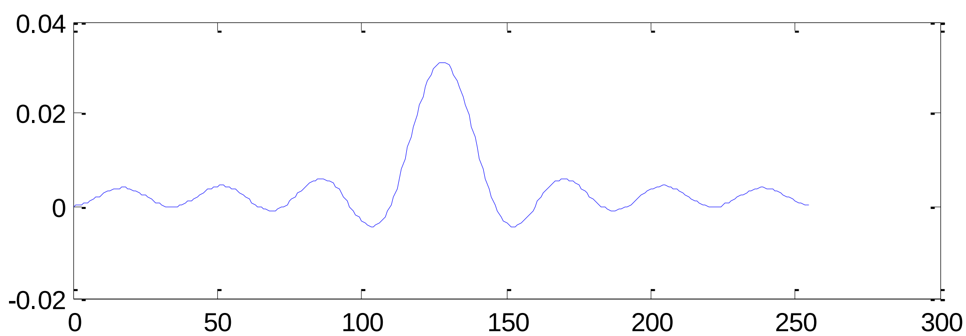



**Figure 5.28 – The spectrum of the rectangular window.**

This function is zero at all k integer values except at k = 0. It has a sinc-like shape and N-1 zero crossing. Now we compute the DTFT of the truncated signal by convolution of the two transforms.

$$\begin{aligned} X_m(\omega) &= X(\omega) * W(\omega) \\ &= \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0) \;\; * e^{-j\omega\left(\frac{N-1}{2}\right)} \frac{\sin N\frac{\omega}{2}}{\sin\frac{\omega}{2}} \end{aligned} \qquad (1.31)$$

If you look carefully you can figure out what this convolution will do. Procedures with delta functions are fairly easy. So the first delta function copies the sinc function at $-\omega$ and the second delta function creates another copy at $+\omega$. these are superimposed and create a smeared double peak spectrum as shown below.

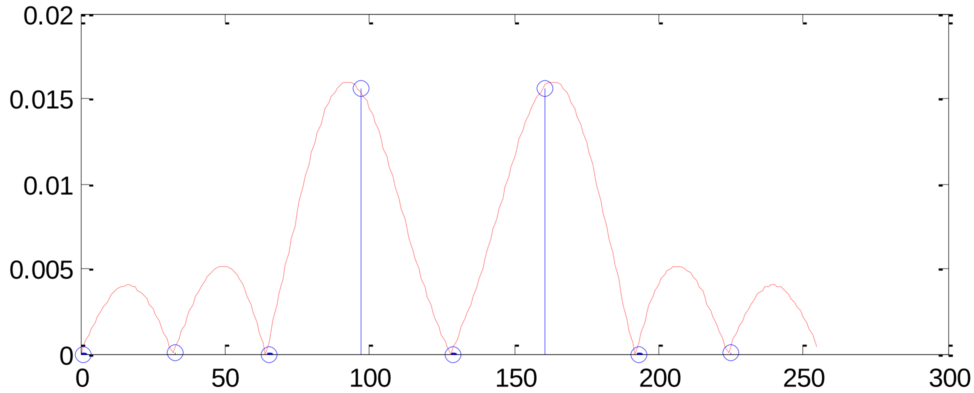



**Figure 5.29 – The DTFT and DFT of the windowed cosine.**

The DTFT of the window function is shown in the red line. It is a continuous function. The DFT however, is sampled values of this function, the 8 values in keeping with the original 8 sample data. If we did not plot the DTFT, all you would see are the two impulses and a bunch of zeros, exactly what you would expect as the DFT of a cosine signal.
The figure below shows what happens if the window is 16 samples long. Notice that the peaks are closer together. We sample this function at 16 places and again get a pretty decent looking DFT. It's a pure sinusoid, only the two impulses. Nothing else.

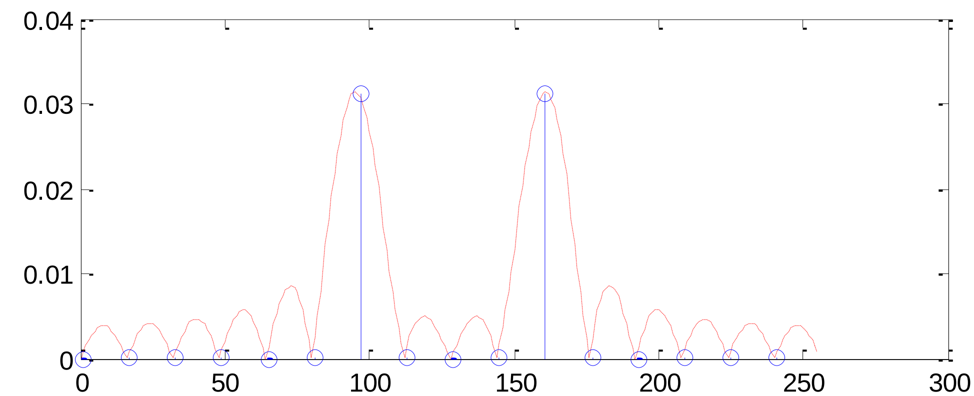



**Figure 5.30 – The DTFT and DFT of the windowed cosine, with N = 256**

But wait, let's try a window of size of 12 instead of 8 samples. We again do a DFT of size 256 points.

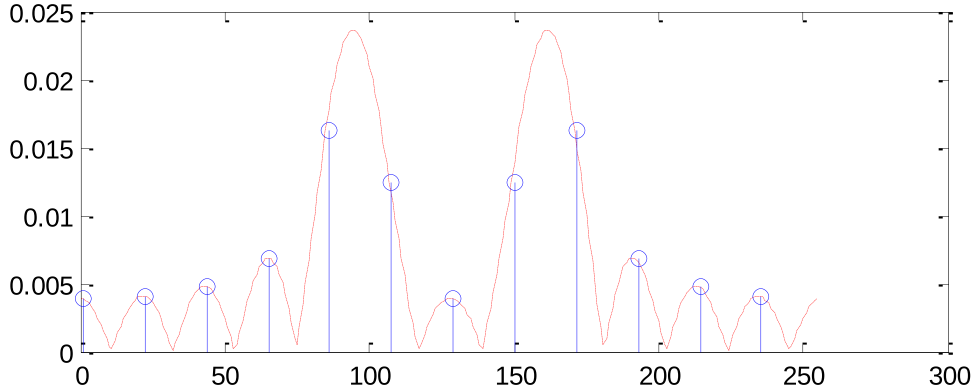



**Figure 5.31 – The DTFT and DFT of the windowed cosine, with window size = 12 samples, N = 256**

Oh no, whatever happened to the clean looking DFT of Fig. 5.27. The DFT points now seem to be at the peak of the side lobes not so in the main lobes. The larger number of points has ended up ruining the form of the DFT. This one does not have two impulses and a bunch of zeros as the previous two cases.  The underlying signal is still the sinusoid, it did not change, in fact, we gave the DFT more points, 12 vs. 8, so why are we getting this mish-mash?

Note that the DTFT has not changed, it looks the same as the one before, what has changed is the place where we sample the DTFT, which is the DFT. This effect is due to the window size and it has happened for one simple reason: the window size is not an integer multiple of the signal period, N. The period of the signal is 8 but we have used 12 points of the signal. Looking at the 12 points, in isolation, we see that if we string together the 12 samples of the cosine, the pieces would not join at the end. The signal has lost its periodicity and hence its DFT looks awry.

In reality, when we get a signal, we do not know the underlying signal period. We pick the length of signal on which to do the analysis based on numerical convenience. So rectangular windowing is a natural part of the problem of finite length signals. It not only truncates the signal but also introduces discontinuities at the edges in most cases. Unless of course, we get lucky and happen to pick a length that exactly matches an integer multiple of the signal period. This is of course not likely. So what can we do? Do we have to accept a messed up DFT? Well no, we can apply another window to the truncated signal, one that will reduce the effect of the truncation (in reality the rectangular window.). So is there some other window we could have used that would allow us to obtain a DFT that is closer to the real sinusoid underneath? Yes, there are very many windows. We will discuss these in the next chapter. Most of them work by forcing the edge points to zero, so the ends always match and we get a forced periodic signal

DFT is something is generally done in software using the matrix method shown. Using the general matrix method, it is possible to do a DFT of any size.

## Fast Fourier Transform (FFT)

The DFT although clear and easy to compute, requires a good many calculations. For each harmonic, we have N+1 multiplications and we do this N times, giving us $N^2 + N$ calculations. A 256 point DFT would require 65792 calculations. The algorithm for Fourier Transform existed for more than 200 years before it came into widespread use mostly because we could not cope with such large number of computations. The algorithm waited for the development of the microprocessor.

In 1948, Cooley and Tukey and came up with a computational breakthrough called the Fast Fourier Transform algorithm. This was an ingenious manipulation of the inherent symmetry of the calculations. It now allowed the computation of a N point DFT as a function only 2N instead of the $N^2$. So a 256 point DFT would only require 512 calculations, a huge improvement from 65792 calculations doing it the laborious way. The algorithm was quickly and widely adopted and is the basis of all modern signal processing.

Most DSP books spend a lot of time going through the mechanism of the FFT and how it is computed. Huge butterfly figures in our books help to confuse and confound us as to what is actually going on. Most of us just give up on it. Let me alleviate your guilt. Although extremely important in itself, the understanding of the mechanism of the FFT is not all that important. So, I am skipping over the details of its implementation.

The method has been programmed in all sort of software and we can safely skip it without impacting our understanding of the ***application*** of the DFT and the FFT. In Matlab, you can do a FFT of any size. However in most hardware, efficiency is important so the FFT algorithm (and there are several versions) is based on signal size which is of length N, such that N is a number generated by.., with k an integer. We will assume that the implementation is a black-box, into which we feed the data. As long you understand what this box is doing, understanding how efficiently it is doing its job is not that important. Only a decade or so ago, most hardware FFT size was limited to about 9000 points, with better hardware, the size and speed are not a big limitation any more.

The main thing one needs to know about the FFT is that it works only with sample numbers that are powers of 2, such as 16, 32, 64 etc. We cannot do a FFT on an arbitrary number of samples as we can with DFT or FFT in Matlab, which is used mostly for didactic and exploration purposes process. The FFT is a DFT with constraints on the number of input samples.

The other thing about the FFT process to know is that it allows zero-padding. Let's say we have 28 samples and we wish to do a DFT via the FFT, we can do two things, 1. we can do a 16 point FFT and discard the remaining 12 points or we can insert four zeros at the end so we have 32 points. Now we can do a 32 point FFT. The zero-padding provides us better resolution but does not provide any extra information. The frequency detected is still a function of the original N samples and not the zero-padded length, although the FFT does *look* a lot better. And looks do count.

Charan Langton

www.complextoreal.com

Like

Matlab Code

Program 1

```
%DFT of a cosine
clf
clear all
t = 0: .01: 3;
xc = 1*cos(2*pi*3*t);
figure(1)
plot(t, xc)
fs = 21;
k = 0: 20;
xn = 1*cos(2*pi*3*k/fs);
figure(2)
stem(k, xn)

xf = (1/30)*(fft(xn,21));
n = -10: 10
figure(3)
xf2 = abs(fftshift(xf,2))
stem(n, abs(fftshift(xf,2)), 'filled')
xfm2 = [ xf2 xf2 xf2];
figure(41)
n2 = -31: 31;
stem(n2, abs(xfm2), 'filled')
```

Program 2

```
%complex exponential
N = 32;
n = 0: N-1;
x = 0.3*exp(1i*2*pi*4*n/N)+ 0.5*exp(-1i*2*pi*6*n/N);
ND = 32;
xf = (1/32)*fft(x, ND);
figure(1)
n2 = -16: 15;
xfs = abs(fftshift(xf,2));
stem(n2, xfs, 'filled')
figure(2)
x = [x zeros(1,ND-N)]
stem(n2, real(x), 'filled')
xfm2 = [ xfs xfs xfs];
figure(3)
```

```
n2 = -48: 47;
stem(n2, abs(xfm2), 'filled')
axis([-48 48 .5 0])
```

Program 3

```
%Delayed pulse
N = 12;
n = 0: N-1;
set(gca,'box','off')
x = [1 0 0 0 0 0 0 0 0 0 0 0 ];
ND = 12;
xf = fft(x, ND)/N;
xfs =fftshift(xf)
figure(1)
n2 = 0: length(xf)-1;
stem(n2, abs(xfs), 'filled')
figure(2)
x = [x zeros(1,ND-N)]
stem(n2, real(x), 'filled')
```

Program 4

```
%modulation
clf
clear all
t = 0: .01: 3;
xc = 1*cos(2*pi*1*t)- .7*cos(2*pi*4*t);
figure(1)
plot(t, xc)
fs = 50;
k = 0: fs-1;
xn = 1*cos(2*pi*1*k/fs)- .7*cos(2*pi*4*k/fs)
figure(2)
stem(k, xn)

xm = cos(2*pi*10*k/fs).*xn
%xm = x;
xf = (1/fs)*(fft(xn,fs));
xfm = (1/fs)*(fft(xm,fs));
figure(3)
t = 0:  length(xm)-1;
stem(t, xm, 'filled')
n = -fs/2: fs/2-1
figure(4)
stem(n, abs(fftshift(xf,2)), 'filled')
figure(5)
stem(n, abs(fftshift(xfm,2)), 'filled')
xfm3 = abs(fftshift(xfm,2));
xfm2 = [ xfm3 xfm3 xfm3];
figure(6)
n2 = -3*fs/2: 3*fs/2-1;
stem(n2, abs(xfm2), 'filled')
```

Program 5

```
%square pulse
N = 32;
x = ones(1,N);
n = 0: length(x)-1;
ND = 32;
xf = fft(x, ND)
xfs = fftshift(xf)
figure(1)
n2 = 0: length(xf)-1;
stem(n2, abs(xfs), 'filled')
figure(2)
x = [x zeros(1,ND-N)]
stem(n2, x, 'filled')
```

Program 6

```
T = 0.25;
t = [0:T:(20-T)]';
N = length(t);
shift = 40*ones(N,1);
y = sin (pi*t - pi*T*shift) ./ (pi*t - pi*T*shift);
y(41)=1;
figure(2)
subplot(1,2,1);
plot(t,y)
%title('x(t), T=0.25 & P=20 sec');
xlabel('TIME, seconds'); ylabel('AMPLITUDE')
Y = T*fft(y);
MY = abs(Y);
fd = 1/(N*T);
f = [0:fd:(N/2-1)*fd]';
f2 = (-(N/2): (N/2)-1);
%mag2 = fftshift(MY,2);
mag = MY(1:(N)/2); magt = mag(end:-1:1);
mag2 = cat(1, magt, mag);
subplot(1,2,2);
plot(f2, mag2)
%stem(f2,mag2, 'marker', 'none')
%title('MAGNITUDE SPECTRUM');
xlabel('FREQUENCY, HZ');
ylabel('MAGNITUDE')
axis([-30 30 0 1.5])
```

Program 7

```
%DTFT and DFT of a window
clf
set(gca,'box','off')
S = 12;% size of window
n = 0: 47; %for showing time domain signal
x = cos(2*pi*n/8); %the signal with N = 8
stem(n, x)
figure (1)
win = ones(1,S) % window of size S, al ones
%win = [.1 .3 .5 1 1 .5 .3 .1]
```

```
N = S;
xw =(1/N)* fft(x, N);
s = 0: S-1;
N2 = 256;
winw = (1/N2)* fft(win, N2); %FFT of the window
stem(s, xw) %fft of signal
figure(2)
n2 = 0:  255;
plot(n2, (fftshift(winw))); %fft of window
figure(3)
sd = x(1:S)
mult = sd.*win(1:S); %multiply two signals
multw = (1/N2)* fft(mult, N2); %fft of both
plot(n2, (fftshift(abs(multw))), '-.r');
hold on
A = (fftshift(abs(multw)));
tvar = 256/S;
sd =  A(1 : 256/S : end);
n5 = 1: 256/S: 256;
stem(n5, sd)
```