Chapter 9

Currency Options (2): Hedging and Valuation

Overview

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Binomial Models—What & Why?

Binomial Model

- given S_t, there only two possible values for S_{t+1}, called "up" and "down".

Restrictive?—Yes, but ...

- the distribution of the total return, after many of these binomial price changes, becomes bell-shaped
- the binomial option price converges to the BMS price
- the binomial math is much more accessible than the BMS math
- BinMod can be used to value more complex derivatives that have no closed-form Black-Scholes type solution.

Ways to explain the model—all very similar:

	via hedging	via replication
in spot market	(not here)	(not here)
forward	yes	yes

Binomial Models—What & Why?

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Outline

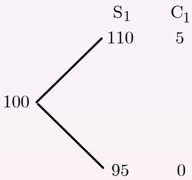
Our Example

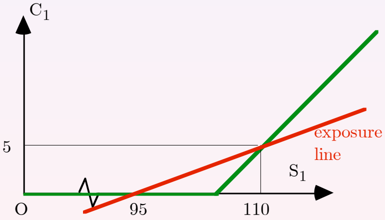
◇ Data

- ▷ S_0 = INR/MTL 100, $r = 5\% p.p.; r^* = 3.9604\%$. Hence:

$$F_{0,1} = S_0 \frac{1 + r_{0,1}}{1 + r^*_{0,1}} = 100 \frac{1.05}{1.039604} = 101.$$

- ▷ S_1 is either $S_{1,\mathrm{u}} = 110$ (“up”) or $S_{1,\mathrm{d}} = 95$ (“down”).
- ▷ 1-period European-style call with X=INR/MTL 105





- ▷ slope of exposure line (*exposure*):

$$\text{exposure} = \frac{C_{1,u} - C_{1,d}}{S_{1,i} - S_{1,d}} = \frac{5 - 0}{110 - 95} = 1/3$$

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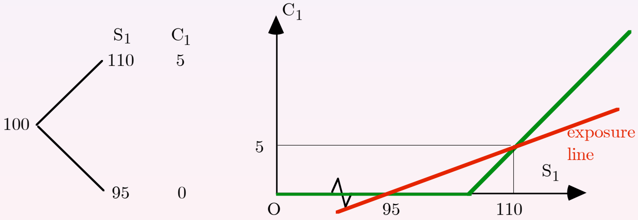

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The Replication Approach

◇ **Step 1** Replicate the payoff from the call—[5 if *u*] and [0 if *d*]:

	(a) = forward contract (buy MTL 1/3 at 101)	(b) = deposit, V1=20	(a)+(b)
$S_1 = 95$	$1/3 \times (95 - 101) = -2$	+2	0
$S_1 = 110$	$1/3 \times (110 - 101) = +3$	+2	5

◇ **Step 2** Time-0 cost of the replicating portfolio:

- forward contract is free
- deposit will cost INR 2/1.05 = INR 1.905

◇ **Step 3** Law of One Price: option price = value portfolio

$$C_0 = \text{INR } 1.905$$

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The Hedging Approach

Replication:	call = forward position + riskfree deposit
Hedging:	call – forward position = riskfree deposit

◇ **Step 1** Hedge the call

	(a) = forward hdege (sell MTL 1/3 at 101)	(b) = call	(a)+(b)
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◇ **Step 2** time-0 value of the riskfree portfolio

$$\text{value} = \text{INR } 2/1.05 = \text{INR } 1.905$$

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$$C_0 + [\text{time-0 value of hedge}] = \text{INR } 1.905 \Rightarrow C_0 = \text{INR } 1.905$$

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- extracts a risk-adjusted probability "up" from the forward market,
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The Risk-adjusted Probabilities

- **Step 1** Extract risk-adjusted probability from F:
  - Ordinary expectation: $E_0(\tilde{S}_1) = p \times 110 + (1-p) \times 95$
  - Risk-adjusted expectation: $\text{CEQ}_0(\tilde{S}_1) = q \times 110 + (1-q) \times 95$
  - We do not know how/why the market selects q, but q is revealed by $F_{0,1}$ (= 101):

$$101 = 95 + q \times (110 - 95) \Rightarrow q = \frac{101 - 95}{110 - 95} = \frac{6}{15} = 0.4$$

- **Step 2** CEQ of the call's payoff:

$$CEQ_0(\tilde{C}_1) = (0.4 \times 5) + (1 - 0.4) \times 0 = 2$$

- **Step 3** Discount at r:

$$C_0 = \frac{CEQ_0(\tilde{C}_1)}{1 + r_{0,1}} = \frac{2}{1.05} = 1.905$$

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Currency Options (2): Hedging and Valuation

P. Sercu, *International Finance: Theory into Practice*

Outline

Multiperiod Pricing: Notation

- **Subscripts: n,j** where
  - n says how many jumps have been made since time 0
  - j says how many of these jumps were *up*
- **General pricing equation:**

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\begin{aligned}
C_{t,j} &= \frac{C_{t+1,u} \times q_t + C_{t+1,d} \times (1-q_t)}{1+r_{t,\text{1period}}}, \\
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\tag{1}
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Assumptions

- **A1 (r and r^*)** : The risk-free one-period rates of return on both currencies are constant
  - denoted by unsubscripted r and r^*
  - Also assumed in Black-Scholes.
- **A2 (u and d) :** The multiplicative change factors, u and d, are constant.
  Also assumed in Black-Scholes:
  - no jumps (sudden de/revaluations) in the exchange rate process, and
  - a constant variance of the period-by-period percentage changes in S.
- **Implication of A1-A2:** q_t is a constant.
- **A2.01 (no free lunch in F):**

$$d < \frac{1+r}{1+r^*} < u \Leftrightarrow S_{t+1,d} < F_t < S_{t+1,u} \Leftrightarrow 0 < q < 1$$

Q: what would you do if S_1=[95 or 110] while F=90? 115?

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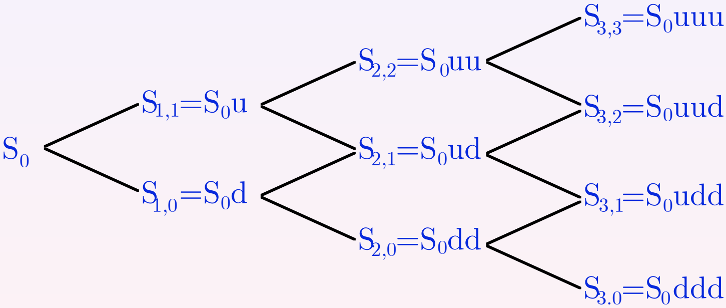
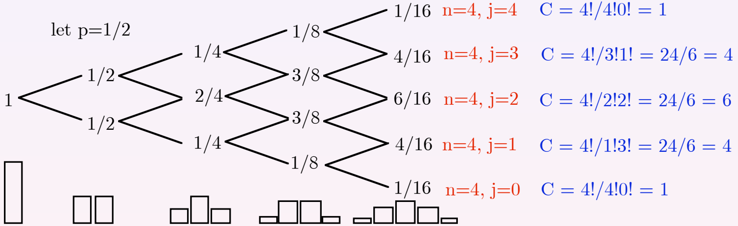
How such a tree works



The Emerging Bellshape

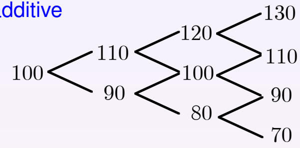


The emerging bell-shape

What Emerging Bellshape?

◇ **Chosing between two oversimplifications:**

additive



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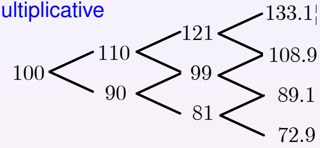

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– non-negative prices: with a multiplicative, the exchange rate can never quite reach zero even if it happens to go down all the time.

– invertible: we get a similar multiplicative process for the exchange rate as viewed abroad, $S^* = 1/S$ (with $d^* = 1/u, \quad u^* = 1/d$).

◇ **Corresponding Limiting Distributions:**

– additive: $\tilde{S}_n = S_0 + \sum_{t=1}^{n} \tilde{\Delta}_t$ where $\tilde{\Delta} = \{+10, -10\}$
$\Leftrightarrow \tilde{S}_n$ is normal if n is large (CLT)

– multiplicative: $\tilde{S}_n = S_0 \times \prod_{t=1}^{n}(1 + \tilde{r}_t)$ where $\tilde{r}$ = {+10%, -10%}
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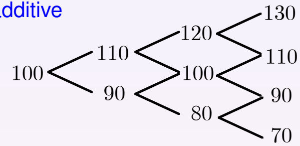

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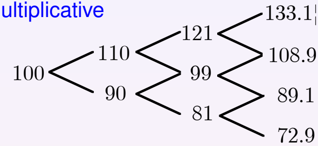

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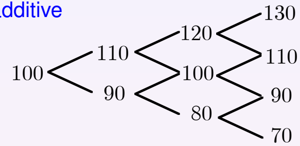

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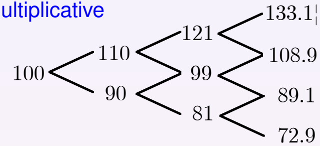

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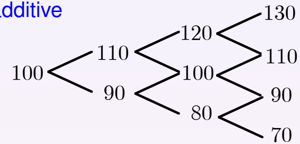
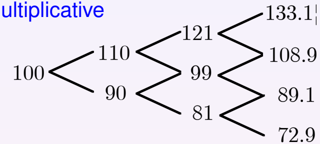


multiplicative



- **cents v percent:** we prefer a constant distribution of *percentage* price changes over a constant distribution of *dollar* price changes.
- **non-negative prices:** with a multiplicative, the exchange rate can never quite reach zero even if it happens to go down all the time.
- **invertible:** we get a similar multiplicative process for the exchange rate as viewed abroad, $S^* = 1/S$ (with $d^* = 1/u, \quad u^* = 1/d$).

◇ Corresponding Limiting Distributions:

- additive: $\tilde{S}_n = S_0 + \sum_{t=1}^{n} \tilde{\Delta}_t$ where $\tilde{\Delta} = \{+10, -10\}$
  $\Leftrightarrow \tilde{S}_n$ is normal if n is large (CLT)
- multiplicative: $\tilde{S}_n = S_0 \times \prod_{t=1}^{n}(1+\tilde{r}_t)$ where $\tilde{r}$ = {+10%, -10%}
  $\Leftrightarrow \ln \tilde{S}_n = \ln S_0 + \sum_{t=1}^{n} \tilde{\rho}_t$ where $\tilde{\rho} = \ln(1+\tilde{r}) = \{+0.095, -0.095\}$
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What Emerging Bellshape?

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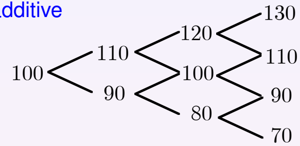

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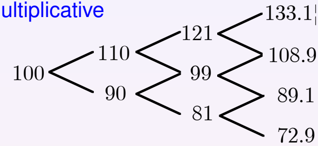

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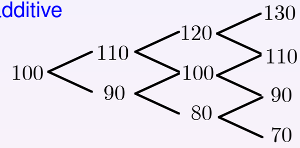

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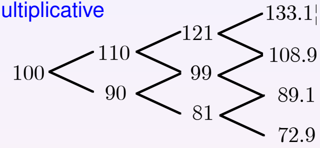

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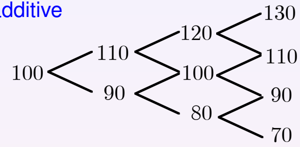

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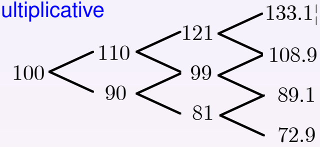

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Outline

An N-period European Call: The Problem

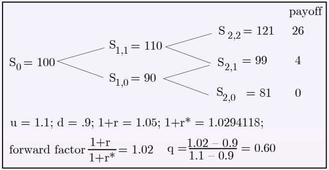
			payoff
		$S_{2,2} = 121$	26
	$S_{1,1} = 110$		
$S_0 = 100$		$S_{2,1} = 99$	4
	$S_{1,0} = 90$		
		$S_{2,0} = 81$	0

$u = 1.1;\ d = .9;\ 1{+}r = 1.05;\ 1{+}r^* = 1.0294118;$

$$\text{forward factor } \frac{1+r}{1+r^*} = 1.02 \qquad q = \frac{1.02 - 0.9}{1.1 - 0.9} = 0.60$$

- **A4**. At any discrete moment in the model, investors can trade and adjust their portfolios of HC-FC loans.
  Black-Scholes: trading is continuous

Backward Pricing, Dynamic Hedging



◇ **if we land in node (1,1):**

$$b_{1,1} = \frac{26-4}{121-99} = 1$$

$$C_{1,1} = \frac{(26 \times 0.6) + (4 \times 0.4)}{1.05} = 16.38$$

◇ **if we land in node (1,0):**

$$b_{1,0} = \frac{4-0}{99-81} = .222$$

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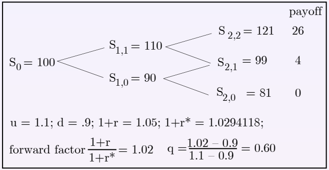

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Backward Pricing, Dynamic Hedging

$$\Rightarrow C_1 = \begin{cases} 16.38 & \text{if } S_1 = 110 \\ 2.29 & \text{if } S_1 = 90 \end{cases}$$

◇ **at time 0 we do have a two-point problem:**

$$\Rightarrow b_0 = \frac{16.38 - 2.29}{110 - 90} = 0.704$$

$$C_{1,1} = \frac{(16.38 \times 0.6) + (2.29 \times 0.4)}{1.05} = 10.23$$

Summary:

- we hedge dynamically:
  - start the hedge at time 0 with 0.704 units sold forward.
  - The time-1 hedge will be to sell forward 1 or 0.222 units of foreign currency, depending on whether the rate moves up of down.
- we price backward, step by step

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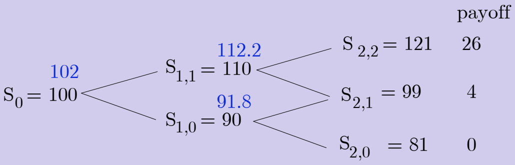
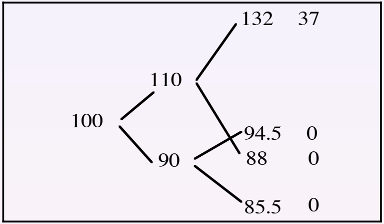
Hedging Verified



▷ **at time 0:**	invest 10.23 129 at 5%, buy fwd MTL 0.704 762 at $100 \times 1.02 = 102$	
— value if up:	$10.23129 \times 1.05 + 0.704762 \times (110 - 102) =$	16.380 95
— value if down:	$10.23129 \times 1.05 + 0.704762 \times (90 - 102) =$	2.295 71
▷ **if in node (1,1):**	invest 16.380 95 at 5%, buy fwd MTL 1 at $100 \times 1.02 = 112.2$	
— value if up:	$16.38095 \times 1.05 + 1.000000 \times (121 - 112.2) =$	26.000 00
— value if down:	$16.38095 \times 1.05 + 1.000000 \times (99 - 112.2) =$	4.000 00
▷ **if in node (1,0):**	invest 2.295 71 at 5%, buy fwd MTL 0.222 222 at $90 \times 1.02 = 91.8$	
— value if up:	$2.29571 \times 1.05 + 0.222222 \times (99 - 91.8) =$	4.000 00
— value if down:	$2.29571 \times 1.05 + 0.222222 \times (81 - 91.8) =$	0.000 00

What can go wrong?

Everything can and will go wrong:



Change of risk: $\pm 20\%$ if up, $\pm 5\%$ if down, instead of the current $\pm 10\%$:

$$C_{1,1} = \frac{37 \times 0.55 + 0}{1.05} = 19.36, \text{ not } 16.38,$$

$$C_{1,0} = \frac{0 + 0}{1.05} = 0.00, \text{ not } 2.29,$$

You would have mishedged:

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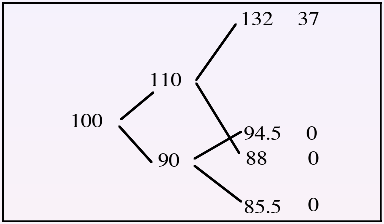

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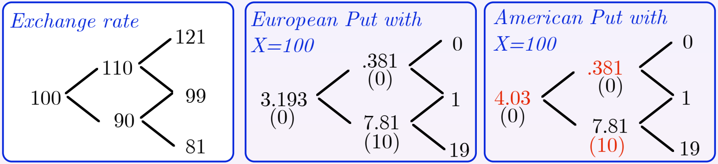
American-style Options



u=1.1, d=0.9, r=5%, (1+r)/(1+r*) = 1.02, q=0.60

- **Node (1,1)** In this node the choices are
  - ▷ PV of later exercise (0 or 1): 0.381
  - ▷ Value of immediate exercise: 0 — so we wait; $V_{1,1} = .381$
- **Node (1,0)** Now the choices are
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- **Node (0)** We now choose between
  - ▷ PV of later exercise (0 or 1 at time 2, or 10 at time 1):

$$P_0^{alive} = \frac{0.381 \times 0.60 + 10 \times 0.40}{1.05} = 4.03$$

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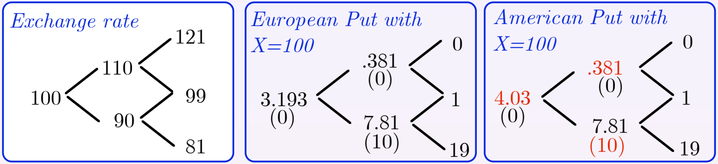

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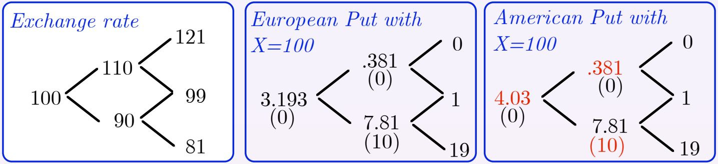

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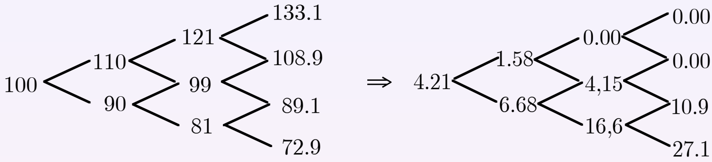
Outline

Straight-Through-Pricing a 3-period Put



The long way:

$$C_{2,2} = \frac{0.00 \times 0.6 + 0.00 \times 0.4}{1.05} = 0.00,$$

$$C_{2,1} = \frac{0.00 \times 0.6 + 10.9 \times 0.4}{1.05} = 4.152,$$

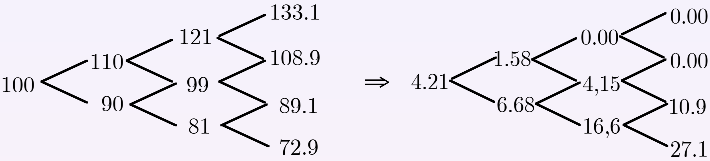
$$C_{2,0} = \frac{10.0 \times 0.6 + 27.1 \times 0.4}{1.05} = 16.55,$$

$$C_{1,1} = \frac{0.000 \times 0.6 + 4.152 \times 0.4}{1.05} = 1.582,$$

$$C_{1,0} = \frac{4.152 \times 0.6 + 16.55 \times 0.4}{1.05} = 8,678,$$

$$C_0 = \frac{1.582 \times 0.6 + 8,678 \times 0.4}{1.05} = 4.210.$$

Straight-Through-Pricing a 3-period Put



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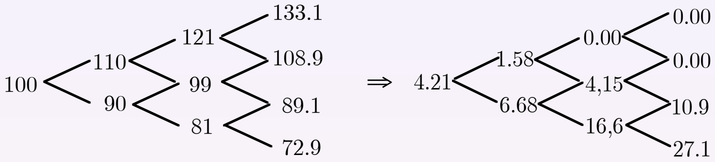

The fast way:

- $pr_3 = \ldots$
- $pr_2 = \ldots$
- $pr_1 = \ldots$
- $pr_0 = \ldots$
- The (risk-adjusted) chance of ending in the money is ...
- $C_0 = \frac{\quad \times \quad + \quad \times \quad + \quad \times \quad + \quad \times \quad}{\qquad} = 4.21.$

Straight-Through-Pricing: 2-period Math

$$
\begin{aligned}
C_{1,1} &= \frac{q\,C_{2,2} + (1-q)C_{2,1}}{1+r}, \\
C_{1,0} &= \frac{q\,C_{2,1} + (1-q)C_{2,0}}{1+r}, \\
C_0 &= \frac{q\,C_{1,1} + (1-q)C_{1,0}}{1+r}, \\
&= \frac{q\left[\frac{q\,C_{2,2}+(1-q)C_{2,1}}{1+r}\right] + (1-q)\left[\frac{q\,C_{2,1}+(1-q)C_{2,0}}{1+r}\right]}{1+r}, \\
&= \frac{q^2\,C_{2,2} + 2q\,(1-q)C_{2,1} + (1-q)^2\,C_{2,0}}{(1+r)^2}
\end{aligned}
$$

Straight-Through-Pricing: 3-period Math

$$C_{1,1} = \frac{q^2\, C_{3,3} + 2q\,(1-q)C_{3,2} + (1-q)^2\, C_{3,1}}{(1+r)^2}$$

$$C_{1,0} = \frac{q^2\, C_{3,2} + 2q\,(1-q)C_{3,1} + (1-q)^2\, C_{3,0}}{(1+r)^2},$$

$$\begin{aligned} C_0 &= \frac{q\, C_{1,1} + (1-q)C_{1,0}}{1+r}, \\ &= \frac{q\left[q^2\, C_{3,3} + 2q\,(1-q)\, C_{3,2} + (1-q)^2\, C_{3,1}\right] + (1-q)\left[q^2\, C_{3,2} + 2q\,(1-q)C_{3,1} + (1-q)^2\, C_{3,0}\right]}{(1+r)^3}, \\ &= \frac{q^3 C_{3,3} + 3q^2(1-q)C_{3,2} + 3q(1-q)^2 C_{3,1} + (1-q)^3 C_{3,0}}{(1+r)^3} \end{aligned}$$

Toward BMS 1: two terms

Let $pr_{n,j}^{(Q)}$ = risk-adjusted chance of having j *ups* in n jumps

$$= \underbrace{\frac{n!}{j!\,(n-j)!}}_{\text{\# of paths with } j \text{ ups}} \times \underbrace{q^j(1-q)^{N-j}}_{\text{prob of such a path}} = \binom{N}{j} q^j (1-q)^{N-j}$$

and let a : $\{j \geq a\} \Leftrightarrow \{S_{n,j} \geq X\}$;

$$\text{then } C_0 = \frac{\sum_{j=0}^{N} pr_{n,j}^{(Q)} C_{n,j}}{(1+r)^N} = \frac{\text{CEQ}_0(\tilde{C}_N)}{\text{discounted}},$$

$$= \frac{\sum_{j=0}^{N} pr_{n,j}^{(Q)} (S_{n,j}-X)_+}{(1+r)^N},$$

$$= \frac{\sum_{j=a}^{N} pr_{n,j}^{(Q)} (S_{n,j}-X)}{(1+r)^N},$$

$$= \frac{\sum_{j=a}^{N} pr_{n,j}^{(Q)} S_{n,j}}{(1+r)^N} - \frac{X}{(1+r)^N} \sum_{j=a}^{N} pr_{n,j}^{(Q)}. \quad (2)$$

Toward BMS 2: two probabilities

$$\text{Recall: } C_0 = \frac{\sum_{j=a}^{N} pr_{n,j}^{(Q)} S_{n,j}}{(1+r)^N} - \frac{X}{(1+r)^N} \sum_{j=a}^{N} pr_{n,j}^{(Q)}.$$

We can factor out S_0, in the first term, by using

$$S_{n,j} = S_0 u^j d^{N-j}.$$

We also use

$$\frac{1}{(1+r)^N} = \frac{1}{(1+r^*)^N} \left(\frac{1+r^*}{1+r}\right)^j \left(\frac{1+r^*}{1+r}\right)^{N-j}$$

$$\frac{\sum_{j=a}^{N} pr_{n,j}^{(Q)} S_{n,j}}{(1+r)^N} = \frac{S_0}{(1+r^*)^N} \sum_{j=a}^{N} \binom{N}{j} \left(q\,\frac{1+r^*}{1+r}\right)^j \left((1-q)\,\frac{1+r^*}{1+r}\right)^{N-j}$$

$$= \frac{S_0}{(1+r^*)^N} \sum_{j=a}^{N} \binom{N}{j} \pi^j (1-\pi)^{N-j}$$

$$\text{where } \pi := q\,\frac{1+r^*}{1+r} \Rightarrow 1-\pi = (1-q)\,\frac{1+r^*}{1+r}.$$

Towards BMS 3: the limit

$$C_0 = \underbrace{\frac{S_0}{(1+r^*)^N}}_{\text{price of the underlying FC PN}} \overbrace{\sum_{j=a}^{N} \binom{N}{j} \pi^j (1-\pi)^{N-j}}^{\text{a “}j \geq a\text{” probability-like expression}} - \underbrace{\frac{X}{(1+r)^N}}_{\text{discounted strike}} \overbrace{\sum_{j=a}^{N} pr_{n,j}^{(Q)}}^{\text{prob}^{(Q)} \text{ of } j \geq a} . \quad (3)$$

- Special case $a = 0$:
  - “$a = 0$” means that ...
  - so both probabilities become ...
  - and we recognize the value of ...
- In the limit for $N \to \infty$ (and suitably adjusting u, d, r, r^*)
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  - first prob typically denoted $\mathrm{N}(d_1)$, $d_1 = \frac{\ln(F_{t,T}/X)+(1/2)\sigma^2_{t,T}}{\sigma_{t,T}}$, with $\sigma_{t,T}$ the effective stdev of $\ln \tilde{S}_T$ as seen at time t
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The Delta of an Option

- **Replication:** in BMS the option formula is still based on a portfolio that replicates the option (over the short time period $\mathrm{d}t$):
  - a fraction $\sum_{j=a}^{n} \pi_j$ or $N(d_1)$ of a FC PN with face value unity, and
  - a fraction $\sum_{j=a}^{n} pr_j$ or $N(d_2)$ of a HC PN with face value X.

- **Hedge:** since hedging is just replication reversed, you can use the formula to hedge:

version of formula	hedge instrument	unit price	size of position
$C_0 = \frac{S_0}{1+r^*_{0,T}} N(d_1) - \dots$	FC PN expiring at T	$\frac{S_0}{1+r^*_{0,T}}$	$N(d_1)$
$C_0 = S_0 \frac{N(d_1)}{1+r^*_{0,T}} - \dots$	FC spot deposit	S_0	$\frac{N(d_1)}{1+r^*_{0,T}}$
$C_0 = F_{0,T} \frac{N(d_1)}{1+r_{0,T}} - \dots$	Forward expiring at T	$F_{0,T}$	$\frac{N(d_1)}{1+r_{0,T}}$

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What have we learned in this chapter?

Why binomial?

- does basically the same as the BMS pde, but ...
- is much simpler

One-period problems

- hedging/replication gets us the price without knowing the true p and the required risk correction in the discount rate
- but that's because we implicitly use q instead:
- the price is the discounted risk-adjusted expectation

Multiperiod models

- basic model assumes constant u, d, r, r^*
- we can hedge dynamically and price backward
- for American-style options, we also compare to the value dead

Black-Merton-Scholes

- For European-style options, you can Straight-Through-Price the option
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