A simple electronic circuit to demonstrate bifurcation and chaos

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Chaos has generated much interest recently, and many of the important features of nonlinear dynamical systems can be demonstrated using the simple electronic circuit described here.

The study of chaos has generated enormous interest in exploring the complexity of behaviour shown by nonlinear circuits and other dynamical systems (Rodgers 1992, Chacón *et al* 1992). A simple electronic circuit, which could form an introduction to project work in the field of chaos, is Chua's autonomous circuit.

Chua's circuit can be used to demonstrate many of the important features of nonlinear dynamical systems, such as *bifurcation*, *period doubling*, *attractors* and *chaos*. We present here a simple description of the operation of this circuit, a practical electronic implementation, and discuss some of the experimental observations that can be made with it.

Chua's circuit

The basic circuit, shown in figure 1, is reciprocal and consists of only one nonlinear element. The parallel combination of C_2 and L constitutes a lossless resonant circuit. The conductance G provides the coupling between this, the active nonlinear resistor R, and C_1.

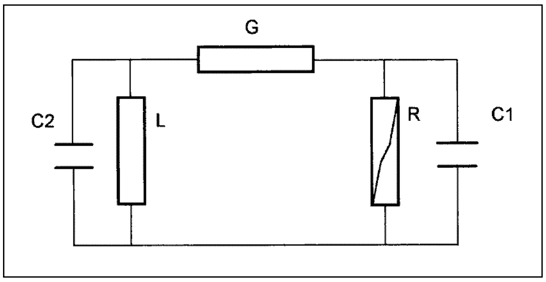

Figure 1. *Schematic diagram of Chua's basic circuit.*

Since R is active, it behaves as a power source. The idealized nonlinear element R is a three-segment piecewise-linear resistor with the transfer characteristic shown in figure 2.

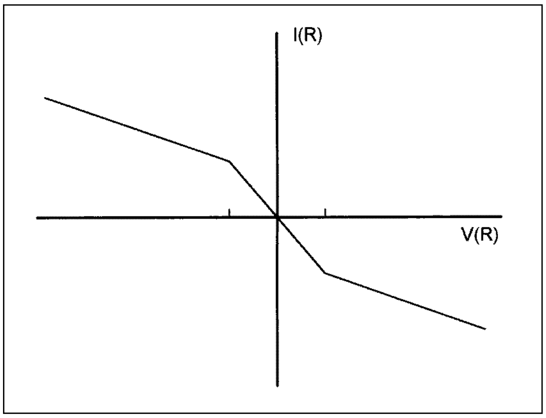

Figure 2. *Idealized transfer characteristic of the nonlinear resistor R.*

The circuit dynamics is described by

$$C_1 \frac{dv_{C1}}{dt} = G(v_{C2} - v_{C1}) - g(v_{C1})$$

$$C_2 \frac{dv_{C2}}{dt} = G(v_{C1} - v_{C2}) + i_L$$

$$L \frac{di_L}{dt} = -v_{C2}$$

where v_{C1}, v_{C2} and i_L denote the voltages and currents across and through the passive components, and g denotes the transfer function of the nonlinear resistor.

We refer the interested reader to the technical papers by Matsumoto *et al* (1985, 1986) for a detailed investigation of these equations.

Practical implementation

The only non-trivial circuit element to implement is the nonlinear resistor R. There are several techniques that have been used in practical circuits. It can be synthesized by a single op-amp, two diodes and resistors (Matsumoto *et al* 1985), two bipolar transistors, two diodes and resistors (Matsumoto *et al* 1986), or two op-amps and six resistors (Kennedy 1993). We have found that the simplest circuit to implement in practice is that of Kennedy using two identical op-amps. This circuit, shown in figure 3, has been used in our undergraduate teaching laboratories for demonstration purposes. The choice of op-amp is not critical, the tolerance on the values of the capacitors and the inductor is $\pm$10%, and the fixed resistors' tolerance should be $\pm$5% or better. In an experiment the value of the conductance, implemented as a series pair of two non-inductive variable resistors, is slowly changed.

The X and Y channels of an oscilloscope are connected to the non-earthed ends of C_1 and C_2. By this means one can observe the change from a period-one attractor, via a sequence of period-doubling bifurcations, to a spiral attractor, periodic windows and a double-scroll chaotic attractor.

A qualitative description of the behaviour demonstrated by Chua's circuit

The role of the *control parameter* in this circuit is taken by the conductance G. The active, or driving, part of the circuit has a natural frequency, ω, associated with it which depends on the specific reactance of its components. In addition, the I–V characteristic of the active, forcing part of the circuit is symmetric under a rotation of 180 degrees, about the origin, which has implications for the steady state solutions observed. Either the voltage across, or the current through, the passive part of the circuit may be measured as a function of time. Alternatively, both these quantities can be displayed simultaneously along the x–y axes of an oscilloscope.

When the apparatus is switched on, initially, one observes a *transient*, which settles to a *steady state* response. This steady state response is called an *attractor* of the system. This means that the responding, or passive, part of the system can take many different initial conditions which will all lead to the same steady state behaviour. For small or large values of the conductance we observe that the

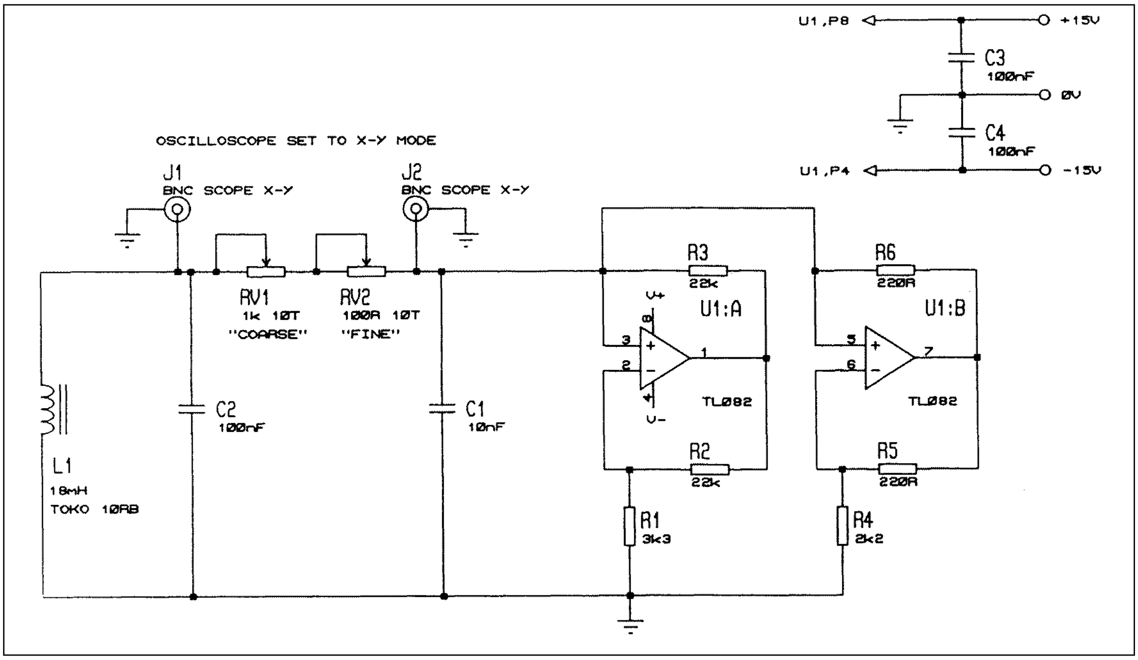

Figure 3. *A practical implementation of Chua's circuit.*

values of both the current and the voltage in the passive part of the circuit return to the same values periodically. In fact, they repeat with the same frequency, ω, as that of the driving force, and thus we say that we observe a *period-one attractor*. As the conductance is smoothly increased we observe a discontinuous change in the response, or a *bifurcation*. The period-one solution period doubles and thus the period of time over which the motion repeats is twice as long.

Period doubling is a phenomenon that is considered to be universal in nonlinear dynamical systems. The same behaviour is observed in many physical systems, for example the periodically forced simple pendulum and mathematical models, for example predator–prey dynamics. A *period doubling cascade* is a common route to the observation of chaos. Here, in Chua's circuit, as we smoothly increase the conductance we see the period-2 period double to a period-4 (figure 4(*a*)), similarly, the period-4 period doubles to a period-8. With infinite precision, we would expect to see an endless series of period-doublings as successively smaller and smaller increases in the control are made, with the limit being a cycle of infinite period, displaying all the characteristics of a *chaotic* attractor (figure 4(*b*)).

In practice, we have found that we can easily observe the period-2, period-4 and period-8. A further small change yields a single scroll chaotic attractor, its shape characteristic of Chua's circuit. An interesting, and counterintuitive, property of chaos is that coexisting with the chaos are an infinite number of periodic solutions of all periods. Careful tuning of the control parameter may result in the abrupt disappearance of the chaotic attractor, which is replaced, for a small interval, by a periodic solution. The chaotic attractor will suddenly reappear at the end of this interval. This phenomenon is known as the appearance of a *periodic window*. In Chua's circuit, we can observe a period-3 window (figure 4(*c*)), which itself period doubles, and three separate period-5 windows. The observation of these windows in itself is evidence that we have a chaotic solution rather than noise. As mentioned above, the nonlinear restoring force is symmetric under a rotation, about the origin, of 180 degrees. This means that two periodic attractors are available to the system. Which one the initial transient eventually settles to depends on the initial condition. As our control is increased we observe one final bifurcation. The attractor suddenly spreads to fill the entire region previously occupied by the two smaller attractors,

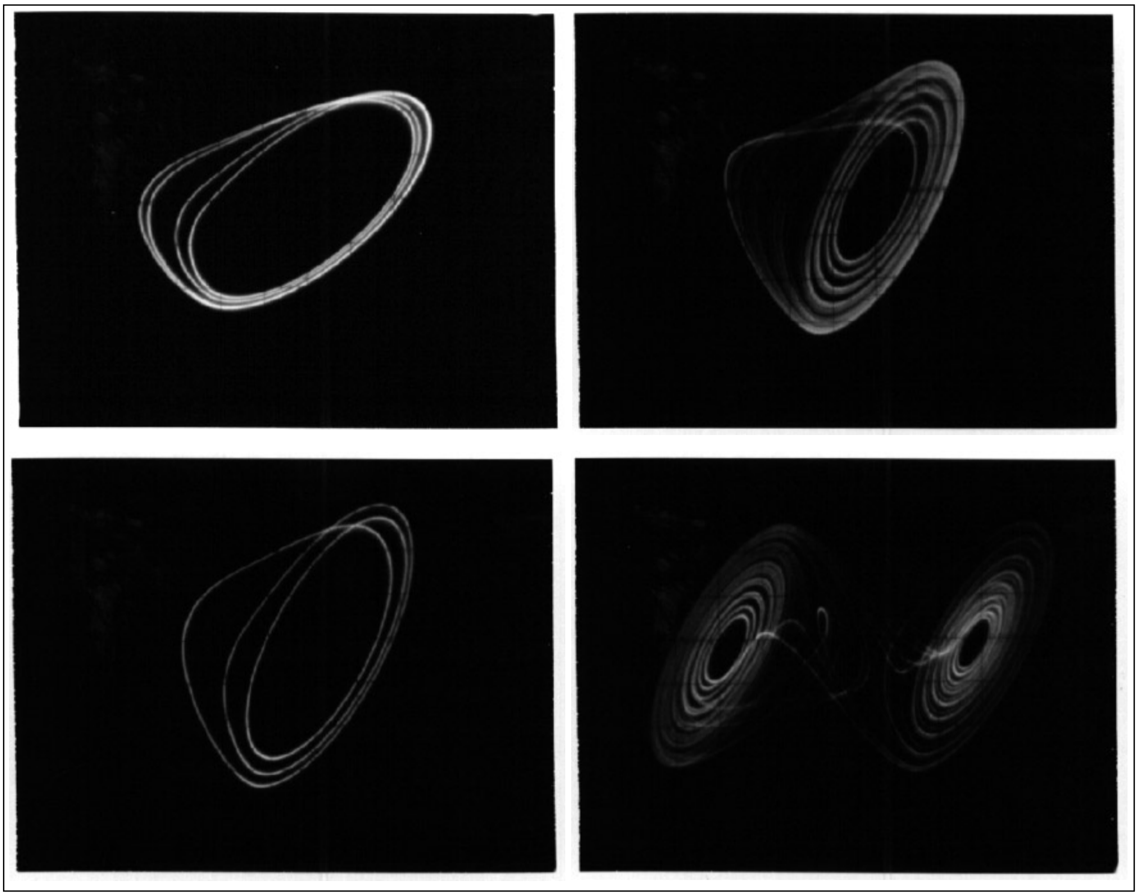

Figure 4. *(a) A period-4 attractor. (b) The single scroll attractor. (c) A period-3 attractor. (d) The double scroll attractor.*

producing the *double scroll chaotic attractor* (figure 4(*d*)).

Conclusions

Nonlinearity is a universal feature of the real world. In most cases the mathematical description of a physical system must be linearized (by a Taylor expansion for example) in order to give a set of equations that can be solved. It is thus valuable to illustrate in a simple system the important features that nonlinearity introduces. We have found Chua's circuit to be easy to implement, stable and robust in use, and an excellent experimental introduction to many aspects of chaotic dynamics. The circuit is a valuable teaching resource which illustrates all of the phenomena that arise in simple nonlinear systems.

A simple extension of this circuit (Murali and Lakshmanan 1992) enables the student to study the effect of an external sinusoidal excitation, and to map out a bifurcation diagram in the excitation-frequency versus excitation-amplitude plane. By pulsing the intensity of the oscilloscope beam (*Z*-modulation) at the driving source frequency, the phase-space is stroboscopically sampled, generating a display known as a Poincaré map.

References

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