Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 458701, 8 pages
doi:10.1155/2012/458701

Research Article

Cubic B-Spline Collocation Method for One-Dimensional Heat and Advection-Diffusion Equations

Joan Goh, Ahmad Abd. Majid, and Ahmad Izani Md. Ismail

Pusat Pengajian Sains Matematik, Universiti Sains Malaysia, 11800 Pulau Pinang, Malaysia

Correspondence should be addressed to Joan Goh, joangoh.usm@gmail.com

Received 12 January 2012; Accepted 9 March 2012

Academic Editor: Ram N. Mohapatra

Copyright © 2012 Joan Goh et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Numerical solutions of one-dimensional heat and advection-diffusion equations are obtained by collocation method based on cubic *B*-spline. Usual finite difference scheme is used for time and space integrations. Cubic *B*-spline is applied as interpolation function. The stability analysis of the scheme is examined by the Von Neumann approach. The efficiency of the method is illustrated by some test problems. The numerical results are found to be in good agreement with the exact solution.

1. Introduction

The combination of advection and diffusion is important for mass transport in fluids. It is well known that the volumetric concentration of a pollutant, $u(x,t)$, at a point x $(a \le x \le b)$ in a one-dimensional moving fluid with a constant speed β and diffusion coefficient α in x direction at time t $(t \ge 0)$ is given by the one-dimensional advection-diffusion equation, which is in the form

$$u_t + \beta u_x = \alpha u_{xx}, \quad a \le x \le b,\ t \ge 0, \tag{1.1}$$

subject to the initial condition

$$u(x,0) = \phi(x), \quad x \in [a,b], \tag{1.2}$$

and the boundary conditions

$$u(a,t) = g_0(t), \tag{1.3a}$$

$$u(b,t) = g_1(t), \qquad t \in [0,\mathrm{T}], \tag{1.3b}$$

where g_0 and g_1 are assumed to be smooth functions. It should be noted that, when $\beta = 0$, the advection-diffusion equation will be reduced to the one-dimensional heat equation in the case of thermal diffusion.

Advection-diffusion equation arises very frequently in transferring mass, heat, energy, and vorticity in chemistry and engineering. Thus, it has been of interest to many authors. A third-degree B-spline function has been used by Caglar et al. for solving one-dimensional heat equation with a nonlocal initial condition [1]. Mohebbi and Dehghan [2] have presented a fourth-order compact finite difference approximation and cubic C^1-spline collocation method for the solution with fourth-order accuracy in both space and time variables, $O(h^4, k^4)$. In [3], Dag and Saka concluded that collocation scheme is easy to implement compared to other numerical methods with giving a better result.

In this paper, a combination of finite difference approach and cubic B-spline method would be considered to solve the one-dimensional heat and advection-diffusion equation. Forward finite difference approach would be used for discretizing the derivative of time, while cubic B-spline would be applied to interpolate the solutions at time t. Von Neumann approach would be used to prove the unconditionally stable property of the method. Finally, the approximated solutions and the numerical errors would be presented to demonstrate the efficiency of the method.

2. Collocation Method

In this paper, cubic B-splines are used to construct the numerical solutions to solve the problems. Consider a partition of $[a,b]$ that is equally divided by knots x_i into n subinterval $[x_i, x_{i+1}]$, where $i = 0, 1, \ldots, n-1$ such that $a = x_0 < x_1 < \cdots < x_n = b$. Hence, an approximation $U(x,t)$ to the exact solution $u(x,t)$ based on collocation approach can be expressed as [4]

$$U(x,t) = \sum_{i=-3}^{n-1} C_i(t) B_{3,i}(x), \tag{2.1}$$

where $C_i(t)$ are time-dependent quantities to be determined and $B_{3,i}(x)$ are third-degree B-spline functions which are defined by the relationship [5]

$$B_{3,i}(x) = \frac{1}{6h^3} \begin{cases} (x - x_i)^3, & x \in [x_i, x_{i+1}], \\ h^3 + 3h^2(x - x_{i+1}) + 3h(x - x_{i+1})^2 - 3(x - x_{i+1})^3, & x \in [x_{i+1}, x_{i+2}], \\ h^3 + 3h^2(x_{i+3} - x) + 3h(x_{i+3} - x)^2 - 3(x_{i+3} - x)^3, & x \in [x_{i+2}, x_{i+3}], \\ (x_{i+4} - x)^3, & x \in [x_{i+3}, x_{i+4}], \end{cases} \tag{2.2}$$

Table 1: Values of B_i, B_i', and B_i''.

	x_i	x_{i+1}	x_{i+2}	x_{i+3}	x_{i+4}
B_i	0	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{1}{6}$	0
B_i'	0	$\frac{1}{2h}$	0	$-\frac{1}{2h}$	0
B_i''	0	$\frac{1}{h^2}$	$-\frac{2}{h^2}$	$\frac{1}{h^2}$	0

where $h = (b-a)/n$. The approximation U_i^k at the point (x_i, t_k) over the subinterval $[x_i, x_{i+1}]$ can be simplified into

$$U_i^k = \sum_{j=i-3}^{i-1} C_j^k B_{3,j}(x), \tag{2.3}$$

where $i = 0, 1, \ldots, n$. To obtain the approximations of the solutions, the values of $B_{3,i}(x)$ and its derivatives at the knots are needed. Since the values vanish at all other knots, they are omitted from Table 1.

The approximations of the solutions of (1.1) at t_{j+1}th time level can be considered by [6]:

$$(U_t)_i^k + (1-\theta) f_i^k + \theta f_i^{k+1} = 0, \tag{2.4}$$

where $f_i^k = \beta (U_x)_i^k - \alpha (U_{xx})_i^k$ and the superscripts k and $k+1$ are successive time levels, $k = 0, 1, 2, \ldots$. Now, discretizing the time derivative by a first-order accurate forward difference scheme and rearranging the equation, we obtain

$$U_i^{k+1} + \theta \Delta t f_i^{k+1} = U_i^k - (1-\theta) \Delta t f_i^k, \tag{2.5}$$

where Δt is the time step. Note that the system becomes an explicit scheme when $\theta = 0$, a fully implicit scheme when $\theta = 1$, and a mixed scheme of Crank-Nicolson when $\theta = 0.5$ [6]. Here, Crank-Nicolson approach is used. Hence, (2.5) takes the form

$$U_i^{k+1} + 0.5 \Delta t f_i^{k+1} = U_i^k - 0.5 \Delta t f_i^k \tag{2.6}$$

for $i = 0, 1, \ldots, n$ at each level of time. Therefore, a linear system of order $(n+1)$ is obtained with $(n+3)$ unknowns $\mathbf{C}^{k+1} = (C_{-3}^{k+1}, C_{-2}^{k+1}, \ldots, C_{n-1}^{k+1})$ at the level time $t = t_{k+1}$. To solve the system, two additional linear equations are needed. Thus, (2.3) is applied to the boundary conditions (1.3a)-(1.3b) to obtain

$$U_0^{k+1} = g_0(t_{k+1}), \tag{2.7a}$$

$$U_n^{k+1} = g_1(t_{k+1}). \tag{2.7b}$$

Equations (2.6), (2.7a)-(2.7b) lead to a $(n+3)\times(n+3)$ tridiagonal matrix system, which can be solved by the Thomas algorithm. Once the initial vector $\mathbf{C}^0$ has been calculated from the initial conditions [7], the approximation solution U_i^{k+1} at each level of time t_{k+1} can be determined by the vector $\mathbf{C}^{k+1}$ which is found by solving the recurrence relation repeatedly.

The initial vector $\mathbf{C}^0$ can be obtained from the initial condition and boundary values of the derivatives of the initial condition as the following expressions [6]:

(1) $(U_i^0)_x = \phi'(x_i)$, $i = 0$,

(2) $U_i^0 = \phi(x_i)$, $i = 0, 1, \ldots, n$,

(3) $(U_i^0)_x = \phi'(x_i)$, $i = n$.

This yields a $(n+3)\times(n+3)$ matrix system where the solution can be found by Thomas algorithms.

3. Stability Analysis

Von Neumann stability method is applied for analyzing the stability of the proposed scheme. This type of stability analysis had been used by many researchers [3, 8–10]. Consider the trial solution (one Fourier mode out of the full solution) at a given point x_m

$$C_m^k = \delta^k \exp(i\eta m h), \tag{3.1}$$

where $i = \sqrt{-1}$ and η is the mode number. By substituting (2.3) into (2.5) and rearranging the equation, it leads to

$$p_1 C_{m-3}^{k+1} + p_2 C_{m-2}^{k+1} + p_3 C_{m-1}^{k+1} = p_4 C_{m-3}^{k} + p_5 C_{m-2}^{k} + p_6 C_{m-1}^{k}, \tag{3.2}$$

where

$$\begin{aligned}
p_1 &= \frac{1}{6} + \frac{\theta\Delta t\beta}{2h} - \frac{\theta\Delta t\alpha}{h^2},\\
p_2 &= \frac{4}{6} + \frac{2\theta\Delta t\alpha}{h^2},\\
p_3 &= \frac{1}{6} - \frac{\theta\Delta t\beta}{2h} - \frac{\theta\Delta t\alpha}{h^2},\\
p_4 &= \frac{1}{6} - \frac{(1-\theta)\Delta t\beta}{2h} + \frac{(1-\theta)\Delta t\alpha}{h^2},\\
p_5 &= \frac{4}{6} - \frac{2(1-\theta)\Delta t\alpha}{h^2},\\
p_6 &= \frac{1}{6} + \frac{(1-\theta)\Delta t\beta}{2h} + \frac{(1-\theta)\Delta t\alpha}{h^2}.
\end{aligned} \tag{3.3}$$

Inserting the trial solution (3.1) into (3.2) and simplifying the equation give

$$\delta = \frac{A + iB}{C + iD}, \tag{3.4}$$

where

$$\begin{aligned}A &= \frac{1}{3}(2+\cos\eta h) - \frac{2(1-\theta)\Delta t\alpha}{h^2}(1-\cos\eta h),\\ B &= \frac{(1-\theta)\Delta t\beta}{h}\sin\eta h,\\ C &= \frac{1}{3}(2+\cos\eta h) + \frac{2\theta\Delta t\alpha}{h^2}(1-\cos\eta h),\\ D &= -\frac{\theta\Delta t\beta}{h}\sin\eta h.\end{aligned} \tag{3.5}$$

If the amplification factor $|\delta| \leq 1$, then the proposed scheme is stable, or else the approximations grow in amplitude and become unstable. As $\theta = 0.5$ is used in the proposed scheme, thus substitute the θ value into (3.4) and after some algebraic manipulation, it can be noticed that

$$a^2 + b^2 \leq c^2 + d^2 \quad \text{or} \quad |\delta|^2 = \frac{a^2+b^2}{c^2+d^2} \leq 1. \tag{3.6}$$

Thus, this had been proved that the presented numerical scheme for the advection-diffusion equation is unconditionally stable.

4. Numerical Results

4.1. Problem 1

Suppose the heat equation is as follows [11]:

$$u_t = u_{xx}, \quad 0 < x < 1,\ t > 0, \tag{4.1}$$

with initial and boundary conditions

$$u(x,0) = \sin(\pi x), \qquad u(0,t) = u(1,t) = 0. \tag{4.2}$$

The exact solution is known to be $u(x,t) = \exp(-\pi^2 t)\sin(\pi x)$. This problem is tested by different values of h and Δt to show the capability of the presented method for solving one-dimensional heat equation. The final time is chosen as $T = 1$. The maximum absolute errors of the method are compared with those obtained by Crank-Nicolson (CN) scheme and compact boundary value method (CBVM) in [11]. The numerical errors are presented in Table 2. Although the fourth-order compact boundary value method gives a much more better solution, the present method is still well compared with the Crank-Nicolson scheme.

Table 2: Maximum absolute error obtained for problem 1.

$h = \Delta t$	CN [11]	CBVM [11]	Present method
1/5	1.1×10^{-1}	2.8×10^{-2}	1.4145×10^{-1}
1/10	3.0×10^{-2}	3.8×10^{-3}	3.7195×10^{-2}
1/20	6.9×10^{-3}	2.7×10^{-4}	8.4588×10^{-3}
1/40	1.7×10^{-3}	1.3×10^{-5}	2.0698×10^{-3}
1/80	4.2×10^{-4}	5.1×10^{-7}	5.1473×10^{-4}

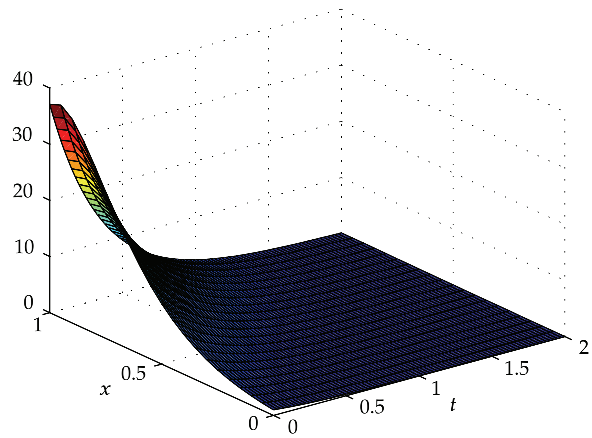

Figure 1: Spatial-time approximations for problem 2 with $h = 0.05$ and $\Delta t = 0.5h$.

4.2. Problem 2

Consider the advection-diffusion equation in (1.1) with $\beta = 1$, $\alpha = 0.1$, as follows [2]:

$$u_t + u_x = 0.1u_{xx}, \quad 0 < x < 1,\ t > 0, \tag{4.3}$$

where the initial condition is given by

$$u(x,0) = \exp(5x)\left[\cos\left(\frac{\pi}{2}x\right) + 0.25\sin\left(\frac{\pi}{2}x\right)\right] \tag{4.4}$$

and the exact solution

$$u(x,t) = \exp\left(5\left(x - \frac{t}{2}\right)\right)\exp\left(-\frac{\pi^2}{40}t\right)\left[\cos\left(\frac{\pi}{2}x\right) + 0.25\sin\left(\frac{\pi}{2}x\right)\right]. \tag{4.5}$$

The boundary conditions at $x = 0$ and $x = 1$ can be obtained from the exact solution. Table 3 shows the absolute errors of the approximations at the grid points when $T = 2$. It can be noticed that the present method is comparable with cubic C^1-spline collocation method. The approximations of the solutions over a time period $t \in [0, 2]$ along x is depicted in Figure 1.

Table 3: Absolute error obtained with $\Delta t = 2h$ at $T = 2$ for problem 2.

Grid point	$h = 0.02$		$h = 0.01$	
	C^1-spline [2]	Present method	C^1-spline [2]	Present method
0.1	7.1744×10^{-6}	8.2212×10^{-6}	1.8035×10^{-6}	2.0556×10^{-6}
0.2	1.1019×10^{-5}	2.2566×10^{-5}	2.7685×10^{-6}	5.6432×10^{-6}
0.3	1.6596×10^{-5}	4.5188×10^{-5}	4.1679×10^{-6}	1.1298×10^{-5}
0.4	2.4579×10^{-5}	7.7748×10^{-5}	6.1705×10^{-6}	1.9435×10^{-5}
0.5	3.5871×10^{-5}	1.2011×10^{-4}	9.0026×10^{-6}	3.0020×10^{-5}
0.6	5.1637×10^{-5}	1.6809×10^{-4}	1.2955×10^{-5}	4.2001×10^{-5}
0.7	7.3208×10^{-5}	2.1002×10^{-4}	1.8360×10^{-5}	5.2464×10^{-5}
0.8	1.0163×10^{-4}	2.2264×10^{-4}	2.5476×10^{-5}	5.5602×10^{-5}
0.9	1.3624×10^{-4}	1.6833×10^{-4}	3.4134×10^{-5}	4.2039×10^{-5}

5. Conclusions

A numerical method based on collocation of cubic B-spline had been described in the previous section for solving one-dimensional heat and advection-diffusion equations. A finite difference scheme had been used for discretizing time derivatives and cubic B-spline for interpolating the solutions at each time level. From the test problems, the obtained results show that the presented method is capable for solving one-dimensional heat and advection-diffusion equations accurately with a promised stability.

Acknowledgment

The authors would like to acknowledge with thanks the financial support from Malaysian Government in the form of Fundamental Research Grant Scheme (FRGS) of number 203/PMATHS/6711150.

References

[1] H. Caglar, M. Ozer, and N. Caglar, "The numerical solution of the one-dimensional heat equation by using third degree B-spline functions," *Chaos, Solitons & Fractals*, vol. 38, no. 4, pp. 1197–1201, 2008.

[2] A. Mohebbi and M. Dehghan, "High-order compact solution of the one-dimensional heat and advection-diffusion equations," *Applied Mathematical Modelling*, vol. 34, no. 10, pp. 3071–3084, 2010.

[3] I. Dag and B. Saka, "A cubic B-spline collocation method for the EW equation," *Mathematical & Computational Applications*, vol. 9, no. 3, pp. 381–392, 2004.

[4] P. M. Prenter, *Splines and Variational Methods*, Wiley Classics Library, John Wiley & Sons, New York, NY, USA, 1989.

[5] C. de Boor, *A Practical Guide to Splines*, vol. 27 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1978.

[6] I. Dag, D. Irk, and B. Saka, "A numerical solution of the Burgers' equation using cubic B-splines," *Applied Mathematics and Computation*, vol. 163, no. 1, pp. 199–211, 2005.

[7] J. Goh, A. A. Majid, and A. I. M. Ismail, "A numerical solution of one-dimensional wave equation using cubic B-splines," in *Proceedings of the National Conference of Application Science and Mathematics (SKASM '10)*, pp. 117–122, December 2010.

[8] H. Fazal, I. Siraj, and I. A. Tirmizi, "A numerical technique for solution of the MRLW equation using quartic B-splines," *Applied Mathematical Modelling*, vol. 34, no. 12, pp. 4151–4160, 2010.

[9] S. M. Hassan and D. G. Alamery, "B-splines collocation algorithms for solving numerically the MRLW equation," *International Journal of Nonlinear Science*, vol. 8, no. 2, pp. 131–140, 2009.

[10] A. K. Khalifa, K. R. Raslan, and H. M. Alzubaidi, "A collocation method with cubic B-splines for solving the MRLW equation," *Journal of Computational and Applied Mathematics*, vol. 212, no. 2, pp. 406–418, 2008.

[11] H. Sun and J. Zhang, "A high-order compact boundary value method for solving one-dimensional heat equations," *Numerical Methods for Partial Differential Equations*, vol. 19, no. 6, pp. 846–857, 2003.


Advances in Operations Research
Advances in Decision Sciences
Journal of Applied Mathematics
Algebra
Journal of Probability and Statistics
The Scientific World Journal
International Journal of Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of Combinatorics
Advances in Mathematical Physics
Journal of Complex Analysis
Journal of Mathematics
Mathematical Problems in Engineering
Abstract and Applied Analysis
Discrete Dynamics in Nature and Society
International Journal of Mathematics and Mathematical Sciences
Journal of Discrete Mathematics
Journal of Function Spaces
International Journal of Stochastic Analysis
Journal of Optimization