Math 241: Multivariable calculus, Lecture 29

Curl and Div,
Section 16.5

go.illinois.edu/math241fa17

wednesday, November 8th, 2017

Review Green's Theorem

Green's Theorem. Suppose D is a region bounded by a simple closed curve and $\mathbf{F} = \langle P, Q \rangle$ is a vector field with continuous second order partial derivatives. Then integral over the boundary is related to double integral over the interior:

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint\limits_{D} (Q_x - P_y)\, dA$$

Green's Theorem: General form

Green's Theorem: General form

If D is bounded by several curve $C_1, \ldots, C_n$, we write ∂D for $C_1 \cup \ldots \cup C_n$ where

Green's Theorem: General form

If D is bounded by several curve $C_1, \ldots, C_n$, we write ∂D for $C_1 \cup \ldots \cup C_n$ where

- The "outer curve" is given the counterclockwise orientation

Green's Theorem: General form

If D is bounded by several curve $C_1, \ldots, C_n$, we write ∂D for $C_1 \cup \ldots \cup C_n$ where

- The "outer curve" is given the counterclockwise orientation
- All others given clockwise orientation.

Green's Theorem: General form

If D is bounded by several curve $C_1, \ldots, C_n$, we write ∂D for $C_1 \cup \ldots \cup C_n$ where

- The "outer curve" is given the counterclockwise orientation
- All others given clockwise orientation.

Green's Theorem. Suppose D is a region bounded by a finite set of simple closed curves and $\mathbf{F} = \langle P, Q \rangle$ is a vector field with continuous second order partial derivatives. Then

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D (Q_x - P_y)\, dA$$

Review Conservative Vector Fields

Theorem 0.1

A vector field is conservative ($\mathbf{F} = \nabla(f)$) *if and only if* $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ *for any simple closed loop.*

Review Conservative Vector Fields

Theorem 0.1

A vector field is conservative ($\mathbf{F} = \nabla(f)$) if and only if $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any simple closed loop.

Theorem 0.2

$\mathbf{F} = \langle P, Q \rangle \colon U \to \mathbb{R}^2$ is a vector field defined on a connected, simply connected (no holes) open set U, and suppose P and Q have continuous partial derivatives. If $P_y = Q_x$, then $\mathbf{F}$ is conservative and so $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any simple closed loop C in U.

Review Conservative Vector Fields

Theorem 0.1

A vector field is conservative ($\mathbf{F} = \nabla(f)$) *if and only if* $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ *for any simple closed loop.*

Theorem 0.2

$\mathbf{F} = \langle P, Q \rangle \colon U \to \mathbb{R}^2$ *is a vector field defined on a connected, simply connected (no holes) open set* U*, and suppose* P *and* Q *have continuous partial derivatives. If* $P_y = Q_x$*, then* $\mathbf{F}$ *is conservative and so* $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ *for any simple closed loop* C *in* U*.*

Review Conservative Vector Fields

Theorem 0.1

A vector field is conservative ($\mathbf{F} = \nabla(f)$) if and only if $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any simple closed loop.

Theorem 0.2

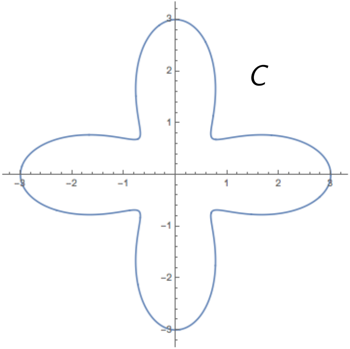
$\mathbf{F} = \langle P, Q \rangle : U \to \mathbb{R}^2$ is a vector field defined on a connected, simply connected (no holes) open set U, and suppose P and Q have continuous partial derivatives. If $P_y = Q_x$, then $\mathbf{F}$ is conservative and so $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any simple closed loop C in U.

Proof: Use Green's thm!

Problem: Let C be the oriented closed curve parameterized by $\mathbf{r}(t) = \langle (2+\cos(4t))\cos(t), (2+\sin(4t))\sin(t)\rangle$ for $t \in [0, 2\pi]$, and let

$$\mathbf{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$.



Solution Hints Problem of the day.

$$\mathbf{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle = \langle P, Q \rangle.$$

Solution Hints Problem of the day.

$$\mathbf{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle = \langle P, Q \rangle.$$

- Is $Q_x = P_y$?

Solution Hints Problem of the day.

$$\mathbf{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle = \langle P, Q \rangle.$$

- Is $Q_x = P_y$?
- Is $\mathbf{F}$ conservative?

Solution Hints Problem of the day.

$$\mathbf{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle = \langle P, Q \rangle.$$

- Is $Q_x = P_y$?
- Is $\mathbf{F}$ conservative?
- Is $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$?

Solution Hints Problem of the day.

$\mathbf{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle = \langle P, Q \rangle$.

- Is $Q_x = P_y$?
- Is $\mathbf{F}$ conservative?
- Is $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$?
- Let C be the flower loop in the problem, and let $\tilde{C}$ be the unit circle, and R the region between C and $\tilde{C}$. What is $\iint_R (Q_x - P_y)\, dA$?

Solution Hints Problem of the day.

$\mathbf{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle = \langle P, Q \rangle.$

- Is $Q_x = P_y$?
- Is $\mathbf{F}$ conservative?
- Is $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$?
- Let C be the flower loop in the problem, and let $\tilde{C}$ be the unit circle, and R the region between C and $\tilde{C}$. What is $\iint_R (Q_x - P_y)\, dA$?
- What is the relation between $\oint_C \mathbf{F} \cdot d\mathbf{r}$ and $\oint_{\tilde{C}} \mathbf{F} \cdot d\mathbf{r}$?

Solution Hints Problem of the day.

$\mathbf{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle = \langle P, Q \rangle$.

- Is $Q_x = P_y$?
- Is $\mathbf{F}$ conservative?
- Is $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$?
- Let C be the flower loop in the problem, and let $\tilde{C}$ be the unit circle, and R the region between C and $\tilde{C}$. What is $\iint_R (Q_x - P_y)\, dA$?
- What is the relation between $\oint_C \mathbf{F} \cdot d\mathbf{r}$ and $\oint_{\tilde{C}} \mathbf{F} \cdot d\mathbf{r}$?
- Calculate $\oint_{\tilde{C}} \mathbf{F} \cdot d\mathbf{r}$.

A second problem...

Calculate the integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where C is the arc of the parabola $y = 1 - x^2$ from $(-1, 0)$ to $(1, 0)$ and

$$\mathbf{F} = \langle e^x + y, e^y - x \rangle.$$

Solution Hints Second Problem

$\mathbf{F} = \langle e^x + y, e^y - x \rangle.$

Solution Hints Second Problem

$\mathbf{F} = \langle e^x + y, e^y - x \rangle$. Idea: Close the loop!

Solution Hints Second Problem

$\mathbf{F} = \langle e^x + y, e^y - x \rangle$. Idea: Close the loop!

- Let $\tilde{C}$ be the curve along the x-axis from $(1, 0)$ to $(-1, 0)$. What is $\int_{\tilde{C}} \mathbf{F} \cdot d\mathbf{r}$?

Solution Hints Second Problem

$\mathbf{F} = \langle e^x + y, e^y - x \rangle$. Idea: Close the loop!

- Let $\tilde{C}$ be the curve along the x-axis from $(1, 0)$ to $(-1, 0)$. What is $\int_{\tilde{C}} \mathbf{F} \cdot d\mathbf{r}$?
- Let R be the region bounded by C and $\tilde{C}$. What is $\iint_R (Q_x - P_y)\, dA$?

Solution Hints Second Problem

$\mathbf{F} = \langle e^x + y, e^y - x \rangle$. Idea: Close the loop!

- Let $\tilde{C}$ be the curve along the x-axis from $(1, 0)$ to $(-1, 0)$. What is $\int_{\tilde{C}} \mathbf{F} \cdot d\mathbf{r}$?
- Let R be the region bounded by C and $\tilde{C}$. What is $\iint_R (Q_x - P_y)\, dA$?
- What is $\int_C \mathbf{F} \cdot d\mathbf{r}$?

Operations on a vector field in $\mathbb{R}^3$: curl

Question: Where does in Green's Thm the funny $Q_x - P_y$ come from?

Operations on a vector field in $\mathbb{R}^3$: curl

Question: Where does in Green's Thm the funny $Q_x - P_y$ come from?

Question: What about conservative vector fields in $\mathbb{R}^3$?

Operations on a vector field in $\mathbb{R}^3$: curl

Question: Where does in Green's Thm the funny $Q_x - P_y$ come from?

Question: What about conservative vector fields in $\mathbb{R}^3$?

Suppose $\mathbf{F} = \langle P, Q, R \rangle$.

Operations on a vector field in $\mathbb{R}^3$: curl

Question: Where does in Green's Thm the funny $Q_x - P_y$ come from?
Question: What about conservative vector fields in $\mathbb{R}^3$?
Suppose $\mathbf{F} = \langle P, Q, R \rangle$. Define the *curl* of $\mathbf{F}$ by

$$\mathrm{curl}(\mathbf{F}) = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle.$$

Operations on a vector field in $\mathbb{R}^3$: curl

Question: Where does in Green's Thm the funny $Q_x - P_y$ come from?
Question: What about conservative vector fields in $\mathbb{R}^3$?
Suppose $\mathbf{F} = \langle P, Q, R \rangle$. Define the *curl* of **F** by

$$\mathrm{curl}(\mathbf{F}) = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle.$$

Write "$\mathrm{curl} = \nabla \times$":

Operations on a vector field in $\mathbb{R}^3$: curl

Question: Where does in Green's Thm the funny $Q_x - P_y$ come from?
Question: What about conservative vector fields in $\mathbb{R}^3$?
Suppose $\mathbf{F} = \langle P, Q, R \rangle$. Define the *curl* of $\mathbf{F}$ by

$$\mathrm{curl}(\mathbf{F}) = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle.$$

Write "$\mathrm{curl} = \nabla \times$":

$$\mathrm{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Operations on a vector field in $\mathbb{R}^3$: curl

Question: Where does in Green's Thm the funny $Q_x - P_y$ come from?
Question: What about conservative vector fields in $\mathbb{R}^3$?
Suppose $\mathbf{F} = \langle P, Q, R \rangle$. Define the *curl* of $\mathbf{F}$ by

$$\mathrm{curl}(\mathbf{F}) = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle.$$

Write "$\mathrm{curl} = \nabla \times$":

$$\mathrm{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Geometric meaning, later: Infinitesimal rotation.

Operations on a vector field in $\mathbb{R}^3$: curl

Operations on a vector field in $\mathbb{R}^3$: curl

Example. For $\mathbf{F} = \langle P(x, y), Q(x, y), 0 \rangle$, what is $\mathrm{curl}(\mathbf{F})$?

Operations on a vector field in $\mathbb{R}^3$: curl

Example. For $\mathbf{F} = \langle P(x,y), Q(x,y), 0 \rangle$, what is $\operatorname{curl}(\mathbf{F})$?

Example. For a function f, what is $\operatorname{curl}(\nabla f)$?

Operations on a vector field in $\mathbb{R}^3$: curl

Example. For $\mathbf{F} = \langle P(x,y), Q(x,y), 0\rangle$, what is $\mathrm{curl}(\mathbf{F})$?

Example. For a function f, what is $\mathrm{curl}(\nabla f)$?

Theorem. Given a vector field $\mathbf{F}$ on $\mathbb{R}^3$ with continuous partial derivatives, $\mathbf{F}$ is conservative if and only if $\mathrm{curl}(\mathbf{F}) = \mathbf{0}$.

Operations on a vector field in $\mathbb{R}^3$: curl

Example. For $\mathbf{F} = \langle P(x,y), Q(x,y), 0\rangle$, what is $\mathrm{curl}(\mathbf{F})$?

Example. For a function f, what is $\mathrm{curl}(\nabla f)$?

Theorem. Given a vector field $\mathbf{F}$ on $\mathbb{R}^3$ with continuous partial derivatives, $\mathbf{F}$ is conservative if and only if $\mathrm{curl}(\mathbf{F}) = \mathbf{0}$.

Also true for $\mathbf{F}$ on a "simply connected" open subsets $U \subset \mathbb{R}^3$... won't discuss this as it is more complicated here.

Operations on a vector field in $\mathbb{R}^3$: divergence

Suppose $\mathbf{F} = \langle P, Q, R \rangle$.

Operations on a vector field in $\mathbb{R}^3$: divergence

Suppose $\mathbf{F} = \langle P, Q, R \rangle$. Define the *divergence* of $\mathbf{F}$ by

$$\mathrm{div}(\mathbf{F}) = P_x + Q_y + R_z.$$

Note: this is a function.

Operations on a vector field in $\mathbb{R}^3$: divergence

Suppose $\mathbf{F} = \langle P, Q, R \rangle$. Define the *divergence* of $\mathbf{F}$ by

$$\mathrm{div}(\mathbf{F}) = P_x + Q_y + R_z.$$

Note: this is a function.

Write "$\mathrm{div} = \nabla \cdot$ ":

Operations on a vector field in $\mathbb{R}^3$: divergence

Suppose $\mathbf{F} = \langle P, Q, R\rangle$. Define the *divergence* of $\mathbf{F}$ by

$$\mathrm{div}(\mathbf{F}) = P_x + Q_y + R_z.$$

Note: this is a function.

Write "$\mathrm{div} = \nabla \cdot$ ":

$$\mathrm{div}(\mathbf{F}) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R\rangle = P_x + Q_y + R_z$$

Operations on a vector field in $\mathbb{R}^3$: divergence

Suppose $\mathbf{F} = \langle P, Q, R \rangle$. Define the *divergence* of $\mathbf{F}$ by

$$\mathrm{div}(\mathbf{F}) = P_x + Q_y + R_z.$$

Note: this is a function.

Write "$\mathrm{div} = \nabla \cdot$ ":

$$\mathrm{div}(\mathbf{F}) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle = P_x + Q_y + R_z$$

Geometric meaning, later: Expansion of volume.

Operations on a vector field in $\mathbb{R}^3$: divergence

Operations on a vector field in $\mathbb{R}^3$: divergence

Example. For a vector field $\mathbf{F}$, what is $\operatorname{div}(\operatorname{curl}(\mathbf{F}))$?

Operations on a vector field in $\mathbb{R}^3$: divergence

Example. For a vector field $\mathbf{F}$, what is $\operatorname{div}(\operatorname{curl}(\mathbf{F}))$?

Theorem. For any vector field on $\mathbb{R}^3$ with continuous derivatives, $\mathbf{F}$ is the curl of some vector field if and only if $\operatorname{div}(\mathbf{F}) = 0$.

Operations on a vector field in $\mathbb{R}^3$: divergence

Example. For a vector field $\mathbf{F}$, what is $\mathrm{div}(\mathrm{curl}(\mathbf{F}))$?

Theorem. For any vector field on $\mathbb{R}^3$ with continuous derivatives, $\mathbf{F}$ is the curl of some vector field if and only if $\mathrm{div}(\mathbf{F}) = 0$.

Another important operator on functions:

$$\nabla^2 f = \nabla \cdot \nabla f = f_{xx} + f_{yy} + f_{zz}.$$

This is called the *Laplacian*, and is often also denoted Δf.