Differential Geometry
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Curves

Differential Geometry Introduction. Differential geometry is a mathematical discipline that uses methods of multivariable calculus and linear algebra to study problems in geometry. In this course, we will study **curves** and **surfaces** and, later in the course, their generalizations **manifolds**.

A pivotal concept that we shall develop both for curves and for surfaces is that of **curvature**. The long term goal is to provide the mathematical background necessary for understanding Einstein's theory of relativity as a geometric theory of space-time in which the gravitation manifests as the curvature of space-time.

We begin by studying curves and their properties such as velocity and acceleration vectors, curvature and torsion, and the arc length. Then we introduce an apparatus that completely describes all the relevant properties of a curve.

After that, we move on to surfaces and their properties. We shall study the first and second fundamental forms, geodesics, and curvature – an apparatus that completely describes a surface. We shall formulate Gauss's **Theorema Egregium** (Remarkable Theorem) that allows the concept of curvature to be generalized to curvature of higher dimensional manifolds and enables you to understand the language used in special and general relativity. Along the way, we also discuss the concept of a tensor.

Curves. Intuitively, we think of a curve as a path traced by a moving particle in space. This approach is formalized by considering a curve as a **function** of a parameter, say t. Thus, the **domain** of a curve is an interval (a, b) (possibly $(-\infty, \infty)$) consisting of all possible values of a parameter t. The **range** of a curve is contained in the three dimensional space.

To simplify the notation, consider that $\mathbb{R}$ is frequently used to denote the set of real numbers. If we think of points in the two dimensional space as the set of ordered pairs (x, y) where each coordinate is in $\mathbb{R}$, the product of $\mathbb{R}$ with itself is considered and is denoted by $\mathbb{R}^2$. Following this reasoning, the three-dimensional space is denoted by $\mathbb{R}^3$.

Thus, a curve $\boldsymbol{\gamma}$ is a mapping of an interval (a, b) into $\mathbb{R}^3$. This is denoted by $\boldsymbol{\gamma} : (a, b) \to \mathbb{R}^3$. Using the standard vector representations of points in $\mathbb{R}^3$ as (x, y, z), we can represent $\boldsymbol{\gamma}$ as a vector function:

$$\boldsymbol{\gamma}(t) = (x(t),\ y(t),\ z(t))$$

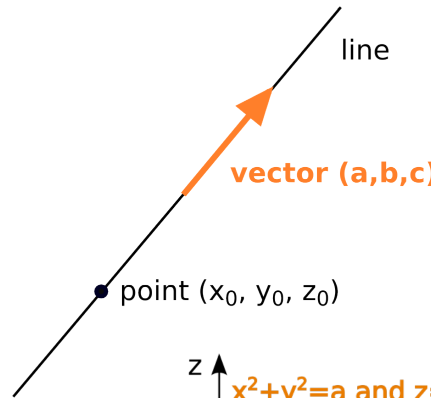
or using the **parametric equations** $x = x(t)$, $y = y(t)$, and $z = z(t)$. The variable t is called the **parameter**.

Examples.

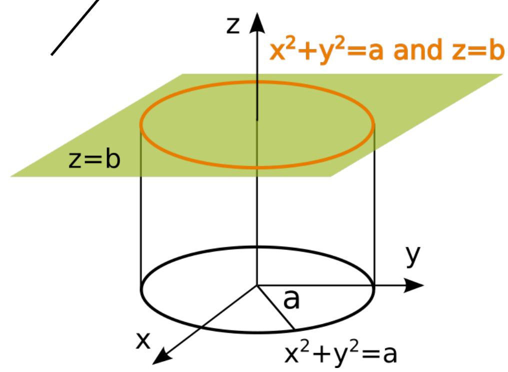
1. **Line.** A line in space is given by the equations

$$x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct$$

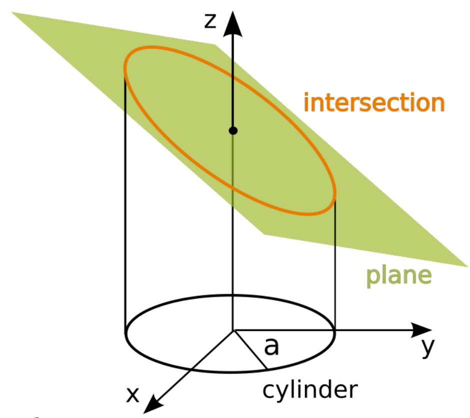
where (x_0, y_0, z_0) is a point on the line and (a, b, c) is a vector parallel to it. Note that in the vector form the equation $\boldsymbol{\gamma} = \boldsymbol{\gamma}(0) + \mathbf{m}\,t$ for $\boldsymbol{\gamma}(0) = (x_0, y_0, z_0)$ and $\mathbf{m} = (a, b, c)$, has exactly the same form as the well known $y = b + mx$.



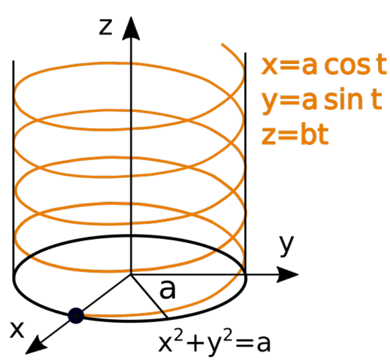
2. **Circle in horizontal plane**. Consider the parametric equations $x = a\cos t \quad y = a\sin t \quad z = b$. Recall that the parametric equation of a circle of radius a centered in the origin of the plane $\mathbb{R}^2$ are $x = a\cos t$, $y = a\sin t$. Recall also that $z = b$ represents the horizontal plane passing b in the z-axis.
Thus, the equations $x = a\cos t \quad y = a\sin t \quad z = b$ represent the circle of radius a in the horizontal plane passing $z = b$ on z-axis.



3. **Ellipse in a plane**. Consider the intersection of a cylinder and a plane. The intersection is an ellipse. For example, if we consider a cylinder with circular base $x = a\cos t$, $y = a\sin t$ and the equation of the plane is $mx + ny + kz = l$ with $k \neq 0$, the parametric equations of ellipse can be obtained by solving the equation of plane for z and using the equations for x and y to obtain the equation of z in parametric form. Thus $z = \frac{1}{k}(l - mx - ny)$ and so $x = a\cos t \quad y = a\sin t \quad z = \frac{1}{k}(l - ma\cos t - na\sin t)$.



4. **Circular helix.** A curve with equations $x = a\cos t \quad y = a\sin t \quad z = bt$ is the curve spiraling around the cylinder with base circle $x = a\cos t$, $y = a\sin t$.



5. **Plane curves**. All the concepts we develop for space curves correspond to plane curves simply considering that $z = 0$.

Tangent Vector and the Arc Length

Velocity vector. If $\boldsymbol{\gamma}(t) = (x(t), y(t), z(t))$ is a space curve, the vector tangent to the curve at the point where $t = t_0$ is

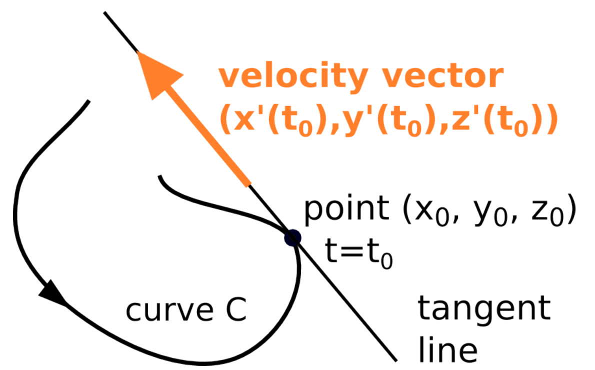
$$\boldsymbol{\gamma}'(t_0) = (x'(t_0),\ y'(t_0),\ z'(t_0)).$$

So, the equation of the **line tangent** to the curve $\boldsymbol{\gamma}$ at the point where $t = t_0$ is

$$x = x(t_0) + x'(t_0)t \quad y = y(t_0) + y'(t_0)t \quad z = z(t_0) + z'(t_0)t$$

or, using the shorter notation,

$$\boldsymbol{\gamma}(t_0) + \boldsymbol{\gamma}'(t_0)t.$$



The derivative vector $\boldsymbol{\gamma}'(t) = (x'(t), y'(t), z'(t))$ is sometimes also called the **velocity vector**. The length of this vector is called the **speed**, sometimes also referred to as instantaneous speed and denoted by $\frac{ds}{dt}$ so

$$\frac{ds}{dt} = |\boldsymbol{\gamma}'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}.$$

Also note that $\qquad ds = |\boldsymbol{\gamma}'(t)|dt.$

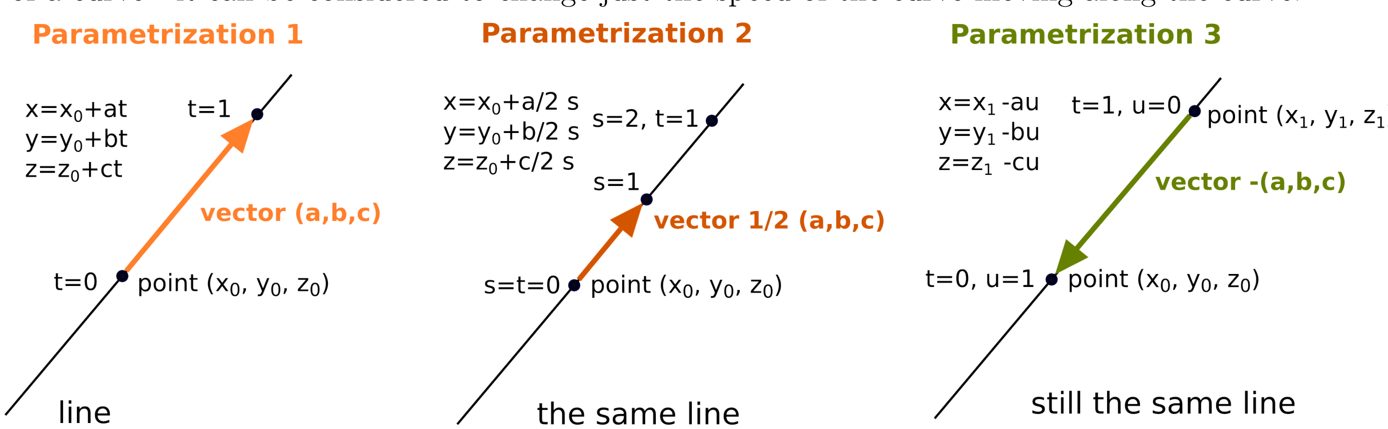
Derivative vector of unit length is especially important. It called the **tangent vector** $\mathbf{T}(t)$ and is obtained by dividing the derivative vector by its length

$$\mathbf{T}(t) = \frac{1}{|\boldsymbol{\gamma}'(t)|}(x'(t), y'(t), z'(t)) = \frac{\boldsymbol{\gamma}'(t)}{|\boldsymbol{\gamma}'(t)|}.$$

The **total length** of the curve $\boldsymbol{\gamma}$ on interval $a \leq t \leq b$ can be obtained by integrating the length element ds from a to b. Thus,

$$L = \int_a^b ds = \int_a^b |\boldsymbol{\gamma}'(t)|dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$$

Reparametrization. For a one-to-one mapping of an interval (c, d) onto interval (a, b) given by $s \mapsto t(s)$, the curve $\boldsymbol{\gamma}(t) = (x(t), y(t), z(t))$ can be given by $\boldsymbol{\gamma}(s) = (x(t(s)), y(t(s)), z(t(s)))$. In this case, $\boldsymbol{\gamma}(s)$ is said to be a reparametrization of $\boldsymbol{\gamma}(t)$. A reparametrization *does not change the graph* of a curve - it can be considered to change just the speed of the curve moving along the curve.



Example 1. The line with parametric equations $\begin{array}{l} x = 1+2t \\ y = 2+4t \\ z = 3-2t \end{array}$ can also be represented as $\begin{array}{l} x = 1+\ s \\ y = 2+2s \\ z = 3-\ s \end{array}$ with $s = 2t$ being the relation between the two parameters just as in the figure above. Further, it can also be represented as $\begin{array}{l} x = 3-2u \\ y = 6-4u \\ z = 1+2u \end{array}$ with $u = 1-t$ being the relation between the parameters in the first and the last parametrization, again just as in the figure above.

This shows that seemingly different parametric equations can be describing the **same line**. When thinking of a line as the trajectory of an object and the parameter as the time, the different parametrization represent the fact that different particles traveling on the same (straight) path but with different speeds and possibly in the opposite orientation are still moving along the same path.

With this in mind, the particle in parametrization $\begin{array}{l} x = 1+2t \\ y = 2+4t \\ z = 3-2t \end{array}$ travels twice as fast than the particle in parametrization $\begin{array}{l} x = 1+\ s \\ y = 2+2s \\ z = 3-\ s. \end{array}$ A third particle with the trajectory given by $\begin{array}{l} x = 3-2u \\ y = 6-4u \\ z = 1+2u \end{array}$ has the different initial point and it is traveling in different orientation than the first two but all three particles are traveling along the same path.

Example 2. Consider the cycle centered at the origin of radius a. These parametric equations $x = a\cos\frac{t}{2}$ and $y = a\sin\frac{t}{2}$ equally well represent the circle as the usual $x = a\cos t$ and $y = a\sin t$. The difference is that it takes a point 4π to make one full rotation using the first parametrization while it takes just 2π to make one full rotation using the second parametrization. So, the point moves twice as slow in the first representation than in the second.

The same conclusion can be obtained when comparing speeds (lengths of the velocity vectors) in both representations. In the first one, the velocity vector is $(\frac{-a}{2}\sin\frac{t}{2}, \frac{a}{2}\cos\frac{t}{2})$ and its length is $\frac{a}{2}$ while in the second the velocity vector is $(-a\sin t, a\cos t)$ and its length is a, twice as large speed than in the first parametrization.

More generally, the parametrization $x = a\cos\frac{t}{b}$ and $y = a\sin\frac{t}{b}$ where b is a positive constant, has the velocity $(\frac{-a}{b}\sin\frac{t}{b}, \frac{a}{b}\cos\frac{t}{b})$ and speed $\frac{a}{b}$. By taking $b = a$, we obtain the parametrization of **unit speed**.

$$x = a\cos\frac{t}{a} \text{ and } y = a\sin\frac{t}{a}$$

One would prefer to be able to obtain the parametrization with unit speed for every curve $\boldsymbol{\gamma}$. In such parametrization, the velocity vector has unit length and so $\boldsymbol{\gamma}' = \mathbf{T}$. We refer to curve parametrized in that way as a **unit-speed curve**.

The unit-speed parametrization can be obtained as a reparametrization of any given parametrization using the substitution $t = t(s)$ such that

$$s(t) = \int_a^t ds = \int_a^t |\boldsymbol{\gamma}'(t)|dt = \int_a^t \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$$

This parametrization is also referred as the **parametrization by the arc length** and s is used to denote the parameter in this case. Thus,

$$\boxed{\mathbf{T}(s) = \boldsymbol{\gamma}'(s) \quad \text{and} \quad |\boldsymbol{\gamma}'(s)| = 1}$$

If the curve is parametrized by arc length, the length L for $a \leq t \leq b$ which corresponds to $c \leq s \leq d$ in the arc-length parametrization, can also be found as

$$L = \int_a^b |\boldsymbol{\gamma}'(t)|dt = \int_c^d |\boldsymbol{\gamma}'(s)|ds = \int_c^d 1ds = (d-c).$$

Example 1. Consider the line $(1+t, 2+2t, 3-t)$ from the earlier example. To parametrize it by the arc length, we calculate the parametrization $s = s(t)$ to be $\int_0^t \sqrt{1+4+1}dt = \sqrt{6}t$. Thus, $t = \frac{1}{\sqrt{6}}s$ and we have $(1 + \frac{1}{\sqrt{6}}s, 2 + \frac{2}{\sqrt{6}}s, 3 - \frac{1}{\sqrt{6}}s)$. Note also that $(\frac{1}{\sqrt{6}} \frac{2}{\sqrt{6}} \frac{-1}{\sqrt{6}})$ is the normalization of the original velocity vector $(1, 2, -1)$.

Example 2. We can verify our earlier conclusion that the substitution $t = \frac{s}{a}$ gives the unit-speed parametrization of the circle $x = a\cos t$, $y = a\sin t$ by calculating that $s = \int_0^t \sqrt{a^2}dt = at$. Thus $t = \frac{s}{a}$ and the unit-speed representation is $x = a\cos\frac{s}{a}$, $y = a\sin\frac{s}{a}$.

Note that unit-speed parametrization is not always possible – in some cases the antiderivative $\int \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$ cannot be found as an elementary function. In some other cases, it may not be possible to solve $s = s(t) = \int_a^t \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$ for t. A class of curves that always allows the parametrization by arc length includes the curves we consider next.

Regular curves. A curve $\boldsymbol{\gamma}(t) = (x(t), y(t), z(t))$ is said to be **regular** if the functions $x(t)$, $y(t)$, and $z(t)$ are continuous with continuous derivatives and such that the tangent vector $(x'(t), y'(t), z'(t))$ is not zero.

Considering regular curves rules out some pathological cases. In particular,

- Continuity requirement guarantees no holes or jumps.
- Continuity of the derivative guarantees no corners or sharp turns (the graph of the absolute value function has a corner and the graph of $y = x^{2/3}$ has a sharp turn).
- This condition guarantees that the tangent vector of the **unit length** can be found at every point. This condition also implies that the reparametrization by arc length will be possible.

The same curve can have a regular and a non-regular parametrization. For example, the x-axis can be given as $(t, 0, 0)$ but also as $(t^3, 0, 0)$. The first parametrization is regular while the second is not since the derivative $(3t^2, 0, 0)$ is 0 ad $t = 0$.

Practice problems.

1. Find an equation of the line through the point $(-2, 4, 10)$ and parallel to the vector $(3, 1, -8)$. Check if $(4, 6, -6)$ and $(1, 4, -4)$ are on the line.
2. Find an equation of the line through the points $(3, 1, -1)$ and $(3, 2, -6)$.

3. Find the equation of tangent line at the point where $t = 0$. Find also the unit tangent vector $\mathbf{T}$ as function of parameter t.

   (a) Line $x = 1 + t \quad y = 2 - 2t \quad z = 1 + 2t$.

   (b) Circle in a horizontal plane $x = \cos t \quad y = \sin t \quad z = 2$.

   (c) Circular helix $x = \cos t \quad y = \sin t \quad z = t$.

   (d) Ellipse in the intersection of the cylinder $x^2 + y^2 = 1$ with the plane $y + z = 2$ (find parametric equations before finding the tangent line).

4. Consider the curve C which is the intersection of the surfaces $x^2 + y^2 = 9$ and $z = 1 - y^2$.
   (a) Find the parametric equations that represent the curve C.
   (b) Find the equation of the tangent line to the curve C at point $(0, 3, -8)$.

5. Consider the curve C which is the intersection of the surfaces $y^2 + z^2 = 16$ and $x = 8 - y^2 - z$.
   (a) Find the parametric equations that represent the curve C.
   (b) Find the equation of the tangent line to the curve C at point $(-8, -4, 0)$.

6. Consider the helix $x = a\cos t$, $y = a\sin t$, $z = at$.

   (a) Find the arc length of the helix for $0 \leq t \leq 2\pi$ using the given parametrization.

   (b) Parametrize the helix by the arc length.

   (c) Find the arc length of the helix for $0 \leq t \leq 2\pi$ using the arc-length parametrization.

7. Parametrize the curve $x = e^t \cos t$, $y = e^t \sin t$, $z = e^t$ by the arc length.

Solutions. 1. $x = -2+3t \quad y = 4+t \quad z = 10-8t$. Yes. No. 2.$x = 3 \quad y = 1+t \quad z = -1-5t$ (or $x = 3 \quad y = 2+t \quad z = -6-5t$).

3. (a) Line; the tangent line is the same as the line itself. $\mathbf{T} = \frac{1}{3}(1, -2, 2) = (\frac{1}{3}, \frac{-2}{3}, \frac{2}{3})$ (b) Circle; the tangent line is $x = 1 \quad y = t \quad z = 2$. $\mathbf{T} = (-\sin t, \cos t, 0)$ (c) Helix; the tangent line is $x = 1 \quad y = t \quad z = t$. $\mathbf{T} = \frac{1}{\sqrt{2}}(-\sin t, \cos t, 1)$ (d) Ellipse, the parametric equations are $x = \cos t, \quad y = \sin t, \quad z = 2 - \sin t$. The tangent line is $x = 1 \quad y = t \quad z = 2 - t$. $\mathbf{T} = \frac{1}{\sqrt{1+\cos^2 t}}(-\sin t, \cos t, -\cos t)$

4. (a) $x = 3\cos t$, $y = 3\sin t$, $z = 1 - y^2 = 1 - 9\sin^2 t$. (b) $(0, 3, -8)$ corresponds to $t = \pi/2$. Plugging $\pi/2$ in derivative gives you $(-3, 0, 0)$. Tangent line: $x = -3t \ y = 3 \ z = -8$.

5. (a) $y = 4\cos t$, $z = 4\sin t$, $x = 8 - y^2 - z = 8 - 16\cos^2 t - 4\sin t$. (b) $(-8, -4, 0)$ corresponds to $t = \pi$. Plugging π in derivative gives you $(4, 0, -4)$. Tangent line: $x = 4t - 8 \ y = -4 \ z = -4t$.

6. (a) $L = \int_0^{2\pi} |\gamma'| dt = \int_0^{2\pi} \sqrt{(-a\sin t)^2 + (a\cos t)^2 + a^2} dt = \int_0^{2\pi} \sqrt{2a^2} dt = \sqrt{2}a \int_0^{2\pi} dt = 2\pi\sqrt{2}a$.

(b) $s = \int_0^t \sqrt{a^2 + a^2} dt = \sqrt{2}at \longrightarrow t = \frac{s}{\sqrt{2}a}$. Thus the arc-length parametrization is $x = a\cos\frac{s}{\sqrt{2}a}$, $y = a\sin\frac{s}{\sqrt{2}a}$, $z = \frac{s}{\sqrt{2}}$.

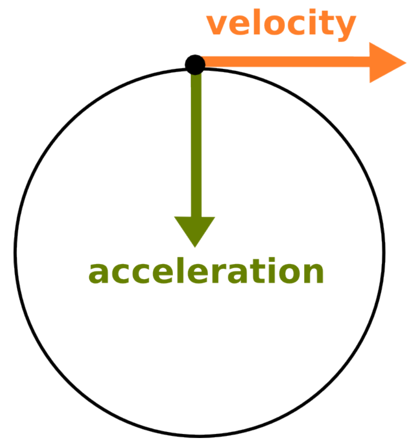
(c) First note that when $t = 0$, $s = 0$ as well and when $t = 2\pi$, then $s = 2\pi\sqrt{2}a$. Thus the bounds for s are 0 and $2\pi\sqrt{2}a$. So, using the arc-length parametrization $L = \int_0^{2\pi\sqrt{2}a} |\gamma'(s)|\, ds = \int_0^{2\pi\sqrt{2}a} 1\, ds = 2\pi\sqrt{2}a$. Since we got the same answer as in part a), we can hypothesize that the length of a curve should not be dependent on specific parametrization.

7. $s = \int_0^t \sqrt{2e^{2t} + e^{2t}} dt = \sqrt{3}e^t - \sqrt{3} \longrightarrow \frac{1}{\sqrt{3}}(s + \sqrt{3}) = e^t \longrightarrow t = \ln(\frac{1}{\sqrt{3}}(s + \sqrt{3}))$. Thus the arc-length parametrization is obtained by substituting $\ln(\frac{1}{\sqrt{3}}(s + \sqrt{3}))$ for t in the given parametrization.

Acceleration Vector and Curvature

The derivative of velocity vector is called **acceleration vector** $\boldsymbol{\gamma}''(t) = (x''(t), y''(t), z''(t))$.

This vector has the same direction as the force needed to keep the particle on the track of $\boldsymbol{\gamma}$ (i.e. make the particle traverse the curve). This force makes a particle traveling along $\boldsymbol{\gamma}$ to stay on this course. Without this force, such particle would continue the motion as indicated by the velocity vector and not stay on the course of $\boldsymbol{\gamma}$.

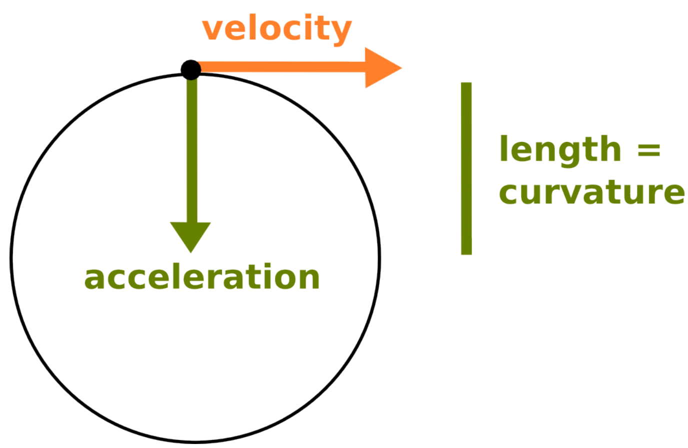


Example. The acceleration of the circle $x = a\cos t$, $y = a\sin t$ is $x'' = -a\cos t$, $y = -a\sin t$ so $\boldsymbol{\gamma}''(t) = -\boldsymbol{\gamma}(t)$ in this case. Thus, the acceleration is directed towards the center and of constant magnitude.

Consider the special case when the curve is parametrized with respect to arc length s. In this case, *the length of the acceleration vector* is called **the curvature**

$$\kappa(s) = |\mathbf{T}'(s)| = |\boldsymbol{\gamma}''(s)|.$$

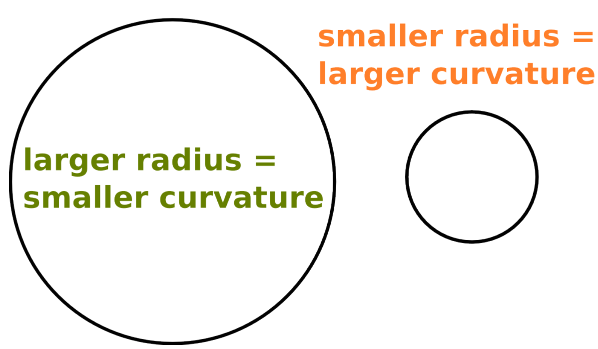
Thus, the curvature measures the intensity of the force needed to keep the particle on the track of $\boldsymbol{\gamma}$. This formula also illustrates that the curvature measures how quickly the curve changes direction since it measures the rate the of change of direction vector as the value of parameter s changes.



Example 1. To find the curvature of line $\boldsymbol{\gamma} = (x_0 + at, y_0 + bt, z_0 + ct)$, compute that $s = \int_0^t |\boldsymbol{\gamma}'(t)|dt = \sqrt{a^2 + b^2 + c^2}\,t$ so that $\mathbf{T} = \boldsymbol{\gamma}'(s) = \frac{1}{\sqrt{a^2+b^2+c^2}}(a, b, c)$. Since this is a constant vector, $\mathbf{T}' = (0, 0, 0)$ and so $\kappa = |\mathbf{T}'| = 0$. Thus, the curvature of a line is 0. This agrees with the intuitive idea of the curvature: *a straight line does not curve at all.*

Example 2. To find the curvature of the circle $\boldsymbol{\gamma} = (a\cos t, a\sin t)$, recall that the arc-length parametrization is given by $t = \frac{s}{a}$ so that $\boldsymbol{\gamma} = (a\cos\frac{s}{a}, a\sin\frac{s}{a})$.

Compute that $\mathbf{T} = \boldsymbol{\gamma}' = (-\sin\frac{s}{a}, \cos\frac{s}{a})$ and so $\mathbf{T}' = \boldsymbol{\gamma}'' = (-\frac{1}{a}\cos\frac{s}{a}, -\frac{1}{a}\sin\frac{s}{a})$ which gives us $\kappa = |\mathbf{T}'| = \sqrt{\frac{1}{a^2}\cos^2\frac{s}{a} + \frac{1}{a^2}\sin^2\frac{s}{a}} = \sqrt{\frac{1}{a^2}} = \frac{1}{a}$. Thus, a circle of *small radius* has *large curvature* and a circle of *large radius* has a *small curvature.*



This example also agrees with the intuitive idea of the curvature: the circle curves equally at every point. So, *the curvature of a circle is constant.*

These two examples generalize so that a fairly straight curve has the direction of derivative vector changing very slowly, so that curvature is small. If a curve bends or twists sharply, the direction of derivative vector changes quickly.

The curvature in non-unit-speed parametrizations. When the unit-speed parametrization leads to complex formulas, it may be easier to compute the curvature using any given parametrization. The formula for computing κ is

$$\boxed{\kappa = \frac{|\gamma' \times \gamma''|}{|\gamma'|^3}}$$

and can be obtained as follows.

1. Start by observing that $\boldsymbol{\gamma}'(t)$ and $\mathbf{T}(s)$ are related by

$$\boldsymbol{\gamma}'(t) = \frac{d\boldsymbol{\gamma}}{dt} = \frac{d\boldsymbol{\gamma}}{ds}\frac{ds}{dt} = \mathbf{T}\frac{ds}{dt}.$$

2. Find derivative of the equation from (1) with respect to t.

$$\boldsymbol{\gamma}'' = \frac{d\mathbf{T}}{dt}\frac{ds}{dt} + \mathbf{T}\frac{d^2s}{dt^2} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt}\frac{ds}{dt} + \mathbf{T}\frac{d^2s}{dt^2} = \mathbf{T}'(\frac{ds}{dt})^2 + \mathbf{T}\frac{d^2s}{dt^2}.$$

3. Cross multiply this equation by $\boldsymbol{\gamma}' = \mathbf{T}\frac{ds}{dt}$, to get

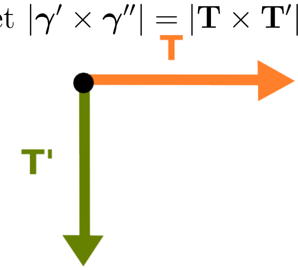
$$\boldsymbol{\gamma}' \times \boldsymbol{\gamma}'' = \mathbf{T} \times \mathbf{T}'(\frac{ds}{dt})^3 + \mathbf{T} \times \mathbf{T}\frac{d^2s}{dt^2}\frac{ds}{dt}.$$

Since $\mathbf{T} \times \mathbf{T} = \mathbf{0}$,

$$\boldsymbol{\gamma}' \times \boldsymbol{\gamma}'' = \mathbf{T} \times \mathbf{T}'(\frac{ds}{dt})^3.$$

4. Consider the length of both sides of the last equation and get $|\boldsymbol{\gamma}' \times \boldsymbol{\gamma}''| = |\mathbf{T} \times \mathbf{T}'||\frac{ds}{dt}|^3$.

Now note that the fact that length of $\mathbf{T}$ is 1 implies that $1 = |\mathbf{T}| = \sqrt{\mathbf{T}\cdot\mathbf{T}} \Rightarrow \mathbf{T}\cdot\mathbf{T} = 1$. Differentiating $\mathbf{T}\cdot\mathbf{T} = 1$ we have that $\mathbf{T}'\cdot\mathbf{T} + \mathbf{T}\cdot\mathbf{T}' = 0 \Rightarrow 2\mathbf{T}'\cdot\mathbf{T} = 0 \Rightarrow \mathbf{T}'\cdot\mathbf{T} = 0$. Thus $\mathbf{T}'$ and $\mathbf{T}$ are orthogonal.



So the length of the cross product $|\mathbf{T} \times \mathbf{T}'|$ can be computed as $|\mathbf{T}'||\mathbf{T}|\sin(\frac{\pi}{2})$ which is equal to $|\mathbf{T}'| = \kappa$ since $|\mathbf{T}| = 1$. Also note that $|\frac{ds}{dt}| = |\boldsymbol{\gamma}'|$. Thus we have that $|\boldsymbol{\gamma}' \times \boldsymbol{\gamma}''| = |\kappa||\boldsymbol{\gamma}'|^3$. Solving for κ we obtain the formula for the curvature to be $\kappa = \frac{|\gamma' \times \gamma''|}{|\gamma'|^3}$.

The argument we use to show that $\mathbf{T}\cdot\mathbf{T}=1$ implies $\mathbf{T}'\cdot\mathbf{T}=0$ (i.e. a unit-length vector is orthogonal to its derivative) is applicable to other unit-length, in fact to other *constant-length* vectors. In fact, we shall need to use this argument for $\mathbf{N}$ also. In addition, the converse holds as well: if a vector and its derivative are orthogonal at every point, the length of the vector is constant. Hence, we can show the claim below.

Claim 1. Let $\mathbf{v}(t)$ be a vector function and c be a constant. Then $\mathbf{v}$ has constant length if and only if $\mathbf{v}$ and $\mathbf{v}'$ are orthogonal. This statement can be written by formulas as

$$|\mathbf{v}| = \text{ constant} \quad \text{ if and only if } \quad \mathbf{v}'\cdot\mathbf{v}=0.$$

Before proving Claim 1, we review some techniques of proving mathematical statements.

Some methods of proving mathematical statements. Many mathematical statements we encounter in this course are formulated as implications or equivalences.

(1) **Implication**. An implication is an expression of the form "if p, then q" (equivalently, p implies q or q follows from p). In symbols, this is expressed as $p\Rightarrow q$. To prove an implication of this form, you can assume that p holds and deduce that q holds in this case as well.

Another way to prove an implication "p implies q" is to use its **contrapositive** "not p implies not q." For example, a contrapositive of the statement "If a polygon has three sides, it is a triangle." is "If a polygon is not a triangle, then it does not have three sides".

Yet another way to prove an implication of the form "p implies q" is to show that simultaneously assuming p and the opposite of q yields a contradiction. This form of the proof is called **reductio ad absurdum** (Latin: "reduction to the absurd"). For example, let us show that if $a\neq 0$, an equation of the form $ax=b$ has only one solution. Let us assume the opposite, that there are two different solutions x_1 and x_2. Since both solutions satisfy the equation, we have that $ax_1=b$ and $ax_2=b$. If we subtract the second equation from the first, we get $a(x_1-x_2)=0$. When we divide by $a\neq 0$ we have that $x_1-x_2=0$ so $x_1=x_2$. But this is in contradiction with the assumption that x_1 is different than x_2.

(2) **Equivalence**. If p implies q and q implies p as well, we say that p and q are equivalent. In symbols, this is written as $p\Leftrightarrow q$. In this case, p is true if and only if q is true. A statement of the form "p if and only if q" can be proven in two steps: (1) show that q holds assuming p and, (2) show that p holds assuming q.

(3) **Chain of Equivalences.** A statement of the form "p, q and r are all equivalent", can be shown by demonstrating that $p\Leftrightarrow q$ and $q\Leftrightarrow r$ because then $p\Leftrightarrow r$ also. Alternatively, if a closed chain of implications $p\Rightarrow q$, $q\Rightarrow r$, and $r\Rightarrow p$ is shown to hold, then all three conditions are equivalent. We will use this argument to prove Corollary 2.

Proof of Claim 1. Let us use c to denote the constant from the statement. First, let us show direction $\Rightarrow$ meaning that we want to assume that $|\mathbf{v}|=c$ and show that $\mathbf{v}'\cdot\mathbf{v}=0$. We can follow the same arguments as in case when $\mathbf{v}=\mathbf{T}$ and $c=1$. Since $\mathbf{v}\cdot\mathbf{v}=|\mathbf{v}|^2$, we have that $\mathbf{v}\cdot\mathbf{v}=c^2$. By differentiating, we have that $\mathbf{v}'\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{v}'=0\Rightarrow\mathbf{v}'\cdot\mathbf{v}+\mathbf{v}'\cdot\mathbf{v}=0\Rightarrow 2\mathbf{v}'\cdot\mathbf{v}=0$. Dividing the first and last expression by 2, we obtain that $\mathbf{v}'\cdot\mathbf{v}=0$.

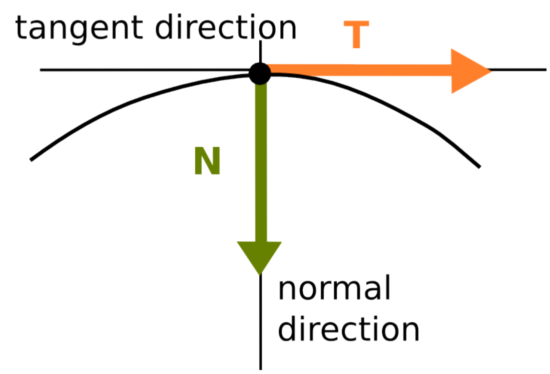
Let us show the converse $\Leftarrow$ now. Assume that $\mathbf{v}'\cdot\mathbf{v}=0$ and try to use the steps of the previous direction in reverse: multiply by 2 to get $0=2\mathbf{v}'\cdot\mathbf{v}=\mathbf{v}'\cdot\mathbf{v}+\mathbf{v}'\cdot\mathbf{v}=\mathbf{v}'\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{v}'$. Note that this last expression is the product rule applied to $(\mathbf{v}\cdot\mathbf{v})'$ so $(\mathbf{v}\cdot\mathbf{v})'=0$. Integrating both sides, we obtain that $\mathbf{v}\cdot\mathbf{v}$ is constant. If we denote this constant by c^2, we obtain that $|\mathbf{v}|^2=\mathbf{v}\cdot\mathbf{v}=c^2$ and so $|\mathbf{v}|=c$. This completes the proof.

Normal and Binormal Vectors. Torsion

The unit vector in the direction of $\mathbf{T}'$ is called the **principal normal vector** and is denoted by $\mathbf{N}$. Thus,

$$\mathbf{N}(s) = \frac{\mathbf{T}'(s)}{|\mathbf{T}'(s)|} = \frac{\mathbf{T}'(s)}{\kappa(s)}$$

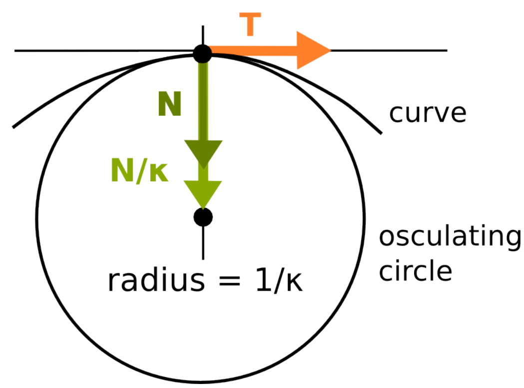
Recall that $\mathbf{T}'$ and $\mathbf{T}$ are orthogonal and so **N and T are orthogonal**.



The plane determined by vectors $\mathbf{T}$ and $\mathbf{N}$ is called the **osculating plane**. If curve is planar, this plane is the plane in which curve lies. The name osculating comes from the following construction. At a point on the curve, consider the circle of radius $\frac{1}{\kappa}$ with the same tangent as the curve and with the center on the normal direction line. Thus, it lies in the osculating plane.

This circle is called the **osculating circle.** The name comes from Latin word osculari means to kiss (osculum meaning kiss) and refers to the fact that curve and osculating circle are tangential curves.

The osculating circle is an approximation of the curve with a circle at a point - it has the same curvature, tangent and normal vector as the curve at a point. The center of the circle can be computed by the formula



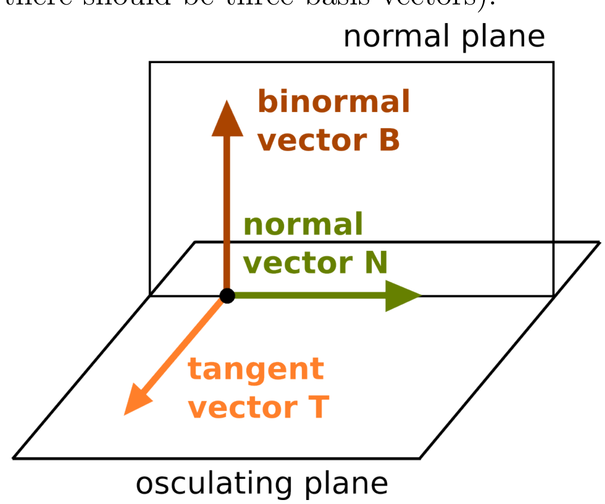
$$\gamma + \frac{1}{\kappa}\mathbf{N}.$$

To complete an apparatus that fully describes the space curves, we need one more vector besides $\mathbf{T}$ and $\mathbf{N}$ (this should not be surprising considering that there should be three basis vectors).

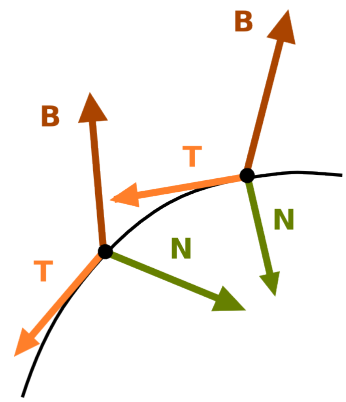
Consider the cross product of $\mathbf{T}$ and $\mathbf{N}$.

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$

This vector is called the **binormal vector**. By definition, $\mathbf{B}$ is perpendicular both to $\mathbf{T}$ and $\mathbf{N}$ and has unit length also since $\mathbf{T}$ and $\mathbf{N}$ are of unit length. The plane determined by $\mathbf{N}$ and $\mathbf{B}$ is called the **normal plane**. All the lines in the normal plane are perpendicular to the tangent vector.



Vector $\mathbf{B}$ indicates **the direction in which the curve departs from being a planar curve.**



The triple $\mathbf{T}, \mathbf{N}, \mathbf{B}$ is call the **moving frame**. Watch the animation on Wikipedia (do a search for "Frenet-Serret formulas") to understand the moving frame better.

The moving frame $\mathbf{T}, \mathbf{N}, \mathbf{B}$ is an orthonormal basis meaning that each vector is of unit length, that each two are perpendicular and that every other vector can be uniquely represented as a linear combination of these three. This means that they take over the role of the usual basis vectors $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, $\mathbf{k} = (0, 0, 1)$, at a point on the curve. The standard basis may be unfitted for describing geometry of the curve. Thus, the change of basis is needed to obtain a basis better fitted for the curve.

Torsion. While $\mathbf{B}$ indicates the direction in which the curve departs from being a planar curve, the **torsion** τ measures the **extent at which the curve departs from being a plane curve**, that is, it measures **how much the curve departs from the osculating plane.** Torsion τ can be computed via $\mathbf{N}$ and $\mathbf{B}$ as follows.

Recall that $\mathbf{B}$ is defined as $\mathbf{T}\times\mathbf{N}$. Differentiating with respect to s we obtain $\mathbf{B}' = \mathbf{T}'\times\mathbf{N}+\mathbf{T}\times\mathbf{N}'$. Substituting that $\mathbf{T}' = \kappa\mathbf{N}$ and noting that $\mathbf{N}\times\mathbf{N} = \mathbf{0}$, we obtain that $\mathbf{B}' = \mathbf{0} + \mathbf{T}\times\mathbf{N}'$. Thus, $\mathbf{B}'$ is orthogonal to both $\mathbf{T}$ and $\mathbf{N}'$. On the other hand, $\mathbf{N}$ is also orthogonal to $\mathbf{T}$ and $\mathbf{N}'$. We already noted that $\mathbf{N}$ is orthogonal to $\mathbf{T}$, and $\mathbf{N}$ and $\mathbf{N}'$ are orthogonal by Claim 1. So, $\mathbf{B}'$ and $\mathbf{N}$ are orthogonal to the same two vectors and that makes them colinear. Hence, $\mathbf{B}'$ is a **multiple of N**. The proportionality constant is denoted by $-\tau(s)$ and so

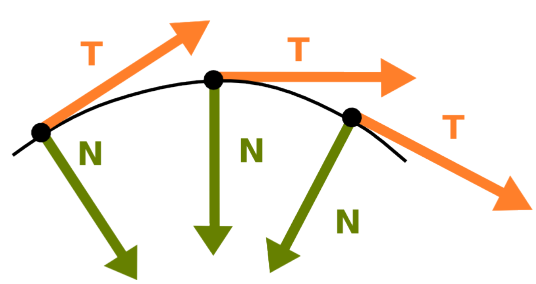
$$\mathbf{B}' = -\tau\mathbf{N}$$

Let us (dot) multiply each side of the equation $\mathbf{B}' = -\tau\mathbf{N}$ by $\mathbf{N}$. This gives us that the torsion τ can be computed as

$$\boxed{\tau = -\mathbf{B}'\cdot\mathbf{N}}$$

For a plane curve, the curve lies in the osculating plane, thus $\tau = 0$. If $\tau > 0$, the curve twists towards the side that $\mathbf{B}$ points to. If $\tau < 0$, the curve twists away from the side that $\mathbf{B}$ points to.

Example. Reparametrizing by the arc length first, compute the curvature, torsion and the moving frame of the helix $(a\cos t, a\sin t, at)$.



Solution. The arc-length parametrization of the helix is given by $t = \frac{s}{\sqrt{2}a}$ (see problem 6 of previous section.) Thus $\boldsymbol{\gamma}(s) = (a\cos\frac{s}{\sqrt{2}a}, a\sin\frac{s}{\sqrt{2}a}, \frac{s}{\sqrt{2}})$ so that $\mathbf{T} = \boldsymbol{\gamma}' = \frac{1}{\sqrt{2}}(-\sin\frac{s}{\sqrt{2}a}, \cos\frac{s}{\sqrt{2}a}, 1)$ and $\mathbf{T}' = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}a}(-\cos\frac{s}{\sqrt{2}a}, -\sin\frac{s}{\sqrt{2}a}, 0)$.

So $\kappa = |\mathbf{T}'| = \frac{1}{2a}$. Thus, the helix has constant curvature of $\frac{1}{2a}$.
$\mathbf{N} = \frac{\mathbf{T}'}{|\mathbf{T}'|} = (-\cos\frac{s}{\sqrt{2}a}, -\sin\frac{s}{\sqrt{2}a}, 0)$ and $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{1}{\sqrt{2}}(\sin\frac{s}{\sqrt{2}a}, -\cos\frac{s}{\sqrt{2}a}, 1)$.

From here we find $\mathbf{B}' = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}a}(\cos\frac{s}{\sqrt{2}a}, \sin\frac{s}{\sqrt{2}a}, 0) = \frac{1}{2a}(\cos\frac{s}{\sqrt{2}a}, \sin\frac{s}{\sqrt{2}a}, 0)$ and $\tau = -\mathbf{B}'\cdot\mathbf{N} = \frac{1}{2a}$. Thus the torsion is also constant.

Non-unit-speed parametrization formulas. In cases when it is hard to find the unit-speed parametrizaton, it may be useful to have the formulas for $\mathbf{T}, \mathbf{N}, \mathbf{B}$, κ and τ given in terms of arbitrary parametrization. We have seen that $\mathbf{T} = \frac{\gamma'}{|\gamma'|}$ and that $\kappa = \frac{|\gamma'\times\gamma''|}{|\gamma'|^3}$ in this case. The formulas for $\mathbf{N}$, $\mathbf{B}$ and τ in any parametrization are given below.

$$\mathbf{T} = \frac{\gamma'}{|\gamma'|}, \quad \kappa = \frac{|\gamma'\times\gamma''|}{|\gamma'|^3}, \quad \mathbf{B} = \frac{\gamma'\times\gamma''}{|\gamma'\times\gamma''|}, \quad \mathbf{N} = \mathbf{B}\times\mathbf{T}, \quad \text{and} \quad \tau = \frac{(\gamma'\times\gamma'')\cdot\gamma'''}{|\gamma'\times\gamma''|^2}$$

The formula for τ can be shortened using the following notation. Let $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ denotes the value of $3\times$ 3 determinant with columns $\mathbf{a}, \mathbf{b}, \mathbf{c}$. Then $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ and so $\tau = \frac{[\gamma', \gamma'', \gamma''']}{|\gamma'\times\gamma''|^2}$.

Example. Using the given parametrization, compute the curvature, torsion and the moving frame of the helix $(a\cos t, a\sin t, at)$.

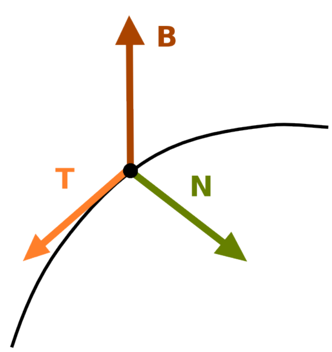
Solution. Here $\gamma' = (-a\sin t, a\cos t, a)$, $|\gamma'| = \sqrt{2a^2} = \sqrt{2}a$, $\gamma'' = (-a\cos t, -a\sin t, 0)$ and $\gamma'\times\gamma'' = a^2(\sin t, -\cos t, 1)$. Thus $|\gamma'\times\gamma''| = \sqrt{2}a^2$. This gives us $\mathbf{T} = \frac{1}{\sqrt{2}}(-\sin t, \cos t, 1)$ and $\kappa = \frac{\sqrt{2}a^2}{(\sqrt{2})^3a^3} = \frac{1}{2a}$ that agrees to what we calculated before.

Then calculate that $\mathbf{B} = \frac{1}{\sqrt{2a^2}}a^2(\sin t, -\cos t, 1) = \frac{1}{\sqrt{2}}(\sin t, -\cos t, 1)$, $\mathbf{N} = \mathbf{B}\times\mathbf{T} = (-\cos t, -\sin t, 0)$ and $\tau = \frac{1}{2a}$ that also agrees what we calculated before.

Frenet-Serret apparatus

The moving frame $\mathbf{T}, \mathbf{N}, \mathbf{B}$ together with curvature κ and torsion τ provide a complete list of geometrical properties of the curve. The three vector functions $\mathbf{T}, \mathbf{N}, \mathbf{B}$ and two scalar functions κ and τ are known as the *Frenet-Serret apparatus.*

The fundamental theorem of differential geometry of curves, known also as the Frenet-Serret theorem states that **a curve is completely determined by its curvature and torsion** alone. Specifically, this means that the moving frame $\mathbf{T}, \mathbf{N}, \mathbf{B}$ is completely determined by the two functions κ and τ. This statement follows from two facts.



1. The derivatives $\mathbf{T}', \mathbf{N}', \mathbf{B}'$ can be represented as

$$\begin{aligned} \mathbf{T}' &= \kappa\mathbf{N} \\ \mathbf{N}' &= -\kappa\mathbf{T} + \tau\mathbf{B} \\ \mathbf{B}' &= -\tau\mathbf{N} \end{aligned}$$

The fist equations follows from the definition of the normal vector $\mathbf{N}$ and the third follows from the definition of torsion τ.

To obtain the second equation, note that the orthonormal system $\mathbf{T}, \mathbf{N}, \mathbf{B}$ is such that

$$\mathbf{T}\times\mathbf{N} = \mathbf{B}, \quad \mathbf{N}\times\mathbf{B} = \mathbf{T}, \quad \text{and} \quad \mathbf{B}\times\mathbf{T} = \mathbf{N}$$

Differentiating the last formula, we have $\mathbf{N}' = \mathbf{B}' \times \mathbf{T} + \mathbf{B} \times \mathbf{T}' = -\tau \mathbf{N} \times \mathbf{T} + \mathbf{B} \times \kappa \mathbf{N} = \tau \mathbf{B} - \kappa \mathbf{T}$ which is the second equation.

This system of first order differential equations is known as Frenet-Serret system.

2. The theory of first order differential equations guarantees that the above system has a unique solution for any value of initial conditions and any values of κ and τ.

The Frenet-Serret formulas were first obtained by Frenet in 1847 in his dissertation. Serret first published them in 1851. Their importance lies in their applications in several different areas. In **kinematics**, the Frenet-Serret formulas describe the kinematic properties of a particle which moves along a curve. In **the life sciences**, particularly in models of microbial motion, considerations of the Frenet-Serret frame have been used to explain the mechanism by which a moving organism in a viscous medium changes its direction. In **relativity theory** a classical coordinate system is not convenient for a trajectory. In cases like that the Frenet-Serret frame is used. In **aerodynamics**, aircraft maneuvers can be expressed in terms of the moving frame. Other areas of applications include **materials science, elasticity theory, and computer graphics.**

The Frenet-Serret theorem has the following corollaries.

- A curve with zero curvature and torsion is a straight line or a line segment. (Moreover, it can be shown that a curve with constant velocity is a straight line).
- A curve with zero torsion is a planar curve.
- A curve with constant curvature and zero torsion is a circle or a part of it.
- A curve with constant curvature and constant torsion is a helix or a part of it.

To illustrate the methods for proving these corollaries, we prove an extended version of the second statement.

Corollary 2. If $\boldsymbol{\gamma}$ is a regular curve, the following conditions are equivalent.

1. $\boldsymbol{\gamma}$ is a planar curve (i.e. there is a plane that contains $\boldsymbol{\gamma}$).
2. The torsion τ is identically zero.
3. The binormal vector $\mathbf{B}$ is constant.

Proof of Corollary 2. We shall show that the three conditions are equivalent by showing that (1) implies (2), (2) implies (3) and (3) implies (1).

Assume that $\boldsymbol{\gamma}$ lies in the plane $ax + by + cz = d$. Then this plane is the osculating plane of the curve $\boldsymbol{\gamma}$. So the vector $\mathbf{B}$ is orthogonal to this plane at every point on $\boldsymbol{\gamma}$. Since $\mathbf{v} = (a, b, c)$ is vector perpendicular to the plane, the vector $\mathbf{B}$ at every point is either $\frac{1}{|\mathbf{v}|}\mathbf{v}$ or $\frac{-1}{|\mathbf{v}|}\mathbf{v}$. In both cases, $\mathbf{B}$ is a constant vector. Thus $\mathbf{B}' = 0$ and so $\tau = -\mathbf{B}' \cdot \mathbf{N} = 0$. Thus, the torsion is identically zero. This proves that (1) implies (2).

Assume now that (2) holds and show (3). If τ is identically zero, then $\mathbf{B}' = -\tau \mathbf{N}$ is also identically zero. Thus $\mathbf{B}$ is a constant vector.

Finally, let us show that (3) implies (1). Recall that an equation describing any point $\mathbf{x} = (x, y, z)$ of a plane passing $\mathbf{x}_0 = (x_0, y_0, z_0)$ perpendicular to a vector $\mathbf{v} = (a, b, c)$ is $(\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{v} = 0$. So, to show that $\boldsymbol{\gamma}$ lies in a plane, it is sufficient to show that the dot product $(\boldsymbol{\gamma}(s) - \boldsymbol{\gamma}(0)) \cdot \mathbf{B} = 0$.

Consider the derivative of this dot product

$$((\boldsymbol{\gamma}(s) - \boldsymbol{\gamma}(0)) \cdot \mathbf{B})' = \boldsymbol{\gamma}'(s) \cdot \mathbf{B} + (\boldsymbol{\gamma}(s) - \boldsymbol{\gamma}(0)) \cdot \mathbf{B}'$$

The second term $(\boldsymbol{\gamma}(s) - \boldsymbol{\gamma}(0)) \cdot \mathbf{B}'$ is zero since $\mathbf{B}' = 0$. The vector $\boldsymbol{\gamma}'(s)$ is $\mathbf{T}$ and $\mathbf{T}$ is perpendicular to $\mathbf{B}$ so the first term $\boldsymbol{\gamma}'(s) \cdot \mathbf{B}$ is zero as well. Hence $(\boldsymbol{\gamma}(s) - \boldsymbol{\gamma}(0)) \cdot \mathbf{B}$ is constant for every s. But since for $s = 0$ this expression is 0, this constant has to be zero for every s. So, $(\boldsymbol{\gamma}(s) - \boldsymbol{\gamma}(0)) \cdot \mathbf{B} = 0$. This shows that $\boldsymbol{\gamma}$ lies in the plane passing $\boldsymbol{\gamma}(0)$ that is perpendicular to $\mathbf{B}$. So (3) implies (1) and this finishes the proof.

Practice Problems

1. Find the curvature of the twisted cubic (t, t^2, t^3).

2. Find the formula calculating the curvature of the parabola $y = x^2$ in terms of x. Use it to calculate the curvature at $x = 0$ and $x = 1$.

3. Find the osculating circle for parabola $y = x^2$ at the origin.

4. Find the Frenet-Serret apparatus for the curve $(\frac{5}{13}\cos s, \frac{8}{13} - \sin s, \frac{-12}{13}\cos s)$. Note first that the curve is parametrized by the arc length.

5. Find the osculating plane of the curve from the previous problem.

6. Find equation of one of the cylindrical surfaces that intersecting the plane from (5) creates the curve from (4).

7. Find the Frenet-Serret apparatus for the twisted cubic (t, t^2, t^3).

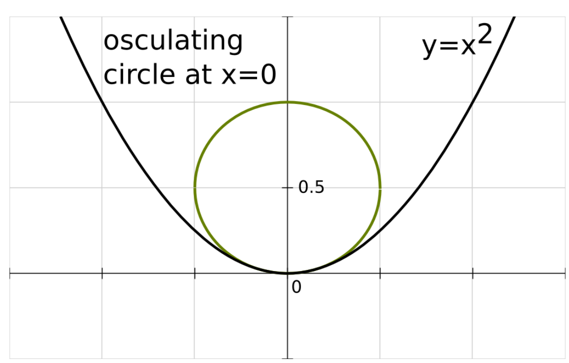
Solutions.

1. $\boldsymbol{\gamma}' = (1, 2t, 3t^2)$, $|\boldsymbol{\gamma}'| = \sqrt{1 + 4t^2 + 9t^4}$, $\boldsymbol{\gamma}'' = (0, 2, 6t)$, $\boldsymbol{\gamma}' \times \boldsymbol{\gamma}'' = (6t^2, -6t, 2)$, $|\boldsymbol{\gamma}' \times \boldsymbol{\gamma}''| = \sqrt{36t^4 + 36t^2 + 4} = 2\sqrt{9t^4 + 9t^2 + 1} \longrightarrow \kappa = \frac{2\sqrt{9t^4+9t^2+1}}{(1+4t^2+9t^4)^{3/2}}$.

2. Considering the parametrization $(x, x^2, 0)$, we find $\boldsymbol{\gamma}' = (1, 2x, 0)$, $|\boldsymbol{\gamma}'| = \sqrt{1 + 4x^2}$, $\boldsymbol{\gamma}'' = (0, 2, 0)$, $\boldsymbol{\gamma}' \times \boldsymbol{\gamma}'' = (0, 0, 2)$. Thus, $\kappa = \frac{2}{(1+4x^2)^{3/2}}$. $\kappa(0) = 2$, $\kappa(1) = 0.18$.

3. The radius of the circle is $\frac{1}{\kappa(0)} = \frac{1}{2}$ by previous problem. The center is given by $\boldsymbol{\gamma}(0) + \frac{1}{\kappa(0)}\mathbf{N}(0)$. Since $\mathbf{N}$ is orthogonal to $\mathbf{T}$ and the tangent to parabola at the origin is the x-axis, $\mathbf{N}$ is parallel to y-axis and so $\mathbf{N} = (0, 1)$.

   $\boldsymbol{\gamma}(0) = (0, 0)$ and so the center is at $(0, 0) + \frac{1}{2}(0, 1) = (0, \frac{1}{2})$. Hence the circle is given by $(\frac{1}{2}\cos t, \frac{1}{2} + \frac{1}{2}\sin t)$.



4. The curve is unit-speed since $\boldsymbol{\gamma}' = (\frac{-5}{13}\sin s, -\cos s, \frac{12}{13}\sin s)$ and $|\boldsymbol{\gamma}'| = \sqrt{\frac{25}{169}\sin^2 s + \cos^2 s + \frac{144}{169}\sin^2 s} = \sqrt{\sin^2 s + \cos^2 s} = \sqrt{1} = 1$. Thus, $\mathbf{T} = \boldsymbol{\gamma}'$. $\mathbf{T}' = \boldsymbol{\gamma}'' = (\frac{-5}{13}\cos s, \sin s, \frac{12}{13}\cos s)$ $\kappa = |\mathbf{T}'| = \sqrt{\frac{25}{169}\cos^2 s + \sin^2 s + \frac{144}{169}\cos^2 s} = \sqrt{\cos^2 s + \sin^2 s} = \sqrt{1} = 1$, thus $\mathbf{N} = \frac{\mathbf{T}'}{1} = \mathbf{T}'$. Finally, $\mathbf{B} = \mathbf{T} \times \mathbf{N} = (\frac{-12}{13}, 0, \frac{-5}{13})$ so $\mathbf{B}' = \mathbf{0}$ and thus $\tau = 0$.

   Note that the results $\kappa = 1$ and $\tau = 0$ tells us that, although not obvious at first, the curve is a circle in a plane.

5. From problem 4. we know that the curve lies in a plane. This plane is equal to the osculating plane and so the osculating plane does not depend on the choice of the point on the curve. The vector $\mathbf{B} = (\frac{-12}{13}, 0, \frac{-5}{13})$ is the normal vector of the osculating plane. For the point on the plane, we can use any point on the curve, for example the point corresponding to $s = \pi/2$ (to make the most terms zero) and so we have $(0, \frac{8}{13} - 1, 0)$. We obtain the equation of the plane $\frac{-12}{13}(x-0) + 0(y + \frac{5}{13}) - \frac{5}{13}(z-0) = 0 \longrightarrow \frac{-12}{13}x - \frac{5}{13}z = 0 \longrightarrow -12x - 5z = 0 \longrightarrow z = \frac{-12}{5}x$.

   Note that without calculating τ or $\mathbf{B}$, it is impossible to tell that the curve lies in a plane simply by looking at the parametrization.

6. One of the simplest solutions is to consider the projections of the curve in xy plane. We obtain the ellipse $x = \frac{5}{13}\cos s$ and $y = \frac{8}{13} - \sin s$. This ellipse can be described by $\frac{x^2}{(5/13)^2} + (y - \frac{8}{13})^2 = 1$. This is an ellipse centered at $(0, \frac{8}{13})$ with semiaxes $\frac{5}{13}$ and 1.

7. In problem 1., we have found that $\boldsymbol{\gamma}' = (1, 2t, 3t^2)$, $|\boldsymbol{\gamma}'| = \sqrt{1 + 4t^2 + 9t^4}$, $\boldsymbol{\gamma}' \times \boldsymbol{\gamma}'' = (6t^2, -6t, 2)$, $|\boldsymbol{\gamma}' \times \boldsymbol{\gamma}''| = \sqrt{36t^4 + 36t^2 + 4} = 2\sqrt{9t^4 + 9t^2 + 1}$, and $\kappa = \frac{|\boldsymbol{\gamma}' \times \boldsymbol{\gamma}''|}{|\boldsymbol{\gamma}'|^3} = \frac{2\sqrt{9t^4+9t^2+1}}{(1+4t^2+9t^4)^{3/2}}$.

   $\mathbf{T} = \frac{\boldsymbol{\gamma}'}{|\boldsymbol{\gamma}'|} = \frac{1}{\sqrt{1+4t^2+9t^4}}(1, 2t, 3t^2)$.

   $\mathbf{B} = \frac{\boldsymbol{\gamma}' \times \boldsymbol{\gamma}''}{|\boldsymbol{\gamma}' \times \boldsymbol{\gamma}''|} = \frac{1}{2\sqrt{9t^4+9t^2+1}}(6t^2, -6t, 2)$.

   $\mathbf{N} = \mathbf{B} \times \mathbf{T} = \frac{1}{2\sqrt{1+4t^2+9t^4}\,\sqrt{9t^4+9t^2+1}}(-18t^3 - 4t, 2 - 18t^4, 12t^3 + 6t)$, $\boldsymbol{\gamma}''' = (0, 0, 6)$.

   $\tau = \frac{(\boldsymbol{\gamma}' \times \boldsymbol{\gamma}'') \cdot \boldsymbol{\gamma}'''}{|\boldsymbol{\gamma}' \times \boldsymbol{\gamma}''|^2} = \frac{12}{4(9t^4+9t^2+1)} = \frac{3}{9t^4+9t^2+1}$.